

FOURTH EDITION

Mechanical Vibration

Analysis, Uncertainties, and Control



Haym Benaroya • Mark Nagurka • Seon Han

 **CRC Press**
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CRC Press

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Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

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CRC Press is an imprint of Taylor & Francis Group, an Informa business

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Printed on acid-free paper

International Standard Book Number-13: 978-1-4987-5294-7 (Hardback)

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PREFACE

Introductory Comments

From the natural world to the physical world, cycles are everywhere. We live our lives according to cycles as the days turn into weeks, the weeks into months, and the months into years. Cycles repeat at regular intervals and it is this repetitiveness that underlies the concept of *vibration*.

Humans are drawn to vibration as a source of comfort – from our earliest days being rocked to sleep to our later years rocking back and forth in a rocking chair. But, vibration can also be a source of great discomfort¹ with the negative effects of numbness, motion sickness, injury, and even death. Speaking subjectively, vibration can be good or bad depending on the circumstance.

When we don't want a cell phone to ring – an example of acoustic vibration – we put it in a mode called “vibrate.” We intentionally want it to vibrate. Yet, when our car vibrates too much we know from the shaking that something is wrong. It could be out of alignment; it could be the shock absorbers are shot; it could be the engine mounts are bad; it could be the engine is misfiring. A trained mechanic will be able to diagnose the problem based on the vibration signature. In this case, we intentionally want to eliminate the vibration, although we know the best we can do is minimize it.

Vibration is pervasive. In fact, it is challenging to find examples that are *not* related to vibration. Applications of vibration cross disparate disciplines, well beyond engineering. The following list is only a subset:

biology (pulse rate, respiration rate, balance, tremors, biorhythms, just to name a few²)

physics (waves, sound, quantum mechanics)

chemistry (atomic vibration,³ spectroscopy)

astronomy (planetary orbits,⁴ sunspot cycles)

geology (seismic tremors, earthquakes, volcanic eruptions)

oceanography (ocean waves, deep sea currents)

meteorology (climate and weather cycles)

zoology (predator-prey populations, host-parasite cycles, ecosystem cycles⁵)

psychology (seasonal cycles in behavior, sleep cycles, manic-depressive cycles)

economics (financial cycles, business cycles)

agronomy (agricultural cycles)

history (war and peace cycles, governance style cycles)

religion (holidays, life-cycle events)

philosophy (cycles in Eastern philosophy)

parapsychology (astrological cycles)

The word *vibration* is a common English word that means motion in oscillation. In a mechanical system, vibration can be viewed as a give-and-take (tug-of-war) between forces tending toward a balance. The forces are fighting each other, causing motion that seesaws back and forth, all the time trying to achieve equilibrium. This simplistic explanation is developed in much more technical rigor in this book.

We will learn that vibration occurs in a dynamic system that has at least two independent energy storage

¹Vibration exposure is more than just a nuisance. It can be a serious health hazard. Constant exposure to vibration has been known to cause a range of medical problems such as back pain, carpal tunnel syndrome, and vascular disorders. Vibration related injury is especially prevalent in occupations that require outdoor work, such as construction, farming, transportation, shipping, and forestry.

There are two classifications for vibration exposure: whole-body vibration and hand-and-arm vibration. Whole-body vibration is vibration transmitted to the entire body via the seat or the feet, or both, often through driving or riding in vehicles (including trucks, tractors, trains, and off-road vehicles) or by standing on vibrating floors (for example, near power presses in a stamping plant or near vibrating heavy machinery; pumps, compressors, air handling units and other equipment all contribute to the excitation of the floors). Hand-and-arm vibration, on the other hand, is limited to the hands and arms and usually results from the use of power hand tools (reciprocating and impact tools, jackhammers, grinders, woodchippers, etc.) and from vehicle controls.

²Almost everything in the body is rhythmic. ECG and EEG signals are rhythmic for a healthy person. Nerve action-potentials are cyclic. Eye blinking is cyclic, although the rate is not constant. Some rhythms in the body can be varied; many cannot.

³At an atomic level vibration means there is temperature.

⁴A year is the time for the Earth to rotate about the Sun once. A month is the time for the Moon to rotate about the Earth once. A day is the time for the Earth to rotate about its axis once.

⁵There are many interesting examples: snakes move by vibration and they sense by vibration; dogs shake water off their fur by vibration and they detect danger by vibration; hummingbirds flap their wings at known frequencies.

elements. We will develop models for vibration starting with a mass – which stores kinetic energy – and a spring – which stores potential energy. It is the interplay between the energies of dynamic systems that is at the root of vibration, even if we might not normally think about it this way.

We close these introductory remarks with a fact of life. Vibration is life.⁶ The absence of vibration is death. The more vibration, the more life; the less vibration, the less life.⁷ We wish you, the reader, a healthy, meaningful, and happy life, something that can only happen if it is filled with vibration!

Another Vibration Book?

The decision that another engineering textbook is needed must be made with great humility. That we have come to such a conclusion is in no way meant to be a rejection of other books. In fact, other books offer ideas and context that we do not. We have chosen to write this book in the format, content, and depth of description that we would have liked when we learned the subject for the first time.

Audience

Engineering requires several skills, including two of which are most fundamental: (a) the ability to read⁸ well and (b) a knowledge of mathematics.⁹

We have written the book assuming that the reader has mastered the first skill and has a basic knowledge of dynamics, mechanics of materials, differential equations, and some knowledge of matrix algebra. A review of some relevant mathematics is presented in [Appendix A](#).

Following our belief that textbooks written for students should present material with sufficient detail to be followed easily, we have included significant details in the formulations and in the explanations. The book is written at the level of the senior engineering student and intended for both undergraduate and graduate students

⁶How do you know if a person is alive? You check for a pulse, you check for breath. Similarly, in the inanimate world, vibration tells us if something is working. How do you know if a car is running? You check for engine vibration or noise from the engine and vehicle.

⁷Health is measured by vibrations in normal ranges. Illness is identified by vibrations out of the normal ranges, typically slowing (falling pulse, dropping temperature) and sometimes growing (racing pulse).

⁸The ability to read with comprehension is a fundamental skill for the engineer that takes effort. We encourage you to develop the habit of reading technical material with passion and to struggle, if necessary, to fully understand what is written. Only through deep reading will you be able to conquer new material and truly learn it. **We believe in reading.**

⁹Without a mastery of mathematics, you will struggle becoming and working as an engineer. (You will also be challenged in dealing with your personal finances.)

(in mechanical, civil, aerospace, and other engineering departments).

Although written primarily for use as a textbook for engineering students, it is also a useful reference for practicing engineers. The material is organized so that considerable flexibility is offered in arranging for course level, content, and self-study.

A considerable amount of thought, feedback, and effort has gone into preparing this revision. We have tried to make it straightforward to read and follow. We do not wish to imply, however, that the reader can delve into this book as if it were a novel. To derive any benefit from it, each page must be studied slowly and carefully.

Coverage

The purpose of this textbook is to present comprehensive coverage of the fundamental principles of vibration theory, with discussions and examples of the applications of these principles to practical engineering problems. In dealing with the subject of vibration, the engineer must also consider the effects of uncertainties in the analysis and methods of vibration control. As such, this book includes the subjects of modeling of uncertainties and vibration control. Of course, the focus of this text is vibration, and specialized texts on randomness and control are available.

Revisions

There are many changes and additions to this fourth edition. The written word has been examined many times to make the book more readable and clear. Derivations have been clarified and more details presented for completeness. New example problems and homework problems as well as new appendices on damping and MATLAB¹⁰ have been added. These, taken together with other enhancements, have made this a new book.

In rewriting this book our objective has been to present the subject in a clear and thorough way. We hope that we have succeeded.

Examples and Problems

The book has many examples. They have been carefully chosen and are presented at strategic points so that the reader will have a more complete understanding of the subject. Some of the example problems are relatively simple and their purpose is to illustrate new concepts. Some are more elaborate and designed to address more realistic and complicated problems.

A wide selection of problems is provided at the end of each chapter, grouped together by section. They range

¹⁰MATLAB is a registered trademark of The MathWorks, Inc.

from simple to challenging. Since engineers must be familiar with SI and U.S. customary systems of units, both systems are used in the examples and problems.

Biographies

The book includes detailed biographies of famous personalities who have contributed to the subject of vibration. We share these in the hope that the reader will appreciate that these individuals, like all of us, faced many challenges throughout their lives. Despite their hardships (most are never known¹¹) they were successful in making significant contributions. We view these individuals as role models and our teachers, even if we know them only by their contributions and through their biographies.

Many of the biographies were from the collection of Professors E.F. Robertson and J.J. O'Connor, School of Mathematical and Computer Sciences, University of St. Andrews, St. Andrews, Scotland. Their website is: <http://www-history.mcs.st-and.ac.uk/history/BiogIndex.html>. Other biographies were acquired from multiple sources, including Wikipedia.

We have benefitted from reading and preparing the biographies. We urge the reader to learn of the glorious history of the personalities of their profession.

Further Comments about the Book

This text is essentially self-contained. The reader may start at the beginning and continue to the end with rare need to refer to other works, except to find additional perspectives on the subject. But, no one text can cover all aspects of a subject as broad as vibration. Often, other works are cited where the reader will find additional information.

Although there is no separate list of references, the footnotes provide the reader with references to articles and texts. The references are a starting point; they are not meant to be all inclusive.

To indicate the end of an example, we have inserted the symbol ■. To distinguish between the text and the biographies, we have used horizontal lines and a slightly smaller font for the biographies. We have also separated

¹¹For example, Max Planck is known as the father of the quantum theory in physics. He introduced a quantum hypothesis to achieve agreement between his theoretical equations, which were based on the second law of thermodynamics and experimental data. Planck had a long and successful career in physics, and was awarded the Nobel Prize in Physics in 1918 “in recognition of services he rendered to the advancement of physics by his discovery of energy quanta.” But, Planck’s personal life was clouded by tragedy. His two daughters died giving birth, one son was killed in action in World War I, and another son, to whom he had been particularly close, was executed for his part in an assassination attempt on Hitler in July 1944.

stories and case studies from the text using horizontal artwork.

Writing this book has been an exceptional privilege and an enormous learning experience. We have spent a significant part of our professional careers learning the topics of the book. We remain lifelong learners and hope we are granted the gifts of mental facility, physical stamina, and time to continue studying this subject – and engineering in general – for the rest of our lives. And, we hope this learning leads to meaningful contributions to our profession and society.

Instructional Options

This book includes material that can be covered at undergraduate and graduate course levels. A first course is likely to omit the more advanced subjects such as variational approaches of [Chapter 5](#) and stochastic processes in [Chapter 9](#). A logical sequence of material has been presented in the chapters so that the instructor can leave out sections that do not fit into the course syllabus. These omitted topics can be studied in a second course, where more advanced topics can provide a broader perspective on vibration.

In particular, an undergraduate course could cover most of the introductory and background [Chapter 1](#), the single degree-of-freedom topics of [Chapters 2 to 4](#), and multi degree-of-freedom systems of [Chapter 6](#). [Chapter 9](#) on randomness and [Chapter 10](#) on feedback control provide the instructor with resources that permit a customized syllabus.

A second, usually graduate, course could briefly review [Chapters 1](#) through [4](#), introduce the subjects of randomness and control in [Chapters 9](#) and [10](#), and then focus on the variational techniques of [Chapter 5](#), and the multi degree and continuous systems of [Chapters 6](#) through [8](#), and [Chapter 11](#) on concepts of nonlinear vibration. The choices and emphasis depend on the level of preparation of the students and the curriculum of the institution. All the material contained in the book can be covered in a rigorous two-semester sequence.

A Special Note to Students

Like most things of meaning in life, the subject of vibration is not easy. Our goal in writing this textbook has been to help you learn the subject – but the book must be read and studied if the material is going to sink in. Nothing beats working through the examples and solving the problems to conquer the subject of vibration. Going through challenging material and struggling to understand it can be frustrating, but they are necessary steps in learning.

Little would be gained if this book could be read once – like a novel – and fully absorbed without much thought. It is only through the process of grappling with fundamental concepts of vibration that you can gain a level of understanding that will make the subject meaningful. Vibrations has many practical applications and we have tried to convey that sense throughout the book. Engineers, after all, work in the realworld, solving real problems that help real people.

It might seem that vibration is an abstract or theoretical discipline, especially seeing how much math there is here. Some of you may even be dismayed by the advanced level of math needed. Recognizing the value of math as a cornerstone of engineering is an important message that we hope the reader takes away. As we progress through the book, we will rely on differential equations to model vibrating systems. Elementary models built of discrete components (masses, springs, dampers) will be couched in terms of ordinary differential equations. Later models assuming continuous components (beams, shafts, rods) will be represented by partial differential equations.

MATLAB

Many of the examples and problems of the book have been solved using the software package MATLAB. For those interested, a set of MATLAB programs is included in [Appendix C](#). They were written for the first edition by Dr. Stephen Kuchnicki, and cover vibration applications introduced in the text. The code has been updated for this edition.

Closing

Despite our best attempts to catch all errors, there are bound to be some in the book. We welcome your feedback about errors, so they can be corrected in subsequent printings, as well as general comments, questions, and suggestions.

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Acknowledgments

This work has the digital fingerprints of a long list of helpful, insightful, and generous individuals to whom we owe deep gratitude. We are grateful to all who helped us along the way, including family, friends, as well as teachers and students, both past and present. We express our sincere appreciation to students who prompted us with thought-provoking questions. And, we are indebted to the authors of many books from which we began to learn and understand the concepts needed for a career in engineering.

We acknowledge and thank individuals who contributed to this edition: Dr. Vincent Prantil (Milwaukee School of Engineering, for suggesting many technical changes, meaningful feedback, and editing), Jonathan Slightam (Marquette, for proofreading and updating the MATLAB programs), Evan Kane (engineer, for information and photographs of vibration absorbers used on pipelines in Alaska), and Douglas Taylor (Taylor Devices, Inc., for generously providing information and photographs of large fluid dampers installed on buildings and bridges).

We have received a number of helpful suggestions from those who we taught from earlier editions of the text. We are especially grateful to Prof. Dr. Metin Gürgöze, Istanbul Teknik Üniversitesi, Istanbul, for his effort to provide us with corrections and suggestions.

We thank CRC Press for inviting us to pursue this fourth edition and our editors, Jonathan Plant and Nora Konopka, for their strong interest and support.

Acknowledgments: H. Benaroya

No project of this magnitude can be completed without the explicit and implicit assistance of others. My father Alfred, one of the best engineers I knew, spent many hours helping to make earlier editions of this book useful and readable to engineers, and provided valuable insights and suggestions. What I am today is, to a large measure, a result of his efforts and those of my mother, Esther. I am very grateful.

This work would not have been possible without the supportive environment provided by the Department of Mechanical and Aerospace Engineering and Rutgers University, for which I am appreciative.

My exposure to vibration began as a student with the

text **Introduction to Structural Dynamics** by J.M. Biggs, McGraw-Hill, 1964, and with the first edition of **Dynamics of Structures** by R.W. Clough and J. Penzien, McGraw-Hill, 1975. As a teacher, my learning continued with the following texts: **Elements of Vibration Analysis** by L. Meirovitch, 2nd ed., McGraw-Hill, 1986, **Analytical Methods in Vibrations** by L. Meirovitch, MacMillan, 1967, **Vibration Problems in Engineering** by S. Timoshenko, D.H. Young, and W. Weaver, Jr., 4th ed., John Wiley, 1974, and **Engineering Vibration** by D.J. Inman, Prentice Hall, 1994.

All these authors have set a standard for technical thoroughness, style, notation, and content against which I measure the success of our efforts. In particular, certain notation and approaches have been adopted that best provide the reader with an understanding and an appreciation of the subject. Since vibration is considered a classical subject, it is impossible to fully attribute the work of each author who has had a hand in the development of the discipline. But, this has been done as much as possible, and we regret any unintentional oversight.

For the third edition I welcomed as coauthor my lifelong friend Mark Nagurka. We have known each other for almost forty years and have often compared notes on what our professional and personal lives have offered us. I am grateful for this friendship and consider Mark to be the brother I never had. With the fourth edition we warmly welcome Seon Han, who coauthored with me our first edition of the book *Probabilistic Models*. Seon is a very talented dynamicist, and a good friend.

I thank my family for their love and support.

Acknowledgments: M. Nagurka

I have been blessed with a lifetime of friendships and professional collaborations that have been a source of immeasurable meaning in my life. The opportunity to join in the third edition was extended by my very special friend, Haym Benaroya, who I first met at the University of Pennsylvania in 1978. I could not have asked for a closer friend, and I am indebted and grateful for this lifelong connection.

Professional colleagues and students at Marquette University have provided me with an opportunity to grow and contribute. I thank my colleagues, especially my

close friend Richard Marklin Jr., and students for pushing me to learn more and, in so doing, grow professionally and personally.

I thank family and friends for their love and support.

Acknowledgments: S. Han

I thank Texas Tech University for providing me with a learning environment. I especially thank our former chairwoman, Jharna Chaudhuri, for her support for the last twelve years. I also thank my former mentor Jordan Berg for showing me by example that learning never stops and for getting me started with the Controls Laboratory and Mechatronics. I also thank Mechanical Engineering students at Texas Tech for filling me with inspiration by their passion for engineering.

I thank Haym Benaroya for my start in engineering and for the opportunity to work together on this fourth edition, whose subject matter means a lot to me.

I thank my family for their love that keeps me grounded.

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We “thank” the legal profession for subconsciously forcing us to add this section, and hope the reader understands that you may have to waste a lot of time on legal junk to be an engineer!

Read this book – and apply it – at your own risk.
Caveat emptor.



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Dedication

H. Benaroya

In honor and love of my mother Esther Benaroya, and the memory of my father, Alfred Benaroya, who devoted themselves to their children. Thank you.

In love of my wife, Shelley, and our children, Ana Faye and Adam Nathaniel, for all the wonderful times, and now our extended family, Liz and Tiffany.

In friendship and love of my sister, Dahlia, for all the years of shared experiences, her husband, Ron, and their son, Max, for being there. (And Zak too.)

Three of us

We also dedicate this book to all who pursue goodness, especially those who selflessly fight for our freedom. Everything we do, all that we treasure, ultimately rests on the shoulders of those who not only support freedom and its institutions with their words and actions – but with their lives.

M. Nagurka

In memory of my mother, Calma M. Nagurka, *z"l*, who always helped others and showered us with kindness. You are deeply missed.

In honor of my father, Jack Nagurka, a bigger-than-life role model of courage, respect, and seeing the positive, who instilled in us the pursuit of knowledge, justice, and faith.

In love of my wife, Anita L. Nagurka, for her compassion, devotion, and affection, and Hadar, Zohar, Adi, Jarrod, Bennett, and Reid and their partners and children for filling my life with inordinate happiness.

S. Han

In love of my family, Andrew Mosedale, James Mosedale, and Mark Mosedale.



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Chapter 1

Introduction and Background

“For it is wise to start simply.”

You are starting to read a book on the subject of engineering vibration. Vibration refers to the oscillatory dynamic response of mechanical systems, such as machines and mechanisms, and of structures, such as buildings and aircraft. Our concern is how such systems behave when undergoing motion, especially oscillatory motion.¹

What is Engineering?

Engineering is a profession of immense creativity and unsung contribution. Those who work in the profession have given us innumerable gifts – gifts that we all too often take for granted and on which we totally rely. As we look around almost all things we see and use are the results and contributions of engineers. Sadly, we rarely know the names of those who gave us these gifts.

Engineers use scientific knowledge to safely and economically solve practical problems to meet human needs. Engineers apply technical and scientific knowledge and utilize natural laws and physical resources in order to design and build structures, machines, devices, systems, and processes that safely meet desired objectives or criteria – for the benefit of people.

The broad discipline of engineering encompasses a range of specialized subdisciplines, each with a specific emphasis on certain fields of application and particular areas of technology. The marvel of engineering is one that is hard to capture in words. We decided to draw on three views that hopefully help complete the picture:

- “Engineering is one of the oldest examples of applied art – a unique union of specific need and specific design in a process that yields an engineered work or product. The role of engineering in providing for mankind’s material needs is as old as civilization and had its origin in the non-literate ages of man’s antiquity. Archaeology constantly provides us with

¹ A student once defined vibration as motion that cannot decide which way to go.

newly discovered examples of man’s early ability to provide for his own and society’s well-being through the creative act of design.”²

- “Engineering problems are under-defined, there are many solutions, good, bad and indifferent. The art is to arrive at a good solution. This is a creative activity, involving imagination, intuition and deliberate choice.”³
- “Scientists study the world as it is, engineers create the world that never has been.”⁴

Engineering is many things to many people. It involves analysis. It involves design. It is all about problem solving. Because engineers solve real-life problems, their ultimate motivation is to work toward making life better for everyone!

What is Vibration?

The subject of engineering vibration is a subset of the discipline of mechanics, the science of motion, forces, and time. Mechanics is comprised of statics – in which time is not a factor – and dynamics – in which time is a factor. Dynamics includes the special areas of kinematics and kinetics. Kinematics is the branch of classical mechanics that describes the motion of objects without consideration of the causes leading to their motion. It is called “the geometry of motion.” The other branch is kinetics, which studies the relationship between the motion of objects and its causes. The field of vibration encompasses both kinematic and kinetic problems of engineering mechanics, as depicted in [Figure 1.1](#).

² **A History of Engineering and Technology: Artful Methods**, E. Garrison, CRC Press, 1998.

³ This quote is attributed to Ove Arup, also known as Sir Ove Nyquist Arup (April 16, 1895-February 5, 1988). He was a leading Anglo-Danish engineer and the founder of the global firm Arup.

⁴ This quote is attributed to Theodore von Kármán. His biography appears in [Chapter 2](#).

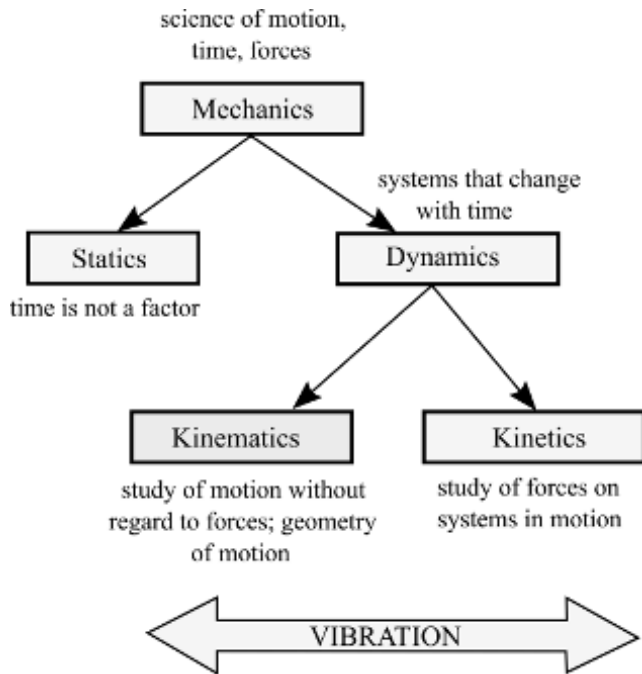


Figure 1.1: This high-level view of the discipline of mechanics shows the relation to the field of vibration.

The subject of vibration has a rich history. Its *modern* formal study is often attributed to Rayleigh⁵ who authored a two part monograph in 1877 and 1878.⁶ Many aspects of vibration, and the overlapping field of acoustics, predate Rayleigh and can be viewed as classical, having been pursued by mathematicians, physicists, and even physicians during earlier centuries.⁷

1.1 Challenges and Examples

Despite the subject's long history, problems from the world of vibration continue to challenge engineers. While the basic concepts of vibration are known, new applications force us to reconsider earlier and simpler formulations that were suitable for less demanding problems.

Consider, for example, the difficulty of designing structures to survive vibration in extreme environments such as in the ocean, in outer space, or in seismically active

regions. Or, consider next-generation machines made of advanced materials that are to operate in environments with very high temperatures and pressures. Will they work? Will they operate for a limited time and then fail? Will they vibrate out of control? Engineers continually push the envelope for performance of systems and structures in extreme environments. For successful designs that stand the test of time, engineers rely on computer simulation studies based on models as well as experimental tests.⁸

Engineers face a wide array of real-world challenges. To help motivate the subject, we identify several examples that “push the envelope” of design:

- Large offshore structures are designed for ocean depths over 300 m.⁹ An example is the Troll A platform shown in Figure 1.2. Some offshore structures use cables and tethers that extend through 1 km of ocean. These structures are designed to withstand very large forces due to ocean waves, currents, and winds and survive for many years of operational life. The analysis and design of such structures requires an in-depth (*no pun intended*) understanding of vibration to predict structural behavior. The ocean and wind environments are modeled using random vibration probabilistic models.
- Aircraft and spacecraft are extremely complex structures that present multiple engineering challenges. They are designed to minimize weight and maximize strength to withstand the severe forces of the atmosphere and space, respectively. Aircraft contain thousands to millions of mechanical, hydraulic, and electronic components under computer control that must work in concert, within the design specifications, to provide safe air travel.
- Since the 1950s spacecraft development has been an exciting and challenging pursuit for engineers.¹⁰ Structures that are designed to operate in space must be protected from the Earth's environment during their route into orbit and able to withstand severe temperature gradients in space. Figure 1.3 depicts the aerodynamics around the space shuttle. The vibrational demands on spacecraft have pushed engineering creativity in the areas of materials science and control. Engineers also ensure the integrity of the spacecraft on its way from the manufacturing facility to the launchpad and beyond.

⁵Lord Rayleigh's real name was John William Strutt. A biography appears later in Chapter 8.

⁶**The Theory of Sound**, Lord Rayleigh, Vol.1, 1877, Vol.2, 1878. Rayleigh's monograph, **The Theory of Sound**, was republished by Dover in 1945. Two books on the history of mechanics that include details on vibration may also be of interest: **A History of Mechanics**, R. Dugas, Dover, 1968, and **History of Strength of Materials**, S.P. Timoshenko, Dover, 1983. (Dover publishes inexpensive copies of older and out-of-print scientific and mathematical works.)

⁷A recommended short article on the relation between the natural sciences, mathematics, and engineering is by M.A. Biot, “Science and the Engineer,” *Applied Mechanics Reviews*, Vol.16, No.2, Feb.1963, pp.89-90. It remains timely and is interesting.

⁸Statistical methods for design are often a core part of engineering models. These methods are especially important for design in extreme environments due to the inherent complexities and uncertainties that cannot be reduced to definitive design constraints.

⁹In this textbook, we use both the SI system of units and English units. A discussion of units appears later in this chapter.

¹⁰**Dream Machines: An Illustrated History of the Space-**



Figure 1.2: The Troll A concrete platform for production of gas is located off the west coast of Norway. The platform was a televised sensation when it was towed into the North Sea in 1996, where it is operated by Statoil. Not only is it among the largest and most complex engineering projects in history, it is the largest object ever to be moved across the surface of the Earth. Troll A has an overall height of 472 m and weighs 656,000 tons. The bottom of the platform is on the seafloor 303 m below the surface. The walls of Troll A's legs are over 1 m thick. (Courtesy of Helge Hansen/StatoilHydro)



Figure 1.3: Computational fluid dynamics image of the space shuttle during launch showing surrounding flow. (Courtesy of NASA Ames Research Center)

- Structures and machines designed for the Earth's seismic regions must operate safely when subjected to earthquakes. For the most severe quakes, homes and buildings must not fail in a way that will harm occupants. A significant challenge is that the designer does not know in advance when an earthquake will occur or what its magnitude will be. The time of the *loading event* is of importance for structures such as office buildings that have different occupancy levels at different times of day. For an economical and safe design, engineering computer programs that include statistical methods are needed. Historical records are used to estimate input characteristics for a particular region in any earthquake-resistant design. Figure 1.4 shows damping elements that are installed on each floor of a building to dissipate energy.
- An example of an earthquake-resistant design is the 17-story Buddhist Headquarters building, shown in the photograph of Figure 1.5 taken during its construction in 2002 in Taipei, Taiwan. Vibration dampers were installed at each floor to dissipate energy in the event of an earthquake.¹¹

Problems in the field of vibration are ubiquitous. Applications crisscross the entire spectrum of engineering disciplines. One goal of this book is to introduce some of these problems and provide fundamental methods of engineering vibration analysis to solve them.

Vibration sensing is an important aspect of measurement. Seismology is the study of vibration and is primarily used for detecting and monitoring ground vibrations or earthquakes. Seismology is also used to study bomb blasts to determine signatures and locations, for example, to verify nuclear test ban treaties. In addition to detecting earthquakes deep in the earth, vibration sensing can be used to detect vibrations in engines, motors, and industrial machinery. Vibration sensing can also be used for diagnosis by monitoring the vibration history. By measuring vibration over time it is possible to identify problems that may be developing, before they become catastrophic.

There are many excellent textbooks on vibration including Meirovitch,¹² Weaver, Timoshenko and Young,¹³

ship in Art, Science, and Literature, R. Miller, Krieger Publishers, 1993.

¹¹ Each of the 60 dampers installed has a stroke of ± 178 mm and is capable of resisting a 981 kN load.

¹² **Elements of Vibration Analysis**, L. Meirovitch, 2nd ed., McGraw-Hill, 1990.

¹³ **Vibration Problems in Engineering**, W. Weaver, S.P. Timoshenko, D.H. Young, 5th ed., Wiley-Interscience, 1990.

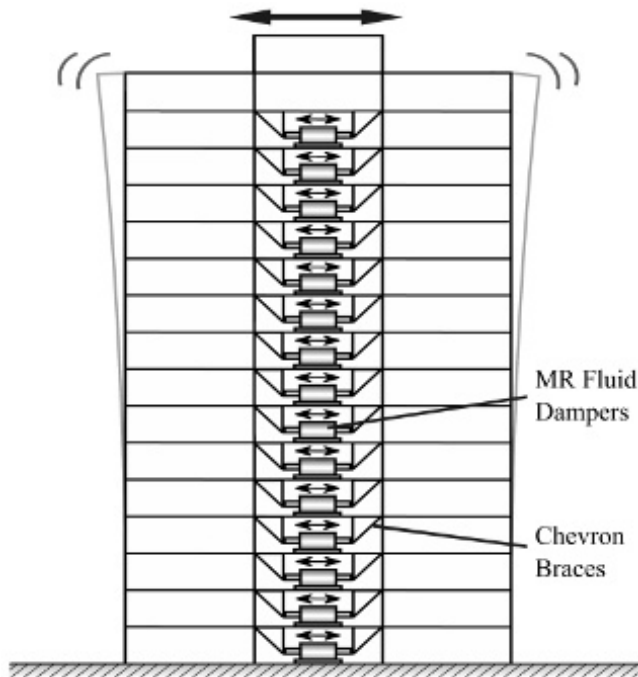


Figure 1.4: Building showing damping elements between floors. A magnetorheological (MR) damper contains a suspension of micrometer-sized magnetic particles in a carrier fluid. When subjected to a magnetic field, the fluid apparent viscosity increases greatly. The yield stress of the fluid can be controlled by varying the magnetic field intensity. Thus, the MR damper's ability to resist forces can be controlled with an electromagnet, offering many possible control-based applications. (Courtesy of Lord Corporation, Cary, North Carolina)

Inman,¹⁴ Thompson,¹⁵ and Kelley.¹⁶ Others are cited in the text. This book is written with an effort to build on these classics. The study of vibration in this book includes, in addition to classical vibration analysis, two much younger disciplines that are connected to vibration: *uncertainty modeling* and *vibration control*.

The study of classical vibration includes many simplifying assumptions. One such assumption is determinism, which means that parameters and system models are known. Although determinism is reasonable in most engineering applications, it cannot be used in problems where complex behavior exists, for example, in seismic analysis or the estimation of aerodynamic loads. In such cases, statistical scatter of data precludes a deterministic analysis. We do not know when an earthquake will occur or when a peak gust will appear. The best we can hope for is to be able to estimate the probability of occurrence.

¹⁴Engineering Vibration, D.J. Inman, Prentice Hall, 1994.

¹⁵Theory of Vibration with Applications, W.T. Thompson, 4th ed., Prentice Hall, 1993.

¹⁶Advanced Vibration Analysis, S.G. Kelley, CRC Press, 2007.



Figure 1.5: Large dampers installed on each floor at the Buddhist Headquarters in Taipei, Taiwan. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)

With structures being designed for extreme environments and for new applications, it has become necessary to consider how to minimize excessive vibration. Thus, the need to *control* vibrational motion has become increasingly important. Active control of structures is a relatively new field; it builds on the discipline of feedback control, transcending many fields. Much of the development of vibration control evolved from earlier developments in the aerospace and space communities, where control system design was integrated into the design of aircraft and rockets to meet performance requirements.

Our studies begin with simple deterministic models. These models permit us to understand the fundamentals of vibrating systems and to predict key features of their behavior. We then continue to more realistic and complex models. For example, we begin with the simple harmonic motion of a structure that can be represented by a single degree-of-freedom model. Later, we examine the effects of random loading and the need to control the vibratory behavior by passive and active means. We then investigate more detailed models involving multiple degrees of freedom leading to our final chapters on continuous systems (which theoretically have an infinite number of degrees of freedom) and on nonlinear vibration.

This book includes numerous example problems to help fix these ideas more firmly. With this grounding, more advanced problems become accessible. The remainder of this chapter provides a qualitative introduction to the topics of the following chapters.

1.2 Systems and Structures

The terms *systems* and *structures* are sometimes distinguished as follows: *structures* are used to represent particular elements such as beams and rods, mechanical components such as rotors, or a large-scale grouping of assembled components. The term *systems* is meant to be more abstract and general. All structures are systems, but not all systems are structures.

A *system* may be defined as a group of integrated items, behaving as a unit, such as automobiles, industrial machines, and computers. Systems may be engineered or they may exist naturally. Humans, plants, and animals are all natural systems, and are much more complicated and sophisticated than engineered systems. Understanding natural systems requires knowledge of biology, chemistry, and physics.

Engineering systems can range in scale from the miniscule, arising from nanotechnology, to the enormous, such as stadiums, bridges, and large structures. Oftentimes, the terms *mechanical systems* and *structural systems* are used interchangeably.¹⁷ We use both terms to represent structures or more general assemblies signifying that simple structural models can be utilized in studies of complex interconnected groups of components.¹⁸

System *modeling* is the process of approximating physical characteristics in terms of mathematical expressions, generally consisting of one or more equations. Mathematical analyses can then be brought to bear to “solve” the equations.¹⁹ Solving the equations eventually results in a prediction of model behavior. The engineer’s task is then to study these predictions and make sure they make sense physically. It is also to make sure that a design based on the predictions can be created and is safe.

Implicit in these definitions is the separation of the system from its environment. An *environment* is generally

viewed as external to the system, interacting with the system and being the source of inputs to the system that affect its behavior. The development of *linear* system models generally proceeds independently of environmental models. Once the vibrational characteristics of the system are determined, it is possible to examine how the system behaves in various environments of differing characteristics. Simple models are studied in [Chapters 2, 3, and 4](#). *Nonlinear* system models that require that system and environmental aspects be considered simultaneously are treated in [Chapter 11](#).

Linearity is significant because it simplifies the formulation and solution of problems. The assumption (that must be verified) that a system behaves linearly allows the engineer to define many characteristics about that structure.

The environment exerts loads or forces on the system or structure. We use the terms loads and forces interchangeably. However, a load is a more general term that refers to the input to the system.

1.3 Basic Concepts of Vibration

We will learn to idealize vibrating systems into reduced-order mathematical models, and to occasionally account for uncertainties in these models. Once the basics are understood, we will begin to understand how to modify the behavior of the system, or control its vibratory characteristics, so that it can perform as desired.

For example, what if an analysis predicts that a structure displaces so much that it contacts its surroundings? The analyst or designer has two options. One is to redesign the structure with new dimensions, masses, and stiffnesses. The other option is to generate forces — known as control forces — that act on the structure to balance the existing forces, resulting in an acceptable total displacement. These concepts are introduced in the early chapters, and discussed in more detail in [Chapter 10](#) on vibration control.

As analysts and designers,²⁰ our interests are with *physical* structures and systems. The key word here is physical, in the sense that the complicating aspects of the structure cannot be ignored. In design, we are challenged with a process that first requires a conceptualization of the necessary structure to be built. A prelim-

¹⁷Some engineering textbooks distinguish between the words *system* and *structure*, reserving *structure* to mean rigid, that is, no part is in motion relative to any other part. This special case, a useful model in statics and dynamics, is referred to as zero degrees of freedom. In reality, there are no perfectly rigid structures, as at some scale and load, there are always finite deformations.

¹⁸In addition to structural systems, there are “nonstructural” systems, such as those based on fluidic or electromagnetic principles.

¹⁹System modeling is generally very challenging and requires creativity and intuition. Modeling is as much an art developed after much practice and experience as it is a science. It is *not* following a cookbook!

²⁰An analyst is traditionally one who performs a mathematical analysis of a structure without necessarily specifying how the structure should be dimensioned or built. The designer has traditionally been the one who makes use of analytical results, whether from a computer analysis or by utilizing a design code, to specify dimensions and construction sequence. Sometimes analysis has taken on an air of being more sophisticated, and design more routine, but those familiar with both recognize the challenges of both. The actual situation is that design is integral to engineering and its greatest challenge. One may make a case that analysis is a subset of design.

inary concept that appears to satisfy the needs of the customer is stipulated by the designer. Then some approximate analyses are performed to get a sense of how well the structure satisfies the major constraints placed on its design, constraints that force it to be viable in its intended mission. If the conceptual structural design is still acceptable, then a full-scale analysis and design are required. Realistic and important aspects of the problem are retained here.

If the conceptual design points to some poor characteristics that may compromise the mission of the structure, then a redesign is needed, with appropriate changes made to alleviate any shortcomings. Experiments on scale models are usually necessary to gather data as well as to gain confidence that the mathematical model accurately predicts actual behavior.



Figure 1.6: A Ferris wheel rider experiences periodic motion in terms of altitude. The car undergoes pendular motion during the ride. (Photograph by M. Nagurka)

1.3.1 Modeling for Vibration

We have only hinted at how an engineer approaches a new problem. Many of the problems presented in this book do not reflect those initially encountered by a practicing engineer.

Problems in engineering practice are typically vague and ill-defined, and often formulated by a non-technical person who has a need for a product or a process. It is the engineer's task to take the ill-defined problem and turn it into a simpler, representative problem that can be solved using known or newly developed techniques. That formulation is used as the basis for a design, from which the system is built or manufactured.

1.3.2 Idealization and Formulation

The process of understanding how the physical world behaves and how it can be modeled generally begins with an

understanding of how much simpler systems and components behave. Such simpler systems are called *idealized systems* and the process by which they are determined and formulated requires a sophisticated understanding of the actual system. The idealized system is used to gain insight into the behavior of the real or physical system.

Idealization of the *physical system* is required because the real system is too complex to be modeled and analyzed, either mathematically or computationally. The analyst and designer must distill the essential properties of the original system in the creation of the idealized model. This model needs to be representative of the physical characteristics and behavior of the real system. We develop an understanding of the behavior of the real system based on the analysis and observation of the idealized system.

The following four examples provide an intuitive view of the process of model idealization. In these examples, the full structure is shown first, and then some possible idealized models are drawn alongside. The simplified models can be used to better understand some aspects of the behavior of the actual system.

Tower Structure Figure 1.7(a) shows a tower structure as it might appear and then its idealizations in three simplified models. The reference to degree of freedom (DOF) signifies the number of coordinates needed to define the position of all the masses in the idealization. Also in these simplified models are schematic elements that represent structural stiffness and damping, concepts that are introduced in this chapter. Models (b) is studied in Chapters 2 to 4, model (c) in Chapter 6, and model (d) in Chapters 7 and 8.

Automobile Figure 1.8(a) shows a schematic of an automobile, as well as four possible idealized models that are suitable for preliminary analyses. Models (b), (c), and (d) include separate modeling of the tire-suspension system. Model (b) is studied in Chapters 2 to 4. Models (c) and (d) are studied in Chapter 6, and models such as in (e) are studied in Chapters 7 and 8.

Rocket A rocket structure along with possible idealizations are shown in Figure 1.9. These simplified models represent axial extension as well as a bending motion of the rocket structure. Models of the type shown in (b) are studied in Chapter 6, and those such as (c) are studied in Chapters 7 and 8.

Turbine A schematic of a turbine is presented with its idealized model in Figure 1.10. The idealized model is conceived of as a rotating shaft supporting several rigid rotors. The rotors and the shaft initially may be assumed

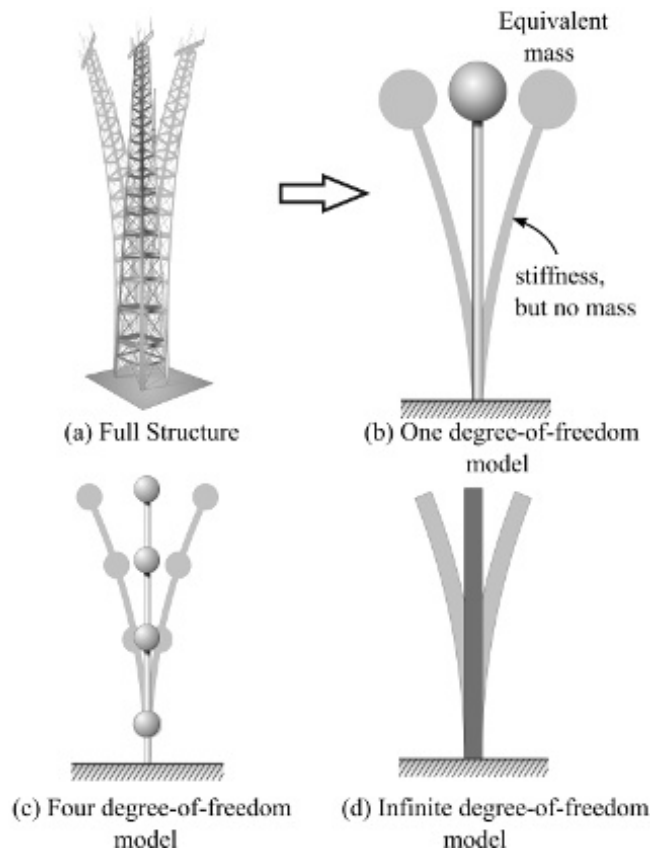


Figure 1.7: A tower structure and its idealized models.

to be rigid; a more realistic model may include elastic effects. Chapters 3, 4, and 6 explore such models.

These examples highlight an important engineering concept: more than one idealized model is suitable for studying the behavior of a full-scale structure. In engineering practice, simple models are created first in order to understand the general behavior of the full structure and provide preliminary design insights. Once the simpler models are understood, more complex and realistic models are used to begin conceptual designs of the full structure. The process is one of developing more realistic models, if needed, to match experimental data of the actual structure or machine.

World's Tallest Buildings

The tallest buildings in the world, both current and under construction, are listed in Table 1.1. Heights are measured to the structural height, which includes architectural elements, but not communication spires or antennas.²¹ (All measurements are according to the Coun-

²¹Until 1996, the world's tallest building was defined by the height to the top of the tallest architectural element, including spires but not antennas. This led to a rivalry between the Bank of Manhattan Building (now The Trump Building) and the Chrysler

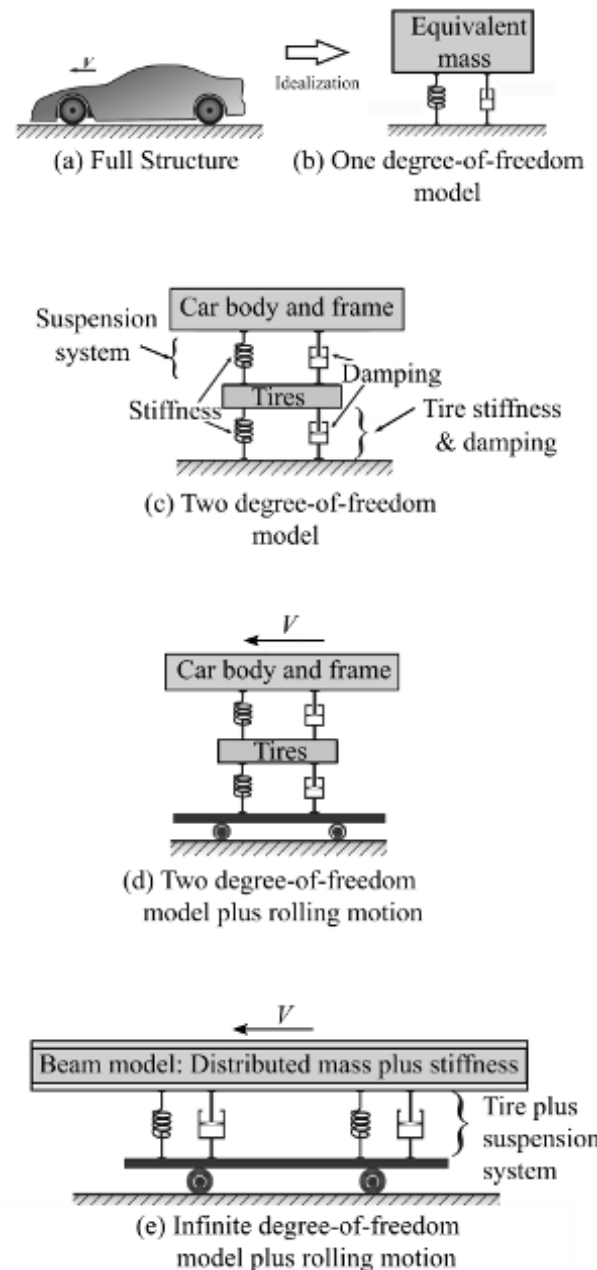


Figure 1.8: An automobile and its idealized models.

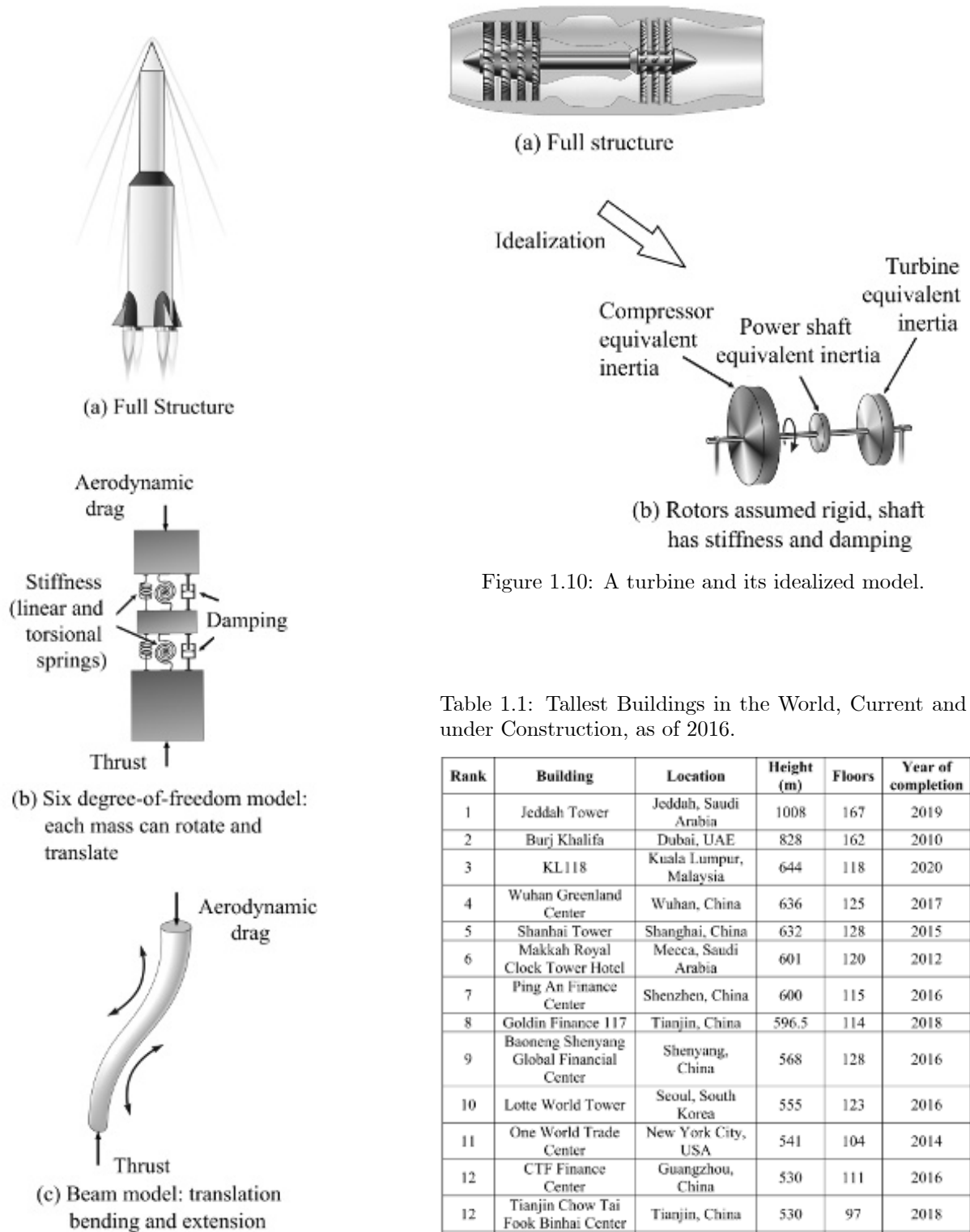


Figure 1.10: A turbine and its idealized model.

Table 1.1: Tallest Buildings in the World, Current and under Construction, as of 2016.

Rank	Building	Location	Height (m)	Floors	Year of completion
1	Jeddah Tower	Jeddah, Saudi Arabia	1008	167	2019
2	Burj Khalifa	Dubai, UAE	828	162	2010
3	KL 118	Kuala Lumpur, Malaysia	644	118	2020
4	Wuhan Greenland Center	Wuhan, China	636	125	2017
5	Shanghai Tower	Shanghai, China	632	128	2015
6	Makkah Royal Clock Tower Hotel	Mecca, Saudi Arabia	601	120	2012
7	Ping An Finance Center	Shenzhen, China	600	115	2016
8	Goldin Finance 117	Tianjin, China	596.5	114	2018
9	Baoneng Shenyang Global Financial Center	Shenyang, China	568	128	2016
10	Lotte World Tower	Seoul, South Korea	555	123	2016
11	One World Trade Center	New York City, USA	541	104	2014
12	CTF Finance Center	Guangzhou, China	530	111	2016
12	Tianjin Chow Tai Fook Binhai Center	Tianjin, China	530	97	2018
13	CenterZhongguo Zun	Beijing, China	528	108	2018
14	Dalian Greenland	Dalian, China	518	88	2018
15	Taipei 101	Taipei, Taiwan	508	101	2004

Figure 1.9: A rocket and its idealized models.



Figure 1.11: One World Trade Center (tallest building in center) as seen from the Hudson River in New York City. (Photograph by Joe Mabel, licensed under the Creative Commons Attribution - Share Alike 2.0 Generic license)



Figure 1.12: China will be home to 27 of the world's 50 tallest skyscrapers by 2020, with Shanghai being home to China's current tallest building. (Photograph by Pierre Selim, licensed under the Creative Commons Attribution - 3.0 Unported license)

cil on Tall Buildings and Urban Habitat.²²) The Jeddah Tower in Saudi Arabia at 1008 m (3307 ft) will become the tallest building in the world when completed in 2019. The 828 m (2717 ft) tall Burj Khalifa in Dubai has been the tallest building in the world since 2008. The One World Trade Center in New York is shown in Figure 1.11 and the buildings of Shanghai are shown in Figure 1.12.

There are significant engineering challenges to erect a kilometer tall building that is strong enough to withstand wind gusts. At that height, enormous wind loads would sway a conventional skyscraper like a tree branch and turbulent vortices would shake it so violently it could collapse. Creative designs are needed solve the problem of wind whipping around the uppermost stories and vibrating the building. The effect of these winds could be catastrophic. One solution to help mitigate the aerodynamic loads is to include vertical ailerons that run the length of the tower and are actively controlled to redirect the changing winds around the structure and scatter the vortices, thereby reducing vibrations.

The effect of vibration of a tall building can be studied using mathematical models.

THE FIVE SENSES. The five traditionally recognized physical senses are touch, hearing, sight, smell, and taste. Of these methods of perception, the first three are directly related to vibration. Much of what we experience in the world is conveyed through vibration.

Hearing. The sounds we hear are the result of molecules in the air banging into each other. So how is it that we hear these molecules? The eardrum vibrates when colliding molecules hit it. Tiny bones connect to the eardrum and transmit these vibrations along to the cochlea, a structure in the inner ear that contains fluid. The vibrations exert pressure on the fluid within the cochlea, and the organ of Corti, another structure within the inner ear, translates these changes in pressure into electrical impulses that travel along the auditory nerve to the brain, which then interprets these signals as sound.

Sound travels faster through liquids and solids than it does

Building. The Bank of Manhattan Building was 283 m (927 ft) tall with a short spire and a much higher top occupied floor. In contrast, the Chrysler Building was 319 m (1048 ft) including its 38 m (125 ft) spire secretly assembled inside the building to claim the title of world's tallest building. It had a lower top occupied floor and a shorter height when both buildings' spires were excluded. In terms of the highest usable floor, the Bank of Manhattan Building was the world's tallest. The observation deck in the Bank of Manhattan Building was nearly 30 m (100 ft) above the top floor in the Chrysler Building, whose spire was strictly ornamental and inaccessible.

At present, the Burj Khalifa tops the list by some margin, regardless of which criterion is applied.

²²The international non-profit organization Council on Tall Buildings and Urban Habitat (CTBUH) was formed in 1969 and announces the title of "The World's Tallest Building" and sets the standards by which buildings are measured. It maintains a list of the 100 tallest completed buildings in the world.

through gases. Since the molecules in liquids and solids are packed closer together than they are in gases, collisions between molecules happen faster. At 0°C (32°F), sound travels through air at 331 m/s, or approximately 740 miles per hour. Sound travels at 1450 m/s through liquid mercury and at 5640 m/s through solid glass. Sound can not travel through space because molecules are spread so far apart that they cannot interact. Without the molecular collisions there can be no sound.

1.3.3 Inertia, Stiffness, and Damping

The dynamic behavior of engineering systems results from the exchange and dissipation of energies. Dynamic forces transfer their energy to the system, which responds via several mechanisms, such as bending or extending. Dynamic behavior can be predicted using mathematical models. The best known method is Newton's second law of motion, discussed in a subsequent section. If the external force is static (or *quasi-static*²³), system forces develop to create an equilibrium. External dynamic forces are balanced in a more complex way with inertial, damping, and stiffness forces.

Inertia is the resistance of an object to a change in its state of motion. In common usage, the term “inertia” is used to refer to an object's “resistance to a change in velocity” – which is equivalent to its mass – or sometimes to its momentum (mass times velocity). In translational systems we use the term inertia synonymously with mass.

This is consistent with “the principle of inertia” as described by Newton in his First Law of Motion. (Newton's laws are discussed later in this chapter.) Expressed simply, the first law states that an object in motion that is not subject to any net external force moves at a constant velocity. The object will continue moving at its current speed and in its current direction until some force causes its speed and/or direction to change. The first law applies to an object that is not in motion, which will remain at rest until some force causes it to move.

Stiffness, a property of a spring, denotes the capacity of a system to store elastic or strain energy. The stiffness force follows *Hooke's law*,²⁴ $F = kx$, where the stiffness constant k is expressed in units of force per unit length and x is the elongation of the spring from its undeformed configuration. This is a linear model where spring extension is measured with respect to the undeformed length. More complicated laws exist, for example, the nonlinear

relation (see Section 1.5.1), $F = k(x)x^a$, where the exponent a depends on the particular material being modeled, and the stiffness parameter $k(x)$ is a function of the spring elongation x .

*Damping*²⁵ defines the ability of a system or structure to dissipate energy.²⁶ For an oscillatory system, damping is a measure of how much energy is dissipated by the system during an oscillation cycle.

1.3.4 Properties of Keyboard Keys

We all use computer keyboards, but probably have not given much thought to the mechanical properties of the keys. In this section,²⁷ we focus on computer keyboard keys. They serve as a readily-available example from which we can better understand the meaning of inertia, stiffness, and damping. The keys provide the tactile “feel” for typing.

The “feel” is often represented by the effective stiffness, damping, and mass of the key. The mass of the key is due to the key cap, whereas the stiffness and damping properties are based on the design underlying the key cap. The most common design, called a rubber-dome key, has a monolithic rubber dome under the key cap. It produces an initial resistance force against key depression, a restoring force after key release, and tactile feedback through a toggling or buckling-like action during which the rubber-dome gives way. Due to its behavior, this type key is also called a “snap action” key.

A complicated nonlinear relationship exists between key force and displacement for a rubber-dome key, as de-

²⁵ “Although both words (damping or dampening) are used interchangeably, it is commonly understood that when one needs to *dampen* a structure, he or she should reach for the nearest garden hose. But, if it is the conversion of mechanical energy into thermal energy for the purpose of reducing the mechanical vibration of a structure, *damping* is what you need. That there is confusion over this rather trivial grammatical point suggests a basic lack of understanding on the subject of damping among technical professionals.” (Paul Macioce, “Viscoelastic Damping 101,” *Sound and Vibration*, Vol.37, No.4, April 2003, p.10.)

²⁶ Certain nonlinear systems exhibit negative damping, where energy is actually supplied to the system. This situation is discussed in Chapter 11 on nonlinear vibration. An excellent book for further study of the subject is **Nonlinear Vibrations in Mechanical and Electrical Systems**, J.J. Stoker, originally published in 1950, and available through Wiley-Interscience in a 1992 edition. A more modern introduction is offered in **Nonlinear Oscillations**, A.H. Nayfeh and D.T. Mook, Wiley, 1979.

²⁷ The authors are grateful to Richard Marklin (Marquette University) for information related to computer keyboards. The following two papers provide more information about the studies and results presented in this section.

(1) M.L. Nagurka and R.M. Marklin, “Measurement of Stiffness and Damping Characteristics of Computer Keyboard Keys,” *ASME Journal of Dynamic Systems, Measurement and Control*, Vol.127, June 2005, pp.283-288.
 (2) M.J. Bufton, R.M. Marklin, M.L. Nagurka, and G.G. Simoneau, “Effect of Keyswitch Design of Desktop and Notebook Keyboards Related to Key Stiffness and Typing Force,” *Ergonomics*, Vol.49, No.10, August 2006, pp.996-1012.

²³ In quasi-static loading there may be time dependency but it is slow enough that inertial effects can be ignored.

²⁴ Hooke's law is a simple *constitutive law*, a mathematical model defining the relationship between material characteristics, force, and displacement. It is valid for linear elastic behavior.

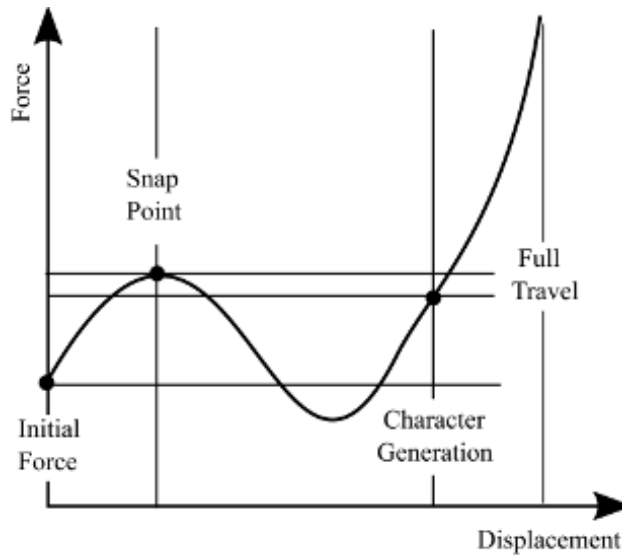


Figure 1.13: Relationship between force and displacement of a key on a rubber-dome computer keyboard.

pictured in Figure 1.13.²⁸ The force-displacement characteristic exhibits a regime of negative stiffness, indicated by decreasing force with increasing travel. Negative stiffness is not commonly encountered in mechanical springs, and is due here to the rubber-dome. Negative stiffness occurs after reaching the snap point, corresponding to the displacement at which the force has a local maximum. Electrical actuation of the key leading to character generation occurs at a force less than or equal to the snap point force. Eventually the key bottoms out when full deflection is reached and the force increases dramatically.

Static force-displacement graphs – giving the effective nonlinear stiffness – have been the primary tool to compare computer key characteristics. It is also possible to determine the force vs. velocity characteristics of keys, giving the effective damping. Figure 1.14 shows an example of the experimental results of a depression test of an Enter key on a rubber-dome keyboard. The best-fit line suggests a viscous damping relationship.

Damping may be important because typing is a dynamic activity. An individual typing at 90 words per minute (reasonable for a trained touch typist) is depressing and releasing each key in approximately 133 ms (assuming no time for finger travel between keys). Although an individual does not type at constant velocity, it is still possible to calculate an average key velocity. Typing at 90 words per minute corresponds to 60 mm/s on a keyboard with an assumed key travel of 4 mm. These calculations provide a lower bound for the key velocity in actual typing.

The results suggest that velocity-dependent damping

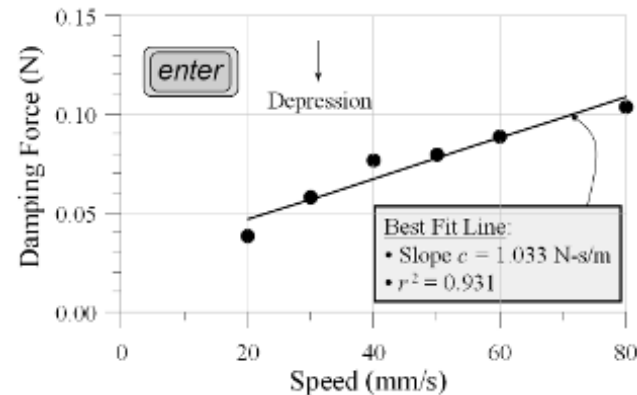


Figure 1.14: Damping force vs. speed for a depression test of the Enter key on a rubber-dome keyboard. The best-fit line has a slope of 1.033 N-s/m with a correlation coefficient of 0.931, suggesting that a straight line model is appropriate. A viscous damper has a linear relationship between force and velocity.

forces of keys augment the static (stiffness) forces and are more evident during higher speed typing. If the key mechanism follows a linear damping model, as implied by Figure 1.14, these forces would play a more dominant role at higher typing speeds. If this is the case, damping of the key mechanism would dissipate energy imparted to the key from the fingertip and would require that a typist exert greater contact force as the speed of depression increases.

To prevent key ringing and minimize unwanted oscillations, some damping may be desirable in the key mechanism. This damping would dissipate the energy imparted to the key suspension and the energy associated with accelerating the key mass during key strike and return. Significant damping would not be desirable, since it would increase the force exerted by the fingers during fast keystrokes. There may be a range of damping that offers a compromise in minimizing key vibration and muscle force for activation.

Key force is related to both key displacement and speed, as depicted in the three-dimensional graph of Figure 1.15. The relationship, summarizing the results of a series of experiments, is clearly nonlinear, making the mechanical modeling of computer keyboard keys challenging.

1.3.5 Computational Aspects

Engineers are called upon to tackle some of society's most difficult technical problems, each of which has considerable economic constraints. The complexities of such problems require engineers of many disciplines to work together to find effective, implementable solutions. So many factors and so much information must be tracked

²⁸ The figure is adapted from ISO/DIS 9241/4.2 Standard, 1998.

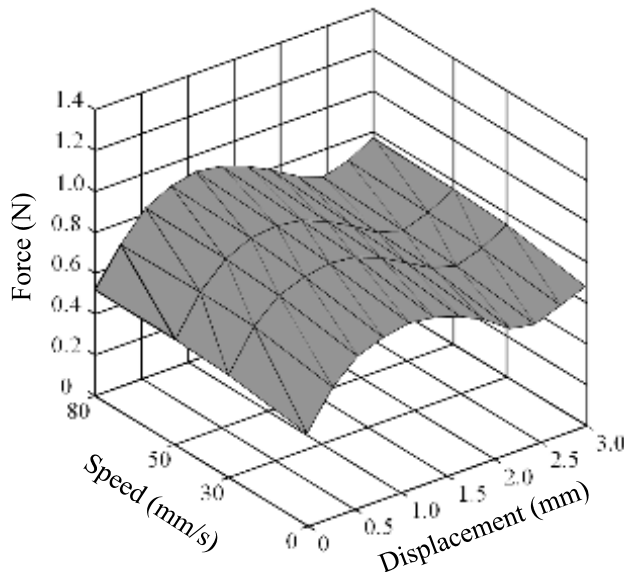


Figure 1.15: Key force as a function of both displacement and speed for depression tests of an Enter key on a rubber-dome keyboard. The three-dimensional surface summarizes the results of many tests and indicates the complicated nature of modeling the key force.

and integrated that the analysis, design, and management of such projects require computational assistance.

This book will provide the conceptual foundations of the subject of vibration. From this the engineer will be able to utilize computational tools, and then be able to interpret the computational results. We will occasionally hint at computational aspects of the subject under study. Certain computational tools that are readily available will be utilized²⁹ to generate some results presented in this book.

DISCOMFORT DUE TO VIBRATION. Vibration can often be heard as well as felt. It can be considered a type of pollution that can cause severe discomfort. The discomfort due to acoustic vibration, that is, noise, depends on its intensity and frequency, as well as other factors. People report different levels of discomfort to noise. (Even the disposition of a person influences the perceived discomfort of sounds.) Despite the subjectiveness, there are known intensities, frequencies, and durations of exposure that cause discomfort in most people.

Vibration transmitted to the human body by contact can also give rise to discomfort. The discomfort depends on the frequency and the duration of exposure, as well as other factors, such as the part of the body in contact with the source. Studies show that the discomfort is not as much a result of the amplitude of the vibration as the peak value (or the root

mean square value) of the acceleration.

1.3.6 Is Vibration Good or Bad?

When first studying engineering vibration, there may be an implicit belief that vibration is bad and must be removed at all costs. Certainly too much vibration – vibration with excessive amplitudes of oscillation – can be detrimental, even fatal. In general, vibration is not a desirable attribute in an engineering system. But, there are many positive examples of vibration.

Perhaps the most positive example of vibration is music, which is *acoustic vibration*.³⁰ Grandfather clocks rely on timing from the period of an oscillating pendulum. Jackhammers operate by putting vibration to good use in breaking up roads and concrete slabs. The vibration and noise produced by rumble strips are effective alarms for drivers leaving the roadway or needing to slow down. Paint shaking machines vibrate cans vigorously to mix paint at the point of purchase. And a vibrating massage chair makes us feel good.

In nature, we find examples of natural structures that vibrate under load. In a strong wind a tree oscillates with large amplitudes. In so doing, it survives the large loads by flexing. Had it been rigid, the tree might crack and fall. Instead it *goes with the flow*.

Engineers design structures so that they survive all anticipated environments and uses, and do so reliably for a specified period of time. This often involves designing structures so that they *go with the flow* – for example, designing tall buildings that intentionally sway due to wind gusts. If the future cannot be anticipated to an acceptable margin, and/or the system cannot be made to perform or operate as desired, then it may be necessary to build in the means to alter system properties to achieve the desired behavior automatically. This we do using a *control system*. We will see examples of structures which were designed with passive and active controllers later in the book.

Next, we look at manufacturing examples that intentionally use vibration to their advantage.

Vibratory Feeding and Finishing Parts orienting³¹ is often important in automation systems. In many mechanized production processes, parts produced by one machine are dropped randomly into a bin and then must be oriented properly before being fed into another machine

²⁹MATLAB programs for solving vibration problems are provided in [Appendix C](#). MATLAB is a registered trademark of Mathworks, Inc.

³⁰The simplest musical tones are sine waves at particular frequencies. More complicated musical sounds arise from overlaying sine waves of different frequencies at the same time.

³¹The authors are grateful to Vikram Cariapa and Thomas Silman (Marquette University) for information, photographs, and open access to the Mass Finishing Laboratory.

for further processing or assembly. For small parts one of the most common ways to accomplish this task is to use a vibratory feeder. Vibratory bowl feeders³² exploit vibration in a positive way to orient and sort parts.

In a vibratory bowl feeder, the parts are placed in the bottom of a bowl. By vibration of the bowl the parts spiral up the inside curved wall of the bowl. As each part climbs, it has to pass various stages which are designed to pass only single parts in the desired orientation.

The bowl employs an internal spiral track leading from the bottom up to the top edge. The bowl is vibrated vertically with an acceleration greater than that of gravity and at the same time given a rotational oscillation. The result is that parts are thrown up and land back on the track a little higher than the point at which they left. A system of gates and trapdoors near the top of the track ensures that any parts which are oriented incorrectly or are of incorrect dimension are thrown off the track to return to the bottom of the bowl. Only parts of the correct orientation and size are allowed to emerge. The rest are rejected back to the supply part of the bowl.

There are two types of vibratory bowl feeder drives: electromagnetic drives and free-piston pneumatic drives. Although there are distinct differences in performance, they each convey parts through vibration.

Watching a vibratory bowl feeder in operation is mesmerizing. The parts spiraling up seem to defy gravity. The added vibrational energy causes the part to go up the spiral each cycle more than it goes down. The key to the net upwards motion is a combination of the frictional coefficients, the bowl inclination angle, and the bowl vibrational amplitude and frequency.

Some products cannot be vibrated successfully due to their geometry. Others should not be vibrated since they will be damaged in the process. The vibratory bowl feeder is applicable to parts which are small, of suitable shape (for example, geometries that are not prone to get tangled or locked together), and able to withstand the considerable jostling that occurs while they are at the bottom of the bowl.

A related manufacturing operation which relies on vibration for its success is vibratory bowl finishing. Here parts are “finished” (ground, polished, cleaned) by abrading with or without media.³³ Vibratory bowl finishing can be used in final finishing (before electroplat-



Figure 1.16: A vibratory bowl finisher (Spiratron ST-1, Roto-Finish, Kalamazoo, MI) in the Mass Finishing Laboratory, Marquette University. (Courtesy of Vikram Cariapa. Photograph by M. Nagurka)

ing), to produce a matte surface (before chemical treatments), and to remove corrosion or residual paint (after stripping operations).

Vibratory finishing utilizes a rotating, eccentric weight system that dynamically oscillates an attached open-topped tub or bowl mounted on springs. Most machines employ an adjustable eccentric weight system, shown in Figure 1.16 as the squares above the center shaft to the right and left. By adding or removing weights the amount of vibrational energy available in an individual finishing operation can be adjusted. The vibrating action of the bowl causes the workload – the media and part(s) – to “flow” around the bowl in a twisting toroidal path. Figure 1.17 captures the vibratory bowl finisher in operation.

The forces generated by the vibrating action of the bowl and the resulting tumbling motion cause the media to rub against and abrade the exposed surfaces of the part(s). While it is a relatively simple machine requiring very little operator intervention during operation, it can be difficult to find the right combination of media, machinery settings for amplitude and frequency, and part sizes to operate effectively.

1.3.7 Vibration Control

Vibration control is the study of how to modify the response of a system to meet desired specifications. If a response exceeds certain tolerances, *appropriate forces*, called *feedback forces*, are needed to bring the response within acceptable levels.

Engineering systems, such as machines, and structures are designed to behave acceptably. This generally means displacement amplitudes must remain below a partic-

³²Design of Automatic Machinery, S.J. Derby, CRC Press, 2004.

³³The term *media* refers to the abrasive or nonabrasive consumable elements used in mass-finishing processes. The main function of media is to abrade or burnish the edges and surfaces of components to the desired finish. Media also helps keep the workpieces from impinging on each other and serve as carriers for any compounds used.

The media may be anything from ground corn cob to ceramic pieces (shown in photographs). The ratio of media to parts is a variable selected by the operator. Sometimes fluid compounds are added as well.



Figure 1.17: Vibratory bowl finishing in operation. Here a test was conducted to image the average dilated height of the media. It allowed the average density of the media to be determined during operation. (Courtesy of Thomas Silman and Vikram Cariapa. Photograph by T. Silman in the Mass Finishing Laboratory, Marquette University)

ular value, and operating frequencies must remain far from natural frequencies. Since operating conditions of a machine or structure may change with time, it becomes impossible to incorporate all the necessary constraints in the design *a priori* and feedback control may be the only option for an effective design.

Vibration control is a vast subject to which numerous books and papers are dedicated. Chapter 10 provides an introduction.

1.4 Types of Vibration

1.4.1 Signal Classification

Signals can be classified as either *deterministic* or *nondeterministic*. A deterministic signal can be described by an explicit mathematical relation. Its future behavior is predictable. A nondeterministic signal is a random signal. It has a time history that is unique, and its future behavior cannot be determined exactly but only to within some confidence limits.

Deterministic signals can be subclassified into *static* and *dynamic* signals, as shown in Figure 1.18.³⁴ In static signals time is not a factor. Static signals are steady in time; their amplitude remains constant. In dynamic signals time is a factor. The amplitudes are either *periodic* or *aperiodic*. A periodic signal repeats itself in

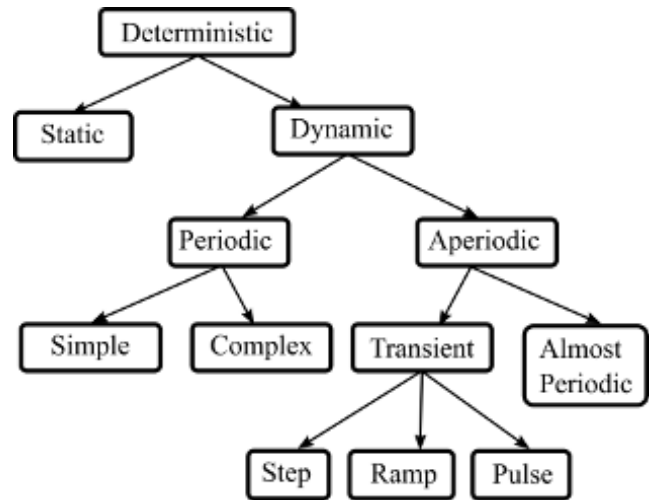


Figure 1.18: Classification of deterministic signals.

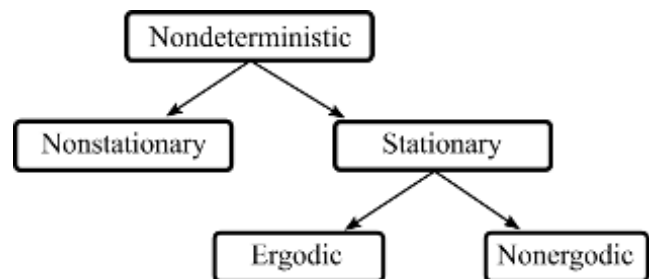


Figure 1.19: Classification of nondeterministic signals.

regular intervals; an aperiodic signal does not. A *simple periodic* signal has one period. A *complex periodic* signal has more than one period and can be built up of simple periodic signals. An *almost periodic* signal is comprised of two or more simple periodic signals with arbitrary frequencies. If the ratios of all possible pairs of frequencies are rational numbers, then an almost periodic signal is complex periodic.

Nondeterministic signals can be subclassified into *nonstationary* and *stationary* signals, as shown in Figure 1.19. In stationary signals, the statistical properties, such as the mean value and variance, do not change over time. The probability distribution is the same at all times. Stationary signals are either *ergodic* or *nonergodic*. In ergodic signals, statistical sampling can be performed at one instant across a group of identical signals or sampled over time on a single signal with no change in the measured result. In practice many random signals representing stationary physical phenomena are assumed ergodic.

1.4.2 Deterministic Approximations

For many applications, the material or geometric uncertainties or inaccuracies are extremely small when compared to their average values. In these cases, it is rea-

³⁴Figures 1.18 and 1.19 are adapted from two sources: (1) **Random Data: Analysis and Measurement Procedures**, J.S. Bendat and A.G. Piersol, 2nd ed., John Wiley, 1986, and (2) **Measurement and Data Analysis for Engineering and Science**, P.F. Dunn, McGraw-Hill, 2005, p.368.

sonable to use the average value, such as the average material modulus or average dimension, in a calculation. Errors resulting from ignoring any deviations from the average value will not appreciably affect the results of the analysis. Most of the structures we rely on have been designed in this way.

Sometimes it is not possible to distinguish in advance between uncertainties that can be ignored and those that must be included. Testing is required to provide a more complete understanding.

1.4.3 Probability

Probability is a much misunderstood and maligned subject that practitioners may try to avoid.³⁵ To some extent this is a reasonable reaction to a discipline that tends to be very formal and mathematical with connections rarely made to experience. The language is strange, and the outcomes are, after all, not certain.

The probabilistic approach is the general framework for the analysis of systems and structures since it explicitly accounts for all the possible loads and material properties. The analysis uses the *probability density function*, a concept introduced in [Chapter 9](#) as the fundamental building block of a probabilistic analysis.

The deterministic case is one of the cases built into the probabilistic model. The probabilistic framework provides a way by which uncertainties can be formalized and quantified so that vibration responses in the presence of uncertainties can be estimated.

A probabilistic analysis provides a measure of the underlying uncertainties, and, very importantly, helps determine the need for additional information or data before progressing to a design. In particular, we can develop confidence bounds on the values of parameters, and then the structural response. Such confidence bounds are a statement that parameter values are in a certain range with a certain probability.

1.4.4 System Model Uncertainty

All system models have associated uncertainties to bound the errors which occur in measurement and calculation. Uncertainties result from inaccuracies in manufacturing tools, instruments that measure data, and from an incomplete understanding of the laws governing the behavior of natural phenomena, such as wind and earthquakes.

Two basic approaches are available to the analyst confronted by a lack of information. The first approach assumes a deterministic model, where all parameters and dimensions are assumed to be known exactly. One then

takes into account the known imperfections by *overdesigning*, for example, by assuming less material strength than actually exists.

The second approach requires the use of probabilistic models to explicitly incorporate inaccuracies into the mathematical models that are utilized in analysis and design. This approach will be explored in more detail in subsequent chapters, in particular [Chapter 9](#).

Both approaches have their place in engineering practice. By far, the deterministic approach is most widely used. The need for very complex structures that must operate in severe environments has necessitated the introduction of probabilistic tools in analysis and design. Earthquake design codes, for example, are based on probabilistic criteria of structural behavior as well as environmental characteristics.

1.4.5 Random Vibration

Random vibration has evolved over the past half century as the discipline focused on the estimation of structural dynamic behavior in response to complex environments that are modeled based on probability. The mathematical techniques of the field are drawn from multiple disciplines including physics and electrical engineering, and were quickly adapted in aerospace engineering.

The need to minimize the mass of aircraft structures while designing them to be strong and reliable challenged engineers due to the nature of the airborne environment. Atmospheric characteristics were exceedingly difficult to characterize, being complex and unpredictable. Probabilistic tools were developed to model atmospheric behavior and random forces.

In time, these tools began to be applied to the design of structures in seismic zones where base excitation due to earthquakes had uncertainties in magnitude and timing, to skyscrapers responding to complex wind-related forces, and to offshore drilling platforms subjected to wind, wave, and current forces of sizable magnitude and uncertainty. There are many engineering applications.

1.5 Types of System Models

System models vary depending on the way a physical problem is idealized and the information required for the analysis. Models may be discrete or continuous, and linear or nonlinear. The nonlinear model will govern a broader range of dynamic behavior, but can require intricate analytical techniques. The continuous model is more realistic than the discrete one, but much more difficult to solve. Linear discrete and linear continuous models are considered in this text. [Chapters 7](#) and [8](#) introduce linear continuous vibrating systems such as strings, beams, membranes, and plates.

³⁵ Perhaps this is because probability is rarely introduced without confusing mathematical jargon and legalistic discussion.

1.5.1 Linear Approximation

Engineering analysis and design are heavily based on linear models. Two very good reasons for this are *the principle of linear superposition* and *linear systems theory*.

The principle of linear superposition³⁶ allows analysts to solve complex linear problems by breaking them down into simple linear problems. Linearity permits the addition of the solutions of these simple problems in order to obtain the complete solution.

Linear system theory is the general framework for the analysis of linear systems and structures. Thus, once it is determined that a system, whether mechanical, electrical, or otherwise, behaves linearly, then all the tools of linear system theory become available for analysis.

Linearity is generally a local phenomenon, meaning that for only small displacements the behavior can be viewed as approximately linear. For example, Hooke's law is valid as long as the spring is not stretched beyond a certain fraction of its initial length. The term "approximately linear" depends on the problem and the error that one can accept depends on the application. In some problems a nonlinear model is appropriate. One difficult aspect in the analysis of nonlinear systems is that, while there are unifying principles, many solution techniques are required, each for a particular problem. Linear solution techniques are applicable to any linear problem.

1.5.2 Dimensionality

Any model contains implicit (unstated) as well as explicit (stated and justified) assumptions. When a one-dimensional model is adopted, implicit assumptions are made that one coordinate is sufficient to describe the motion and that additional coordinates are not needed to understand the behavior of interest.

For example, the deformation of a rod in its axial direction is accompanied by a proportional contraction of its cross section that is given by Poisson's ratio. If we are primarily interested in tensile stresses within the elastic range, secondary effects such as the change in rod cross section can be ignored. For larger stresses, the reduction in cross section becomes a significant factor in the calculation of axial stresses.

The important conclusion is that one must always be aware of defined and *hidden* assumptions in any formulation.

³⁶This principle serves as the basis for the theory of linear differential equations. See [Appendix A](#) for a review.

1.5.3 Discrete Models

Computers³⁷ operate in discrete time and therefore require discretized models. Common methods used for such analyses are the *finite* and *boundary element* methods. Our initial studies focus on discrete models which help us to understand modeling, and rely on ideal characteristics such as inertia, stiffness, and damping.

Single Degree-of-Freedom Models

In a single degree-of-freedom model one coordinate is sufficient to describe the response of a system to an input. It reflects the dominance of one response parameter over all others. While such a system rarely exists in application, it proves to be a useful idealized model for learning the concepts of vibration as well as for gathering initial insights into the character of more complicated dynamic systems. Single degree-of-freedom models help us define and understand many of the key characteristics of a vibrating system and therefore prove to be valuable in preliminary studies. We will begin our studies of vibration with basic single degree-of-freedom models in [Chapter 2](#), and more advanced models in [Chapters 3](#) and [4](#).

Multi Degree-of-Freedom Models

If more than one coordinate is needed to define the behavior of a system, multi degree-of-freedom models are required. Generally, it is necessary to include as many degrees of freedom as there are distinct motions in a system. A continuous system has an infinite number of degrees of freedom in theory that are often modeled in practice using a finite number of degrees of freedom.

In many applications, it suffices to model the first several degrees of freedom to help understand the system. Other times, where very detailed behavior is required, many more degrees of freedom are required. The determining factor will be whether the analysis is used for preliminary studies or for a detailed design. For example, aircraft structures may have hundreds of thousands of components and, therefore, hundreds of thousands of degrees of freedom. To gain a basic understanding of aircraft structural vibration, models of less than ten degrees of freedom may be sufficient. We will study such models in [Chapter 6](#).

1.5.4 Continuous Models

Other engineering systems can only be represented with parameters that are distributed in space. Distributed parameter, or continuous, system models are necessary when the physical characteristics of a structure cannot be lumped accurately at discrete locations. Distributed

³⁷Historically, there were analog computers and digital computers. Today, references to computers imply digital computers.

parameter models of more than one variable are generally expressed mathematically in the form of partial differential equations. However, it should be noted that computer solutions of such models have to be discretized via specialized numerical techniques. We will also discover that many of our solutions for continuous models utilize results from discrete solutions. Such models are studied in [Chapters 7 and 8](#).

1.5.5 Nonlinear Models

This text focuses on models of systems where materials and geometries remain within linear ranges. Generally this means that oscillation amplitudes remain “small” and materials are within linear elastic bounds. Such models are very powerful, and carry with them a tremendous body of mathematical analysis for their solution.

However, there are important cases where a linear model will not be able to predict real-world behavior. Some applications cannot be approximated as linear and exhibit behaviors such as limit cycles, hysteresis, and chaotic oscillations that do not exist even as approximations in the linear domain. These phenomena require at least a “small” amount of nonlinearity in their governing equations.

Such problems are introduced in [Chapter 11](#), where the issue of stability is examined as well. Stability is a characteristic of the oscillatory behavior of a system, where the term stable implies that the system remains close in some sense to an equilibrium. Unstable implies that oscillations grow without bounds and make the system fail.

1.6 Basic Dynamics

1.6.1 Statics and Equilibrium

A *static* system, as distinguished from a dynamic system, is one that does not change as a function of time. Many engineering problems begin by considering the static behavior of the system. The concept of *equilibrium* implies a static situation, that is, one which is not time-dependent, although dynamic equilibrium is sometimes a useful concept.³⁸ A static structure resists external forces by deforming itself and/or by transmitting forces through connections to its environment. Dynamic implies time-dependent behavior.

1.6.2 The Equations of Motion

The step that follows the idealization of a system is the formulation or derivation of the equations of motion. The

³⁸D’Alembert’s principle, which we explore in [Chapter 5](#), is essentially a dynamic equilibrium form of Newton’s second law of motion.



Figure 1.20: The load carried by a crane displays pendular motion. The load exhibits oscillatory motion when the crane is moving the load. (Brookfield, WI; photograph by M. Nagurka)

simplicity of developing the governing equations and the ease in solving them are related directly to the system complexity and the effectiveness of its idealization. Thus, a delicate balance exists between physical reality and mathematical solvability. Small changes in the formulation of the idealized model may result in significantly more difficult governing equations.

Two distinct approaches are used in the derivation of the system governing equations: *Newton’s second law of motion* and methods based on the consideration of *kinetic and potential energies*. Each has its advantages. The first method is useful for simpler problems that permit visualization of the interaction between the forces acting on a system and the resulting internal stiffness and damping reaction forces. *Energy methods* are generally simpler to apply, especially for systems with many components that interact in an intricate fashion. Both approaches are used in this book.

Newton’s First and Third Law of Motion

Newton’s First Law of Motion states that every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.³⁹ New-

³⁹Newton’s first law of motion is often stated as: An object at rest tends to stay at rest and an object in motion tends to stay

ton's first law of motion predicts the behavior of objects for which all existing forces are balanced. From the first law – sometimes referred to as the law of inertia – if the forces acting on an object are balanced, then there is no acceleration of that object. Objects in equilibrium (the condition in which all forces balance) will not accelerate.

Newton's Second Law of Motion is treated in the following section in more depth as it is a cornerstone method of vibration analysis.

Newton's Third Law of Motion states that for every action there is an equal and opposite reaction. This statement means that in every interaction there is a pair of forces acting on the two interacting objects. The magnitude of the force on the first object equals the magnitude of the force on the second object. The direction of the force on the first object is opposite to the direction of the force on the second object. Forces always come in pairs – equal and opposite action-reaction force pairs.

Newton's Second Law of Motion

Newton's second law of motion is one of the most important physical laws and is fundamental to many disciplines. It states that the acceleration of an object is proportional to the applied force and inversely proportional to the object's mass. The mathematical form of this law, $F = ma$, is verified by way of experiments. Accordingly, an object will accelerate only if there is a net or unbalanced force acting on it. The presence of an unbalanced force will accelerate an object – changing either its speed, its direction, or both.

Consider the following ideal experiment⁴⁰ in which an object is subjected to a force F and the acceleration a is measured. Assume that all measurements are exact. The experiment is conducted in an *inertial* setting, that is, the experimental apparatus is not accelerating. If this experiment is carried out on the same object n times, we find

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \dots = \frac{F_n}{a_n} = \text{constant.}$$

The experiments for the same object yield the same constant quantity for the ratio between force and acceleration. This property of the object is called its *inertia*. It can be understood physically as the resistance of the object to a rate change in its velocity. For a given force, an object with a larger inertia has a smaller acceleration. For the same force, but for an object with a

in motion with the same speed and in the same direction unless acted upon by an unbalanced force. There are two parts to this statement: one predicts the behavior of stationary objects and the other predicts the behavior of moving objects. The behavior of all objects can be described by saying that objects tend to “keep on doing what they are doing” (unless acted upon by an unbalanced force).

⁴⁰This discussion is based on the well-developed introduction in **Engineering Mechanics: Dynamics**, J.L. Meriam and L.G. Kraige, 5th ed., Wiley, 2002.

smaller inertia, the acceleration is larger. The quantitative measure of inertia is the mass m .

Another conclusion drawn from the experiments is that the acceleration is always in the direction of the applied force. Mathematically, the equation relating force and acceleration is a vector relation. Newton's second law of motion takes the form

$$\mathbf{F} = m\mathbf{a},$$

where the bold notation indicates a vector. For one-dimensional motion, vector notation is not needed. If there are many forces, $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$, acting on the object then a vector summation of all the forces is taken,

$$\sum_{i=1}^n \mathbf{F} = m\mathbf{a}.$$

For an object of mass m with position defined by vector $\mathbf{r}$, the instantaneous velocity at time t is given by the limit

$$\mathbf{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt},$$

and the instantaneous acceleration at time t is given by

$$\mathbf{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} = \frac{d^2 \mathbf{r}}{dt^2}.$$

Velocity is the time rate of change of position and acceleration is the time rate of change of velocity.

Example 1.1 The V-2 Rocket⁴¹

The V-2 military rocket, used by Germany in World War II, weighed about 12 tons (that is, a mass of 12,000 kg) loaded with fuel and 3 tons (3000 kg) empty. Its rocket engine created a thrust of 240,000 N. What was the acceleration of the V-2 at launch and at burn-out, just before it ran out of fuel?

Solution: Let the upwards direction be positive, and the downwards direction negative. Using this convention, we can work with signed numbers rather than vectors. At launch, two forces act on the rocket: a thrust of +240,000 N, and the weight of the loaded rocket, $mg = -120,000$ N (if the thrust were less than 120,000 N, the rocket would not lift off). The total upwards force is therefore

$$F = +240,000 \text{ N} - 120,000 \text{ N} = +120,000 \text{ N},$$

and the initial acceleration, by Newton's second law of motion, is

$$a = \frac{F}{m} = \frac{120,000 \text{ N}}{12,000 \text{ kg}} = 10 \text{ m/s}^2 \simeq 1 g.$$

Thus, the rocket starts rising with the same acceleration as a stone falling. As the fuel is used up, the mass

⁴¹Courtesy of David P. Stern, <http://www.phy6.org/>.

m decreases whereas the force does not, so we expect a to grow. At burn-out, $mg = -30,000$ N and we have $F = +240,000$ N $-30,000$ N $= +210,000$ N, giving

$$a = \frac{F}{m} = \frac{210,000 \text{ N}}{3,000 \text{ kg}} = 70 \text{ m/s}^2 \simeq 7 g.$$

The fact that acceleration increases as fuel is burned is particularly important in manned spaceflight, when the “payload” includes living astronauts. The body of an astronaut given an acceleration of $7 g$ will experience a force up to 8 times his or her weight (gravity still contributes!), creating excessive stress ($3\text{--}4 g$ is probably the limit without special suits). It is hard to control the thrust of a rocket, but a rocket with several stages can drop the first stage before the acceleration gets too large, and continue with a smaller engine. Or, as with the space shuttle and the original Atlas rocket, some rocket engines are shut off or dropped, while others continue operating. ■

1.6.3 Linear Momentum and Impulse

The *linear momentum* of an object of mass m is defined as $\mathbf{p} = m\mathbf{v}$. Newton’s second law of motion can then be written for constant mass m subjected to n external forces as

$$\sum_{i=1}^n \mathbf{F}_i = \frac{d}{dt} (m\mathbf{v}).$$

If the sum of the forces equals zero, then the change in momentum is also zero. This is a statement of the *principle of conservation of linear momentum*. Equivalently, this equation can be written as

$$\int_{t_1}^{t_2} \mathbf{F} dt = m(\mathbf{v}_2 - \mathbf{v}_1), \quad (1.1)$$

where $\mathbf{F} = \sum_{i=1}^n \mathbf{F}_i$.

The concept of a large force acting over a short period of time, shown in the graph of Figure 1.21, has important applications. One example is the collision between elastic bodies. Collisions happen over time durations that are short compared to timescales of other dynamic characteristics. During collisions large forces act resulting in almost instantaneous changes in velocity and therefore in linear momentum.

Applying Equation 1.1 for the change in momentum due to an impulsive force during a collision yields

$$\begin{aligned} \int_{t=0^-}^{t=0^+} \mathbf{F} \delta(t) dt &= \Delta \mathbf{p} \\ &= m\mathbf{v}(0^+) - m\mathbf{v}(0^-), \end{aligned}$$

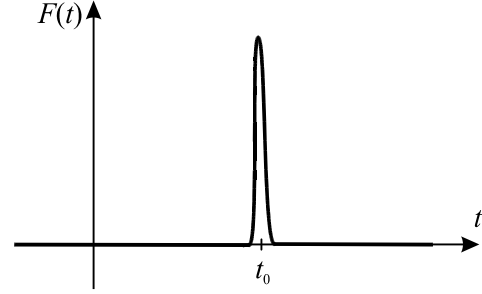


Figure 1.21: Large force acting over a short time duration.

where $\delta(t)$ is the Dirac delta function,

$$\int_{t=0^-}^{t=0^+} \delta(t) dt = 1,$$

and zero otherwise. Therefore,

$$\Delta \mathbf{p} = m\mathbf{v}(0^+) - m\mathbf{v}(0^-),$$

and recognizing that $\mathbf{v}(0^+) - \mathbf{v}(0^-) = \mathbf{v}(0)$, the initial velocity of the system,

$$\mathbf{v}(0) = \frac{\Delta \mathbf{p}}{m}.$$

The impulse response is the response of the system due to the initial velocity condition.⁴² This is discussed further in Section 4.3 in the context of an oscillator.

1.6.4 Principles of Work and Energy

The concepts of work and energy are fundamental to the study of dynamics and vibration. We consider energy formulations throughout this book, with many details provided in Chapter 5. Here some key definitions are introduced. Consider a particle of mass m moving along a curve C under the action of a force $\mathbf{F}$ as shown in Figure 1.22. (We follow the convention of showing vectors as boldface variables.)

The position of the particle with respect to origin O of coordinate system xyz with unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively, is given by the vector $\mathbf{r}$, which is a function of time. The work necessary to move the mass a distance $d\mathbf{r}$ is the scalar $dW = \mathbf{F} \cdot d\mathbf{r}$. The *work done* to move the particle from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ is the scalar W_{12} , defined as

$$W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}.$$

Assuming the mass of the particle is constant, Newton’s second law of motion can be written as

$$\mathbf{F} = m \frac{d\dot{\mathbf{r}}}{dt} = m \frac{d}{dt} \left(\frac{d\mathbf{r}}{dt} \right).$$

⁴²In classical mechanics, impulse is the integral of a force over the time interval for which it acts. Since force is a vector, impulse is also a vector in the same direction.

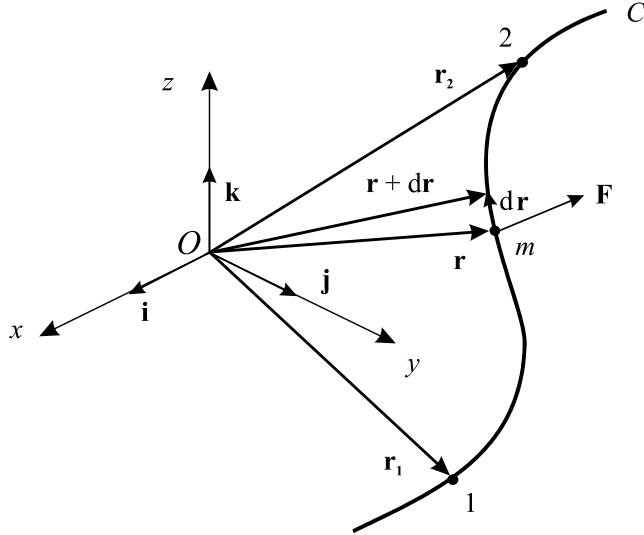


Figure 1.22: Path of particle of mass m due to force $\mathbf{F}$. Unit vectors, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are aligned along coordinate axes x , y , z , respectively.

To relate force, work, and energy, we use $d\mathbf{r} = \dot{\mathbf{r}}dt$, and the above equations, to find

$$\begin{aligned}
 W_{12} &= \int_{t_1}^{t_2} m \frac{d\dot{\mathbf{r}}}{dt} \cdot \dot{\mathbf{r}} dt \\
 &= \frac{1}{2} \int_{t_1}^{t_2} m \frac{d}{dt} (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) dt \\
 &= \frac{1}{2} m [(\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2) - (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1)] \\
 &= \frac{1}{2} m (\dot{r}_2^2 - \dot{r}_1^2) \\
 &= T_2 - T_1,
 \end{aligned}$$

where the limits of integration have been transformed from $\mathbf{r}$ to t , and T is the kinetic energy of the particle, $T = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$. As expected, we started with a scalar, the work, and ended with a scalar, the change in kinetic energy. The kinetic energy of a body is defined as the total work that must be done on the particle to bring it from a state of rest to a velocity $\dot{\mathbf{r}}$. For $v = \dot{r}$,

$$\begin{aligned}
 T_2 - T_1 &= \int_{v_1}^{v_2} m v dv \\
 &= \frac{1}{2} m (v_2^2 - v_1^2).
 \end{aligned}$$

Next, the work done by the force is related to the respective change in position of the mass. To do this, we define a *conservative force*, $\mathbf{F}_c$, as one that is a function of position, $\mathbf{F}_c = \mathbf{F}(\mathbf{r})$. The work done by a conservative force depends only on the *initial* and the *final* positions of the particle and is *independent* of the path connecting these positions. A *conservative force field* is one where the forces are conservative. Examples of conservative

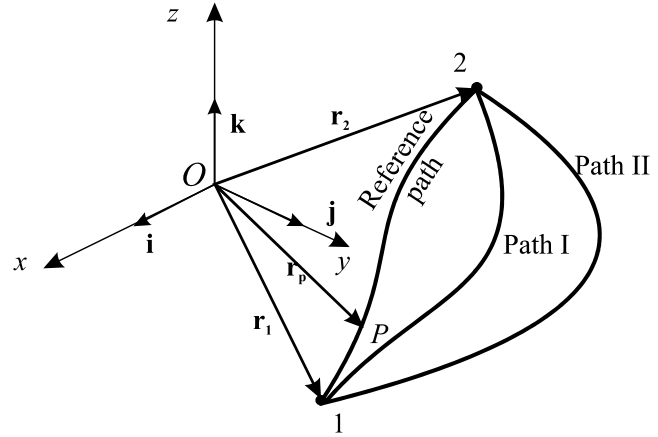


Figure 1.23: Path in a conservative force field.

forces are gravity, mg , and the spring force, kx . The work done by *nonconservative* forces is path-dependent and the change in total energy equals the work done by the nonconservative force.

From Figure 1.23, any path within the conservative force field which connects points 1 and 2 can be selected, and the work done bringing the particle from 1 to 2 will be the same, and is denoted by

$$W_{12c} = \underbrace{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c(\mathbf{r}) \cdot d\mathbf{r}}_{\text{Path I}} = \underbrace{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c(\mathbf{r}) \cdot d\mathbf{r}}_{\text{Path II}},$$

because by definition $\mathbf{F}_c$ is a function of at most $\mathbf{r}$.

The *potential energy* $V(\mathbf{r}_1)$ is associated with position $\mathbf{r}_1$ and is defined as the work done by a conservative force moving a particle from position $\mathbf{r}_1$ to a *reference position* $\mathbf{r}_p$,

$$V(\mathbf{r}_1) = - \int_{\mathbf{r}_p}^{\mathbf{r}_1} \mathbf{F}_c(\mathbf{r}) \cdot d\mathbf{r}.$$

We next relate the work done moving a particle in a conservative force field to the potential energy of the particle. To do this, we consider W_{12c} again but choose an arbitrary path through reference position $\mathbf{r}_p$. Then

$$\begin{aligned}
 W_{12c} &= \int_{\mathbf{r}_1}^{\mathbf{r}_p} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{r}_p}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \\
 &= \int_{\mathbf{r}_1}^{\mathbf{r}_p} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathbf{r}_2}^{\mathbf{r}_p} \mathbf{F} \cdot d\mathbf{r} \\
 &= -[V(\mathbf{r}_2) - V(\mathbf{r}_1)] \\
 &= -(V_2 - V_1).
 \end{aligned}$$

Thus, *the work done in a conservative force field is the negative of the change in potential energy*. From vector calculus, a conservative force equals the negative of the gradient of the potential energy function.

Gravitational potential energy is defined as the work done against the gravitational field to elevate a body of

mass m a distance h above an arbitrary reference (datum),

$$W_g = mgh,$$

where positive work is performed in the direction of the gravitational field.

If we denote W_{12nc} as the nonconservative work, then

$$\begin{aligned} W_{12nc} &= W_{12} - W_{12c} \\ &= (T_2 - T_1) + (V_2 - V_1) \\ &= (T_2 + V_2) - (T_1 + V_1) \\ &= E_2 - E_1, \end{aligned}$$

where E_i denotes the total energy in state i . Therefore, W_{12nc} is a measure of the change in particle energy due to dissipation, and if $W_{12nc} = 0$ then $E_2 = E_1$. In this case, the energy of the particle is constant and this is the *principle of conservation of energy*.

1.7 Units

1.7.1 Mars Orbiter Loss

The Mars Climate Orbiter⁴³ probe was launched on December 11, 1998 by NASA from Cape Canaveral in Florida. The complete burn sequence lasted 42 minutes bringing the spacecraft into a transfer orbit, with a final velocity of 5.5 km/s relative to Mars, and sending the probe into a 669 million kilometer trajectory.

NASA lost the Mars Climate Orbiter on September 23, 1999 because one engineering team used metric units while another used English units for a key spacecraft operation.⁴⁴ Information failed to transfer between the Mars Climate Orbiter spacecraft team at Lockheed Martin in Colorado and the mission navigation team in California. Lockheed Martin built the spacecraft.

“People sometimes make errors,” said Edward Weiler, NASA’s Associate Administrator for Space Science in a written statement. “The problem here was not the error, it was the failure of NASA’s systems engineering, and the checks and balances in our processes to detect the error. That’s why we lost the spacecraft.” This unfortunate event demonstrates the importance of working in consistent units, a lesson for all engineers.

An internal peer review panel at NASA’s Jet Propulsion Laboratory found that the failed information transfer scrambled commands for maneuvering the spacecraft to place it in orbit around Mars. “Our inability to recognize and correct this simple error has had major implications,” said JPL Director Edward Stone.

The spacecraft completed a nearly 10-month journey to Mars before it was lost. The navigation mishap

⁴³ “Mars Climate Orbiter Fact Sheet.” NASA-JPL. October 3, 2012.

⁴⁴ Based on web-posted story by CNN on September 30, 1999.

Table 1.2: SI and English/U.S. Units for Key Physical Parameters.

Parameter	SI	English/U.S.
Mass	kg	2.2046 lb _m = 0.0685 slug
Length	m	3.2808 ft
Acceleration	m/s ²	3.2808 ft/s ²
Force	N (kg·m/s ²)	0.2248 lb
Spring constant	N/m	6.8543 × 10 ⁻² lb/ft
Pressure	Pa (N/m ²)	1.4504 × 10 ⁻⁴ psi (lb/in ²)
Damping constant	N·s/m	6.8543 × 10 ⁻² lb·s/ft
Mass moment of inertia	kg·m ²	0.7375 lb·ft s ²

pushed the spacecraft dangerously close to the planet’s atmosphere where it presumably burned and broke into pieces, ending the mission on the day when engineers had expected to celebrate the craft’s entry into Mars’ orbit.

The total cost of the mission was \$328 million: \$193 million for spacecraft development, \$92 million for launching it, and \$43 million for mission operations.

1.7.2 U.S. Customary and SI Systems

All physical parameters have units that tie them to a particular system. There are two primary systems of units, the *English/U.S. System*, sometimes called the *U.S. Customary System*, and the *SI System*, where SI stands for *Système International*. The SI units, commonly known as metric units, are considered modern and used seamlessly across the globe. In this book, both systems are adopted, which is the practice in the United States.

In Table 1.2, the English and SI systems of units are shown for some physical parameters that we will encounter in this text. In Section 1.7.4, we will learn how the dimensions of parameters relevant to a particular problem can be used to derive equations that relate those parameters, and help direct the design of experiments.

In the SI system, the unit of force (Newton, N) is derived using Newton’s second law of motion from the base unit of mass (kilogram, kg) times the acceleration (meter per second per second, m/s²). Thus, the Newton has units N = kg·m/s². Acceleration due to Earth’s gravitational field is approximately 9.8 m/s². In the U.S. or English system, the unit of mass (slug) is derived from the unit of force (pound force, lb) divided by the unit of acceleration (foot per second per second, ft/s²). Thus, the mass units are slugs (lb·s²/ft). One slug equals 14.59 kg.

If we apply a horizontal force to an object sitting on a

frictionless horizontal plane, it will accelerate. A force of 1 N on a body of mass 1 kg accelerates it at 1 m/s^2 . A force of 1 lb acting on a body of mass 1 slug accelerates it at 1 ft/s^2 . A force of 1 lb (sometimes denoted lb_f for clarity) on a body of mass 1 pound mass (denoted lb_m) accelerates it at 32.2 ft/s^2 . Generally, to avoid confusion, the term lb is used for force and slug is used for mass in the English/U.S. system of units.

1.7.3 The Second

In 1967, the official definition of the second, also called the *atomic second*, was:

The duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.⁴⁵

The second was defined based on counting the time duration of something that vibrates or oscillates 9.2 billion times.⁴⁶ (With a frequency of 9.193×10^9 cycles per second or Hz, each period of radiation is 1.088×10^{-10} s.) The problem is that the definition of time seems to rely on the concept of time, making it circular.^{47,48}

During the 1970s it was realized that gravitational time dilation⁴⁹ caused the second produced by each

atomic clock to differ depending on its altitude. A uniform second was produced by correcting the output of each atomic clock to mean sea level, lengthening the second by about 1×10^{-10} s, or one period. This correction was applied at the beginning of 1977 and formalized in 1980.

The definition of the second was refined in 1997 to include the statement:

This definition refers to a cesium atom at rest at a temperature of 0 K.

The revised definition implies that the ideal atomic clock would contain a single cesium atom at rest emitting a single frequency. In practice, however, the definition means that for highest accuracy, atomic clocks should compensate for the effects of the ambient temperature (black-body radiation).

According to Newton, writing in the late 17th century, “time flows equably without reference to anything external,” meaning that the universe is equipped with a kind of built-in clock that ticks off seconds identically, regardless of location or epoch. This is an intuitive perspective of time.

In the early part of the 20th century, Einstein proposed a counter intuitive paradigm-shift that modified the Newtonian view of time. According to Einstein, the passage of time depends on circumstance and environment. He showed that the wristwatches worn by two individuals moving relative to one another, or experiencing different gravitational fields, tick off time at different rates. The passage of time, according to Einstein, is in the eye of the beholder.⁵⁰

Numerous experiments and astronomical observations leave no doubt that Einstein was right. Nevertheless, because Einstein’s concept of time becomes meaningful only at speeds near the speed of light or in strong gravitational fields (near a black hole), in the real world – in which engineers work – we adopt Newton’s concept of time and we rely on it for measurement of vibration.⁵¹

⁴⁵There are 54 electrons spinning around the nucleus of the cesium-133 atom. As the lone electron in the outermost shell spins it creates a tiny magnetic field. The nucleus spins also, creating its own magnetic field. When the two magnetic fields are spinning in the same direction, the total energy within the atom is slightly higher than when the two spins are in opposing directions. When the lonely electron flips its magnetic direction relative to the nucleus it emits or absorbs a tiny quantum of energy in the form of radiation with a frequency of 9,192,631,770 cycles per second. (*Rhythms of Life*, R.G. Foster and L. Kreitzman, Profile Books, 2004.)

⁴⁶It is interesting that we use oscillations to track the flow of continuous time. Using oscillatory motion to track time seems reminiscent of using a pendulum to measure time. As noted by Landes, “One would have expected something very different – that time, which is itself continuous, even and unidirectional would be best measured by some other continuous, even and unidirectional phenomenon.” (*Revolution in Time: Clocks and the Making of the Modern World*, D.S. Landes, Belknap Press, 2000.)

⁴⁷“The problem is that it is difficult, perhaps even impossible, to come up with a definition of time without using the word ‘time’ itself.” (*Rhythms of Life*, R.G. Foster and L. Kreitzman, Profile Books, 2004, paraphrasing B. Greene.)

⁴⁸“Perfectly regular cycles of motion implicitly involve a notion of time, since regular refers to equal time durations elapsing for each cycle.” (*The Elegant Universe: Superstrings, Hidden Dimensions, and the Quest for the Ultimate Theory*, B. Greene, W.W. Norton & Company, 2003.)

⁴⁹Gravitational time dilation is the effect of time passing at different rates in regions of different gravitational potential; the higher the local distortion of space-time due to gravity, the more slowly time passes. Einstein originally predicted this effect in his theory of relativity. It has been confirmed by tests that have demonstrated that atomic clocks at different altitudes (and thus different gravitational potential) show different times. The effects detected in such

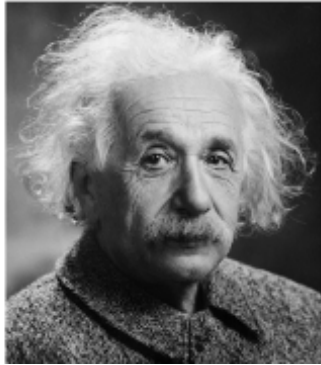
experiments are extremely small, with differences being measured in nanoseconds.

⁵⁰B. Greene, “The Time We Thought We Knew,” *The New York Times*, January 1, 2004.

⁵¹The Earth’s spin is not constant; it is gradually slowing down. (The Moon’s gravitational pull and other factors such as strong earthquakes affect how long the Earth takes to rotate on its axis.) Leap seconds are added occasionally to the Coordinated Universal Time (UTC), the global measure of time, to ensure that atomic clocks match the Earth’s rotation. Since 1972, the International Earth Rotation and Reference Systems Service (IERS) keeping track of the world’s time has added 26 leap seconds.

ALBERT EINSTEIN

(March 14, 1879 – April 18, 1955)



Contribution: Albert Einstein was a German-born theoretical physicist. He developed the general theory of relativity, one of the two pillars of modern physics (alongside quantum mechanics). Einstein's work is also known for its influence on the philosophy of science. Einstein is best known in popular culture for his mass-energy equivalence formula $E = mc^2$ (which has been dubbed "the world's most famous equation"). He received the 1921 Nobel Prize in Physics for his "services to theoretical physics," in particular his discovery of the law of the photoelectric effect, a pivotal step in the evolution of quantum theory

Albert Einstein was the embodiment of genius and the pre-eminent scientist of the modern age. His theories and discoveries have profoundly affected the way people view and understand the world. Einstein was also a philosopher and humanist who was keenly interested in the affairs of the world.

Life Sketch: Albert Einstein was born in Ulm, in the Kingdom of Württemberg in the German Empire in 1879. His parents were Hermann Einstein, a salesman and engineer, and Pauline Koch. In 1880, the family moved to Munich, where Einstein's father and his uncle Jakob founded Elektrotechnische Fabrik J. Einstein & Cie, a company that manufactured electrical equipment based on direct current (DC).

The Einsteins were a non-observant Jewish family. Albert attended a Catholic elementary school in Munich from the age of five for three years. In addition to his violin lessons, which he had from age six to age thirteen, he was given some religious education at home where he was taught Judaism. At the age of eight, he was transferred to the Luitpold Gymnasium (now known as the Albert Einstein Gymnasium), where he received advanced primary and secondary school education until he left the German Empire seven years later. He studied mathematics, in particular calculus, beginning around 1891.

In 1894, Hermann and Jakob's company lost a bid to supply the city of Munich with electrical lighting because they lacked the capital to convert their equipment from DC to the more efficient alternating current (AC) standard. The loss

forced the sale of the Munich factory. In search of business, the Einstein family moved to Italy, first to Milan and a few months later to Pavia. Einstein stayed in Munich to finish his studies at the Luitpold Gymnasium, where his father intended for him to pursue electrical engineering.

Einstein clashed with authorities and resented the school's regimen and teaching method. He later wrote that the spirit of learning and creative thought was lost in strict rote learning. At the end of December 1894, he traveled to Italy to join his family in Pavia, convincing the school to let him go by using a doctor's note. During his time in Italy he wrote a short essay with the title "On the Investigation of the State of the Ether in a Magnetic Field."

In 1895, at the age of 16, Einstein sat the entrance examinations for the Swiss Federal Polytechnic in Zürich (later the Eidgenössische Technische Hochschule, ETH). Passing would have allowed him to study for a diploma as an electrical engineer. Although he obtained exceptional grades in physics and mathematics, he failed to reach the required standard in the general part of the exam, foiling his plans.

On the advice of the principal of the Polytechnic, he attended the Argovian cantonal school (gymnasium) in Aarau, Switzerland in 1895-96 to complete his secondary schooling. His plan was to use this route to enter the ETH in Zürich. While at Aarau he wrote an essay (for which he was given barely passing marks!) that included his plans for the future, "If I were to have the good fortune to pass my examinations, I would go to Zürich. I would stay there for four years in order to study mathematics and physics. I imagine myself becoming a teacher in those branches of the natural sciences, choosing the theoretical part of them. Here are the reasons which lead me to this plan. Above all, it is my disposition for abstract and mathematical thought, and my lack of imagination and practical ability." Also while at Aarau he fell in love with Marie Winteler, the daughter of the family with whom he was lodging.

In January 1896, with his father's approval, Einstein renounced his citizenship in the German Kingdom of Württemberg to avoid military service. He was stateless for a number of years, until applying for Swiss citizenship in 1899 with citizenship being granted in 1901. In September 1896, he passed the Swiss Matura with mostly good grades, including top grades in physics and mathematics. Though only 17, he enrolled in the four-year mathematics and physics teaching diploma program at the Zürich Polytechnic.

Einstein's future wife, Mileva Marić, also enrolled at the Polytechnic, being the only woman among the six students in the mathematics and physics section of the teaching diploma course. Over the next few years, Einstein and Marić's friendship developed into romance, and they read books together on physics in which Einstein was taking an increasing interest. In 1900, Einstein was awarded the Zürich Polytechnic teaching diploma, but Marić failed the examination with a poor grade in the mathematics component.

Einstein graduated in 1900 as a teacher of mathematics and physics. Einstein tried to obtain a post after graduation, but was not successful. Three of Einstein's fellow students (including one named Grossman) were appointed assistants at ETH in Zürich. It appears Einstein had not impressed the faculty sufficiently to be offered an opportunity. In 1900 and 1901 he was writing to universities to obtain a job, and was not successful.

He did manage to avoid Swiss military service on the grounds that he had flat feet and varicose veins. By mid-1901 he had a temporary job teaching mathematics at the Technical High School in Winterthur. Around this time he wrote, "I have given up the ambition to get to a university."

Another temporary position teaching in a private school in Schaffhausen followed. The father of one of Einstein's fellow students (Grossman) tried to help Einstein get a job by recommending him to the director of the patent office in Bern. Einstein was appointed as a technical expert third class.

Einstein worked in this patent office from 1902 to 1909, holding a temporary post when he was first appointed. He evaluated patent applications for a variety of devices including a gravel sorter and an electromechanical typewriter. By 1904 the position was made permanent and in 1906 he was promoted to technical expert second class. Much of his work at the patent office related to questions about transmission of electric signals and electrical-mechanical synchronization of time, two technical problems that show up conspicuously in the thought experiments that eventually led Einstein to his radical conclusions about the nature of light and the fundamental connection between space and time.

While in the Bern patent office he completed an astonishing range of theoretical physics publications, written in his spare time without the benefit of close contact with scientific literature or colleagues.

Einstein and Marić married in January 1903. In May 1904, their first son, Hans Albert Einstein, was born in Bern, Switzerland. Their second son, Eduard, was born in Zürich in July 1910. In April they moved to Berlin. After a few months his wife returned to Zürich with their sons, after learning that Einstein's chief romantic attraction was his first and second cousin Elsa. They divorced on February 14, 1919, having lived apart for five years. Eduard, whom his father called "Tete" (for *petit*), had a breakdown at about age 20 and was diagnosed with schizophrenia. His mother cared for him and he was also committed to asylums for several periods, finally being committed permanently after her death.

In letters revealed in 2015, Einstein wrote to his early love, Marie Winteler, about his marriage and his still-strong feelings for her. In 1910 he wrote to her that "I think of you in heartfelt love every spare minute and am so unhappy as only a man can be" while his wife was pregnant with their second child. Einstein spoke about a "misguided love" and a "missed life" regarding his love for Marie.

Einstein married Elsa Löwenthal in 1919, after having had a personal relationship with her since 1912. She was a first

cousin maternally and a second cousin paternally. In 1933, they emigrated to the United States. In 1935, Elsa Einstein was diagnosed with heart and kidney problems; she died in December 1936.

In 1905 Einstein earned a doctorate from the University of Zürich for a thesis *On a new determination of molecular dimensions*. (His advisor was Alfred Kleiner, Professor of Experimental Physics, and he dedicated his thesis to Grossmann.) That same year, Einstein published four groundbreaking papers, on the photoelectric effect, Brownian motion, special relativity, and the equivalence of mass and energy, which were to bring him to the notice of the academic world, at the age of 26.

In the first of the papers, Einstein examined the phenomenon discovered by Max Planck, according to which electromagnetic energy seemed to be emitted from radiating objects in discrete quantities. The energy of these quanta was directly proportional to the frequency of the radiation. This seemed to contradict classical electromagnetic theory, based on Maxwell's equations and the laws of thermodynamics that assumed that electromagnetic energy consisted of waves containing a small amount of energy. Einstein used Planck's quantum hypothesis to describe the electromagnetic radiation of light.

Einstein's second 1905 paper proposed what is today called the special theory of relativity. He based his new theory on a reinterpretation of the classical principle of relativity, namely that the laws of physics had to have the same form in any frame of reference. As a second fundamental hypothesis, Einstein assumed that the speed of light remained constant in all frames of reference, as required by Maxwell's theory.

The third paper concerned statistical mechanics, a field that had been studied by Ludwig Boltzmann and Josiah Gibbs. Later in 1905 Einstein showed how mass and energy were equivalent. Einstein was not the first to propose all the components of special theory of relativity. His contribution is unifying important parts of classical mechanics and Maxwell's electrodynamics.

After 1905 Einstein continued working in the areas described above. He made important contributions to quantum theory, but he sought to extend the special theory of relativity to phenomena involving acceleration. The key appeared in 1907 with the principle of equivalence, in which gravitational acceleration was held to be indistinguishable from acceleration caused by mechanical forces. Gravitational mass was therefore identical with inertial mass.

In 1908 Einstein became a lecturer at the University of Bern after submitting his Habilitation thesis *Consequences for the constitution of radiation following from the energy distribution law of black bodies*. The following year he became professor of physics at the University of Zürich, having resigned his lectureship at Bern and his job in the patent office in Bern. By 1909 Einstein was recognized as a leading scientific thinker.

Einstein became a full professor at the German Karl-Ferdinand University in Prague in April 1911, accepting Austrian citizenship in the Austro-Hungarian Empire to do so. In 1911 Einstein made preliminary predictions about how a ray of light from a distant star, passing near the Sun, would appear to be bent slightly, in the direction of the Sun due to the Sun's gravity. This would be highly significant as it would lead to the first experimental evidence in favor of Einstein's theory. During his Prague stay, Einstein wrote 11 scientific works, five of them on radiation mathematics and on the quantum theory of solids. In July 1912, he moved from Prague to Zürich, returning to his alma mater in Zürich. From 1912 until 1914, he was professor of theoretical physics at the ETH Zurich, where he taught analytical mechanics and thermodynamics. He also studied continuum mechanics, the molecular theory of heat, and the problem of gravitation, the latter on which he worked with mathematician and friend Marcel Grossmann, expressing his work in terms of tensor calculus.

In 1914, Einstein returned to the German Empire after being appointed director of the Kaiser Wilhelm Institute for Physics (1914–1932), although he did not reapply for German citizenship. He accepted an impressive offer: a research position in the Prussian Academy of Sciences together with a chair (with no teaching duties) at the University of Berlin. He was also offered the directorship of the Kaiser Wilhelm Institute of Physics in Berlin, which was about to be established. In 1916 he was appointed president of the German Physical Society.

After a number of false starts Einstein published, late in 1915, the definitive version of the general theory of relativity. His prediction that light from another star should be bent by the Sun's gravity was confirmed by Sir Arthur Eddington during the solar eclipse of May 29, 1919. Those observations were published in the international media, making Einstein world famous. Einstein was idolized by the popular press. *The London Times*, the leading British newspaper, ran the headline on November 7, 1919: "Revolution in science - New theory of the Universe - Newtonian ideas overthrown."

In 1920, Einstein's lectures in Berlin were disrupted by demonstrations which, although officially denied, were almost certainly anti-Jewish. There were strong feelings expressed against his works during this period. Einstein replied in the press quoting Lorentz, Planck, and Eddington as supporting his theories and stating that certain Germans would have attacked them if he had been "... a German national with or without swastika instead of a Jew with liberal international convictions... ."

During 1921 Einstein made his first visit to the United States. One reason was to raise funds for the planned Hebrew University of Jerusalem. He received an official welcome by the mayor of New York City, followed by three weeks of lectures and receptions. He delivered lectures at Columbia University and Princeton University. At a lecture in a large hall overflowing with people at Princeton, he is reported to have commented, "I never realized that so many Americans

were interested in tensor analysis." In Washington he accompanied representatives of the National Academy of Science on a visit to the White House, and during his visit he received the Barnard Medal.

In 1922, Einstein was awarded the 1921 Nobel Prize in Physics for his 1905 work on the photoelectric effect, not for his work on relativity. He was not present in December 1922 to receive the prize, being on a voyage to Japan. The official citation was "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect." The general theory of relativity was still considered somewhat controversial! For some it was considered outlandish and it did not receive universal acceptance until the 1924 derivation of the Planck spectrum by S.N. Bose.

Around this time Einstein made many international visits. He visited Paris earlier in 1922 and Palestine in 1923. After making his last major scientific discovery on associating waves with matter in 1924, he visited South America in 1925.

Einstein's life had been hectic and in 1928 he collapsed brought on by overwork. He made a full recovery despite having to take life easier throughout 1928.

By 1930 he was making international visits again, back to the United States. A third visit to the United States in 1932 was followed by the offer of a post at Princeton. The idea was for Einstein to spend seven months a year in Berlin and five months at Princeton. Einstein accepted and left Germany in December 1932. The following month the Nazis came to power and Einstein was never to return to Germany.

While at American universities in early 1933, he undertook his third two-month visiting professorship at the California Institute of Technology in Pasadena. He and his wife Elsa returned to Belgium by ship in March, and during the trip they learned that their cottage was raided by the Nazis and his personal sailboat confiscated. Upon landing in Antwerp on March 28, he immediately went to the German consulate and turned in his passport, formally renouncing his German citizenship. A few years later, the Nazis sold his boat and turned his cottage into a Hitler Youth camp.

During 1933 Einstein traveled in Europe visiting Oxford, Glasgow, Brussels, and Zürich. Offers of academic posts, which he found so hard to get in 1901, came from Jerusalem, Leiden, Oxford, Madrid, and Paris.

What was intended only as a visit became a permanent arrangement. In 1935 he applied and was granted permanent residency in the United States. At Princeton his work attempted to unify the laws of physics. However, he was attempting problems of great depth and he wrote, "I have locked myself into quite hopeless scientific problems — the more so since, as an elderly man, I have remained estranged from the society here... ." Einstein was affiliated with the Institute for Advanced Study in Princeton until his death.

In 1940 Einstein became a citizen of the United States, retaining his Swiss citizenship. He made many contributions to peace during his life. In 1944 he made a contribution to the war effort by hand writing his 1905 paper on special

relativity and putting it up for auction. It raised six million dollars. The manuscript today is in the Library of Congress.

By 1949 Einstein was not well. A spell in a hospital helped him recover. He began to prepare for death by drawing up his will in 1950. He left his scientific papers to the Hebrew University in Jerusalem, a university which he had raised funds for on his first visit to the U.S., served as a governor of the university from 1925 to 1928, and had turned down the offer of a post in 1933.

On April 17, 1955, Albert Einstein experienced internal bleeding caused by the rupture of an abdominal aortic aneurysm, which had been reinforced surgically in 1948. He took the draft of a speech he was preparing for a television appearance commemorating the State of Israel's seventh anniversary with him to the hospital, but he did not live long enough to complete it. Einstein refused surgery, saying "I want to go when I want. It is tasteless to prolong life artificially. I have done my share, it is time to go. I will do it elegantly."

He died in Princeton Hospital early the next morning at the age of 76, having continued to work until near the end. Einstein was cremated at Trenton, New Jersey at 4 pm on April 18, 1955, the day of his death. His ashes were scattered at an undisclosed place.

Noteworthy: Einstein developed an appreciation of music at an early age, and later wrote: "If I were not a physicist, I would probably be a musician. I often think in music. I live my daydreams in music. I see my life in terms of music... I get most joy in life out of music."

Near the beginning of his career, Einstein thought that Newtonian mechanics was no longer enough to reconcile the laws of classical mechanics with the laws of the electromagnetic field. This led to the development of his special theory of relativity. He realized, however, that the principle of relativity could also be extended to gravitational fields, and with his subsequent theory of gravitation in 1916, he published a paper on general relativity. He continued to deal with problems of statistical mechanics and quantum theory, which led to his explanations of particle theory and the motion of molecules. He also investigated the thermal properties of light which laid the foundation of the photon theory of light. In 1917, Einstein applied the general theory of relativity to model the large-scale structure of the universe.

In July 1921, Einstein published an essay, "My First Impression of the U.S.A.," in which he tried briefly to describe some characteristics of Americans: "What strikes a visitor is the joyous, positive attitude to life The American is friendly, self-confident, optimistic, and without envy."

In 1922, his travels took him to Asia and later to Palestine (for 12 days, his only visit), as part of a six-month excursion and speaking tour. After his first public lecture in Japan, he met the emperor and empress at the Imperial Palace, where thousands came to watch. In a letter to his sons, Einstein described his impression of the Japanese as being modest, intelligent, considerate, and having a true feel for art.

Einstein's aversion to war led him to befriend author Upton Sinclair and film star Charlie Chaplin, both noted for their pacifism. He had an instant rapport with Chaplin, who on his first meeting at Universal Studios invited Einstein and his wife, Elsa, to his home for dinner. Chaplin said Einstein's outward persona, calm and gentle, seemed to conceal a "highly emotional temperament," from which came his "extraordinary intellectual energy."

In 1933, Einstein's works were among those targeted by the German Student Union in the Nazi book burnings, with Nazi propaganda minister Joseph Goebbels proclaiming, "Jewish intellectualism is dead." One German magazine included him in a list of enemies of the German regime with the phrase, "not yet hanged," offering a \$5,000 bounty on his head. In a subsequent letter to physicist and friend Max Born, who had already emigrated from Germany to England, Einstein wrote, "... I must confess that the degree of their brutality and cowardice came as something of a surprise." After moving to the U.S., he described the book burnings as a "spontaneous emotional outburst" by those who "shun popular enlightenment," and "more than anything else in the world, fear the influence of men of intellectual independence."

Einstein asked Winston Churchill to help bring Jewish scientists out of Germany. British historian Martin Gilbert notes that Churchill responded immediately, and sent his physicist friend Frederick Lindemann to Germany to seek out Jewish scientists and place them in British universities. Churchill later observed that as a result of Germany having driven the Jews out, they had lowered their "technical standards" and put the Allies' technology ahead of theirs.

On the eve of World War II, he endorsed a letter to President Franklin D. Roosevelt alerting him to the potential development of "extremely powerful bombs of a new type" and recommending that the U.S. begin similar research. This eventually led to what would become the Manhattan Project. Einstein supported defending the Allied forces, but largely denounced the idea of using the newly discovered nuclear fission as a weapon. Later, with the British philosopher Bertrand Russell, Einstein signed the Russell-Einstein Manifesto, which highlighted the danger of nuclear weapons.

For Einstein, "war was a disease ... [and] he called for resistance to war." By signing the letter to Roosevelt, he went against his pacifist principles. In 1954, a year before his death, Einstein said to his old friend, Linus Pauling, "I made one great mistake in my life—when I signed the letter to President Roosevelt recommending that atom bombs be made; but there was some justification—the danger that the Germans would make them"

Einstein's views about religious belief have been collected from interviews and original writings. He called himself an agnostic, while disassociating himself from the label atheist. He said "I believe in Spinoza's God, who reveals himself in the lawful harmony of all that exists" (*New York Times*, April 25, 1929). He said he believed in the "pantheistic" God of Baruch Spinoza, but not in a personal god.

Einstein published more than 300 scientific papers along with over 150 non-scientific works. On December 5, 2014, universities and archives announced the release of Einstein's papers, comprising more than 30,000 unique documents. Einstein's intellectual achievements and originality have made the word "Einstein" synonymous with "genius."

Among many honors that Einstein received were the Copley Medal of the Royal Society in 1925, the Gold Medal of the Royal Astronomical Society in 1926, and the Max Planck Medal in 1929. He was selected by *Time* magazine as the "Person of the Century" in 1999.

Not long after settling into his career at the Institute for Advanced Study in Princeton, he expressed his appreciation of the meritocracy in American culture when compared to Europe. He recognized the "right of individuals to say and think what they pleased," without social barriers, and as a result, individuals were encouraged, he said, to be more creative, a trait he valued from his own early education.

Einstein was a passionate, committed antiracist and joined National Association for the Advancement of Colored People (NAACP) in Princeton, where he campaigned for the civil rights of African Americans. He considered racism America's "worst disease," seeing it as "handed down from one generation to the next." As part of his involvement, he corresponded with civil rights activist W.E.B. Du Bois and was prepared to testify on his behalf during his trial in 1951. When Einstein offered to be a character witness for Du Bois, the judge decided to drop the case.

The discovery and publication in 1987 of an early correspondence between Einstein and his future wife Marić revealed that they had had a daughter, called "Lieserl," born in early 1902 in Novi Sad where Marić was staying with her parents. Marić returned to Switzerland without the child, whose real name and fate are unknown. Einstein probably never saw his daughter. The contents of his letter to Marić in September 1903 suggest that the girl was either adopted or died of scarlet fever in infancy.

There have been claims that Einstein collaborated with his wife Marić on his celebrated 1905 papers, but historians of physics who have studied the issue find no evidence that she made any substantive contributions.

After the death of Chaim Weizmann, the first president of Israel in 1952, Prime Minister David Ben-Gurion offered Einstein the position, a mostly ceremonial post. The offer was presented by Israel's ambassador in Washington, Abba Eban, who explained that the offer "embodies the deepest respect which the Jewish people can repose in any of its sons." Einstein declined, and wrote in his response that he was "deeply moved" and "at once saddened and ashamed" that he could not accept it.

One week before his death Einstein signed his last letter. It was a letter to Bertrand Russell in which he agreed that his name should go on a manifesto urging all nations to give up nuclear weapons. It is fitting that one of his last acts was to argue, as he had done all his life, for international peace.

During his autopsy, the pathologist of Princeton Hospital removed Einstein's brain for preservation without the permission of his family, in the hope that neuroscience in the future would be able to discover what made Einstein so intelligent.

Einstein's wise and humorous quotations, letters, and articles are widely used throughout popular culture as well as in historical and academic works. His name and image are instantly recognizable everywhere in the world.

Here are a few quotes from Einstein:

"Imagination is more important than knowledge."

"The important thing is not to stop questioning. Curiosity has its own reason for existing."

"Two things are infinite: the universe and human stupidity; and I'm not sure about the universe."

"Anyone who has never made a mistake has never tried anything new."

"Science without religion is lame, religion without science is blind."

"The significant problems we have cannot be solved at the same level of thinking with which we created them."

"Try not to become a man of success, but rather try to become a man of value."

"The most beautiful thing we can experience is the mysterious. It is the source of all true art and science."

"Great spirits have always encountered violent opposition from mediocre minds."

"Everything should be made as simple as possible, but not simpler."

1.7.4 Dimensional Analysis

Physical variables possess units. Therefore, for any equation to be true it must have consistent units. Just as the numbers on both sides of the equality must be equal to each other, the same is true for the units. *Dimensional analysis* is a method that seeks to identify significant *dimensionless ratios* or variables in a problem. These dimensionless groups are used because relationships derived for such ratios are independent of a particular set of units. Sometimes the process of identifying these groups leads to important but unnoticed relationships linking particular variables. Examples include the *Reynolds number* in fluid mechanics and the *viscous damping factor*, introduced in the next chapter, a very important parameter in vibration studies.

It is possible to use the units of variables that we expect to be part of a particular analysis to derive analytical relationships between those variables. This may appear remarkable, but, as mentioned above, the units must also satisfy the equations. Essentially, the units of the variables provide us with additional equations that must be satisfied.

The problems we study in this book, and in the physical sciences in general, are formulated in terms of only

a few physical variables. Examples include mass, velocity, density, force, momentum, temperature, and stress. Other disciplines, such as electricity and magnetism, have their own variables. Each variable has associated units, and we examine how these units can be used to derive key nondimensional parameters for particular problems. Dimensional analysis, also called *similitude theory*, is useful in the planning and design of experiments because key parameters are identified and scaling relations can be derived. It enables one to focus on the important variables of a problem when designing an experiment and deciding which parameters need to be measured.

In the following example we demonstrate how the relevant parameters of an oscillator problem are defined on purely dimensional considerations. In designing an experiment to determine key parameters required for the design of a rocket, we may wish to answer the question: If the rocket is $1/5^{th}$ scale, then what scale factors apply to the aerodynamic load and thrust? Variables scale differently according to their units.

Example 1.2 Parameters of an Oscillator⁵²

Stiffness and mass are two key parameters that define an oscillatory system's behavior. Use dimensional analysis to derive some dimensionless parameters considering only units relevant to the modeling of a vibrating system.

Solution: For an undamped oscillator, the following parameters are required to model its behavior: maximum deflection X , spring constant K , and mass M . The forcing can be represented by peak magnitude P , with an associated time constant T . The time constant could be representative of frequency or time to peak amplitude.

The basis for this procedure is the *Buckingham-II Theorem*, which states that any complete physical relation can be expressed in terms of a set of independent dimensionless products composed of the relevant physical parameters, called Π (pi) terms. Because Π terms are products or quotients of zero dimension, a general dimensionless equation expressing this is

$$X^a K^b M^c P^d T^e = M^0 L^0 T^0, \quad (1.2)$$

where the equality implies dimensional equality, and superscripts are powers. The right-hand side is a product of mass, length, and time, each taken to the zero power, indicating that both sides of the equation must be dimensionless. In order to treat Equation 1.2 as an algebraic equation, we need to substitute appropriate units for the variables on the left-hand side, obtaining

$$L^a \left(\frac{M}{T^2} \right)^b M^c \left(\frac{ML}{T^2} \right)^d T^e = M^0 L^0 T^0.$$

Combining like terms on the left-hand side results in the simplified equation

$$M^{(b+c+d)} L^{(a+d)} T^{(e-2b-2d)} = M^0 L^0 T^0.$$

This equation is equivalent to three equations since the units of each dimensional quantity must satisfy the equality. Equating the powers of like units,

$$\begin{aligned} b + c + d &= 0 \\ a + d &= 0 \\ e - 2b - 2d &= 0. \end{aligned}$$

These three equations can be solved for any three of the five constants in terms of the other two. Here, we solve for b, c , and d in terms of a and e to find

$$b = a + \frac{1}{2}e, \quad c = -\frac{1}{2}e, \quad d = -a.$$

Substituting the expressions for b, c , and d into Equation 1.2 we find

$$X^a K^{(a+e/2)} M^{(-e/2)} P^{-a} T^e = M^0 L^0 T^0.$$

Collecting terms with the same exponents,

$$\left(\frac{XK}{P} \right)^a \left(\sqrt{\frac{K}{M}} T \right)^e = M^0 L^0 T^0.$$

The Π dimensionless terms are

$$\frac{X}{P/K}, \quad \sqrt{\frac{K}{M}} T.$$

The Buckingham-II theorem further states that there exists a function of these dimensionless terms such that

$$f \left(\frac{X}{P/K}, \sqrt{\frac{K}{M}} T \right) = 0,$$

meaning that $\frac{X}{P/K}$ is a function of $\sqrt{\frac{K}{M}} T$. These parameters turn out to be important in describing an oscillator. P/K is the static displacement, $\frac{X}{P/K}$ is the maximum displacement normalized with respect to the static displacement, and $\sqrt{K/M}$ is the natural frequency of oscillation. In an experiment, data could be gathered to plot $\frac{X}{P/K}$ as a function of $\sqrt{\frac{K}{M}} T$. This is much easier and more useful than plotting all the combinations of the original five parameters with which we started.

These ideas can be used to tackle more complex problems in which the key parameters may not be known. The Buckingham-II Theorem is a very useful tool for identifying the important parameters of a new problem. ■

⁵² **Similarity Methods in Engineering Dynamics**, W.E. Baker, P.S. Westine, and F.T. Dodge, Elsevier, 2nd ed., 1991, presents interesting applications of dimensional analysis.

1.8 Concepts Summary

We have introduced the subject matter of this text on vibration. An effort has been made to provide the reader with some basic concepts of vibration and dynamics, modeling, and the concerns of engineers working with problems of vibration.

The next chapter begins our study of the simplest vibration problems, those amenable to single degree-of-freedom mathematical models. Such models lead us to the important vibration parameters of natural frequency of oscillation and eventually the viscous damping factor.

1.9 Quotes

- “Art without engineering is dreaming; engineering without art is calculating.” (Steven K. Roberts)
- “Scientists investigate that which already is; engineers create that which has never been.” (Theodore von Kármán)
- “Why do two bodies attract each other? Why should there be this force at all? What does it consist of? And the answer is: We don’t know. Newton’s mathematical formula simply describes the phenomenon, but why that phenomenon should exist in the first place or how it operates, we don’t know. In this great age of technology, we don’t even understand what it is that keeps our feet on the ground.” (M. Scott Peck, **Further Along The Road Less Traveled**, 1993, p.73)
- “Wherever you may go in the world, engineering is an intimate part of our lives – from waking up to the alarm clock, to turning off the lights at night, to getting to and from work or school, to using a computer. No teacher ever needs to look far as the study of engineering is the study of the accomplishments that surround us everywhere.” (Mark Nagurka)
- “There is not much we can do about the past, and the present is so fleeting, as we experience it, that it is transformed into the past as we touch it.” (Dostoevsky)
- “For every action, there is an equal and opposite criticism.”
- “It is not the critic who counts; not the man who points out how the strong man stumbles, or where the doer of deeds could have done them better. The credit belongs to the man who is actually in the arena, whose face is marred by dust and sweat and blood; who strives valiantly; who errs, who comes short again and again, because there is no effort

without error and shortcoming; but who does actually strive to do the deeds; who knows great enthusiasms, the great devotions; who spends himself in a worthy cause; who at the best knows in the end the triumph of high achievement, and who at the worst, if he fails, at least fails while daring greatly, so that his place shall never be with those cold and timid souls who neither know victory nor defeat.” (Theodore Roosevelt, excerpt from the speech “Citizenship In A Republic” delivered at the Sorbonne in Paris, France on April 23, 1910)

- “I’ve been lucky. There’s a saying that we use in golf: ‘I’d rather be lucky than good.’ Of course, to be lucky and good is the ideal. If you study hard, you can get good.” (Clint Eastwood)
- “When I went to school, they asked me what I wanted to be when I grew up. I wrote down ‘happy.’ They told me I didn’t understand the assignment and I told them they didn’t understand life.” (John Lennon)



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Chapter 2

Single Degree-of-Freedom Undamped Vibration

“The simplest model, exactly known.”

We begin our exploration of dynamic systems with models in which one coordinate completely describes the motion of the system. Despite its simplicity, the study of single degree-of-freedom systems is very useful since many of the key principles developed here are applicable to all problems of dynamics and vibration. We can explore the behavior of simple models and learn how to derive and solve their governing equations. The concept of oscillation frequency is introduced as the most important measure of vibratory motion.

In this chapter, we learn how to model and solve for the response of theoretical oscillators where there is no loss of energy through damping. In [Chapter 3](#), damping is introduced as are more realistic force models and more complex vibratory environments. An oscillator is generally taken to be a single degree-of-freedom system undergoing repetitive motion.

This chapter and the next two can be viewed as parts one, two, and three of the study of single degree-of-freedom oscillators. The split has been created for pedagogical reasons. Several application examples are introduced first to motivate the subsequent reading.

2.1 Motivating Examples

Most engineering systems are examples of vibrating systems. Examples include bridges, vehicles, and rotating machinery. Our studies begin with relatively simple models that are single degree-of-freedom idealizations.

The student may wonder how it is possible to derive useful results about complex systems with only a single degree of freedom. Although one coordinate is not enough for a detailed study of a complicated system, it is useful to capture the essential behavior of such systems for preliminary analysis and design. It is also important

to note that the idealized model may not have any physical resemblance to the actual system. Rather, the model is a mathematical idealization that reflects key behavioral characteristics of the physical system.

2.1.1 Transport of a Satellite

A primary objective of vibration studies is isolating and protecting a system from the potentially destructive effects of vibration. Since isolation is generally only partially successful, a secondary objective is to design structures so that they can withstand vibration. Both goals are typically considered in analysis and design.

An important application is a shipping container for satellites. Since satellites cannot survive the rigors of transport from the manufacturing site to the launch pad in conventional vehicles (for example, rail cars or flat bed trucks), it becomes necessary to design a container that, among many other purposes, isolates the contents from the vibrational loading. This problem is considered in more detail at the end of this chapter.

Satellites are designed to orbit the Earth in a micro-gravity, atmosphere-free environment. However, they experience very different loads from their point of departure at the factory until they are operational in orbit. Competing design constraints govern the need to simultaneously survive shipment and be launched into orbit for space operations. The design requirement for a shipping container is to deliver satellites in undamaged condition to the launch pad.

When the space shuttle was being prepared for launch, there were many components that were designed and manufactured at different locations and then shipped to the vehicle assembly building. An example is the external tank shown in [Figure 2.1](#). Following final assembly of the shuttle and all components, the completed shuttle was driven by a specially-designed crawler to the launching pad, as shown in [Figure 2.2](#).¹

¹Between March and July 2011 NASA retired its fleet of three



Figure 2.1: The shuttle's external tank sits on a transport vehicle and is backed into the vehicle assembly building at Cape Canaveral. In this building all the components are mated to the shuttle for the mission. (NASA Image)

2.1.2 Rocket Propulsion

A rocket is propelled into space by *thrust* created by gases from the combustion of its fuel, resulting in a decreasing fuel mass within its fuel tanks over time. Thus, a rocket is an example of a system with *variable mass*. An analogy with a cannon is useful to help understand how thrust is generated. As the cannon is fired, it recoils in the direction opposite that of the cannonball, satisfying the *principle of conservation of linear momentum*. If the cannon is fired rapidly, it picks up speed with each shot. The cannon expels mass in one direction and moves in the opposite direction, moving faster when it shoots out more mass over a shorter period of time.

The rocket is comprised of several stages, with at least one stage within which fuel is burned and ejected at high speed, accelerating it in a direction opposite that of the fuel ejection. The acceleration is a function of the rate of fuel consumption and the speed with which combustion gases are ejected. The total mass of the rocket decreases with time.

Figure 2.3 depicts a futuristic rocket entering the Martian atmosphere, which is one-hundredth as thin as the Earth's. The forces on this structure are the thrust, Martian gravity (about 0.38 that of Earth gravity), and buffeting due to the thin Martian atmosphere. (The temperature at Mars' equator is similar to the dry valleys of the Antarctic.)

We consider the dynamics of a rocket later in this chapter. In Example 2.3, the momentum form of Newton's second law of motion is used to derive the variable mass equation of motion.

space shuttles: Discovery, Endeavor, and Atlantis.



Figure 2.2: The shuttle sits on a crawler that travels 5.6 km (3.5 mi) to the launch pad at Cape Canaveral – seen at the left in the distance. It took between 6 and 8 h to make the trip. (NASA Image)

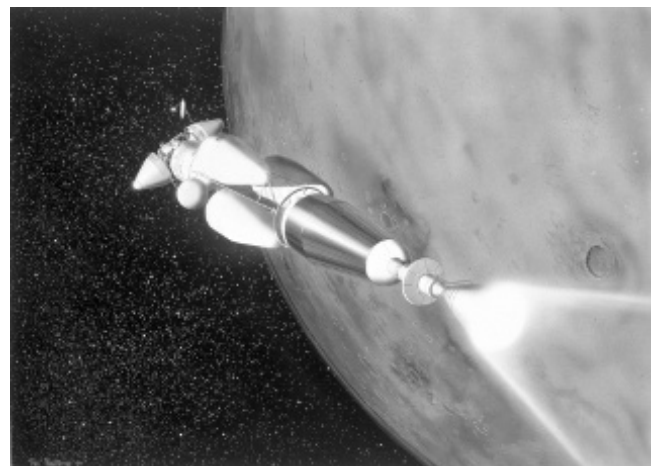


Figure 2.3: An artist's concept of a nuclear thermal rocket fired upon arrival in the vicinity of Mars to insert the transfer vehicle into orbit. Nuclear propulsion would shorten interplanetary trip times and reduce the mass launched from Earth. As the primary transfer propulsion system, the spacecraft's reactor would remain inactive until departure from Earth orbit. (NASA artwork by Pat Rawlings, February 1995)

2.2 Deterministic Modeling

Mathematical modeling is the process of developing a mathematical framework to represent the physical behavior of a component or system, as discussed in [Chapter 1](#). Modeling provides the analyst with a systematic approach for specifying constitutive behavior, deriving equations of motion, and classifying external inputs, such as forces.

2.2.1 Problem Idealization

Most real engineering problems are exceptionally complicated. If too many aspects of the problem are included in the formulation, the model can become unnecessarily unwieldy. Fortunately, not all aspects are equally important in predicting how the system responds to its environment. Most often, only a few aspects are needed to determine the system response with reasonable accuracy. Many others can be ignored.

A judicious choice of the key parameters of a system and its environment enables us to derive a simpler model. This process of *idealization* is one that is difficult, if even possible, to capture with a set of simple rules. In a sense, idealization is an art with which one becomes more proficient with more practice. The idealized model is appropriate if it has predictive capability, that is, if it predicts results that can be validated by experimental data. The range of situations that it can predict are a direct result of the type and number of idealizations made in its derivation. The process of experimental verification and idealization is iterative and generally requires several stages of model refinement.²

If little is known about how a system will behave, we test a prototype. From the test, data are gathered that help us develop a mathematical model of the system. This model is then used to predict system behavior for which data do not exist. To validate the model, additional tests are run for more comparisons. Only after a series of such iterations comparing test results and model predictions can we be confident that the model is representative of the actual system.

The idealized model does not necessarily look like the system it represents. What is paramount is that the idealized model (mathematically) represents the key characteristics of the system.

The Linear Approximation

Even physical systems that can be represented by a nonlinear model behave linearly over some range of inputs, even if it is a small range. As an example, consider a

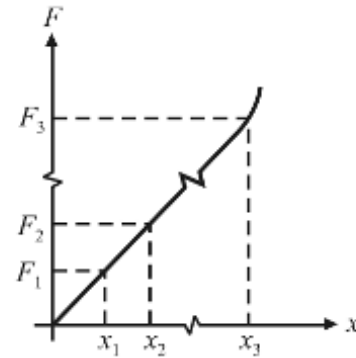


Figure 2.4: Force-displacement characteristic of a spring in tension.

mass suspended by a spring. A small vertical displacement of the mass from its static equilibrium position, when released, results in an oscillation about the equilibrium position. The spring develops a tensile force that is linearly related to its stretch. The plot in [Figure 2.4](#) of force F vs. displacement x about the equilibrium position is a straight line for $x \leq x_3$. (This is a graphical representation of Hooke's law.) However, if the mass undergoes a larger displacement, say $x > x_3$, then the spring force no longer remains linear with displacement. In [Figure 2.4](#), the nonlinear behavior would be indicative of a hardening spring, which gets stiffer with increasing deflection.

Why is linearity so important? The answer is based on the *principle of linear superposition*. This principle allows us to add solutions in the special case of a linear model. Suppose a structure is loaded by a wind force $F_{wind}(t)$ and responds with displacement $x_{wind}(t)$. Subsequently, the structure is loaded by an earthquake force $F_{eq}(t)$ with response $x_{eq}(t)$. From the principle of linear superposition, if the structure is loaded with the combined force $F_{wind}(t) + F_{eq}(t)$, then it will displace $x_{wind}(t) + x_{eq}(t)$. This remarkable but intuitive result is due to the linearity of the model, even if $F_{wind}(t)$ and $F_{eq}(t)$ are nonlinear functions. If the model included nonlinear stiffness or nonlinear damping, then superposition would not be valid.

If a governing equation is nonlinear, it is sometimes *linearized* using the Taylor series. Any results obtained using the linearized equation are valid only for a model that retains its linear characteristics. In [Section A.3](#) we discuss the concept of *linearization*. The idea is to expand a nonlinear function in a Taylor series and retain only the linear terms.

²The accuracy of the model is related to our understanding of the physical processes being modeled.

BROOK TAYLOR

(August 18, 1685 – November 30, 1731)



Contribution: Brook Taylor was an English mathematician, known for Taylor's theorem and Taylor series. He is credited with the "calculus of finite differences," which contained Taylor's expansion, and with integration by parts.

In calculus, Taylor's theorem gives a sequence of approximations of a differentiable function around a given point by polynomials (the Taylor polynomials of that function) whose coefficients depend only on the derivatives of the function at that point. The theorem also gives precise estimates on the size of the error in the approximation. Although the theorem is named after Taylor, who stated it in 1712, the result was first discovered in 1671 by James Gregory.

The Taylor series is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point. It may be regarded as the limit of the Taylor polynomials. If the series is centered at zero, the series is also called a Maclaurin series, named after Scottish mathematician Colin Maclaurin.

Life Sketch: Taylor was born in Edmonton, England. He entered St. John's College, Cambridge as a fellow-commoner in 1701, and earned degrees of LL.B. and LL.D. in 1709 and 1714, respectively.

He wrote many articles in *Philosophical Transactions* that discussed the motion of projectiles, centers of oscillation, and forms taken by liquids when raised by capillary action.

Taylor published his book, *Methodus Incrementorum Directa et Inversa*, in London in 1715. It was the earliest to identify theorems on the change of the independent variable. Taylor applied calculus to various questions in his book. In particular, he addressed the theory of transverse vibrations of strings, a problem which had troubled previous investigators. His book added a new branch to higher mathematics, now designated as the "calculus of finite differences."

In 1719, Taylor published a treatise on perspective that contained the earliest general form of the principle of vanishing points. In 1719, he abandoned the study of mathematics.

His marriage in 1721 led to an estrangement from his fa-

ther, which ended in 1723 after his wife's death giving birth to a son, who also died. He spent the next two years with his family, and in 1725 he remarried (this time with his father's approval). His second wife died in childbirth in 1730; in this case, the child, a daughter, survived.

Taylor's fragile health gave way and he died in 1731 at age 46 in Somerset House, London. He was buried in London on December 2, 1731, near his first wife, in the churchyard of St. Anne's, Soho.

Noteworthy: Prior to graduation from St. John's College in 1708 he obtained a remarkable solution of the problem of the "center of oscillation." Although it remained unpublished until May 1714 (*Philosophical Transactions*, Vol. xxviii), his claim to discovery was unjustly disputed by Johann Bernoulli.

This was the first of several cases in which his work was given due credit long after it was completed. His Taylor's expansion – presented in his 1715 book – went unnoticed until it was recognized as the basic principle of the differential calculus in 1772 by Lagrange. (Lagrange realized its powers and termed it "the main foundation of differential calculus.")

Taylor was elected a fellow of the Royal Society early in 1712, and in the same year sat on the committee for adjudicating the claims of Newton and Leibnitz. He was a secretary to the society from January 1714 to October 1718.

From 1715 his studies took a philosophical and religious bent. Found among his papers after his death were unfinished treatises, *On the Jewish Sacrifices* and *On the Lawfulness of Eating Blood*, written in 1719.

Much of his writing suffered from brevity and obscurity, and was elucidated by others. A great part of his brilliance was lost through his failure to express his ideas clearly.

2.2.2 Mass, Damping, and Stiffness

In [Section 1.3.3](#) the concepts of mass, damping, and stiffness were presented. How a mechanical system responds to its loading environment depends on its *mass*, *damping*, and *stiffness* characteristics. Of the three, damping is the most difficult to quantify, even for relatively simple structures.

Newton's second law of motion relates the forces acting on a body to its acceleration, with the constant of proportionality being the body's mass. The forces acting on a body can be characterized as *external* or *internal*. External forces are forces from the environment, such as those due to wind, ocean, impact, or actuators. Internal forces are due to stiffness or damping.

A free-body diagram is used to visualize all the forces acting on a body. [Figure 2.5](#) shows a schematic of a system and its *free-body*. It is particularly useful in formulating the equation of motion of a body using Newton's second law of motion, as will be shown in [Section 2.2.4](#).

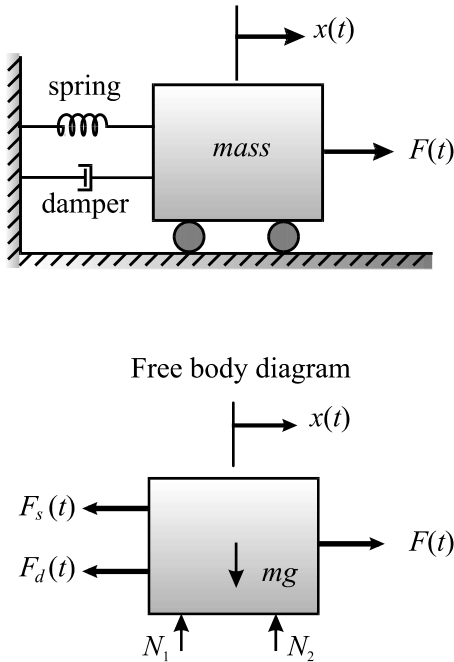


Figure 2.5: A mass-spring-damper system acted on by an external force is shown at the top. The free-body diagram of the mass is shown at the bottom. Here, x is shown in the assumed positive direction of motion.

Stiffness forces are related to the deformation of a representative “spring” element according to Hooke’s law, $F_s(t) = kx(t)$, where the “rest length” is taken to be zero. A more general expression of Hooke’s law is $F_s(t) = k[x(t) - x_0]$, where x_0 is the rest length of the spring.

Damping forces are related to the relative velocity between a body and its attachment for viscous damping, $F_d(t) = c\dot{x}(t)$, where the “overdot” implies differentiation with respect to time. The *damping constant* c has units of force per velocity. Other damping laws exist such as Coulomb damping, aerodynamic drag, and other nonlinear relationships.

In this book, our focus is on viscous damping models, although we address other models as well. In the next chapter and in [Appendix B](#) we discuss damping models in more detail.

2.2.3 Deterministic Approximation

A deterministic parameter is one that is known exactly, that is, it possesses the same numerical value regardless of how many times it is measured. Although slight variability may occur in practice, it is generally ignored. When significant variability occurs, it needs to be included in the problem formulation and solution.

Possible sources of uncertainties for a single degree-of-freedom system are the external forces and the system properties: mass, damping, and stiffness. If all these

have parameter values of sufficient accuracy for analysis and design, then we may assume that the system model is *deterministic* and the governing equation will provide accurate results.³

In some applications, the loading⁴ has a *broad frequency band* (the loading is comprised of energies at many frequencies) and/or the amplitudes of the loading are *widely distributed*. Testing is necessary, both to establish that the behavior of the mathematical model is representative to that of the actual system, and also to gather data on the loading frequencies, amplitudes, and material properties of the system. Where such variability exists, *nondeterministic* or probabilistic methods are necessary. We will study these in [Chapter 9](#).

In the following section, we proceed to formulate the governing equation of an oscillating system assuming a deterministic model.

2.2.4 Equations of Motion

Two general approaches are available for obtaining the equations of motion of a dynamic system. The first is based on *Newton’s second law of motion*. The second is based on an understanding of the kinetic and potential energies of a system and any dissipation of such energies. For systems with one or only a few degrees of freedom, Newton’s second law of motion can easily be used to arrive at the equations of motion. For systems with many degrees of freedom, that is, more complicated and realistic systems, energy methods may be better suited for the derivation of the governing equations. This approach is introduced in [Section 2.2.5](#) and in later chapters.

For a single degree-of-freedom model, it is relatively straightforward to derive the equation of motion based on the free-body diagram introduced earlier. This tool permits us to visualize the *forces* acting on a *free* or *isolated body*, as seen in [Figure 2.5](#). Newton’s second law of motion states that the sum of the external forces on a free-body equals the product of the mass of the body and the acceleration of its center of mass,

$$\sum_{i=1}^n F_i(t) = ma(t), \quad (2.1)$$

where we have written the scalar form of the vector equation $\mathbf{F} = m\mathbf{a}$. A sign convention needs to be established so that the directions of the forces and motions in [Equation 2.1](#) are properly included. For a body in rotational

³It was not until the 1950s, when the aerospace industry began to design aircraft that were subjected to unpredictable aerodynamic forces, that uncertainties regarding such forces were included explicitly in the vibration analysis. This was the formal start to the discipline of *random vibration*.

⁴The terms *loading* and *forcing* are used interchangeably.

motion, the corresponding equation of motion is

$$\sum_{i=1}^n M_i(t) = I\alpha(t), \quad (2.2)$$

where externally applied moments are summed about a fixed point or the center of mass. (This equation is sometimes called the *Newton-Euler* equation or the *Euler* equation of motion.) In this case, the inertial property is the mass moment of inertia I ,⁵ and the acceleration is the angular acceleration $\alpha(t)$. Mathematically, these second-order linear differential equations are equivalent.

The governing equation of motion for a single body with linear stiffness and viscous damping properties, undergoing rectilinear motion, can be written using Equation 2.1 as

$$F(t) - c\dot{x}(t) - kx(t) = m\ddot{x}(t), \quad (2.3)$$

where $F(t)$ is the magnitude of the vector sum of external forces in the coordinate direction of motion, and $a \equiv \ddot{x}(t)$. (While we retain the time argument in these equations, we omit them where it is clear.) In the standard or *canonical* form, Equation 2.3 becomes⁶

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t). \quad (2.4)$$

By dividing by the mass m , Equation 2.4 can be written with a unity coefficient for the highest derivative term,

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = \frac{1}{m}F(t), \quad (2.5)$$

where

$$\zeta = \frac{c}{2m\omega_n} \quad (2.6)$$

$$\omega_n = \sqrt{\frac{k}{m}}, \quad (2.7)$$

where ζ is the dimensionless *viscous damping factor* and ω_n is the undamped *natural frequency*⁷ of oscillation with units of rad/s. Equation 2.5 is the equation of motion for a dynamic system idealized by a single degree-of-freedom model subjected to external forcing. The viscous damping factor ζ is also known as the *viscous damping ratio* or just *damping ratio* since it can be written as the ratio

$$\zeta = \frac{c}{c_{cr}},$$

⁵The term moment of inertia was introduced by Leonhard Euler in 1765. It is represented in classical references by both the letters I and J . Most often, I denotes mass moment of inertia and J is used for area moment of inertia. But, this is not consistent.

⁶If positive $x(t)$ is taken in the opposite direction, then applying Newton's second law of motion yields

$$-F(t) + c\dot{x}(t) + kx(t) = -m\ddot{x}(t),$$

which is the same equation of motion, as it should be.

⁷The natural frequency, ω_n , is sometimes referred to as the natural circular frequency or the natural angular frequency, as distinguished from the natural cyclic frequency, f_n , where $\omega_n = 2\pi f_n$ and the units of f_n are hertz (Hz).

Table 2.1: Ranges of Damping Factor Values.

Damping Mechanism	ζ
Material damping	0.1 to 1
Air radiation	0.1 to 2
Joints	2 to 5
Equipment	2 to 8
Special damping materials	1 to 50

where c_{cr} is the critical damping constant defined as $c_{cr} = 2m\omega_n$, the value of c that yields $\zeta = 1$. Some values for the viscous damping factor ζ are listed in Table 2.1.

In this chapter we neglect the effect of damping. This means that in our models the second term in Equations 2.4 and 2.5 is assumed absent. (We consider the effect of damping in the next chapter.) In the following we present two examples showing the derivation of the undamped equations of motion.

Example 2.1 Torsional Vibration

Derive the equation of motion and determine the natural frequency for the torsional vibration system shown in Figure 2.6.

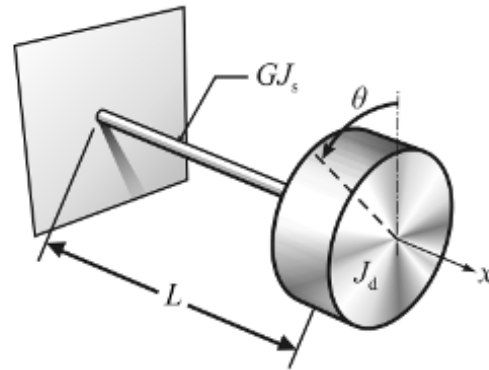


Figure 2.6: Torsional vibration of a disk of mass moment of inertia J_d mounted on a shaft of torsional rigidity GJ_s , where G is the shear modulus of the material and J_s is the area moment of inertia of the shaft of length L .

Solution: The governing equation for torsional oscillation of a single degree-of-freedom system has the same mathematical form as a single degree-of-freedom system in translational motion. The parameters have different physical meanings, for example, moment of inertia for torsion vs. mass for translation.

The shaft in Figure 2.6 is assumed to be uniform, with a torsional stiffness of GJ_s/L , where J_s is the area moment of inertia of the shaft about the axis of rotation, G is the shear modulus, and GJ_s is the torsional rigidity. The disk is assumed to be rigid with mass moment of inertia J_d . Summing the moments about the x axis of rotation for small angles of rotation $\theta(t)$, and using

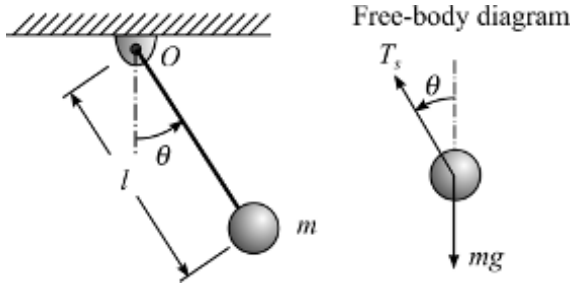


Figure 2.7: In a simple pendulum mass m is suspended on a massless string of length l . The position of the pendulum is defined by the angular displacement θ . The free-body diagram is shown to the right, where T_s is the tension in the string.

the rotational form of Newton's second law of motion, Equation 2.2 results in

$$-\frac{GJ_s}{L}\theta(t) = J_d\ddot{\theta}(t).$$

In standard form, the equation of rotational motion is

$$\ddot{\theta}(t) + \frac{GJ_s}{LJ_d}\theta(t) = 0. \quad (2.8)$$

The natural frequency of torsional oscillation is

$$\omega_n = \sqrt{\frac{GJ_s}{LJ_d}}. \quad (2.9)$$

Comparing Equations 2.5 and 2.8 indicates that this single degree-of-freedom model for torsional vibration does not include a damping term or a forcing term. Also, Equation 2.9 is analogous to $\omega_n = \sqrt{k/m}$ of Equation 2.7 for a single degree-of-freedom translational system. ■

Example 2.2 Simple Pendulum

Figure 2.7 shows a schematic of a simple pendulum and its free body. The pendulum mass is assumed to be “lumped” at a point (so that the pendulum's rotational inertia can be ignored). Derive the equation of motion in two ways: (a) by summing forces and (b) by summing moments. Then show that the total energy of the system is constant.

Solution: The example of a simple pendulum demonstrates how a nonlinear model of the motion can be linearized under certain assumptions.

(a) By summing forces in the instantaneous direction of motion – perpendicular to the string – Newton's second law gives

$$m(l\ddot{\theta}(t)) = -mg \sin \theta(t),$$

where the tangential acceleration is $l\ddot{\theta}(t)$. The equation of motion can be written in standard form as

$$\ddot{\theta}(t) + \frac{g}{l} \sin \theta(t) = 0. \quad (2.10)$$

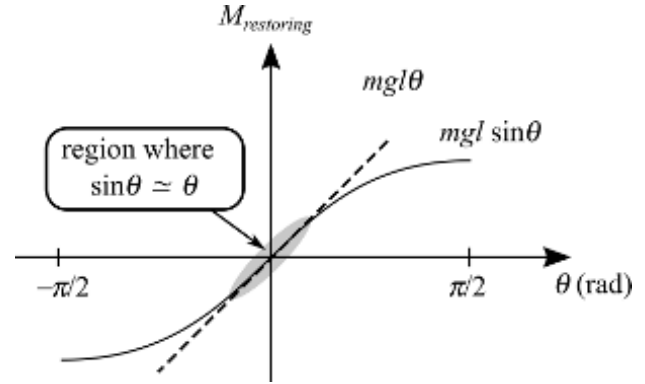


Figure 2.8: Pendulum restoring moment vs. angular displacement showing shaded region of valid linear approximation (dashed line).

(b) By taking moments about the point of rotation O , the equivalent governing equation is

$$I_O\ddot{\theta}(t) = -mgl \sin \theta(t),$$

where $l \sin \theta(t)$ is the moment arm for the weight mg , $\ddot{\theta}(t)$ is the angular acceleration, and I_O is the mass moment of inertia about O , which can be expressed as $I_O = ml^2$. The equation of motion in standard form then becomes

$$\ddot{\theta}(t) + \frac{g}{l} \sin \theta(t) = 0, \quad (2.11)$$

which is the same as Equation 2.10. This equation is nonlinear due to the $\sin \theta$ term. Following our previous discussions on Taylor series and linearization, we can approximate $\sin \theta \simeq \theta$ for small θ ,

$$\ddot{\theta}(t) + \frac{g}{l} \theta(t) = 0. \quad (2.12)$$

Figure 2.8 shows the shaded region where the linear approximation is valid.

We can recast Equation 2.11 in terms of energies. To do this, multiply both sides by $m\dot{\theta}$ and integrate

$$m \int \ddot{\theta} \dot{\theta} dt + \frac{mg}{l} \int \dot{\theta} \sin \theta dt = \text{constant}$$

$$m \int \frac{d}{dt} \left(\frac{1}{2} \dot{\theta}^2 \right) dt - \frac{mg}{l} \int \frac{d}{dt} (\cos \theta) dt = \text{constant},$$

or

$$\frac{1}{2} m \dot{\theta}^2 - \frac{mg}{l} \cos \theta = \text{constant}$$

$$T + V = \text{constant},$$

where $T = m\dot{\theta}^2/2$ equals the kinetic energy and $V = -mg \cos \theta/l$ equals the potential energy of the mass in a gravity field. The sum of the kinetic and potential energies equals the total energy, which is constant over time for a system without damping. ■

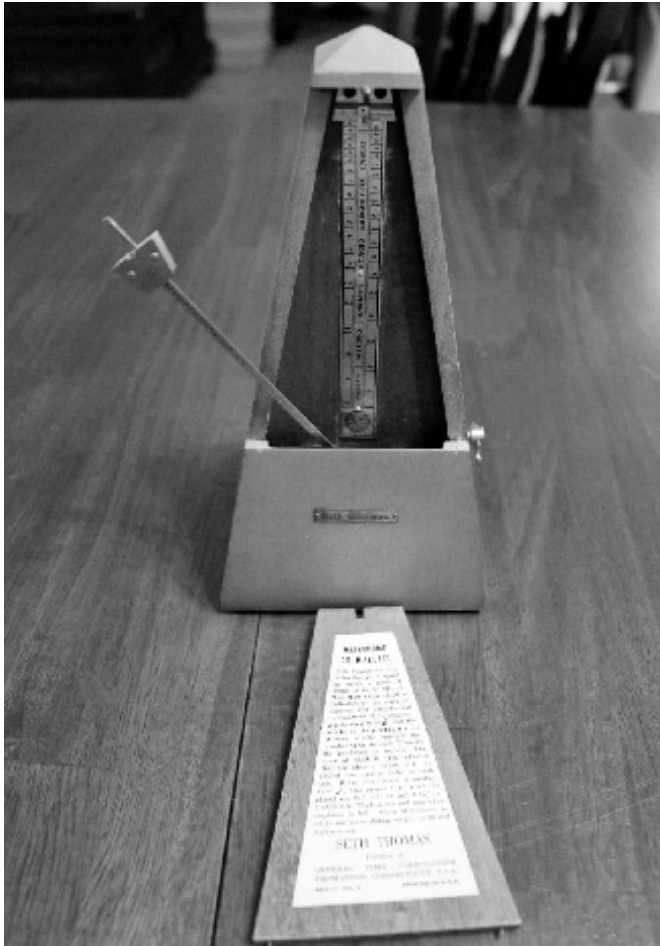


Figure 2.9: Mechanical metronome by Seth Thomas. (Photograph by M. Nagurka)

METRONOME. A metronome is an instrument that produces regularly timed click sounds at a fixed number of beats per minute (BPM). It is used by musicians as they practice playing any instrument, such as a guitar, piano, clarinet, etc.

In 1581, Galileo Galilei discovered the isochronism of pendulums⁸ – pendulums of any length oscillate in the same time. Almost a century passed before pendulums were successfully applied to clocks by Christian Huyghens (circa 1659) and George Graham (circa 1715).

In 1696, Etienne Loulie made the first recorded attempt to apply the pendulum to a metronome. His “machine” was merely an adjustable pendulum with calibrations. He was followed by a long list of inventors whose attempts were unsuccessful owing to the great pendulum lengths required to beat the lower tempos used in music (40 to 60 BPM).

⁸The plural of pendulum is pendulums or pendula. According to The Oxford English Dictionary (2nd ed.) pendula is a rare and archaic plural. A web search for “pendula” shows many hits – although not as many as for “pendulums” – and articles in physics and engineering journals use “pendula” as the plural.

In 1812, Dietrik Winkel (1780-1826) found that a double weighted pendulum (a weight on each side of the pivot) would beat low tempos, even when made of short length. Johann Maelzel appropriated Winkel’s idea and in 1816 started manufacturing “Maelzel’s” metronome.

Mechanical metronomes, such as the one shown in Figure 2.9, consist of a pendulum that swings on a pivot and are actuated by a hand-wound mechanism whose escapement makes a ticking sound. Below the pivot there is a fixed weight and above it, a sliding weight. A scale of numbers indicates how many oscillations per minute occur when the sliding weight is moved to a position on the pendulum. The notation “M.M. (Maelzel’s metronome) {half note} = 60” indicates that at 60 oscillations per minute the half note will receive one beat.

The advent of alternating current (A.C.) made possible the invention of the Franz electric metronome (1938). In this metronome, a synchronous motor drives a tempo-beating hammer through a mechanical reduction that is adjustable from 40 to 208. These electromechanical units were produced until about the early 1990s. In the 1950s, electronic versions based on oscillator type circuits were introduced. In the late 1970s digital electronic metronomes that increased the accuracy by a factor of 10 or more were on the market. Microprocessors made it possible to add tuning pitches, accented beats, and other functions.

We will solve equations of the form of Equation 2.12 when studying various model applications in the following sections. Before proceeding, we consider the following question. *In modeling, how do we know that reasonable assumptions have been made?* The answer is that we do not know until comparisons are made between the predictions of the mathematical model and experimental data. Reasonable comparisons provide assurances that, for similar circumstances, the model is an accurate predictor. Otherwise, we must go back to the beginning of the problem formulation to rethink the assumptions made in the derivation.

Example 2.3 System of Variable Mass

A schematic of a rocket with its ejected fuel is shown in Figure 2.10. (a) Derive the equation of motion (for velocity). (b) Solve for the relation between rocket mass, m , and velocity, v .

Solution: (a) The mass of the rocket decreases by the quantity of ejected fuel that is burned and ejected. We wish to obtain the equation of motion of the rocket, which includes a term reflecting the variable mass. Let $m(t)$ be the mass of the rocket and the fuel inside at time t , and m_i be the initial mass before ignition.

Consider the rocket-fuel system at two time instances Δt apart. Assume that initially, at time t , the rocket and all its fuel have mass m with velocity v . (Since the rocket travels in one direction in this example, vectors

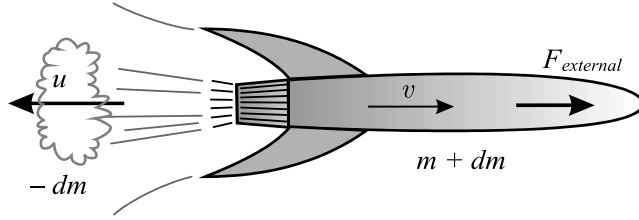


Figure 2.10: Schematic of rocket in flight. The external force is the thrust imparted by ejected combusted fuel. There are also other external forces such as aerodynamic and thermal loads. The absolute speed of the ejected fuel is u and the absolute speed of the rocket is v .

are shown as scalars.) At a later time $t + \Delta t$, a quantity of fuel Δm has been ejected. Denoting this loss of mass as a negative quantity,⁹ the rocket mass is $m + \Delta m$. The rocket is traveling at an increased velocity, $v + \Delta v$. The ejected fuel has mass $-\Delta m$ and is traveling at $-u$. The velocity of the ejected fuel can also be written as $u = v_{rel} - v$ to the left. It is reasonable to expect that $|u| \gg |v|$.

Consider next the momentum of the total system. Recalling Newton's second law of motion in momentum form,

$$F = \frac{dp}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta p}{\Delta t}, \quad (2.13)$$

where p is the momentum and F are external forces such as due to gravity, aerodynamic drag, etc. During time interval Δt , the change in momentum is

$$\Delta p = p_f - p_i, \quad (2.14)$$

that is, the final minus the initial momenta, where, from the previous discussion,

$$\begin{aligned} p_i &= mv \\ p_f &= (m + \Delta m)(v + \Delta v) + (-\Delta m)(-u) \\ &= (m + \Delta m)(v + \Delta v) + (v_{rel} - v)\Delta m. \end{aligned}$$

Equation 2.14 then becomes

$$\begin{aligned} \Delta p &= (m + \Delta m)(v + \Delta v) + (v_{rel} - v)\Delta m - mv \\ &= m\Delta v + \Delta m\Delta v + v_{rel}\Delta m. \end{aligned}$$

From Equation 2.13,

$$\begin{aligned} F &= \lim_{\Delta t \rightarrow 0} \frac{\Delta p}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left[m \frac{\Delta v}{\Delta t} + v_{rel} \frac{\Delta m}{\Delta t} + \Delta v \frac{\Delta m}{\Delta t} \right] \\ &= m \frac{dv}{dt} + v_{rel} \frac{dm}{dt}, \end{aligned} \quad (2.15)$$

⁹It is customary to denote this loss of mass as a negative quantity. As such, for a loss of mass, Δm is negative, and for a gain of mass, Δm is positive.

since $\Delta v \rightarrow 0$ as $\Delta t \rightarrow 0$. The effect of the variable mass is reflected in the second term, which is $v_{rel} dm/dt$, an extra term in the equation of motion. Equation 2.15 can be written as

$$m \frac{dv}{dt} = F - v_{rel} \frac{dm}{dt}. \quad (2.16)$$

In this form, the second term on the right-hand side represents an equivalent force acting on the system due to the movement of mass. For a rocket, this term is called the thrust. To maximize the thrust, designers attempt to make each term, v_{rel} and dm/dt , as large as possible.¹⁰ Equation 2.16 can be solved for the relation between mass and velocity.

(b) It is assumed that the only force acting on the rocket is the thrust. Other forces act on the rocket structure, such as aerodynamic forces, and they cannot be ignored in an actual design, but we do so here in order to show an analytical solution. Making fewer simplifying assumptions would mean that a computational method would be needed to obtain the solution. Equation 2.16 becomes

$$m \frac{dv}{dt} = -v_{rel} \frac{dm}{dt}. \quad (2.17)$$

The negative sign indicates that as the mass of the rocket decreases, the velocity of the rocket increases. Equation 2.17 can be rewritten as

$$dv = -v_{rel} \frac{dm}{m}, \quad (2.18)$$

during which some fuel is remaining. Integrating both sides of Equation 2.18, we find

$$\begin{aligned} \int_{v_i}^v dv &= -v_{rel} \int_{m_i}^m \frac{dm}{m} \\ v - v_i &= -v_{rel} \ln \left(\frac{m}{m_i} \right), \end{aligned}$$

where m_i is the combined mass of the rocket and the fuel when the velocity is v_i , and m is the combined mass when the velocity is v . If the rocket starts from rest, $v_i = 0$ and

$$m(v) = m_i e^{-v/v_{rel}}$$

is the relation between the combined mass of the rocket and the remaining fuel, m , and the velocity, v . This equation can be used to estimate the mass of fuel needed to attain a certain velocity. Figure 2.11 shows the plot of the remaining mass as a function of the velocity of the rocket. For example, for the propellant mass fraction of

¹⁰A simple approximation for thrust is given by

$$u_o \frac{dm}{dt} \simeq A_o \Delta P,$$

where u_o is the speed of gas through the orifice of the rocket, A_o is the area of the orifice, and ΔP is the pressure difference between the inside of the rocket and the atmospheric pressure.

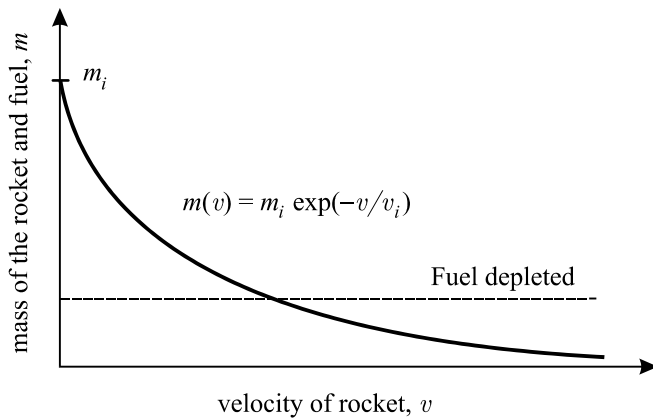


Figure 2.11: Rocket mass as a function of velocity.

0.8 (that is, the 80% of the total initial mass is fuel), the fuel runs out when the velocity is

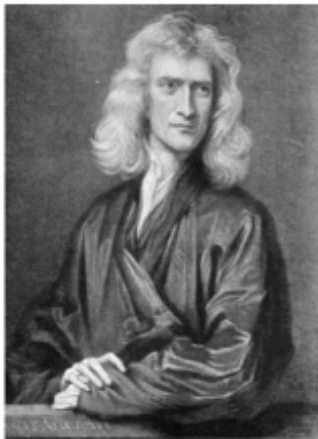
$$0.2m_i = m_i e^{-v/v_{rel}}$$

$$v/v_{rel} = 1.609,$$

or 1.6 times the relative fuel ejection velocity. The relative fuel ejection velocity is also called the effective exhaust velocity and is a characteristic of a rocket. ■

ISAAC NEWTON

(January 4, 1643 – March 31, 1727)



Contribution: Isaac Newton was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian whose work revolutionized the fields of physics and celestial mechanics. His *Philosophiæ Naturalis Principia Mathematica* (or *Principia*), published in 1687, is considered to be the most influential book in the history of science. In this work, Newton described universal gravitation and the three laws of motion, laying the groundwork for classical mechanics, which dominated the scientific view of the Universe

for the next three centuries and is the basis for modern engineering.

Newton showed that the motions of objects on Earth and of celestial bodies are governed by the same set of natural laws. He demonstrated the consistency between Kepler's laws of planetary motion and his theory of gravitation, thus removing the last doubts about heliocentrism and advancing the scientific revolution.

In mechanics, Newton is credited with the principles of conservation of linear and angular momenta. In thermodynamics, he formulated an empirical law of cooling. In acoustics, he studied the speed of sound. In optics, he built the first practical reflecting telescope, developed a theory of color based on the observation that a prism decomposes white light into a visible spectrum, and investigated diffraction and chromatic aberration (still a problem with lenses in today's digital cameras). To explain some of his observations he used a wave theory of light in conjunction with his corpuscular theory.

In mathematics, Newton developed differential and integral calculus, sharing the credit with Leibnitz. Newton wrote his "method of fluxions" in 1671 but it did not appear in print until an English translation from Latin was produced in 1736. He also demonstrated the generalized binomial theorem, developed "Newton's method" for approximating the zeros of a function, and contributed to the study of power series.

Newton was highly religious (though unorthodox), producing more work on Biblical hermeneutics (interpretation of written texts) than the science he is known for today.

Life Sketch: Newton was born in Woolsthorpe, England. His life can be divided into three periods. The first concerns his boyhood days and lasted from 1643 until his graduation in 1669. In the second period, from 1669 to 1687, he was highly productive and was the Lucasian professor at Cambridge. In the third period (following a nervous breakdown) Newton was a highly paid government official in London with little further interest in mathematics.

At the time of Newton's birth, England had not adopted the latest papal calendar, and, therefore, his date of birth was recorded as Christmas Day 1642. Newton was born prematurely three months after the death of his father. When Newton was three, his mother remarried, leaving her son in the care of his maternal grandmother. Newton disliked his stepfather and had enmity towards his mother for marrying him.

From the age of about twelve until he was seventeen, Newton was educated at The King's School, Grantham (where his signature can still be seen on a library window sill). Newton's school reports described him as "idle" and "inattentive."

He was removed from school, and by October 1659 he was at Woolsthorpe-by-Colsterworth, where his mother, widowed by now for a second time, attempted to make him a farmer. He hated farming. The master at the King's School persuaded his mother to send him back to school so that he might complete his education. This he did at the age of eighteen, achieving an admirable final report.

An uncle decided that he should be prepared for the university, and he entered his uncle's old college, Trinity College, Cambridge, in June 1661 as a sizar (a sort of work-study role). Newton's aim at Cambridge was a law degree.

Although the college's teachings were based on those of Aristotle, Newton preferred to read the more advanced ideas of modern philosophers such as Descartes and astronomers such as Copernicus, Galileo, and Kepler. In 1665, he discovered the generalized binomial theorem and began to develop a mathematical theory that would later become calculus. Soon after Newton had obtained his degree in August 1665, the University closed as a precaution against the Great Plague.

Although he had been undistinguished as a Cambridge student, Newton's scientific genius emerged when he had to return to his home in Woolsthorpe. There, in a period of less than two years, while Newton was still less than 25, he began revolutionary advances in mathematics, optics, physics, and astronomy.

Newton, at age 27, was appointed the Lucasian chair (a professor of mathematics) in 1669. In that day, any fellow of Cambridge or Oxford had to be an ordained Anglican priest. However, the terms of the Lucasian professorship required that the holder not be active in the church (presumably so as to have more time for science). Newton argued that this should exempt him from the ordination requirement. Charles II, whose permission was needed, accepted this argument, thus averting a conflict between Newton's religious views and Anglican orthodoxy.

His first work as the Lucasian professor was on optics. During the two plague years Newton reached the conclusion that white light was a mixture of many different types of rays refracted at slightly different angles. He argued that each different ray produced a given spectral color. He investigated the refraction of light, demonstrating that a prism could decompose white light into a spectrum of colors, and that a lens and a second prism could recombine the multicolored spectrum into white light.

He also showed that colored light does not change its properties. Newton noted that regardless of whether it was reflected or scattered or transmitted, it stayed the same color. Thus, he observed that color is the result of objects interacting with already-colored light rather than objects generating the color. This is known as Newton's theory of color.

From this work he concluded that the lens of any refracting telescope would suffer from the dispersion of light into colors – chromatic aberration. Using a mirror as the objective to bypass that problem, he constructed a telescope, the first known functional reflecting telescope called a Newtonian telescope. By grinding his own mirrors, in February 1669 he was able to produce an instrument without chromatic aberration.

In 1671 the Royal Society asked for a demonstration of his reflecting telescope. Their interest encouraged him to publish his notes *On Colour*, which he later expanded into *Opticks*. When Robert Hooke criticized some of Newton's ideas, Newton was so offended that he withdrew from public debate.

The two men remained enemies until Hooke's death.

Newton was elected a fellow of the Royal Society in 1672. Also in 1672 Newton published his first scientific paper on light and color in the *Philosophical Transactions of the Royal Society*. Newton's paper was well received, but Hooke and Huygens objected to Newton's attempt to prove by experiment that light consists of the motion of small particles rather than waves.

Newton is credited with the generalized binomial theorem, valid for any exponent. He discovered Newton's identities, Newton's method, classified cubic plane curves (polynomials of degree three in two variables), made substantial contributions to the theory of finite differences, and was the first to use fractional indices and coordinate geometry to derive solutions to Diophantine equations. He approximated partial sums of the harmonic series by logarithms (a precursor to Euler's summation formula), and was the first to use power series. He also discovered a new formula for calculating π .

In 1677, Newton returned to his work on mechanics, in particular, gravitation and its effect on the orbits of planets with reference to Kepler's laws of planetary motion. He published his results in 1684 and they contained the beginnings of the laws of motion.

The *Philosophiæ Naturalis Principia Mathematica* (known as the *Principia*) was published July 1687 with encouragement and financial help from Edmond Halley. In this work Newton stated the three universal laws of motion. He used the Latin word *gravitas* (weight) for the effect that would become known as gravity, and defined the law of universal gravitation. In the same work he presented the first analytical determination, based on Boyle's law, of the speed of sound in air. Newton's postulate of an invisible force able to act over vast distances led to him being criticized for introducing "occult agencies" into science.

With the *Principia*, Newton became internationally recognized. He acquired a circle of admirers, including the Swiss-born mathematician de Duillier, with whom he formed a close relationship that lasted until 1693; it ended abruptly when Newton suffered a nervous breakdown.

In the 1690s, Newton wrote a number of religious tracts dealing with the literal interpretation of the Bible. A manuscript he sent to John Locke in which he disputed the existence of the Trinity was never published. Later works – *The Chronology of Ancient Kingdoms Amended* (1728) and *Observations Upon the Prophecies of Daniel and the Apocalypse of St. John* (1733) – were published after his death. He also devoted a great deal of time to alchemy.

Most modern historians believe that Newton and Leibnitz developed infinitesimal calculus independently, using their own unique notations. According to Newton's inner circle, Newton had worked out his method years before Leibnitz, yet he published almost nothing about it until 1693, and did not give a full account until 1704. Meanwhile, Leibnitz began publishing his methods in 1684. Whereas Leibnitz's notebooks show the advancement of the ideas from early stages

until maturity, there is only the end product in Newton's known notes. Newton claimed that he had been reluctant to publish his calculus because he feared being mocked for it.

Starting in 1699, members of the Royal Society (of which Newton was a member) accused Leibnitz of plagiarism, and the dispute broke out in full force in 1711. The Royal Society claimed that it was Newton who was the true discoverer and labeled Leibnitz a fraud. This claim was cast into doubt when it was later found that Newton wrote the study's concluding remarks on Leibnitz. Thus began the bitter Newton *vs.* Leibnitz calculus controversy, which marred the lives of both men until Leibnitz's death in 1716.

Newton was a member of the Parliament of England from 1689 to 1690 and in 1701. His only recorded comments were to complain about a cold draft in the chamber and request that the window be closed.

Newton moved to London to take up the post of warden of the Royal Mint in 1696, a position that he had obtained through the patronage of Charles Montagu, 1st Earl of Halifax, then Chancellor of the Exchequer. He took charge of England's great recoinage. Newton became the best-known Master of the Mint in 1699, a position he held until his death. These appointments were intended as sinecures (an office which requires little or no responsibility, labor, or active service), but Newton took them seriously, retiring from his Cambridge duties in 1701, and exercising his power to reform the currency and punish counterfeiters.

As Master of the Mint in 1717 Newton unintentionally moved the Pound Sterling from the silver standard to the gold standard by setting the bimetallic relationship between gold coins and the silver penny in favor of gold. This caused silver sterling coin to be melted and shipped out of Britain.

Newton was made President of the Royal Society in 1703 and an associate of the French Academy of Sciences. In his position at the Royal Society, Newton made an enemy of John Flamsteed, the Royal Astronomer, by prematurely publishing Flamsteed's star catalog, which he had used in his studies.

In April 1705 Queen Anne knighted Newton during a royal visit to Trinity College, Cambridge. The knighthood is likely to have been motivated by political considerations connected with the Parliamentary election in May 1705, rather than any recognition of Newton's scientific work or services as Master of the Mint.

Newton died in London, and was buried in Westminster Abbey. Having never married and with no children, he had divested much of his estate to relatives in his last years and died intestate.

After his death, Newton's body was discovered to have had massive amounts of mercury in it, probably resulting from his alchemy pursuits. Mercury poisoning could explain Newton's eccentricity in late life.

Noteworthy: His inspiring works led many to regard him as one of the two most revolutionary figures in science (Albert Einstein being the other). Newton's stature among scientists remains at the very top rank. In a 2005 survey

of scientists in Britain's Royal Society asking who had the greater effect on the history of science, Newton or Albert Einstein. Newton was deemed the more influential.

The brachystochrone problem was presented as a challenge to mathematicians throughout Europe by Johann Bernoulli in 1696. (The problem is to find the fastest path for a body to fall from one point to another lower point with gravity as the only force affecting the body.) Jacob Bernoulli, L'Hôpital, Leibnitz, and others submitted solutions. Newton submitted anonymously. About this solution Johann Bernoulli is said to have remarked: "You can tell the lion from his claws!" The solution is a cycloid curve.

Newton argued that light is composed of particles or corpuscles, but he had to associate them with waves to explain the diffraction of light. Later physicists favored a purely wavelike explanation of light to account for diffraction. In today's quantum mechanics, photons and the idea of wave-particle duality bear only a minor resemblance to Newton's understanding of light.

In his *Hypothesis of Light* of 1675, Newton posited the existence of ether to transmit forces between particles. After a renewed interest in alchemy, he replaced the ether with occult forces based on Hermetic ideas of attraction and repulsion between particles. J.M. Keynes, who acquired many of Newton's writings on alchemy, stated that "Newton was not the first of the age of reason: he was the last of the magicians." Newton's interest in alchemy cannot be isolated from his contributions to science. (This was at a time when there was no clear distinction between alchemy and science.) Had he not relied on the occult idea of action at a distance, across a vacuum, he might not have developed his theory of gravity.

Newton himself told the story that he was inspired to formulate his theory of gravitation by watching an apple fall from a tree. "It fell straight down — why was that?" It is known from his notebooks that Newton was grappling in the late 1660s with the idea that terrestrial gravity extends, in an inverse-square proportion, to the Moon; it took him two decades to develop the full-fledged theory.

Although Newton pursued his studies of physics in England, he was well aware of scientific developments flowering throughout Europe.

There are several famous mottoes attributed to Newton, including "I do not invent hypotheses." He said, "Gravity explains the motions of the planets, but it cannot explain who set the planets in motion. God governs all things and knows all that is or can be done." Newton warned against using the laws of motion and universal gravitation to view the Universe as a mere machine, as if akin to a great clock. (For this Leibnitz lampooned him: "God Almighty wants to wind up his watch from time to time: otherwise it would cease to move. He had not, it seems, sufficient foresight to make it a perpetual motion.")

In 1662 Newton wrote out a list of all the sins he could remember having committed. Addressed directly to God, the list gives a fascinating glimpse into Newton's conscience. Per-

haps the two most striking features of the list are how short it is and how innocuous most of the “sins” now seem. Reflecting his puritanical upbringing, Newton many years after the event felt guilty about several minor instances of Sabbath-breaking, including “Squirting water on Thy day” and “Making pies on Sunday night,” and the misdeeds of “Idle discourse on Thy day and at other times.”

In his lifetime, Newton wrote more on religion than on natural science. He placed the crucifixion of Jesus at April 3, 33, which agrees with one traditionally accepted date. He attempted, unsuccessfully, to find hidden messages within the Bible. In a 1704 manuscript, he describes his attempts to extract scientific information from the Book of Daniel.

He estimated that the world would end no earlier than 2060. In predicting this he wrote, “This I mention not to assert when the time of the end shall be, but to put a stop to the rash conjectures of fanciful men who are frequently predicting the time of the end, and by doing so bring the sacred prophesies into discredit as often as their predictions fail.”

He was a scholar of deep faith. He detailed the precise dimensions of the Jewish temple in Jerusalem and wrote treatises on the daily practices in the temple. He penned phrases in careful Hebrew. Newton interpreted Biblical prophesies to mean that the Jewish people would return to the Holy Land before the world ends. The end of days will see “the ruin of the wicked nations, the end of weeping and of all troubles, the return of the Jews captivity and their setting up a flourishing and everlasting Kingdom.”

Newton saw evidence of design in the system of the world: “Such a wonderful uniformity in the planetary system must be allowed the effect of choice.” But Newton insisted that divine intervention would eventually be required to reform the system, due to the slow growth of instabilities.

A one sentence summary of his life is: “Newton was an English physicist and mathematician, who in addition to developing classical mechanics and gravitation, invented calculus (along with Leibnitz, 1646-1716) and showed that white light contains the colors of the rainbow, but he was entirely without humor and retired from research, spending the last third of his life as a government official following a nervous breakdown in 1693.” (**Physics Demonstrations**, J.C. Sprott, University of Wisconsin Press, 2006, p.1.)

In summing up Newton’s brilliant career, Alexander Pope (generally regarded as the greatest English poet of the eighteenth century,) wrote in Newton’s honor: “Nature and Nature’s laws lay hid in night; God said, Let Newton be! And all was light.”

2.2.5 Energy Formulation

Newton’s second law of motion is a cumbersome way to derive the equations of motion for a system having more than a few degrees of freedom. In [Chapter 5](#), we intro-

duce *Lagrange’s* energy approach for the derivation of governing equations. Here, an early glimpse is provided into the use of energy to develop the equations of motion.¹¹

In a vibratory system there is generally an interplay of *kinetic* and *potential* energies. We are familiar with gravitational potential energy. *Strain energy* is another kind of potential energy stored in an element such as a spring. If the system is *conservative*, meaning that energy is conserved without dissipation or losses, the total energy remains constant. The sum of the kinetic T and potential/strain V energies is

$$T + V = \text{constant.} \quad (2.19)$$

For a spring element, the energy stored in the spring due to an elongation x equals the area under the F vs. x curve. In a linear force-deflection relation of slope k , this area equals $kx^2/2$. If the spring has an initial stretch x_0 , then the total strain energy stored is $k(x + x_0)^2/2$. A change in the vertical position x of the mass m results in a change in the gravitational potential energy equal to $-mgx$.¹²

In a mass-spring system with a massless spring, the kinetic energy of the system is due to the motion of the mass and equals $m\dot{x}^2/2$. Substituting into [Equation 2.19](#) gives

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(x + x_0)^2 - mgx = \text{constant.}$$

Differentiating this equation with respect to time yields

$$(m\ddot{x} + kx)\dot{x} + (kx_0 - mg)\dot{x} = 0.$$

Recognizing that in static equilibrium $kx_0 = mg$ represents the force balance in the free-body diagram of [Figure 2.12](#), we have

$$(m\ddot{x} + kx)\dot{x} = 0.$$

Since $\dot{x}(t)$ cannot be zero for all time t , it must be that

$$m\ddot{x} + kx = 0.$$

This is the governing equation of motion derived from only energy considerations. For a system with damping, the procedure is not as simple. Energy loss due to dissipation is discussed in [Chapter 3](#).

From our knowledge of differential equations, we know that second-order differential equations without damping

¹¹In using Newton’s second law of motion for a system with many connected masses, internal forces between masses will appear in the equations of motion. We do not generally need these forces, but they are among the unknowns of the problem.

¹²Gravitational potential energy is defined as the work done against the gravitational field g to elevate a body of mass m a distance x above, in this case, an arbitrary reference plane called the *datum*.

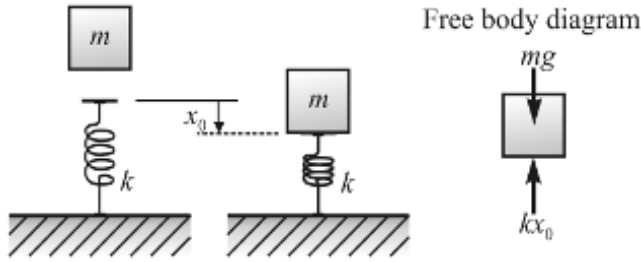


Figure 2.12: At the left, the position of a body of mass m and a spring of constant k are shown before they come into contact. In the middle, the static equilibrium position of the mass is shown. This is the rest position after the mass is placed on the spring. At the right, the free-body diagram of the mass in static equilibrium is shown.

have harmonic solutions of the form $A \sin \omega t$. In Section 2.3 we will learn how to solve such differential equations. Here it is also possible to find the natural frequency ω_n of this system directly from energy considerations. For conservative systems with mass m and stiffness k , the maximum kinetic energy equals the maximum potential energy since no dissipation occurs,

$$T_{max} = V_{max}.$$

The displacement of an oscillatory system may be represented by the harmonic function $x = A \sin \omega t$. Its velocity is then $\dot{x} = A \omega \cos \omega t$, and

$$V_{max} = \frac{1}{2} k x_{max}^2 = \frac{1}{2} k (A \sin \omega t)_{max}^2 = \frac{1}{2} k A^2$$

$$T_{max} = \frac{1}{2} m \dot{x}_{max}^2 = \frac{1}{2} m (A \omega \cos \omega t)_{max}^2 = \frac{1}{2} m A^2 \omega^2,$$

and therefore

$$\frac{1}{2} m A^2 \omega^2 = \frac{1}{2} k A^2,$$

or $\omega^2 = k/m$, the square of the natural frequency. We will see a more sophisticated and useful version of this idea when multiple degree-of-freedom systems are studied and *Rayleigh's quotient* is introduced in Chapter 6.

Vertical Motion vs. Pendulum Motion

The vertical motion of a body suspended on a spring is governed by the same mathematical equation as the linearized equation for the angular motion of a pendulum oscillating about its equilibrium position. There is a physical difference though. When the suspended body is in its static equilibrium position, there is energy stored in the spring since the body's weight is balanced by the spring force kx_0 . When the pendulum mass is at static equilibrium, since the cable or rod to which it is attached is not elastic, there is no stored energy in the system.¹³

¹³ Some refer to the effect of gravity as gravitational stiffness for pendulum problems.

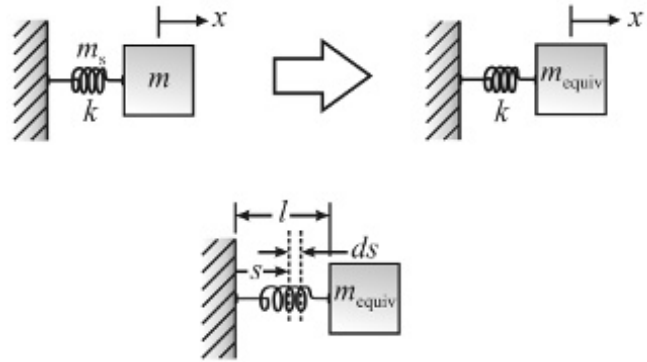


Figure 2.13: Equivalent mass-spring system accounting for inertia of the spring. On the left, the original system is composed of mass m and spring of mass m_s and stiffness k . At the right, the lumped equivalent mass m_{equiv} is determined so that the behavior of the system with a massless spring is the same as that of the original mass m with spring of mass m_s . At the bottom, an element of the spring length used in the calculation is shown.

Gravitational potential energy can be taken to be zero at this location. While in motion both the pendulum body and the suspended body have their maximum kinetic energies as they pass through equilibrium.

All of our models have assumed lumped-parameter properties – the spring has only the property of stiffness and the mass has only the property of inertia. This means that the spring is massless. Physical springs, of course, have mass, stiffness, and even damping properties.

The next example demonstrates the importance of including the mass in analyzing a physical spring.

Example 2.4 Springs with Mass

In general, we ignore the mass of a spring when deriving an equation of motion with the justification that the mass of the body connected to the spring is much larger than the mass of the spring. Sometimes, the assumption is unrealistic. Derive the equations of motion for a mass-spring system accounting for the mass of the spring.

Solution: If the mass of the spring is comparable to the mass of the object to which it is attached, then the combined system is most appropriately modeled as a continuous system. (This is addressed in Chapter 7.)

If the mass of the spring cannot be ignored, then a single degree-of-freedom model can be developed that includes the inertial effect of the spring. This case is considered here. In effect, we increase the mass of the body so that it represents both the mass of the spring and body, as shown schematically in Figure 2.13.

For the system, the kinetic energy is $T_{actual} = T_s + (1/2)m\dot{x}^2$, and for the equivalent system, $T_{equiv} = (1/2)m_{equiv}\dot{x}^2$, where T_s is the kinetic energy of a spring

of mass m_s and m_{equiv} is an equivalent mass of both the spring and the body. By making both kinetic energies equal, the value of T_s can be used to determine m_{equiv} .

Consider the unstretched spring of length l . An element of this spring of length ds has mass dm_s . If the body is displaced a distance x , then a point on the linear spring at location s will be displaced a proportionate distance

$$u(s) = \frac{x}{l}s, \quad (2.20)$$

where $u(s)$ represents the deflection of the spring at any location s . If the point on the spring is at the fixed end, then $u(0) = 0$, and for the point at the end connected to the body, $u(l) = x$. The kinetic energy of the element ds is $dT_s = (1/2)dm_s \dot{u}^2$, where $\dot{u} = (\dot{x}/l)s$ and $dm_s = (m_s/l)ds$. Therefore,

$$\begin{aligned} T_s &= \int dT_s = \int_0^l \frac{1}{2} \frac{m_s}{l} \left(\frac{s}{l} \dot{x} \right)^2 ds \\ &= \frac{1}{2} \frac{m_s}{l} \left(\frac{\dot{x}}{l} \right)^2 \int_0^l s^2 ds \\ &= \frac{m_s \dot{x}^2}{6}. \end{aligned}$$

Equating the original kinetic energy to the equivalent kinetic energy yields the equivalent mass m_{equiv} ,

$$\begin{aligned} T_s + \frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m_{equiv}\dot{x}^2 \\ \frac{m_s \dot{x}^2}{6} + \frac{m\dot{x}^2}{2} &= \frac{m_{equiv}\dot{x}^2}{2} \\ \Rightarrow m_{equiv} &= m + \frac{m_s}{3}. \end{aligned}$$

Thus, for a spring that behaves according to the linear displacement model of Equation 2.20, one-third of the mass of the spring is added to the mass of the attached body in order to capture the effect of spring inertia. ■

Example 2.5 Ship Vibration¹⁴

This example investigates a vibration problem encountered in a large luxury liner. The ship was driven by four propellers, each having a mass of 12,200 kg and each rotated by a long hollow shaft 71.6 m long with a 0.56 m OD (outer diameter) and 0.28 m ID (inner diameter). The shaft was made of plated steel with a density of 7.50 kg/m³ and modulus of elasticity of 2.05×10^{11} N/m². At the ship's cruising speed, the propellers were rotating at 258 rpm. Determine the natural frequency of longitudinal vibration of the propeller and shaft. What will happen if each shaft drives a four-bladed propeller? What

could be done to correct a possible vibration problem?

Solution: We focus on one propeller shaft and model it as having both longitudinal (that is, axial) stiffness and mass. The effective spring constant of the shaft is

$$k_s = \frac{P}{\Delta},$$

where P is the axial load on the shaft and Δ is the longitudinal deflection of the shaft given by

$$\Delta = \frac{Pl}{AE},$$

where l is the length of the shaft, A is the cross-sectional area of the shaft, and E is the modulus of elasticity of the shaft material. Combining these, the effective stiffness of the propeller shaft is

$$k_s = \frac{AE}{l},$$

or with numbers, where $A = \pi [(0.56)^2 - (0.28)^2] / 4 = 0.185$ m²,

$$\begin{aligned} k_s &= \frac{(0.185 \text{ m}^2) (2.05 \times 10^{11} \text{ N/m}^2)}{71.6 \text{ m}} \\ &= 5.30 \times 10^8 \text{ N/m}. \end{aligned}$$

The mass of the propeller shaft is its density times its volume,

$$m_s = (7.50 \text{ kg/m}^3) (0.185 \text{ m}^2) (71.6 \text{ m}) = 99,200 \text{ kg},$$

which is an order of magnitude larger than the mass of the propeller. Thus, calculating the natural frequency without considering the mass of the effective spring – the propeller shaft – would result in a significant error.

Accounting for the mass of the shaft, and using the results of the previous example, the natural frequency for longitudinal vibration is

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_s}{m + m_s/3}} = \sqrt{\frac{5.30 \times 10^8 \text{ N/m}}{12,200 + 99,200/3 \text{ kg}}} \\ &= 108 \text{ rad/s}, \end{aligned}$$

or, equivalently,

$$f_n = \frac{\omega_n}{2\pi} = 17.2 \text{ Hz} = 1032 \text{ cycles/min} = 1302 \text{ rpm}.$$

At the cruising speed corresponding to 258 rpm of the propeller shaft, one blade of the four-bladed propeller passes by a critical area between the propeller and the hull $4 \times 258 = 1032$ times/min, or every 0.058 s. This matches the natural frequency (1032 cycles/min). Thus, we predict a serious vibration problem at the normal cruising speed, which did occur. The problem was corrected by using three-bladed propellers. ■

¹⁴This problem is presented and solved in **An Introduction to Mechanical Vibrations**, R.F. Steidel, Jr., 3rd ed., Wiley, 1989, pp.89-90. Steidel indicates that the entire problem was chronicled in the *Journal of the Society of Naval Architects and Marine Engineers*.

2.2.6 Representing Harmonic Motion

Most motion due to vibration is periodic. Periodic motion repeats itself at equal time intervals. The simplest form of periodic motion is harmonic motion represented by a sine or cosine function, and is called *simple periodic motion*. More complicated periodic motion represented by the summation of sine and cosine functions – harmonic components – is called *complex periodic motion*. Although harmonic motion is always periodic, periodic motion is not always harmonic.

Consider harmonic motion of a mass-spring system with its displacement represented by

$$x(t) = a \cos \omega t.$$

Differentiating with respect to t gives the velocity and acceleration

$$\frac{dx}{dt} = -a\omega \sin \omega t = a\omega \cos \left(\omega t + \frac{\pi}{2} \right)$$

and

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos \omega t = a\omega^2 \cos (\omega t + \pi).$$

We can draw several conclusions.

- If the displacement is harmonic at a given frequency, then the velocity and acceleration are also harmonic at the same frequency.
- The amplitude of the velocity is $a\omega$ and the amplitude of the acceleration is $a\omega^2$. The velocity amplitude is thus a linear function of the frequency, whereas the acceleration amplitude is a quadratic function of the frequency. This means that if the frequency is high, then the acceleration amplitude can be quite high even if the displacement amplitude is small.
- The velocity and acceleration are $\pi/2$ rad = 90° and π rad = 180° , respectively, ahead of the displacement. These are the phase angles. We say that the velocity leads the displacement by 90° and the acceleration leads the displacement by 180° . Thus, the velocity, with a magnitude $a\omega$, has a phase angle that is 90° ahead of that of displacement. The acceleration, with magnitude $a\omega^2$, has a phase angle that is 90° ahead of that of velocity.
- The magnitudes of the stiffness, damping, and inertia forces are frequency-dependent since stiffness is related to x , damping to $\dot{x}$, and inertia to $\ddot{x}$. Therefore, with low-frequency loading, the dominant force is due to the stiffness, and with high-frequency loading, the dominant force is due to the inertia.

We will revisit these concepts in our study of mass-spring-damper systems, when we consider the magnitude and phase of the response as a function of frequency.

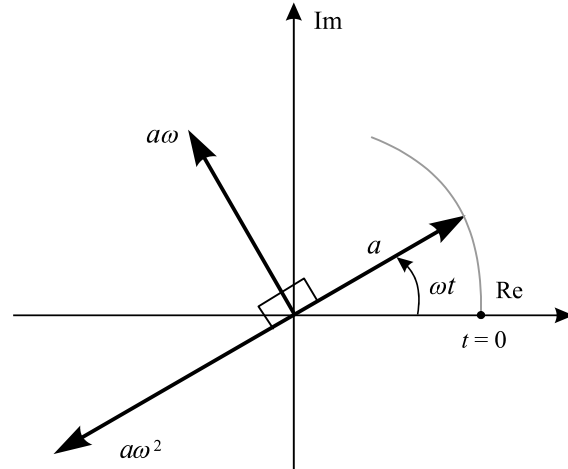


Figure 2.14: Vector representation of harmonic motion. The length of a vector is the amplitude, and the angle of the vector measured in the positive counterclockwise direction from the real axis is the phase angle.

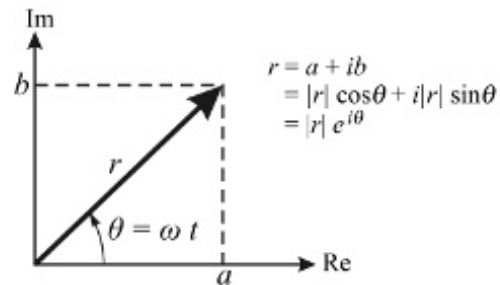


Figure 2.15: A representation of a complex number.

Rotating Vector Representation

Harmonic or periodic motion of a mass-spring system may be easily represented by a rotating vector. Consider the displacement $x(t)$ represented graphically by a vector with magnitude $x(t) = a \cos \omega t$. As ωt increases in the range $0 \leq \omega t \leq 2\pi$ rad, $x(t)$ prescribes a circle of radius a .

The angle of the vector with the horizontal is ωt , as shown in Figure 2.14, from which we obtain the term *circular frequency* for variable ω , in units of *radians per second*. Since one *cycle* or revolution is equivalent to 2π rad, the frequency can be defined equivalently as $f = \omega/2\pi$ *cycles per second* or *Hertz* (Hz). Figure 2.14 also shows the velocity vector of magnitude $a\omega$ leading the displacement by 90° and the acceleration vector of magnitude $a\omega^2$ leading the velocity by 90° .

*Complex numbers*¹⁵ of the form $a + ib$, where a and b are real-valued constants, can be used to represent rotating vectors as well and the imaginary number $i = \sqrt{-1}$. Complex numbers can be portrayed as coordinates in the *complex plane*, shown in Figure 2.15.

¹⁵See Appendix A for more details on complex numbers.

Consider the complex number r drawn as a vector from the origin to the coordinate point (a, b) . It has magnitude $|r|$, and can be written as $r(\cos \omega t + i \sin \omega t)$ for an angle of ωt with the horizontal. For harmonic motion $r \cos \omega t$ is equivalent to the horizontal component of the rotating vector or *the real part of the complex number*. Note that *Euler's formula*, $e^{i\omega t} = \cos \omega t + i \sin \omega t$, can be used to represent harmonic motion in the compact complex exponential form, where either the real or imaginary part of the expression is retained depending on whether the cosine or sine term is desired.

Work Done by Harmonic Loading

It is of interest to calculate the *work* done by a harmonic force $F(t) = F_0 \sin(\omega t + \phi)$ acting on a body that is responding harmonically to this force according to $x(t) = x_0 \sin \omega t$.¹⁶ This could be the work done by a linear actuator driving a load to create reciprocating motion at the same frequency.¹⁷ The work done in moving a body an increment dx in the direction of the force is $F(t)dx$. Considering one cycle of x motion, the argument ωt varies from 0 to 2π rad. To evaluate the work done over one oscillation period, let $F(t)dx = F(t)\frac{dx}{dt}dt$, with $0 \leq t \leq 2\pi/\omega$. Then, with the appropriate change of variables, the work equals

$$\begin{aligned} W &= \int_0^{2\pi/\omega} F(t) \frac{dx}{dt} dt \\ &= \frac{1}{\omega} \int_0^{2\pi} F(t) \frac{dx}{dt} d(\omega t) \\ &= F_0 x_0 \int_0^{2\pi} \sin(\omega t + \phi) \cos \omega t d(\omega t) \\ &= F_0 x_0 \cos \phi \int_0^{2\pi} \sin \omega t \cos \omega t d(\omega t) \\ &\quad + F_0 x_0 \sin \phi \int_0^{2\pi} \cos^2 \omega t d(\omega t). \end{aligned} \quad (2.21)$$

The first integral in Equation 2.21 equals zero, and the second integral¹⁸ equals π . Therefore,

$$W = \pi F_0 x_0 \sin \phi. \quad (2.22)$$

We can interpret this result by noting that Equation 2.21 contains two parts. In the first part, the force is out of phase with the velocity. In the second part, the force is *in phase* with the velocity. *A force does work only with that component in phase with the velocity.*

¹⁶ A cosine function could have equivalently been chosen for both the force and displacement.

¹⁷ Chapter 10 discusses the use of actuators for vibrational control.

¹⁸ The term $\cos^2 x$ varies between 0 and 1, with an average value of $1/2$ over a range of 2π . Therefore, the integral (area) equals $(1/2) \times (2\pi)$, or π .

Example 2.6 Work Done by a Harmonic Force with a Different Frequency¹⁹

Consider the case in which the frequency of the force is different than that of the displacement response. Assume that the force is $F(t) = F_0 \sin n\omega t$ and the harmonic response to this force is $x(t) = x_0 \sin(m\omega t + \phi)$. Show that the work done by this force during a period is zero, where $T = 2\pi/\omega$, and m and n are integers with $m \neq n$.

Solution: The work done per cycle is $\int F(t)dx$, or

$$\begin{aligned} W &= \int_0^T F(t) \frac{dx}{dt} dt \\ &= F_0 x_0 m\omega \int_0^T \sin n\omega t \cos(m\omega t + \phi) dt. \end{aligned}$$

When the cosine is expanded, two integrals result of the form

$$\int_0^T \sin n\omega t \sin m\omega t dt, \quad \int_0^T \sin n\omega t \cos m\omega t dt,$$

as before, except that both are now equal to zero. Thus, the work done by a harmonic force of a frequency different than the response frequency does no work.

If $m = n$,

$$\int_0^T \sin n\omega t \cos m\omega t dt = \frac{T}{2},$$

so that $W = -\pi F_0 x_0 n \sin \phi$, which can be evaluated for particular parameter values. ■

To conclude:

- The work done by a harmonic force of one frequency that creates a harmonic displacement or velocity of a different frequency is zero if the time interval includes an integer number of force cycles and a different integer number of velocity cycles.
- The work done by a harmonic force that is $\pi/2$ rad or 90 degrees out of phase with a harmonic velocity of the same frequency is zero during a full cycle.
- The work done by a harmonic force of amplitude F_0 and frequency ω , in phase with a harmonic velocity $v_0 = x_0\omega$ of the same frequency, is $\pi F_0 v_0/\omega = \pi F_0 x_0$ over a full cycle.

¹⁹ This example is adapted from *Mechanical Vibrations*, J.P. Den Hartog, Dover, 1984, p.15.

GOTTFRIED WILHELM von LEIBNITZ

(July 5, 1646 – November 14, 1716)



Contribution: Leibniz was a German mathematician and philosopher who wrote primarily in Latin and French. He invented infinitesimal calculus independently of Newton, and developed the present day notation for differential and integral calculus, including $\int f(x)dx$ and $d(x^n) = nx^{n-1}dx$.

Leibniz made major contributions in physics and technology, and anticipated notions that surfaced much later in biology, medicine, geology, probability theory, psychology, linguistics, and information science. He was a strong proponent of the binary system, the foundation of computer architectures.

Leibniz devised a new theory of dynamics, based on kinetic energy and potential energy, which posited space as relative, whereas Newton felt strongly space was absolute.

Leibniz was a serious inventor, engineer, and applied scientist, with great respect for practical life. He argued that theory be combined with practical application, and has been called the father of applied science. He designed wind-driven propellers and water pumps, mining machines to extract ore, hydraulic presses, lamps, submarines, clocks, etc. He invented a steam engine (with Denis Papin) and proposed a method for desalinating water.

He claimed that the earth has a molten core, a finding of modern geology.

In philosophy, he is remembered for optimism, that is, his conclusion that our universe is, in a restricted sense, the best possible one God could have made. He, along with René Descartes and Baruch Spinoza, was one of the three greatest 17th-century rationalists, yet his philosophy looked back to the scholastic tradition and anticipated modern logic and analysis.

He wrote on politics, law, ethics, theology, history, philology, and even verse. His contributions to this vast array of subjects are scattered in journals and in tens of thousands of letters and unpublished manuscripts.

Life Sketch: Leibniz was born in Leipzig, Saxony, now Germany. His father, a Professor of Moral Philosophy at the University of Leipzig, died when he was six. He learned his

religious and moral values from his mother. These would exert a profound influence on his philosophical thought in later life.

His father left a personal library to which Leibniz was granted free access from age seven onwards. By 12, he had taught himself Latin, which he used freely all his life, and had begun studying Greek.

He entered his father's university at age 14 and completed university studies by 20, specializing in law and mastering the standard university courses in classics, logic, theology, and scholastic philosophy. However, his education in mathematics was not up to the French and British standards. In 1666 he published his first book *On the Art of Combinations*.

When the University of Leipzig declined to assure him a position teaching law upon graduation, Leibniz submitted the thesis he had intended to submit to the University of Altdorf instead, and obtained his doctorate in law in five months in 1667. He then declined an offer of academic appointment at Altdorf, because he had "very different things in view." He spent the rest of his life in the service of two major German noble families.

Leibniz's first position was as a salaried alchemist in Nuremberg, even though he knew nothing about the subject. He soon met Johann Christian von Boineburg (1622–1672), the dismissed chief minister of the Elector of Mainz. Von Boineburg hired Leibniz as an assistant, and shortly thereafter reconciled with the Elector and introduced Leibniz to him. Leibniz then dedicated an essay on law to the Elector in the hope of obtaining employment. The strategy worked; the Elector asked Leibniz to assist with the redrafting of the legal code for his Electorate. In 1669, Leibniz was appointed Assessor in the Court of Appeal. Although von Boineburg died late in 1672, Leibniz remained under the employment of his widow until she dismissed him in 1674.

Von Boineburg did much to promote Leibniz's reputation, and the latter's memoranda and letters began to attract favorable notice. Leibniz's service to the Elector soon followed a diplomatic role. The main European geopolitical reality during Leibniz's adult life was the ambition of Louis XIV of France, backed by French military and economic might. Meanwhile, the Thirty Years' War had left German-speaking Europe exhausted, fragmented, and economically backward.

Leibniz proposed to protect German-speaking Europe by distracting Louis as follows. France would be invited to take Egypt as a stepping stone towards an eventual conquest of the Dutch East Indies. In return, France would agree to leave Germany and the Netherlands undisturbed. This plan obtained the Elector's cautious support. In 1672, the French government invited Leibniz to Paris for discussion, but the plan was soon overtaken by events and became irrelevant. (Napoleon's failed invasion of Egypt in 1798 was an unwitting implementation of Leibniz's plan.)

Leibniz spent several years in Paris, where he met Dutch physicist and mathematician Christian Huygens and realized that his own knowledge of mathematics and physics was

spotty. With Huygens as mentor, he began a program of self-study that soon pushed him to making major contributions to both subjects, including inventing his version of differential and integral calculus. He met the leading French philosophers of the day, and studied the writings of Descartes and Pascal.

When it became clear that France would not implement its part of Leibnitz's Egyptian plan, the Elector sent his nephew, escorted by Leibnitz, on a related mission to the English government in London early in 1673. There Leibnitz demonstrated to the Royal Society a calculating machine he had been designing and building since 1670. It was the first such machine that could execute all four basic arithmetical operations. The Society made him an external member. The mission ended when news of the Elector's death arrived, whereupon Leibnitz promptly returned to Paris.

The sudden deaths of Leibnitz's two patrons in the same winter meant that Leibnitz had to find work. In this regard, a 1669 invitation from the Duke of Brunswick to visit Hanover proved fateful. Leibnitz declined the invitation, but began corresponding with the Duke in 1671. In 1673, the Duke offered him the post of Counsellor which Leibnitz reluctantly accepted two years later, after it became clear to him that there was no employment in Paris, where he relished intellectual stimulation.

Leibnitz managed to delay his arrival in Hanover until the end of 1676, after making one more short journey to London, where he possibly was shown some of Newton's unpublished work on the calculus. (This was deemed evidence supporting the accusation, made decades later, that he had stolen the calculus from Newton.) On the journey from London to Hanover, Leibnitz stopped in The Hague where he met Leeuwenhoek, the discoverer of microorganisms. He also spent several days in intense discussion with Spinoza, who had just completed his masterwork, the *Ethics*. Leibnitz respected Spinoza's intellect, but was dismayed by his conclusions that contradicted both Christian and Jewish orthodoxy.

In 1677, he was promoted, at his request, to Privy Counsellor of Justice, a post he held for the rest of his life. Leibnitz served three consecutive rulers of the House of Brunswick as historian, political adviser, and most consequentially as librarian of the ducal library. He wrote on various political, historical, and theological matters involving the House of Brunswick; the resulting documents form a valuable part of the historical record for the period.

Leibnitz founded the Berlin Academy in 1700 and was its first president. He became a recluse in his later years in Hanover, where he died at age 70.

Noteworthy: The last years of his life were embittered by controversy over whether he had discovered the differential calculus independently of Newton. Newton claimed with justification that "not a single previously unsolved problem was solved" by Leibnitz. But the formalism of Leibnitz's approach was to prove vital in the development of calculus. Leibnitz never thought of the derivative as a limit. (This first appears in the work of d'Alembert.)

In 1684, Leibnitz published details of his differential calculus in *Acta Eruditorum*, a journal established in Leipzig two years earlier. The paper contained the familiar d notation for derivative, the rules for computing the derivatives of powers, products, and quotients. However, it contained no proofs and the Bernoullis called it an enigma rather than an explanation.

In 1686 Leibnitz published in *Acta Eruditorum* a paper dealing with the integral calculus with the first appearance in print of the $\int$ notation. Newton's *Principia* appeared the following year. Newton's "method of fluxions" was written in 1671, but Newton failed to get it published. It did not appear in print until John Colson produced an English translation from the Latin in 1736, resulting in a dispute with Leibnitz.

Leibnitz's *vis viva* (Latin for living force) was mv^2 , twice the modern kinetic energy. He realized that the total energy would be conserved in certain mechanical systems, so he considered it an innate motive characteristic of matter. Here too his thinking gave rise to another regrettable dispute. His *vis viva* was seen as rivaling the conservation of momentum championed by Newton in England and by Descartes in France; hence academics in those countries tended to neglect Leibnitz's idea. Eventually *vis viva* was found useful, and the two approaches were seen as complementary.

The Duke of Brunswick tolerated the enormous effort Leibnitz devoted to intellectual pursuits unrelated to his duties as a courtier, pursuits such as perfecting the calculus, writing about mathematics, logic, physics, and philosophy, and keeping up a vast correspondence. He began working on the calculus in 1674; the earliest evidence of its use in his surviving notebooks is 1675. By 1677 he had a coherent system, but did not publish it until 1684. Leibnitz's most important mathematical papers were published between 1682 and 1692.

The Elector Ernst August commissioned Leibnitz to write a history of the House of Brunswick, going back to the time of Charlemagne or earlier, hoping that the resulting book would advance his dynastic ambitions. From 1687 to 1690, Leibnitz traveled extensively in Germany, Austria, and Italy, seeking and finding archival materials bearing on this project. Decades went by but no history appeared; the next Elector became quite annoyed. Leibnitz never finished the project, in part because of his huge output on many other fronts, but also because he insisted on writing a meticulously researched and erudite book based on archival sources. They never knew that he had carried out a fair part of his assigned task. When the material Leibnitz had written and collected for his history of the House of Brunswick was finally published in the 19th century, it filled three volumes.

Even though Leibnitz was a life member of the Royal Society and the Berlin Academy of Sciences, neither organization saw fit to honor his passing. His grave went unmarked for more than 50 years.

Leibnitz never married. He complained on occasion about money, but the fair sum he left to his sole heir, his sister's stepson, proved that the Brunswicks had paid him well.

The product rule of differential calculus is still called “Leibnitz’s law.” In addition, the theorem that tells how and when to differentiate under the integral sign is called the Leibnitz integral rule.

Until the discovery of subatomic particles and the quantum mechanics governing them, many of Leibnitz’s speculative ideas about aspects of nature made little sense. For instance, he anticipated Einstein by arguing, against Newton, that space, time, and motion are relative, not absolute.

In 1934, Norbert Wiener claimed to have found in Leibnitz’s writings a mention of the concept of feedback, central to Wiener’s later cybernetic theory.

In 1677, Leibnitz called for a European confederation, governed by a council or senate, whose members would represent entire nations and would be free to vote their consciences. In doing so, he anticipated the European Union. He believed that Europe would adopt a uniform religion. He reiterated these proposals in 1715.

Leibnitz devoted considerable intellectual and diplomatic effort to what would now be called ecumenical endeavor, seeking to reconcile first the Roman Catholic and Lutheran churches, and later the Lutheran and Reformed churches. These efforts involved Leibnitz in a fair bit of theological controversy.

2.2.7 Solving the Equations of Motion

The next section begins to develop the solution of the governing equation of motion of a mass-spring system.²⁰ We solve Equation 2.5, from which the different classes of vibratory behavior can be understood. We also address how to interpret the solutions as functions of the loading, initial conditions, and damping (discussed in the next chapter).

The solution of the governing equation of motion can be carried out in several ways. In one approach called the *time-domain* solution,²¹ one first solves the homogeneous differential equation (right-hand side set to zero). Then, the particular problem (solution for the specific right-hand side) is found. Next, the two solutions are added, and finally the initial conditions are satisfied in order to specify the arbitrary constants.

An alternate approach is to derive the response characteristics such as amplitude as functions of the ratio of the driving frequency to the natural frequency. These solutions are in the *frequency-domain* for which the independent variable is the driving frequency. The frequency-domain solution includes the use of the *Laplace transform*,²² where the differential equation is transformed

into an algebraic equation. One major difference between the time-domain approach and the frequency-domain approach is that the Laplace transform satisfies the initial conditions as part of the solution. Vibration with damping and the Laplace transform method are discussed in Chapter 3.

2.3 Undamped Free Vibration

The undamped forced oscillator is representative of a mass-spring system in which an external force acts on the mass and damping plays a negligible role in the response. It is governed by the equation

$$\ddot{x}(t) + \omega_n^2 x(t) = \frac{1}{m} F(t), \quad (2.23)$$

where $x(t)$ is the displacement, and $F(t)$ is the external force. To solve for the “free” vibration, we set $F(t)$ equal to zero. The free response depends only on the *initial conditions* of the system, that is, $x(0)$ and $\dot{x}(0)$, the initial displacement and initial velocity, respectively.

Since the left-hand side of Equation 2.23 includes the displacement $x(t)$ and its second derivative $\ddot{x}(t)$, the solution, when differentiated twice, is a function that reappears: $\ddot{x}(t) = -\omega_n^2 x(t)$. Functions with this property are harmonic functions, such as *sines* and *cosines*. Therefore, we assume a solution of the form

$$x(t) = C_1 \sin \omega t + C_2 \cos \omega t, \quad (2.24)$$

where ω needs to be determined and constants C_1 and C_2 can be found by satisfying the initial conditions. Double differentiating and substituting the assumed solution into the governing equation results in

$$(-\omega^2 + \omega_n^2)C_1 \sin \omega t + (-\omega^2 + \omega_n^2)C_2 \cos \omega t \equiv 0,$$

which must be satisfied *identically*²³ for all t . Thus,

$$-\omega^2 + \omega_n^2 = 0,$$

meaning ω equals the natural frequency of oscillation, and, therefore, $\omega = \omega_n = \sqrt{k/m}$ for the mass-spring system. (The negative value of the square root is discounted since there are no negative frequencies.) Figure 2.16 shows how the natural frequency varies as a function of the ratio of stiffness to mass.

By replacing ω by $\sqrt{k/m}$, Equation 2.24 becomes

$$x(t) = C_1 \sin \sqrt{\frac{k}{m}} t + C_2 \cos \sqrt{\frac{k}{m}} t. \quad (2.25)$$

²⁰The solution of this equation is developed in a first course in ordinary differential equations. A review is provided in Appendix A.

²¹The time-domain solution is simply the response of the system represented by the differential equation as a function of time.

²²A review of the Laplace transform is included in Appendix A.

²³The term *identically* is represented using the notation $\equiv$ and means that the equality holds for each part of the relation. Thus, the coefficient of the sine term on the left-hand side of the equation must equal that on the right-hand side. The same is true for the coefficient of the cosine term. In this case, the right-hand side is zero.

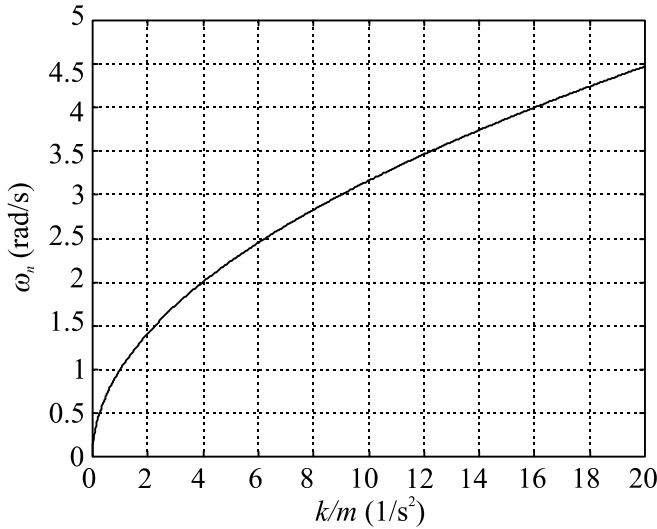


Figure 2.16: Natural frequency ω_n as a function of the ratio k/m .

The constants of integration C_1 and C_2 can be evaluated from the initial displacement $x(0)$ and initial velocity $\dot{x}(0)$. From Equation 2.25, we write equations for $x(0)$ and $\dot{x}(0)$ to find that $C_2 = x(0)$ and $C_1 = \dot{x}(0)/\omega_n$. The general response in terms of the initial conditions is then

$$x(t) = x(0) \cos \omega_n t + \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t. \quad (2.26)$$

For the undamped system the oscillation continues for all time with the period $T = 2\pi/\omega_n = 2\pi\sqrt{m/k}$. The response is a function of the initial conditions, with the frequency of oscillation ω_n . Since there is no decrease in amplitude with time, this case is purely theoretical. All physical systems dissipate energy and displacements decay.

Equation 2.26 can be expressed equivalently in terms of an amplitude, frequency, and phase of vibration by writing

$$x(t) = A \cos(\omega_n t - \phi), \quad (2.27)$$

where the *amplitude* is A and the *phase angle* is ϕ .²⁴ These can be found by expanding the cosine term²⁵ and equating it to Equation 2.26,

$$A = \sqrt{[x(0)]^2 + \left(\frac{\dot{x}(0)}{\omega_n}\right)^2} \quad (2.28)$$

$$\phi = \tan^{-1} \left(\frac{\dot{x}(0)}{x(0) \omega_n} \right). \quad (2.29)$$

In this form we can see the effects of the initial conditions on the response amplitude and phase. In Equation 2.29

²⁴ The phase angle is shown with a negative sign in Equation 2.27 in anticipation of the damped response, in which the phase of the system response *lags* behind the forcing.

²⁵ $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$.

care must be taken in calculating ϕ since the value of the *arctan* depends on the signs of the argument numerator and denominator to know the quadrant.

GALILEO'S OBSERVATION. The story is told that the great Italian scientist Galileo Galilei (February 15, 1564–January 8, 1642) used to amuse himself by watching the swinging of the great bronze chandelier that hung from the ceiling of the cathedral in Pisa. The rope suspending the chandelier from the ceiling was very long and so the chandelier swung backwards and forwards very slowly.

Galileo observed what happened on many occasions. On each occasion, the chandelier swung differently; sometimes making only very small swings, and sometimes larger ones. He keenly realized that the period of time taken by the swinging chandelier to complete a single out and back swing was the same regardless of how far it swung. Given a large push, it went further than if it were given a small one. If it went further, it went faster and took the same time to get back to where it started from as if it were pushed very gently.

Galileo's acute observation led to the idea of the pendulum clock. The first working version was made in 1656 by Dutch physicist Christiaan Huygens (April 14, 1629–July 8, 1695).

Example 2.7 Effect of Initial Conditions, Case I

Plot the displacement response time history of Equation 2.26 for the case $x(0) = 1$ m, $\dot{x}(0) = 1$ m/s, and $\omega_n = 1$ rad/s.

Solution: From ω_n the cyclic natural frequency is $f_n = \omega_n/2\pi = 0.159$ Hz and thus the period is $T = 1/f_n = 2\pi$ s or 6.28 s. The response is of the form

$$x(t) = C_1 \cos t + C_2 \sin t.$$

Applying initial conditions, we find

$$x(t) = 1 \cos t + 1 \sin t \text{ m.}$$

If we wish to express the response in terms of cosine with an amplitude and phase such that

$$x(t) = A \cos(t - \phi),$$

we can use the cosine sum identity given by

$$A \cos(t - \phi) = A \cos \phi \cos t + A \sin \phi \sin t.$$

Comparing the two expressions for the response, we find that

$$A \cos \phi = 1 \text{ and } A \sin \phi = 1,$$

or

$$A = \sqrt{2} \text{ and } \tan \phi = 1.$$

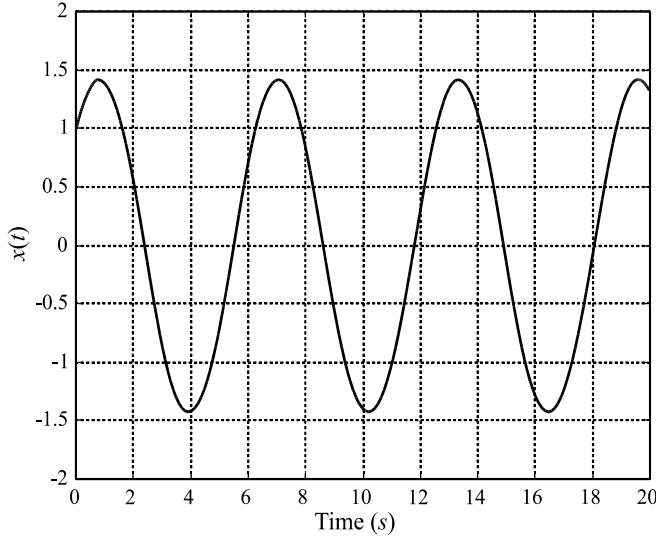


Figure 2.17: Response to initial displacement $x(0) = 1$ m, initial velocity $\dot{x}(0) = 1$ m/s, and natural frequency $\omega_n = 1$ rad/s.

We need to take care that the phase angle is in the correct quadrant. The quadrant where $\cos \phi$ is positive and $\sin \phi$ is positive is the first quadrant, and thus the correct phase angle is

$$\phi = \pi/4 \text{ rad.}$$

The response is given by

$$x(t) = \sqrt{2} \cos(t - \pi/4) \text{ m.}$$

It is possible to express the response in terms of a sine function with an amplitude and phase. Using the sine sum formula, the response is given by

$$\begin{aligned} x(t) &= 1 \cos t + 1 \sin t \\ &= A \sin(t - \phi) \\ &= A \cos \phi \sin t - A \sin \phi \cos t. \end{aligned}$$

Then,

$$A \cos \phi = 1 \text{ and } A \sin \phi = -1.$$

We find

$$A = \sqrt{2} \text{ and } \phi = -\pi/4 \text{ rad.}$$

The response is given by

$$x(t) = \sqrt{2} \sin(t + \pi/4) \text{ m.}$$

The response is graphed in Figure 2.17. The initial conditions are satisfied as indicated by the displacement and slope at $t = 0$. There is no decay in the solution since there is no damping. ■

Example 2.8 Effect of Initial Conditions, Case II
Find the displacement response time history of Equation

2.26 for the case $x(0) = -1$ m, $\dot{x}(0) = -1$ m/s, and $\omega_n = 1$ rad/s. Rewrite the expression in terms of cosine function with an amplitude and phase angle.

Solution: Applying initial conditions, we find

$$x(t) = -1 \cos t - 1 \sin t \text{ m,}$$

which can be rewritten as

$$\begin{aligned} x(t) &= A \cos(t - \phi) \\ &= A \cos \phi \cos t + A \sin \phi \sin t. \end{aligned}$$

Then, A and ϕ must satisfy

$$\begin{aligned} A \cos \phi &= -1 \\ A \sin \phi &= -1. \end{aligned}$$

With positive amplitude, A , the phase angle is in the third quadrant since both cosine and sine are negative. Then,

$$A = \sqrt{2} \text{ and } \phi = \frac{5}{4}\pi \text{ rad,}$$

giving

$$x(t) = \sqrt{2} \cos\left(t - \frac{5}{4}\pi\right) \text{ m.}$$

Note that if the initial conditions are substituted into Equation 2.29 directly, a calculator may give the phase angle in the first quadrant. Some calculators have a built-in function that takes two arguments, numerator and denominator, to yield the correct phase angle. ■

2.3.1 Alternate Formulation

Instead of the harmonic form of Equation 2.24, an alternate approach in solving the equation of motion is to assume the form $x(t) = Ce^{\lambda t}$, where C and λ are to be determined. The exponential form of the solution is often used in the solution of differential equations. In vibration studies it is used because it leads to functions that are important for characterizing a system. An example is the frequency response function, derived in the next chapter.

Differentiating the assumed form and substituting into the governing equation leads to $\lambda^2 + \omega_n^2 = 0$ or $\lambda_{1,2} = \pm i\omega_n$, and thus

$$\begin{aligned} x(t) &= D_1 e^{i\omega_n t} + D_2 e^{-i\omega_n t} \\ &= D_1 (\cos \omega_n t + i \sin \omega_n t) + D_2 (\cos \omega_n t - i \sin \omega_n t) \\ &= (D_1 + D_2) \cos \omega_n t + i(D_1 - D_2) \sin \omega_n t \\ &= B_1 \cos \omega_n t + B_2 \sin \omega_n t, \end{aligned} \tag{2.30}$$

which is the same form as Equation 2.25 since D_1 and D_2 must be complex numbers²⁶ in order for $x(t)$ to be real-valued. The constants of integration, B_1 and B_2 , are real and are evaluated by satisfying the initial conditions,

$$x(0) = B_1 \text{ and } \dot{x}(0) = B_2\omega_n. \quad (2.31)$$

Therefore, the general solution, in terms of the initial conditions and the natural frequency, matches Equation 2.26.

Example 2.9 Springs in Series and in Parallel

Consider two springs of stiffness k_1 and k_2 in the configurations shown in Figure 2.18. (a) Determine the effective stiffness of a system with the two springs in series and of a system with the two springs in parallel. (b) Assuming that a mass m is suspended from these springs, find the general relations for the respective frequencies of oscillation. (c) For the case where $k_1 = 200$ N/m, $k_2 = 400$ N/m, and $m = 50$ kg, calculate the values of the frequencies.

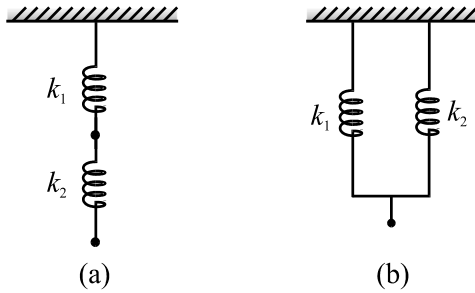


Figure 2.18: Two springs (a) in series and (b) in parallel.

Solution: (a) Extend the series configuration of the two springs by applying a static force F_s at the end. Let x_1 be the deflection of a point in between the two springs, and x_2 be the deflection of the end. Then

$$x_1 = \frac{F_s}{k_1}, \quad x_2 = \frac{F_s}{k_2},$$

since the same F_s acts on each spring. The total deflection is the sum,

$$x = x_1 + x_2 = F_s \left(\frac{1}{k_1} + \frac{1}{k_2} \right) = F_s \frac{1}{k_s},$$

where k_s is the equivalent or effective series spring constant,

$$k_s = \frac{k_1 k_2}{k_1 + k_2}.$$

²⁶Not only must D_1 and D_2 be complex numbers, they must be complex conjugates. If $D_1 = a_1 + ib_1$ and $D_2 = a_2 + ib_2$, then $B_1 = D_1 + D_2$ can be a real number, as it must, only if $ib_1 + ib_2 = 0$, that is, $b_1 = -b_2$. Similarly, $B_2 = i(D_1 - D_2)$ can only be a real number if $a_1 - a_2 = 0$, that is, $a_1 = a_2$. Thus, we have shown that D_1 and D_2 must be complex conjugates. We can easily show that the following equalities hold: $B_1 = 2a_1 = 2a_2$, $B_2 = -2b_1 = 2b_2$.

For the parallel configuration, a static force F_p is applied at the end resulting in the same deflection x of each spring. Each spring resists a part of the force,

$$F_{p1} = k_1 x, \quad F_{p2} = k_2 x.$$

The total force equals the sum of the two forces,

$$F_p = F_{p1} + F_{p2} = (k_1 + k_2)x = k_p x,$$

where k_p is the equivalent or effective parallel spring constant, $k_p = k_1 + k_2$.

(b) The frequencies of oscillation are obtained from the equation, $\omega_n = \sqrt{k/m}$. For the series arrangement,

$$\omega_{ns} = \sqrt{\frac{k_s}{m}} = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}$$

and for the parallel arrangement,

$$\omega_{np} = \sqrt{\frac{k_p}{m}} = \sqrt{\frac{k_1 + k_2}{m}}.$$

(c) For the parameter values given,

$$k_s = \frac{200 \cdot 400}{200 + 400} = 133 \text{ N/m}$$

$$k_p = 200 + 400 = 600 \text{ N/m}$$

$$\omega_{ns} = \sqrt{\frac{133}{50}} = 1.63 \text{ rad/s}$$

$$\omega_{np} = \sqrt{\frac{600}{50}} = 3.46 \text{ rad/s}.$$

The effective stiffness and natural frequency depend on the configuration of the two springs. For springs in series, the effective stiffness is always less than the stiffness of the softest spring. For springs in parallel, it is always greater than the stiffness of the stiffest spring. ■

Example 2.10 Discontinuous Forces²⁷

In the mass-spring system shown in Figure 2.19, the mass is connected to a spring on the left side. In addition, there is a spring on the right side that is not connected to the mass. It acts on the mass if the mass displaces a distance greater than the gap d . Determine the equation or equations of motion that can account for the possibility of the mass coming in contact with the right spring. Also determine the period for a cycle. The parameter values are $m = 20$ kg, $k_1 = 2.5$ N/mm, $k_2 = 6.0$ N/mm.

²⁷This example considers a nonlinear system (because of the gap). The problem is introduced to highlight the utility of piecewise linear solutions. Chapter 11 introduces the subject of nonlinear vibration.

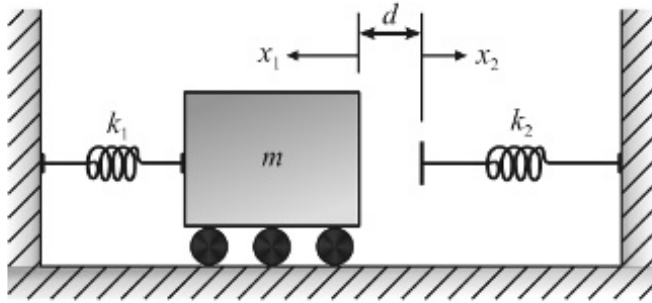


Figure 2.19: Mass-spring system subjected to possible discontinuous spring force.

The mass is moved 80 mm to the left from static equilibrium and released from rest. At equilibrium the mass is $d = 30$ mm from the right spring.

Solution: The behavior is analyzed in two segments. In the first segment, the mass moves only under the action of the left spring force. Defining x_1 as the position of the mass, with positive direction to the left, the equation of motion is

$$m\ddot{x}_1 + k_1x_1 = 0.$$

From Equation 2.25, the displacement response is

$$x_1(t) = A \sin \omega_n t + B \cos \omega_n t,$$

where $\omega_n = \sqrt{k_1/m} = \sqrt{2.5 \cdot 1000/20} = 11.18$ rad/s. Since the initial conditions are $x_1(0) = 80$ mm and $\dot{x}_1(0) = 0$,

$$x_1(t) = 0.08 \cos 11.18t \text{ m.}$$

The next step is to calculate the time it takes to arrive at the right spring, traveling to position $x_1(t_1) = -0.03$ m,

$$-0.03 = 0.08 \cos 11.18t_1$$

giving

$$t_1 = 0.175 \text{ s.}$$

When the mass touches the right spring, we reset the time to zero and define the deflection of both springs in terms of x_2 , with positive direction to the right. The equation of motion now includes the forces of both springs, and is given by

$$m\ddot{x}_2 + k_1(x_2 + 0.03) + k_2x_2 = 0,$$

or

$$20\ddot{x}_2 + 2500(x_2 + 0.03) + 6000x_2 = 0$$

or

$$\ddot{x}_2 + 425x_2 = -3.75.$$

The solution is

$$x_2(t) = C \sin 20.62t + D \cos 20.62t - 0.00882 \text{ m.}$$

To find C and D we satisfy the initial conditions, $x_2(0) = 0$ and $\dot{x}_2(0) = -\dot{x}_1(0.175)$, where

$$\dot{x}_1(t) = -0.08(11.18) \sin 11.18t.$$

Therefore, $\dot{x}_2(0) = 0.829$ m/s. Thus, $D = 0.00882$ and $C = 0.0402$, yielding

$$x_2(t) = 0.0402 \sin 20.62t + 0.00882 \cos 20.62t - 0.00882 \text{ m.}$$

Finally, we need to calculate the time at which the mass stops moving, t_2 , which, when added to t_1 , yields the half period of oscillation. The value of t_2 is found by setting $\dot{x}_2(t) = 0$,

$$\dot{x}_2(t_2) = 0.829 \cos 20.62t_2 - 0.181 \sin 20.62t_2 = 0,$$

or

$$\begin{aligned} \tan 20.62t_2 &= \frac{0.829}{0.181} \\ t_2 &= \frac{1.356}{20.62} = 0.0658 \text{ s.} \end{aligned}$$

Therefore, the period of oscillation is

$$\begin{aligned} T &= 2(t_1 + t_2) = 2(0.175 + 0.0658) \\ &= 0.482 \text{ s,} \end{aligned}$$

and the cycles continue without slowing down since there is no damping in the system. ■

2.3.2 Phase Plane

Another way to envision the motion of an unforced harmonic oscillator is in the *phase plane*. The phase plane is created by replacing the second-order governing equation $\ddot{x}(t) + \omega_n^2 x(t) = 0$ by two equivalent first-order differential equations,

$$\begin{aligned} \frac{dx(t)}{dt} &= y(t) \\ \frac{dy(t)}{dt} &= -\omega_n^2 x(t). \end{aligned}$$

Substituting the derivative of the first equation into the second equation recovers the original second-order equation.

For a mass-spring system, the path of the mass in the xy plane can be found as follows:

$$\begin{aligned} \frac{dy/dt}{dx/dt} &= -\frac{\omega_n^2 x}{y} \\ \frac{dy}{dx} + \frac{\omega_n^2 x}{y} &= 0 \\ y dy + \omega_n^2 x dx &= 0 \\ y^2 + \omega_n^2 x^2 &= c, \end{aligned} \tag{2.32}$$

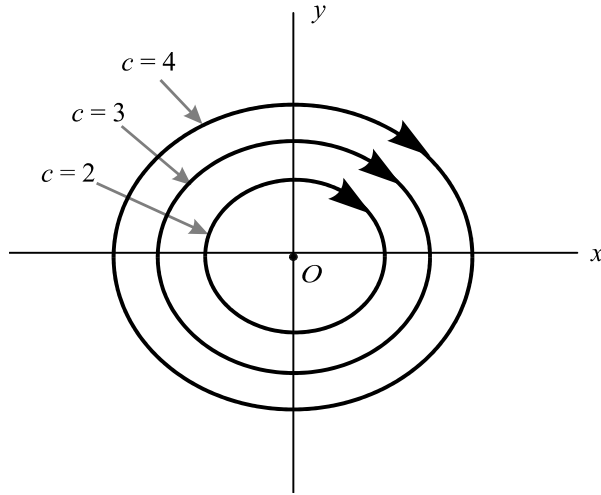


Figure 2.20: Phase plane plot for $\ddot{x} + \omega_n^2 x = 0$. The paths are concentric ellipses governed by $y^2 + \omega_n^2 x^2 = c$.

where the last equation is obtained by integrating the third equation and c is the constant of integration. (In these equations, we have left off the explicit dependency on time.)

Since x is the displacement and y is the velocity, x^2 is related to potential energy and y^2 is related to kinetic energy. Thus, Equation 2.32 can be viewed as an equation for constant energy and c is related to the total system energy. Curves for different values of c are shown in Figure 2.20. The arrows indicate the direction of motion, which can be determined by selecting a quadrant and evaluating the rate of change of a parameter. For example, in the first quadrant, $y = \dot{x} > 0$, then $y > 0$ and x increases with time.

This perspective for depicting the trajectory of oscillatory motions is useful for understanding the behavior of linear systems and especially of nonlinear systems studied in Chapter 11.

Next we consider the response of a dynamic system to a harmonic load, the most important loading case found in practice.



Figure 2.21: Rowing can be viewed as periodic forcing. (Seattle, WA; photograph by M. Nagurka)

2.4 Harmonically Forced Vibration

We begin with governing Equation 2.23 with $F(t) = A \cos \omega t$ with amplitude A and driving frequency ω , where $\omega \neq \omega_n$. For the particular response, assume a solution of the form²⁸

$$x_p(t) = B_1 \cos \omega t.$$

Differentiating the assumed solution twice and substituting the appropriate expressions into the governing equation $\ddot{x} + \omega_n^2 x = (A/m) \cos \omega t$ gives

$$B_1(-\omega^2 + \omega_n^2) \cos \omega t = \frac{A}{m} \cos \omega t,$$

from which

$$B_1 = \frac{A/m}{-\omega^2 + \omega_n^2} = \frac{A/k}{1 - (\omega/\omega_n)^2}, \quad (2.33)$$

where in the last equality, both numerator and denominator were divided by ω_n^2 . The particular response is then

$$x_p(t) = \frac{A/k}{1 - (\omega/\omega_n)^2} \cos \omega t. \quad (2.34)$$

The term A/k in the numerator is the static deflection under a constant force A . Defining $x_{st} = A/k$, the dimensionless displacement is given by

$$\frac{x_p(t)}{x_{st}} = \frac{1}{1 - (\omega/\omega_n)^2} \cos \omega t. \quad (2.35)$$

The coefficient $1/[1 - (\omega/\omega_n)^2]$ represents a *dynamic amplification factor* and is a function of the frequency ratio ω/ω_n . Von Kármán²⁹ called it a *resonance factor*.³⁰ This factor acts like a *filter* that allows signals of certain frequencies to pass through it and blocks signals of other frequencies. We will learn more about this factor later.

The complete solution (free plus forced vibration) is then

$$x(t) = C_1 \sin \omega_n t + C_2 \cos \omega_n t + \frac{x_{st}}{1 - (\omega/\omega_n)^2} \cos \omega t, \quad (2.36)$$

where C_1 and C_2 are determined by satisfying the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$. Using Equation 2.36 to evaluate $x(0)$ and its derivative $\dot{x}(0)$, we can find the amplitudes

$$C_1 = \frac{v_0}{\omega_n}, \quad C_2 = x_0 - \frac{x_{st}}{1 - (\omega/\omega_n)^2},$$

²⁸If we selected the force $F(t) = A \sin \omega t$, then we would assume a response $x(t) = B_1 \sin \omega t$ and arrive at the same value for B_1 as in Equation 2.33. One expects the same response amplitude since the only difference between the sine and cosine is a phase difference of 90 deg. Also, if we would have assumed $x(t) = B_1 \cos \omega t + B_2 \sin \omega t$, the analysis would have led to $B_2 = 0$ and B_1 as in Equation 2.33.

²⁹A biography of von Kármán appears later in this chapter.

³⁰**Mathematical Methods in Engineering**, T. von Kármán and M.A. Biot, McGraw-Hill, 1940, p.134.

where C_2 is a function of the frequency ratio ω/ω_n . Therefore, the response to harmonic loading with arbitrary initial conditions is

$$\begin{aligned} x(t) &= \frac{v_0}{\omega_n} \sin \omega_n t + \left[x_0 - \frac{x_{st}}{1 - (\omega/\omega_n)^2} \right] \cos \omega_n t \\ &\quad + \frac{x_{st}}{1 - (\omega/\omega_n)^2} \cos \omega t \\ &= \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t \\ &\quad + \frac{x_{st}}{1 - (\omega/\omega_n)^2} [\cos \omega t - \cos \omega_n t]. \end{aligned} \quad (2.37)$$

Even though the load is harmonic, the response is not harmonic since it is the difference between two harmonic functions of different frequencies.



MICRO SCALE RESONATORS³¹

An exciting new field of research and development involving structural vibrations is that of resonant microelectromechanical systems, resonant MEMS.³² Put more simply, a MEMS resonator is a tiny mechanical structure (dimensions at the micrometer scale) fabricated in an integrated manner with electrical circuitry, the latter providing a means of actuating the mechanical structure into a resonant state and/or monitoring its resonant frequency. These structures are often designed as simple cantilever beams due to the ease of creating this simple geometry using standard microfabrication technologies.

We describe two important applications of cantilever-based MEMS resonators – chemical/biosensing and energy harvesting – and remark on how the underlying principles for the operation of these state-of-the-art devices are related to elementary vibration theory. Also of note is that the successful operation of these devices relies on the existence of resonance, which can occur due to the closeness to the natural frequency and the relatively small damping in the system. This differs from the more common situation in conventional mechanical/structural systems (automobiles, buildings, and bridges) in which resonance is to be avoided.

Chemical/Biosensing: A major advantage of a small-scale structure is that its resonant frequency can be quite sensitive to small perturbations in its environment. Stated less technically, a small entity can “see” or “feel” other small things that would not be perceptible to a larger entity; hence the advantage of microscale cantilever resonators. In particular, if the surface of the cantilever is “functionalized” in such a way that a particular target substance (the “analyte”), when present, will adhere to or be absorbed by the cantilever, the increase in the structure’s mass will result in a measurable decrease in the resonant frequency.

³¹The authors are grateful to Stephen M. Heinrich of Marquette University for this contribution.

³²**Resonant MEMS. Fundamentals, Implementation and Application**, O. Brand, I. Dufour, S.M. Heinrich, and F. Josse (eds.), Wiley-VCH Verlag, Weinheim, Germany, 2015.

A firm grasp of the underlying mechanics of the system enables one to correlate the characteristics of the analyte to the device’s resonant frequency shift. For example, tracking the resonant frequency shift may enable one to measure small concentrations of a toxin in the air or the growth history of a particular strain of bacteria when exposed to an experimental drug.

In terms of fundamental vibration theory, the cantilever’s response may be modeled as that of a simple mass-spring oscillator so that its resonant frequency is $\omega = \sqrt{k/m}$. Defining the mass sensitivity S of the cantilever as the change in resonant frequency with respect to the change in mass (due to the analyte), we can show through differentiation that $S = -\omega/2m$. Thus, the device is more sensitive to added mass if the device mass (m) is decreased or the resonant frequency (ω) is increased. Both of these are achieved with device miniaturization, the latter due to the fact that reduction of scale causes the beam’s mass to decrease more rapidly than its stiffness.

Energy Harvesting: In many microsensor applications the detection device is deployed in a remote or inaccessible location, making it difficult, expensive, or impossible to perform on-site repair. One way to make these devices more autonomous from a power standpoint is to “harvest” or “scavenge” kinetic energy from ambient vibrations and convert it into useful electrical energy needed to power the sensor. In this case a MEMS cantilever may be used as the energy harvester.

To function effectively the energy harvester must (1) have a natural frequency close to the frequency of ambient vibrations, and (2) include a transductive component (e.g., a coating) that converts energy from the mechanical domain to the electrical domain. Since most environmental vibrations are low frequency (usually no more than 100-200 Hz) and microcantilevers typically have much higher resonant frequencies, the first criterion may be addressed by using a flexible material for the harvester (reducing k) and adding a large mass onto the cantilever’s end (increasing m) to tune the resonant frequency to a level consistent with ambient vibrations. The transductive component noted in the second criterion may involve one of several “smart” materials (e.g., piezoelectric, magnetostrictive, or electrostrictive materials or an electroactive polymer) to convert the deformation energy caused by the ambient-induced harvester vibration into electrical energy.

The research community continues to explore various ways to maximize the deformation of energy harvesting cantilevers, including the design of optimally tapered cantilevers, to maximize the electrical power generated per unit volume of the device. Potential applications for small-scale energy harvesters are numerous, ranging from medicine to public security. One example is in the realm of structural health monitoring, in which the vibrations of a bridge due to traffic or wind may be converted into energy that can power bridge-mounted microsensors for measuring bridge deformation, temperature, moisture, or material degradation, thereby providing a criti-

cal early warning should structural performance be compromised.



Figure 2.22: To get rid of water in their fur, dogs shake at a frequency of about 4 to 6 Hz. Doing so, they remove about 70 percent of the water in their fur.

2.4.1 A Note on Terminology

The definitions of the terms transient, steady-state, free, and forced solutions in vibration as well as the term total solution warrant clarification. The summation of the transient and the steady-state solutions is the total solution:

1. The transient solution is the part of the solution that decays with time.
2. The steady-state solution is the part of the solution that persists over time.

The summation of the free and forced solutions is also the total solution:

1. The free solution is the part of the solution due to the initial conditions.
2. The forced solution is the part of the solution that is due to the external force.

In the undamped case, the particular solution is *not* the steady-state solution or the forced solution. For example, consider the undamped forced system,

$$\ddot{x} + \omega_n^2 x = \frac{A}{m} \cos \omega t.$$

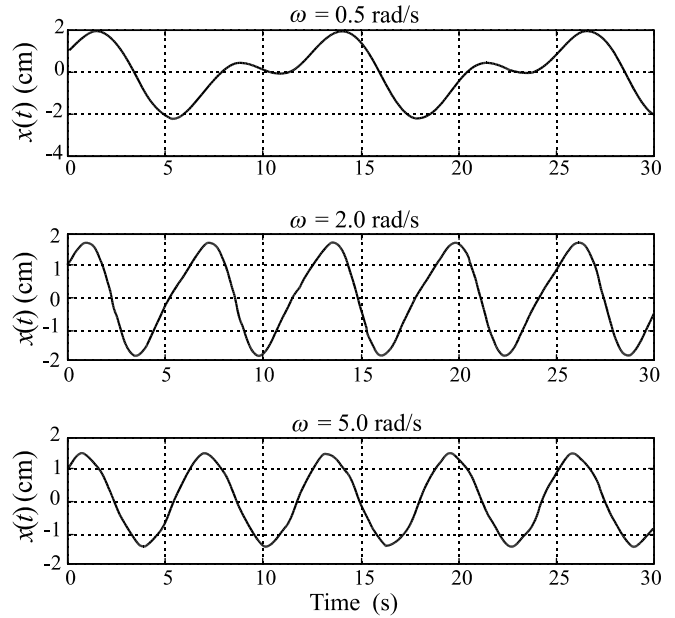


Figure 2.23: Displacement responses for the cases $\omega = 0.5, 2.0$, and 5.0 rad/s in [Example 2.11](#).

The total solution is given by [Equation 2.37](#), which can be written as

$$x(t) = \underbrace{\frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t}_{x_{\text{FREE}}} + \underbrace{\frac{x_{st}}{1 - (\omega/\omega_n)^2} [\cos \omega t - \cos \omega_n t]}_{x_{\text{FORCED}}}$$

or

$$x(t) = \underbrace{\frac{v_0}{\omega_n} \sin \omega_n t + \left(x_0 - \frac{x_{st}}{1 - (\omega/\omega_n)^2} \right) \cos \omega_n t}_{x_{\text{HOMOGENEOUS}}} + \underbrace{\frac{x_{st}}{1 - (\omega/\omega_n)^2} \cos \omega t}_{x_{\text{PARTICULAR}}}$$

In this case $x(t)$ is solely a steady-state solution since with no damping there is no transient response.

Example 2.11 Free Plus Forced Vibration

For the case where $x_0 = 1$ cm, $v_0 = 1$ cm/s, $\omega_n = 1$ rad/s, and $x_{st} = 1$ cm, plot the displacement responses for loading frequencies: $\omega = 0.5, 2$, and 5 rad/s.

Solution: The displacement time histories $x(t)$ are given for three loading frequencies $\omega = 0.5, 2$, and 5 rad/s in [Figure 2.23](#). We can see from the plots that for the higher driving frequency the maximum amplitude of the response is lower. We will explore this property in more depth later; it occurs because the

denominator in the third term in Equation 2.37 becomes larger. There are significant differences in the displacement responses depending on the loading frequencies. ■

Example 2.12 Driving Sinusoidal Motion

A rigid beam AB is pinned at A and driven at its free end with a small displacement $y = A \sin \omega t$, as shown in Figure 2.24. At the midpoint of the beam, point C , a mass m is suspended via a spring of constant k . (a) Derive a general expression for the maximum steady-state force on the beam at C due to the motion of the mass. (b) Calculate this force for the parameter values $m = 0.75$ kg, $A = 10$ mm, $l = 1$ m, $\omega = 10$ rad/s, and $k = 150$ N/m.

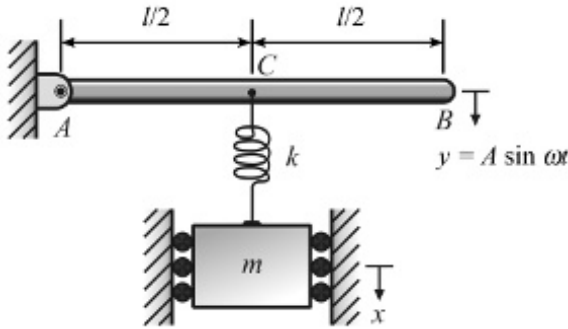


Figure 2.24: Mass driven by sinusoidal motion.

Solution: The equation of motion for the mass is

$$m\ddot{x} + k(x - x_c) = 0,$$

where $x_c = y/2$ is the displacement of point C on the beam. Since point C is midpoint on the beam, it oscillates at half the amplitude of the end. Therefore,

$$m\ddot{x} + kx = \left(\frac{kA}{2}\right) \sin \omega t.$$

The steady-state response is assumed to be $x(t) = D \sin \omega t$. Substituting this into the governing equation yields

$$-m\omega^2 D + kD = \left(\frac{kA}{2}\right),$$

which can be solved for the amplitude,

$$D = \frac{kA/2}{-m\omega^2 + k}.$$

(a) The maximum force on the beam occurs when the mass is at its maximum displacement. The maximum elongation of the spring equals $D - A/2$. The force on the beam equals k times the maximum elongation, added to the weight of the mass,

$$F_{\max} = mg + k(D - A/2).$$

(b) For the given parameters,

$$D = \frac{(150)(0.005)}{-(0.75)(10)^2 + 150} = 0.010 \text{ m}$$

and

$$\begin{aligned} F_{\max} &= (0.75)(9.81) + 150(0.010 - 0.005) \\ &= 8.11 \text{ N.} \end{aligned}$$

These numerical results could be used in a preliminary design to determine if the beam material is appropriate and whether a resizing is required. ■

2.4.2 Resonance

The denominator in Equation 2.34, $[1 - (\omega/\omega_n)^2]$, is a positive quantity for $\omega/\omega_n < 1$ and a negative quantity for $\omega/\omega_n > 1$. Physically, this means that when the natural frequency is greater than the forcing frequency, the force and motion are in phase, and when the forcing frequency is greater than the natural frequency, they are out of phase. The value of the denominator affects the magnitude of the steady-state response.

Vibration becomes dramatic when the forcing frequency is close to the natural frequency since the factor $1/[1 - (\omega/\omega_n)^2]$ in Equation 2.35 becomes very large when $\omega \simeq \omega_n$. When $\omega = \omega_n$ the theoretical prediction is infinite displacement amplitude. In real systems, damping effects are always present, limiting the displacement.

To avoid significant amplitudes, preliminary engineering designs attempt to keep natural frequencies as far as possible from driving frequencies. This is true regardless of the number of degrees of freedom. It becomes almost impossible to achieve for very large structures with many degrees of freedom since each degree of freedom has an associated natural frequency.

Consider again nondimensional Equation 2.35, which is plotted in Figure 2.25 for the case $x_{st} = 1.0$ cm and $\omega_n = 1.0$ rad/s. The driving frequencies are $\omega = 0.01, 0.5, 0.95$, and 10 rad/s. The set of graphs shows that the further away the driving frequency is below or above $\omega_n = 1.0$ rad/s, the smaller the amplitude of oscillation. (Note that the vertical scales have different values.) For lower driving frequencies, the response is almost static with very long periods. For higher driving frequencies, there is more rapid oscillation with very small oscillation amplitudes. The inertial effects preclude the system from keeping up with the higher forcing frequency.

In order to understand how the dynamic amplification factor $1/[1 - (\omega/\omega_n)^2]$, which is plotted in Figure 2.26, varies as a function of the driving frequency, we consider³³ three cases: (a) $0 < \omega/\omega_n \ll 1$, (b) $\omega/\omega_n \gg 1$, and (c) $\omega/\omega_n = 1$.

³³This approach is that of Den Hartog, p.44.

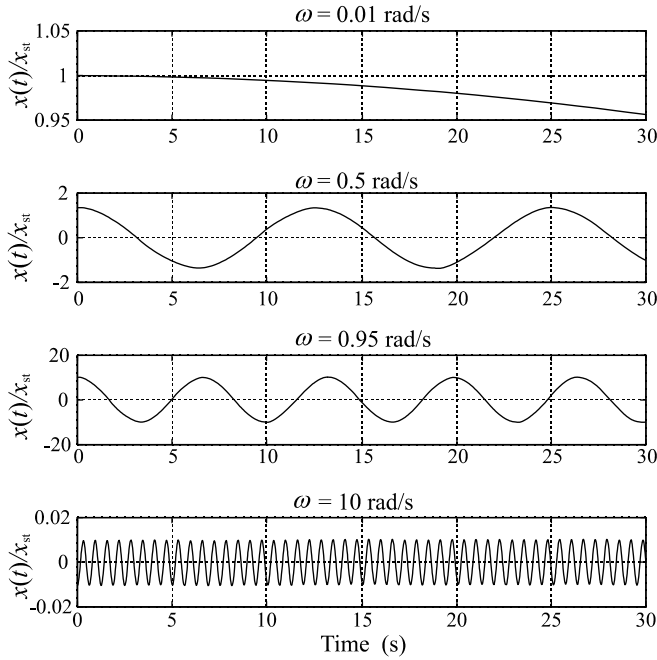


Figure 2.25: Response curves for Equation 2.35 for the case $x_{st} = 1.0$ cm and $\omega_n = 1.0$ rad/s. The driving frequencies are $\omega = 0.01, 0.5, 0.95$, and 10 rad/s. The largest displacement ratio (approximately 15) is seen for the driving frequency $\omega = 0.95$ rad/s, which is close to the natural frequency of 1.0 rad/s.

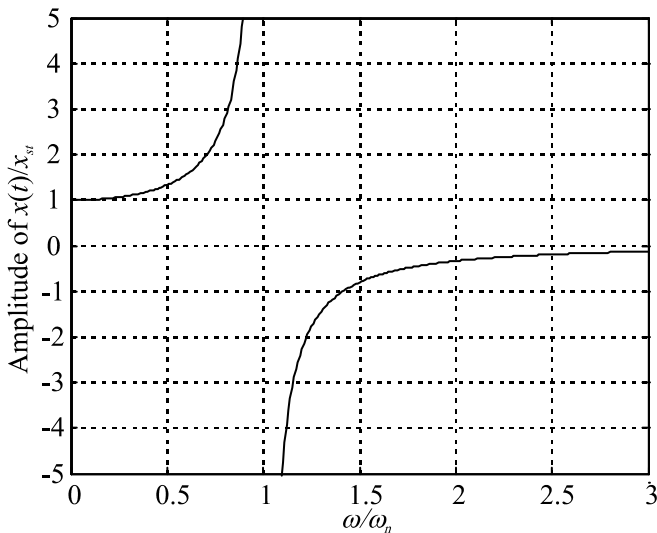


Figure 2.26: Dynamic amplification factor vs. frequency ratio for an undamped system.

For $\omega/\omega_n \ll 1$, the forcing frequency is very slow and the mass will be displaced essentially to its static deflection with very minor perturbations. In the limit as $\omega \rightarrow 0$, the static displacement is the response and $x(t)/x_{st} \rightarrow 1$, as Figure 2.26 shows.

For $\omega/\omega_n \gg 1$, the forcing frequency is very high and the mass cannot follow the rapidly oscillating force. Therefore, since the average value of the force is zero, in the limit for large ω , the displacement approaches zero, and $x(t)/x_{st} \rightarrow 0$.

The most interesting and important case is when $\omega/\omega_n = 1$, which means that the forcing frequency coincides exactly with the system's natural frequency. Theoretically, the response is infinite. It will be shown shortly that the response grows linearly requiring infinite time for infinite response. Practically, the force is continuously pushing the mass in the direction of motion, adding energy with each cycle. The system vibrates with ever-increasing amplitudes since there is no damping to dissipate energy. A small force can eventually make the amplitude very large, a phenomenon called *resonance*.³⁴ For the case of no damping, the natural frequency is the *resonant frequency*.

It is customary to plot Figure 2.26 as the absolute value of the amplification factor

$$\left| \frac{1}{1 - (\omega/\omega_n)^2} \right|$$

vs. frequency ratio (as we do in Figure 3.46) where the negative curve is flipped about the horizontal axis.

For the case where the driving frequency is equal to the natural frequency, $\omega = \omega_n$, the equation of motion is

$$\ddot{x} + \omega_n^2 x = \frac{A}{m} \cos \omega_n t. \quad (2.38)$$

From the theory of linear differential equations, the solution of a nonhomogeneous equation requires the sum of two *linearly independent* solutions, the homogeneous solution plus the particular solution. In assuming a particular solution for Equation 2.38, we cannot choose $x(t) = B_1 \cos \omega_n t$ since this would not be linearly independent of the homogeneous solution as it *is* the homogeneous solution. Therefore, we multiply the nonresonant homogeneous solution, Equation 2.24, by the independent variable t in order to generate the linearly independent solution

$$x(t) = B_1 t \cos \omega_n t + B_2 t \sin \omega_n t,$$

where both sine and cosine terms have been included.

To solve, we differentiate the assumed solution twice, substitute the required expressions into the governing

³⁴Resonance may be understood as a situation in which energy can most easily be exchanged between components of dynamic systems.

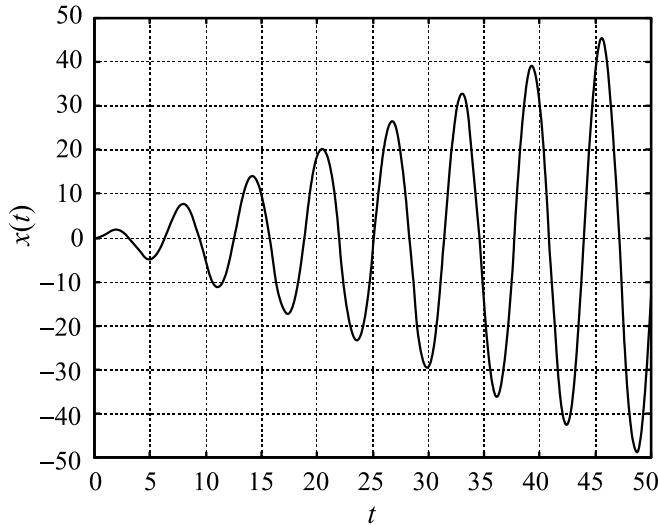


Figure 2.27: Secular oscillation for $x(t) = t \sin t$. When the driving frequency equals the natural frequency, each cycle of the oscillation leads to a larger peak amplitude.

equation of motion, factor common $\cos \omega_n t$ and $\sin \omega_n t$ terms, and find

$$\begin{aligned} \cos \omega_n t [-B_1 t \omega_n^2 + 2B_2 \omega_n + \omega_n^2 B_1 t] \\ + \sin \omega_n t [-B_2 t \omega_n^2 - 2B_1 \omega_n + \omega_n^2 B_2 t] \equiv \frac{A}{m} \cos \omega_n t. \end{aligned}$$

Satisfying the identity leads to two equations, which can be solved for the constants,

$$\begin{aligned} B_1 &= 0 \\ B_2 &= \frac{A}{2\omega_n m} = \frac{A}{2\sqrt{km}}. \end{aligned}$$

The response is then

$$x(t) = \frac{A}{2\sqrt{km}} t \sin \omega_n t, \quad (2.39)$$

where $x(t)$ grows without bound.

Expressions where time t appears as a factor are called *secular*. Since all systems possess some damping, Equation 2.39 is a theoretical solution. Even with damping a system can be driven to excessive amplitudes and failure if the driving frequency is close to the natural frequency. If $x(t)$ becomes large enough, the linearity assumption upon which the solution depends becomes invalid. Figure 2.27 depicts how a secular oscillation behaves for the special case of $x(t) = t \sin t$. If we suppose the stress at a point in the system depends on $x(t)$, then at some time the stress will exceed the yield strength of the material and the structure will fail.

Had the force been $(A/m) \sin \omega_n t$, then following the above procedure, the response would be $x(t) = -(A/2\sqrt{km}) t \cos \omega_n t$.

Example 2.13 Limit to Find Resonance Response Rather than solve the resonance response as a separate problem, consider the general solution, Equation 2.37, and solve for the response by taking a limit as ω approaches ω_n .

Solution: Assuming zero initial conditions, the general solution becomes

$$x(t) = \frac{x_{st}}{1 - (\omega/\omega_n)^2} [\cos \omega t - \cos \omega_n t].$$

We can rewrite this equation in the form

$$x(t) = \frac{x_{st}}{1 - (\omega/\omega_n)^2} \left[2 \sin \frac{(\omega_n + \omega)t}{2} \sin \frac{(\omega_n - \omega)t}{2} \right], \quad (2.40)$$

and define $2\varepsilon = \omega_n - \omega$, where $\varepsilon \ll 1$, to find

$$\begin{aligned} x(t) &= \frac{2x_{st}}{1 - (\omega/\omega_n)^2} [\sin(\varepsilon + \omega)t \sin \varepsilon t] \\ &= \frac{x_{st}\omega_n^2}{2\varepsilon(\varepsilon + \omega)} [\sin(\varepsilon + \omega)t \sin \varepsilon t]. \end{aligned} \quad (2.41)$$

There are two cases of interest: (a) resonance, where $\omega = \omega_n$, and (b) beating, where $\omega \simeq \omega_n$, i.e., very close but not equal to the natural frequency.

(a) If we take the limit of Equation 2.41 as $\varepsilon \rightarrow 0$, the result is indeterminate: $0/0$. Therefore, we use L'Hôpital's³⁵ rule to find

$$\lim_{\varepsilon \rightarrow 0} x(t) = x_{st}\omega_n^2 \left[\frac{t \sin \omega_n t}{2\omega_n} \right] = \frac{A}{2\sqrt{km}} t \sin \omega_n t,$$

where $x_{st} = A/k$ and $\omega \rightarrow \omega_n$. This is the same result found before in Equation 2.39.

(b) In Equation 2.40, we rewrite

$$\frac{1}{1 - (\omega/\omega_n)^2} = \frac{\omega_n^2}{(\omega_n + \omega)(\omega_n - \omega)}.$$

When the natural frequency and the driving frequency are close to each other, the factor $(\omega_n - \omega)$ in the denominator becomes very small, equaling 2ε , making the ratio very large as $\varepsilon \ll 1$. The phenomenon is called *beating*. When it occurs a harmonic motion with angular frequency $(\omega_n + \omega)/2 \simeq \omega_n$ has a periodic but slowly varying amplitude of frequency $(\omega_n - \omega)/2$. The beat frequency is twice the slow frequency or $\omega_{beat} = |\omega_n - \omega|$. The number of beats per second equals $|(\omega_n - \omega)/2\pi|$ with ω in units of rad/s.

At resonance, the period of beating becomes infinite and the buildup is continuous. Therefore, beating may be viewed as a near-resonance phenomenon. We will study beating in more detail for a two mass system in Chapter 6. ■

³⁵A biography of l'Hôpital is presented later in this chapter.



ROCKET BOOSTER VIBRATION

An interesting problem had to be solved in the development of the Ares I booster³⁶ called “thrust oscillation.” Unlike the liquid fuel rockets of the 1960s where sloshing occurred, with a solid fuel rocket the fuel starts burning at the bottom of the booster while the combustion works its way to the top. When the fuel is depleted, all that is left is the rocket structure (the hollow metal tube).

Initial reports stated that the Ares I natural frequency was approximately 12-15 Hz and the full stack of stages – the booster, second stage, and the Orion crew vehicle – also had a natural frequency of 15 Hz. So what appeared to exist was a source of energy from the empty booster driving the remaining structure at its natural frequency, resulting in a resonance condition.

Additional studies suggested that the two frequencies might be more than 10% apart, meaning that resonance may not actually have been a problem. Full scale tests were needed to resolve this potentially hazardous issue. If there actually was a problem, a redesign would be warranted so that the natural frequency could be shifted by changing the stiffness or mass properties of the structure. It was suggested that a damper be added between the first and second stages, or some of the mass be redistributed, to shift the resonant frequency.

MARCHING ON BRIDGES

The Angers Bridge, also called the Basse-Chaine Bridge, was a suspension bridge over the Maine River in Angers, France. It was built between 1836 and 1839. The bridge spanned 102 m, with two wire cables carrying a 7.2 m wide deck. Its towers consisted of cast iron columns 5.47 m tall.

The bridge collapsed on April 16, 1850, when 478 French soldiers (part of the Battalion of the 11th light infantry regiment) marched across it in lockstep. Because the soldiers were marching together, they caused the bridge to vibrate and twist from side to side, dislodging an anchoring cable from its concrete mooring.

A commission of inquiry established that the bridge had been constructed according to accepted norms and showed no defects in terms of construction or maintenance. Though a thunderstorm occurred during the collapse, the report indicated that the collapse was principally due to the soldiers instead of the storm. Of the 478 soldiers, 226 died in the river below the bridge.

The disaster was due to a combination of factors, principally dynamic load and corrosion of the anchors for the main cables. Gusts of wind caused the bridge to sway. The swaying was worsened by the almost spontaneous synchrony of the troops marching in step with the rhythm of the initial swaying motions giving rise to resonance.

As a result of the bridge’s collapse, soldiers were advised not to walk in step when crossing bridges, a reiteration of

much earlier advice given to Roman soldiers – to “break step” when marching over a bridge. Ironically, even crowds of people that do not intend to walk in step often do so to match the swaying of the bridge (as happened in the Angers Bridge failure) to help maintain their balance. The transfer of momentum from the pedestrians to the bridge structure leads to resonance. The same problem occurred in 2000 at the recently completed Millennium Bridge in London, when over 600 people walked across the bridge causing lateral vibration. The problem was corrected by installing large dampers to absorb the excess vibration of the structure. This is addressed in an example in [Chapter 3](#).

On the night of August 14, 2003, during New York City’s electrical blackout, the Brooklyn Bridge swayed as pedestrians trudged across it. It was attributed to pedestrian movement interacting with the bridge. The action and reaction of pedestrian traffic on bridges is a complex phenomenon, and bridges are not always designed for periodic loading from people traversing them.

One way to shield a bridge from the effects of pedestrian traffic is to include passive vibration absorbers – tuned mass-spring-damper systems. [Figure 2.28](#) shows an example of one such system installed on the Cumberland River Pedestrian Bridge in Nashville, Tennessee in 2007. Five absorbers were used to control lateral and vertical vibrations that might be caused by pedestrian traffic. We discuss vibration absorbers in [Section 6.4.3](#).

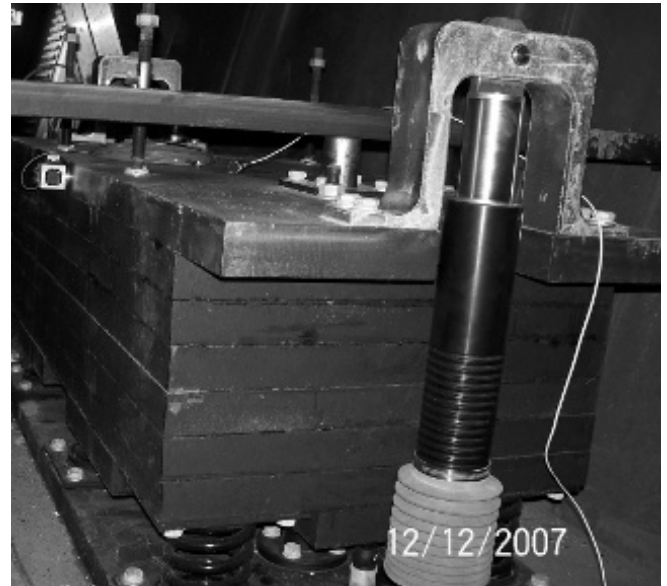


Figure 2.28: Tuned vibration absorber to mitigate vertical vibrations on the Cumberland River Bridge in Nashville, TN. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)

³⁶This example is based on news reports from February 2008.

AN AUTOMOTIVE VIBRATION PROBLEM³⁷

An automotive OEM³⁸ had developed a new model for introduction into the marketplace which was a major restyling of the previous model. The basic architecture of the vehicle was unchanged from the previous model, but the product managers decided that a more aerodynamic look was appropriate. One of the many features of this new “aero look” was a very long and shallow sloped back window (or “backlight” as it is called at the OEMs). This change alone directly resulted in a most unexpected steering wheel vibration problem which was not discovered until prototype vehicles were built, at which point making major changes was nearly impossible.

The vibration problem was steering wheel vibration at idle speed, especially while in gear and with a full accessory load (air-conditioning, headlights, blower motor, for example). This particular vehicle had a V6 engine which fired at 3 times the crank rotation speed (known as 3rd order). With an engine idle speed of 650 rpm, this resulted in an excitation frequency of 30 Hz ($600 \text{ rpm} \div 60 \text{ cycles/s} \times 3$). This will become important later in the story.

When the vibration problem was first discovered on the prototypes, no one really understood how this could have happened since the basic underbody and the general architecture of the vehicle were the same as those in the previous model, and it did not exhibit this idle vibration problem. It was only after a lengthy investigation that the connection to the styling change in the rear window was discovered. The following is a basic explanation of the root cause of this vibration problem.

The styling change to the rear window created a much larger surface area which required additional defroster heating elements. The increased heat load demand forced the alternator (generator) to put out more power when the rear defroster was turned on. This created a problem for the battery as explained below.

Automotive OEMs have a requirement regarding the vehicle charging system which forces the system to prevent the battery from discharging while the engine is running. This means that under no circumstances can the alternator output be allowed to drop below the demands of the electrical load in the vehicle. If this were permitted, the vehicle could theoretically drain the battery over time, even with the vehicle running! The worst case scenario for this is a vehicle sitting at idle with all major electrical systems on full: headlights on high beams, windshield wipers on high, blower motor on high, air-conditioner compressor engaged, rear defroster on, and rear brake lights on. This is a common experience for those who live in a cold, wet climate.

Back to the vibration problem. It was discovered by the team responsible for the charging system that the additional load from the rear defroster increased demand and would exceed the alternator’s output at idle under full accessory load conditions. They evaluated two options: (1) change the alternator pulley diameter, effectively speeding up the alternator and creating more power for the same engine speed, or (2) increase the engine speed.

Option 1 was ruled out due to durability concerns of the alternator at high engine speeds and also due to past experience with unacceptable alternator noise at high engine speeds. Option 2 was chosen without much fanfare. After conducting tests, the charging team increased the engine idle speed from 600 rpm to 650 rpm.

This decision took place during the normal “engine calibration” phase of the vehicle program which generally occurs late in the design. The idle speed change was accompanied by a myriad of other routine engine calibration tweaks. What no one realized was that lurking in that decision to change the idle speed was a vibration problem just waiting to be unleashed.

The new 650 rpm idle speed increased the excitation frequency of the engine at idle from 30.0 Hz to 32.5 Hz, which was the exact frequency at which a major bending mode of the body structure was found. This body bending resonance had a very specific mode shape which pivoted about the “cowl” (the structure at the base of the windshield) to which the steering column was attached. This pivoting motion acted like a crank on the steering column and drove the steering column into a vertical up/down motion as the body went through its bending mode, finally creating the steering vibration.

In summary, the sequence was as follows: the decision for a larger rear window created the need for additional defrosting elements. This resulted in an increased load on the alternator, which was solved by an increase in idle speed. The increased idle speed lined up the engine 3rd order idle excitation frequencies with a major bending mode of the body structure which drove the steering column into unwanted vibration. Who would have guessed?

The irony in this story is that body structural engineers are always trying to achieve higher stiffness (higher modal frequencies), and during the development of the vehicle architecture, great effort and cost were put into achieving a body bending mode above 30 Hz. The body engineers were rightfully proud that they had achieved this (they made it all the way to 32.5 Hz), but they had no idea that the decision for a larger rear window was coming back to bite them.

As a matter of interest, the solution to this problem was a tuned mass-damper (a chunk of lead on rubber springs) mounted behind the airbag module in the steering wheel that was tuned to exactly 32.5 Hz.

The moral of this story is that engineering a complex system like an automobile requires the engineer to be on the lookout for system interactions which lurk outside what is known.

³⁷The authors are grateful to Greg Goetchius (NVH Senior Specialist at Atieva, Menlo Park, CA) for sharing this true story which occurred in 1997. References to the specific model and vehicle have been removed.

³⁸An original equipment manufacturer or OEM is a company that uses a component made by a second company in its own product, or sells the product of the second company under its own brand.

To be successful, the engineer must have a solid foundation in the fundamentals of the system, a heavy dose of experience (once burned, twice shy), and an open mind.



**GUILLAUME FRANÇOIS ANTOINE
MARQUIS DE L'HÔPITAL**
(1661 – February 2, 1704)



CONTRIBUTION: L'Hôpital wrote the first textbook on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (literal translation: Analysis of the Infinitely Small to Understand Curved Lines), published in 1696. Although L'Hôpital acknowledged his indebtedness to Leibnitz, Jacob Bernoulli, and Johann Bernoulli in the introduction, he regarded the foundations as his own ideas.

He is best known for developing the rule which bears his name for finding the limit of a rational function whose numerator and denominator tend to zero at a point. L'Hôpital's rule uses derivatives to help evaluate limits involving indeterminate forms. Application (or repeated application) of the rule often converts an indeterminate form to a determinate form, allowing easy evaluation of the limit. L'Hôpital's rule first appeared in his book.

Life Sketch: Guillaume François Antoine, Marquis de L'Hôpital was born in Paris, France. As a child, L'Hôpital had no talent for subjects like Latin, but he developed strong abilities and a real passion for mathematics.

L'Hôpital had planned a military career and served as a cavalry officer, but did not give up his interest in mathematics. It is reported that he studied geometry even in his tent. He resigned from the army due to nearsightedness and from then on directed his attention to mathematics.

He was taught calculus by the Swiss mathematician Johann Bernoulli in 1691. He eventually solved the brachistochrone problem. (The brachistochrone problem was presented as a challenge to mathematicians throughout Europe by Johann Bernoulli in 1696. The problem is to find the curve in which

a body, starting from a given point, and descending solely by the force of gravity, will reach another given point in the shortest time. This is sometimes called the curve of quickest descent.) This problem was solved independently by Newton, Leibnitz, and Jacob Bernoulli, putting L'Hôpital in very good company if the solution was indeed due to him.

His textbook published in 1696 included the lectures of his teacher, Johann Bernoulli, and discussed the indeterminate form $0/0$. The method for solving such indeterminate forms through repeated differentiation bears his name.

L'Hôpital considered publishing a book on integration but dropped his plans on learning that Leibnitz was going to publish on the topic. A manuscript of a book was discovered following his death; it was published in 1707 with a second edition in 1720.

L'Hôpital married and had one son and three daughters. He died in Paris, France at the age of 43.

Noteworthy: L'Hôpital devoted much of his life to scientific writing. His book was an extremely important contribution and was used for a long time, with new editions produced until 1781. It served as a model for the next generation of calculus books.

L'Hôpital possessed a very attractive personality, being, among other things, modest and generous, two qualities which were not widespread among mathematicians of his time.

L'Hôpital is commonly spelled as both "l'Hôpital" and "L'Hôpital." The Marquis spelled his name with an "s"; however, the French language has since dropped the silent "s" and added a circumflex to the preceding vowel.

There is controversy surrounding the origin of L'Hôpital's rule. In 1694 L'Hôpital forged a deal with Johann Bernoulli. The deal was that L'Hôpital paid Bernoulli 300 Francs a year to tell him of his discoveries. In 1704, after L'Hôpital's death, Bernoulli revealed the deal to the world, claiming that many of the results in L'Hôpital's book were due to him.

In 1921 a manuscript copy of the course given by Johann Bernoulli to L'Hôpital came to light. It was seen how closely the book followed the course notes. Bernoulli had not been in a position to complain when L'Hôpital's book was published because of the agreement between them. But, it seems likely that L'Hôpital's rule was discovered by Johann Bernoulli. As a result, L'Hôpital's rule is also called Bernoulli's rule to give him credit.

To give Guillaume de L'Hôpital's full name would take a whole paragraph so we give just a much shortened version: Guillaume-François-Antoine Marquis de L'Hôpital, Marquis de Sainte-Mesme, Comte d'Entremont and Seigneur d'Ouques-la-Chaise. The family had been prominent in France going back to around the 12th century.

Example 2.14 Resonance in Rotating Equipment

This example addresses the phenomenon of resonance in rotating equipment and how it can be alleviated. Rotating equipment includes pumps, motors, and compressors.

Although these dynamic systems possess many natural frequencies, we assume that the majority of the energy occurs at the fundamental frequency. Discuss the design of such a system.

Solution: The design of rotating equipment generally assumes that it will not operate at the natural frequency where large amplitude motion is predicted to occur. If a preliminary design leads to a natural frequency that might be close to an operating frequency, it is generally necessary to redesign the pump or the compressor impellers, or change the shaft diameter, bearing stiffness, spans, or the mass of the unit. We know this since frequency is directly proportional to the $\sqrt{\text{stiffness}}$ and inversely proportional to the $\sqrt{\text{mass}}$. Any change in geometry will alter the stiffness properties of the system. The addition or removal of mass will lower or raise, respectively, the natural frequency of the rotating machine.

If the resonant frequency, or natural frequency, is close to the operating speed, its speed amplitude increases. If the resonance cannot be moved away from the operating speed, *trim balancing* (dynamic balancing) the rotor can reduce the resonant effect. Such balancing alters the operating speed by a change in mass, thus moving it further from the resonant frequency.

Resonances close to the operating speed can normally be moved above the operating speed by increasing the stiffness of the mounting hardware, clamping the feet, removing distortion at the feet by shimmying, or using reduced bearing clearances. The example offered by Campbell³⁹ is that of a number of identical vertical pump and motor units detrimentally affected by resonance. The decision was to lower the resonant frequency by adding mass to the unit at the motor flange, rather than the more difficult process of stiffening the unit. The first step was to decide how far the resonant frequency was to be shifted. This decision led to the amount of mass to be added.

In a particular case, the resonant frequency was measured to be approximately 14.8 Hz and the motor, shroud, drive shaft, and filled pump weighed 1650 lb. Since $m = W/g$, the natural frequency in Hz is

$$f = \frac{1}{2\pi} \sqrt{\frac{kg}{W}},$$

which can be rearranged to find the stiffness of the system,

$$k = \frac{\omega^2}{m} = \frac{f^2 4\pi^2 W}{g} = 36,900 \text{ lb/in.}$$

If the weight is increased by 400 lb to 2050 lb, the resonant frequency is 13.3 Hz, a drop of 1.5 Hz or about 10%. In practice, when weight was added to the

motor flange, the resonant frequency was reduced approximately 1.5 Hz from the motor's original frequency. The effect was to reduce the vibration amplitude by approximately 80% at the top of the motor. ■

2.4.3 Vibration of a Structure in Water

Fluid-structure interaction is a very important and quite complex subject.⁴⁰ Most of the problems considered in this text assume that the medium surrounding the vibrating structure is of such low density – compared to that of the structure – that it can be ignored. Generally this is true. However, if the fluid is water or air flowing at high speed, its density cannot be ignored. An important engineering application where the fluid characteristics are important to the problem formulation and solution is in the analysis and design of offshore drilling structures. These structures, ranging from hundreds of meters to thousands of meters tall, experience forces due to gravity, ocean waves, currents, buoyancy forces, and other effects.

The question is how to incorporate the *added effects* of the fluid when there is a structure vibrating. For a static structure, the only fluid force is due to the variation of hydrostatic pressure. This is the *buoyancy force*, which is equal to the weight of the liquid displaced by the body and acts in the direction opposite of gravity. The buoyancy force balances the weight of a static structure, that is, in equilibrium there is a balance between buoyancy and gravity forces. Ship structures are designed so that the equilibrium is at an appropriate floating position.

If the structure is displaced from its equilibrium position, then an imbalance of forces leads to a structural oscillation about the equilibrium. Such an oscillation is affected significantly by the water surrounding the structure. When the structure oscillates, it causes motion in the surrounding fluid. In general, the fluid dampens the structural motion as a result of drag between it and the structure, and the structure *entrains* or pulls with it some of the surrounding fluid as it moves. This means that the structure has an effective mass that is larger than its actual value. Therefore, it is necessary to include this *added mass* in any dynamic analysis. The fluid motion past the structure also creates a drag force that is a function of friction and the shape of the structure. If the fluid flow is accelerating, then in addition to the added mass, an inertial force is exerted by the fluid on the submerged structure.

An important flow effect occurs when *non-*

³⁹This example is based on the article by W.R. Campbell, "Practical Solution of Resonance of Electric Motors," *Shock & Vibration Digest*, Vol.26, No.1, Jan/Feb 1994.

⁴⁰Sea Loads on Ships and Offshore Structures, O.M. Faltinsen, Cambridge University Press, 1993.

A Course in Ocean Engineering, S. Gran, Elsevier, 1992.

Dynamics of Offshore Structures, J.F. Wilson, Wiley-Interscience, 1986.

aerodynamic structures, called *bluff* or *blunt bodies*, are placed in steady flows. These bodies may be cylindrical in cross section or have sharp edges, but are not tailored like a wing. When a fluid flows past such a body, it separates on the far side producing a trailing wake, creating vortices that are shed *alternately* with forces perpendicular to the fluid flow. These vortices are called von Kármán vortices. Figure 2.29 depicts this effect for a circular cylinder. Fluid interaction effects are very complicated, requiring experimental data for their quantification.



Figure 2.29: Vortices shed alternately from a submerged cylindrical structure, shown in cross section. The flow is moving from left to right with the structural cylinder responding in the transverse or lift direction.

THEODORE VON KÁRMÁN

(May 11, 1881 – May 7, 1963)



Contribution: Von Kármán was a Hungarian-German-American engineer known for his pioneering work in the use of mathematics and the basic sciences in aeronautics and astronautics. He was responsible for many key advances in aerodynamics, notably in characterizing supersonic and hypersonic airflow. His work in aerodynamics and rocket research helped shape both scientific and political history.

His early research focused on fluids and the compression of structures. In 1906, he published *The theory of buckling and compression: tests on long slender columns*. His interest later turned to applying mathematics to aeronautics.

In 1911, using results from a wind tunnel, he made an analysis of the alternating double row of vortices behind a

flat body in a fluid flow which is now known as von Kármán's vortex street. The name came from the way that the vortices alternated in position on the two sides of the double row in an arrangement similar to that of street lights on the two sides of a street. A von Kármán vortex street is an accepted term in fluid dynamics for a repeating pattern of swirling vortices caused by the unsteady separation of flow of a fluid over bluff bodies. It is responsible for such phenomena as the "singing" of suspended telephone or power lines, the vibration of a car antenna at certain speeds, and the fluttering of Venetian blinds as the wind passes through them.

His contributions include theories of non-elastic buckling, unsteady wakes in flow around bluff cylinders, stability of laminar flow, turbulence, airfoils in steady and unsteady flow, boundary layers, and supersonic aerodynamics. He made additional contributions in other fields, including elasticity, vibration, heat transfer, and crystallography.

He collaborated with Max Born in research on the vibration of atoms. This work focused on lattice dynamics and identified the degrees of freedom of a crystal with its normal modes.

Life Sketch: Theodore von Kármán (original Hungarian name Szöllőskislaki Kármán Tódor) was born in Budapest, Hungary. His mother was from a leading Bohemian family, and his father was a leading professor of philosophy and education. As a young boy he was tutored at home and his early education was dominated by his father, who required him to study geography, history, and literature. (It appears he was consciously steered away from mathematics out of his father's fear that his son, who had a remarkable ability to perform large mental calculations, would end up as a freak performing mathematical tricks!)

When he was nine years old, von Kármán entered the Minta Gymnasium in Budapest. Even though he won the Eötvös Prize for the best student in mathematics and science in Hungary, his father urged him away from mathematics toward engineering. As a result, he studied engineering at the Palatine Joseph Polytechnic in Budapest (later the Technical University of Budapest and known today as Budapest University of Technology and Economics).

He graduated in 1902 with a degree in mechanical engineering, having written a dissertation on *The motion of a heavy rod supported on its rounded end by a horizontal plane*. After graduating, von Kármán served as an artillery cadet in the Austro-Hungarian army. Then, in 1903, he was appointed as an assistant in hydraulics at the Palatine Joseph Polytechnic, a position he held for three years. His research was on fluids and the compression of structures.

In 1906 he was awarded a two-year fellowship from the Hungarian Academy of Sciences and he left Budapest to study at Göttingen, where he was greatly influenced by Hilbert and Prandtl. He developed mathematical models for the buckling of large structures, using data obtained through experiments with a large hydraulic press. For this research von Kármán was awarded his doctorate in 1908 and in the same year he

accepted a position at Göttingen. His interest in aeronautics grew into a research area with the construction at Göttingen of a wind tunnel for the Zeppelin airship company.

In February 1913 von Kármán accepted a post as director of the Aeronautical Institute at Aachen in Germany and also the chair of aeronautics and mechanics at the Technical University in Aachen. He built a strong research team that built and improved the equipment, including a wind tunnel. He began theoretical work on aircraft design and the team experimentally tested his designs.

When World War I broke out in 1914, he returned to his native Hungary, where he eventually became head of research in the Austro-Hungarian Army Aviation Corps. During this time he designed an early version of the helicopter, which was intended to solve problems of observation balloons then in use. His work resulted in the development of a prototype helicopter, the first to maintain hovering flight.

When the war ended in 1918 von Kármán decided to stay in Hungary and promote teaching of science. However, the political situation in Hungary was extremely difficult and educational reforms were not realistic.

In 1919, von Kármán returned to his previous positions in Aachen. With financial assistance from the leading German manufacturers of aircraft, he began an intense program of studying the resistance of fluids, turbulence, and lift, all leading to better aircraft design. He did not realize at the time that he was laying the foundations of a new German air force, the Luftwaffe of World War II.

Von Kármán visited the USA in 1926 to advise on the design of a wind tunnel. By 1928 he was spending six months each year at Caltech and six months at Aachen. In 1930 he was asked to be the full-time director of the Aeronautical Laboratory at California Institute of Technology. Despite his love for Aachen, the political events in Germany and in particular the rising anti-Semitism persuaded him to accept the offer. His mother and younger sister went to California with him. Kármán never married and his mother, then sister, looked after his home for him.

The head of Caltech asked him to make the Aeronautical Laboratory a research center to draw the U.S. aviation industry to Southern California and to bring to the campus national pre-eminence in aeronautics. In 1933 he founded the U.S. Institute of Aeronautical Sciences continuing his research on fluid mechanics, turbulence theory, and supersonic flight. He studied applications of mathematics to engineering, aircraft structures, and soil erosion.

His work turned towards research on rockets. When Germany was seen to have developed rockets for military purposes during World War II, the United States government put large sums of money into rocket research. In November 1944 the funding was used to set up the Jet Propulsion Laboratory at Caltech with von Kármán as director. This laboratory later made major contributions to the space program.

In 1949 he resigned his two positions of director and be-

came professor emeritus at Caltech. He was still very active in giving advice to the U.S. Air Force and NATO and played a major role in international conferences on aeronautics.

Von Kármán died while on a visit to Aachen, Germany, in 1963 at the age of 81.

Noteworthy: Von Kármán's fame was in the use of mathematical tools to study fluid flow, and the interpretation of those results to guide practical designs. He was instrumental in recognizing the importance of the swept-back wings that are used in modern jet aircraft. Despite von Kármán's great theoretical talents he was not a skilled experimenter.

Von Kármán was awarded many honors. He received the U.S. Medal for Merit in 1946, the Franklin Gold Medal in 1948, and was the first to be awarded the National Medal for Science in 1963. He also received honors from France, the Vatican, Germany, Greece, the United Kingdom, Spain, and the Netherlands.

At age 81 von Kármán was the recipient of the first National Medal of Science, bestowed in a White House ceremony by President Kennedy. He was recognized, "For his leadership in the science and engineering basic to aeronautics; for his effective teaching and related contributions in many fields of mechanics, for his distinguished counsel to the Armed Services, and for his promoting international cooperation in science and engineering."

In 1992 the United States Postal Service issued a 29 cent stamp with his image as an "Aerospace Scientist." Craters on Mars and the Moon are named in his honor. In Irvine, CA there is a 5 mile street that runs through the business center named after him.

Von Kármán remained unabashed about his lifelong association with military authorities, first in Austro-Hungary, then in Germany, and finally in the U.S. and NATO. He thought that "scientists as a group should not try to force or even persuade the government to follow their decisions."

In 1946 he became the first chairman of the Scientific Advisory Group which studied aeronautical technologies for the U.S. Army Air Forces. He also helped found AGARD, the NATO aerodynamics research oversight group (1951), the International Council of the Aeronautical Sciences (1956), the International Academy of Astronautics (1960), and the Von Kármán Institute for Fluid Dynamics in Brussels (1956).

Each year since 1960 the American Society of Civil Engineers has awarded to an individual the Theodore von Kármán Medal, "in recognition of distinguished achievement in engineering mechanics."

Von Kármán was known as a colorful, multilingual character. He left us with intriguing quotes and stories, including:

"Science is the study of what is, engineering builds what will be."

"The scientist merely explores that which exists, while the engineer creates what has never existed before."

After the collapse of the Tacoma Bridge, the governor of the state of Washington made an emotional speech, in which he declared "We are going to build the exact same bridge,

exactly as before.” Upon hearing this, von Kármán sent a telegram to the governor stating “If you build the exact same bridge exactly as before, it will fall into the exact same river exactly as before.”

Single Degree-of-Freedom Flow-Oscillator

Single degree-of-freedom flow-oscillator models⁴¹ are the simplest mathematical models of the interaction between vortex shedding and structural response. A single ordinary differential equation can describe the behavior of the structural oscillator.

The general form of such models is given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{1}{m}F(x, \dot{x}, \ddot{x}, \omega_s t),$$

where m is the mass of cylinder, $x(t)$ is the transverse (lift direction) displacement, ω_s is the Strouhal⁴² frequency, and F is a nonlinear *aeroelastic* forcing function. The influence of the wake dynamics is incorporated into this equation via the appropriate function F .

Aeroelasticity is the science which studies the interaction among inertial, elastic, and aerodynamic forces. Aeroelastic phenomena arise when structural deformations induce changes in aerodynamic forces. In dimensional analysis, the Strouhal number St is a dimensionless number describing oscillating flow mechanisms. It is often given as

$$St = \frac{fL}{v},$$

where f is the frequency of vortex shedding, L is the characteristic length in the structure (for example diameter), and v is the velocity of the fluid.

An example would be the wind-induced large amplitude response of a tower supported by cable guy wires at several locations along its length. As a first step, a single degree-of-freedom model is developed in which the effects of the flowing fluid are characterized as the fluid damping and stiffness, and these parameters are in turn dependent on oscillation amplitude, flow velocity, and Reynolds number. In fluid mechanics, the Reynolds number is the ratio of inertial forces to viscous forces; it quantifies the relative importance of these two types of forces for given flow conditions. It is one of the most significant dimensionless numbers in fluid mechanics, and is given by

$$Re = \frac{vL}{\nu},$$

⁴¹ The following paper provides details of various modeling approaches for flow-oscillators: R.D. Gabbai and H. Benaroya, “Review: An overview of modeling and experiments of vortex-induced vibration of circular cylinders,” *J. of Sound and Vibration*, Vol.282, 2005, pp.575-616.

⁴² **Vincenc Strouhal** (April 10, 1850-January 26, 1922) was a Czech physicist specializing in experimental physics.

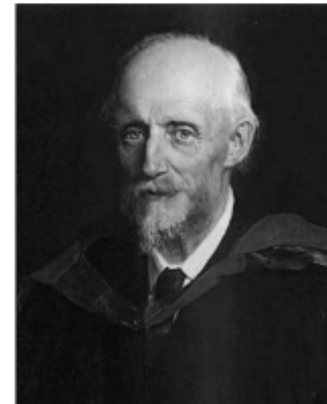
where v (m/s) is the mean fluid velocity, L (m) is the characteristic length, and ν (m²/s) is the kinematic viscosity.

Fluid-structure interaction studies investigate whether resonances can occur between the frequencies associated with the water movement and those associated with the tower oscillations. Identifying these resonance frequencies allow the engineer to redesign the tower so it can resist the wave forces. Wave speed, cable tension, and damping are found to affect the resonances. The resonances in the cables may be reduced significantly by changing their natural frequencies through careful adjustment of the tensions.

Similar flow-induced vibration issues affect structures in air and water where the fluid flow interacts with the structure. This happens with tall buildings, offshore drilling towers, ships and submarines, and aircraft and spacecraft in the atmosphere.

OSBORNE REYNOLDS

(August 23, 1842 – February 21, 1912)



Contribution: Reynolds was a prominent innovator in the understanding of fluid dynamics and his contributions were of major significance. He studied the flow in a pipe when it changes from laminar to turbulent flow. In 1886 he formulated a theory of lubrication. Three years later he produced an important theoretical model for turbulent flow, which is still the standard mathematical framework used in the study of turbulence.

Reynolds' work on hydrodynamic stability was published in 1883 and 1895. In the 1883 paper (titled “An experimental investigation of the circumstances which determine whether the motion of water in parallel channels shall be direct or sinuous and of the law of resistance in parallel channels”), he proposed a dimensionless number (the ratio of inertial forces to viscous forces) – now known as the Reynolds number – for dynamic similarity. Reynolds also proposed what is now known as Reynolds-averaging of turbulent flows, where quantities such as velocity are expressed as the sum of mean and fluctuating

components. Such averaging allows for a “bulk” description of turbulent flow, for example, using the Reynolds-averaged Navier-Stokes equations.

He also conducted studies of heat transfer between solids and fluids that brought improvements in boiler and condenser designs.

Life Sketch: Reynolds was born in Belfast, Ireland, although soon afterward moved to Dedham, Essex. His father worked as a school headmaster and clergyman. He was also a very able mathematician with a keen interest in mechanics, and he held a number of patents for improvements to agricultural equipment. Reynolds credited his father with being his chief teacher as a boy.

He began his schooling at Dedham, where his father was headmaster, and then received private tutoring to complete his secondary education. He then took an apprenticeship with an engineering firm in 1861. He obtained practical experience in the manufacture and fitting of coastal steamers, and thus gained an early appreciation of the practical value of fluid dynamics.

Following the apprenticeship he began his formal studies of mathematics at Cambridge. He graduated in 1867 as Seventh Wrangler in the Mathematical Tripos (ranked seventh in the list of First Class students) from Cambridge.

For a year following his graduation he worked as a practicing civil engineer in the London sewage system.

In 1868 he was appointed a professor of engineering at Owens College in Manchester. The 25-year-old Reynolds was the first professor of engineering in Manchester (and the second in England). He remained at Owens College for the rest of his career. (In 1880 the college was renamed University of Manchester.)

Reynolds produced first-class papers on a wide array of physical and engineering phenomena. His contributions to engineering science were pivotal. His early work was on magnetism and electricity. He concentrated on hydraulics and hydrodynamics. He also worked on electromagnetic properties of the sun and of comets, and considered tidal motions in rivers.

Reynolds published about seventy science and engineering research reports. Towards the end of his career these were republished by Cambridge University Press as a collection that filled three volumes. In addition to fluid dynamics, topics covered thermodynamics, kinetic theory of gases, condensation of steam, screw-propeller-type ship propulsion, turbine-type ship propulsion, hydraulic brakes, hydrodynamic lubrication, and a laboratory apparatus for better measurement of Joule’s mechanical equivalent of heat.

By the beginning of the 1900s Reynolds’ health began to fail and he retired in 1905. He deteriorated physically and mentally, although barely 60 years old, and died in Watchet, England.

Noteworthy: Reynolds became a Fellow of the Royal Society in 1877 and, 11 years later, won their Royal Medal. In 1884 he was awarded an honorary LL.D. degree by the Univer-

sity of Glasgow. He was elected President of the Manchester Literary and Philosophical Society in 1888, and received the Dalton Medal in 1903.

To study flow regimes Reynolds injected a thin trace of ink into water flowing along a glass pipe in a tank. By varying the diameter of the pipe and the velocity of the water in the tube, Reynolds developed a relationship between these quantities and the viscosity which governs the laminar/turbulent switch.⁴³ His later and perhaps even more fundamental work established the equations which govern turbulence. These are still used in computational fluid mechanics studies.

Reynolds’ contributions to fluid mechanics were not lost on ship designers (naval architects). The ability to make a small scale model of a ship, and extract useful predictive data with respect to a full size ship, depends directly on the Reynolds’ turbulence principles to friction drag computations, along with William Froude’s theories of gravity wave energy and propagation. Reynolds published a number of papers concerning ship design in the *Transactions of the Institution of Naval Architects*.

In addition to Reynolds’ role in the unraveling and understanding of the science of fluid mechanics, he is also considered the founder of the science of lubrication, the father of tribology (friction, lubrication, and wear).

Reynolds was a scholarly man with high standards. Engineering education was new to English universities at that time, and Reynolds had definite ideas about its proper form. He believed that all engineering students, no matter what their specialty, should have a common background in mathematics, physics, and the fundamentals of classical mechanics.

Despite his strong interest in education, he was not known as a great lecturer. His lectures were difficult to follow, and he frequently wandered among topics with little or no connection.

Early in his career at Manchester, Reynolds gave a lecture called “The Progress of Engineering with Respect to the Social Conditions of the Country,” in which he said, “The results, however, of the labor and invention of this century are not to be found in a network of railways, in superb bridges, in enormous guns, or in instantaneous communication. We must compare the social state of the inhabitants of the country with what it was. The change is apparent enough. The population is double what it was a century back; the people are better fed and better housed, and comforts and even luxuries that were only within the reach of the wealthy can now be obtained by all classes alike. . . . But with these advantages there are some drawbacks. These have in many cases assumed national importance, and it has become the province of the engineer to provide a remedy.”

⁴³ An illustration of the switch between smooth laminar flow and turbulent eddy-like motion is observed on seeing the smoke rise from a cigarette in a draft-free room. At a certain height the smooth thin column of rising smoke breaks up into a wider collection of eddies with ‘curly’ motion. This effect was demonstrated by Reynolds.

In keeping with this theme, many of his publications addressed everyday practical problems; “Sewer Gas and How to Keep it Out of the House” was particularly relevant to poor slum housing in Manchester and “Improved Means of Signaling and Communication Between the Passengers, Guard and Engine Driver of Railway Trains in Motion” was obviously a response to the frequent railway accidents of the 1860s.

Lamb, who knew Reynolds well as a fellow worker in fluid dynamics, wrote: “The character of Reynolds was like his writings, strongly individual. He was conscious of the value of his work, but was content to leave it to the mature judgment of the scientific world. For advertisement he had no taste, and undue pretension on the part of others only elicited a tolerant smile. To his pupils he was most generous in the opportunities for valuable work which he put in their way, and in the share of cooperation. Somewhat reserved in serious or personal matters and occasionally combative and tenacious in debate, he was in the ordinary relations of life the most kindly and genial of companions.”

He had chosen to study mathematics at Cambridge because, in his own words in his 1868 application for the professorship, “From my earliest recollection I have had an irresistible liking for mechanics and the physical laws on which mechanics as a science are based . . . my attention drawn to various mechanical phenomena, for the explanation of which I discovered that a knowledge of mathematics was essential.”

Reynolds’ style was illustrated by this account of one of his lectures, given by his most famous pupil, Sir J.J. Thompson, later Nobel Laureate, President of the Royal Society and Master of Trinity: “He was one of the most original and independent of men and never did anything or expressed himself like anybody else. The result was that it was very difficult to take notes at his lectures so that we had to trust mainly to Rankine’s text books. Occasionally in the higher classes he would forget all about having to lecture and, after waiting for ten minutes or so, we sent the janitor to tell him that the class was waiting. He would come rushing into the door, taking a volume of Rankine from the table, open it apparently at random, see some formula or other and say it was wrong. He then went up to the blackboard to prove this. He wrote on the board with his back to us, talking to himself, and every now and then rubbed it all out and said it was wrong. He would then start afresh on a new line, and so on. Generally, towards the end of the lecture he would finish one which he did not rub out and say that this proved Rankine was right after all.”

2.5 Concepts Summary

In this chapter we have developed the theory of linear elastic vibration of an undamped single degree-of-freedom system. We have considered the formulation of the equation of motion using Newton’s second law of mo-

tion and using an energy-based approach. The problems of free vibration and forced harmonic vibration have been solved assuming single degree-of-freedom models without damping. In the next chapter damping is investigated.

2.6 Quotes

- “Don’t say you don’t have enough time. You have exactly the same number of hours per day that were given to Helen Keller, Pasteur, Michaelangelo, Mother Teresa, Leonardo da Vinci, Thomas Jefferson, and Albert Einstein.” (H. Jackson Brown)
- “When I am working on a problem I never think about beauty. I only think about how to solve the problem. But when I have finished, if the solution is not beautiful, I know it is wrong.” (Buckminster Fuller)
- “For every complex problem, there is a solution that is simple, neat, and wrong.” (H. L. Mencken)
- “I have not failed. I’ve just found 10,000 ways that won’t work.” (Thomas Edison)
- “Design is the essence of engineering. Creativity is the essence of design.” (Pradip Sheth)
- “Inventions have long since reached their limit, and I see no hope for further development.” (Julius Frontinus, highly respected engineer in Rome, 1st century)
- “For every problem that you cannot solve, there is a simpler problem that you cannot solve.” (Dennis Bernstein)
- “Coincidence is just God’s way of remaining anonymous.” (Albert Einstein)
- “You have to be odd to be number one.” (Dr. Seuss)
- “Nothing is impossible, the word itself says ‘I’m possible’!” (Audrey Hepburn)

2.7 Problems

Problems for [Section 2.2](#) – Deterministic Modeling

1. The beam in [Figure 2.30](#) vibrates as a result of loading not shown. State the necessary assumptions to reduce this problem to a one degree-of-freedom oscillator. Then derive the equation of motion.

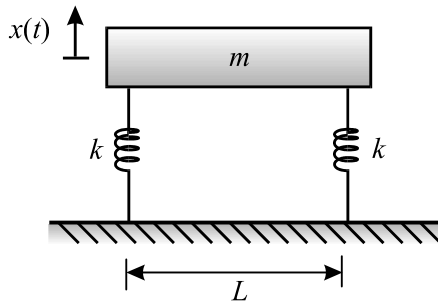


Figure 2.30: Vibrating beam of mass m supported by springs of stiffness k separated by length L .

2. If a beam is supported continuously on a foundation, as shown in Figure 2.31, damping must be added to an idealized model to represent the viscous effects of the mat foundation. How would you idealize this system as a one degree-of-freedom oscillator? Derive the equation of motion.



Figure 2.31: Vibrating beam on mat foundation.

3. An idealized one degree-of-freedom model is tested many times in order to estimate its natural frequency. It is relatively straightforward to measure its mass m , but stiffness k can only be measured approximately. How might one use the natural frequency data to estimate k ?
4. The cantilever beam in Figure 2.32 undergoes harmonic oscillation, being driven by a force of amplitude A and nominal frequency ω . An examination of a long time-history of the response shows slight fluctuations about an exact harmonic response. If we divide this long time-history into segments of one period ($2\pi/\omega$) and superpose these, we obtain the set of curves shown in Figure 2.33. How serious are such fluctuations in the response of the beam? Is there a way to relate the magnitude of the fluctuation to the maximum response? What would be a reasonable way to specify the value of ω ?

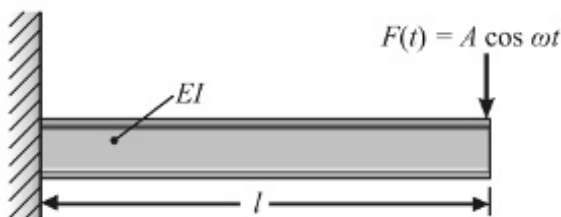


Figure 2.32: Cantilever beam vibrating under harmonic loading.

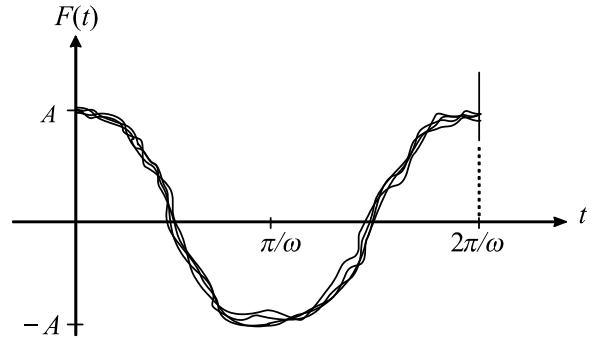


Figure 2.33: Overlapping time-histories.

5. For each nonlinear equation of motion, linearize the equation about the indicated equilibrium position and discuss the range of validity of the linearized equation. Plot the relative error for the linearized equation of motion. Analytically solve the linearized equation of motion.
- $\ddot{\theta} + 3 \cos \theta = 0$, $\theta_{eq} = -\pi/2$, and zero initial conditions.
 - $\ddot{\theta} + 3 \sin \theta = 0$, $\theta_{eq} = 0$, $\theta(0) = 0.5$, and $\dot{\theta}(0) = 0$.
 - $\ddot{\theta} + 3 \cos^2 \theta = 0$, $\theta_{eq} = -\pi/2$, and zero initial conditions.
 - $\ddot{\theta} + 3\theta^3 = 3$, $\theta_{eq} = 1$, and zero initial conditions.
 - $\ddot{\theta} + \sqrt{\theta} = 1$, $\theta_{eq} = 1$, and zero initial conditions.
6. Consider the mass m suspended by a nonlinear hardening spring as shown in Figure 2.34. The spring obeys $F = kx^3$ and has an original length of L . The coordinate x is the stretch of the spring, and y is the displacement of the mass measured from its static equilibrium position. Derive the linearized equation of motion about its equilibrium position in terms of x and also in terms of y . Find the natural frequency of this system.

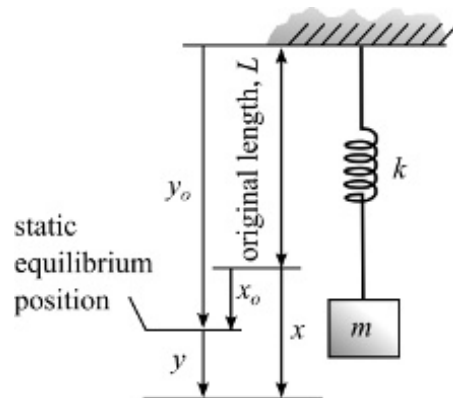


Figure 2.34: A mass-spring system with a hardening spring.

7. For each idealized model in Figures 2.35 to 2.38, draw a free-body diagram and derive the equation of motion using (a) Newton's second law of motion and (b) the energy method. The block in Figure 2.36 slides on a frictionless surface. State whether the oscillations are linear or nonlinear. Determine the natural frequency of each model.

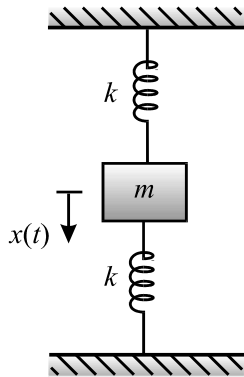


Figure 2.35: Mass m oscillating in the vertical direction about equilibrium.

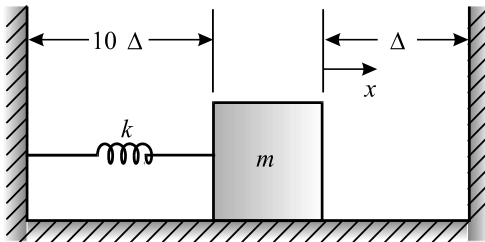


Figure 2.36: Sliding oscillation of mass m on a frictionless surface with possibility of impact.

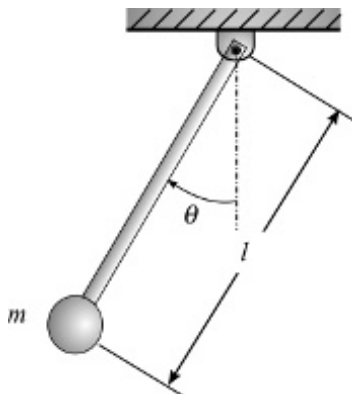


Figure 2.37: Oscillation of a simple pendulum of mass m and length l .

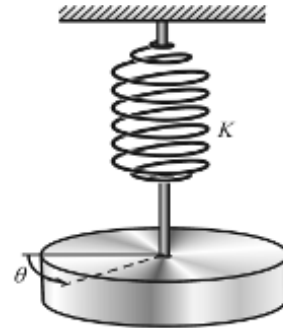


Figure 2.38: System under torsional vibration.

8. A disk of mass m is mounted between two shafts with different properties, as shown in Figure 2.39. (a) What is the natural frequency of the system? (b) If the disk is rotated θ , where $\theta \ll 1$ rad, and then released, what will its angular position be at an arbitrary time t ?

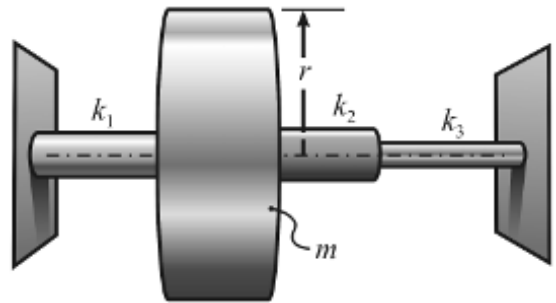


Figure 2.39: A disk mounted between two shafts.

9. Calculate the equivalent torsional spring constant for the stepped shaft shown in Figure 2.40. The two shafts have negligible mass and are made of a material with shear modulus G_1 and G_2 , respectively. At the end of the shaft is a rigid disk of mass inertia I_G . Express your answers in terms of known quantities.

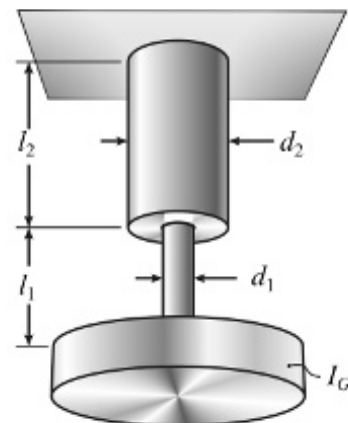


Figure 2.40: Stepped shaft.

10. The valve mechanism in Figure 2.41 is in equilibrium when the rocker arm is horizontal. The system is assumed to be frictionless. Use an energy method to determine the natural frequency for small vibration about equilibrium in terms of k , m_B , m_R , m_V , a , and b .

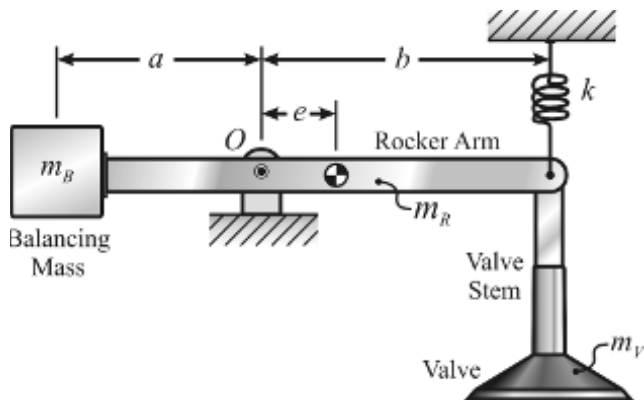


Figure 2.41: A rocker-arm valve system.

11. A rod is supported on two rotating grooved rollers, as depicted in Figure 2.42. The rollers rotate in opposite directions and the coefficient of friction between the rollers and the rod is μ_k . Find the natural frequency of the system and describe the behavior of the rod if it is disturbed in the horizontal direction.

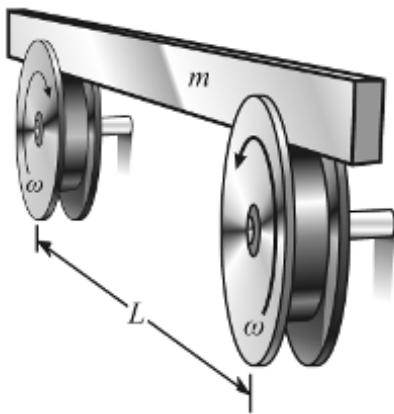


Figure 2.42: Rod supported by two rotating grooved rollers.

12. A 25-kg block is suspended by two cables, as depicted in Figure 2.43. Assume small displacements. (a) What is the frequency of oscillation in Hz of the block in the x direction if it is displaced slightly in this direction? (b) What is the period of oscillation in the z direction if the block is displaced slightly in this direction?

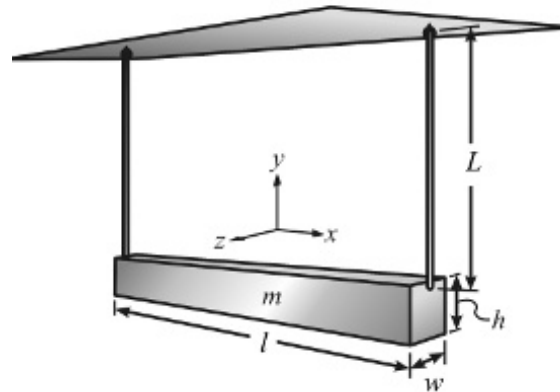


Figure 2.43: A suspended block.

13. Derive the equation of motion and natural frequency for a mass m on the string that is under constant tension T as shown in Figure 2.44. Assume small displacements and that m is much larger than the mass of the string.

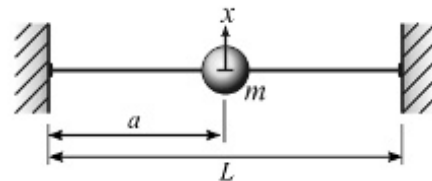


Figure 2.44: Vibration of a mass on a string under tension.

14. Continuing Problem 13, the string is stretched to the position shown in Figure 2.45. Calculate the natural frequency of the system using the following parameter values: $mg = 2$ lb, $T = 50$ lb, and $l = 4$ ft, where T is the tension in the string for the configuration shown.

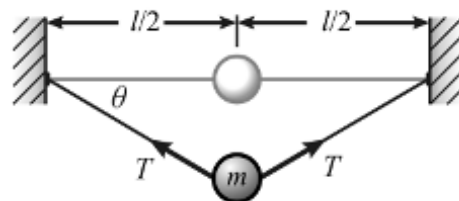


Figure 2.45: Mass on a stretched string.

15. Derive the equation of motion for a uniform stiff rod restrained from vertical motion by a torsional spring of stiffness K and two translational springs each of stiffness k , as shown in Figure 2.46. The torsional spring constant is determined by the application of a moment M and the measurement of the angular displacement θ , that is, $M = K\theta$. Calculate the natural

frequency of oscillation. Let J be the mass moment of inertia of the rod about the left restraint. State any assumptions.



Figure 2.46: Restrained rigid rod.

16. A uniform rigid and massless rod is pinned at one end and connected to ground via a spring at the other end. At midpoint on the rod, a spring is connected to a mass which is connected to a fixed point via another spring, as shown in Figure 2.47. (a) Derive the equation of free vibration for the system. (b) Find the natural frequency. (c) Calculate the natural frequency given the following parameters: $k_1 = 20$ lb/in, $k_2 = 30$ lb/in, $k_3 = 40$ lb/in, $mg = 50$ lb, and $l = 8$ ft.

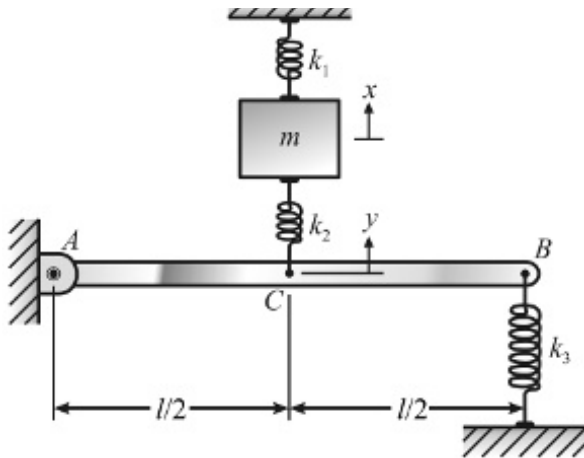


Figure 2.47: A coupled rod-mass system.

17. A component of uniform material and nonuniform cross section is pinned at one end and supported by two springs as shown in the two configurations of Figure 2.48. The component is displaced slightly from equilibrium and released and observed to oscillate with a period of $T = 1.5$ s. (a) For the system in the top figure, calculate the moment of inertia of the component with respect to the hinge axis of rotation if $k = 500$ N/m and $l = 1$ m. (b) For the system in the bottom figure, calculate the period of oscillation assuming the same parameter values.

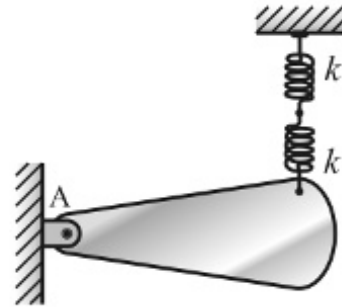
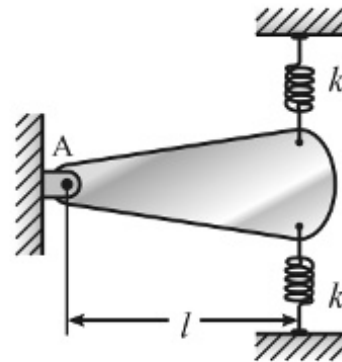


Figure 2.48: A rigid component supported by two springs in two configurations.

18. A body of mass m is suspended by a spring of constant k and attached to an elastic beam of length l , as shown in Figure 2.49. When the mass is attached to the spring, measurements are taken of the spring extension δ_s and the end deflection of the beam x_b . Neglect the masses of the beam and the spring. (a) Estimate the natural frequency of the system. (b) Calculate the natural frequency if the mass is attached to the beam directly without a spring. (c) Calculate the natural frequency if the beam is assumed to be rigid. (d) Calculate the respective numerical values given $\delta_s = 12$ mm, $x_b = 2$ mm, and $l = 0.5$ m.

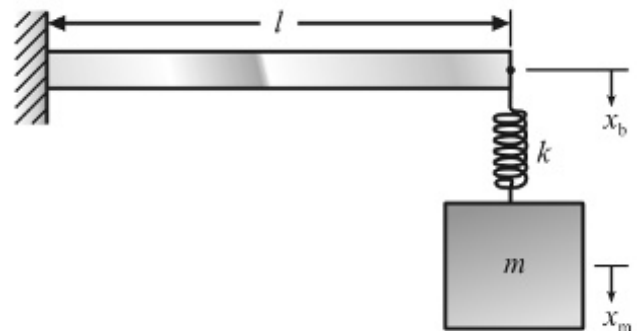


Figure 2.49: Mass suspended by a spring from a beam.

19. A solid cylinder floating in equilibrium in a liquid of specific gravity γ is depressed slightly and released. A schematic is shown in Figure 2.50. Find the equilibrium position and solve for the natural frequency of oscillation assuming the cylinder remains upright at all times. Suppose the assumption that the cylinder remains upright is not reasonable. Then what difficulties do you foresee in the calculations, and how might they be resolved?

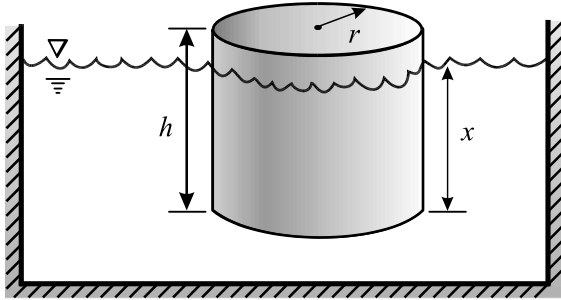


Figure 2.50: Solid body oscillating in a liquid.

20. The uniform rod is restrained by four translational springs and a torsional spring as shown in Figure 2.51. Determine the natural frequency of the system using the energy approach.

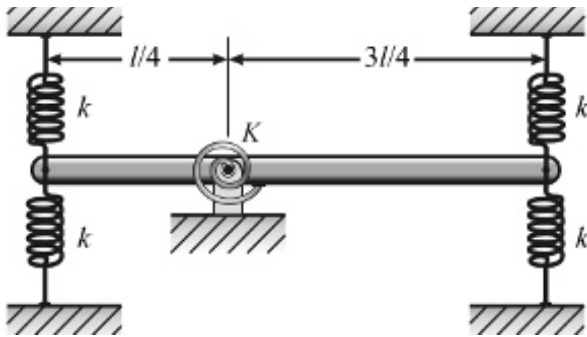


Figure 2.51: A rigid rod supported by four translational springs and a torsional spring.

21. Derive Equation 2.26,

$$x(t) = x(0) \cos \omega_n t + \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t.$$

22. Derive the equation of motion and natural frequency for a disk in motion that is constrained to move vertically, but can rotate, as shown in Figure 2.52. The coordinate y is the displacement of its center of mass, measured from the static equilibrium position, and the coordinate x is the displacement of the center of mass measured from its location when the spring is unstretched. Let θ be the rotation of the disk measured from the static equilibrium position.

The mass of the disk is m , and the mass moment of inertia for the disk is $mR^2/2$. Assume that the cable is inextensible and the spring constant is k .

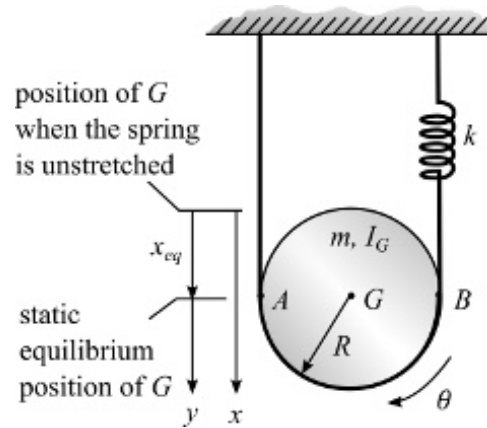


Figure 2.52: A disk suspended by an inextensible cable and a spring.

23. Derive the equation of motion and natural frequency for mass m shown in Figure 2.53. Assume the pulleys are massless, the cable is inextensible, and let the spring constant be k . The coordinate y is the displacement of its center of mass, measured from the static equilibrium position, and the coordinate x is the displacement of the center of mass measured from its location when the spring is unstretched.

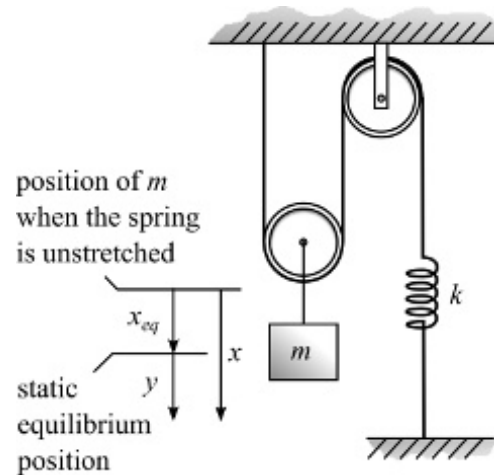


Figure 2.53: A mass suspended in a pulley system.

24. For the two-mass system shown in Figure 2.54 derive the equation of motion in terms of x_1 . Assume the pulleys are massless, the cable is inextensible, and let the spring constant be k . The coordinates y_1 and y_2 are the displacements of the center of mass of m_1 and m_2 , respectively, measured from the static equilibrium position. The coordinates x_1 and x_2 are the displacements of the center of mass of m_1 and

m_2 , respectively, measured from its location when the spring is unstretched.

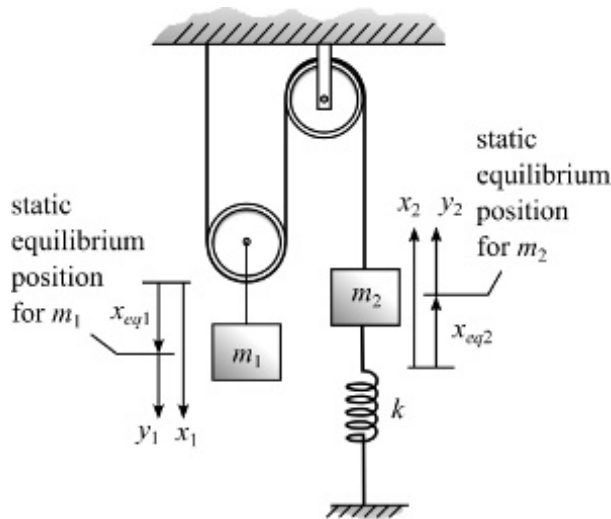


Figure 2.54: Two masses suspended in a pulley system.

25. Derive the equation of motion and natural frequency for the disk shown in Figure 2.55. Assume the disk is rolling without slip.

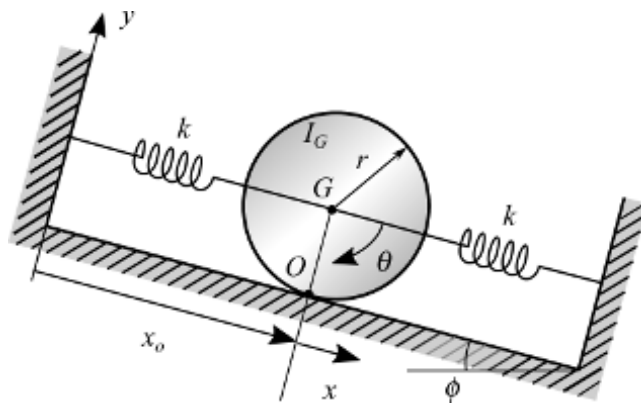


Figure 2.55: Disk on an inclined plane connected to two springs. It is assumed that the disk rolls without slipping, and there is no energy loss from friction.

26. Derive the equation of motion and natural frequency for the disk shown in Figure 2.56. Assume the disk is rolling without slip, and the cable is wrapped around the disk.

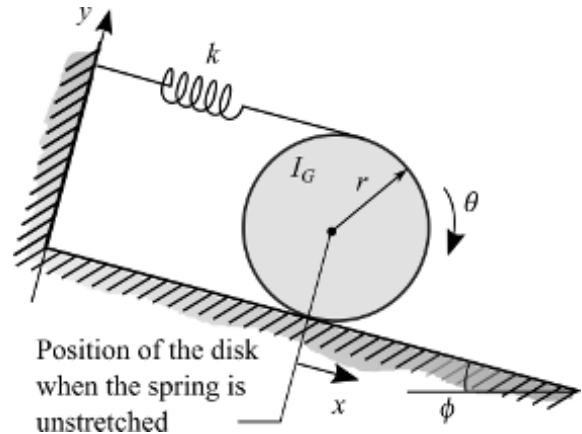


Figure 2.56: A rolling disk on an incline.

27. Derive the equation of motion and natural frequency for mass m shown in Figure 2.57. Assume the pulleys are massless, the cable is inextensible, and let the spring constant be k .

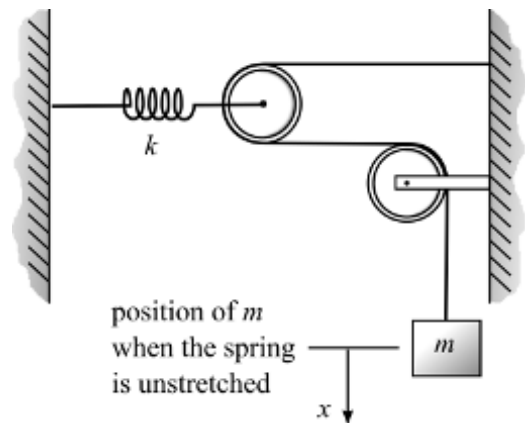


Figure 2.57: A mass suspended in a pulley system.

28. Derive the equation of motion and natural frequency for mass m shown in Figure 2.58. Assume the pulleys are massless, and let the spring constants be k_1 and k_2 . Let x_1 and x_2 be the displacements of the pulley and the mass measured from their locations when the springs are unstretched. Use x_2 for the equation of motion.

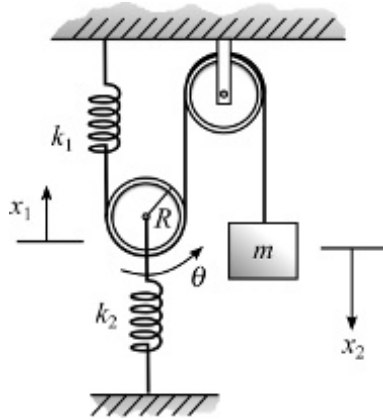


Figure 2.58: A mass suspended by a pulley and springs.

Problems for Section 2.3 – Undamped Free Vibration

29. Derive Equation 2.30,

$$x(t) = B_1 \cos \omega_n t + B_2 \sin \omega_n t,$$

beginning with the equation of motion.

30. Show that the period of free vibration of a load weighing W suspended from two parallel springs, as shown in Figure 2.59, is given by T ,

$$T = 2\pi \sqrt{\frac{W}{g(k_1 + k_2)}},$$

and show that the equivalent stiffness is $k = k_1 + k_2$. Discuss the need to hang the weight asymmetrically, that is $a_1 \neq a_2$ if $k_1 \neq k_2$, so that the extension of the springs is identical and that the ratio $a_1/a_2 = k_2/k_1$.

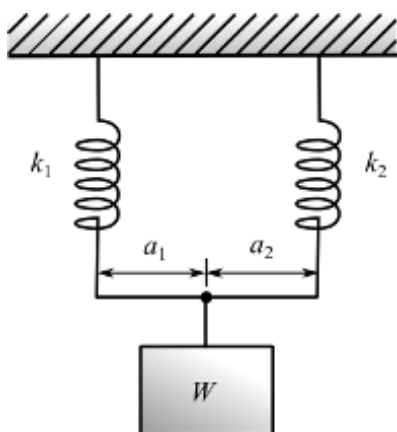


Figure 2.59: Weight hanging from two parallel springs.

31. For a body suspended between two springs as in Figure 2.60, show that the period of oscillation is

$$T = 2\pi \sqrt{\frac{W}{g(k_1 + k_2)}}.$$

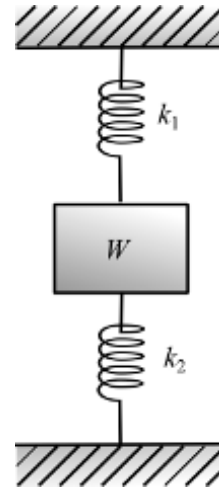


Figure 2.60: Body suspended between two springs.

32. A compound pendulum in the shape of a rectangle is supported at point O and allowed to oscillate. The dimensions of the rectangle are given in Figure 2.61. Calculate the natural frequency for small oscillations.

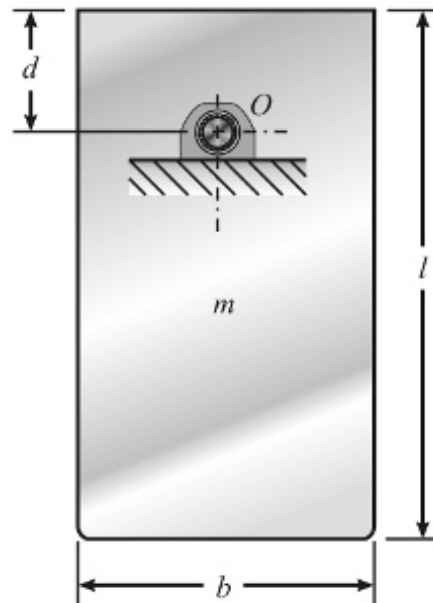


Figure 2.61: A compound pendulum.

33. Two springs in series support weight W , as shown in Figure 2.62. If these springs are replaced by a single spring, find the equivalent stiffness as well as the period of oscillation. The solution is

$$k = \frac{k_1 k_2}{k_1 + k_2}, \quad T = 2\pi \sqrt{\frac{W(k_1 + k_2)}{g k_1 k_2}}.$$

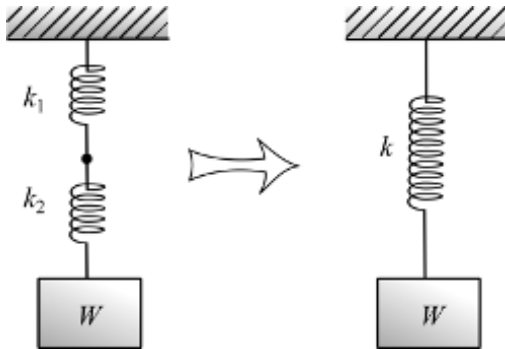


Figure 2.62: Two springs in series on left, and equivalent spring on right.

34. A pendulum of mass m and mass moment of inertia I_O is suspended from a hinge, as shown in Figure 2.63. The center of gravity is located a distance h from the hinge. For small oscillations, the period equals T . Suppose the pendulum swings with an amplitude of angle θ from the vertical. Find the force exerted on the hinge when the pendulum is (a) at its maximum rotation and (b) at its vertical position.



Figure 2.63: A compound pendulum.

35. A bifilar pendulum of length $2a$ is suspended with two vertical strings, each of length l , as shown in Figure 2.64. (Bifilar means fitted with or involving

the use of two threads or wires.) Assuming small rotations of the strings, such that the bar is essentially horizontal with half its weight supported by each string, show that the period is

$$T = \frac{2\pi a}{b} \sqrt{\frac{l}{3g}}.$$

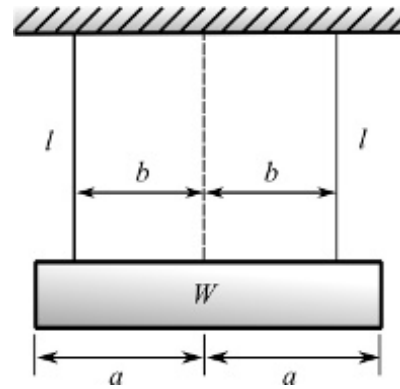


Figure 2.64: A bifilar compound pendulum.

36. An inverted hinged pendulum with a mass m at the top is suspended between two springs with constants k , as shown in Figure 2.65. The rod can be assumed rigid and massless, and in the vertical position the springs are unstretched. For small motion, the springs can be assumed to remain horizontal. Show that the period of oscillation is

$$T = \frac{2\pi}{\sqrt{\frac{2k}{m} - \frac{g}{l}}}.$$

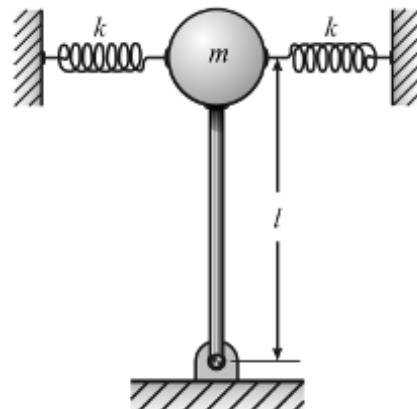


Figure 2.65: Stabilized inverted pendulum.

37. Consider the inverted simple pendulum shown in Figure 2.66. Initially the pendulum is in a perfectly vertical position. If it is displaced very slightly from

the vertical, what are the stability characteristics of the system? What kind of motion is expected and what is the equation of motion? Discuss the long-term behavior.



Figure 2.66: Inverted pendulum.

38. For the system of Figure 2.67, what should the value of k_3 be if $k_1 = 2k_2 = 3k_3$, for a period of free vibration of 400 ms for $m = 2.5$ kg?

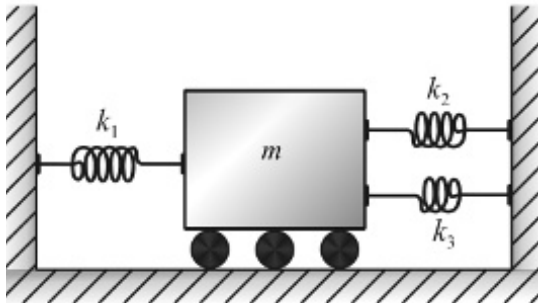


Figure 2.67: Mass connected to ground through three springs.

39. A slender bar of mass m and length l is supported at its base by a torsional spring of stiffness K , as per Figure 2.68. The bar rests in the vertical position when in equilibrium with the spring unstretched. Show that the differential equation of rotation θ from equilibrium is

$$\frac{ml^2}{3}\ddot{\theta} + K\theta - \frac{mgl}{2}\sin\theta = 0.$$

For small vibration, that is, $\theta \ll 1$, show that the natural frequency is given by

$$\omega_n = \frac{(K - mgl/2)}{ml^2/3}.$$

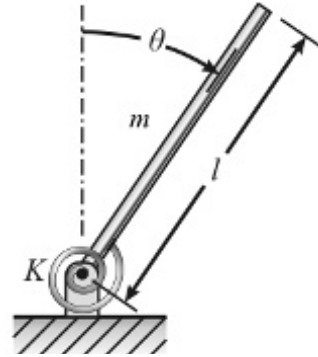


Figure 2.68: A slender bar of mass m and length l is supported at its base by a torsional spring of stiffness K .

40. Solve Problem 39 for the initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = \dot{\theta}_0$ for two cases of the stiffness:

$$K > \frac{mgl}{2} \quad \text{and} \quad K < \frac{mgl}{2}.$$

What is the solution for $K = mgl/2$?

41. A bar supported by a hinge at its base is held in place by a spring connected to a collar, as per Figure 2.69. The spring is unstretched when the bar is vertical. As the bar is displaced from equilibrium by an angle θ , the collar slides on the frictionless bar so that the spring remains horizontal. Derive the equation of motion using an energy method, and then linearize the model for small amplitude motion. Identify the frequency of oscillation.

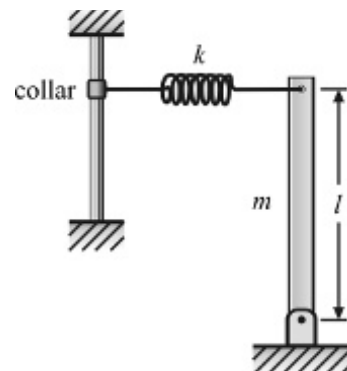


Figure 2.69: An inverted bar is held in place by a spring connected to a collar.

42. A simple pendulum of initial length l_0 and initial angle θ_0 is released from rest. If the length is a function of time according to $l = l_0 + \varepsilon t$, find the position (l, θ) of the pendulum at any time assuming small oscillations. The governing equation of motion will turn out to be

$$(l_0 + \varepsilon t)\ddot{\theta} + 2\varepsilon\dot{\theta} + g\theta = 0,$$

and in transformed Bessel form,

$$x^2\theta'' + 2x\theta' + \frac{xg}{\varepsilon^2}\theta = 0,$$

where $x = l_0 + \varepsilon t$. (This problem requires the use of Bessel functions.)

43. A homogeneous disk of weight W and radius r is supported by two identical cylindrical steel shafts of length l , as shown in Figure 2.70. From solid mechanics, for each shaft the relation between the moment on the disk M and the angle of rotation of this disk θ is

$$M = \frac{GJ}{l}\theta,$$

where G is the shear modulus of the material, J is the polar moment of inertia of the cross section of each shaft, and GJ is known as the torsional rigidity. Suppose this system is being designed for a particular application that requires the frequency of vibration to be f . Find the value of r in terms of the parameters to satisfy this requirement.

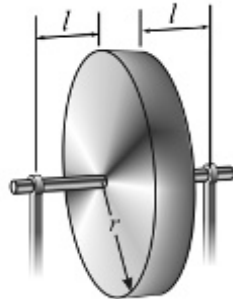


Figure 2.70: A homogeneous disk of weight W and radius r is supported by two identical cylindrical steel shafts of length l between support and face of disk.

44. When a manned craft is sent into space it is necessary to know the mass properties of the system, including the moment of inertia of astronauts on the flight, in order to accurately calculate its trajectory and fuel expenditures. The device in Figure 2.71 is one way to measure an astronaut's moment of inertia. The horizontal platform is pinned at O and supported a distance l away by a spring with constant k . When the astronaut is not present, the frequency of small vibration of the platform about O is measured to be f . When the astronaut is lying on the platform, the frequency of small vibration is measured to be f_a . (a) Find the astronaut's moment of inertia about the z axis. (b) Assuming reasonable parameter values for the system and an astronaut of height 2.0 m, calculate the moment of inertia.

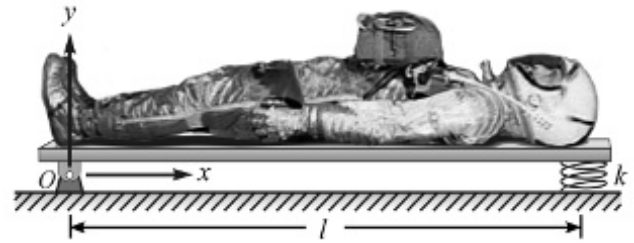


Figure 2.71: System to measure astronaut's moment of inertia.

45. A small pendulum is mounted in a rocket that is accelerating up at a rate of $4g$, as sketched in Figure 2.72. The pendulum is composed of a massless rod of length $l = 1$ m that supports a block of mass $m = 0.5$ kg. Assuming small oscillations, what is the rotational natural frequency of the pendulum?

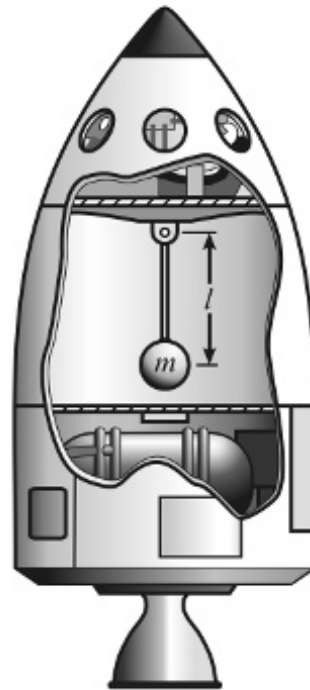


Figure 2.72: An accelerating rocket with a pendulum inside.

46. For the undamped rocker arm sketched in Figure 2.73, determine the natural frequency of the system undergoing small amplitude oscillation using Newton's second law of motion. Assume that the mass of the T-bar is negligible compared to the mass m of the block.

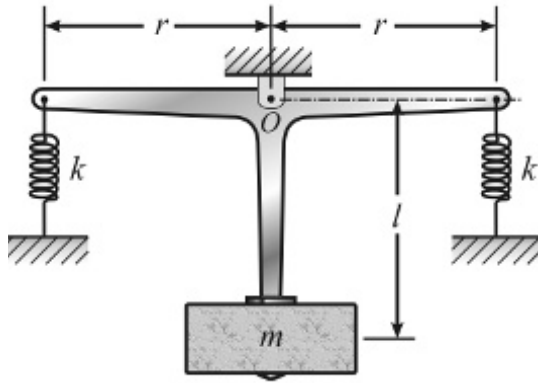


Figure 2.73: Vibrating rocker arm.

47. For the system of Figure 2.74 with spring stiffness k , bar mass M and length l , and end mass m , derive the equation of motion for small oscillations about the horizontal equilibrium position assuming the support pin is frictionless. Based on the equation of motion, determine the effective mass and effective stiffness of the system.

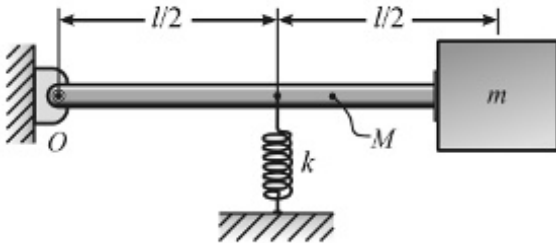


Figure 2.74: Pinned bar-spring system.

48. Consider a generalization of Example 2.10, where a mass moves to the left with speed v on a platform, disconnected from two springs, as shown in Figure 2.75. Assuming that there is no friction, find the period of oscillation of the mass, given the data: $k_1 = 36$ N/cm, $k_2 = 18$ N/cm, $m = 25$ kg, $b = 30$ cm, $d = 100$ cm, and $v = 6$ m/s.

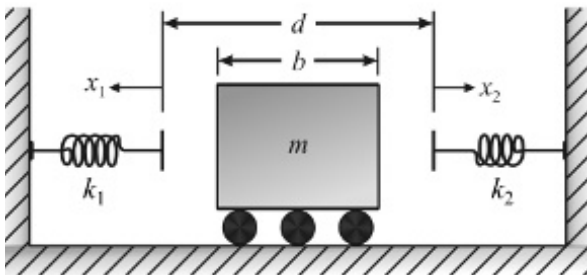


Figure 2.75: Mass oscillating between springs.

49. A block of mass m is suspended from the ceiling by a spring of stiffness k . The equation of motion is

$$m\ddot{x} + kx = 0,$$

where x is measured from the static equilibrium position. Assume the initial conditions are $x(0) = 1$ in and $\dot{x}(0) = 1$ in/s. Solve for the response $x(t)$ for two cases: (a) the spring is assumed massless, and (b) the mass of the spring is included.

Plot both solutions and comment on the importance of including the mass of the spring. Assume $mg = 1$ lb and a spring mass that is 10% of the block mass for $k = 0.1, 1$, and 10 lb/in. What conclusions can be drawn?

50. Two systems with discontinuities are shown in Figure 2.76. In each system the mass oscillates and comes into contact with springs at some distance $\pm x_c$. For each system, derive the equation of motion for a complete cycle.

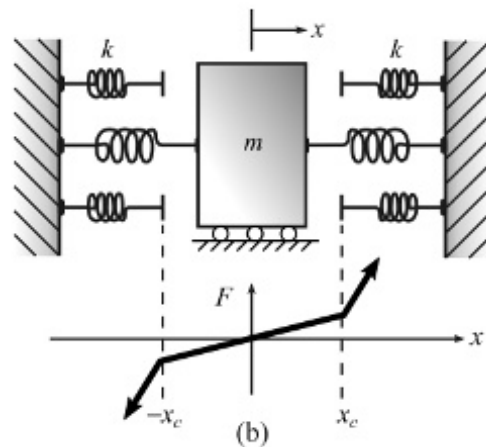
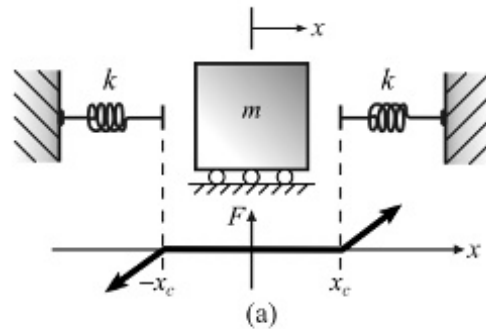


Figure 2.76: (a) A mass with springs engaging at a distance $\pm x_c$. (b) A mass with multiple springs engaging at a distance $\pm x_c$.

Problems for Section 2.4 – Harmonic Forcing with No Damping

51. Derive Equation 2.37,

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t + \frac{x_{st}}{1 - (\omega/\omega_n)^2} [\cos \omega t - \cos \omega_n t].$$

52. Solve the equation of motion,

$$\ddot{x} + x = \frac{1}{m} F(t),$$

for: (a) $F(t) = \cos 0.5t$, (b) $F(t) = \cos 0.99t$, (c) $F(t) = \cos t$, and (d) $F(t) = \cos 2t$. Plot the responses and compare the results. Let $m = 1$ with appropriate units.

53. For the oscillator that is beating according to the equation

$$\ddot{x} + 16x = \frac{A}{m} \sin(4 - \varepsilon)t,$$

where ε is small but not equal to zero, what happens as $\varepsilon \rightarrow 0$? Relate the rate of growth of the response amplitude to the value of ε or to the value of $4 - \varepsilon$.

54. The block shown in Figure 2.77 is acted on by the force

$$F(t) = 100 + 25 \sin 75t \text{ N}$$

and, after the transients have died out, it oscillates with an amplitude of 0.6 mm about a position 55 mm to the right of the static equilibrium position corresponding to the condition when no force is present. What is the mass of the block?

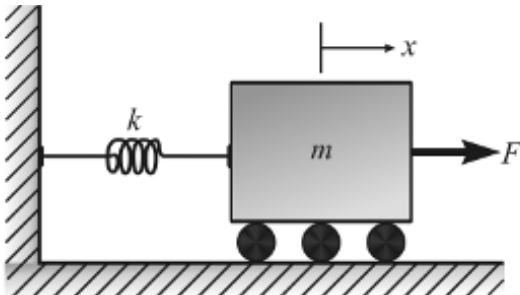


Figure 2.77: Block subjected to external force.

55. For Example 2.12, what range of frequencies of the motion $y(t)$ must be excluded to keep the maximum force at C less than 10 N?

56. A gantry crane system is shown schematically in Figure 2.78. The crane is a simple pendulum of length l and mass m . The gantry oscillates with the displacement $x(t) = A \sin \omega t$. If A is very small, what

ω gives the crane an amplitude of motion equal to $2A$? Assume $l = 1$ m.

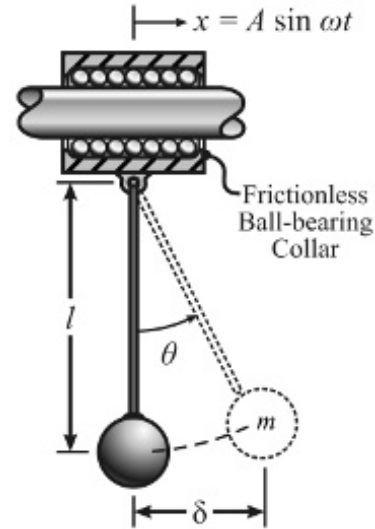


Figure 2.78: A model of a gantry crane.

57. Solve for the response of the equation of motion,

$$\ddot{x} + 9x = 3 \sin t + \cos 3t,$$

assuming zero initial conditions.

58. Solve for the response of the equation of motion,

$$\ddot{x} + \omega_n^2 x = \frac{A}{m} \sin \omega_n t,$$

for arbitrary initial conditions.

59. Derive Equation 2.41,

$$x(t) = \frac{x_{st} \omega_n^2}{2\varepsilon(\varepsilon + \omega)} [\sin(\varepsilon + \omega)t \sin \varepsilon t].$$

60. A cylinder of mass m is mounted in a water tunnel as shown in Figure 2.79 with the cylinder axis transverse to the flow direction. When there is no flow, a vertical force of F_{static} on the cylinder results in a deflection of x . With flow in the tunnel, shedding vortices impart alternating forces on the cylinder. The velocity of the water is v in the x direction, the distance between vortices is d , and the magnitude of the lateral forces is F_v . The lateral forces can be modeled by the harmonic function $F(t) = F_v \sin \omega t$. Find the amplitude of lateral steady-state response of the cylinder.

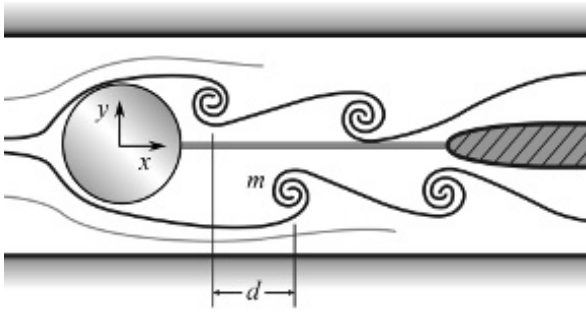


Figure 2.79: Cross section of an elastically mounted cylinder in fluid flow, excited by shedding vortices.

Chapter 3

Single Degree-of-Freedom Damped Vibration

“The simplest model with energy dissipation.”

We continue the development of single degree-of-freedom models of dynamic systems started in [Chapter 2](#). More realistic models are developed accounting for system damping as well as more complicated loading functions. With damping, system models respond to loading in a manner that better matches our intuition and experimental results.

All motion, including vibratory motion, is accompanied by energy dissipation in one form or another. Whether or not this energy loss is significant depends on the type and magnitude of the damping and the timescale of interest. Systems in motion over longer timescales are kept in motion by external forcing.

In the current chapter, as well as subsequent chapters, we primarily investigate viscous damping as a way to estimate the effects of material and structural damping. Such an approach is approximate and leads the way to more sophisticated and accurate damping models. We also consider to a lesser extent Coulomb damping, which has nonlinear characteristics due to the change of the direction of the damping force as the oscillations change direction. The subject of damping is addressed in more depth in specialized textbooks¹ with some advanced concepts presented in [Appendix B](#).²

3.1 Overview

This chapter introduces the concept of energy dissipation, or damping, for vibrating systems. Damping is a

complex physical phenomena (it is temperature and pressure dependent) and there are various sources of dissipation in a system. We will develop mathematical models of damping that are commonly used in equations governing oscillatory motion. Our focus is on linear damping models.

As with any study of real systems, we are forced to idealize and approximate in order to begin to understand complex behavior. When considering vibrating systems, we first idealize by discretizing real continuous systems into discrete approximate systems. Further, we assume small motions so that linear theory applies, and assume some components to be rigid and some to be elastic. Just as linearity is always an approximation, in many instances leading to almost exact results, elasticity is also always an approximation.

We will see that damping introduces a phase lag in system response. Damping, in addition to accounting for energy dissipation, implies that structural displacement lags forcing and strain lags stress. We also know from material behavior experiments that strain is not a single-valued function of stress. “Solids creep when a sufficiently high stress is applied, and the strain is a function of time.”³ *Anelasticity* is the term used to denote such creep and time-dependence. The response of a solid to a force can be viewed as having two components, one part that is (ideally) elastic and instantaneous, and one part that is anelastic and time-dependent, or delayed. As per Anderson, “The anelastic part contains information about temperature, stress and the defect nature of the solid.”

Energy is lost either externally to the system or internally within the system. There is always energy dissipation in any real system. We sometimes ignore damping in order to study idealized (undamped) systems because they exhibit key oscillatory characteristics that are useful to our fundamental understanding.

¹The literature on damping is extensive. One recommended reference is **Vibration Damping of Structural Elements**, C.T. Sun and Y.P. Lu, Prentice Hall, 1995.

²Advanced concepts can be helpful to the reader interested in the study of composites and other engineered materials as well as situations in which simple damping models are not sufficient.

³**Theory of the Earth**, D.L. Anderson, Blackwell Scientific Publications, 1989.

Damping mechanisms are not as well understood as are other aspects of vibratory systems. There is no universal model for damping. “There are excellent reasons why the stiffness and inertia properties of a general discrete system, executing small vibration around a position of stable equilibrium, may be approximated via the familiar stiffness and mass matrices. These simply represent the first non-trivial terms which do not vanish when the potential and kinetic energy functions are Taylor-expanded for small amplitudes of motion. Nothing so simple can be done to represent damping, because it is not in general clear which *state variables* the damping forces will depend on.”⁴

Damping mechanisms involving structures can be grouped into three types: (1) *material damping* where energy is dissipated within the body, (2) *boundary damping* where dissipation occurs at junctures between elements of a structure, and (3) *viscous damping* due to contact between a structure and a fluid environment where energy is dissipated into the fluid. Damping is further subdivided in the next section. Viscous damping is commonly used in mechanical vibration and in this book.

Material damping is induced by microstructural strains and relative motion, in particular, due to imperfections in the atomic lattice structure of the material. This discrete reality is usually represented by a viscoelastic continuum model in applications. Viscoelastic models are discussed in [Appendix B](#).

Less well defined is boundary damping. Built-up structures are comprised of hundreds to many thousands of parts that are welded, riveted, glued, or fastened in other ways. At these interfaces significant friction forces develop as a result of oscillation and differential movements. As per Woodhouse, “When damping is measured on a built-up structure like a ship or a building, [friction forces are] commonly found to be at least an order of magnitude higher than the intrinsic material damping of the main components of the structure.” *Because of the low levels of damping found in most engineering structures, viscous damping force models are found to yield acceptably accurate results for most vibration studies.*

Viscous damping resulting from fluid viscosity is commonly used for cases where there is no fluid. Such models are based on the assumption that such damping forces are proportional to the instantaneous velocities. Again, since most structural systems have low levels of damping, viscous damping force models are acceptably accurate. Viscous damping models are used extensively in this text.

It is important to distinguish between effective mathematical models that lead to accurate results and physical models that are representative of the underlying physics

of the behavior. We always desire that the mathematical models be representative of the physics, but that is not always the case. Sometimes, a mathematical model that is not derived from the underlying physics can be predictive because it contains key elements of the observed behavior.

Finally, we remind the reader that the primary mathematical framework for the study of dynamic and vibrating systems is the differential equation. For discrete systems it is the second-order differential equation obtained from Newton’s second law of motion, or via a variational principle, introduced in a subsequent chapter. Terms in such differential equations are generally forces, for example, a stiffness force (related to displacement) and a damping force (generally related to velocity). In continuous systems we will see partial differential equations.

Examples of Large Dampers

Large dampers⁵ are used extensively in buildings, bridges, and a host of large structures. For example, the Sutong Changjiang River Bridge ([Figure 3.1](#)) is the world’s second longest cable-stayed bridge and employs special spring-damper systems ([Figure 3.2](#)) on the main span to control seismic and wind movements. Eight spring-dampers were installed in 2007, each with ± 850 mm stroke and capable of resisting a 6580 kN load.

In 2004 a new Los Angeles Regional Transportation Management Center was constructed with base isolation. Special fluid dampers ([Figure 3.3](#)) function as seismic energy absorbers. Twenty-five dampers were installed, each with ± 660 mm stroke and capable of resisting a 1450 kN load.

In 1999 a new Airport Rail Transit and Bay Area Rapid Transit structure ([Figure 3.4](#)) was constructed at the San Francisco International Airport. Ten fluid dampers were installed for earthquake energy dissipation. Two different types were employed, all with a ± 508 mm stroke, some capable of resisting a 3115 kN load and others a 4225 kN load.

The Poplar Street Bridge ([Figure 3.5](#)), a large highway bridge over the Mississippi River in St. Louis, uses 64 fluid dampers to control longitudinal earthquake movement while allowing for free thermal movement. Two different types of dampers were used, some with a ± 183 mm stroke and capable of resisting a 1334 kN load and others with a ± 229 mm stroke and capable of resisting a 2224 kN load.

⁴J. Woodhouse, “Linear Damping Models for Structural Vibration,” *Journal of Sound and Vibration*, Vol.215, No.3, 1998, pp.547-569.

⁵The authors are indebted to Douglas P. Taylor of Taylor Devices, Inc. (North Tonawanda, NY) for information and photographs. At the end of the Cold War in 1990, Taylor conceived and promoted the concept of using large damping devices from U.S. Ballistic Missile Programs for earthquake protection for buildings and bridges. This technology transfer to the private sector has proven highly successful. Taylor holds over 30 patents on large damper systems.



Figure 3.1: The Sutong Changjiang River Bridge in China is a cable-stayed bridge with a span of 1.1 km (the world's longest from 2008-2012). Its two side spans are 300 m each, and the two towers of the bridge are 306 m high. The total bridge length is 8.2 km. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)



Figure 3.2: Spring-dampers installed under the Sutong Changjiang River Bridge in China. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)



Figure 3.3: Large damper installed at the Los Angeles Regional Transportation Management Center. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)



Figure 3.4: Dampers installed at the San Francisco International Airport. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)



Figure 3.5: Vibration dampers are used under the Poplar Street bridge near the Riverfront in St. Louis. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)

3.2 Introduction to Damping

Damping refers to the removal of energy from vibratory motion and is primarily associated with the irreversible transformation of mechanical energy into thermal energy. Damping is the energy dissipation property of materials and structures undergoing time-dependent deformation and/or displacements.⁶ A deformation occurs when a body's shape changes as a result of forces acting on it, whereas a displacement is a change in location of a point or points on a body. In most instances, both occur. Lumped-parameter models idealize this situation by concentrating their mass, damping, and stiffness properties at discrete locations.

Depending on the mechanism of energy dissipation, damping can be classified according to the following overlapping characteristics:

- **Viscous Damping** – In viscous damping the force is linearly proportional to the relative velocity. The constant of proportionality is the damping constant that is determined experimentally. Viscous damping is the simplest model of damping. It is used extensively in vibration models, and it represents a mix of material and structural damping.
- **Coulomb Damping** – Coulomb damping, also called dry damping, is a result of the sliding of two dry surfaces. The damping force is assumed to be independent of the relative velocity of the two surfaces, and is the product of the normal force between the two bodies and the coefficient of kinetic friction.
- **Material Damping** – In material damping energy is dissipated by deformation in a medium (such as irreversible intercrystal heat flux or grain boundary viscosity). Dissipation of energy is an inherent property of the material and is due to material defects or due to rotation or sliding within the atomic lattice. Such damping can be called internal friction. Material damping is sometimes modeled as viscoelastic.
- **Structural Damping** – Damping in assembled structures can include material damping in members, frictional losses (microslip and macrosip) at contact surfaces (bolted, riveted, damped, welded connections), dissipation in a medium between surfaces in relative motion (gas pumping, squeeze film damping, lubricated bearing).
- **Viscoelastic Damping** – In viscoelastic damping the damping forces are a function of the history of the behavior to the present time. The material has “memory” and it is necessary to integrate that

behavior to the present time to model the present behavior. Anelastic damping is a special case of a viscoelastic material that fully recovers its original state with the removal of the load.

- **Radiation Damping** – In radiation or geometric damping energy is radiated into a surrounding medium.
- **Active/Passive Damping** – In active/passive damping there is damping with/without external energy and control. Except for systems with active feedback control, the damping is passive. Passive damping includes material damping and *system damping*.
- **Internal/External Damping** – In internal/external damping, damping occurs inside/outside a defined system boundary.

Two simple mechanical models are commonly used to represent anelastic behavior. These are the *Kelvin-Voigt* and the *complex modulus*, both introduced in [Appendix B](#). From Graesser,⁷ “Anelastic material behavior requires that the following conditions be imposed on stress, strain, and equilibrium in a material:

1. For each value of stress in a material there must be an equilibrium value of strain (as a corollary, this condition requires a complete recovery of strain upon unloading to zero stress).
2. The equilibrium response is arrived at following some sufficient time delay (self-adjustment or relaxation).
3. A linear stress-strain relationship is required.”⁸

While a discussion of the many parameters used to measure damping is beyond the scope of this chapter and book,⁹ for completeness we mention the parameters used in vibration analyses: tangent of the phase lag ($\tan \phi$), the damping ratio (ζ), the specific damping capacity (ψ), the loss factor (η), the inverse quality factor (Q^{-1}), and the logarithmic decrement (δ). These parameters can be derived from the Kelvin-Voigt and the complex modulus models. Some of these are introduced in this and subsequent chapters.

⁷E.J. Graesser, C.R. Wong, *The Relationship of Traditional Damping Measures with High Damping Capacity*, DTRC-SME-91/05, January 1991.

⁸Continuing from Graesser, “This definition differs from that of an ideally elastic material only in the condition imposed by Item 2. For an ideally elastic material the equilibrium response is instantaneous and thus the difference between an ideally elastic material and an anelastic material is based solely on the condition of instantaneity.” For example, steel is almost ideally elastic.

⁹The report by Graesser and Wong includes an in-depth discussion of different damping measures, how they are related, and their limitations.

⁶L. Gaul, “The Influence of Damping on Waves and Vibrations,” *Mechanical Systems and Signal Processing*, Vol.13, No.1, 1999, pp.1-30.

3.3 Damping Models

In vibration studies, the viscous damping model is the most widely adopted model. Not only is it simple, under certain conditions a decoupling of the equations of motion in multi degree-of-freedom systems is possible. Several commonly used damping models for elastic behavior are special cases of the following,¹⁰

$$f(\dot{x}) = a\dot{x}|\dot{x}|^{\theta-1},$$

where $f(\dot{x})$ is the damping force and a is the damping coefficient. The value of θ determines the damping model. For $\theta = 1$ and $a = c$, we have the linear viscous damping model,

$$f(\dot{x}) = c\dot{x}.$$

For $\theta = 0$ and $a = \mu N$, we have Coulomb damping, used to represent contact friction losses,

$$f(\dot{x}) = \mu N \frac{\dot{x}}{|\dot{x}|} = \mu N \text{sgn}(\dot{x}),$$

where $\text{sgn}(\dot{x})$ is defined as ± 1 for $\dot{x} \gtrless 0$ and equals 0 for $\dot{x} = 0$. For $\theta = 2$ and $a = q$, we have quadratic damping,

$$f(\dot{x}) = q\dot{x}|\dot{x}|,$$

a more complex form of which is used to represent *hydrodynamic damping*. These models are all functions of velocity only; other damping models are functions of velocity as well as displacement.

3.3.1 Viscous Damping and Loss Factor

Damping extracts energy from a system, and for oscillatory systems undergoing free (unforced) vibration, the peak potential energy represents the total energy in the system prior to dissipation. Therefore, for a particular cycle, one measure of dissipation is the ratio of the energy lost, W , in that cycle and the peak potential energy, V , in that cycle. The energy lost equals the work done by the dissipation forces. A system does not generally lose the same amount of energy in each cycle. The loss depends on amplitude, frequency, and temperature. The above-mentioned ratio is called the *loss factor*, η ,

$$\eta = \frac{W}{2\pi V},$$

a dimensionless quantity. Most structural and mechanical systems of practical importance have a small amount of damping. Thus, “values for loss factors that are encountered in practice range from about $\eta = 10^{-5}$ to

¹⁰ A. Kareem and K. Gurley, “Damping in Structures: Its Evaluation and Treatment of Uncertainty,” *Journal of Wind Engineering and Industrial Aerodynamics*, Vol.59, 1996, pp.131-157.

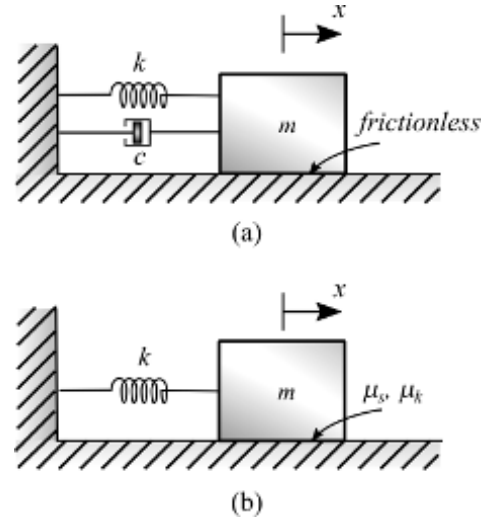


Figure 3.6: The schematic shows (a) a body subjected to viscous damping and (b) a body subjected to Coulomb damping.

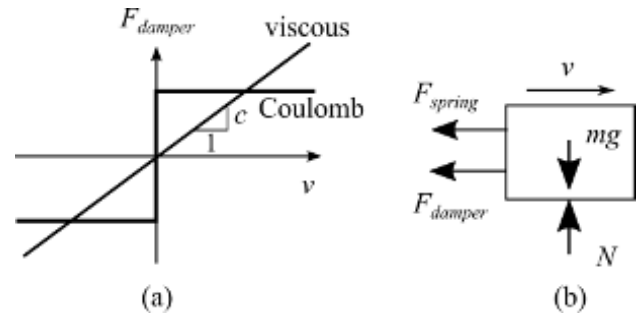


Figure 3.7: (a) Damping force vs. velocity for viscous and Coulomb damping. (b) Free-body diagram of the mass in Figure 3.6(a).

$\eta = 0.2$, although larger values of η are found in instrument mechanisms, transducers and vehicle suspensions.”¹¹

In vibrating systems, an ideal linear dashpot¹² is introduced for modeling purposes, resulting in a force that is opposite in direction and proportional to the (relative) velocity between the ends of the dashpot, as shown in Figures 3.6(a) and 3.7(a). The dashpot, or damping, force is given by $F_d = c\dot{x}$. If the structural displacement is given by x , then $v = \dot{x}$ and $F_d = c\dot{x}$. This is the commonly used *viscous damping model*.

The loss factor in the instance of a sinusoidal oscillation is derived next. The energy dissipated in one cycle equals the work done by the damping force,

$$W_d = \oint F_d dx. \quad (3.1)$$

¹¹ S.H. Crandall, “The Role of Damping in Vibration Theory,” *Journal of Sound and Vibration*, Vol.11, No.1, 1970, pp.3-18.

¹² The term “dashpot” refers to a mechanical device, such as a piston in a cylinder, that resists motion via viscous friction.

For harmonic forcing, $F = F_0 \sin \omega t$, the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t,$$

with the steady-state response given by

$$x(t) = X \sin(\omega t - \phi), \quad (3.2)$$

where X is the response amplitude and ϕ is the phase angle. Both will be discussed in full in the subsequent chapter. Equation 3.1 becomes

$$\begin{aligned} W_d &= \oint c\dot{x} dx = \oint c\dot{x}^2 dt \\ &= c(\omega X)^2 \oint \cos^2(\omega t - \phi) dt, \end{aligned}$$

where the period of forced vibration is $T = 2\pi/\omega$. Integrating over the period yields

$$W_d = \pi c \omega X^2. \quad (3.3)$$

This equation indicates that the dissipation is proportional to the excitation frequency and to the square of the response amplitude. At the natural frequency, $\omega = \omega_n = \sqrt{k/m}$,

$$\begin{aligned} W_{dn} &= \pi c \omega_n X^2 \\ &= \pi c \sqrt{k/m} X^2 \\ &= 2\pi \zeta k X^2, \end{aligned}$$

where $\zeta = c/2\sqrt{km}$ is dimensionless and called the viscous damping ratio or damping factor. The damping constant c , which may be frequency-dependent, turns out to be an accurate measure of the actual damping for many applications.

The loss factor is then

$$\eta = \frac{W_d}{2\pi V} = \frac{\pi c |\omega| X^2}{2\pi(kX^2/2)} = \frac{c|\omega|}{k}, \quad (3.4)$$

where c is assumed positive and the absolute value sign about $|\omega|$ is used to signify that the direction of motion $x(t)$ does not change the energy loss per cycle. The loss factor must be even, real, non-negative and lead to behavior that is *causal*. Causality is discussed further in Section B.3.

In many applications where damping is light, its effect is greatest near the natural frequency, and at other frequencies of oscillation it will not significantly affect the overall dynamic response. At the natural frequency, the loss factor η_n is given by

$$\eta_n = \frac{c\omega_n}{k} = \frac{c}{\sqrt{km}}.$$

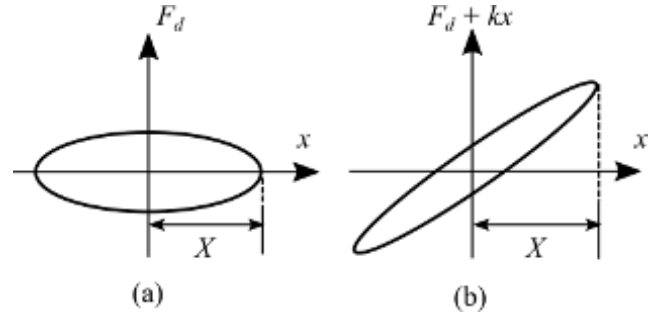


Figure 3.8: Hysteresis loops for linear viscous damping.

The damping force can be written as

$$\begin{aligned} F_d &= c\dot{x} = c\omega X \cos(\omega t - \phi) \\ &= c\omega X \sqrt{1 - \sin^2(\omega t - \phi)} \\ &= c\omega \sqrt{X^2 - x^2}, \end{aligned}$$

or, rearranging,

$$\left(\frac{x}{X}\right)^2 + \left(\frac{F_d}{c\omega X}\right)^2 = 1,$$

which is an ellipse on the F_d - x plane, as shown in Figure 3.8(a). The area enclosed by the ellipse equals the work done, or energy dissipated, during one cycle, $\pi c \omega X^2$. For the case of nonlinear, damped oscillation, the curve is still closed but not elliptical as for the linear case. These closed curves are also called *hysteresis loops*.

A similar ellipse of the same enclosed area can be drawn by plotting $(F_d + kx)$ vs. x , as shown in Figure 3.8(b), except that the ellipse now has a positive-sloped major axis. This represents a dashpot in parallel with a spring, which is the *Voigt model* discussed in Appendix B.

Carfagni et al.¹³ note the following: “The loss factor is undoubtedly the most general of the measurement indexes. Whereas other indexes such as the damping ratio ζ are defined on the grounds of the linear single degree-of-freedom viscous model, the loss factor can be successfully applied in the case of nonlinear systems. ... η can be used in material testing or in evaluating a composite structure. It directly measures dissipation, with no reference to the physical mechanisms involved - undoubtedly, a recognition of its generality.”

Equivalent Viscous Damping Models

Since viscous damping is the easiest to use in vibration models, Equation 3.3 can be used to derive *equivalent*

¹³M. Carfagni, E. Lenzi, and M. Pierini, *The Loss Factor as a Measure of Mechanical Damping*, IMAC XVI 16th International Modal Analysis Conference, 1998.

viscous damping constants c_{eq} by solving for the viscous damping parameter,

$$c_{eq} = \frac{W_d}{\pi\omega X^2}, \quad (3.5)$$

where W_d is now the work done in one cycle by a different loss mechanism. Of course, for a viscous damper, $c_{eq} = c$.

Without details, we state the following results. For a simple Coulomb friction model that dissipates $W_c/4 = F_c X$ energy in each quarter cycle the equivalent viscous damper is given by

$$c_{eq} = \frac{4\mu N}{\pi\omega X}.$$

For the case of quadratic damping, a useful model for shock absorbers and fluid-structure systems, the damping force is $F_d = c_q \operatorname{sgn}(v) \cdot v^2$, where $\operatorname{sgn}(v)$ is defined here as ∓ 1 for $v \gtrless 0$ and equals 0 for $v = 0$. Here, $v = \dot{x}$, the oscillator speed, given by the derivative of Equation 3.2. The energy dissipated in one cycle is

$$\begin{aligned} W_d &= \oint c_q \dot{x}^2 dx = c_q \operatorname{sgn}(\dot{x}) \oint \dot{x}^3 dt \\ &= 4c_q \omega^3 X^3 \int_0^{\pi/2\omega} \cos^3(\omega t - \phi) dt \\ &= \frac{8}{3} c_q \omega^2 X^3. \end{aligned}$$

Substituting W_d into Equation 3.5 yields

$$c_{eq} = \frac{8}{3\pi} c_q \omega X.$$

We note that for the two equivalent damping models here, one is a linear function of driving frequency ω and the other is an inverse function of ω , resulting in very different characteristics.



DAMPERS. A viscous damper can be made using two parallel plates with a viscous fluid between them. If one plate is fixed and the other plate translates in its plane at a velocity, this will produce a shearing action on the fluid. The result is a horizontal force on the moving plate in a direction that opposes the motion and that is proportional to the velocity.

Another viscous damper can be constructed using a cylinder with a loose-fitting piston. If the piston rod is translated at a velocity, the fluid trapped on one side of the piston squeezes through the annular space between the piston and the cylinder. As in the case of the parallel plate damper, the fluid action will oppose the motion with a force proportional to the velocity.

For a nonlinear damper, a plot of force vs. velocity would not be a straight line. The equivalent linear damper can be approximated by the best straight line within a given range. For a very small range, the slope is the damping constant.

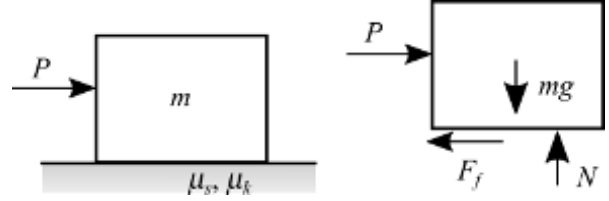


Figure 3.9: A body in static equilibrium with an external horizontal force resisted by a force due to Coulomb friction. The free-body diagram is shown at the right.

3.3.2 Coulomb Damping

Here we discuss Coulomb damping, also called dry friction, whose schematic is shown in Figure 3.6(b), where μ_s and μ_k are the static and kinetic coefficients of friction, respectively. Before analyzing the dynamic motion of the mass, we consider a mass in static equilibrium. The mass is resting on a horizontal surface and subject to a horizontal force P , as shown in Figure 3.9.

For a small value of P , the body will remain in static equilibrium, and the static friction force is same as the applied force P . As the force P is increased further, the static friction force can no longer equilibrate the force and the mass will start to move. This maximum static friction force is found experimentally to be proportional to the normal force between the mass and the floor. The proportionality constant is called the coefficient of static friction, μ_s , such that

$$F_f = P \text{ as long as } |F_f| \leq \mu_s N \text{ for a body at rest.}$$

The direction of the friction force always opposes the motion. The value of μ_s depends on many factors and is determined experimentally for different materials and conditions.

When the static friction is overcome, the mass moves and the friction force takes on the constant value,

$$|F_f| = \mu_k N,$$

where μ_k is the coefficient of kinetic friction. This coefficient is typically smaller than the coefficient of static friction. Once the mass is in motion, the friction force is constant at $\mu_k N$ even if the force P drops below what was necessary to start the motion.

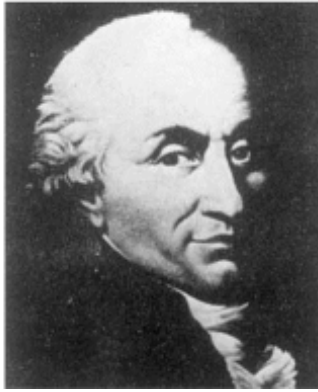
If the mass is restrained by a spring, it will oscillate; it will move in one direction, slow down, stop, and then move in the opposite direction. Once it stops, the friction force is again governed by the static friction coefficient, and there will be a short period of time when it will “stick” (not move) before the external force is large enough again to overcome the static friction force. The mass stops permanently once the spring force is not large enough to overcome the maximum static friction.

Figure 3.7 depicts the viscous and Coulomb damping forces as a function of velocity. The free-body diagram is

also shown. We will solve for the response of the system with Coulomb damping in [Section 3.5](#).

CHARLES AUGUSTIN DE COULOMB

(June 14, 1736 – August 23, 1806)



Contribution: Coulomb was a French physicist who wrote fundamental treatises on electricity and magnetism. He developed a theory of attraction and repulsion between bodies of the same and opposite electrical charge. He demonstrated an inverse square law for such forces – later named after him as Coulomb’s law – and studied electrical properties of conductors and dielectrics. He suggested that there was no perfect dielectric, proposing that every substance has a limit above which it will conduct electricity.

His papers on electricity and magnetism were only a small part of the work he undertook. He provided a simple, elegant solution to the problem of torsion in cylinders and his use of the torsion balance in physical applications was important to numerous physicists. Coulomb developed a theory of torsion in thin silk and hair threads that he used to show how torsion suspension could provide an accurate method of measuring extremely small forces. He developed a generalized sliding wedge theory of soil mechanics that remains in use.

Life Sketch: Coulomb was born in Angoulême, France. Both of his parents came from prominent families. After being brought up in southwestern France, Coulomb’s family moved to Paris. He entered the Collège Mazarin, where he received a classical education in language, literature, and philosophy, and the best available teaching in mathematics, astronomy, chemistry, and botany.

Due to unsuccessful financial speculations, Coulomb’s father lost all his money and moved from Paris to Montpellier. Coulomb’s mother remained in Paris. Due to a disagreement with her over the direction of his career, he left Paris to live with his father. Coulomb’s interests were mainly in mathematics and astronomy; in Montpellier he joined the Society of Sciences and in 1757 read several papers on these topics to the Society.

Coulomb wanted to enter the École du Génie at Mézières

but realized that to succeed in passing the entrance examinations he needed to be tutored. In October 1758, Coulomb went to Paris and studied for several months. He passed the examinations and entered the École du Génie at Mézières in February 1760.

Coulomb graduated in November 1761. He was now a trained engineer with the rank of lieutenant in the Corps du Génie. Over the next twenty years he was posted to a variety of different places and worked in engineering, structural design, fortifications, soil mechanics, and many other areas.

After he left the school in 1761, he took part in the survey for the British coastal charts. In February 1764, he was sent to Martinique in the West Indies. Martinique fell under the sovereignty of France under Louis XIV in 1658. Martinique was attacked by a number of foreign fleets over the following years. The Dutch attacked in 1674 but were driven off, as were the English in 1693 and the English again in 1759. Martinique was finally captured by the English in 1762 and was returned to France under the terms of the Treaty of Paris in 1763. The French then made attempts to make the island more secure by building a new fort.

Coulomb was put in charge of building this new fort, Fort Bourbon, and this task occupied him until June 1772. During this period, the practical side of his engineering skills were needed to organize the construction. His experiences would play a major role in the later theoretical memoirs he wrote on mechanics. As for Coulomb’s health, these were difficult years. Coulomb contracted tropical fever and suffered other illnesses in Martinique, which left him in poor health for the rest of his life.

On his return, at the rank of Captain, he was employed at La Rochelle, the Isle of Aix and Cherbourg. In 1781, he was stationed permanently in Paris. Coulomb began to write important works on applied mechanics. He presented his first work to the French Academy of Sciences in Paris in 1773. He used a combination of mathematics and physics in studying the influence of friction and cohesion in problems of statics. In this work Coulomb used calculus of variations to solve engineering problems.

His work was highly valued by the French Academy of Sciences. In 1777 he submitted a famous paper on the magnetic compass for the Grand Prix of the French Academy of Sciences. This paper won Coulomb a share of the prize and it contained his first work on the torsion balance.

Coulomb had developed a reputation as a military engineer with special skills designing fortifications. In 1779 he was sent to Rochefort to collaborate on the construction of a fort made entirely from wood. During his time in Rochefort, Coulomb carried on his research into mechanics using the shipyards there as laboratories for his experiments. His studies led to Coulomb’s major work on friction, which won him the Grand Prix from the French Academy of Sciences in 1781. Coulomb investigated both static and dynamic friction of sliding surfaces and friction in bending of cords and in rolling. From examination of many physical parameters, he developed a

series of two-term equations, the first term a constant and the second term varying with time, normal force, velocity, or other parameters.

In 1781, Coulomb was elected to the mechanics section of the Academy of Sciences, and moved to Paris where he held a permanent post. His work took on more of a scientific rather than engineering bent, although he did serve as an engineering consultant.

Coulomb worked for the respective French governments in varied fields such as education and reform of hospitals. In 1787, he made a trip to England to report on the conditions in the hospitals of London. In July 1784, he was appointed to maintain the royal fountains and a large part of the water supply of Paris.

In February 1790 Coulomb's first son was born. He was not married to the mother of his son.

When the French Revolution began in 1789 Coulomb had been deeply involved with his scientific work. Many institutions were reorganized, not all to Coulomb's liking, and he retired from the Corps du Génie in 1791. He was removed from his role in charge of the water supply and, in December 1793, the weights and measures committee on which he was serving was disbanded. Coulomb retired to a small country estate to pursue scientific research.

The French Academy of Sciences was replaced by the French Institute, and Coulomb returned to Paris when he was elected to the Institute in December 1795. In July 1797, his second son was born. In 1802, he married the mother of his two sons.

Coulomb was involved with services to education, largely between 1802 and 1806 when he was inspector general of public instruction. In that role he was mainly responsible for setting up the lycées (high schools) across France.

His health was already very feeble, and he died in Paris, France in 1806 at age 70.

Noteworthy: He is best known for developing Coulomb's law which defines the electrostatic force of attraction and repulsion. The SI unit of charge, the coulomb, is named after him.

Coulomb's contributions to the science of friction were exceptional. He essentially created this science. He was a pioneer in the field of geotechnical engineering for his contribution to retaining wall design.

In working as an engineering consultant on canal and harbor improvements, he ended up taking the blame when criticisms were made, spending a week in prison in November 1783.

3.4 Free Vibration with Viscous Damping

Dissipation of energy is inherent in all physical systems and it is modeled via damping. From Equation 2.5, the

effect of viscous damping on the free vibration of a linear oscillator is governed by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0, \quad (3.6)$$

where ζ is the damping factor or damping ratio and ω_n is the undamped natural frequency.

To solve differential Equation 3.6, a solution of the form $x(t) = Ae^{rt}$ is assumed,¹⁴ as in Section 2.3. This solution is differentiated and substituted into Equation 3.6 to give

$$A[r^2 + 2\zeta\omega_nr + \omega_n^2]e^{rt} = 0.$$

Since $Ae^{rt} \neq 0$, the quadratic equation in the brackets must equal zero,

$$r^2 + 2\zeta\omega_nr + \omega_n^2 = 0.$$

This algebraic equation is known as the *characteristic equation* for the system. Solving by the quadratic formula, the roots are

$$r_{1,2} = \left[-\zeta \pm \sqrt{\zeta^2 - 1} \right] \omega_n.$$

There are three classes of solutions depending on the value of the radicand $\zeta^2 - 1$. The class that arises for repeated roots is $\zeta = 1$ and is known as *critical damping* c_{cr} . It represents the boundary between aperiodic exponentially decaying motion (*overdamped*, $\zeta > 1$) and exponentially decaying oscillatory motion (*underdamped*, $0 < \zeta < 1$). The critically damped system will approach equilibrium the fastest. Most structures will have much lower values of viscous damping.

Substituting the roots $r_{1,2}$ into the assumed form of the solution gives the general form of the response,

$$x(t) = A_1 \exp \left(- \left[\zeta - \sqrt{\zeta^2 - 1} \right] \omega_n t \right) + A_2 \exp \left(- \left[\zeta + \sqrt{\zeta^2 - 1} \right] \omega_n t \right), \quad (3.7)$$

for $\zeta \neq 1$. (The assumed solution Ae^{rt} is not valid for repeated roots. This case is considered separately in the following section.) The character of the solution depends on the value of the viscous damping factor ζ .

Figure 3.10 shows the response $x(t)$ as a function of ζ and $\omega_n t$. The transition between oscillatory behavior and purely decaying behavior occurs at $\zeta = 1$.

¹⁴ An alternate approach is to assume a trigonometric solution of the form

$$x(t) = A \sin \omega t + B \cos \omega t.$$

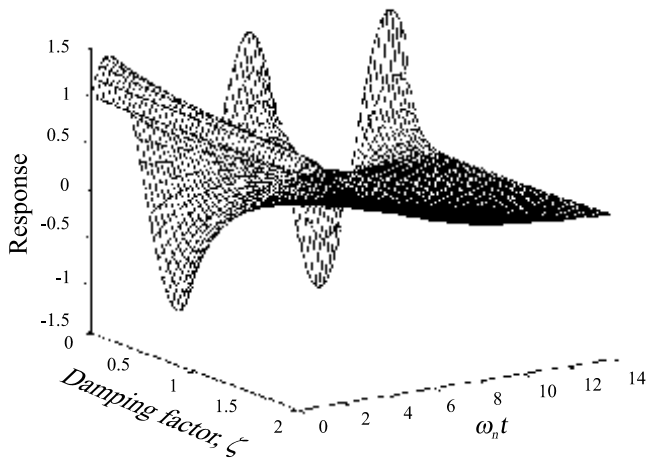


Figure 3.10: Free vibration as a function of ζ and $\omega_n t$.

3.4.1 Critically Damped and Overdamped Systems

For the critically damped system, $\zeta = 1$, the roots of the characteristic equation are equal and real,

$$r_{1,2} = -\omega_n.$$

For differential equations with repeated roots, the solution is

$$x(t) = (A_1 + tA_2) \exp(-\omega_n t),$$

with the second term, $tA_2 \exp(-\omega_n t)$, needed to ensure independent solutions. This term is initially driven by t , i.e., the effect of t in the response dominates. At a later time the exponential decay governs the behavior. From the expression $\zeta = c/2m\omega_n$ for a mass-spring-damper system, for $\zeta = 1$, the viscous damping coefficient assumes its critical value, $c_{cr} = 2m\omega_n = 2\sqrt{km}$. Equivalently, $\zeta = c/c_{cr}$.

For the overdamped case, the response is aperiodic and exponentially decaying with time with the solution given by Equation 3.7. For example, for two damping coefficient values, $\zeta = 1$ and $\zeta = 2$ and assuming $\omega_n = 1$ rad/s, $x(0) = 1$ and $\dot{x}(0) = 1$,¹⁵ the critical damped and overdamped responses, $x_{cr}(t)$ and $x_{od}(t)$, respectively, are given by

$$\begin{aligned} x_{cr}(t) &= (1 + 2t) \exp(-t) \\ x_{od}(t) &= \frac{1}{2} \left(1 + \frac{3}{\sqrt{3}} \right) \exp \left[-(2 - \sqrt{2^2 - 1})t \right] \\ &\quad + \frac{1}{2} \left(1 - \frac{3}{\sqrt{3}} \right) \exp \left[-(2 + \sqrt{2^2 - 1})t \right], \end{aligned}$$

and are shown in Figure 3.11. Note that the critically damped case decays to zero faster.

¹⁵ Although no units are shown, for completeness units are needed: x has units of displacement (m, ft, etc.) and $\dot{x}$ has units of velocity (m/s, ft/s, etc.)

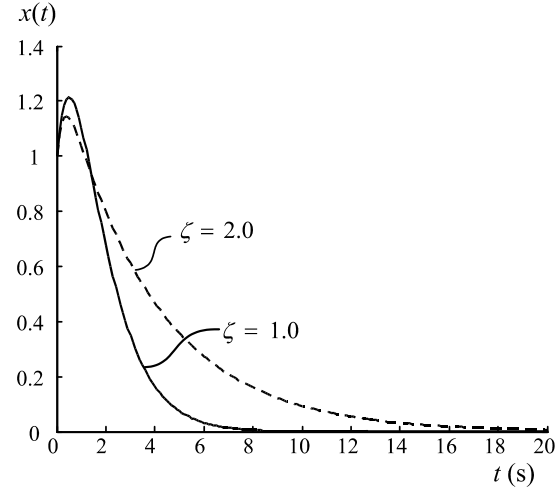


Figure 3.11: Displacement of critical response x_{cr} ($\zeta = 1$) and overdamped response x_{od} ($\zeta = 2$). The critical response quickly becomes larger than the overdamped response and then decays more rapidly to equilibrium.

3.4.2 Some Time Constants

The *characteristic time* is a concept that appears in many physical sciences and is representative of timescales that have intrinsic physical meanings for a structure, fluid, or system. For an undamped second-order dynamic system, a characteristic time is the period,

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{k}}.$$

For a first-order system, the characteristic time is the inverse of the absolute value of the characteristic root, and it is the time it takes for the response to decay to e^{-1} of its original value. Due to the nature of the exponential function, it takes exactly another characteristic time for the response to decay to e^{-1} of the previous value. An underdamped second-order system oscillates with its amplitude decaying exponentially. The envelope has a characteristic time of

$$\tau = \frac{1}{\zeta\omega_n} = \frac{2m}{c}.$$

The characteristic time of an exponential function is also called the time constant or relaxation time since the product $\zeta\omega_n$ characterizes the decay rate of the amplitude of the freely vibrating damped system. An important property of the exponential function is that it takes the same amount of time to increase or decrease by the same factor.

Another time parameter, called the *correlation time*, can be defined as a measure of the “memory” of the motion. It answers the question: How long into the future is the response affected by the history of the process? A probabilistic concept known as correlation function is a measure of this, and it will be studied in Chapter 9.

Example 3.1 Some Typical Timescales

Discuss timescales for typical engineering systems.

Solution: Many engineering systems have a damping ratio in the range $\zeta \simeq 0.01$ to 0.20 . For $\omega_n = 1$ rad/s and $\zeta = 0.05$,

$$\zeta\omega_n = 0.05$$

$$\tau = \frac{1}{\zeta\omega_n} = 20.0 \text{ s.}$$

Similarly, if $\zeta = 0.15$, then $\tau = 6.67$ s. What if instead $\omega_n = 3$ rad/s? Then the characteristic time becomes 6.67 s and 2.22 s for $\zeta = 0.05$ and 0.15 , respectively. A plot of τ vs. $\zeta\omega_n$ shows a hyperbolic relation. Since $\zeta\omega_n = c/2m$, the timescale is proportional to the ratio of inertia to damping, which makes physical sense. ■

Example 3.2 Characteristic Time of an Exponential Function

Find the characteristic times for the exponentially decaying function $f(t) = Ae^{-t/\tau}$ where

- (a) the function is halved every 5 seconds, and
- (b) the function is reduced by 20% every 7 seconds.

Solution: (a) The function at $t = 0$ and $t = 5$ s can be written as

$$f(0) = A \quad \text{and} \quad f(5) = Ae^{-5/\tau},$$

and given that, the function is halved every five seconds, or $f(5) = 0.5f(0)$. Thus,

$$Ae^{-5/\tau} = 0.5A$$

$$-\frac{5}{\tau} = \ln 0.5$$

$$\tau = 5/\ln 2 = 7.21 \text{ s.}$$

The half-life can be converted to the characteristic time by dividing the half-life by $\ln 2$.

(b) Since the function is reduced by 20% every 7 seconds, we can write

$$f(7) = Ae^{-7/\tau}$$

or

$$f(7) = 0.8f(0).$$

Solving, as done in part (a), $\tau = -7/\ln(0.8) = 31.37$ s. ■

Example 3.3 Characteristic Time of an Exponential Function

Consider an exponential function with a characteristic time of τ seconds. By what factor is the function reduced

- (a) every τ seconds, (b) every 2τ seconds, (c) every 3τ

seconds, and (d) every 4τ seconds?

Solution: (a) At $t = \tau$, $f(\tau) = Ae^{-1} = 0.368A$. At τ , the function is reduced to 36.8% of the original value (the function is reduced by 63.2% of what it was initially).

(b) At $t = 2\tau$, the function is reduced to 13.5% of the original value.

(c) At $t = 3\tau$, the function is reduced to 4.98% of the original value.

(d) At $t = 4\tau$, the function is reduced to 0.67% of the original value. The “rule of thumb” is that at $t = 4\tau$ or higher the function is fully decayed. ■

Example 3.4 First-Order Approximation of an Overdamped Second-Order System

Can the following second-order overdamped systems be approximated as first-order systems? Justify your answer and write down the approximate first-order system:

- (a) $\ddot{x} + 6\dot{x} + 5x = 0$
- (b) $\ddot{x} + 3\dot{x} + 2x = 0$.

Solution: (a) $\ddot{x} + 6\dot{x} + 5x = 0$ has the characteristic roots -1 and -5 . The response is given by

$$x(t) = C_1e^{-t} + C_2e^{-5t}.$$

The first term is dominant, and the second term disappears five times faster. Therefore, the response can be approximated by $x(t) \simeq Ce^{-t}$, and the equation of motion by $\dot{x} + x = 0$.

(b) The characteristic roots of $\ddot{x} + 3\dot{x} + 2x = 0$ are -1 and -2 . The dominant term decays only twice as slowly as the fast decaying term. This system cannot be modeled as a first-order differential equation. ■

The critically damped response always decays faster than the overdamped response for systems with the same natural frequency ω_n . The critically damped response exponentially decays with the characteristic time, $\tau_{cr} = 1/\omega_n$. The overdamped response is the summation of the two exponential functions with the characteristic times,

$$\tau_{od1} = \frac{1}{\omega_n (\zeta - \sqrt{\zeta^2 - 1})}$$

and

$$\tau_{od2} = \frac{1}{\omega_n (\zeta + \sqrt{\zeta^2 - 1})}.$$

The part of the response that decays more slowly is dominant and the characteristic time for the overdamped response is governed by the slower response, τ_{od1} . It is generally accepted that an exponential function whose characteristic time is more than four times greater than that of the dominant exponential function can be neglected in the response. For example, an overdamped

response, $x(t) = 10e^{-t} + 100e^{-5t}$, can be approximated as $x(t) \simeq 10e^{-t}$.

The term $(\zeta - \sqrt{\zeta^2 - 1})$ is always less than one, and therefore, the characteristic time for the overdamped response is always greater than that of the critically damped response, assuming both have the same natural frequency.

3.4.3 Underdamped Systems

In vibration studies our interest is in the underdamped case, $0 < \zeta < 1$, since it results in oscillatory behavior.¹⁶ The damping in many engineered structures falls in the range $0.01 < \zeta < 0.20$. For the underdamped case, Equation 3.7 can be written as

$$x(t) = A_1 \exp \left(- \left[\zeta - i\sqrt{1 - \zeta^2} \right] \omega_n t \right) + A_2 \exp \left(- \left[\zeta + i\sqrt{1 - \zeta^2} \right] \omega_n t \right).$$

Defining the *frequency of damped vibration*¹⁷

$$\omega_d \equiv \omega_n \sqrt{1 - \zeta^2}, \quad (3.8)$$

then

$$x(t) = A_1 \exp(-\zeta \omega_n t) \exp(i \omega_d t) + A_2 \exp(-\zeta \omega_n t) \exp(-i \omega_d t),$$

or

$$x(t) = (A_1 e^{i \omega_d t} + A_2 e^{-i \omega_d t}) e^{-\zeta \omega_n t}.$$

Introducing Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ and simplifying terms yields a more useful form of the solution,

$$\begin{aligned} x(t) &= (A_1 [\cos \omega_d t + i \sin \omega_d t] \\ &+ A_2 [\cos \omega_d t - i \sin \omega_d t]) \exp(-\zeta \omega_n t) \\ &= [B_1 \cos \omega_d t + B_2 \sin \omega_d t] \exp(-\zeta \omega_n t), \end{aligned}$$

or

$$x(t) = C e^{-\zeta \omega_n t} \cos(\omega_d t - \phi). \quad (3.9)$$

The sum of a sine and a cosine is equivalent to a single cosine with a phase lag.¹⁸

¹⁶In an automobile, which is subjected to extremely broadband inputs (forcing at many different frequencies), damping ratios of approximately 0.20 to 0.25 are typical in “standard” suspensions, $\simeq 0.30$ is typical for “heavy-duty” suspensions, and $\simeq 0.40$ is typical for a “high-performance” suspension. Damping ratios above 0.40 are very uncomfortable for humans and tend to loosen structural joints. (Douglas P. Taylor, President, Taylor Devices, Inc.)

¹⁷It is also called the natural frequency of damped vibration or just the damped natural frequency.

¹⁸Note that A_1 and A_2 are complex conjugates of each other. This makes sense physically because $x(t)$ is a real physical quantity and therefore $(A_1 + A_2)$ and $i(A_1 - A_2)$ must equal the real numbers B_1 and B_2 .

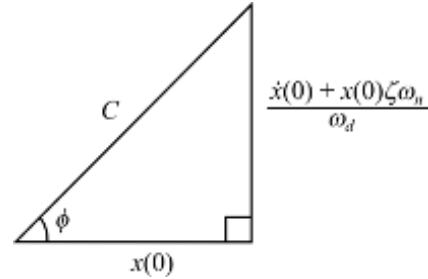


Figure 3.12: Right triangle representing relationships among $x(0)$, $\dot{x}(0)$, C , and ϕ for a given ζ and ω_n .

Equation 3.9 represents oscillatory motion bounded by an exponentially decaying envelope. An example is plotted in Figure 3.14. The initial displacement is equal to the intercept of the curve with the ordinate and the initial velocity is equal to the slope at that point. The factor $C e^{-\zeta \omega_n t}$ gives the decay envelope that has a characteristic time of $1/\zeta \omega_n$. The frequency of vibration is ω_d . The constant C and phase angle ϕ are obtained by satisfying the initial conditions.

From Equation 3.9 and its derivative at $t = 0$,

$$x(0) = C \cos(-\phi) \quad (3.10)$$

$$\dot{x}(0) = -C(-\omega_d \sin \phi + \zeta \omega_n \cos \phi), \quad (3.11)$$

which can be manipulated to give

$$C \cos \phi = x(0) \quad (3.12)$$

$$C \sin \phi = \frac{\dot{x}(0)}{\omega_d} + x(0) \zeta \frac{\omega_n}{\omega_d}. \quad (3.13)$$

Squaring these two equations and adding them gives

$$C = \frac{\sqrt{[\dot{x}(0) + x(0)\zeta\omega_n]^2 + [x(0)\omega_d]^2}}{\omega_d}.$$

An alternate expression for C is obtained after solving Equation 3.10 for C and substituting C into Equation 3.11,

$$C = \frac{x(0)}{\cos \phi}.$$

The phase angle is obtained by dividing Equation 3.13 by Equation 3.12,

$$\phi = \tan^{-1} \left(\frac{\dot{x}(0) + x(0)\zeta\omega_n}{x(0)\omega_d} \right).$$

The relationships governing $x(0)$, $\dot{x}(0)$, C , and ϕ are depicted graphically in Figure 3.12.

Substituting these equations for C , and ϕ , which are explicit functions of the initial displacement and velocity, into Equation 3.9, yields

$$x(t) = \frac{x(0)}{\cos \phi} e^{-\zeta \omega_n t} \cos(\omega_d t - \phi), \quad (3.14)$$

where

$$\begin{aligned}\cos \phi &= \frac{x(0)}{C} \\ &= \frac{x(0)\omega_d}{\sqrt{[\dot{x}(0) + x(0)\zeta\omega_n]^2 + [x(0)\omega_d]^2}}.\end{aligned}\quad (3.15)$$

The phase angle ϕ is a measure of the delay of the response. (We will see that phase is an especially important response parameter for forced structures. It is a measure of the damping-induced lag in the response of the structure to the forcing.)

Substituting Equation 3.15 into Equation 3.14,

$$\begin{aligned}x(t) &= \frac{\sqrt{[\dot{x}(0) + x(0)\zeta\omega_n]^2 + [x(0)\omega_d]^2}}{\omega_d} e^{-\zeta\omega_n t} \\ &\quad \times \cos \left[\omega_d t - \tan^{-1} \left(\frac{\dot{x}(0) + x(0)\zeta\omega_n}{x(0)\omega_d} \right) \right].\end{aligned}\quad (3.16)$$

As an example of the information that can be obtained from Equation 3.14 or 3.16, the time to maximum displacement can be found by differentiating $x(t)$ with respect to t , and setting that expression equal to zero,

$$\frac{dx}{dt} = 0. \quad (3.17)$$

In solving for t there will be (theoretically) an infinite number of solutions. The first positive solution is the time to maximum displacement and it can be substituted into Equation 3.16 to find $x_{\max}$.

Figure 3.13 shows schematics of the free responses for various roots of the characteristic equation. Those in the left-half plane show exponential decay. In the right-half plane there is unstable exponential growth. When the characteristic roots with multiplicity of one are on the imaginary axis, the free response neither grows nor decays. This corresponds to the undamped case that was studied in the previous chapter. If the characteristic roots are on the imaginary axis and are repeated, the free response becomes unstable growing linearly with time.

Underdamped systems have characteristic roots that have negative real parts and non-zero imaginary parts. Overdamped systems have negative and real characteristic roots. Critically damped systems have a negative real characteristic root with multiplicity of two, that is, two identical real roots. Damped systems (with positive damping ratio), whether underdamped, overdamped, or critically damped, have all of their characteristic roots in the left-half plane on the complex plane, shown in Figure 3.13.

Example 3.5 Second-Order Damped Systems

For each of the following mass-spring-damper systems, identify whether it is overdamped, underdamped, or critically damped. Find the natural frequency, the damping

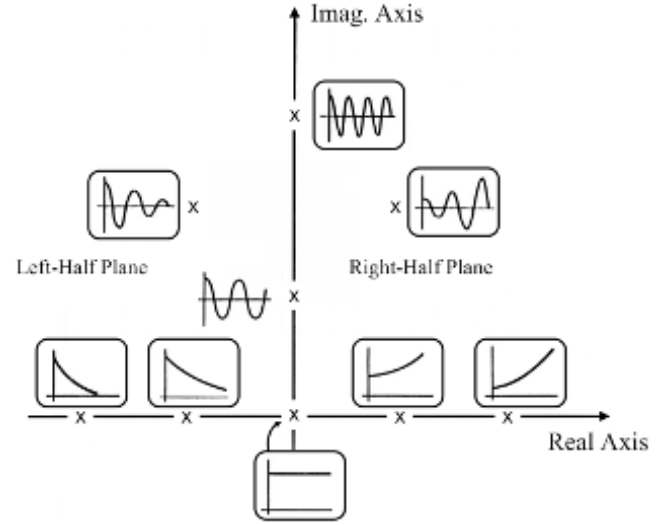


Figure 3.13: Sketches placed on the complex plane that show the free responses corresponding to locations of the roots of the characteristic equation.

ratio, and the characteristic roots. If the system is underdamped, find the damped natural frequency and the characteristic time for the decay envelope:

(a) $\ddot{x} + 4\dot{x} + 5x = 0$

(b) $2\ddot{x} + 5\dot{x} + 3x = 0$

(c) $4\ddot{x} + 4\dot{x} + x = 0$.

Solution: (a) For $\ddot{x} + 4\dot{x} + 6x = 0$, $m = 1$, $c = 4$, and $k = 6$, with appropriate units. The natural frequency and the damping ratio are then

$$\begin{aligned}\omega_n &= \sqrt{\frac{k}{m}} = \sqrt{6} = 2.449 \text{ rad/s} \\ \zeta &= \frac{c}{2\sqrt{mk}} = \frac{4}{2\sqrt{6}} = 0.816.\end{aligned}$$

Since $\zeta < 1$ the system is underdamped. The characteristic equation is given by

$$r^2 + 4r + 5 = 0,$$

which can be determined by inspection of the differential equation. The characteristic roots are $r = -2 \pm i$. The damped natural frequency is the imaginary part of the characteristic root, $\omega_d = 1$ rad/s, and the decay envelope has a characteristic time that is the inverse of the absolute value of the real part, or $\tau = 1/2$ s.

(b) For $2\ddot{x} + 5\dot{x} + 3x = 0$, the natural frequency and the damping ratio are

$$\begin{aligned}\omega_n &= \sqrt{\frac{k}{m}} = \sqrt{\frac{3}{2}} = 1.225 \text{ rad/s} \\ \zeta &= \frac{c}{2\sqrt{mk}} = \frac{5}{2\sqrt{6}} = 1.021.\end{aligned}$$

Since $\zeta > 1$ the system is overdamped. The characteristic equation is given by

$$2r^2 + 5r + 3 = 0,$$

giving the characteristic roots $r = -1$ and -1.5 .

(c) For $4\ddot{x} + 4\dot{x} + x = 0$, the natural frequency and the damping ratio are

$$\omega_n = \sqrt{\frac{k}{m}} = \frac{1}{2} = 0.5 \text{ rad/s}$$

$$\zeta = \frac{c}{2\sqrt{mk}} = 1.$$

Since $\zeta = 1$ the system is critically damped. The characteristic equation is

$$4r^2 + 4r + 1 = 0,$$

giving the repeated characteristic roots $r = -0.5$. ■

Example 3.6 Underdamped Response for Specific Parameters

Given values $C = 1$ cm, $\phi = 1$ rad, $\zeta = 0.1$, and $\omega_n = 1$ rad/s, determine the initial conditions.

Solution: We calculate $\omega_d = 0.995$ rad/s, and the response from Equation 3.9 is given by

$$x(t) = \exp(-0.1t) \cos(0.995t - 1) \text{ cm.} \quad (3.18)$$

Figure 3.14 is the plot of Equation 3.18. The initial displacement is

$$x(0) = \cos(-1) = 0.540 \text{ cm.}$$

The velocity equation is

$$\dot{x}(t) = -0.1 \exp(-0.1t) \cos(0.995t - 1) \\ - 0.995 \exp(-0.1t) \sin(0.995t - 1),$$

with the initial velocity

$$\dot{x}(0) = -0.1 \cos(-1) - 0.995 \sin(-1) = 0.783 \text{ cm/s.}$$

The initial condition for displacement can be read from the graph. ■

Example 3.7 Underdamped Response with Given Initial Conditions

Find the response of a mass-spring-damper system with the equation of motion,

$$\ddot{x} + 2\dot{x} + 5x = 0,$$

with initial conditions, $x(0) = -2$ and $\dot{x}(0) = -3$. Write the response in terms of an exponentially decaying cosine

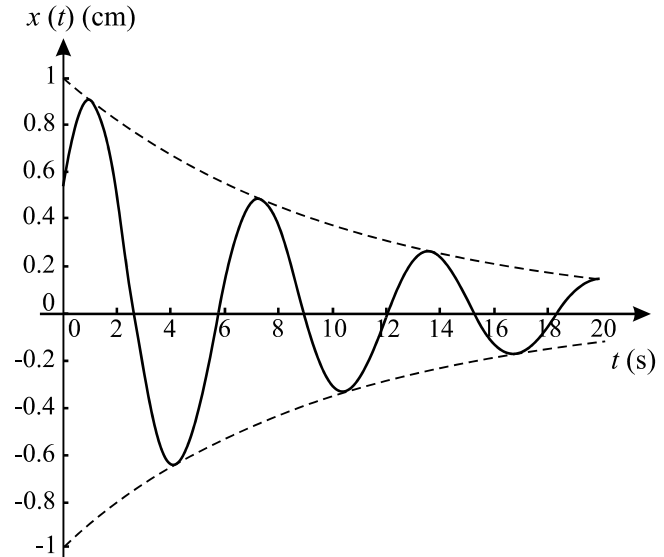


Figure 3.14: Plot of Equation 3.9 for $C = 1$ cm, $\phi = 1$ rad, $\zeta = 0.1$, $\omega_n = 1$ rad/s. The response (solid line) is $x(t) = \exp(-0.1t) \cos(0.995t - 1)$ cm. The dashed lines represent the exponentially decaying boundary or envelope of the response.

function with a phase shift.

Solution: The characteristic equation is

$$r^2 + 2r + 5 = 0,$$

giving the characteristic roots $r = -1 \pm 2i$. The free response has the form

$$x(t) = Ce^{-t} \cos(2t - \phi).$$

The velocity is given by

$$\dot{x}(t) = -Ce^{-t} \cos(2t - \phi) - 2Ce^{-t} \sin(2t - \phi).$$

Applying initial conditions, we find

$$\begin{aligned} -2 &= C \cos(-\phi) \\ -3 &= -C \cos(-\phi) - 2C \sin(-\phi), \end{aligned}$$

which is equivalent to

$$C \cos \phi = 2 \text{ and } C \sin \phi = -5/2.$$

Solving

$$C = \sqrt{10.25} = 3.20 \text{ and } \tan \phi = (-2.5)/(-2),$$

where the phase angle is in the third quadrant where both $\cos \phi$ and $\sin \phi$ are negative. The phase angle is $\phi = -2.24$ rad. The free response is then

$$x(t) = 2.06e^{-t} \cos(2t + 2.24).$$

■

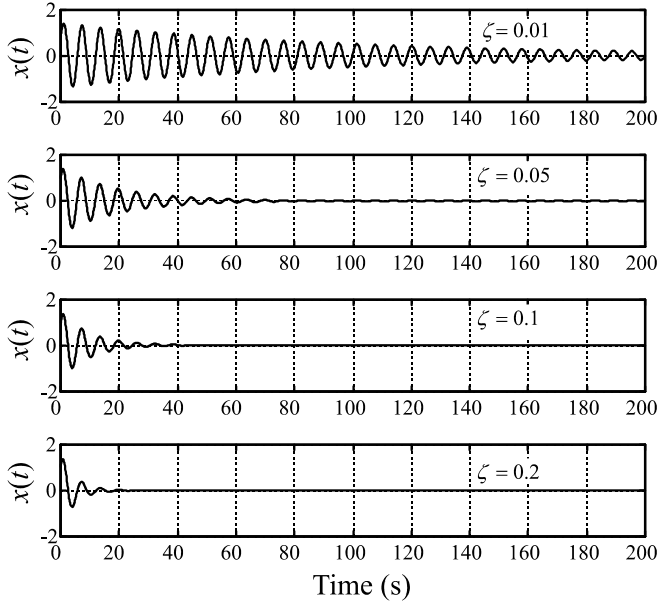


Figure 3.15: Free vibration with damping showing that as ζ increases the oscillations decay faster.

Example 3.8 Numerical Evaluation of Equation 3.14

Investigate the response $x(t)$ of Equation 3.14 to understand the effects of ζ on the response.

Solution: The plots in Figure 3.15 assume $\omega_n = 1$ rad/s, $x(0) = 1$ cm, and $\dot{x}(0) = 1$ cm/s for four different values of ζ : 0.01, 0.05, 0.10, and 0.20. We see that the damping ratio plays a primary role in the character of the response. ■

3.4.4 Logarithmic Decrement

The *logarithmic decrement* method is a common experimental technique used to estimate the damping ratio in underdamped systems.

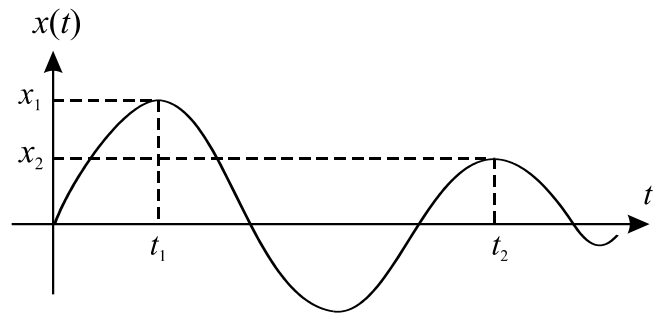


Figure 3.16: A time interval of decaying free vibration. Consecutive oscillation peaks occur at times t_1 and t_2 with the damped period $T_d = t_2 - t_1$.

Consider the measured displacement in a mechanical underdamped system. The first two peaks of displace-

ment are $x_1 = x(t_1)$ and $x_2 = x(t_2)$, as shown in Figure 3.16. The damped period of oscillation $T_d = t_2 - t_1 = 2\pi/\omega_d$. At times t_1 and t_2 , the displacements from Equation 3.9 are

$$\begin{aligned} x(t_1) &= Ce^{-\zeta\omega_n t_1} \cos(\omega_d t_1 - \phi) \\ x(t_2) &= Ce^{-\zeta\omega_n t_2} \cos(\omega_d t_2 - \phi). \end{aligned}$$

The ratio of the two displacements is given by

$$\frac{x(t_2)}{x(t_1)} = e^{-\zeta\omega_n(t_2-t_1)} \frac{\cos(\omega_d t_2 - \phi)}{\cos(\omega_d t_1 - \phi)}. \quad (3.19)$$

Since $T_d = t_2 - t_1$, the relationship of the cosine functions at the two times is

$$\begin{aligned} \cos(\omega_d t_2 - \phi) &= \cos(\omega_d t_1 - \phi + \omega_d T_d) \\ &= \cos(\omega_d t_1 - \phi + 2\pi) \\ &= \cos(\omega_d t_1 - \phi), \end{aligned}$$

and Equation 3.19 becomes

$$\frac{x(t_2)}{x(t_1)} = e^{-\zeta\omega_n T_d}. \quad (3.20)$$

Defining the *logarithmic decrement*, δ , as the natural logarithm of the amplitude ratio $x(t_1)/x(t_2)$, we have

$$\delta \equiv \ln \frac{x(t_1)}{x(t_2)} = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}},$$

or solving for ζ

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}. \quad (3.21)$$

Generalizing Equation 3.20, for peaks n cycles apart, we obtain

$$\begin{aligned} \frac{x(t_1)}{x(t_{n+1})} &= \frac{x(t_1)}{x(t_2)} \frac{x(t_2)}{x(t_3)} \cdots \frac{x(t_j)}{x(t_{n+1})} \\ &= [\exp(\zeta\omega_n T_d)]^n. \end{aligned} \quad (3.22)$$

Taking the natural logarithm of each side and making use of the definition of logarithmic decrement yields

$$\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}},$$

where $x_1 = x(t_1)$ and $x_{n+1} = x(t_{n+1})$.

Example 3.9 Calculating Number of Cycles for a Specific Decay

Derive an expression for the number of cycles n corresponding to a displacement amplitude decay of 70% in a mass-spring-damper underdamped response.

Solution: The logarithmic decrement δ for 70% decay is

$$\delta = \frac{1}{n} \ln \left(\frac{1}{0.30} \right),$$

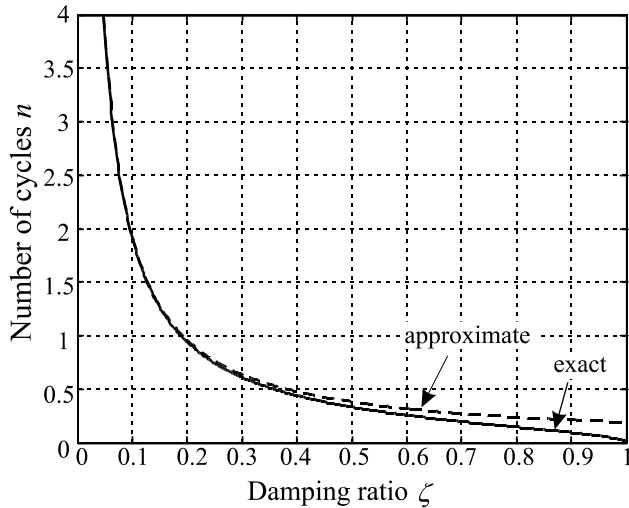


Figure 3.17: Number of cycles n for a 70% reduction in amplitude for small ζ . The approximate curve is given by $n\zeta = 0.192$.

where $\delta = 2\pi\zeta/\sqrt{1-\zeta^2}$. Evaluating the natural logarithm and solving for n yields

$$n = \frac{1.204\sqrt{1-\zeta^2}}{2\pi\zeta}.$$

This relationship between n and ζ is plotted in Figure 3.17. For purposes of comparison, if ζ is small, say $\zeta < 0.20$, then $\sqrt{1-\zeta^2} \simeq 1$ and $n\zeta = 0.192$, giving a hyperbolic relationship that is also graphed in Figure 3.17. We see that this approximate solution, when compared with the exact curve, is accurate for small ζ . From this curve we can answer questions such as: If $0.1 < \zeta < 0.3$, what is the range of values for n ? or, What must the value of ζ be so that $n < 3$? These types of questions are often posed in a preliminary design. ■

Example 3.10 System Specification

Consider the mass-spring-damper system in Figure 3.6(a) subject to initial conditions $x(0) = 1$ cm and $\dot{x}(0) = 0$. The displacement time history is plotted in Figure 3.18. Find the damping ratio and the natural frequency of the system.

Solution: From two peaks, one at $x = 1$ and one at $x = 0.49$, the logarithmic decrement,

$$\begin{aligned}\delta &= \frac{1}{n} \ln \left(\frac{x_1}{x_{n+1}} \right) \\ &= \frac{1}{2} \ln \left(\frac{1}{0.490} \right) = 0.357.\end{aligned}$$

The damping ratio, given by Equation 3.21, is

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.0567.$$

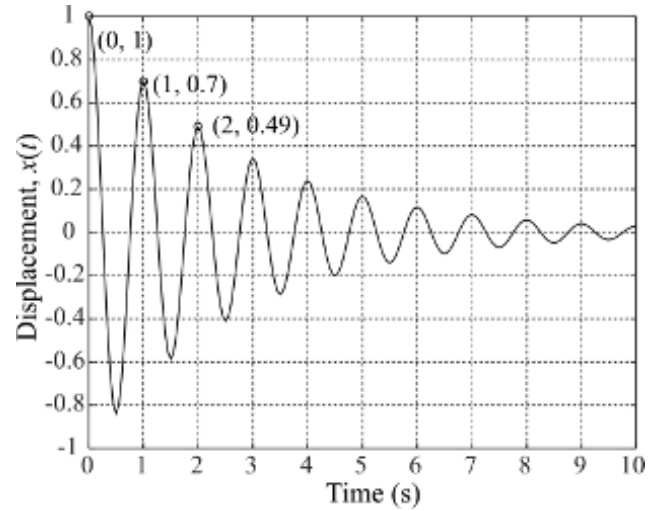


Figure 3.18: Response plot for Example 3.10.

From the graph, the damped period $T_d = 1$ s, giving $\omega_d = 2\pi/T_d = 6.28$ rad/s, and thus the natural frequency

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 6.29 \text{ rad/s}.$$

Although ζ and ω_n can be found using the logarithmic decrement method, unique values of m , c , and k cannot be determined. For each (ζ, ω_n) pair, there is an infinite number of (m, c, k) triplets. ■

LEONHARD EULER

(April 15, 1707 – September 18, 1783)



Contribution: Euler was an exceptionally prolific mathematician who made significant contributions to modern mathematics, science, and engineering. He worked in almost all areas of mathematics – geometry, calculus, trigonometry, algebra, and number theory – as well as physics and astronomy. He was a seminal figure in the history of mathematics.

Euler gave us the notation $f(x)$ for a function (1734), e for the base of natural logs (1727), i for the square root of

-1 (1777), π for pi, $\sum$ for summation (1755), the notation for finite differences Δy and $\Delta^2 y$, and many others.

He made substantial contributions to calculus. He unified Leibnitz's differential calculus and Newton's method of fluxions into mathematical analysis. He introduced beta and gamma functions, and integrating factors for differential equations. He studied continuum mechanics, lunar theory, the three-body problem, elasticity, acoustics, the wave theory of light, hydraulics, and music.

He derived many new mathematical principles in order to solve physically meaningful problems. He laid the foundation of analytical mechanics, in particular the mechanics of rigid bodies. In addition to his contributions to the kinematics and kinetics of rigid bodies, he made analytical contributions in his studies of the vibration of beams and stability of columns (the Euler buckling load).

Specific Contributions: In 1729, Goldbach (a Prussian mathematician) asked Euler if he knew of Fermat's conjecture that the numbers $2^{2^n} + 1$ were always prime if n is a power of 2. Euler verified this for $n = 1, 2, 4, 8$ and 16 and, by 1732, showed that the next case $2^{32} + 1 = 4,294,967,297$ is divisible by 641 and so is not prime.

Euler studied other unproved results of Fermat and in so doing introduced the Euler phi function $\varphi(n)$, the number of integers k with $1 \leq k \leq n$ and k coprime to n . In 1749, he proved another of Fermat's assertions, namely that if a and b are coprime, then $a^2 + b^2$ has no divisor of the form $4n - 1$.

In 1748 Euler made ideas of Johann Bernoulli more precise in defining a function. He stated that mathematical analysis was the study of functions, and based the calculus and the theory of elementary functions. Euler discovered the equation

$$e^{ix} = \cos x + i \sin x,$$

which is known as Euler's formula.

Euler published his full theory of logarithms of complex numbers in 1751. He dealt with logarithms of a variable taking only positive values, although he had discovered the formula $\ln(-1) = \pi$ in 1727.

Analytic functions of a complex variable were investigated by Euler in a number of different contexts, including the study of orthogonal trajectories and cartography. He discovered the Cauchy-Riemann equations in 1777, although d'Alembert had discovered them in 1752 while investigating hydrodynamics.

In 1755 Euler published a book studying the calculus of finite differences. The work makes a thorough investigation of how differentiation behaves under substitutions. In another book (1768-70) Euler made a thorough investigation of integrals that can be expressed in terms of elementary functions. He also studied beta and gamma functions, which he had introduced first in 1729. Legendre called these "Eulerian integrals of the first and second kind," respectively. They were given the names beta function and gamma function by Binet and Gauss, respectively. In addition to investigating double

integrals, Euler considered ordinary and partial differential equations.

Euler made fundamental discoveries in the calculus of variations. His published work in 1740 began the proper study of the calculus of variations.

Problems in mathematical physics had led Euler to a wide study of differential equations. He considered linear equations with constant coefficients, second-order differential equations with variable coefficients, power-series solutions of differential equations, a method of variation of constants, integrating factors, a method of approximating solutions, and many others. When considering vibrating membranes, Euler was led to the Bessel equation, which he solved by introducing Bessel functions.

Euler made substantial contributions to differential geometry, investigating the theory of surfaces and curvature of surfaces. Many unpublished results by Euler in this area were rediscovered by Gauss. Other geometric investigations led him to fundamental ideas in topology such as the Euler characteristic of a polyhedron.

In 1736 Euler published *Mechanica*, which was a major advance in mechanics. The distinguishing feature of Euler's investigations in mechanics as compared to those of his predecessors was his systematic and successful application of analysis. Euler considered the motion of a point mass both in a vacuum and in a resisting medium. He analyzed the motion of a point mass under a central force and also considered the motion of a point mass on a surface.

Mechanica was followed by another important work in rational mechanics, that is, Euler's two volume work on naval science. He applied variational principles to determine optimal ship design and established the principles of hydrostatics. Euler here also began developing the kinematics and dynamics of rigid bodies, introducing in part the differential equations for their motion.

Euler helped develop the Euler-Bernoulli beam equation, used extensively in mechanics of materials. Euler-Bernoulli beam theory is a simplification of the linear theory of elasticity and provides a means of calculating the load-carrying and deflection characteristics of beams. He worked on it circa 1750.

In 1765, Euler published another major work on mechanics in which he decomposed the motion of a solid into a rectilinear motion and a rotational motion. He developed the Euler angles and studied rotational problems which were motivated by the problem of the precession of the equinoxes.

Euler's work on fluid mechanics is also remarkable. He published a number of major articles through the 1750s setting up the main equations including the continuity equation, the Laplace velocity potential equation, and the Euler equations for the motion of an inviscid incompressible fluid.

Euler developed a theory of music in 1739. He tried to make music part of mathematics and deduced in an orderly manner, from correct principles, combinations of tones that sound pleasing. The work was too advanced in its mathemat-

ics for musicians and too musical for mathematicians.

He made advances in the study of modern analytic geometry and trigonometry. He was the first to consider trigonometric terms such as the sine and cosine as functions.

He was extremely prolific, writing 886 books and papers.

Life Sketch: Euler was born in Basel, Switzerland. His father, a Protestant minister who had some mathematical training, taught his son elementary mathematics along with other subjects. Euler's father wanted his son to follow him into the church and sent him to the University of Basel to prepare for the ministry. He entered the University in 1720, at the age of 14, first to obtain a general education before going on to more advanced studies.

In 1723 Euler completed his Master's degree in philosophy having compared and contrasted the philosophical ideas of Descartes and Newton. He began his study of theology in the autumn of 1723, following his father's wishes. Although he was to be a devout Christian all his life, he could not find the enthusiasm for the study of theology, Greek, and Hebrew that he found in mathematics. Euler obtained his father's consent to change to mathematics.

Euler completed his studies at the University of Basel in 1726. His advisor was Johann Bernoulli. By then Euler had already published a short article on isochronous curves in a resisting medium. In 1727 he published another article on reciprocal trajectories and submitted an entry for the 1727 Grand Prize of the Paris Academy on the best arrangement of masts on a ship, in which he won second place.

When Nicolaus (II) Bernoulli died in St. Petersburg in July 1726, Euler was offered his academic post. He accepted but delayed arriving in St. Petersburg until May 1727. He joined the St. Petersburg Academy of Sciences two years after it had been founded by Catherine I, the wife of Peter the Great. Euler was appointed to the mathematical-physical division of the Academy. In St. Petersburg he lived with Daniel Bernoulli.

Euler served as a medical lieutenant in the Russian navy from 1727-1730. He became a professor of physics at the Academy in 1730. This allowed Euler to be a full member of the Academy, and he was able to give up his Russian navy post.

When Daniel Bernoulli left St. Petersburg to return to Basel in 1733, Euler was appointed to his senior chair in mathematics at the Academy. The financial improvement which came from this appointment allowed Euler to marry in January 1734. His wife, like Euler, was from a Swiss family. They had 13 children, although only five survived infancy.

After 1730 he carried out state projects dealing with cartography, science education, magnetism, fire engines, machines, and ship building. The core of his research program was now set in place: number theory, analysis, differential equations, the calculus of variations, and rational mechanics. He viewed these fields as intimately interconnected.

The publication of many articles and his book *Mechanica* (1736-37), which extensively presented Newtonian dynamics

in the form of mathematical analysis for the first time, gave Euler significant exposure.

Euler's health problems began in 1735 when he had a severe fever and almost lost his life. He kept this news from his parents and members of the Bernoulli family in Basel until he recovered. In his autobiographical writings Euler says that his eyesight problems began in 1738 with overstrain due to his cartographic work and that by 1740 he had lost sight in one eye.

By 1740 Euler had earned a very high reputation, having won the Grand Prize of the Paris Academy in 1738 and 1740. Euler's fame led to an offer to go to Berlin. Although he preferred to remain in St. Petersburg, political turmoil in Russia made the position of foreigners particularly difficult and contributed to Euler changing his mind. Euler, at the invitation of Frederick the Great, went to Berlin where an Academy of Science was planned to replace the Society of Sciences. He left St. Petersburg in June 1741.

The Berlin Academy was founded in 1744 with Euler as director of mathematics. Euler undertook an enormous amount of work for the Academy: he supervised the observatory and the botanical gardens, selected the personnel, oversaw various financial matters, and managed the publication of various calendars and geographical maps (their sale was a source of income for the Academy). The king also charged Euler with practical problems, such as the project of correcting the level of the Finow Canal. He supervised the work on pumps and pipes of the hydraulic system at Sans Souci, the royal summer residence. He served as an advisor to the government on state lotteries, insurance, annuities and pensions, and artillery.

During the twenty-five years he spent in Berlin, Euler wrote approximately 380 articles. He wrote books on the calculus of variations and differential calculus, on planetary orbits, on artillery and ballistics, on shipbuilding and navigation, on the motion of the moon, and a popular scientific publication *Letters to a Princess of Germany*. (This book was the result of Euler being asked to tutor the Princess of Anhalt-Dessau, Frederick's niece. Euler wrote her over 200 letters, which became this publication. It contains Euler's views on subjects pertaining to physics and mathematics, as well as offers valuable insights into his personality and religious beliefs.)

In 1759 Euler assumed the leadership of the Berlin Academy, although not the title of President. King Frederick was in overall charge and Euler was not on as good terms as he had been earlier. Euler, who had disagreed with d'Alembert on scientific matters, was disturbed when Frederick offered d'Alembert the presidency of the Academy in 1763. D'Alembert refused to move to Berlin, but Frederick's continued interference with the running of the Academy made Euler decide to leave.

In 1766 Euler returned to St. Petersburg, greatly angering Frederick. His return to Russia was marred by tragedies. Soon after his return, Euler became almost entirely blind after an illness. A fire in St. Petersburg in 1771 cost him his home, and almost his life. In 1773, he lost his wife of 40 years. Three

years after his wife's death, Euler married her half sister. This marriage lasted until his death.

A cataract operation shortly after the fire in 1771 restored his sight for a few days but Euler became almost totally blind. Due to his remarkable memory, he was able to continue his work on optics, algebra, and lunar motion. Amazingly, after his return to St. Petersburg (when Euler was 59), he produced almost half his total works despite being blind.

Euler achieved this remarkable level of output with help from two sons and by members of the Academy. The scientists assisting Euler were not mere secretaries. He discussed the general scheme of the works with them, and they developed his ideas, calculating tables, and compiled examples.

After suffering a brain hemorrhage, Euler died at age 76 in St. Petersburg, Russia. After his death in 1783 the St. Petersburg Academy continued to publish Euler's unpublished work for nearly 50 more years.

Noteworthy: Euler insisted that knowledge is founded in part on the basis of precise quantitative laws. Euler claimed that he made some of his greatest mathematical discoveries while holding a baby in his arms with children playing around his feet.

Euler was a simple religious man and a hard worker. He was very conventional in his beliefs and tastes. He was a devout Christian who believed in the Bible to be inspired.

Euler's eyesight worsened throughout his career. Three years after suffering a near-fatal fever in 1735 he became nearly blind in his right eye, but Euler blamed his condition on the painstaking cartography work he performed for the St. Petersburg Academy. Euler's sight in that eye worsened throughout his stay in Germany, so much so that Frederick referred to him as "Cyclops."

Euler later suffered a cataract in his good left eye, rendering him almost totally blind a few weeks after its discovery in 1766. Even so, his condition appeared to have little effect on his productivity, as he compensated for it with his mental calculation skills and photographic memory. With the aid of his scribes, Euler's productivity in many areas of study actually increased. He produced, on average, one mathematical paper every week in the year 1775.

Euler's identity, $e^{i\pi} + 1 = 0$, was called "the most remarkable formula in mathematics" by Richard Feynman for its use of addition, multiplication, exponentiation, and equality, and of the important constants 0, 1, e , i , and π . By 1772 Euler had proved that $2^{31} - 1 = 2,147,483,647$ is a Mersenne prime. It may have remained the largest known prime until 1867.

Leonhard Euler's father, Paul Euler, had studied theology at the University of Basel and had attended Jacob Bernoulli's lectures there. Paul Euler and Johann Bernoulli had both lived in Jacob Bernoulli's house while undergraduates at Basel.

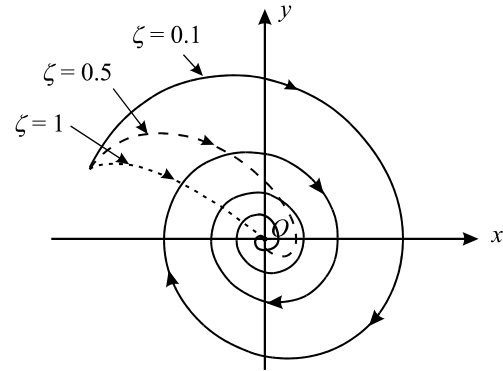


Figure 3.19: Phase paths for the damped oscillator with $x(0) = -10$ cm, $v(0) = 4$ cm/s, $\omega_n = 1$ rad/s, and three values of ζ .

3.4.5 Phase Plane

The second-order differential equation for the damped oscillator can be converted into two first-order differential equations,

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega_n^2 x - 2\zeta\omega_n \dot{x} \\ &= -\omega_n^2 x - 2\zeta\omega_n y,\end{aligned}$$

which can be used to find the phase plane. The equilibrium position is at the origin, where $(x, y) = (0, 0)$. The phase paths are given by

$$\frac{dy}{dx} = \frac{-\omega_n^2 x - 2\zeta\omega_n y}{y}.$$

Example paths are shown in Figure 3.19. Note that for the critical damping case, $\zeta = 1$, the path goes to the equilibrium position $(x, y) = (0, 0)$ without any curving around O . This implies that there is no oscillation, only a decay directly to the equilibrium position. For the underdamped cases $\zeta = 0.1$ and 0.5 , the path encircles O , signifying decaying oscillations. The less damping the system possesses, the more spiraling about O .

3.5 Free Vibration with Coulomb Damping

For a body on a flat surface with Coulomb friction, as shown in Figure 3.6(b), the normal force equals the weight mg of the mass. The equation of motion of a mass-spring system with Coulomb friction can be written as two piecewise linear equations,

$$m\ddot{x} + kx = -\mu_k N \quad \text{for } \dot{x} > 0 \quad (3.23)$$

$$m\ddot{x} + kx = \mu_k N \quad \text{for } \dot{x} < 0, \quad (3.24)$$

which can be rewritten more compactly as

$$m\ddot{x} + \mu_k mg \operatorname{sgn}(\dot{x}) + kx = 0,$$

where $\operatorname{sgn}(\dot{x})$ in the damping term is also known as the *signum*¹⁹ function,

$$\operatorname{sgn}(\dot{x}) = \frac{\dot{x}}{|\dot{x}|} = \begin{cases} -1 & \text{if } \dot{x} < 0 \\ 0 & \text{if } \dot{x} = 0 \\ 1 & \text{if } \dot{x} > 0. \end{cases}$$

Dividing by m , the equation of motion can be written as

$$\ddot{x} + \mu_k g \operatorname{sgn}(\dot{x}) + \omega_n^2 x = 0.$$

The magnitude of the “damping” term is constant except for the change in sign with each half oscillation cycle. If we consider it to be “driving” the system, we can place it on the right-hand side of the equation. Then, it must be solved separately for each half cycle of motion depending on the sign of the velocity, and needs to be solved for the free and forced vibration.

From Chapter 2, the general undamped free vibration response is given by

$$x_f(t) = A \cos \omega_n t + B \sin \omega_n t,$$

where $x(0)$ is the initial displacement and $\dot{x}(0)$ is the initial velocity and coefficients A and B depend on the initial conditions. This part of the solution is the same for both Equations 3.23 and 3.24. The difference comes due to the different directions of the friction force.

In order to simplify the algebra, assume the initial velocity $\dot{x}(0) = 0$. Essentially, this problem is one where the mass is displaced distance $x(0) = x_0$ in a positive direction (that is, to the right). The mass is then let go with zero initial velocity, and is pulled back (to the left) by the stretched spring assuming that the spring force is sufficient to overcome the friction force.

For this first part of the vibration, the velocity is in the negative direction (to the left) or $\dot{x} < 0$, and Equation 3.24 governs the forced response. Assume the steady-state response $x_s = C$, where a constant C has been chosen since the forcing is a constant. Substituting the assumed solution into the equation of motion yields

$$\omega_n^2 C = \mu_k g,$$

or $C = \mu_k g / \omega_n^2$. The complete solution is then

$$x(t) = A \cos \omega_n t + B \sin \omega_n t + \frac{\mu_k g}{\omega_n^2}.$$

Coefficients A and B can be found by satisfying the initial conditions. At $t = 0$,

$$\begin{aligned} x(0) &= x_0 \\ &= A + \frac{\mu_k g}{\omega_n^2}, \end{aligned}$$

or $A = x_0 - \mu_k g / \omega_n^2$. In order to satisfy the zero initial velocity condition, B must equal zero. Thus,

$$x(t) = \left(x_0 - \frac{\mu_k g}{\omega_n^2} \right) \cos \omega_n t + \frac{\mu_k g}{\omega_n^2}, \quad (3.25)$$

which represents harmonic oscillation superposed on the average response $\mu_k g / \omega_n^2$. Equation 3.25 is valid for $0 \leq t \leq t_1$, where t_1 is the time at which the mass velocity is zero at the transition between its left and right movements. The value of t_1 can be found by differentiating Equation 3.25 with respect to time and solving for the first value where the velocity equals zero, $\dot{x}(t_1) = 0$,

$$\dot{x}(t) = -\omega_n \left(x_0 - \frac{\mu_k g}{\omega_n^2} \right) \sin \omega_n t.$$

Then,

$$\dot{x}(t_1) = -\omega_n \left(x_0 - \frac{\mu_k g}{\omega_n^2} \right) \sin \omega_n t_1 = 0,$$

which is satisfied when $\omega_n t_1 = \pi$ rad, resulting in the displacement,

$$\begin{aligned} x(t_1) &= \left(x_0 - \frac{\mu_k g}{\omega_n^2} \right) \cos \pi + \frac{\mu_k g}{\omega_n^2} \\ &= - \left(x_0 - \frac{\mu_k g}{\omega_n^2} \right) + \frac{\mu_k g}{\omega_n^2} \\ &= - \left(x_0 - 2 \frac{\mu_k g}{\omega_n^2} \right). \end{aligned}$$

This displacement is the initial displacement for the next part of the oscillation when the mass moves to the right. The initial velocity is zero. Equation 3.23 governs the motion during this phase. The response is given by

$$x(t) = A \cos \omega_n t + B \sin \omega_n t - \frac{\mu_k g}{\omega_n^2}. \quad (3.26)$$

The constants A and B can be found using Equation 3.26 with $t_1 = \pi / \omega_n$, giving

$$\begin{aligned} - \left(x_0 - 2 \frac{\mu_k g}{\omega_n^2} \right) &= -A - \frac{\mu_k g}{\omega_n^2} \\ 0 &= -B \omega_n. \end{aligned}$$

Thus,

$$x(t) = \left(x_0 - 3 \frac{\mu_k g}{\omega_n^2} \right) \cos \omega_n t - \frac{\mu_k g}{\omega_n^2}, \quad (3.27)$$

for $t_1 \leq t \leq t_2$, where t_2 is the time the mass stops at the right. Here, the amplitude of the harmonic component of the response is smaller by $2\mu_k g / \omega_n^2$, and the constant component has the opposite sign. Equation 3.27 is valid until time t_2 , which can be found as per the procedure

¹⁹In order not to confuse sine and sign, signum is used as shown here. Signum is Latin for sign.

above for t_1 to be $t_2 = 2\pi/\omega_n$, at which time the displacement is

$$x(t_2) = \left(x_0 - 4\frac{\mu_k g}{\omega_n^2} \right).$$

The procedure can be repeated over each subsequent half cycle of oscillation. From the responses for the first two half cycles the Coulomb damping response is bounded with envelopes that decay linearly with time. The envelopes of the oscillations are straight lines, the dashed lines in Figure 3.20, with slopes given by

$$\text{slope} = \pm \frac{4\mu_k g / \omega_n^2}{t_2} = \pm \frac{4\mu_k g / \omega_n^2}{2\pi / \omega_n} = \pm \frac{2\mu_k g}{\pi \omega_n}.$$

This result contrasts with that of the linear viscous damping oscillation that has an exponentially decaying envelope.

The motion stops abruptly when the displacement at the end of the half cycle is not sufficiently large for the restoring force in the spring to overcome the static friction force. Since the displacement magnitude is reduced by $2\mu_k g / \omega_n^2$ with each half cycle, the motion stops at the end of the half cycle for which the amplitude of the harmonic component is smaller than $2\mu_k g / \omega_n^2$. The n^{th} peak amplitude, including the negative peaks, is

$$|x_n| = x_0 - n \frac{2\mu_k g}{\omega_n^2},$$

and is shown in Figure 3.20. The mass stops at the n^{th} half cycle if the spring force is smaller than the maximum static friction, that is,

$$\begin{aligned} k|x_n| &\leq \mu_s mg \\ x_0 - n \frac{2\mu_k g}{\omega_n^2} &\leq \frac{\mu_s g}{\omega_n^2}. \end{aligned}$$

Since the mass does not stop once the mass is already in motion, we choose the smallest integer n rounded up from the solution to the previous inequality. For example, if we find that $n \geq 4.2$, then the mass stops when $n = 5$. The time at which it comes to stop is $n\pi/\omega_n$. The final position is

$$x_{\text{final}} = x_0 - [n] \frac{2\mu_k g}{\omega_n^2},$$

where $[n]$ is defined as the smallest integer not smaller than n .

If we combine the solutions for each half cycle, we obtain the response shown in Figure 3.20.

Example 3.11 Coulomb Damping

Consider the system in Figure 3.6(b) with the following parameter values: $m = 4$ kg, $k = 1000$ N/m, $\mu_k = 0.2$,

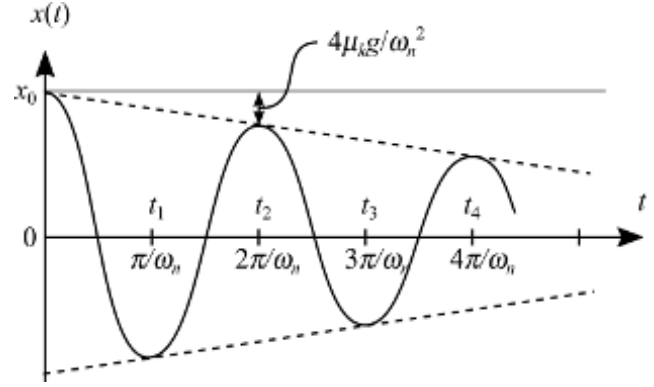


Figure 3.20: Free response of a system with Coulomb damping. The decay of the peaks is linear.

and $\mu_s = 0.3$. Calculate the decay per cycle and the number of half cycles until the mass stops for the initial conditions, $x(0) = 0.2$ m and $\dot{x}(0) = 0$.

Solution: The decay per half cycle is $2\mu_k g / \omega_n^2$,

$$\begin{aligned} \frac{2\mu_k g}{\omega_n^2} &= \frac{2\mu_k g}{k/m} = \frac{2(0.2 \cdot 9.81 \text{ m/s}^2)}{(1000 \text{ N/m}) / 4 \text{ kg}} \\ &= 0.0157 \text{ m}. \end{aligned}$$

The number of half cycles, n , before the mass comes to rest is obtained by

$$\begin{aligned} x_0 - n \frac{2\mu_k g}{\omega_n^2} &\leq \frac{\mu_s g}{\omega_n^2} \\ 0.2 - n \cdot 0.0157 &\leq \frac{0.3(9.81)}{1000/4} \\ n &\geq 11.99 \text{ for mass to stop.} \end{aligned}$$

The mass stops when $n = 12$. When $n = 11$, the spring force is still greater than the maximum static friction. The mass comes to stop at the 12th half cycle or when $t = 12\pi/\omega_n = 2.38$ s. The final position is

$$\begin{aligned} x_{\text{final}} &= x_0 - [n] \frac{2\mu_k g}{\omega_n^2} \\ &= 0.2 - 12(0.0157) = 0.0116 \text{ m}. \end{aligned}$$

■

Figure 3.21 shows the response with the final position of the mass. When the peak position is less than $\mu_s mg/k$ or 0.0118 m, the mass stops.

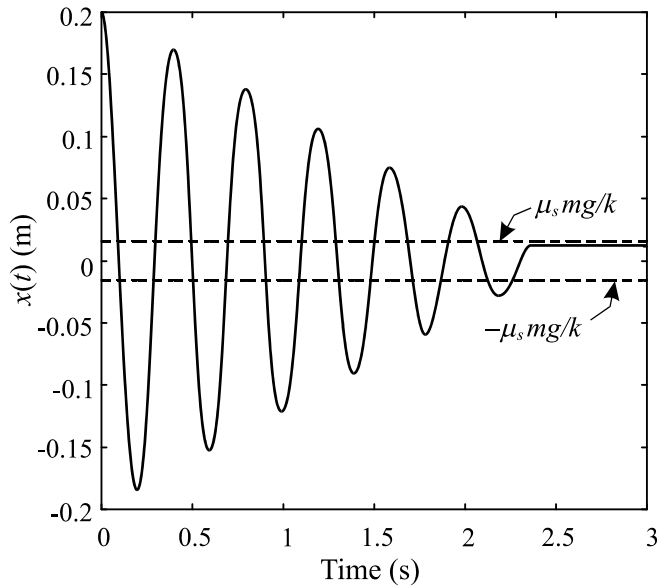


Figure 3.21: The response plot for Example 3.11.



VIBRATION AND POWERED HAND TOOLS

*Vibration Exposure and Usability of Hand Tools.*²⁰ In powered hand tools, vibration can be conducted through the grasping surfaces to the user. The vibration characteristics of these tools depend strongly on the internal drive mechanisms. While the choice of mechanisms can be important to design integrity, it can be equally important to consider the usability of the tool. Research has shown that there is a relationship between the type and duration of vibration exposure and the risk of Hand-Arm Vibration Syndrome (HAVS). Ergonomic factors of the tool, such as its weight, balance, size, and location of grasping surfaces, and body position during use, can also have an effect.

A tool's usability depends on both the magnitude of the vibration exposure and its duration of use for a task. Factors including motor speed, stroke length, and reciprocation frequency help define the magnitude of the vibration. Similarly, these factors have an influence on the mechanism efficiency, which relates to the time it takes to perform an operation, or "trigger on-time." Coupled with an understanding of the number of operations a user may perform in a work day, magnitude of vibration, and trigger on-time can be used to calculate the user's total vibration exposure. This can lead to insight into estimating a user's risk of contracting HVAS.

Consider two powered hand tools that are used to perform the same operation and have the same ergonomics. The product with the higher vibration magnitude, but greater efficiency meaning a shorter trigger on-time, may actually deliver a lower total exposure than a product with a lower magnitude and longer trigger on-time. These are complicated comparisons to run and include extensive vibration testing.

²⁰The authors are grateful to Mark Hickok, Milwaukee Electric Tool Corp., for these comments and insights.

Consultation with an expert may help ensure usability, and ultimately, user acceptance of a tool's design.

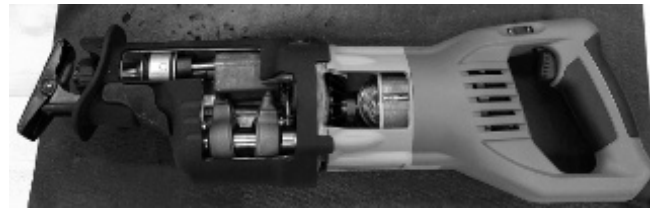


Figure 3.22: A Milwaukee Tool Sawzall with cutouts to see the wobble plate mechanism for converting rotational motion from the motor to translational motion of the blade. One wobble plate moves the shaft for the blade, and one moves the counterbalance in the opposite direction. (Courtesy of Mark Hickok, Milwaukee Electric Tool Corp.; photograph by M. Nagurka)



Figure 3.23: Internal mechanism of a Milwaukee Tool Sawzall that uses a slider-crank mechanism. The counterbalance on the crank is visible. (Courtesy of Mark Hickok, Milwaukee Electric Tool Corp.; photograph by M. Nagurka)

Reciprocating Saws. Power reciprocating saws are commonly used in construction and demolition work. These saws use a reciprocating mechanism driven by a motor to move a blade back and forth. The mechanism converts continuous motion, such as constant speed of a motor shaft, into intermittent motion, driving the blade in alternating directions along rectilinear or orbital paths. Essentially, the blade oscillates back and forth and in so doing the teeth of the blade can cut wood, metal, plastic, drywall, plaster, etc.

The reciprocating action that converts rotational to translational motion may be produced several ways. Examples include a wobble plate or swashplate drive, a slider-crank mechanism, a Scotch yoke drive, or a cam.

A wobble plate or swashplate is essentially a disk attached to a straight shaft at an oblique angle, which causes its edge to appear to oscillate along the shaft's axis (as observed from a non-rotating point of view away from the shaft). Translational motion is achieved by means of a follower that presses against one of its two surfaces near its circumference. The device has many similarities to a cam.

In a reciprocating saw using a wobble plate or swashplate drive the principal vibration is in line with the blade and there is little rotational out of balance. Slider-crank mechanisms, Scotch yoke drives, and cam designs often include

counterbalance weights to reduce vibration in the plane of the rotating element. Even with counterbalance weights, the tools exhibit vibration that may make the saw fatiguing to use for extended time and challenging to control.

Figure 3.22 shows Milwaukee Tool Sawzall²¹ with cutouts to make visible the wobble plates that create the reciprocating action of the blade. (The wobble plates are shown at the lower central part of the photograph.) In this design the wobble plates are mounted on an offset or bent shaft. One wobble plate pushes against the shaft for the blade, and one wobble plate moves the counterbalance in the opposite direction to minimize vibration.

Figure 3.23 shows the internal mechanism of a Sawzall reciprocating saw that operates using a slider-crank mechanism. At the left is the slider that connects to a blade (not shown at the far left) and a connecting rod (toward the center). The connecting rod is connected to the crank (at the center) with a counterbalance to reduce vibration. At the right is the armature and commutator of the motor. The tool can deliver 3000 strokes per minute for high performance cutting of a variety of materials and has a 19 mm ($\frac{3}{4}$ in.) stroke length.

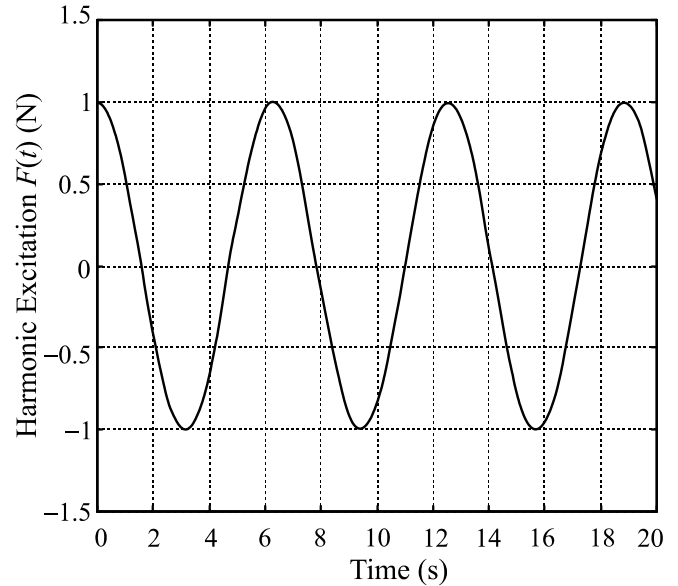


Figure 3.24: Harmonic excitation; $F(t) = A \cos \omega t$, $\omega = 1$ rad/s, $A = 1$ N.

3.6 Forced Vibration with Viscous Damping

Forces on structures are generally very complicated functions of time and space. It is necessary to idealize and simplify them so that their response to specific loads can be studied and understood. For linear systems, it is possible to study a few loading types and use these cases to build more general and challenging loading scenarios. Here, we consider the following excitation cases: (a) harmonic (Figure 3.24), (b) periodic but not harmonic (Figure 3.25), (c) step (Figure 3.26), (d) impulse (Figure 3.27), and (e) arbitrary excitation (Figure 3.28).

The logic in this sequence is to build progressively more sophisticated mathematical models representing the excitation. The next more realistic loading model omitted in the above sequence is a *random* load. This important case, whose solution requires probability theory, is introduced in Chapter 9.

For all the loading cases, the equation of motion is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{1}{m}F(t), \quad (3.28)$$

where $F(t)$ is the forcing function. The undamped case, $\ddot{x} + \omega_n^2x = F(t)/m$, is a special case obtained by setting $\zeta = 0$.

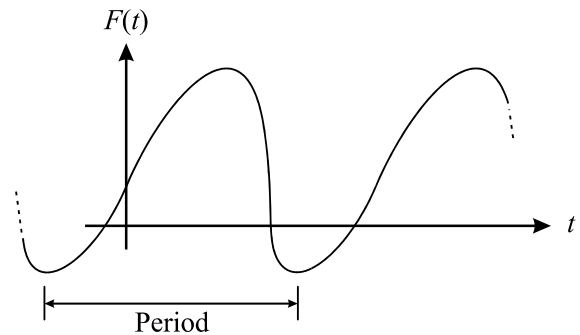


Figure 3.25: Periodic but not harmonic excitation.

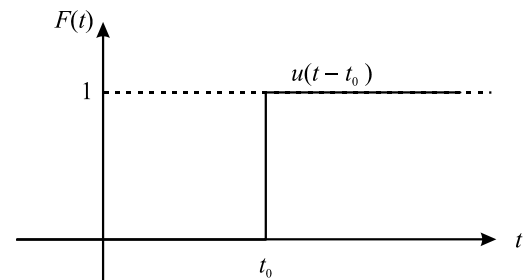


Figure 3.26: Unit step loading: the input amplitude changes from 0 to 1 at $t = t_0$.

²¹Sawzall is a trademark of the Milwaukee Electric Tool Corp., Brookfield, WI.

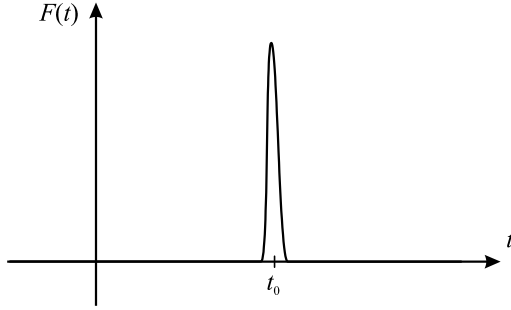


Figure 3.27: Impulse loading: the loading is “large” and acts over a “short” period of time. The unit impulse has an area under the impulse equal to one.

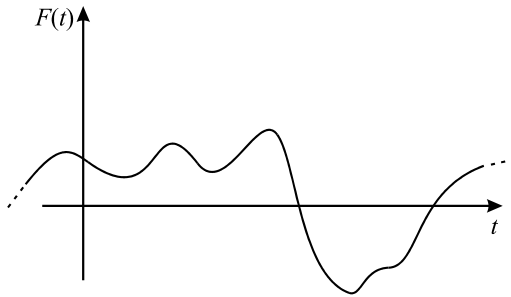


Figure 3.28: Arbitrary loading: a complicated function describes the variations in loading amplitude.

3.7 Forced Harmonic Vibration

The equation of motion is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{1}{m}A\cos\omega t, \quad (3.29)$$

where $A\cos\omega t$ is a harmonic load²² with amplitude A and frequency ω . It is assumed that both the amplitude and frequency of the load are constant.

The primary interest is in finding the *amplitude* and the *phase lag* of the response. It will be shown that the introduction of damping causes the response to lag the input – that is, there is a delay between the input harmonic load and the output harmonic response.

3.7.1 Response to Harmonic Excitation

Assume that the steady-state solution to Equation 3.29 has the form²³

$$x_s(t) = B_1 \cos\omega t + B_2 \sin\omega t. \quad (3.30)$$

²²Equivalently, the load could have been represented by $F(t) = A\sin\omega t$.

²³Both the cosine and the sine are needed in the solution since the governing equation has first and second derivatives. Recall that for the undamped response, only sine or cosine is needed as the governing equation does not contain a first derivative term.

Constants B_1 and B_2 are evaluated by requiring that the assumed solution $x_s(t)$ satisfies Equation 3.29, that is,

$$\begin{aligned} &(-\omega^2 B_1 + 2\zeta\omega_n\omega B_2 + \omega_n^2 B_1) \cos\omega t \\ &+ (-\omega^2 B_2 - 2\zeta\omega_n\omega B_1 + \omega_n^2 B_2) \sin\omega t = \frac{1}{m}A \cos\omega t. \end{aligned}$$

This equation can be satisfied for all values of t only if the expressions in the parentheses vanish identically,

$$\begin{aligned} -\omega^2 B_1 + 2\zeta\omega_n\omega B_2 + \omega_n^2 B_1 &= \frac{1}{m}A \\ -\omega^2 B_2 - 2\zeta\omega_n\omega B_1 + \omega_n^2 B_2 &= 0, \end{aligned}$$

giving two equations in two unknowns from which

$$\begin{aligned} B_1 &= \frac{A(\omega_n^2 - \omega^2)/m}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \\ B_2 &= \frac{A(2\zeta\omega_n\omega)/m}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}. \end{aligned}$$

Substituting into Equation 3.30 gives

$$\begin{aligned} x_s(t) &= \frac{A(\omega_n^2 - \omega^2)/m}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \cos\omega t \\ &+ \frac{A(2\zeta\omega_n\omega)/m}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \sin\omega t. \end{aligned} \quad (3.31)$$

Equation 3.31 is in a form that makes the response difficult to visualize. To obtain a more useful form, the response can be written as

$$x_s(t) = D \cos(\omega t - \theta), \quad (3.32)$$

where, as in Section 2.3, $D = \sqrt{B_1^2 + B_2^2}$ and $\theta = \tan^{-1}(B_2/B_1)$, or,

$$\begin{aligned} D &= \frac{A/m}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \\ &= \frac{A/k}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}, \end{aligned} \quad (3.33)$$

$$\theta = \tan^{-1} \left[\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \right], \quad 0 \leq \theta \leq \pi, \quad (3.34)$$

with θ in the first or second quadrant. Note that the phase lag is independent of the loading amplitude. If $\zeta = 0$, there is no lag and the system responds instantaneously. Damping causes a delay in the system's response to loading.

A nondimensional *magnification factor* β , also called *magnitude ratio*, can be defined as

$$\beta = \frac{1}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}, \quad (3.35)$$

where β represents the frequency-dependent effect in Equation 3.33. Equation 3.32 then can be written as

$$x_s(t) = \frac{A}{k} \beta \cos(\omega t - \theta),$$

where

$$D = \frac{A}{k}\beta,$$

and where A/k is the static deflection.

We can plot the magnification factor β and the phase lag θ as functions of the frequency ratio ω/ω_n for various values of damping ratio ζ . These plots are given in Figures 3.29 and 3.30, respectively. (Such plots, drawn in logarithmic scales, are called Bode plots, named after control engineer Hendrik Wade Bode. Bode plots consist of two plots, one for magnitude and one for phase. Bode plots will be discussed again in Chapter 10.)

From the β curves in Figure 3.29, an increase in the damping factor results in a decrease in the response amplitude for all ω/ω_n . For $\zeta > 0$, the maximum amplitude occurs at a frequency ratio which is slightly less than 1, as the following example shows.

The phase lag as a function of frequency ratio is shown in Figure 3.30. For the case with no damping, for a load with a frequency below resonance, the phase $\theta = 0$, and with a frequency above resonance, the phase $\theta = \pi$ rad. A discontinuity exists at the resonant frequency.

With damping, the sharp transition observed at resonance for the undamped case is softened. For all cases regardless of damping value, the phase is $\pi/2$ rad or 90° at resonance. This property becomes useful in problems of system identification and testing. For example, if a vibration test is performed on a machine with phase data plotted as a function of frequency for a broad band of frequencies, then where this curve passes through the phase value of $\pi/2$ rad is a resonant frequency, indicating that this is one of the natural frequencies. In general, multi degree-of-freedom structures have multiple natural frequencies, one for each degree of freedom.

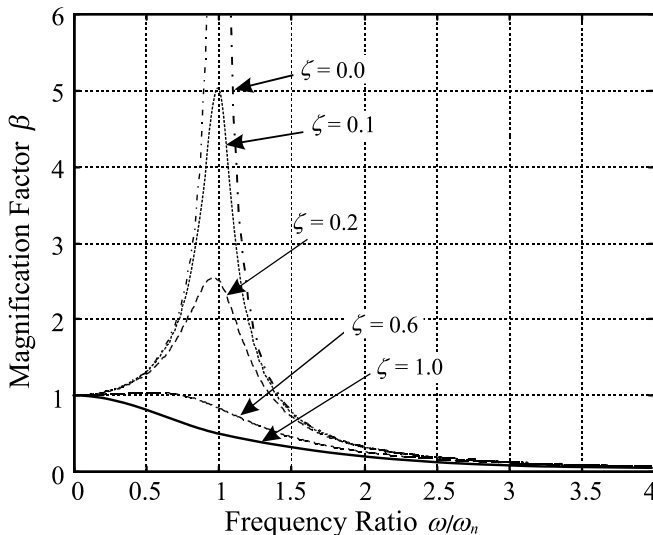


Figure 3.29: Magnification factor β as a function of ω/ω_n for various values of ζ . Driving frequencies ω close to ω_n result in large β for low values of ζ .

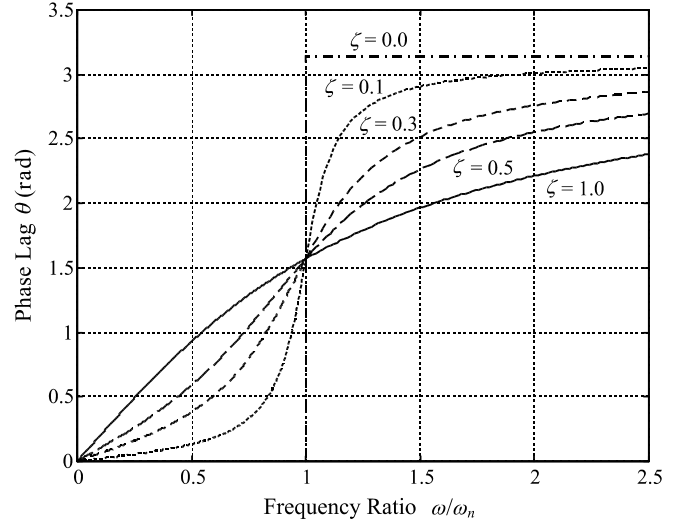


Figure 3.30: Phase lag θ as a function of ω/ω_n for various values of ζ .

Example 3.12 Frequency Ratio for Peak Amplitude

Find the frequency ratio corresponding to the peak amplitude.

Solution: For peak amplification β is a maximum. Taking the derivative of β with respect to the square of the frequency ratio $r = \omega/\omega_n$,

$$r^2 = \frac{\omega^2}{\omega_n^2},$$

and setting the resulting equation to zero,

$$\frac{d\beta}{dr} = 0,$$

leads to the equation for the value of r for which β is a maximum. We denote this value as $\hat{r}$, and find it to be

$$\hat{r} = 1 - 2\zeta^2,$$

or

$$\frac{\omega_r}{\omega_n} = \sqrt{1 - 2\zeta^2}, \quad (3.36)$$

where ω_r is called the resonant frequency. (For $0.01 \leq \zeta \leq 0.30$, the range in frequency ratios at which the peak magnification occurs is $0.90554 \leq (\omega_r/\omega_n) \leq 0.99990$.) Thus, the peak occurs when ω is slightly less than ω_n , not equal to it unless $\zeta = 0$.²⁴ In practice, it is generally assumed that

²⁴Equation 3.36 is similar in form but different from

$$\frac{\omega_d}{\omega_n} = \sqrt{1 - \zeta^2}.$$

Be careful not to confuse ω_d , the damped natural frequency, and ω_r , the resonant frequency.

the peak amplitude occurs at the natural frequency. ■

Example 3.13 Sample Response

Find the amplitude and the phase lag as well as the magnification factor of the steady-state response of a mass-spring-damper system forced harmonically. Use $m = 2$ kg, $c = 10$ N-s/m, $k = 1000$ N/m, and an input force amplitude of 10 N and frequency of 10 Hz.

Solution: The steady-state response amplitude and its phase lag with respect to the input force are given by Equations 3.33 and 3.34. Here, we rewrite them in terms of frequency ratio r ,

$$D = \frac{A/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}},$$

$$\theta = \tan^{-1} \left[\frac{2\zeta r}{1-r^2} \right], \quad 0 \leq \theta \leq \pi.$$

The forcing amplitude is $A = 10$ N, the damping ratio is $\zeta = c/2\sqrt{mk} = 0.1118$, and the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{\omega}{\sqrt{k/m}}$$

$$= \frac{2\pi(10)}{\sqrt{1000/2}} = 0.2810.$$

The steady-state response amplitude is then

$$D = 0.0108 \text{ m}.$$

The displacement for a static load of 10 N is

$$\delta_{st} = \frac{F}{k} = \frac{10 \text{ N}}{1000 \text{ N/m}} = 0.01 \text{ m}.$$

From Equation 3.35 the magnification factor β for $r = 0.2810$ is

$$\beta = 1.0832.$$

Thus, there is 8.3% amplification at the forcing frequency. The magnification factor is a frequency-dependent ratio of the amplitude of the dynamic motion to the static deflection. ■

Den Hartog Force Vector Diagram

To help understand Figures 3.29 and 3.30 Den Hartog²⁵ looked at the vector representation of Equation 3.29 on a complex plane. The magnitude and relative direction of the forces on both sides of Equation 3.29 are depicted in

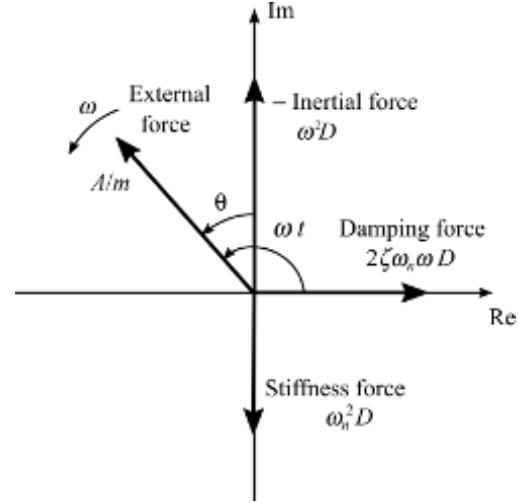


Figure 3.31: Vector diagram for the force components in Newton's second law of motion, $F/m - \ddot{x} - 2\zeta\omega_n\dot{x} - \omega_n^2x = 0$, where A/m is the forcing magnitude of the harmonic input force per mass, θ is the response phase lag with respect to the input force, and D is the response magnitude.

Figure 3.31, where x , $\dot{x}$, and $\ddot{x}$ are obtained from Equation 3.32. The stiffness, damping, and inertia forces per unit mass are, respectively,

$$\begin{aligned} \text{Stiffness force} &= -\omega_n^2 x = -\omega_n^2 D \cos(\omega t - \theta) \\ &= \omega_n^2 D \cos(\omega t - \theta - \pi) \end{aligned}$$

$$\begin{aligned} \text{Damping force} &= -2\zeta\omega_n\dot{x} = -2\zeta\omega_n\omega D \sin(\omega t - \theta) \\ &= 2\zeta\omega_n\omega D \cos(\omega t - \theta - \pi/2) \end{aligned}$$

$$\text{Inertia force} = \ddot{x} = -\omega^2 D \cos(\omega t - \theta).$$

The stiffness force lags behind the input force by $\theta + 180^\circ$, the damping force lags behind the input force by $\theta + 90^\circ$, and the negative of inertia force lags behind the input force by θ . The magnitudes of the normalized forces and the directions are shown in Figure 3.31.

From Figure 3.31, the vertical and horizontal summations of $\frac{A}{m} \cos \omega t - \ddot{x} - 2\zeta\omega_n\dot{x} - \omega_n^2x = 0$, respectively, yield

$$\begin{aligned} \frac{A}{m} \cos \theta - \omega^2 D - \omega_n^2 D &= 0 \\ -\frac{A}{m} \sin \theta + 2\zeta\omega_n\omega D &= 0. \end{aligned}$$

From these two equations, the unknowns D and θ can be solved. The expressions for D and θ are identical to Equations 3.33 and 3.34.

The vector diagram of Figure 3.31 helps to visualize how the amplitude and phase angle vary with driving frequency. For very slow oscillations ($\omega \simeq 0$), the normalized damping and inertia forces are negligible, as discussed earlier, and the normalized stiffness force has a

²⁵Mechanical Vibrations, J.D. Den Hartog, Dover, 1984, pp.48-50.

magnitude $\omega_n^2 D$, where $\theta = 0$. Therefore,

$$\frac{A}{m} \simeq \omega_n^2 D,$$

and the normalized forces in the vertical direction balance. With increasing frequency, the magnitude of the normalized damping force increases according to $2\zeta\omega_n\omega D$, and the normalized inertia force grows even faster as $\omega^2 D$. The phase angle cannot be zero since A/m must have a horizontal component to balance the damping force. The inertia force vector will grow until it becomes as large as the stiffness force. Then θ must equal $\pi/2$ rad and

$$\frac{A}{m} = 2\zeta\omega_n\omega D.$$

This happens at resonance where $\omega^2 D = \omega_n^2 D$, or $\omega = \omega_n$. Thus, at resonance, the phase angle equals $\pi/2$ rad independent of the damping value. Above this frequency, $\omega^2 D$ will grow larger than $\omega_n^2 D$, so that A/m rotates downward and $\theta > \pi/2$ rad. For very high frequencies, the normalized stiffness force $\omega_n^2 D$ is insignificant compared to the normalized inertia force $\omega^2 D$, so that A/m is used to balance the inertia force with $\theta = \pi$ rad.

In summary, at low frequencies the stiffness force overcomes the input force. At high frequencies, the inertia force overcomes the input force. At resonance, the damping force overcomes the input force. The following discussion expands on this topic.

Frequency Regions of Dominant Properties

The behavior of a mechanical oscillator is determined by the property values of the mass, spring, and damper. Figure 3.32 shows approximate frequency regions (bandwidths) for which the mass, stiffness, damping, or a combination of properties governs the vibration. At low frequency, the stiffness dominates. At high frequency, the mass (inertia) dominates. The stiffness and mass play a crucial role in two intermediate regions. The frequency region for resonant behavior is denoted “damping dominated.” Here the magnitude of the resonant peak is determined by the value of damping. In fact, the behavior in that region is determined by the values of the stiffness, mass, and damping.

Complete Solution

To obtain the complete solution, the particular response, given by Equation 3.30, is added to the homogeneous response, given by Equation 3.9,

$$x(t) = Ce^{-\zeta\omega_n t} \cos(\omega_d t - \phi) + B_1 \cos \omega t + B_2 \sin \omega t, \quad (3.37)$$

where B_1 and B_2 were evaluated earlier. Next, the initial conditions $x(0)$ and $\dot{x}(0)$ can be satisfied and the

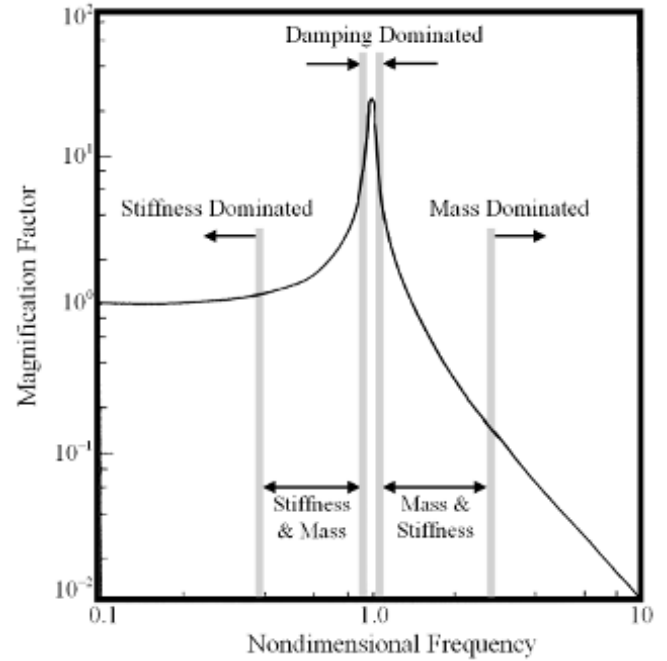


Figure 3.32: Frequencies for dominant mass, stiffness, and damping properties. (Adapted from Figure 2.7, **Passive Vibration Control**, D.J. Mead, Wiley, 2000, p.45.)

constants C and ϕ determined. Following the same procedure as in Section 3.4,

$$C = \frac{x(0) - B_1}{\cos \phi} \quad (3.38)$$

$$\phi = \tan^{-1} \left[\frac{\dot{x}(0) + [x(0) - B_1]\zeta\omega_n - B_2\omega}{[x(0) - B_1]\omega_d} \right]. \quad (3.39)$$

Care must be taken to choose the phase angle ϕ in the correct quadrant. For example, if both numerator and denominators are positive, ϕ is in the first quadrant. If the numerator is negative while the denominator is positive, the phase angle is in the fourth quadrant.

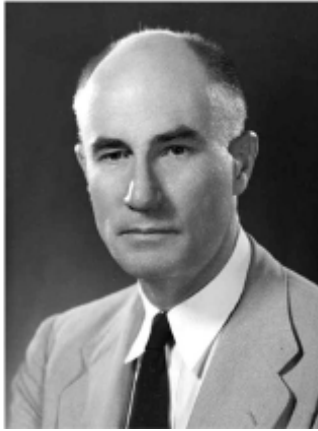
The general response, Equation 3.37, has two distinct components. The first, resulting from the initial conditions, is important only during the early stages of the response. The second, which results from the forcing, is important as long as the force is acting on the system. For example, if the load is an explosion or an impact, then we care only about what happens at the beginning of the response since the largest amplitudes occur during this early time. In this case, the harmonic components of the steady-state response should be added to the free vibration since the effects of the initial conditions have not yet decayed to insignificance.

On the other hand, over long periods of time the influence of the initial conditions becomes small and unimportant in comparison to the forced response. The decaying exponential vanishes leaving no effect of the initial con-

ditions. In this case, one may solve only for the forced response, ignoring initial conditions and free vibration, since the long term behavior is of greater interest.

JACOB PIETER DEN HARTOG

(July 23, 1901 – March 17, 1989)



Contribution: Den Hartog²⁶ was a Dutch-American mechanical engineer and engineering educator. He worked at Westinghouse in Pittsburgh, Pennsylvania under the supervision of Timoshenko. He served in the Navy during World War II, and afterwards was a Professor of Mechanical Engineering at MIT from 1945 to 1967 where he taught dynamics and strength of materials. He was internationally famous as a vibration consultant with an uncanny ability to identify the causes of mysterious vibrations. He had many gifts beyond his deep physical insight into mechanical vibration, such as energetic enthusiasm and sparkling wit.

Life Sketch: Jacob Pieter Den Hartog was born in Ambarawa on the island of Java in the Dutch East Indies on July 23, 1901. His father, Maarten, had been a school teacher in Amsterdam until he was dismissed because of radical activity. Maarten had been an outspoken supporter of Alfred Dreyfus in the early phases of that famous affair when the popular view was strongly against Dreyfus. The family was forced to go to the Indies, and Maarten taught school in the colonial system in Ambarawa, Makassar, and Batavia.

Young “Jaap” grew up speaking Dutch and Malay. He attended elementary school and took violin lessons. When it was time for Den Hartog to enter high school, it was decided that he, his two younger sisters, and his mother, Elizabeth, would return to Amsterdam, while Maarten would remain in Java. Because Holland was a neutral country during the First World War, their ship could sail without interference, although it was necessary in 1916 to bypass the Suez Canal and

sail down around the southern tip of Africa and up around Iceland to reach Holland.

The following eight years were difficult. Maarten died in Java soon after the family returned to Holland, leaving his widow with three children to support. Den Hartog was such an outstanding high school student that some of his relatives undertook to pay his expenses at the Technical University of Delft. Entering Delft in 1919, he decided to become an electrical engineer after seeing a dramatic physics demonstration in which a bolt of lightning jumped from one charged sphere to another. Given his limited financial situation he did not participate in sports or social activities. He received his M.Sc. degree in electrical engineering in 1924.

Economic conditions in Holland were sufficiently bad in 1924 when he graduated that even the best Delft students were not sure of finding a job. Den Hartog applied for two jobs, and when he was rejected, he decided to leave Holland to seek his fortune in the U.S.

Arriving in New York without connections and essentially penniless, Den Hartog took to America with great enthusiasm. He worked briefly at temporary jobs until he learned that Westinghouse was hiring electrical engineers in Pittsburgh. Luckily, he was accepted by Westinghouse just in time to be placed in an in-house training course for new engineers. Among the lecturers was Stephen P. Timoshenko, an emigré Russian professor of mechanics who had been hired by the Westinghouse Research Laboratories the previous year.

Timoshenko was impressed by the eager young Dutchman. When the training course was completed, Timoshenko requested that Den Hartog be assigned to the mechanics section of the research laboratories as his assistant. It was here that Den Hartog served his real professional apprenticeship.

In the next three years Timoshenko converted the young electrical engineer into a mechanical engineer by assigning him a wide variety of vibration problems across the whole spectrum of Westinghouse products: electric motors and generators, steam turbines, hydropower turbines, railroad electrification, etc. While working at the research laboratories during the day, Den Hartog studied mathematics in the evenings at the University of Pittsburgh.

One of his first cases involved the shaft of a motor-generator set that was continuously breaking. Timoshenko recognized the fracture as due to torsional fatigue and suggested that Den Hartog calculate the torsional resonance. It turned out that the torsional critical speed was exactly at the operating speed. Furthermore, only a slight detuning was required to alleviate the situation. Although the detuning could be accomplished equally well by either stiffening or softening the shaft, Den Hartog boldly recommended that the diameter of the shaft be reduced by one-sixteenth of an inch. When this unlikely cure completely solved the problem of broken shafts, his reputation as a vibration expert was launched.

In 1926, after receiving a steady salary for nearly two years, Den Hartog proposed to his childhood sweetheart, Elisabeth F. Stolker. They decided to have the wedding in Amsterdam

²⁶ The authors are indebted to Stephen H. Crandall for his “Biographical Memoir Jacob Pieter Den Hartog, 1901-1989,” National Academy of Sciences, 1995, from which much of this material was drawn.

during his summer vacation and took the ocean voyage back to America as their honeymoon. The young couple called each other by their Dutch diminutives, Jaapie and Beppie, as did most of their friends throughout their lives. They settled in Pittsburgh and bought a piano; Beppie was an accomplished pianist. As enthusiastic emigrés, they made a pact to always speak English to one another. A year later their first son Maarten was born.

While at Westinghouse, Timoshenko was active in the establishment of a separate Applied Mechanics Division within the American Society of Mechanical Engineers (ASME). When the division was finally started in 1927, Den Hartog plunged into its activities with great energy. In the Transactions for that first year are his first three published papers, quite an accomplishment given that this occurred at the same time that he was working on his doctoral dissertation.

When he received his PhD degree from the University of Pittsburgh in 1929, Den Hartog had published a total of eight papers on technical problems he had solved at Westinghouse. His dissertation, *Nonlinear Vibration with Coulomb Damping*, provided material for three more publications.

In 1930, Den Hartog became a naturalized citizen of the United States. Also in 1930, the Applied Mechanics Division was reorganized into two sections: dynamics and materials. Den Hartog headed the dynamics section.

Den Hartog arranged to spend the following sabbatical year as a postdoctoral student in the laboratory of Professor Ludwig Prandtl. He and the family enjoyed this year in Göttingen immensely. For Den Hartog it was an opportunity to meet engineering research leaders in Europe.

Soon after returning to Pittsburgh, he was offered an appointment as assistant professor of mechanical engineering at Harvard University. He had done some lecturing in the Westinghouse training courses for new engineers, and he jumped at the chance to try teaching as a full-time occupation.

In September 1932 Den Hartog arrived at Harvard and started his academic career. Full of enthusiasm, he poured his energies into his vibrations course, getting an extensive collection of demonstration models made and starting to write his famous text *Mechanical Vibrations*. Although only thirty-one years old, he already was widely known as a vibration expert. MIT faculty used to regularly ride the trolley over to Harvard to sit in on Den Hartog's lectures.

During the decade at Harvard, Den Hartog was engaged in a variety of professional activities. Perhaps the most important achievement was the publication of *Mechanical Vibrations* in 1934, with a second edition in 1940. This book remains a classic text. The early chapters follow the grand outline of Rayleigh but have a less mathematical, more appealing, practically motivated style with chapters on vibrations of real machines (reciprocating engines, rotating machinery) and simplified physical explanations for an extensive catalog of self-excited vibration phenomena.

During these years Den Hartog continued his active involvement with the Applied Mechanics Division of ASME.

When the *Journal of Applied Mechanics* began separate publication in 1933, it contained at least one of his contributions each year for the first seven years. He served as division chairman in 1940 and 1941. Another important involvement was with the International Congresses of Applied Mechanics, which renewed and extended his relations with European researchers and engineers. He presented a paper at the fourth congress held in Cambridge, England in 1934 and took an active part in hosting the fifth congress, held in Cambridge, Massachusetts in 1938. He also served as coeditor of the proceedings of the fifth congress.

At the University of Michigan, Timoshenko organized an annual summer school for teachers of mechanics, which had an important influence on mechanics education in America. Den Hartog was a guest lecturer for several summers in this program while at Harvard. It was also during this period that he began to take on consulting jobs. His principal clients were Hamilton Standard, an aircraft propeller manufacturer, and two builders of Mississippi tugboats that had torsional vibration problems in their diesel engine drives.

The decade at Harvard also saw many changes in the family life of the Den Hartogs. They bought a house in Wellesley, Massachusetts. Their second son, named Stephen Ludwig, in honor of Timoshenko and Prandtl, respectively, was born in 1933. In 1938, the Den Hartogs bought a small island in Lake Winnepesaukee, 75 miles north of Boston.

In 1939 Den Hartog volunteered for a commission in the U.S. Naval Reserve. This may seem like a strange move for a successful academic to do in mid-career. Stephen Crandall (MIT Professor) believed he had two major reasons. First, he had come to the conclusion that war was inevitable and he wanted to position himself advantageously for that outcome. Second, he had begun to chafe at the low esteem with which engineering seemed to be regarded by most of the Harvard administration. His own position was safe (tenure had come with his promotion to associate professor in 1936), but he was irked by the policy decision to move away from engineering toward applied science. At any rate, he was given a commission as a lieutenant commander on inactive duty, which he held for two years until June of 1941 when he was called up for active duty. A year later he resigned from Harvard.

The four Navy years proved to be an exciting break in Den Hartog's academic career. Initially, he was assigned to the Taylor Model Basin in Bethesda, Maryland, and the family moved down to nearby Washington, DC. After Pearl Harbor he was transferred to the Bureau of Ships in Washington. The next three years were full of hard work.

The Navy was building ships as fast as possible, and they had many different kinds of vibration problems, from windshield wipers on P.T. boats to propeller shafts on cruisers. Den Hartog shuttled from design reviews on one ship to sea trials on another. He was involved in most of the interesting vibration problems, and he had the opportunity to interact with many of the country's leading engineers. In February 1943 he was promoted to the rank of commander. Later that

year he was approached by MIT administrators and it was decided that he would join their faculty after the war.

In August of 1944 it began to be clear that the Allies were prevailing over the Nazis in Europe. The Navy decided to send a special technical mission to Europe. This group of some forty officers and sixty enlisted men was to follow the advancing Allied forces with the aim of debriefing enemy technicians and capturing interesting technical equipment. Commander Den Hartog was an ideal choice for this mission. He spoke Dutch, Flemish, German, and French and knew many European scientists personally. In his wartime diary Den Hartog called this final year of the war “the most interesting year of my life.”

The technical mission arrived in France shortly after its liberation and set up headquarters in Paris. For most of the year normal operations involved trips in teams of two or three officers to target locations. Typically, the information and equipment could be obtained in a few days, and the team returned to Paris to write up its reports and plan the next trip. Travel close to the front lines was always difficult and involved many frustrations with interservice red tape. Den Hartog’s targets in the winter of 1944-45 were in Belgium, France, and England. In the spring his targets were in a sequence of German cities, including Göttingen, Nürnberg, and Buchenwald as the Nazis retreated.

When the Germans surrendered in Holland and Denmark, he immediately went to Holland and was in Amsterdam on V-E day. The technical mission continued in high gear for an additional three or four months. In Denmark Den Hartog stumbled on a number of very small submarines still in shipping containers and was able to arrange to have five of them, plus five trained German operators, sent back to the States for careful appraisal of their capabilities.

The technical mission worked hard for a year. It also played hard. Evenings back in Paris often involved dates, dinner and dancing, and theaters or concerts. Den Hartog was an enthusiastic participant in these extracurricular activities. He organized a string quartet and rented instruments for them to play. He had connections with well-placed French families who were only too happy to be invited to the officers’ dining room. He also enjoyed striking up friendships from chance encounters. With his command of languages he was often the one who provided dates for his colleagues. In his diary he wrote, “The upper class and the lower class know how to enjoy themselves, but Lord deliver me from the uptight middle class.”

In September 1945 Captain Den Hartog was deactivated, and Professor Den Hartog took up his new post as professor of mechanical engineering at MIT. After the excitement of the previous year, it took a while to get back into the academic groove. He began by teaching one class and working on the third edition of *Mechanical Vibrations*. Gradually, Den Hartog began to expand his activities.

He took on the task of acting as graduate student registration officer for the department and started again to supervise

doctoral students. He also began to play in the violin section of the MIT orchestra.

The buildings on the island in Lake Winnepesaukee were expanded to facilitate entertaining, and an isolated study cabin was built with a huge desk looking out on the water. It was here that Den Hartog wrote his textbooks on *Mechanics* (1948), *Strength of Materials* (1949), and *Advanced Strength of Materials* (1952).

His consulting practice began to grow. Typically, a newly built structure or machine would be found to be inexplicably vibrating. Den Hartog would be telephoned and would travel to the site by overnight train. He would observe the phenomenon intently and ask many questions. In his mind an image of the underlying phenomenon would develop, along with a diagnosis of the most probable cause. He would then share his surgical insight with the client, with clarity and humor.

In many cases his initial insight was sufficient to provide the basis for a satisfactory solution. In obstinate cases additional tests would be recommended to pinpoint the difficulty. A great many of his clients were satisfied with the results after only one or two days of professional service. An exception was Exxon, which kept him on a retainer for thirty-four years and regularly sought his advice over a wide range of problems.

In the 1950s Den Hartog was at his peak as an educator. He had become an entertaining raconteur with a large store of real case histories that he used to illustrate fundamental concepts. For the graduate students in mechanical engineering, his lectures were the high point of their MIT education. Following in the footsteps of his mentor Timoshenko, he organized his own special summer courses in vibration for engineers in industry.

He and his wife often entertained students and visiting academics on their island. Many foreign visitors were enthusiastically bundled into a car, driven up to New Hampshire, seated in a canoe, and paddled across the lake to spend an idyllic weekend on the island.

In the early 1950s Den Hartog’s son Maarten, now an architect, designed for his parents a spacious house in Concord. The house, only 18 miles from MIT, was used for entertaining, especially during the years 1954-58 when Den Hartog served as head of the Department of Mechanical Engineering.

It was in this period that he went on a sabbatical term to Japan as a Fulbright lecturer in 1955 and went to England in 1957 to give the Thomas Hawksley Lecture, the first American to be so honored. He also made a final improvement of *Mechanical Vibrations*, publishing the fourth edition in 1956. By this time the book had become world famous, with fifteen foreign editions published in eleven languages.

When Den Hartog returned to teaching in 1958, the jet airplane had arrived and was making travel more convenient, especially for lecture tours and consulting visits. He lectured widely in America and abroad, including the Soviet Union in 1960 and 1961. As he began to approach retirement, he was honored with an increasing stream of awards. The Design Division of ASME dedicated the proceedings of its first vibra-

tion conference as a Festschrift to Den Hartog on the occasion of his retirement at age sixty-five in 1967. He continued part-time lecturing as Professor Emeritus and consulting for an additional five years but had to cut back on some activities. Arthritis in his fingers made it impossible to play the violin. In 1972 the Applied Mechanics Division of ASME awarded him the Timoshenko Medal, established in honor of his early mentor. The ceremony took place just a few months after Timoshenko's death at the age of 93.

In the following decade there was a gradual slowdown in Den Hartog's consulting activities and occasional lectures as arthritis ravaged his body. The year that he was eighty (1981-82) was the final year before he became totally bedridden. It was, however, a memorable year. On his birthday MIT's Department of Mechanical Engineering established the Den Hartog Prize for "excellence in teaching." A few months later the British Institution of Mechanical Engineers awarded him the prestigious James Watt Medal. Then the National Academy of Engineering awarded him its top honor, the Founders Award. Finally, the following spring he received the Order of the Rising Sun, signed by the Emperor of Japan.

Den Hartog's arthritis-ravaged body finally surrendered, imprisoning his active spirit in a skeleton he could not move. The Den Hartogs sold their Concord house and purchased a condominium for Beppie next door to a nursing home where he stayed in Hanover, New Hampshire. On Easter day 1985, Den Hartog's wife Beppie passed away peacefully in her sleep.

Den Hartog lived another four years unable to move himself and unable to read or write. He spent his time listening to music, mostly string quartets, and news programs. He never complained. He could look back on a very full life: the excitements of emigration and the wartime adventures, the challenges of strange new consulting problems, and the accomplishments of a beloved teacher.

In his twilight years Den Hartog was visited by many of his students and colleagues. His mind remained clear, and his memory was excellent. He would retell old stories, and he retained his curiosity about the present and the future.

Den Hartog died at age 87 in Hanover, New Hampshire.

Noteworthy: Den Hartog was a consummate teacher. He could hold the attention of a single colleague or a class of a hundred students as he explained a particular mechanism and wrapped his audience in the sheer fun of imagining how it would move and why. He taught dynamics by creating vivid images of particular cases that dramatized general concepts. Generations of students were enriched by his verve, wit, and captivating physical insight.

He had made several major contributions beyond his famous *Mechanical Vibrations* book. In his research papers he made major advances to the theory of vibrations. Of particular importance were his extensions to systems with damping and the theory of dynamic vibration absorbers (1928) and the Holzer method for torsional vibration (1946). He was the first to obtain solutions for vibratory systems with Coulomb

damping (1931). He was also the first to give a quantitative explanation for the phenomenon of galloping of ice-laden transmission lines (1932).

He was awarded the Timoshenko Medal in 1972 "in recognition of distinguished contributions to the field of applied mechanics." In 1987 the Design Division of ASME announced the establishment of the J.P. Den Hartog Award for "sustained meritorious contributions to vibration engineering" at its eleventh vibration conference. The first recipient of the award was Den Hartog himself. This was twenty years after the first vibration conference had been dedicated to him. When the medal was delivered to him he was pleased to think that "his boys still remembered him."

He was a Member of the National Academy of Sciences and of the National Academy of Engineering, a Fellow of the American Academy of Arts and Sciences, and an Honorary member of the American Society of Mechanical Engineers and of the Japan Society of Mechanical Engineers. He was also a Foreign member of the Royal Dutch Academy of Arts and Sciences.

Other awards include the Charles Russ Richards Medal, the Worcester Reed Warner Medal, the Founders Award of the National Academy of Engineering, and the Lamme Medal of the American Society of Engineering Education.



THE LONDON MILLENNIUM BRIDGE.²⁷

In 2000, the new London Millennium Footbridge, shown in Figure 3.33, in the United Kingdom opened with much fanfare, yet was closed after 48 hours due to excessive and dangerous swaying caused by as few as 200 pedestrians walking on the 325-m long structure. The problem was due to what was termed "synchronous lateral footfall," creating mechanical feedback into the bridge structure.

The feedback resulted from people walking on the bridge and their seemingly random walking motions becoming synchronized over time, similar to sailors walking on a rolling ship deck in a storm. The resulting amplified harmonic motion occurred within a frequency bandwidth of 0.5 to 1.1 Hz, with measured lateral accelerations building up to 0.25g, essentially the same magnitude as a moderate earthquake. A study revealed that as many as five separate structural modes were being excited, all of which needed to be suppressed. Structural damping for the as-built structure was found to be in the range of 0.4-0.5% critical, sufficiently low so as to allow dangerous levels of amplification to develop, which conceivably could cause extensive damage to the bridge structure, or even a potential bridge collapse.

The simplest solution to the bridge problem was to add significant damping to each mode, in the range of 20% critical. The advantage of added damping in a structure undergoing forced resonance ($\omega = \omega_n$) is well understood, although used more often by mechanical engineers in the technology fields

²⁷Example and images courtesy Douglas Taylor, August 2016.



Figure 3.33: London Millennium Bridge. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)

of mechanisms and machinery. For a simple mass-damper-spring system, amplitude under steady-state forced resonance is the amplification factor $\beta = 1/2\zeta$. This is found by setting $\omega = \omega_n$ in the general equation for β .

Thus, if a simple first-order system with 0.5% critical damping is excited by forced resonance, the magnification factor is

$$\beta = \frac{1}{2(0.005)} = 100.$$

If damping in the system is elevated to 20% critical, then the magnification reduces substantially to

$$\beta = \frac{1}{2(0.2)} = 2.5.$$

When comparing the above magnification factors to the bridge structure, the as-built bridge was not constructed with a 100:1 safety factor, but was indeed capable of accepting 2.5:1 amplification without any difficulty. Thus, the addition of 20% added damping to the bridge could provide a theoretical reduction in peak g loadings by a factor of 40.

To achieve the desired 20% added damping, a total of thirty-seven fluid damping devices were integrated into the bridge structure at locations where the desired suppression of responses for the individual modes would be optimum. These included seventeen dampers to suppress lateral modes, four angled vertical to ground dampers to suppress vertical and lateral modes, and sixteen pier dampers to suppress lateral and torsional modes.

Three additional design issues existed, the first being the owner's desire for a 50-year zero-maintenance minimum life, equating to 10^9 cycles on the dampers without any repairs, replacement of parts, or adding fluid. The second issue involved the requirement for extremely low damper seal hysteresis, so that the pedestrians would perceive only smooth motions, rather than a stick-slip motion, while walking on the bridge.



Figure 3.34: Long deflection flexible metal bellows damper. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)

For these reasons, a damper design utilized previously by NASA and the U.S. government on spacecraft was selected, which utilized long deflection flexible metal bellows for sealing, rather than conventional sliding elastomeric seals. A photograph of the dampers is shown in Figure 3.34. The metal bellows seals were expected to provide an extremely long life with zero maintenance. In addition, because sealing was achieved by flexing rather than sliding, the seal has essentially zero hysteresis. The longest dampers measured some 8 m in length, requiring internal recirculating ball bearing type bushings that could operate with low hysteresis compared to simple sliding bushings. The third design issue was that all dampers must be hidden, or seamlessly integrated into the bridge without appreciably altering its architecture.

Preliminary testing on the Millennium Bridge with the added fluid dampers took place in 2002, using a group of 700 pedestrians, followed by final tests with 2000 pedestrians at a loading density of 1.5 persons per m^2 . The results showed a reduction in peak measured accelerations from the original $0.25g$ undamped to $0.006g$ for the damped bridge. More importantly, the 20% added damping appeared sufficient to reduce any tendency of the pedestrians to want to walk in cadence while on the bridge, thus eliminating feedback to the bridge structure. Note that the nearly 40:1 reduction in acceleration agreed closely with the magnification factor calculations provided for the simple single degree-of-free system, with 0.5% and 20% damping.

The Millennium Bridge has since become a popular means to cross the River Thames by walking, and is used by upwards of four million people each year. Many who have walked on the bridge comment that it feels extremely solid, almost as if it were made from solid concrete.

Example 3.14 Hinged and Damped Beam

The system shown in Figure 3.35 is comprised of a massless rigid beam of length l that is hinged at one end and carries a mass m at the other end. The mass is connected to a fixed base via a damper with viscous damping constant c and the beam is connected at an interior point to a moving support via a spring of stiffness constant k . The interior point is a distance a from the left end and a distance b from the right end. The support moves vertically in sinusoidal motion given by $y(t) = A \sin \omega t$. When $y = 0$ the beam is horizontal and in static equilibrium. Derive the equation of motion.

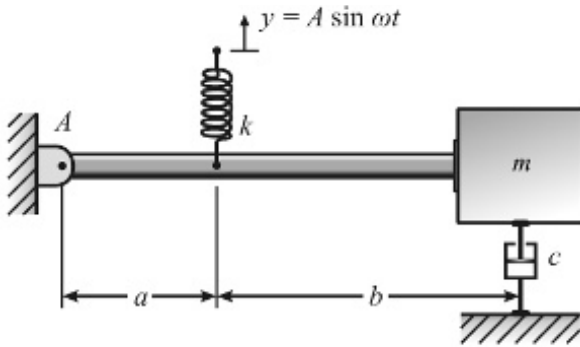


Figure 3.35: A pinned massless beam connected to a mass at its free end and supported by a spring and a damper. The beam rotates with angle θ .

Solution: We apply the Newton-Euler equation of motion,

$$\sum M_A = I_A \frac{d^2\theta}{dt^2},$$

where $\theta(t)$ is the angular displacement of the bar from the horizontal position. The moment due to the elongation of the spring is given by

$$k(a\theta - A \sin \omega t) \cdot a,$$

assuming small angular displacement θ . The moment due to the velocity at the damper is given by

$$c \left(l \frac{d\theta}{dt} \right) \cdot l,$$

where the length $l = a + b$.

The mass moment of inertia of the mass m about the point of rotation A is

$$I_A = ml^2,$$

assuming the mass is a point mass. Substituting these terms into the Newton-Euler equation, with the right-hand rule for moments, yields

$$-ka(a\theta - A \sin \omega t) - cl \left(l \frac{d\theta}{dt} \right) = ml^2 \frac{d^2\theta}{dt^2}$$

or in standard form,

$$ml \frac{d^2\theta}{dt^2} + cl \frac{d\theta}{dt} + \frac{ka^2}{l} \theta = ka \frac{A}{l} \sin \omega t.$$

This equation can be solved using the techniques of this chapter. ■

3.7.2 Harmonic Excitation in Complex Notation

An alternate approach to derive the steady-state response is to write the loading in complex exponential form,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m} A e^{i\omega t}.$$

Here, two equivalent possible complex forms of the steady-state response can be written:

$$x_s(t) = X(i\omega) e^{i\omega t} \quad (3.40)$$

$$x_s(t) = X(\omega) e^{i(\omega t - \phi)} = [X(\omega) e^{-i\phi}] e^{i\omega t}. \quad (3.41)$$

In the first formulation, Equation 3.40, $X(i\omega)$ is a complex quantity that implicitly embodies both amplitude and phase lag information.²⁸ Recall that any complex number can be written as *modulus* times *phase*,

$$X(\omega) = |X(\omega)| e^{-i\phi},$$

where $|X(\omega)|$ is the modulus, or response amplitude, a real number, and $e^{-i\phi}$ is the phase.²⁹ Since the response lags behind the load, the negative sign is introduced for ϕ . The second formulation, Equation 3.41, directly includes amplitude and phase.

One of the goals of this analysis is the derivation of the frequency response function, $G(i\omega)$. Starting with Equation 3.40, we can differentiate twice,

$$\begin{aligned} \dot{x}_s &= i\omega X(i\omega) e^{i\omega t} \\ \ddot{x}_s &= -\omega^2 X(i\omega) e^{i\omega t}, \end{aligned}$$

²⁸We can relate the two expressions as follows: $X(i\omega) = X(\omega) e^{-i\phi}$, so that $|X(i\omega)| = X(\omega)$.

²⁹To see this, let $X(i\omega) = x + iy$, then $|X(i\omega)| = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$. These are essentially geometric constructs that can be easily written down by drawing the coordinate point (x, y) in the complex plane, and noting that $|X(i\omega)|$ is the position vector to the point and ϕ is the angle of that vector measured from horizontal.

and substitute these terms into the governing equation to find

$$X(i\omega) [-\omega^2 + 2i\omega\zeta\omega_n + \omega_n^2] e^{i\omega t} = \frac{A}{m} e^{i\omega t}.$$

Simplifying algebraically, the *frequency response function*, $G(i\omega)$, is defined as³⁰

$$\begin{aligned} G(i\omega) &= \frac{X(i\omega)}{A} = \frac{1}{m(-\omega^2 + 2i\omega\zeta\omega_n + \omega_n^2)} \\ &= \frac{1}{m\omega_n^2 \left(-(\omega/\omega_n)^2 + i2\zeta\omega/\omega_n + 1 \right)} \\ &= \frac{1}{k \left(-(\omega/\omega_n)^2 + i2\zeta\omega/\omega_n + 1 \right)}. \end{aligned} \quad (3.42)$$

Since $G(i\omega) = |G(i\omega)|e^{-i\phi}$, Equation 3.42 gives

$$\begin{aligned} X(i\omega) &= AG(i\omega) \\ &= A|G(i\omega)|e^{-i\phi}, \end{aligned}$$

and substituting the result into Equation 3.40 yields the response

$$x_s(t) = A|G(i\omega)|e^{i(\omega t - \phi)}, \quad (3.43)$$

$$\text{where } |G(i\omega)| = \frac{1}{k\sqrt{(1 - (\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2}} \quad (3.44)$$

$$\text{and } \phi(\omega) = \tan^{-1} \left[\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} \right], \quad 0 \leq \phi \leq \pi. \quad (3.45)$$

The non-dimensional version of the frequency response function is given by

$$\begin{aligned} H(i\omega) &= kG(i\omega) \\ &= \frac{1}{\left(-(\omega/\omega_n)^2 + i2\zeta\omega/\omega_n + 1 \right)}, \end{aligned}$$

where its magnitude $|H(i\omega)|$ is the magnification factor β of Equation 3.35 and $\phi = \theta$ in Equation 3.34.³¹

There are many applications of the theory just developed for harmonic excitation. These include the problems of *base excitation*, *excitation due to rotating unbalance*, and the modeling of *vibration measurement devices*.

Example 3.15 Amplitude and Phase Lag

Consider a system with an equation of motion given by

$$2\ddot{x} + 4\dot{x} + 3x = 5 \sin 7t.$$

³⁰This expression is sometimes called a *transfer function* when expressed in terms of Laplace variable s . It represents the ratio of the Laplace transform of the output to the Laplace transform of the input, and is described in Chapter 10.

³¹Looking ahead, the above is related to the development in Section 4.1 on Laplace transforms, where, if $s = i\omega$ is substituted in Equation 4.4, then $G(s|_{s=i\omega}) = G(i\omega)$.

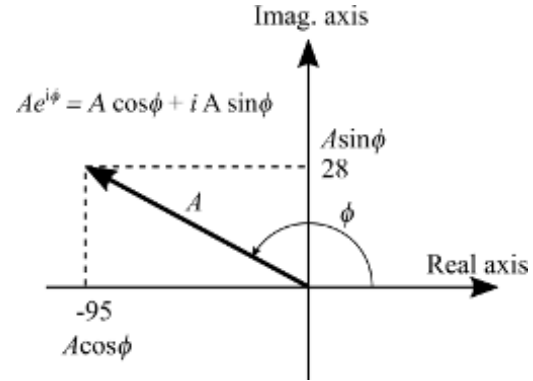


Figure 3.36: Demonstration of Euler's formula for Example 3.15.

Find the amplitude and the phase lag of the steady-state response with respect to the input.

Solution: First, rewrite the forcing function as $5e^{i7t}$. Then, assume the steady-state solution of the form

$$x_s(t) = X(i\omega) e^{i7t}.$$

The actual steady-state solution is the imaginary part since the forcing is a sine function, $x_s(t) = \text{Im}(X e^{i7t})$. Upon substituting the assumed solution into the equation of motion, we find

$$2(-49)X e^{i7t} + 4(i7)X e^{i7t} + 3X e^{i7t} = 5e^{i7t}.$$

Then, the steady-state response amplitude is given by

$$X = \frac{5}{-98 + 28i + 3}.$$

The steady-state response is then given by

$$\begin{aligned} x_s(t) &= \text{Im}(X e^{i7t}) \\ &= \text{Im}\left(\frac{5}{-95 + 28i} e^{i7t}\right). \end{aligned}$$

We can rewrite the complex number $-95 + 28i$ in the denominator in amplitude-phase form using Euler's formula, $Ae^{i\phi} = A(\cos \phi + i \sin \phi)$, such that

$$A \cos \phi = -95 \quad \text{and} \quad A \sin \phi = 28.$$

It is easier to plot $-95 + 28i$ in the complex plane, as shown in Figure 3.36. The amplitude A is the length of the vector and the phase is the angle that the vector makes with the positive real axis. Then,

$$\begin{aligned} A &= \sqrt{95^2 + 28^2} = 99.04 \\ \phi &= \pi - \tan^{-1} \frac{28}{95} = 2.855 \text{ rad}. \end{aligned}$$

The steady-state response is then given by

$$\begin{aligned} x_s(t) &= \text{Im} \left[\frac{5}{99.04e^{i2.855}} e^{i7t} \right] \\ &= \text{Im} \left[0.0505e^{i(7t-2.855)} \right] \\ &= 0.0505 \sin(7t - 2.855). \end{aligned}$$

The response amplitude is $0.0505/5$ or 1.01% of the input amplitude, meaning significant attenuation. ■

3.7.3 Harmonic Base Excitation

In base excitation problems, the forcing applied to a structure or machine is through its foundation or support. Base excitation problems have many real-world applications. These include the vibration of structures on foundations (such as in earthquake engineering, where the loading is through the base), the response of an automobile to road irregularities, and the interaction between a machine and its support. We make the assumption here that the excitation is harmonic, providing us with some of the key characteristics of base-excited systems.

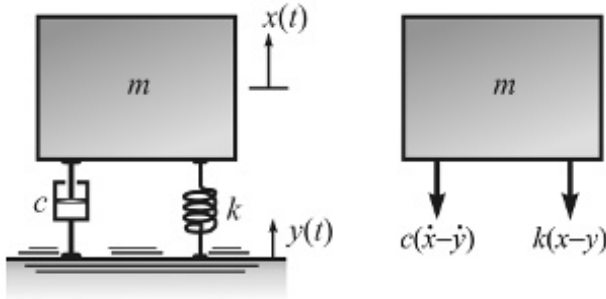


Figure 3.37: A schematic of a mass-spring-damper system for a base-excited system and its free-body diagram.

The design engineer needs to determine the displacement $x(t)$ as well as the force exerted by the base on the system. Consider the idealized model in Figure 3.37 of a structure connected to a base. The structure and the base are connected with stiffness and damping elements. We are interested in how the structure responds. From the free-body diagram, Newton's second law of motion can be written as³²

$$m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y),$$

or

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky.$$

³² We choose relative velocity and relative displacement as $(\dot{x} - \dot{y})$ and $(x - y)$ because it is the motion of mass m that is of interest, and how the damping and stiffness forces, $c\dot{x}$ and kx , respectively, are altered by the motion of the base through y and $\dot{y}$.

The base excitation is modeled by $y(t) = Y \sin \omega_b t$. Substituting, the resulting equation of motion is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 2\zeta\omega_n\omega_bY \cos \omega_b t + \omega_n^2Y \sin \omega_b t. \quad (3.46)$$

To solve for the steady-state response, we proceed as we did for the general solution of a damped oscillator with harmonic excitation. To simplify the algebra, let the right-hand side of Equation 3.46 be written as $A_1 \cos \omega_b t + A_2 \sin \omega_b t$, where $A_1 = 2\zeta\omega_n\omega_bY$ and $A_2 = \omega_n^2Y$. Assuming a steady-state response of the form

$$x_s(t) = B_1 \cos \omega_b t + B_2 \sin \omega_b t, \quad (3.47)$$

and differentiating and substituting into Equation 3.46, we find

$$\begin{aligned} &(-\omega_b^2 B_1 + 2\zeta\omega_n\omega_b B_2 + \omega_n^2 B_1) \cos \omega_b t \\ &+ (-\omega_b^2 B_2 - 2\zeta\omega_n\omega_b B_1 + \omega_n^2 B_2) \sin \omega_b t \\ &= A_1 \cos \omega_b t + A_2 \sin \omega_b t. \end{aligned}$$

Equating coefficients of cosine and sine terms, respectively, we have

$$-\omega_b^2 B_1 + 2\zeta\omega_n\omega_b B_2 + \omega_n^2 B_1 = A_1 \quad (3.48)$$

$$-\omega_b^2 B_2 - 2\zeta\omega_n\omega_b B_1 + \omega_n^2 B_2 = A_2. \quad (3.49)$$

Solving for B_1 and B_2 yields

$$\begin{aligned} B_1 &= \left[\frac{-2\zeta\omega_n\omega_b^3}{(2\zeta\omega_n\omega_b)^2 + (\omega_n^2 - \omega_b^2)^2} \right] Y \\ B_2 &= \left[\frac{\omega_n^2 (\omega_n^2 - \omega_b^2) + (2\zeta\omega_n\omega_b)^2}{(2\zeta\omega_n\omega_b)^2 + (\omega_n^2 - \omega_b^2)^2} \right] Y, \end{aligned}$$

which can be substituted into Equation 3.47 to give the steady-state response.

Following the derivation of Equations 3.33 and 3.34, we write

$$x_s(t) = X \sin(\omega_b t - \theta), \quad (3.50)$$

since the input displacement is given in terms of a sine function, that is, $y(t) = Y \sin \omega_b t$, where

$$\begin{aligned} X &= \sqrt{B_1^2 + B_2^2} \\ \theta &= \tan^{-1} \left(-\frac{B_1}{B_2} \right). \end{aligned}$$

The angle θ must be chosen so that it is in the correct quadrant. Carrying out the algebra, we can find the ratio of the displacement amplitude to the base excitation amplitude, X/Y ,

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta\omega_b/\omega_n)^2}{(1 - \omega_b^2/\omega_n^2)^2 + (2\zeta\omega_b/\omega_n)^2} \right]^{1/2}. \quad (3.51)$$

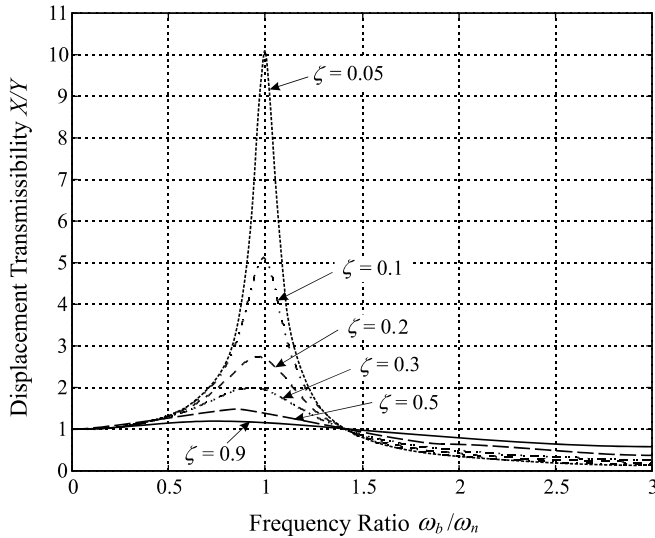


Figure 3.38: Base excitation displacement transmissibility (X/Y) as a function of ω_b/ω_n for various values of ζ . For the frequency ratio range $0 \leq \omega_b/\omega_n \leq \sqrt{2}$, smaller damping results in larger transmissibility values, that is, a larger response.

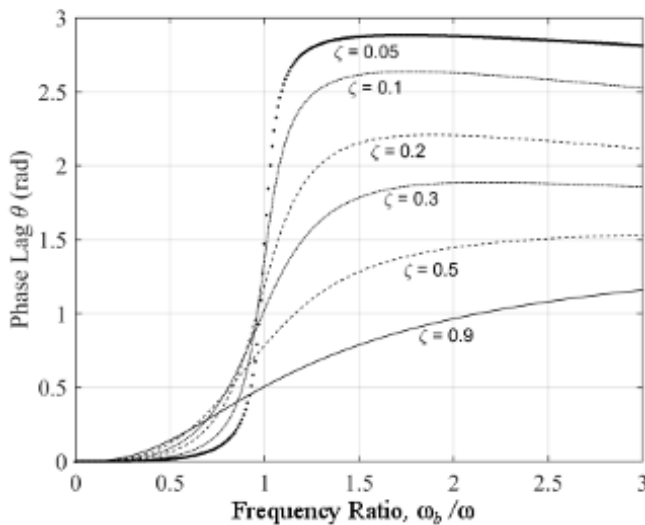


Figure 3.39: Base excitation phase lag θ as a function of ω_b/ω_n for various values of ζ .

This ratio is called the *displacement transmissibility* and is plotted in Figure 3.38. The corresponding phase lag is plotted in Figure 3.39.

It is instructive to compare this result with Equation 3.35 for the magnification factor β . Here there is a second term in the numerator due to the base loading that is carried through the damper. The 1 in the numerator is due to the loading that is carried through the spring.

Figure 3.38 shows that $X = Y$ at $\omega_b/\omega_n = 0$ and $\omega_b/\omega_n = \sqrt{2}$. The curves intersect at these two points,

regardless of the value of ζ . A key phenomenon of base-excited systems is that for the frequency ratio range $0 \leq \omega_b/\omega_n \leq \sqrt{2}$, a decrease in damping results in an increase in transmissibility, that is, less damping leads to a larger response amplitude. On the other hand, for $\omega_b/\omega_n > \sqrt{2}$, smaller amplitudes of response occur for smaller damping, which is counterintuitive.³³

It is important to isolate the vibration and motion of a vibrating machine from its foundation, and vice versa. The term *transmissibility* is used as a measure of the motion isolation achieved.

An effective strategy for vibration isolation so that $X \ll Y$ is to operate far above the natural frequency. For example, consider the design of sensitive equipment to be used in a room exposed to vibration due to foot traffic. By making the natural frequency of sensitive equipment low, one can make the equipment fairly immune to vibration. In particular, if the equipment mass is high and the support stiffness is low so that the combination gives a natural frequency below 2 Hz, then it will be reasonably protected from most vibration generated by walking or dancing, since these occur at frequencies above 20 Hz. If the ratio $\omega_b/\omega_n = 10$, then $X/Y = 0.0416$ for $\zeta = 0.2$, $X/Y = 0.101$ for $\zeta = 0.5$, and $X/Y = 0.140$ for $\zeta = 0.7$. (This is the counterintuitive effect. We do not expect smaller amplitudes for lower damping!)

The other situation is one in which we wish the displacement of the mass (of the machine or suspended part) to track the input, that is, we seek $X = Y$. This can be accomplished if the various components are made very stiff. The natural frequencies are then very high compared to the input frequencies of expected operation, meaning $\omega_b/\omega_n \ll 1$. As a result the desired tracking can be achieved without the need for modifying the input function to compensate for lag or overshoot, for example.

Next, we consider the force transmitted by the base excitation to the system using the equation $F_{transmit}(t) = k(x - y) + c(\dot{x} - \dot{y})$, which is the equal-and-opposite force on the body. From Newton's second law of motion, this force is equivalent to $-m\ddot{x}$. Differentiating Equation 3.50 twice and substituting into the term $-m\ddot{x}$ leads to the expression for the transmitted force. Rewriting $m\omega_b^2 Y$ as $kY(\omega_b/\omega_n)^2$ the magnitude of the transmitted force is

$$F_T = kY \left(\frac{\omega_b}{\omega_n} \right)^2 \left[\frac{1 + (2\zeta\omega_b/\omega_n)^2}{(1 - \omega_b^2/\omega_n^2)^2 + (2\zeta\omega_b/\omega_n)^2} \right]^{1/2}.$$

The ratio

$$\frac{F_T}{kY} = \left(\frac{\omega_b}{\omega_n} \right)^2 \left[\frac{1 + (2\zeta\omega_b/\omega_n)^2}{(1 - \omega_b^2/\omega_n^2)^2 + (2\zeta\omega_b/\omega_n)^2} \right]^{1/2} \quad (3.52)$$

³³ It appears that the frequency ratio $\sqrt{2}$ is only a result of the mathematics rather than any inherent physical significance.

is a dimensionless parameter called the *force transmissibility*. It is the ratio of the magnitude of the transmitted force to the magnitude of the stiffness force imparted at the base. It represents the relative magnitude of force that is transmitted from the base to the structure.

When Equation 3.52 is plotted in the same way as was Equation 3.51 in Figure 3.38, we find the same qualitative result. That is, for $\omega_b/\omega_n < \sqrt{2}$, less damping results in larger transmitted forces, as expected. But for $\omega_b/\omega_n > \sqrt{2}$, less damping results in smaller transmitted forces, again a counterintuitive result.

For $\omega_b/\omega_n > \sqrt{2}$, the transmitted force increases dramatically with frequency ratio.

Example 3.16 Displacement Transmissibility using Frequency Response Function

Find the displacement transmissibility given by Equation 3.51 by writing the loading in complex exponential form as shown in Section 3.7.2.

Solution: Start with the equation of motion given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 2\zeta\omega_n\dot{y} + \omega_n^2y,$$

where

$$y(t) = Y e^{i\omega_b t}.$$

The particular solution or the steady-state response will have the same frequency as the driving frequency ω_b so that we can write

$$x_s(t) = X(i\omega) e^{i\omega_b t},$$

where $X(i\omega)$ is the amplitude of the steady-state response to be determined. Substituting $x_s(t)$ and $y(t)$ and their derivatives into the equation of motion, we have

$$(-\omega_b^2 + i2\zeta\omega_n\omega_b + \omega_n^2) X e^{i\omega_b t} = (i2\zeta\omega_n\omega_b + \omega_n^2) Y e^{i\omega_b t}.$$

The ratio of the output amplitude to the input amplitude is given by

$$\frac{X}{Y} = \frac{(i2\zeta\omega_n\omega_b + \omega_n^2)}{(\omega_n^2 - \omega_b^2 + i2\zeta\omega_n\omega_b)}.$$

Using the frequency ratio $r = \omega_b/\omega_n$, we can further simplify the amplitude ratio,

$$\frac{X}{Y} = \frac{(i2\zeta r + 1)}{(1 - r^2 + i2\zeta r)}.$$

The magnitude of X/Y gives the displacement transmissibility

$$\left| \frac{X}{Y} \right| = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2},$$

which matches Equation 3.51.

The phase lag of the response $x_s(t)$ with respect to the input $y(t)$ can also be found,

$$\begin{aligned} \theta &= -\angle \frac{(i2\zeta r + 1)}{(1 - r^2 + i2\zeta r)} \\ &= -\left(\tan^{-1} \frac{2\zeta r}{1} - \tan^{-1} \frac{2\zeta r}{1 - r^2} \right) \\ &= -\tan^{-1} \frac{2\zeta r}{1} + \tan^{-1} \frac{2\zeta r}{1 - r^2}, \end{aligned}$$

where $\angle$ is the angle of the complex number plotted in the complex plane. ■

Example 3.17 Machine-Induced Base Excitation

A machine imparts a motion of $y(t) = 0.5 \sin \omega_b t$ cm to the floor. Estimate the peak force that would be experienced by a 6000-kg compressor that needs to be placed adjacent to the machine. The compressor is connected to the floor via a mat that has stiffness $k = 80,000$ N/m and damping 1000 N-s/m.

Solution: The peak force occurs at approximately $\omega_b = \omega_n$, thus simplifying the equation for force transmissibility to

$$\frac{F_T}{kY} \simeq \left[\frac{1 + (2\zeta)^2}{(2\zeta)^2} \right]^{1/2}.$$

The magnitude of the transmitted force is then given by the simplified relation

$$F_T \simeq \frac{kY}{2\zeta} (1 + 4\zeta^2)^{1/2}$$

with ζ given by

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{1000}{2\sqrt{80000 \times 6000}} = 0.023,$$

and $Y = 0.005$ m. The transmitted force is then

$$F_T \simeq \frac{(80000)(0.005)}{2(0.023)} (1 + 4(0.023)^2)^{1/2} = 8,705 \text{ N}.$$

(This is slightly less than 2000 lb.) Since ζ is very small, the assumption that the peak force occurs at ω_n is valid.

We note that the transmitted force is approximately 15% of the weight of the compressor, given by $6000 \text{ kg} \times 9.81 \text{ m/s}^2 = 58,860 \text{ N}$. If this peak force is too large, the design of the compressor mat must be changed and the force re-calculated. Otherwise, it may be necessary to stiffen the floor upon which all machines sit. ■



VIBRATION AND NOISE. High vibration levels can cause machinery failure, as well as objectionable noise levels. A common source of objectionable noise in buildings is the vibration of machines that are mounted on floors or walls. Obviously, the best place to mount a vibrating machine is on

the ground floor. Unfortunately, this is not always possible. A typical problem is a rotating machine (such as a pump, air conditioner compressor, blower, engine, etc.) mounted on a roof, or on a floor above the ground floor. The problem is usually most apparent in the immediate vicinity of the vibration source. However, mechanical vibrations can transmit over long distances, and by very circuitous routes through a building, sometimes resurfacing far from the source. A related problem is the isolation of vibration-sensitive machines from the normally occurring disturbances in a building (street traffic, slamming doors, foot traffic, elevators, etc.). Examples of sensitive machines include surgical microscopes, electronic equipment, lasers and laser printers, MRI units, scanning electron microscopes, etc.³⁴



3.7.4 Rotating Unbalance

Unbalanced centrifugal forces – forces that try to pull apart a rotating structure – are due to imperfections in the manufacture of rotating machines.³⁵ Imperfections can be taken to be heavier spots at a location on the machine. Their size and locations are a function of the distribution of imperfections. There may be a number of mass variabilities distributed at different locations.

If there is an unbalanced mass, it can create unwanted stresses that will grow with higher rotating speed, resulting in fatigue or cracking, and eventual failure. The speed at which this occurs is called the *burst speed*. The design of rotors incorporates a factor of safety of between 4 and 10 for the maximum safe speed. That is, the allowable speed is between 1/10 and 1/4 of the burst speed.

The unbalanced force will distort the rotor and will eventually couple the rotor motion to the stationary structure that supports it. The rotating device is coupled to the stationary support structure via a bearing.³⁶ The bearing heats up in the process of transmitting the unbalanced force to the stationary structure.

A large and heavy rotor is relatively insensitive to unbalance due to its large inertia. Light, high-speed rotors are more susceptible and typically need rebalancing more often. While unbalance is generally a radial phenomenon, certain rotors will couple this radial force into the axial direction (due to the way they are mounted giving rise to thrust loads). Bearings not designed for thrust loads do not tolerate axial forces well.

³⁴ Adapted from notes of J.S. Lamancusa, May 28, 2002.

³⁵ This discussion is based on the article V. Wowk, “Balancing and its Effect on Vibration Response,” *Energy-Tech.com*, October 2007, pp.8-12.

³⁶ A bearing permits constrained relative motion between two parts, typically rotation or linear movement. Bearings may be classified broadly according to the motions they allow and according to their principle of operation. Books on machine design describe bearings in more detail.

Roller-element bearings are the main casualty of unbalance. They have small axial and radial clearances and unbalanced eccentricity presses the bearings through the surrounding grease and oil against mating surfaces. Large stresses result due to the small contact area. This leads to fatigue damage, cracking, and spalling. To avoid this, compliant bearings are sometimes used.

The design of the support structure of the machine must account for its vibratory motion. If the support structure is rigidly fixed to the foundation, then it will vibrate less but have to withstand larger forces and stresses. On the other hand, if the support structure is supported on a compliant or soft foundation, then it will have to resist lower forces and stresses but oscillate more. We see the trade-off here in the design, that is, making one adjustment for a positive benefit can result in a negative effect as well.

If the oscillations become large, then the behavior becomes nonlinear and the system can generate frequencies that were not in the original system. They too can cause damage.

Excitation due to Rotating Unbalance

As discussed, for rotating machinery, such as turbines and generators, imperfections in component geometry and irregularities in mass distribution create dynamic rotating unbalances. These result in an effective harmonic load on the structure at the frequency of the rotating component.

An example of this effect is an automobile tire that must be balanced to avoid developing periodic forces. The balancing is performed on a special machine that rotates the tire at high speed and determines the magnitude and location of any “deficit mass.” At these locations, small lead masses of appropriate weights are attached by a mechanic. This process is repeated until the tire is balanced (within a tolerance). Tires require re-balancing because of uneven wear. Similar but much more intricate considerations are part of internal combustion engine design.³⁷

To perform a simple analysis of an unbalanced machine based on our understanding of harmonically loaded structures, we replace the imperfection by an eccentrically rotating mass that has an equivalent effect. Thus, the unbalance in the *idealized* model is due to a mass m_0 rotating at constant angular speed ω_r about a point at some eccentricity e , as depicted in Figure 3.40.³⁸ In general, the imbalance results in forces in all directions. Here we assume that the machine is constrained to move

³⁷ Chapter 8 of *The Internal Combustion Engine in Theory and Practice, Volume 2: Combustion, Fuels, Materials, Design*, C.F. Taylor, MIT Press, revised edition, 1985.

³⁸ Here, ω_r represents the angular speed of the rotating unbalance. It is not the same as the resonant frequency.

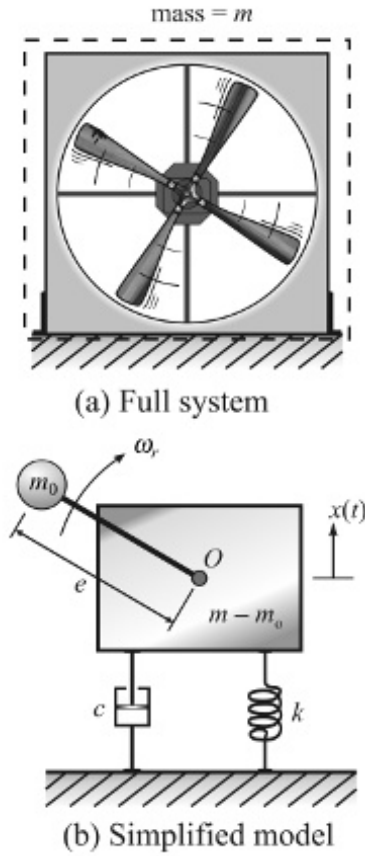


Figure 3.40: Rotating unbalance. The system has a total mass m comprised of a block of mass $(m - m_0)$ and a rotating part of mass m_0 . Figure (a) shows the rotating machine and (b) the simplified model.

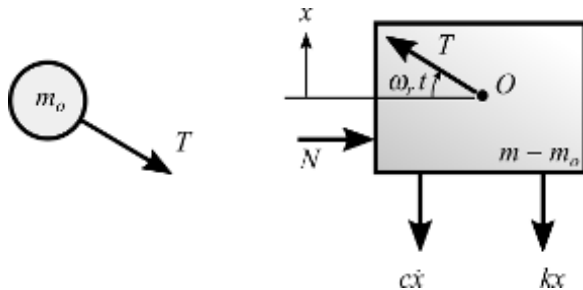


Figure 3.41: Free-body diagrams of the rotating mass and the block.

only in the vertical x direction. The force exerted by the rotating mass can be determined using Newton's second law of motion, in the following way.

The position of the rotating mass with respect to the base is given by the angle $\omega_r t$ and the position of the machine, idealized as a block in the model, is $x(t)$. Therefore, the vertical component of the motion of the rotating mass is $x_r(t) = x(t) + e \sin \omega_r t$.

Figure 3.41 shows a free-body diagram of the block and the rotating mass. The internal force T holds the two bodies together as they move, and is equal in magnitude and opposite in direction in the two diagrams.

Applying Newton's second law of motion to each body in the vertical direction we find

$$-T \sin \omega_r t = m_0 \frac{d^2 x}{dt^2}$$

for the rotating mass and

$$T \sin \omega_r t - c\dot{x} - kx = (m - m_0)\ddot{x}$$

for the block. Combining both equations and eliminating $T \sin \omega_r t$ we have

$$-c\dot{x} - kx = (m - m_0)\ddot{x} + m_0 \frac{d^2}{dt^2}(x + e \sin \omega_r t).$$

Simplifying the right-hand side, rearranging, and dividing by m leads to

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{m_0 e \omega_r^2}{m} \sin \omega_r t.$$

This equation is identical to Equation 3.29 with $A = m_0 e \omega_r^2$. Therefore, the steady-state response is given by

$$x_s(t) = A \sin(\omega_r t - \theta), \quad (3.53)$$

where

$$A = \frac{m_0 e \omega_r^2 / m \omega_n^2}{\sqrt{(1 - \omega_r^2 / \omega_n^2)^2 + (2\zeta\omega_r / \omega_n)^2}},$$

$$\theta = \tan^{-1} \left(\frac{2\zeta\omega_r / \omega_n}{1 - \omega_r^2 / \omega_n^2} \right), \quad 0 \leq \theta \leq \pi.$$

The expression for A can be simplified by setting $m\omega_n^2 = k$, but we leave it in this form since we plot A as a function of frequency ratio ω_r / ω_n .

From the magnitude plot in Figure 3.42, for high speed machinery $\omega_r \gg \omega_n$. The system experiences a resonance with large magnitude response during powering up or down at approximately $\omega_r / \omega_n = 1$. This is true for all systems except those that are highly damped. In a design the goal is for the rotating component to pass through resonance quickly enough that for those brief moments the system vibrates acceptably.

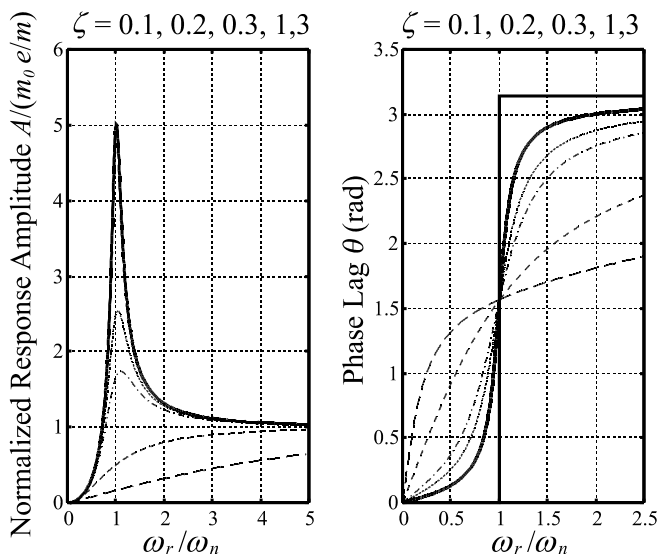


Figure 3.42: (a) Nondimensional magnitude and (b) Phase lag vs. frequency ratio for $\zeta = 0.1$ (—), $\zeta = 0.2$ ($\cdots$), $\zeta = 0.3$ ($\cdot -$), $\zeta = 1.0$ (---), $\zeta = 3.0$ (---).

Example 3.18 Base-Excited Structure with Rotating Unbalance

Combine the effects introduced in Sections 3.7.3 and 3.7.4.

Solution: A schematic of the system is given in Figure 3.43. A free-body diagram for the derivation of the equation of motion would be a combination of free-body diagrams from the two component problems. Linearity permits the superposition of the loadings and responses of the system.

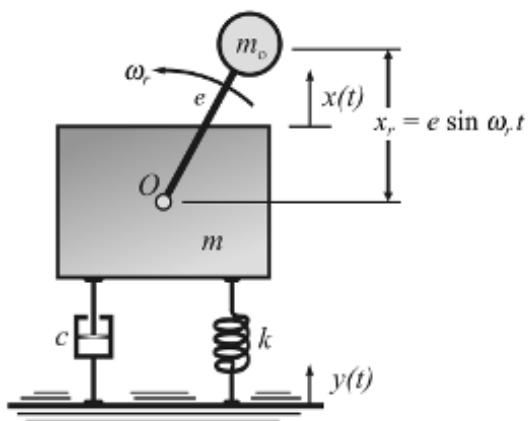


Figure 3.43: The combined problem of a base-excited structure and an unbalanced rotor.

The governing equation is then

$$m\ddot{x} + c\dot{x} + kx = m_0 e \omega_r^2 \sin \omega_r t + c\dot{y} + ky.$$

As before, assuming the harmonic base motion to be

$y(t) = Y \sin \omega_b t$, and dividing through by m leads to

$$\begin{aligned} \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x &= \frac{m_0}{m} e \omega_r^2 \sin \omega_r t + 2\zeta\omega_n\omega_b Y \cos \omega_b t \\ &\quad + \omega_n^2 Y \sin \omega_b t. \end{aligned} \quad (3.54)$$

Assuming a response that includes both harmonic components and then substituting the assumed response into the governing equation allows us to find the coefficients.

A simpler way is to solve for the response to each of the inputs on the right-hand side of Equation 3.54 and then add all three components for the complete solution, $x_s(t) = x_1(t) + x_2(t) + x_3(t)$, where $x_1(t)$, $x_2(t)$, and $x_3(t)$ are governed by

$$\begin{aligned} \ddot{x}_1 + 2\zeta\omega_n\dot{x}_1 + \omega_n^2 x_1 &= \frac{m_0}{m} e \omega_r^2 \sin \omega_r t \\ \ddot{x}_2 + 2\zeta\omega_n\dot{x}_2 + \omega_n^2 x_2 &= 2\zeta\omega_n\omega_b Y \cos \omega_b t \\ \ddot{x}_3 + 2\zeta\omega_n\dot{x}_3 + \omega_n^2 x_3 &= \omega_n^2 Y \sin \omega_b t. \end{aligned}$$

Each of these equations is of the form of Equation 3.29 with the solution given by Equation 3.32. Several resonances are possible. If any of the driving frequencies, ω_r or ω_b , are at or near the natural frequency ω_n , then a significant magnification of the response amplitude may occur. The study of such possible behavior is an important part of any preliminary analysis and design.

■



VEHICLE WHEEL BALANCING.³⁹ In a vehicle such as a car or truck, there are three primary causes of wheel vibration: imbalance, runout, and radial force variation. The goal is to minimize or eliminate all three.

Wheel imbalance is the result of the wheel not being properly balanced both statically and dynamically. Static balance considers only a single plane for wheel imbalance. This is the method that was used in old-fashioned “bubble balancers.” The wheel sits horizontal on a floating spindle and weights are placed around the outside of the rim until the bubble is centered in the glass window in the center of the spindle. Almost any shape object can be balanced this way, whether it is round or not.

Dynamic balance introduces a second plane for weights, which addresses “side-to-side” imbalance, or lateral “wobble” imbalance. This method involves spinning the wheel on a horizontal shaft. The dynamic balancing machine identifies the weights required on both the outboard and inboard rim flanges to cancel both vertical and horizontal imbalances. This method is sometimes advertised as “computer spin-balancing.” Again, almost any object can be balanced this way, whether round or not.

³⁹The authors are grateful to John Hinckley, Hunter Engineering Company, for these notes on wheel vibration.

Runout has two types. In radial runout the rim is out-of-round to some degree relative to its center pilot hole and in lateral runout the rim has some degree of side-to-side misalignment of “wobble” relative to the wheel mounting surface. Either type of runout can cause wheel vibration.

Radial force variation is a third potential cause of vibration. Tires are compressible and have stiffness. Variations in the compressibility and stiffness of the tread, shoulders, and sidewalls of the tire under load cause what is known as radial force variation. It can be measured only while the tire is spinning and the load it normally carries is applied to the tread surface.⁴⁰

A problem plaguing high-speed rotating shafts⁴¹ is unwanted vibration known as *whirling*. In a rotating machine the angular speed corresponding to the natural frequency is called the *critical speed*. The operating speed of the machine should be kept away from the critical speed to avoid shaking and excitation. Small rotating machines are characterized by small mass and high stiffness – their rotors have small mass and their bearings have high stiffness. These machines are relatively small in size and operate at low speed – below their critical speed, called *subcritical operation*.

Large rotating machines – centrifugal compressors, gas and steam turbines, and large electrical generators – pose a different problem. For this type of machine, the mass of the rotor is usually large and the stiffness is limited (there is a practical upper limit to the shaft size that can be used). The machine operates at high speeds, above its critical speed, known as *supercritical operation*. The main problem is that during start-up and shut-down, the machine must pass through its critical speed. To avoid dangerously large amplitudes during these passes, adequate damping must be present in the bearings and foundations.

The excitation in rotating machinery is attributable to rotating unbalanced masses. This can occur from several factors. The center of mass may be different from the center of rotation due to an uneven distribution of mass about the geometric axis of the system. It may be different due to deflection of the shaft caused by the weight of the rotor and exaggerated if the shaft has a bend or bow in it. The bearings may force the shaft to rotate about the geometric center of the bearings, and not the center of mass of the system. These unbalance forces are

⁴⁰Radial force variation issues are addressed on new cars by a process called “match-mounting.” The wheel manufacturer marks the wheel at its lowest point of radial runout, the tire manufacturer marks the tire at its point of highest radial force variation, and those two marks are aligned in the plant when the tire is mounted to the wheel to minimize the radial force variation of the tire/wheel assembly when new.

⁴¹A detailed reference on this subject is **Vibration of Structures and Machines - Practical Aspects**, G. Genta, 2nd ed., Springer-Verlag, 1995.

a function of ω_r^2 and make the design and operation of high-speed machinery a challenge. Balancing is the only method available to tame these excitation forces.

Example 3.19 Rotating Shafts and Whirling⁴²

Derive a simplified governing equation of motion to investigate the whirling phenomenon.

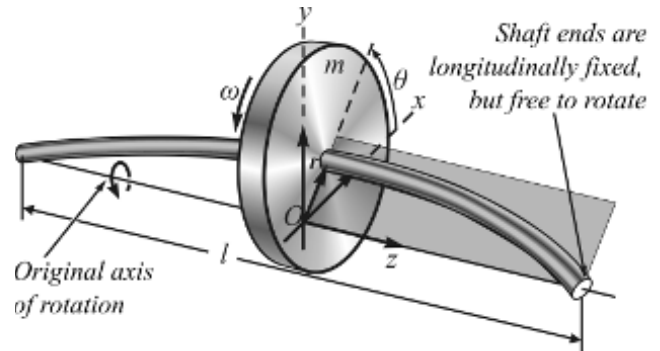


Figure 3.44: An exaggerated schematic of a disk of mass m mounted on a rotating shaft that bends off axis.

Solution: Consider the schematic of a rotating shaft in Figure 3.44 showing a thin disk of mass m attached to a shaft with a mass that is so small when compared to m that it can be ignored. If damping is neglected, the only force acting on m is due to the elastic restoring force of the bending shaft.

The simplest and most useful model in the study of the flexural behavior of rotors is called the Jeffcott⁴³ rotor. The support structure is assumed to be rigid. In addition to the rotation, the shaft will oscillate in bending as a beam, although this behavior is ignored here.

How does the originally straight shaft bend as it rotates due to its operating speed? *Imperfections* in the shaft and the disk can create an unbalance of forces resulting in a net *centripetal* force and a bent shaft. It is the rotation about the bearing axis of the plane that contains the bent shaft, shown shaded in Figure 3.44.

Figure 3.45 shows an exaggerated view of the disk cross section where C_m is the disk center of mass and C_s is the geometric center of the shaft. The distance between them is the eccentricity ε due to imperfections in the disk. If the disk is perfect, the eccentricity equals zero, and, in theory, the shaft does not experience vibration at any speed. In practice, whirling behavior may still occur due to shaft imperfections.

⁴²This example does not involve damping but we include it in this chapter since it is related to the rotating unbalance problem.

⁴³H.H. Jeffcott, “The Lateral Vibration Loaded Shafts in the Neighborhood of a Whirling Speed. The Effect of Want of Balance,” *Philosophical Magazine*, Vol.37, 1919.

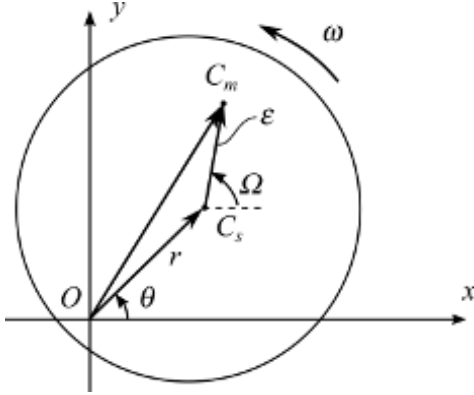


Figure 3.45: An exaggerated cross-sectional view of an eccentric disk. Point C_m is the disk center of mass and point C_s is the geometric center of the shaft.

We define the following position vectors:⁴⁴

$$\begin{aligned}\mathbf{r}_{OC_m} &= (x + \varepsilon \cos \Omega)\mathbf{i} + (y + \varepsilon \sin \Omega)\mathbf{j} \\ \mathbf{r} &= x\mathbf{i} + y\mathbf{j} \\ |\mathbf{r}_{C_s C_m}| &= \varepsilon,\end{aligned}$$

where the unit vectors are $\mathbf{i}$ and $\mathbf{j}$, ω is the rotational speed of the disk, which is considered a general function of time, and $\Omega = \omega t$. Figure 3.45 defines the angular position of C_s by the angle θ , with $\dot{\theta}$ as the rotation speed of the plane formed by the bent shaft and its axis. If $\dot{\theta}$ is a constant then $\theta = \omega t$. In general $\dot{\theta} \neq \omega$.

To derive the equations of motion⁴⁵ in the x and y directions, we use the same procedure as for the rotating unbalance problem, leading to the two governing equations,

$$\begin{aligned}m \frac{d^2}{dt^2} (x + \varepsilon \cos \Omega) + k_x x &= 0 \\ m \frac{d^2}{dt^2} (y + \varepsilon \sin \Omega) + k_y y &= 0,\end{aligned}$$

where k_x and k_y represent the support stiffnesses in the x and y directions, respectively. The two equations governing x and y motions are uncoupled, meaning that in theory motions in the two directions are independent. This is a result of the prior assumptions and may generally not be the case. Carrying out the derivatives on the left-hand side where Ω is a function of time, we find

$$m(\ddot{x} - \varepsilon \dot{\Omega}^2 \cos \Omega - \varepsilon \ddot{\Omega} \sin \Omega) + k_x x = 0 \quad (3.55)$$

$$m(\ddot{y} - \varepsilon \dot{\Omega}^2 \sin \Omega + \varepsilon \ddot{\Omega} \cos \Omega) + k_y y = 0. \quad (3.56)$$

Assuming that ω is constant and that there is *synchronous whirl*, that is, $\dot{\Omega} = \omega$ and therefore $\ddot{\Omega} = \dot{\omega} = 0$,

⁴⁴The notation for the position vector is $\mathbf{r}_{AB}$, representing the displacement from point A to point B .

⁴⁵We follow Genta, pp.218-222, for the derivation and discussion.

Equations 3.55 and 3.56 become

$$m(\ddot{x} - \varepsilon \omega^2 \cos \omega t) + k_x x = 0$$

$$m(\ddot{y} - \varepsilon \omega^2 \sin \omega t) + k_y y = 0.$$

Dividing by m , the equations of motion are

$$\ddot{x} + \omega_{nx}^2 x = \varepsilon \omega^2 \cos \omega t \quad (3.57)$$

$$\ddot{y} + \omega_{ny}^2 y = \varepsilon \omega^2 \sin \omega t, \quad (3.58)$$

which can be solved using the techniques presented in this chapter.

For the case where $k_x = k_y = k$ and therefore $\omega_{nx}^2 = \omega_{ny}^2 = \omega_n^2$, with $z = x + iy$, Equations 3.57 and 3.58 can be added and combined into one complex parametrized equation,

$$\ddot{z} + i\ddot{y} + \omega_n^2(x + iy) = \varepsilon \omega^2(\cos \omega t + i \sin \omega t), \quad (3.59)$$

where each original equation can be recovered by taking the real or imaginary parts of Equation 3.59. From the definition for z and Euler's formula, Equation 3.59 becomes

$$\ddot{z} + \omega_n^2 z = \varepsilon \omega^2 e^{i\omega t},$$

with the particular solution $z = z_0 e^{i\omega t}$, where

$$z_0 = \frac{\varepsilon \omega^2}{\omega_n^2 - \omega^2}. \quad (3.60)$$

The critical value of the disk rotation frequency ω is $\omega_{cr} = \omega_n$. For the undamped case, the *critical flexural speed* is defined as the speed at which the frequency of rotation equals the natural frequency of the system.⁴⁶ If z represents the rotation of the plane of the bent shaft about the bearing axis and rotates with speed ω in the xy plane, Equation 3.60 can be recast as

$$\frac{z_0}{\varepsilon} = \frac{1}{(\omega_n/\omega)^2 - 1},$$

or

$$\frac{z_0}{\varepsilon} = \frac{(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2}.$$

The graph of $|z_0|/\varepsilon$ vs. ω/ω_n is shown in Figure 3.46 and depicts the response magnitude as a function of ω , where the magnitude $|z_0| = \sqrt{x^2 + y^2}$.

In the *subcritical range*, $\omega < \omega_n$, the amplitude z_0 grows from zero and the limit approaches infinity as $\omega \rightarrow \omega_n$ since no damping has been included in the model.

In the *supercritical range*, $\omega > \omega_n$, the amplitude z_0 is always negative, although is shown as positive in Figure 3.46 due to the absolute value. With increasing ω the magnitude of the amplitude decreases monotonically to $-\varepsilon$ as $\omega \rightarrow \infty$. This phenomenon, where the disk begins

⁴⁶The critical speed matches the natural frequency only for the case of no damping.

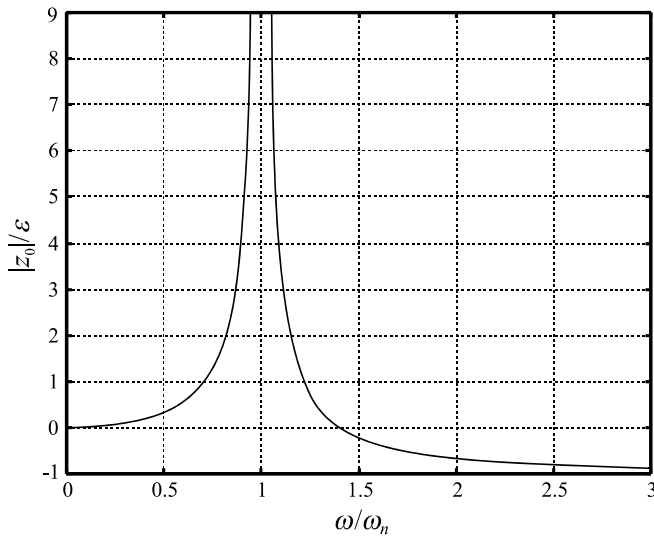


Figure 3.46: Response magnitude $|z_o|/\varepsilon$ vs. frequency ratio ω_n/ω in whirling. Resonance occurs at $\omega = \omega_n$.

to rotate about its mass center, is referred to as *self-centering* since the rotor tends to rotate about its center of mass instead of its geometric center.

When damping is included in the model, there is the added possibility that the system may become unstable due to a phenomenon where energy is fed into the system by the damping, rather than the usual removal of energy. In this case the response amplitude grows exponentially, leading to failure. Such a phenomenon is known as *self-excited whirling*.

An introduction to self-excited oscillations is given in Section 4.9 and Chapter 11. ■

JEAN BAPTISTE JOSEPH FOURIER

(March 21, 1768 – May 16, 1830)



Contribution: Fourier was a French mathematician and physicist best known for functional trigonometric approxima-

tions and their application to problems of heat flow. Fourier's fame is based on his mathematical theory of heat conduction, a theory involving expansions of arbitrary functions in trigonometric series. Although such expansions had been investigated earlier, they bear his name because of his major contributions. Fourier series are used extensively in science and engineering. The Fourier transform is also named in his honor.

Another one of Fourier's contributions is the concept of dimensional homogeneity in equations; that is, an equation can be correct only if the dimensions match on both sides of the equality. Fourier also developed the method of representing physical units, such as velocity and acceleration, by their fundamental dimensions of mass, time, and length, to obtain relations connecting them.

Fourier is generally credited with the discovery of the greenhouse effect.

Life Sketch: Fourier was born in Auxerre, France about 100 miles south of Paris. He was the ninth of twelve children of his father's second marriage. His father was a tailor and had three children from his first marriage. His mother died when he was nine years old and his father died the following year.

Being an orphan, Fourier's life was varied and difficult at times. He studied Latin and French in school and showed great promise. In 1780 he entered a military school run by the Benedictines in Auxerre, the École Royale Militaire of Auxerre, where his talents were in literature but very soon, by the age of thirteen, mathematics became his real passion.

In 1787 Fourier decided to train for the priesthood and entered the Benedictine abbey of St. Benoit-sur-Loire. His interest in mathematics continued and Fourier was unsure if he was making the right decision. In one letter Fourier wrote, "Yesterday was my 21st birthday, at that age Newton and Pascal had already acquired many claims to immortality."

Fourier did not take his religious vows. Having left St. Benoit in 1789, he visited Paris and read a paper on algebraic equations at the French Academy of Sciences. In 1790 he became a teacher at the Benedictine college, École Royale Militaire of Auxerre, where he had studied. Up until this time Fourier had been conflicted about whether he should follow a religious life or one of mathematical research. In 1793 a third element was added to this conflict when he became involved in politics and joined the local Revolutionary Committee.

Fourier took a prominent part in his own district in promoting the French Revolution. He was unhappy about the Terror which resulted from the French Revolution and attempted to resign from the Revolutionary Committee. This proved impossible and Fourier was firmly entangled with the Revolution.

The Revolution was a complicated affair with many factions, with broadly similar aims, violently opposed to each other. Fourier defended members of one faction. In July 1794 he was arrested and imprisoned. He narrowly escaped execution on more than one occasion. After Robespierre (one

of the best-known figures of the French Revolution) went to the guillotine, political changes led to Fourier being freed.

Later in 1794 Fourier was nominated to study at the École Normale in Paris. This institution had been set up for training teachers and it was intended to serve as a model for other teacher-training schools. The school opened in January 1795 and Fourier was the most able of the pupils there. He was taught by Lagrange, Laplace, and other famous individuals.

Fourier began teaching at the Collège de France and, having excellent relations with Lagrange, Laplace, and others, began further mathematical research. He was appointed to a position at the École Centrale des Travaux Publics, which was soon to be renamed the École Polytechnique. However, repercussions of his earlier arrest remained and he was arrested again and imprisoned. His release has been attributed to a variety of causes, including pleas by his pupils, pleas by Lagrange or Laplace, or a change in the political climate.

By September 1795 Fourier was back teaching at the École Polytechnique. In 1797 he succeeded Lagrange as the chair of analysis and mechanics. He was renowned as an outstanding lecturer and it appears he did not undertake original research during this time.

In 1798 Fourier joined Napoleon's army in its invasion of Egypt serving as scientific adviser. Fourier was made governor of Lower Egypt and secretary of the Institut d'Égypte. He acted as an administrator as French-type political institutions were set up. He helped establish educational facilities in Egypt and carried out archaeological explorations.

While in Cairo Fourier helped found the Cairo Institute and was one of the twelve members of the mathematics division. Fourier was elected secretary to the Institute, a position he continued to hold during the entire French occupation of Egypt. Fourier was also put in charge of collating the scientific and literary discoveries made during the time in Egypt. He proposed a grand compendium of all the treasures which had been discovered during the Egyptian campaign. This was the first complete list of its kind ever published.

Napoleon abandoned his army and returned to Paris in 1799. He soon held absolute power in France. Fourier returned to France in 1801 with the remains of the expeditionary force and resumed his post as Professor of Analysis at the École Polytechnique. However, Napoleon had other ideas how Fourier might serve him and appointed him prefect of a department in southern France with headquarters in Grenoble.

Fourier was not happy at the prospect of leaving the academic world and Paris but could not refuse Napoleon's request. He went to Grenoble where his duties were many and varied. His two greatest achievements in this position were overseeing the operation to drain the swamps of Bourgoin and supervising the construction of a new highway from Grenoble to Turin. He also spent time working on the *Description of Egypt*, which was not completed until 1810 when Napoleon made changes, rewriting history in places, before publication. By the time a second edition appeared every

reference to Napoleon was removed.

It was during his time in Grenoble that Fourier did his important mathematical work on the theory of heat. His work on the topic began around 1804 and by 1807 he had completed his important memoir *On the Propagation of Heat in Solid Bodies*. The memoir was read to the Paris Institute in December 1807 and, although very highly regarded today, at the time it caused significant controversy.

The first objection, made by Lagrange and Laplace in 1808, was to Fourier's expansions of functions as trigonometric series, what we now call Fourier series. Further clarification by Fourier still failed to convince them.

The second objection was made by Biot against Fourier's derivation of the equations of transfer of heat. Fourier had not made reference to Biot's 1804 paper on this topic (although Biot's paper was incorrect). Laplace, and later Poisson, had similar objections.

For the 1811 mathematics prize of the Paris Institute, the subject of the competition was the propagation of heat in solid bodies. Fourier submitted his 1807 memoir together with additional work on the cooling of infinite solids and terrestrial and radiant heat. Although Fourier was awarded the prize, his work was given mixed reviews and there was no move in Paris to publish Fourier's work.

When Napoleon was defeated and on his way to exile in Elba, his route should have been through Grenoble. Fourier managed to avoid this difficult confrontation by sending word that it would be dangerous for Napoleon. Following Napoleon's escape from Elba he marched towards Grenoble with an army, Fourier tried to persuade the people of Grenoble to oppose Napoleon and give their allegiance to the King. As Napoleon marched into the town Fourier left in haste.

Fourier returned to Paris and was elected to the Academy of Sciences in 1817. Louis XVIII at first opposed his nomination to the Academy of Sciences because of his association with Napoleon, but relented. Shortly after Fourier became the Secretary to the mathematical section of the Academy in 1822, his prize winning essay *Théorie Analytique de la Chaleur* was published.

In 1824 Fourier described the phenomenon that gases in the atmosphere might increase the surface temperature of the Earth. This effect was later called the greenhouse effect. In 1827 he refined this idea adding that the atmosphere serves to warm a planet. This established the concept of planetary energy balance - that planets obtain energy from a number of sources that cause temperature increase. Planets also lose energy by infrared radiation (that Fourier called "*chaleur obscure*" or "dark heat") with the rate increasing with temperature. A balance is reached between heat gain and heat loss; the atmosphere shifts the balance toward the higher temperatures by slowing the heat loss. Fourier seemed to understand that the rate of infrared radiation increased with temperature, although the Stefan-Boltzmann law which gives the exact form of this dependency (a fourth-power law) was discovered fifty years later.

During Fourier's eight last years in Paris he resumed his mathematical research and published a number of papers in pure and applied mathematics. His life was not without problems, since his theory of heat still provoked controversy. Biot claimed priority over Fourier, a claim which Fourier had little difficulty showing to be false. Poisson attacked both Fourier's mathematical techniques and also claimed to have an alternative theory. Fourier wrote *Historical Précis* as a reply to these claims. Although the work was shown to various mathematicians, it was never published.

The final years of Fourier's life were spent in Paris, where he was Secretary of the Academy of Sciences. He succeeded Laplace as President of the Council of the École Polytechnique. He spent increasingly more time in confinement due to an illness, presumably contracted during his stay in Egypt. Even during this time he continued publishing in mechanics, heat transfer, theory of equations, and statistics. He died in Paris at the age of 62.

Noteworthy: Fourier did his most important scientific work in Grenoble. Since his professional life was almost equally divided between politics and science, his advancement of the frontiers of mathematical science is quite remarkable.

In 1822 he published his *Théorie Analytique de la Chaleur*, in which he based his reasoning on Newton's law of cooling, namely, that the flow of heat between two adjacent molecules is proportional to the extremely small difference of their temperatures. In this work he claimed that any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable. Though this result is incorrect, Fourier's observation that some discontinuous functions are the sum of infinite series was a breakthrough.

Fourier recognized that the Earth primarily gets energy from solar radiation, to which the atmosphere is largely transparent, and that geothermal heat does not contribute much to the energy balance. However, he mistakenly believed that there is a significant contribution of radiation from interplanetary space.

Throughout his career, Fourier won the loyalty of younger friends by his unselfish support and encouragement. Most senior colleagues were impressed with his achievements. A major exception was Poisson, from whom there was continual controversy, criticism, and enmity.

3.8 Forced Periodic Vibration

3.8.1 Harmonic/Spectral Analysis

Any continuous complex periodic motion can be represented by a Fourier series of individual sine and cosine functions which are related harmonically. The summation of these individual sine and cosine waveforms equals – or closely approximates – the original complex wave-

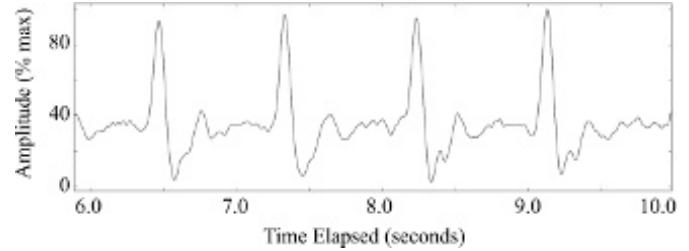


Figure 3.47: The signal from a pulse oximeter is a complex periodic signal that can be decomposed into Fourier components. The graph shows the percent oxygen saturation as a function of time.

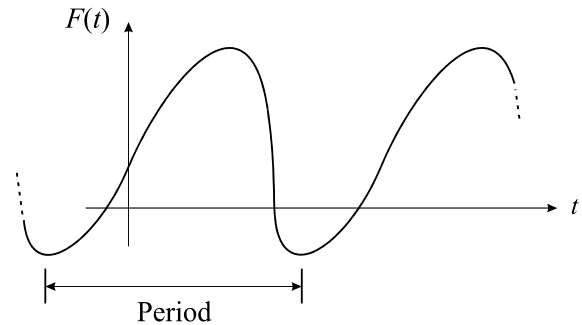


Figure 3.48: Periodic but not harmonic function.

form of the periodic motion.

When the Fourier spectrum is plotted in a graph showing vibration amplitude vs. frequency, one can see which discrete vibration frequencies over the entire frequency spectrum contribute to the vibration signal. Thus, spectral analysis is very useful for troubleshooting vibration problems in mechanical systems. By determining the culprit frequency or frequencies, spectral analysis shows which components of the system may be the source of the vibration problem. Vibration analyzers employ Fast Fourier Transform (FFT) algorithms to perform spectral analysis of vibration signals of very high frequencies.

3.8.2 Fourier Series

Structural loading may be periodic but not harmonic. In such a case, there exists a finite period of time after which the waveform of the loading repeats itself. Although the shape of the waveform is not representable by a single sine or cosine, a *Fourier series* can represent such periodic functions in terms of the summation of an infinite number of sines and cosines.

The Fourier series of an arbitrary, periodic function, $f(t)$, is given by

$$f(t) = \frac{a_0}{2} + \sum_{p=1}^{\infty} (a_p \cos p\omega_T t + b_p \sin p\omega_T t),$$

$$\omega_T = \frac{2\pi}{T},$$

where $p = 1, 2, 3, \dots, \omega_T$ is the fundamental frequency of the function, T is the period of the function, and $a_0/2$ is the average value of the periodic waveform. The Fourier coefficients a_p and b_p are derivable using the orthogonality property of sine and cosine functions:

$$a_p = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos p\omega_T t \, dt \quad p = 0, 1, 2, \dots$$

$$b_p = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin p\omega_T t \, dt \quad p = 1, 2, \dots$$

If the function $f(t)$ is odd, then the coefficients $a_p = 0$. If $f(t)$ is even, then the coefficients $b_p = 0$. In theory, an infinite number of harmonic components are required to duplicate the waveform $f(t)$. In practice, the finite sum of only a few terms (usually less than 5) can accurately represent most periodic functions. The Fourier coefficients are a measure of participation of each harmonic component in the waveform.

By the principle of linear superposition, we can solve problems of periodic forcing using the procedures and results of Section 3.7. The governing equation of motion for the mass-spring-damper system with periodic excitation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$

$$= \frac{1}{m} \left[\frac{a_0}{2} + \sum_{p=1}^{\infty} (a_p \cos p\omega_T t + b_p \sin p\omega_T t) \right],$$

where $F(t)$ is the periodic external force. To solve, we superpose the response due to each harmonic contribution to obtain the complete response.

If we define $x_p(t)$ to be the response for terms with index value p , then the complete steady-state response is the sum over all values of p ,

$$x_s(t) = \frac{1}{k} \frac{a_0}{2} + \sum_{p=1}^{\infty} x_p(t)$$

$$= \frac{1}{k} \frac{a_0}{2} + \sum_{p=1}^{\infty} A_p [a_p \cos(p\omega_T t - \theta_p) + b_p \sin(p\omega_T t - \theta_p)],$$

where

$$A_p = \frac{1/k}{\sqrt{(1 - p^2\omega_T^2/\omega_n^2)^2 + (2\zeta p\omega_T/\omega_n)^2}}$$

$$\theta_p = \tan^{-1} \left[\frac{2\zeta p\omega_T/\omega_n}{1 - (p\omega_T/\omega_n)^2} \right].$$

Since the mass is loaded by harmonics of frequency ω_T , then a resonant-type response occurs if in *one or more* harmonic functions $p\omega_T = \omega_n$. It is sufficient that only

one such equality, or very-near equality, occur for resonance. However, with sufficient damping, the problem may be insignificant. As a practical matter, resonances in the higher frequencies tend to be less important for many applications since they contain less energy than those in the lower frequencies. The application dictates the importance of resonant behavior.

Example 3.20 Response to a Square Wave Using Fourier Series⁴⁷

Consider the simple undamped oscillator $\ddot{x} + \omega_n^2 x = F(t)/m$, where $F(t)$ is the force represented by the *square wave* function in Figure 3.49. The square wave is the simplest of all non-harmonic, periodic functions since the amplitude over each half-period is constant. Solve for the response using the Fourier series representation for $F(t)$, assuming zero initial conditions.

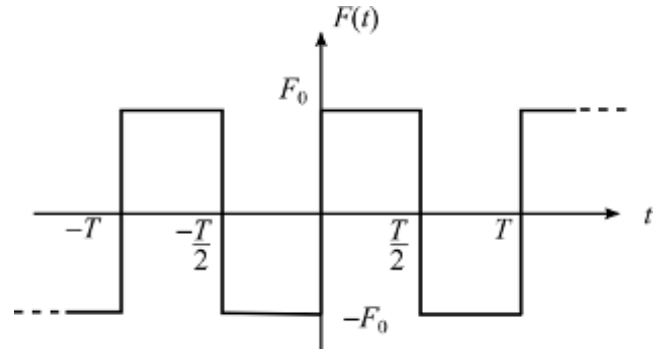


Figure 3.49: A square wave forcing function $F(t)$.

Solution: The square wave is defined by

$$F(t) = \begin{cases} -F_0 & \text{if } -T/2 < t < 0 \\ F_0 & \text{if } 0 < t < T/2 \end{cases}$$

for one period of the function. Since $F(t)$ is an odd function, the a_p coefficients equal zero. The b_p coefficients are evaluated as follows,

$$b_p = \frac{2}{T} \left[\int_{-T/2}^0 F(t) \sin p\omega_T t \, dt + \int_0^{T/2} F(t) \sin p\omega_T t \, dt \right]$$

$$= \frac{4}{T} \int_0^{T/2} F_0 \sin p\omega_T t \, dt$$

$$= \frac{4F_0}{T p \omega_T} [1 - \cos(p\omega_T T/2)],$$

where $\omega_T = 2\pi/T$ and the cosine term can be simplified as $\cos(p\omega_T T/2) = \cos p\pi = (-1)^p$. Therefore,

$$b_p = \frac{2F_0}{p\pi} [1 - (-1)^p] = \frac{4F_0}{p\pi}, \quad p \text{ odd},$$

⁴⁷This undamped problem is included in this chapter to demonstrate the approach and character of the response. With damping the analysis is more complicated.

and zero otherwise. The square wave loading is given by the series

$$F(t) = \frac{4F_0}{\pi} \sum_{p=1,3,\dots}^{\infty} \frac{1}{p} \sin p\omega_T t.$$

The equation of motion is then

$$\ddot{x} + \omega_n^2 x = \frac{4F_0}{m\pi} \sum_{p=1,3,\dots}^{\infty} \frac{1}{p} \sin p\omega_T t.$$

The complete response to this infinite series of harmonic loads is

$$x(t) = c_1 \sin \omega_n t + c_2 \cos \omega_n t + \frac{4F_0}{\pi m \omega_n^2} \sum_{p \text{ odd}} \frac{\sin p\omega_T t}{p[1 - (p\omega_T/\omega_n)^2]}. \quad (3.61)$$

Applying the initial conditions, $x(0) = 0$ and $\dot{x}(0) = 0$, we find that both constants of integration are equal to zero. This is not always true. Let us see how this comes about,

$$\begin{aligned} x(0) &= c_2 = 0 \\ \dot{x}(0) &= \omega_n c_1 + \frac{4F_0}{m\pi} \sum_{p=1,3,\dots} \frac{p\omega_T}{p(\omega_n^2 - p^2\omega_T^2)} \\ c_1 &= -\frac{4F_0}{m\pi} \sum_{p=1,3,\dots} \frac{\omega_T/\omega_n}{(\omega_n^2 - p^2\omega_T^2)}. \end{aligned}$$

In this example, for $\omega_n = 2\omega_T$,

$$c_1 = -\frac{4F_0}{\pi m \omega_n^2} \sum_{p=1,3,\dots} \frac{1}{2} \frac{1}{(1 - p^2/4)},$$

which converges to zero (shown below). Then, with $m\omega_n^2 = k$,

$$\begin{aligned} c_1 &= -\frac{4F_0}{\pi k} \sum_{p=1,3,\dots} \frac{4}{2} \frac{1}{(4 - p^2)} \\ &= -\frac{8F_0}{\pi k} \sum_{p=1,3,\dots} \frac{1}{(4 - p^2)}. \end{aligned}$$

The following calculation shows that $\sum_{p=1,3,\dots} 1/(4 - p^2) = 0$, so that $c_1 = 0$,

$$\sum_{p=1,3,\dots} \frac{1}{(4 - p^2)} = \frac{1}{4} \sum_{p=1,3,\dots} \frac{1}{(2 - p)} + \frac{1}{(2 + p)},$$

where

$$\begin{aligned} \sum_{p=1,3,\dots} \frac{1}{(2 - p)} &= - \sum_{p=1,3,\dots} \frac{1}{(p - 2)} \\ &= - \left(\frac{1}{-1} + \frac{1}{1} + \sum_{p=5,7,\dots} \frac{1}{(p - 2)} \right) \\ &= - \sum_{p=5,7,\dots} \frac{1}{(p - 2)} = - \sum_{p=1,3,\dots} \frac{1}{(p + 2)}. \end{aligned}$$

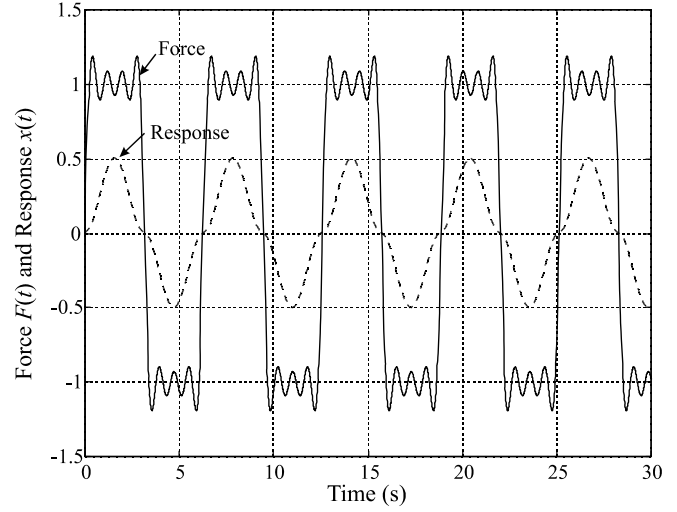


Figure 3.50: Loading and response time histories for the case $\omega_n = 2\omega_T$, $m = 1$ kg, and $k = 4$ N/m using a four term series for both loading and response. Note how the response curve changes its inflection when the force transitions between positive and negative peak values.

Then,

$$\begin{aligned} &\sum_{p=1,3,\dots} \frac{1}{(2 - p)} + \frac{1}{(2 + p)} \\ &= - \sum_{p=1,3,\dots} \frac{1}{(p + 2)} + \sum_{p=1,3,\dots} \frac{1}{(p + 2)} = 0. \end{aligned}$$

Therefore, $c_1 = 0$.

Equation 3.61 shows that resonances occur for $p\omega_T = \omega_n$. Figure 3.50 shows the time histories of the loading and the response using four terms in the respective Fourier series representations. The response was obtained for the case $\omega_n = 2\omega_T$, $m = 1/\pi$ kg, and $k = 4$ N/m. Although only four terms were used in the series for the force, the response turns out to be very close to exact. Had there been a response component at $2\omega_T$, this undamped system would have infinite amplification. (This would not occur with damping.) If $\omega_n = 6\omega_T$, there would be a magnification about that frequency. However, since there is relatively little energy driving the system at the higher frequencies, amplification may not be a concern. ■

3.9 Concepts Summary

This chapter has introduced the physical concept of damping, its sources, and a few of the most commonly used mathematical models of damping in vibration studies. The inclusion of the damping force in differential equations of motion provides a more realistic model of

vibration. We considered two damping force models: the viscous damping model (of primary importance) and the Coulomb damping model.

This chapter has focused on single degree-of-freedom vibration with damping. Underdamped motion is the meaningful case for vibration. Also of importance is the case in which the forcing and natural frequencies are equal or nearly equal, leading to resonance. The forcing cases examined are harmonic loading and periodic, non-harmonic loading. In the latter case, Fourier series representations can be used to express the loading in terms of harmonic functions. Then, by the principle of linear superposition, the response can be found from the summation of harmonic response solutions.

3.10 Quotes

- “When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely in your thoughts advanced to the state of science.” (Lord Kelvin)
- “And in the end, it’s not the years in your life that count. It’s the life in your years.” (Abraham Lincoln)
- “Everything that can be invented has been invented.” (Charles H. Duell, Commissioner, U.S. Office of Patents, 1899)
- “Open-mindedness is very important to the scientist. Without it, there is only a clash of personalities, not ideas. It is important that a scientist be open-minded and skeptical. It is not enough to be skeptical, and it is not enough to be open-minded. Strive to be both.” (Paul G. Hewitt)
- “If everyone is thinking alike, someone isn’t thinking.” (General George Patton Jr.)
- “The one who follows the crowd will usually get no further than the crowd. The one who walks alone is likely to find himself in places no one has ever been.” (Albert Einstein)
- “Creativity is just connecting things. When you ask creative people how they did something, they feel a little guilty because they didn’t really do it, they just saw something. It seemed obvious to them after a while. That’s because they were able to connect experiences they’ve had and synthesize new things.” (Steve Jobs)
- “Sometimes it is better to react with no reaction.”

3.11 Problems

Problems for Section 3.4 – Free Vibration with Viscous Damping

1. Solve Equation 3.6, $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$ for $\zeta > 1$.
2. Derive Equation 3.9, $x(t) = Ce^{-\zeta\omega_nt} \cos(\omega_d t - \phi)$.
3. (a) The response of an oscillator with mass $m = 1$ kg is shown in Figure 3.51. What can you say about the properties of the oscillator and the response? Determine the equation of motion and the initial conditions.

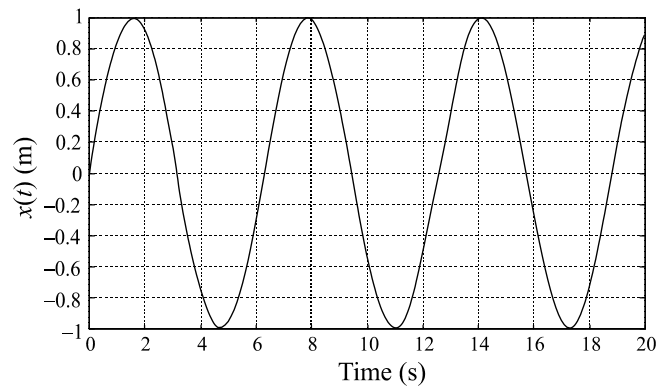


Figure 3.51: Free vibration for Problem 3(a).

- (b) One property is changed, with all others remaining the same. The resulting time history now looks like Figure 3.52. What can you say about the properties of the oscillator and the response? Determine the equation of motion and the initial conditions.

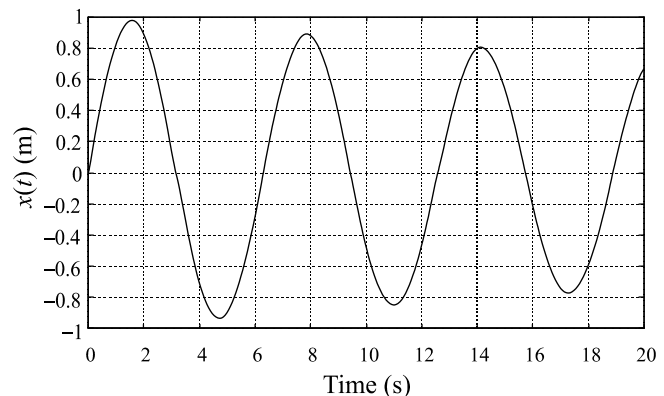


Figure 3.52: Free vibration with one modification for Problem 3(b).

4. (a) Derive the equation of motion for a mass-spring-damper system in free vibration. Solve for the transient response $x(t)$ that is driven by the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. (b) Assume parameter values: $k = 1$ lb/in, weight $mg = 100$ lb,

$x_0 = 0$, $v_0 = 10$ in/s. Vary the damping constant c so that both underdamped and overdamped responses can be demonstrated. Try values of c such that the cases $\zeta = 0.1$ and $\zeta = 0.9$ are obtained. (c) For the case with $\zeta = 0.1$, vary the initial velocity v_0 and study the variation of the first intercept (zero displacement) as a function of initial velocity.

5. A mass-spring-damper system is tested to determine the value of c . Assume $k = 10$ lb/in and $m = 2$ slug. (a) If the vibrational amplitude is observed to decrease to 33% of its initial value after 2 consecutive cycles, what is the value of c ? (b) If k , rather than having the exact value given above, has a range of possible values of 8 to 12 lb/in with equal likelihood, then what is the range of possible values of c ?
6. A periodic torque with a maximum value of 1.0 N-m at a frequency of 5.0 rad/s is applied to a flywheel suspended from a wire. The flywheel has a moment of inertia of 0.10 kg-m² and the wire has a stiffness of 1 N-m/rad. A viscous component applies a damping couple of 0.5 N-m at an angular velocity of 1 rad/s. Calculate (a) the amplitude of the angular displacement from the rest position during the steady-state response, and (b) the amplitude of the moment applied to the viscous component during the steady-state response.
7. A periodic force $F(t) = A \cos \omega t$ is applied to the damped system shown in Figure 3.53. Calculate the work done in one cycle during the steady-state response where $A = 50$ N, $\omega = 10$ rad/s, $k = 6$ kN/m, $c = 200$ N-s/m, and $m = 25$ kg.

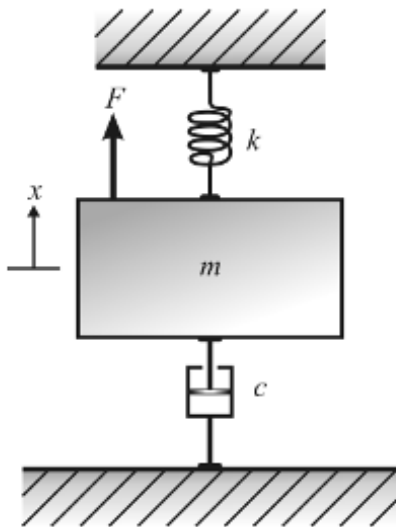


Figure 3.53: Mass suspended between a spring and a damper.

Table 3.1: Cases for t_{max} in Problem 9.

Case	$x(0)$ (in)	$\dot{x}(0)$ (in/s)	ζ	ω_n (rad/s)
1	0	5	0.1	2
2	5	0	0.1	2
3	5	5	0.1	2
4	5	5	0.5	2
5	5	5	1.1	2
6	5	5	0.1	4
7	5	5	0.1	10

quired for a structural oscillation amplitude to decay to $x\%$ of its maximum. (a) What is the expression for the logarithmic decrement in terms of n and x ? (b) Solve for n in terms of ζ and x . (c) Plot n as a function of ζ for (i) $x = 70\%$, (ii) $x = 50\%$, (iii) $x = 20\%$ using the exact formula, and then using the approximate formula for $\zeta = 0.1$ and $\zeta = 0.5$.

9. Solve for t_{max} in Equation 3.17 in general for arbitrary variables. Then evaluate t_{max} numerically for the cases in Table 3.1. Plot each case and discuss trends.
10. Coulomb used the following method to determine the viscosity of liquids. A thin plate of weight W is suspended vertically and set into motion, first in air and then fully submerged in the particular liquid as shown in Figure 3.54. The time duration, t_1 in air and t_2 in the liquid, required for one oscillation is measured. The frictional force between the plate and fluid is estimated as $2Acv$, where $2A$ is the area of both sides of the plate, c is the coefficient of viscosity, and v is the plate velocity. The frictional force between the plate and air is negligible. Find the value of c in terms of the above parameters.

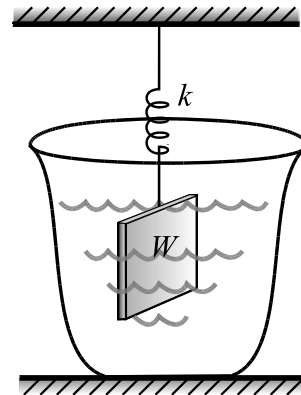


Figure 3.54: Plate suspended in a liquid.

8. This problem investigates the number of cycles n re-
11. The bell-crank mechanism depicted in Figure 3.55 is

rotated slightly and released to oscillate in free vibration. Derive the damped frequency of oscillation ω_d and the critical damping constant c_{cr} . The mass of the frame can be ignored compared to the mass attached to it at the end.

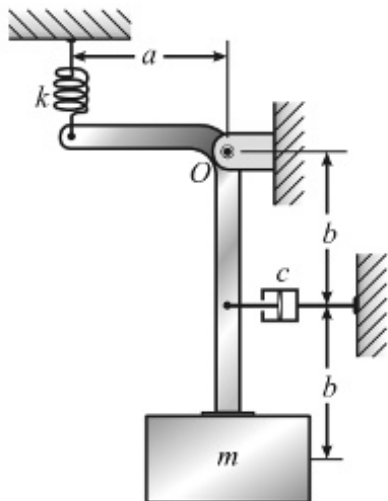


Figure 3.55: Bell-crank mechanism.

12. An automobile's suspension system determines the vehicle's response to inputs from uneven road surfaces. Figure 3.56 shows a drawing of a suspension system that includes a coil spring, a shock absorber, and connecting links to the car frame and the wheel. Next to this artist's rendering is a simple model that can be used to estimate the vertical oscillations of the system. From a static state, the system is given an initial velocity of $\dot{x}(0) = 15$ m/s. Calculate the response $x(t)$ of the simplified system for the following parameter values: $m = 40$ kg, $k = 25$ kN/m, and $c = 2.4$ kN-s/m. There are no external forces acting on the wheel.

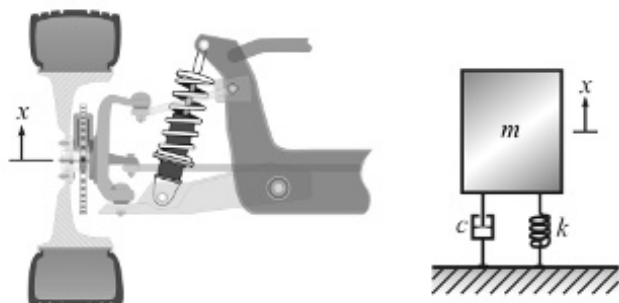


Figure 3.56: A cross section of an automotive suspension system consisting of a coil spring, a shock absorber, and connecting links to the car frame and the wheel. Next to this sketch is a single degree-of-freedom simplified model.

Problems for Section 3.7 – Harmonic Excitation and Damped Response

13. For the system in Figure 3.57, the forcing per unit mass is $F(t)/m = 10 \sin 15t$ ft/s². Solve for the transient response, the steady-state response, and the constants of integration for the following initial conditions.

- (a) $x(0) = 0$ in, $\dot{x}(0) = 5$ in/s
- (b) $x(0) = 5$ in, $\dot{x}(0) = 0$ in/s
- (c) $x(0) = 5$ in, $\dot{x}(0) = 5$ in/s.

For the parameter values $W = 40$ lb, $c = 0.85$ lb-s/in, $k = 12.5$ lb/in, write down the complete response. Plot the transient and steady-state responses separately, then plot both of these with the complete solution on the same graph. Compare the results and discuss the significance of the initial conditions on the total response as time evolves.

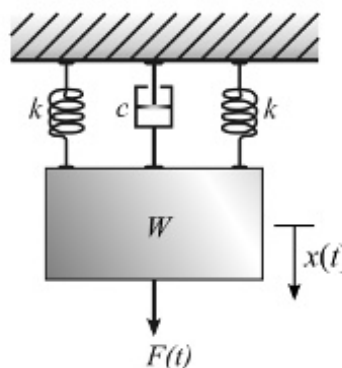


Figure 3.57: Harmonically driven oscillator.

14. For a mass-spring-damper system under harmonic loading per unit mass $F(t)/m = \cos \omega t$ ft/s², solve for the response amplitude for the case where $k = 20$ lb/in, $W = 40$ lb, and then plot β for a broad range of driving frequencies ω . If $0 \leq \omega \leq 3\omega_n$, discuss design considerations for the system as the force is varied from the rest state ($\omega = 0$) to its highest frequency ($\omega = 3\omega_n$). Consider the damping cases: $\zeta = 0.01, 0.1, 0.5$. Discuss parameters of possible concern such as x_{max} and whether material yielding could be of concern.
15. Solve Problem 14 for $k = 40$ lb/in and $W = 20$ lb.
16. Solve Problem 14 for $k = 20$ lb/in and $W = 20$ lb.
17. For Problem 14 where $k = 20$ lb/in and $W = 40$ lb, solve for the steady-state responses in ft/s² to the following excitation forces per unit mass:
- (a) $F(t)/m = 100 \sin(0.9\omega_n t)$
 - (b) $F(t)/m = 100 \sin(\omega_n t)$

- (c) $F(t)/m = 100 \sin(1.1\omega_n t)$
 (d) $F(t)/m = 100 \sin(0.5\omega_n t)$
 (e) $F(t)/m = 100 \sin(2\omega_n t)$.

Plot and discuss each case.

18. For what range of frequency ratios ω/ω_n will the magnification factor β be greater than 1 for the idealized model of Figure 3.58? For what range of frequency ratios will the magnification factor be greater than 86% of the peak response? Let $m = 1$ slug, $k = 9$ lb/in, and $\zeta = 0.15$. Show the cases on a plot of the magnification factor vs. frequency ratio.

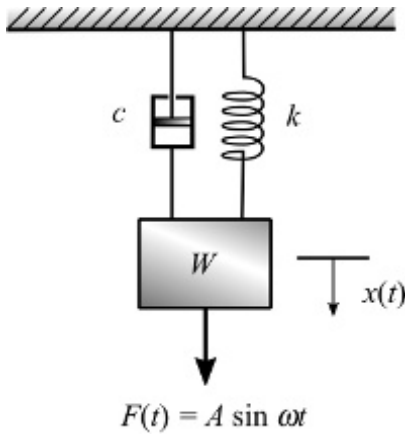


Figure 3.58: Harmonically driven oscillator.

19. For the system of Figure 3.58, what do the initial conditions have to be so that both integration constants C and ϕ equal zero in Equation 3.37?
20. Derive C and ϕ to verify Equations 3.38 and 3.39.
21. For each case in Table 3.2, for the governing equation $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = A \cos \omega t$, evaluate C and ϕ , and B_1 and B_2 in Equation 3.37. Then plot the free and forced responses separately and then in sum. Let $A = 1$ cm/s² and $\omega = 1$ rad/s.
22. Derive Equations 3.44 and 3.45:

$$|G(i\omega)| = \frac{1}{k\sqrt{(1 - (\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

$$\phi = \tan^{-1} \left[\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} \right].$$

23. Derive B_1 and B_2 in Equations 3.48 and 3.49:

$$-\omega_b^2 B_1 + 2\zeta\omega_n\omega_b B_2 + \omega_n^2 B_1 = A_1$$

$$-\omega_b^2 B_2 - 2\zeta\omega_n\omega_b B_1 + \omega_n^2 B_2 = A_2.$$

Table 3.2: Cases for Damped Harmonic Oscillators in Problem 21.

	ω_n (rad/s)	ζ	$x(0)$ (cm)	$\dot{x}(0)$ (cm/s)
(a)	0.5	0.1	1	1
(b)	1	0.1	1	1
(c)	10	0.1	1	1
(d)	0.5	0.1	0	1
(e)	0.5	0.1	10	1
(f)	0.5	0.1	1	0
(g)	0.5	0.1	1	10
(h)	0.5	0.01	1	1
(i)	0.5	0.95	1	1

24. Plot F_T/kY , Equation 3.52, vs. ω_b/ω_n for $\zeta = 0.05, 0.10$, and 0.25 .
25. The base-excited system of Figure 3.59 is driven by a force with frequencies in the range $1.0 \leq \omega_b/\omega_n \leq 2.0$. For parameter values $m = 1$ kg, $k = 9$ N/cm, and $Y = 10$ cm, select a value of c so that the average magnitude of the relative motion equals 1 cm over the frequency range.

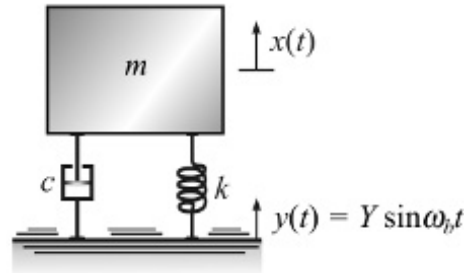


Figure 3.59: A model of a base-excited system.

26. Solve for the response of the base-excited system governed by $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 2\zeta\omega_n\dot{y} + \omega_n^2y$, where $y(t) = A \exp(i\omega_b t)$ for the response $x(t) = X(i\omega) \exp(i\omega t)$. Plot the response for parameter values: $A = 1$, $\zeta = 0.1$, $\omega_b = 1$ rad/s, and $\omega_n = 0.5$ rad/s.
27. The response of a system excited at its base by $y(t) = Y \sin \omega t$ is given by

$$x(t) = X \sin(\omega_b t - \phi),$$

with

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta\omega_b/\omega_n)^2}{(1 - \omega_b^2/\omega_n^2)^2 + (2\zeta\omega_b/\omega_n)^2} \right]^{1/2}.$$

Use Figure 3.38 for displacement transmissibility in answering the following questions. (Do not solve any equations.) Assume negligible damping.

- (a) Assume that the maximum amplitude of the base is $Y = 1$ cm. A design for a rotating machine with this value of Y and with natural frequency $\omega_n = 3$ rad/s is required so that $X \leq 2$ cm across all frequencies ω_b . Which ζ value should be chosen?
- (b) Suppose that instead of the criteria in (a), we require $X \leq 1$ cm and $\zeta < 1.0$. The machine operates at $\omega_b = 9$ rad/s. What are the options for satisfying this criterion?
- (c) What is the equation for the maximum force transmitted to the base in terms of ζ , (ω_b/ω_n) , k , and Y ? For $\zeta = 1.0$, what is the force transmissibility (F_T/kY) when the machine is running at $\omega_b \simeq 0$, $\omega_b = 3\sqrt{2}$ rad/s, and $\omega_b = 9$ rad/s?
28. A vibration sensor has a damping ratio $\zeta = 0.55$ and a natural frequency $f_n = 10$ Hz. Estimate the lowest frequency that can be measured with 1% error.
29. An engine of mass 200 kg is to be supported on four helical springs. When the engine speed is 1000 rpm there is a primary vertical periodic force of maximum value 350 N due to the unbalanced reciprocating masses. Assume that the engine vibrates in the vertical direction with negligible damping, and there is no horizontal or angular movement. (a) Find the stiffness of each spring to limit the maximum total periodic force on the foundation to 25 N. (b) What will be the amplitude of vibration of the engine when its speed is 500 rpm? Use the natural frequency found in part (a).
30. A rotating machine component has an eccentricity of approximately $e = 0.1$ mm in its center of mass resulting in a harmonic load on the structure. Assume, using the notation of the chapter, that $m_0/m = 0.05$, operating frequency $\omega_r = 100$ rad/s, $\zeta = 0.10$, and $\omega_n = 10$ rad/s. (a) Solve for the amplitude and phase of the response in the form of Equation 3.53. (b) Suppose amplitude A must be reduced by 15%. What options exist if actively controlling the structure is not one? Discuss which parameters need to be changed.
31. Derive the general equations of motion, Equations 3.55 and 3.56,

$$m(\ddot{x} - \varepsilon\dot{\Omega}^2 \cos \Omega - \varepsilon\ddot{\Omega} \sin \Omega) + k_x x = 0$$

$$m(\ddot{y} - \varepsilon\dot{\Omega}^2 \sin \Omega + \varepsilon\ddot{\Omega} \cos \Omega) + k_y y = 0.$$

32. A simple model of a seismometer is shown in Figure 3.60. The top view shows how a pen attached to the seismometer mass traces out a pattern on a paper grid that is on a roller. The side view provides more details of the model. The absolute motion of

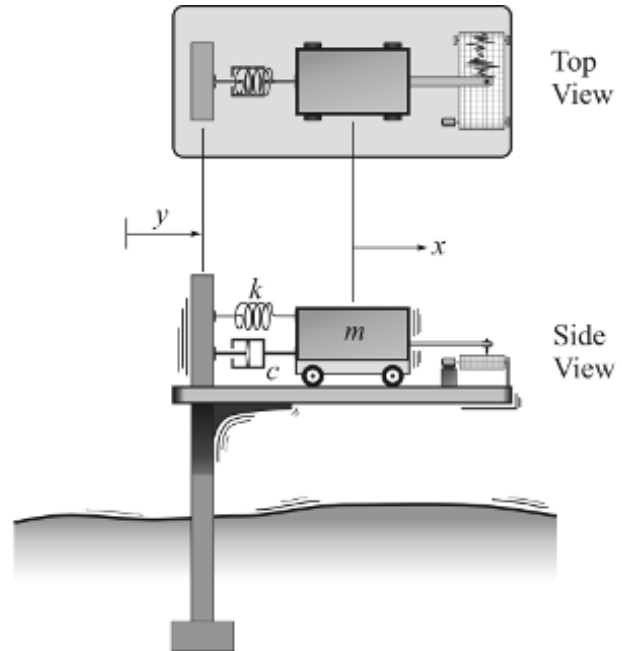


Figure 3.60: A schematic of a seismograph. Ground shaking oscillates the base, which then shakes the body, the motion of which is recorded on the drum.

the ground is defined by $y(t)$, the relative motion of the mass recorded by the seismometer is $x(t)$, and parameter values are $m = 1$ kg, $c = 3$ N-s/m, and $k = 15$ N/m. The seismometer is initially stationary until ground motion initiates motion, governed by the differential equation,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t)/m,$$

where

$$F(t) = -\ddot{y} = a\omega_b^2 \sin \omega_b t,$$

with $a = 15$ mm and $\omega_b = 3$ rad/s. Derive the response in general, and then for the specific parameter values. Discuss the possibilities of resonance.

Problems for Section 3.8 – Periodic but Not Harmonic Excitation

33. A machine is loaded by a periodic “sawtooth” shaped force, as depicted in Figure 3.61. The load is assumed to have existed for a very long time. Model this force using: (a) a one-term Fourier series, (b) a three-term Fourier series, and (c) a five-term Fourier series. In each case solve for the response as a function of time. System properties are $m = 1$ kg, $k = 9$ N/cm, $\zeta = 0.15$, $T = 1$ s, and $A = 1$ cm.

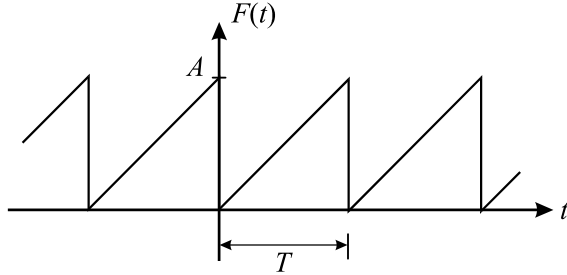


Figure 3.61: Sawtooth loading.

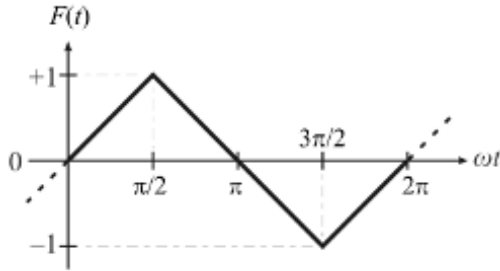


Figure 3.62: Triangle forcing function.

34. Solve Problem 33 with (a) $T = 2$ s, and (b) $T = 0.5$ s.
35. Solve Problem 33 with $A = 2$ cm.
36. Solve Problem 33 with $\zeta = 1.0$.
37. A body is subjected to the two harmonic motions,

$$x_1(t) = 100 \sin\left(\omega t + \frac{\pi}{3}\right) \text{ mm}$$

$$x_2(t) = 50 \sin\left(\omega t + \frac{\pi}{6}\right) \text{ mm}.$$

What additional harmonic motion should be applied to the body to bring it to static equilibrium?

38. An undamped system is forced by the periodic function shown in [Figure 3.62](#). Derive the Fourier series representation for this function, and solve for the response assuming zero initial conditions.
39. For [Example 3.20](#), plot $F(t)$ and $x(t)$ for the case $\omega_n = 6\omega_T$. Compare the results with those of [Figure 3.50](#) and discuss. Assume $T = 1$ s, $F_0 = 1$ N, $k = 4$ N/m, and $\zeta = 0.1$.
40. Solve [Example 3.20](#) where the structure has viscous damping, that is, the governing equation of motion is $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = F(t)$, the forcing $F(t)$ is the same square wave, and $\omega_n = 2\omega_T$ rad/s.
41. Solve [Example 3.20](#), as in Problem 40, with $\omega_n = 4\omega_T$ rad/s.

42. A base-excited structure is governed by the equation $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 2\zeta\omega_n\dot{y} + \omega_n^2y$, where $y(t)$ is the base motion. Solve for the response if the base motion is given by the sawtooth function shown in [Figure 3.61](#).

43. Solve Problem 42 where the input is given by the square wave of [Figure 3.63](#).

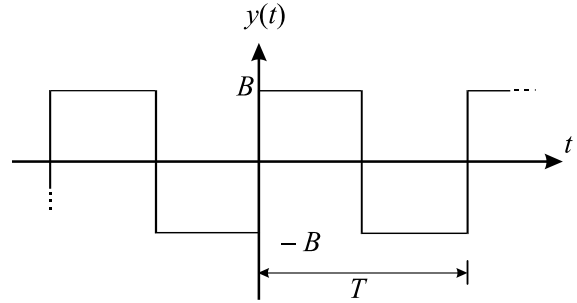
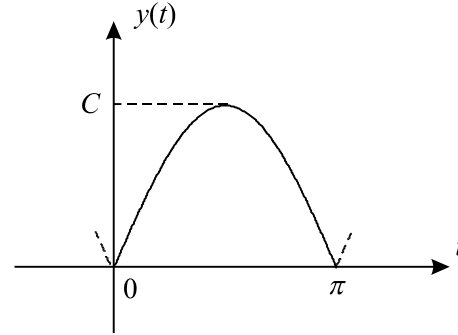


Figure 3.63: Square wave base motion.

44. Solve Problem 42 where the input is given by the function $\sin t$, $0 \leq t \leq \pi$, shown in [Figure 3.64](#), repeated periodically.

Figure 3.64: Base motion driven by $\sin t$.



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Chapter 4

Single Degree-of-Freedom Vibration: General Loading and Advanced Topics

“Complexities begin to arise.”

This chapter continues our study of single degree-of-freedom systems and considers more general loading cases as well as several advanced topics. We go beyond harmonic loading to include impulse, step, and arbitrary loading cases. We introduce the Laplace transform technique as well as the convolution integral. We demonstrate the use of Lagrange’s equation for the derivation of the equation of motion as an alternative to Newton’s second law. We consider how to account for uncertainties in deriving and solving the equation of motion.

The possibility of modifying the response amplitude via the use of control theory is introduced. In addition, the concept of the inverse problem, that is, the estimation of the system parameters or loading based on the response data, is examined.

Each of these topics could fill many volumes. The intent here is to give the reader sufficient background to appreciate that even linear single degree-of-freedom systems can be challenging and very useful in practice.

4.1 Arbitrary Loading: Laplace Transform

The *Laplace transform* is one of a powerful group of techniques known as *transform methods*. The Laplace transform¹ of the time-domain function $x(t)$ is defined by

$$\mathcal{L}[x(t)] \equiv X(s) = \int_0^{\infty} e^{-st} x(t) dt, \quad (4.1)$$

where the Laplace variable, s , is a complex quantity, $s = \sigma + i\omega$ with σ and ω real. Notationally, any function with s as its argument is a Laplace transformed variable. The complex domain showing complex conjugate locations of the Laplace variable s is depicted in Figure 4.1.

¹Appendix A provides details on the Laplace transform.

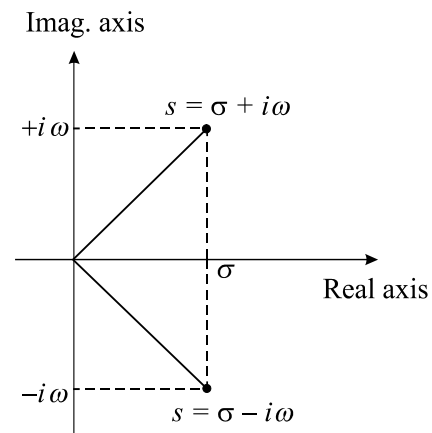


Figure 4.1: Locations of complex conjugate Laplace variable $s = \sigma \pm i\omega$ in the complex plane.

In vibration analysis, the purpose for using the Laplace transform is to transform the governing differential equation of motion into an equivalent *algebraic* equation. This is accomplished by transforming the differential equation from the time-domain to the Laplace-domain. The transform from t to s is relatively straightforward compared to the inverse transform from s to t . In the complex s domain, the algebraic relation is solved for the transform of the response. Then, it is necessary to transform back to the time-domain to obtain the solution in terms of t . The inverse transform can be intricate, requiring *complex variable methods*, but we will rely on tabulated transforms.²

Using the Laplace transform technique to solve an ordinary differential equation also accounts for the initial conditions. Consider the damped oscillator driven by an arbitrary force, $F(t)$, governed by Equation 3.28,

²One should not lose sight of a fundamental nature of problem solving, that is, the *conservation of complexity*: techniques that make one part of the problem easier will generally make another part more difficult. Anyone who finds a method to make all parts easier has found something truly special.

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = F(t)/m$, for $0 \leq \zeta < 1$, with initial conditions $x(0)$ and $\dot{x}(0)$. We transform from the time-domain by taking the Laplace transform of both sides of the equation. Using the definition of the Laplace transform, the following relations are found,

$$\begin{aligned}\mathcal{L}[x(t)] &= X(s) \\ \mathcal{L}[\dot{x}(t)] &= sX(s) - x(0) \\ \mathcal{L}[\ddot{x}(t)] &= s^2X(s) - sx(0) - \dot{x}(0) \\ \mathcal{L}[F(t)] &= \mathcal{F}(s).\end{aligned}$$

Substituting these transformed variables into the governing time-domain equation and solving for the transform of the solution $X(s)$, we find

$$X(s) = \frac{\mathcal{F}(s)/m + \dot{x}(0) + (s + 2\zeta\omega_n)x(0)}{s^2 + 2\zeta\omega_ns + \omega_n^2}, \quad (4.2)$$

where the initial conditions are explicitly included in the numerator. For any load $F(t)$, Equation 4.2 must be inverted to solve for $x(t)$.

Using *Borel's theorem*,³ and a table of Laplace transform pairs provided in Appendix A, Equation 4.2 can be inverted term-by-term:

$$\begin{aligned}\frac{1}{m}\mathcal{L}^{-1}\left[\frac{\mathcal{F}(s)}{s^2 + 2\zeta\omega_ns + \omega_n^2}\right] &= \frac{1}{m\omega_d} \int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau \\ \mathcal{L}^{-1}\left[\frac{\dot{x}(0)}{s^2 + 2\zeta\omega_ns + \omega_n^2}\right] &= \frac{\dot{x}(0)}{\omega_d} e^{-\zeta\omega_nt} \sin \omega_dt \\ \mathcal{L}^{-1}\left[\frac{(s + 2\zeta\omega_n)x(0)}{s^2 + 2\zeta\omega_ns + \omega_n^2}\right] &= \frac{x(0)}{(1-\zeta^2)^{1/2}} e^{-\zeta\omega_nt} \cos(\omega_dt - \psi),\end{aligned}$$

where $0 \leq \zeta < 1$ and

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}, \quad \psi = \tan^{-1} \left[\frac{\zeta}{\sqrt{1 - \zeta^2}} \right].$$

Therefore, the inverse transform of Equation 4.2 gives the solution⁴

$$\begin{aligned}x(t) &= \frac{1}{m\omega_d} \int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau \\ &+ \frac{\dot{x}(0)}{\omega_d} e^{-\zeta\omega_nt} \sin \omega_dt \\ &+ \frac{x(0)}{(1-\zeta^2)^{1/2}} e^{-\zeta\omega_nt} \cos(\omega_dt - \psi).\end{aligned} \quad (4.3)$$

³Borel's theorem states that the inverse Laplace transform of the product of two transforms is equal to the convolution of their inverse transforms. It is presented in Appendix A.

⁴We will find the identical solution when we derive the response to arbitrary loading using the convolution integral in Section 4.4.

The *forced* response to $F(t)$ is given by the first term in Equation 4.3, and the *free* response to the initial conditions is given by the second and third terms on the right-hand side. The effects of the initial conditions on the response diminish with time due to the decaying exponential factor. The integral term in Equation 4.3 is known as the *convolution* integral. We will derive the convolution integral in Section 4.4.

Impedance and Transfer Function Some important definitions are based on ratios of Laplace transformed variables. These ratios are important in the study of feedback control, which is the topic of Chapter 10. The system *impedance* is defined for zero initial conditions as

$$Z(s) = \frac{\mathcal{F}(s)}{X(s)} = m(s^2 + 2\zeta\omega_ns + \omega_n^2).$$

Its reciprocal is called the *admittance*,

$$Y(s) = \frac{X(s)}{\mathcal{F}(s)} = \frac{1}{Z(s)}.$$

In many applications, the admittance is better recognized by the name *system function* or *transfer function*,⁵ and is denoted by $G(s)$,

$$G(s) = \frac{X(s)}{\mathcal{F}(s)} = \frac{1}{m(s^2 + 2\zeta\omega_ns + \omega_n^2)}. \quad (4.4)$$

The transfer function is a powerful means to represent system dynamics. In the s -domain, we can find the response to any input by multiplying the transfer function by the input. From Equation 4.4,

$$X(s) = G(s)\mathcal{F}(s), \quad (4.5)$$

and transforming back to time,

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}[G(s)\mathcal{F}(s)]. \quad (4.6)$$

The inverse Laplace transform can be performed numerically if an analytical solution is difficult. We can extract important information about the system's free response from the denominator of the transfer function.

The inverse Laplace transform of the transfer function is called the impulse response function, and it has a significant meaning. The impulse response function will be discussed in detail in Section 4.3. The impulse response function is used in finding the response to arbitrary loads. This will be shown in Section 4.4.

Substituting $s = i\omega$ into Equation 4.4 gives

$$G(s)|_{s=i\omega} = \frac{1}{m[(i\omega)^2 + 2\zeta\omega_n(i\omega) + \omega_n^2]}, \quad (4.7)$$

⁵The transfer function is defined by the ratio of the Laplace transform of the output to the Laplace transform of the input with zero initial conditions.

which we call the frequency response function as per Equation 3.42. The frequency response function is useful in finding the particular solution to a harmonic input. This was shown in Section 3.7.2.

The *characteristic equation* can be obtained by setting the denominator of the transfer function in Equation 4.4 to zero. The *roots* of the characteristic equation are called the *poles* of the system. In some problems, there will be a polynomial in the numerator as well.⁶ Such a system is said to have numerator dynamics. The roots of the numerator polynomial are called the *zeros* of the system, since $G(s) = 0$ if s is replaced by a zero. The values of the poles and zeros affect a system's dynamic characteristics.

Example 4.1 Mechanical Accelerometer

A mechanical *accelerometer*⁷ is an instrument attached to a structure in order to obtain its acceleration. In this example, the accelerometer is attached to a rocket. The schematic of Figure 4.2 is not to scale and shows the accelerometer as a structure (with mass, damping, and stiffness properties) that is attached to the rocket. Evaluate how *quickly* the accelerometer will provide accurate data on a rocket accelerating at a high rate.

Solution: Here, a rocket of (assumed constant) mass m_s is accelerating at a rate of $\ddot{x}(t)$ due to the thrust force $T(t)$. In general, there is a lag in the response time of the accelerometer such that $\ddot{y}(t) \neq \ddot{x}(t)$, where $y(t)$ is the displacement of mass m measured relative to the rocket.

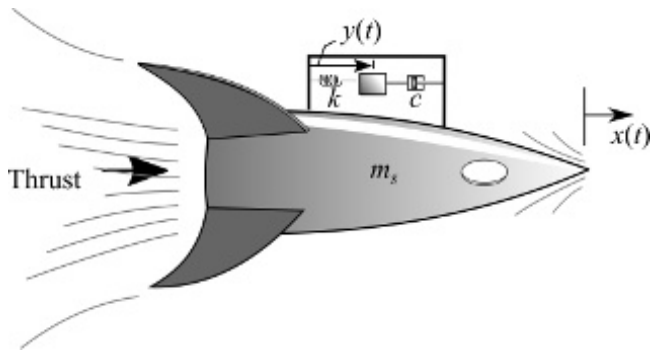


Figure 4.2: Mechanical accelerometer attached to an accelerating rocket structure. The position of the rocket $x(t)$ is measured from an inertial frame of reference.

To derive the equations of motion for the accelerometer and the rocket, free-body diagrams can be drawn for each, assuming that the accelerometer is fixed rigidly to the rocket. Applying Newton's second law, the equation

of motion for the accelerometer mass m is

$$m \frac{d^2}{dt^2}(y + x) + c\dot{y} + ky = 0, \quad (4.8)$$

where the only forces on the mass are due to the accelerometer stiffness and damping, and the inertia force of the accelerating rocket.

It is assumed that the only force acting on the rocket is the thrust $T(t)$. The forces ignored are those due to aerodynamics and the elastic vibration of the structure itself. The thrust-driven equation of motion for the rocket of mass m_s is then

$$m_s \ddot{x} = T(t). \quad (4.9)$$

Substituting $\ddot{x}$ from Equation 4.9 into Equation 4.8 results in a single equation for the accelerometer response,

$$m\ddot{y} + c\dot{y} + ky = -\frac{m}{m_s}T(t). \quad (4.10)$$

Dividing by m , Equation 4.10 becomes

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = -\frac{T(t)}{m_s},$$

with the initial conditions given by $y(0) = y_0$ and $\dot{y}(0) = v_0$. Taking the Laplace transform of each side, we find

$$\begin{aligned} [s^2 Y(s) - sy_0 - v_0] + 2\zeta\omega_n [sY(s) - y_0] \\ + \omega_n^2 Y(s) = -\frac{1}{m_s} \mathcal{T}(s). \end{aligned} \quad (4.11)$$

To continue, the thrust needs to be specified. Assuming that the thrust acts nearly instantaneously, it can be approximated by a step function of magnitude P ,

$$\begin{aligned} T(t) &= Pu(t) \\ \mathcal{T}(s) &= P \frac{1}{s}, \end{aligned}$$

where the unit step function $u(t) = 1$ for $t \geq 0$, and zero for $t < 0$. Then, combining like terms in Equation 4.11 leads to the expression for the transform of the accelerometer response,

$$Y(s) = \frac{-P/m_s + s(s + 2\zeta\omega_n)y_0 + sv_0}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}. \quad (4.12)$$

As a numerical example, assume that the initial conditions are zero, that is, $y_0 = 0$ and $v_0 = 0$, and that $2\zeta\omega_n = 8$ rad/s and $\omega_n^2 = 15$ (rad/s)². With these values, the system is slightly overdamped, with $\zeta = 1.03$, putting it just above critical damping. Equation 4.12 becomes

$$\begin{aligned} Y(s) &= \frac{-P/m_s}{s(s^2 + 8s + 15)} \\ &= \frac{-P/m_s}{s(s + 3)(s + 5)}. \end{aligned} \quad (4.13)$$

⁶Transfer functions with numerator and denominator polynomials will be discussed in more detail in Chapter 10.

⁷In general, the mechanical accelerometer has been superseded by more sophisticated microelectromechanical (MEMS) versions.

To invert $Y(s)$ via transform tables, we expand the above using partial fractions,

$$Y(s) = \frac{p_1}{s} + \frac{p_2}{s+5} + \frac{p_3}{s+3},$$

and express this equation in the form of [Equation 4.13](#) with the numerator set equal to P/m_s . The p_i are found, yielding

$$Y(s) = -\frac{1}{m_s} \left[\frac{P/15}{s} + \frac{P/10}{s+5} + \frac{-P/6}{s+3} \right].$$

From the Laplace transform tables in [Appendix A](#), the inverse transform of both sides yields

$$y(t) = -\frac{1}{m_s} \left[\frac{P}{15}u(t) + \frac{P}{10}e^{-5t} - \frac{P}{6}e^{-3t} \right].$$

This equation tells us that as t increases, the second and third terms representing the transient response decay and $y(t) \rightarrow -P/15m_s$. As expected, for an overdamped system there are no oscillations, just a decay to zero. The first term is the steady-state response to the loading. Once $y(t)$ is known, the steady-state acceleration of the rocket is found to be $\ddot{y}(t)$. ■

Example 4.2 Transfer Functions

Find the transfer functions of the base-excited system whose equation is given by $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$ for the following systems for the specified inputs and outputs.

- Let the displacement $x(t)$ be the output and the ground motion $y(t)$ be the input.
- Let the force $F_{transmit}(t)$ transmitted to the base be the output and $ky(t)$ be the input.

Solution: (a) Take the Laplace transform of the differential equation with zero initial conditions to obtain

$$\frac{X(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k}.$$

- The force transmitted to the base is

$$F_{transmit}(t) = c(\dot{x} - \dot{y}) + k(x - y),$$

which is also equal to $F_{transmit}(t) = -m\ddot{x}$. The Laplace transform of the force transmitted is related to the displacement by $F_{transmit}(s) = -ms^2X(s)$. Then,

$$\begin{aligned} \frac{F_{transmit}(s)}{kY(s)} &= \frac{-ms^2X(s)}{kY(s)} \\ &= -\frac{ms^2}{k} \frac{cs + k}{ms^2 + cs + k}. \end{aligned}$$

■

Example 4.3 Response to Step Input

Find the equation of motion and the response due to a unit step input of a system whose transfer function is given by

$$G(s) = \frac{X(s)}{\mathcal{F}(s)} = \frac{1}{s^2 + 1}.$$

Show that the initial conditions equal zero.

Solution: The equation of motion is

$$\ddot{x} + x = F(t).$$

From [Equation 4.6](#), the response $x(t)$ is given by

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}[G(s)\mathcal{F}(s)] \\ &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1} \times \frac{1}{s}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) \\ &= u(t) - \cos t, \end{aligned}$$

where $1/s$ is the Laplace transform of the unit step function. At $t = 0$, $x(t) = 0$ and $\dot{x}(t) = 0$. ■

Example 4.4 Response to Harmonic Input

Find the forced response of the system in [Example 4.3](#) to $F(t) = \sin 2t$ with $x(0) = 0$ and $\dot{x}(0) = 0$.

Solution: The transfer function of the system is given by

$$G(s) = \frac{X(s)}{\mathcal{F}(s)} = \frac{1}{s^2 + 1},$$

where $F(t) = \sin 2t$. The response $x(t)$ is obtained by

$$\begin{aligned} x_{forced}(t) &= \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}[G(s)\mathcal{F}(s)] \\ &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1} \cdot \frac{2}{s^2 + 4}\right) \\ &= \mathcal{L}^{-1}\left(\frac{2}{3} \frac{1}{s^2 + 1} - \frac{1}{3} \frac{2}{s^2 + 4}\right) \\ &= \frac{2}{3} \sin t - \frac{1}{3} \sin 2t. \end{aligned}$$

■

Example 4.5 Response to Harmonic Input

Find the particular solution of a base-excited system where the base motion is given by $y(t) = 2 \sin 5t$ m. Let $\zeta = 2$ and $\omega_n = 3$ rad/s. Use the method described in [Section 3.7.2](#).

Solution: The equation of motion of the system is

$$\ddot{x} + 12\dot{x} + 9x = 12\dot{y} + 9y.$$

The transfer function is given by

$$G(s) = \frac{X(s)}{Y(s)} = \frac{12s + 9}{s^2 + 12s + 9}.$$

The frequency response function is given by

$$G(s = i\omega) = \frac{9 + 12i\omega}{(9 - \omega^2) + 12i\omega}.$$

The frequency response at the forcing frequency of 5 rad/s is

$$G(i5) = \frac{9 + 60i}{-16 + 60i}.$$

As shown graphically in Figure 4.3, we can use Euler's formula to write the numerator of $G(i5)$ as

$$9 + 60i = \sqrt{9^2 + 60^2} \exp(i \tan^{-1}(60/9)).$$

Similarly, the denominator of $G(i5)$ is given by

$$-16 + 60i = \sqrt{(-16)^2 + 60^2} \exp(i \tan^{-1}(60/-16)).$$

Then, the frequency response at the forcing frequency

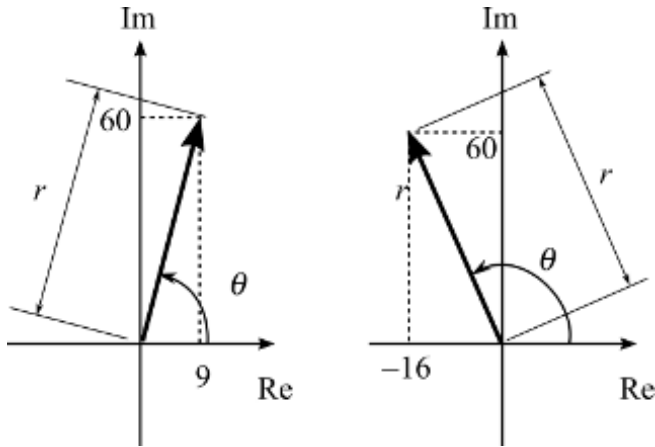


Figure 4.3: Demonstration of Euler's formula to express a complex number in the amplitude-phase form.

can be written as

$$G(i5) = |G(i5)| \exp(i \angle G(i5)),$$

where

$$\begin{aligned} |G(i5)| &= \frac{\sqrt{9^2 + 60^2}}{\sqrt{(-16)^2 + 60^2}} = 0.977 \\ \angle G(i5) &= \tan^{-1}(60/9) - \angle \tan^{-1}(60/-16) \\ &= 1.4219 - 1.8314 \\ &= -0.4095. \end{aligned}$$

The particular solution is then

$$\begin{aligned} x_p(t) &= 2 |G(i5)| \sin(5t + \angle G(i5)) \\ &= 1.9541 \sin(5t - 0.4095). \end{aligned}$$

In this case, the particular solution is also the steady-state solution as the homogeneous solution decays with time due to the damping term. ■

Example 4.6 Particular Solution vs. Steady-State Solution

Consider a system governed by the equation of motion

$$\ddot{x} + 4x = 2F(t).$$

Find the particular solution when $F(t) = 3 \cos(5t + 1)$. Also find the complete solution.

Solution: Take the Laplace transform with zero initial conditions to find

$$(s^2 + 4) X(s) = 2\mathcal{F}(s).$$

The transfer function, $X(s)/\mathcal{F}(s)$, is

$$G(s) = \frac{X(s)}{\mathcal{F}(s)} = \frac{2}{s^2 + 4}.$$

The particular solution has the same form as the input with its amplitude and phase modified such that

$$x_p(t) = 3 |G(i5)| \cos(5t + 1 + \angle G(i5)),$$

where 3 is the amplitude of $F(t)$ given in the problem. The frequency response function evaluated at the input frequency is

$$G(i5) = \frac{2}{(i5)^2 + 4} = \frac{2}{-24} = -\frac{1}{12}.$$

Figure 4.4 shows $-1/12$ as a vector on the complex plane.

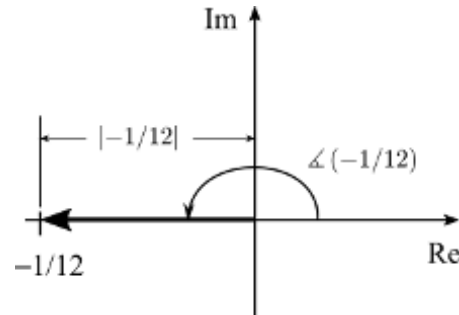


Figure 4.4: Euler formula for $G(i5) = -1/12$.

The magnitude is the length of the vector and the phase is the angle measured from the positive real axis. Then,

$$\begin{aligned} |G(i5)| &= \frac{1}{12} \\ \angle G(i5) &= \pi \text{ rad}. \end{aligned}$$

Then,

$$x_p(t) = \frac{1}{12} \cos(5t + 1 + \pi).$$

The homogeneous solution of this problem is given by

$$x_h(t) = C_1 \cos 2t + C_2 \sin 2t.$$

The complete solution is given by

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{12} \cos(5t + 1 + \pi),$$

with no decay as there is no damping in the governing equation. Therefore, the total solution is the steady-state solution, and there is no transient response. ■

4.2 Step Loading

In the last section, the step load and its Laplace transform had been introduced by way of example. We now consider the response of a single degree-of-freedom oscillator to a step load. A unit step load is shown graphically in Figure 4.5. The unit step response is of interest for its own sake, and also as a building block for more complicated forcing functions.

The unit step function is defined as

$$u(t - t_0) = \begin{cases} 1 & \text{if } t \geq t_0 \\ 0 & \text{otherwise.} \end{cases}$$

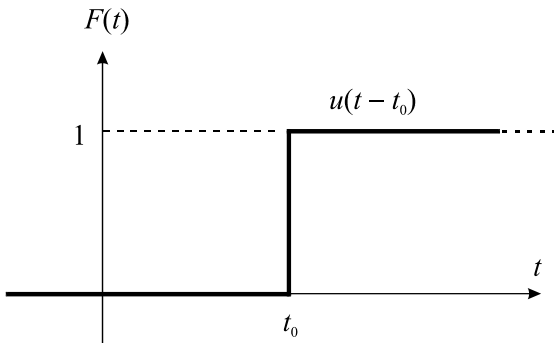


Figure 4.5: Unit step load.

The unit step response $v(t)$ is defined as the response of a system with zero initial conditions to a unit step input at $t_0 = 0$. Here, we evaluate the step response of a single degree-of-freedom oscillator using the Laplace transform. The governing equation of motion is

$$\ddot{v} + 2\zeta\omega_n \dot{v} + \omega_n^2 v = \frac{1}{m} u(t). \quad (4.14)$$

Taking the Laplace transform of both sides gives

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) V(s) = \frac{1}{m} U(s), \quad (4.15)$$

which, from Equation 4.4, can be written as

$$\begin{aligned} V(s) &= \frac{1}{m} \frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} U(s) \\ &= G(s)U(s). \end{aligned} \quad (4.16)$$

The Laplace transform of $u(t)$ is given by

$$U(s) = \frac{1}{s}. \quad (4.17)$$

The unit step response is then

$$\begin{aligned} v(t) &= \mathcal{L}^{-1}[V(s)] \\ &= \frac{1}{m} \mathcal{L}^{-1} \left[\frac{1}{s} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right]. \end{aligned}$$

To transform this equation to the time-domain, we use Borel's theorem, where

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{1}{s} \right] &= u(t) \\ \mathcal{L}^{-1} \left[\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right] &= \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t, \end{aligned}$$

and, using a convolution integral, the response becomes

$$v(t) = \frac{1}{m\omega_d} \int_0^t u(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau.$$

The step function $u(\tau)$ equals unity as it “turns on” at $\tau = 0$.

To solve this integral, we introduce the change of variables $\xi = t - \tau$ with $d\xi = -d\tau$, along with appropriate changes in the limits of integration to obtain

$$v(t) = -\frac{1}{m\omega_d} \int_0^t e^{-\zeta\omega_n \xi} \sin \omega_d \xi d\xi. \quad (4.18)$$

One straightforward way to integrate Equation 4.18 is by parts. Another way is to use the identity

$$\sin \omega_d \xi = \frac{e^{i\omega_d \xi} - e^{-i\omega_d \xi}}{2i}$$

in the integral, and, after some algebra, obtain the desired solution for $t \geq 0$,

$$v(t) = \frac{1}{m\omega_n^2} \left[1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t \right) \right], \quad (4.19)$$

where $m\omega_n^2 = k$.

After the step load is applied to the oscillator and the transients die down, the oscillator responds only to the constant unit load. Thus, we expect the response Equation 4.19 to approach the static response. As t becomes large, $v(t) \rightarrow 1/k$, which is the static deflection of a mass restrained by a spring of constant k under unit load.⁸

⁸Suppose the governing equation is

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = F/m,$$

For the case where the step is applied at $t = t_0$, the Laplace transform of $u(t - t_0)$ is

$$\int_0^\infty e^{-st} u(t_0) dt = \frac{e^{-st_0}}{s},$$

so that

$$v(t) = \frac{1}{m} \mathcal{L}^{-1} \left[\frac{e^{-st_0}}{s} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right]. \quad (4.20)$$

We would then proceed as above to find $v(t)$ explicitly.

Example 4.7 One Cycle of Cosine Forcing

An undamped⁹ system is driven by the function,

$$\ddot{x} + 16x = \frac{F(t)}{m} = \frac{1}{m} \begin{cases} \cos 4t & \text{if } 0 \leq t < \pi \\ 0 & \text{elsewhere.} \end{cases}$$

The initial conditions are $x(0) = 0$ m and $\dot{x}(0) = 1$ m/s. Solve for the response.

Solution: We cannot use the usual harmonic solution approach because the loading function does not act over all time. The term “harmonic function” implicitly means *harmonic over all time*. We can solve this problem using Laplace transforms.

Two shifting theorems for Laplace transforms are useful. They are provided here without proof:

$$\begin{aligned} \mathcal{L}[F(t)e^{at}] &= \mathcal{F}(s-a) \\ \mathcal{L}[F(t-a)u(t-a)] &= e^{-as}\mathcal{F}(s). \end{aligned}$$

To solve the equation above using Laplace transforms, we need to use the second shifting theorem. We can rewrite the loading as

$$F(t) = \cos 4t - \cos 4t u(t-\pi) = \cos 4t - \cos 4(t-\pi) u(t-\pi),$$

where $u(t)$ is the unit step function, and $\cos 4t = \cos(4t - 4\pi)$. Taking the Laplace transform of each side of the governing equation, we obtain

$$\begin{aligned} s^2 X(s) - sx(0) - \dot{x}(0) + 16X(s) \\ = \frac{1}{m} \left[\frac{s}{s^2 + 16} - \frac{s}{s^2 + 16} e^{-\pi s} \right]. \end{aligned}$$

Substituting the initial conditions and solving for $X(s)$, we find

$$X(s) = \frac{1}{s^2 + 16} + \frac{1}{m} \left[\frac{s}{(s^2 + 16)^2} - \frac{s}{(s^2 + 16)^2} e^{-\pi s} \right],$$

and the force F is a constant. Taking a solution in the form of the forcing, then $y = b$, a constant, and thus $\dot{y} = \ddot{y} = 0$. Substituting into the governing equation, we have

$$0 + 0 + \omega_n^2 b = F/m.$$

Then, $b = F/\omega_n^2 = F/k$ is the static displacement.

⁹Without damping this example is not realistic, but is included to demonstrate the method.

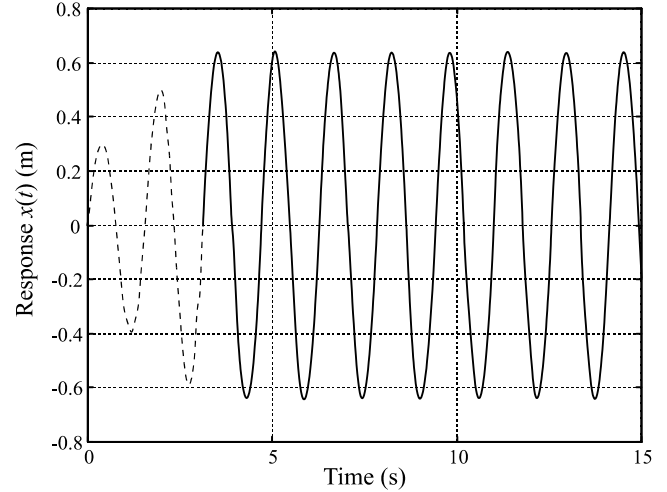


Figure 4.6: Response to short duration harmonic forcing. The transient response is denoted by the dashed line. At about the third cycle, the transient has decayed and the steady-state response continues.

from which

$$\begin{aligned} x(t) &= \frac{1}{4} \mathcal{L}^{-1} \left[\frac{4}{s^2 + 16} \right] + \frac{1}{8m} \mathcal{L}^{-1} \left[\frac{8s}{(s^2 + 16)^2} \right] \\ &\quad - \frac{1}{8m} \mathcal{L}^{-1} \left[\frac{8s}{(s^2 + 16)^2} e^{-\pi s} \right] \\ &= \frac{1}{4} \sin 4t + \frac{1}{8m} t \sin 4t \\ &\quad - \frac{1}{8m} (t - \pi) \sin 4(t - \pi) u(t - \pi). \end{aligned}$$

This solution can be written more clearly as

$$x(t) = \begin{cases} \frac{1}{4} \sin 4t + \frac{1}{8m} t \sin 4t, & 0 \leq t < \pi \\ \frac{1}{4} \sin 4t + \frac{\pi}{8m} \sin 4t, & t \geq \pi. \end{cases}$$

The plot of this response is shown in [Figure 4.6](#) for the case $m = 1$ kg.

Note that the response grows during the loading phase, and then oscillates in steady-state after the load disappears. Because there is no damping, the oscillation continues without end. ■

PIERRE-SIMON DE LAPLACE

(April 23, 1749 – March 5, 1827)



Contribution: Laplace was a French mathematician and astronomer who made major contributions to difference equations and differential equations. Laplace pursued applications to mathematical astronomy and to the theory of probability, two topics he worked on throughout his life. His work on mathematical astronomy included the inclination of planetary orbits, how planets were perturbed by their moons, the motions of the planets, as well as the stability of the solar system.

He restated and developed the nebular hypothesis of the origin of the solar system and was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.

He summarized and extended the work of his predecessors in his five volume *Mécanique Céleste* (Celestial Mechanics) (1799-1825). This seminal work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the Bayesian interpretation of probability was mainly developed by Laplace.

He formulated Laplace's equation, and invented the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming.

He is remembered as one of the greatest scientists of all time, sometimes referred to as the French Newton or Newton of France.

Life Sketch: Laplace was born in Normandy, France. His father was comfortably well off in the cider trade. His mother came from a fairly prosperous farming family. Many accounts of Laplace say he came from a family of "poor farming people" or "peasant farmers" but this seems to be inaccurate.

Laplace attended a Benedictine priory school between the ages of 7 and 16. His father expected him to make a career in the Church and indeed either the Church or the army was the usual destination of pupils at the priory school.

At the age of 16 Laplace entered Caen University. As he was still intending to enter the Church, he enrolled to study theology. However, during his two years at the University,

Laplace discovered his mathematical talents and his love of the subject.

Realizing that mathematics was his calling, Laplace left Caen without completing a degree. He went to Paris and took with him a letter of introduction to d'Alembert. Although Laplace was only 19 years old he quickly impressed d'Alembert, who began to direct Laplace's mathematical studies and helped him find a position. Laplace was soon appointed as professor of mathematics at the École Militaire.

Laplace began producing a steady stream of remarkable mathematical papers, the first presented to the Academy of Sciences in Paris on March 28, 1770. This paper was on maxima and minima of curves, improving on methods given by Lagrange. His next paper for the Academy followed soon afterwards, and on July 18, 1770 he read a paper on difference equations.

Laplace's first paper to appear in print was one on the integral calculus which he translated into Latin and published in 1771 in the *Nova Acta Eruditorum* (the first scientific journal of the German lands, published from 1682 to 1782). Six years later Laplace republished an improved version, apologizing for the 1771 paper and blaming errors contained in it on the printer. Laplace also translated the paper on maxima and minima into Latin and published it in the *Nova Acta Eruditorum* in 1774. In 1771 Laplace submitted another paper that contained important equations in mechanics and physical astronomy.

Laplace's reputation steadily increased during the 1770s. But it was in the 1780s that Laplace produced the depth of results which made him one of the world's most famous and influential scientists.

Applying quantitative methods to a comparison of living and nonliving systems, Laplace and the chemist Antoine Lavoisier in 1780, with the aid of an ice calorimeter that they had invented, showed respiration to be a form of combustion.

Although Laplace soon returned to his study of mathematical astronomy, his work with Lavoisier marked the beginning of an important area of research for Laplace, namely his work on the theory of heat which he worked on towards the end of his career.

In 1784 Laplace was appointed as examiner at the Royal Artillery Corps, and in this role in 1785, he examined and passed the 16 year old Napoleon Bonaparte. His position enabled him to write reports on the cadets that he examined. As a result Laplace became well known to the ministers of the government and others in positions of power in France.

Laplace served on many of the committees of the Academy of Sciences. For example, he served on a committee set up to investigate the largest hospital in Paris and he used his expertise in probability to compare mortality rates at the hospital with those of other hospitals in France and elsewhere.

Laplace was promoted to a senior position in the Academy of Sciences in 1785. Two years later Lagrange left Berlin to join Laplace as a member of the Academy of Sciences. Thus, the two great mathematical geniuses came together in Paris

and, despite a rivalry between them, each benefited from their interactions.

Laplace was made a member of the committee of the Academy of Sciences to standardize weights and measures in May 1790. This committee worked on the metric system and advocated a decimal base. In 1793 the Reign of Terror began and the Academy of Sciences, along with the other learned societies, was suppressed. The weights and measures commission was the only one allowed to continue, but soon Laplace, together with others (including Lavoisier and Coulomb), was thrown off the commission (those on the committee had to be worthy “by their Republican virtues and hatred of kings”).

Before the Reign of Terror, Laplace together with his wife and two children left Paris and lived 50 km southeast of Paris. He did not return to Paris until after July 1794. Although Laplace managed to avoid the fate of some of his colleagues during the Revolution, such as Lavoisier who was guillotined in May 1794 while Laplace was out of Paris, he did have some difficult times.

He was consulted, together with Lagrange, over the new calendar for the Revolution. Laplace knew that the proposed scheme did not really work because the length of the proposed year did not fit with the astronomical data. However, he was wise enough not to try to overrule political dogma with scientific facts. He also conformed, perhaps more happily, to the decisions regarding the metric division of angles into 100 subdivisions.

In 1795 the Academy of Sciences was reopened as the Institut National des Sciences et des Arts. Also in 1795 the Bureau des Longitudes was founded with Lagrange and Laplace as the mathematicians among its founding members. Laplace went on to lead the Bureau and the Paris Observatory. Although some were impressed with him in these posts, others criticized Laplace for being too theoretical.

Laplace presented his famous nebular hypothesis in 1796, which viewed the solar system as originating from the contracting and cooling of a large, flattened, and slowly rotating cloud of incandescent gas. His *Exposition* consisted of five books: the first was on the apparent motions of the celestial bodies, the motion of the sea, and also atmospheric refraction; the second was on the actual motion of the celestial bodies; the third was on force and momentum; the fourth was on the theory of universal gravitation and included an account of the motion of the sea and the shape of the Earth; the final book gave an historical account of astronomy and included his famous nebular hypothesis.

In view of modern theories it is particularly interesting to see Laplace’s remarkably modern view of impacts of comets on the Earth: “...the small probability of collision of the Earth and a comet can become very great in adding over a long sequence of centuries. It is easy to picture the effects of this impact on the Earth. The axis and the motion of rotation have changed, the seas abandoning their old position..., a large part of men and animals drowned in this universal deluge, or destroyed by the violent tremor imparted to the terrestrial

globe.”

In 1786 Laplace proved that the eccentricities and inclinations of planetary orbits to each other always remain small, constant, and self-correcting. These and many other of his earlier results formed the basis for his great work, the *Traité de Mécanique Céleste*, published in five volumes, the first two in 1799.

The first volume is divided into two books, the first on general laws of equilibrium and motion of solids and also fluids, and the second on the law of universal gravitation and the motions of the centers of gravity of bodies in the solar system. In it he developed differential equations and solved them to describe the motions. The second volume deals with mechanics applied to a study of the planets. In it Laplace included a study of the shape of the Earth which included a discussion of data obtained from several different expeditions, and Laplace applied his theory of errors to the results. Another topic studied here by Laplace was the theory of the tides.

In the *Mécanique Céleste* Laplace’s equation appears (although we name this equation after Laplace, it was known earlier). The Legendre functions also appear and were known for many years as the Laplace coefficients. The *Mécanique Céleste* does not attribute many of the ideas to the work of others, but Laplace was heavily influenced by Lagrange and by Legendre and used methods which they had developed.

Under Napoleon, Laplace was a member, then chancellor, of the Senate, and received the Legion of Honor in 1805. However, Napoleon in his memoirs says he removed Laplace from the office of Minister of the Interior, which he held in 1799, after only six weeks: “...because he brought the spirit of the infinitely small into the government.”

The first edition of Laplace’s *Théorie Analytique des Probabilités* was published in 1812 and was dedicated to Napoleon-le-Grand. The dedication was removed in later editions. The work consisted of two books and a second edition two years later saw an increase in the material by about an extra 30 percent.

The first book studied generating functions and also approximations to various expressions occurring in probability theory. The second book contains Laplace’s definition of probability, Bayes’s rule (so named by Poincaré many years later), and remarks on moral and mathematical expectation. The book continues with methods of finding probabilities of compound events when the probabilities of their simple components are known, then a discussion of the method of least squares, Buffon’s needle problem, and inverse probability. Applications to mortality, life expectancy, and the length of marriages are given and finally Laplace looks at moral expectation and probability in legal matters.

Later editions of the *Théorie Analytique des Probabilités* contain supplements which consider applications of probability to: errors in observations; the determination of the masses of Jupiter, Saturn, and Uranus; triangulation methods in surveying; and problems of geodesy, in particular the determina-

tion of the meridian of France. Much of this work was done by Laplace between 1817 and 1819 and appears in the 1820 edition of the *Théorie Analytique*. A fourth supplement, which returns to the first topic of generating functions, appeared with the 1825 edition. This final supplement was presented to the Institute by Laplace at age 76.

Around 1804 Laplace developed an approach to physics which would be highly influential for some years. Laplace wrote: "...I have sought to establish that the phenomena of nature can be reduced in the last analysis to actions at a distance between molecule and molecule, and that the consideration of these actions must serve as the basis of the mathematical theory of these phenomena."

The fourth volume of the *Mécanique Céleste*, which appeared in 1805, contained a study of pressure and density, astronomical refraction, barometric pressure and the transmission of gravity based on this new philosophy of physics.

After the publication of the fourth volume of the *Mécanique Céleste*, Laplace continued to apply his ideas of physics to other problems, such as capillary action (1806-07), double refraction (1809), the velocity of sound (1816), the theory of heat, in particular the shape and rotation of the cooling Earth (1817-1820), and elastic fluids (1821). However, during this period, his dominant position in French science came to an end and others with different physical theories began to grow in importance.

Many of Laplace's physical theories (including the corpuscular theory which he supported and developed in contrast to the wave theory of light as proposed by Fresnel around 1815) were attacked. His caloric theory of heat was at odds with the work of Petit and of Fourier. However, Laplace did not concede that his physical theories were wrong and kept his belief in the fluids of heat and light, writing papers on these topics past the age of 70.

Laplace married on May 15, 1788. His wife was 20 years younger than the 39-year-old Laplace. They had two children.

At the time that his influence was waning, personal tragedy struck Laplace. His only daughter died giving birth in 1813. The child, however, survived and it is through her that there are descendants of Laplace. Laplace's son lived to the age of 85 but had no children.

On the morning of March 5, 1827 Laplace died at age 77. Few events would cause the Academy to cancel a meeting but they did on that day as a mark of respect.

Noteworthy: Laplace became a count of the First French Empire in 1806 and was named a marquis in 1817, after the Bourbon Restoration.

Laplace's fame was not achieved with solid relationships with his colleagues. For example, although d'Alembert had been proud to have considered Laplace as his protégé, he began to feel that Laplace was rapidly making much of his own life's work obsolete. Laplace tried to ease the pain for d'Alembert by stressing the importance of d'Alembert's work since he undoubtedly felt well disposed towards d'Alembert

for the help and support he had given.

It does appear that Laplace was not modest about his abilities and achievements. He considered himself the best mathematician in France. Laplace had a wide knowledge of all sciences and dominated discussions in the Academy.

Laplace changed his views with the changing political events of the time, modifying his opinions to fit in with the frequent political changes. This way of behaving added to his success in the 1790s and 1800s but certainly did nothing for his personal relations with his colleagues who saw his changes of views as merely attempts to win favor.

His name is affiliated with many mathematical operators and phenomena: Laplace's equation, Laplacian differential operator, Laplace transform, Laplace distribution, Laplace-Beltrami operator, Laplace's demon, Laplace expansion, Young-Laplace equation, discrete Laplace operator, Laplace-Runge-Lenz vector, two-sided Laplace transform, Laplace's law, Laplace-Stieltjes transform, Laplace number, Laplace limit, De Moivre-Laplace theorem, Laplace invariant, log-Laplace distribution, and Laplace principle.

He is one of only seventy-two people to have their name engraved on the Eiffel Tower. Asteroid 4628 Laplace is named for him. The European Space Agency's working title for the international Europa Jupiter System Mission is "Laplace."

Many quotes are attributed to Laplace, including:

"All the effects of nature are only the mathematical consequences of a small number of immutable laws."

"What we know is not much. What we do not know is immense."

"I had no need of that hypothesis." ("Je n'avais pas besoin de cette hypothèse-là," as a reply to Napoleon, who had asked why he hadn't mentioned God in his book on astronomy. Napoleon, greatly amused, exclaimed, "Ah! c'est une belle hypothèse; ça explique beaucoup de choses." ["Ah, it is a fine hypothesis; it explains so many things."])

"It is therefore obvious that..." (frequently used in the *Celestial Mechanics* when he had proved something and mislaid the proof, or found it clumsy. Notorious as a signal for something true, but hard to prove.)

"The weight of evidence for an extraordinary claim must be proportioned to its strangeness." (Also known as the Principle of Laplace.)

4.3 Impulsive Excitation

We investigate the oscillator response to an "impulse." The unit impulse, shown in Figure 4.7, is defined as

$$\delta(t - t_0) = 0, \quad t \neq t_0$$

$$\lim_{t \rightarrow t_0} \delta(t - t_0) = +\infty,$$

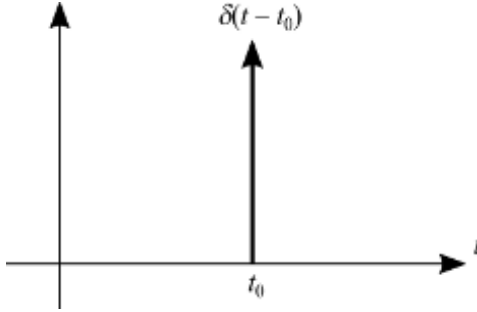


Figure 4.7: Unit impulse. The area under the function is unity.

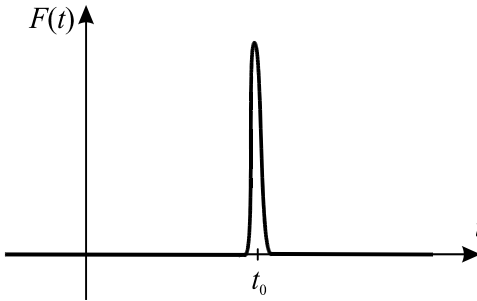


Figure 4.8: Impulsive force which may be modeled as a Dirac delta function.

where $\delta(t)$ is known as the *Dirac delta function* or the *unit impulse function* and is constrained such that

$$\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1.$$

The Dirac delta function $\delta(t)$ has units s^{-1} . Therefore, its integral over time is dimensionless with unity magnitude.¹⁰ It is useful in modeling impulsive phenomena shown in Figure 4.8, where very large forces act over very short intervals of time. Examples are a hammer blow on an anvil, an airplane making a hard landing, and a ship being hit by a slamming wave. The impulse function is one mathematical way to represent such a force. It is also important for the analysis of a response to an arbitrary force, as we will see in the next section.

Let $g(t)$ be defined as the *impulse response* of a system with zero initial conditions to a unit impulse at $t_0 = 0$. The equation of motion is

$$\ddot{g} + 2\zeta\omega_n\dot{g} + \omega_n^2 g = \frac{1}{m}\delta(t), \quad (4.21)$$

for a single degree-of-freedom oscillator under unit impulsive load. It should be noted that the right-hand side is the force divided by the mass. Taking the Laplace transform of both sides, and using the notation

$$G(s) = \frac{1}{m(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

¹⁰It is interesting to note that the function takes on a value of ∞ over zero time interval in such a way that the area under the curve equals one.

yields

$$\mathcal{L}[g(t)] = G(s) \int_0^{\infty} e^{-st} \delta(t) dt = G(s), \quad (4.22)$$

since the impulse function is nonzero only at $t = 0$, and the integral equals one. This is an important result: the impulse response function $g(t)$ and the system transfer function $G(s)$ are a Laplace transform pair.¹¹ The impulse response can be found by taking the inverse transform,

$$g(t) = \mathcal{L}^{-1}[G(s)].$$

Alternatively, and more simply, we can compare Equation 4.16 and 4.22 and find

$$\mathcal{L}[g(t)] = sV(s),$$

where $sV(s)$ is the Laplace transform of the time derivative of $v(t)$ with $v(0) = 0$. Taking the inverse transform of both sides,

$$\mathcal{L}^{-1}[\mathcal{L}[g(t)]] = \mathcal{L}^{-1}[sV(s)]$$

$$g(t) = \frac{dv}{dt}. \quad (4.23)$$

Thus, the unit impulse response equals the time derivative of the unit step response. This result is expected since the unit impulse is the time derivative of the unit step input.

Taking the derivative dv/dt of Equation 4.19, and substituting $\omega_d^2 = \omega_n^2(1 - \zeta^2)$, yields the desired result,

$$g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t u(t), \quad (4.24)$$

since $v(0)\delta(t) = 0$. The impulse response function will be used to derive the response of an oscillator to an arbitrary load.

Example 4.8 Response of Underdamped System

Find the response of an underdamped single degree-of-freedom system subjected to a 10,000 N force applied for 0.0002 seconds at $t = 0$. What is the response at $t = 0.1$ s? Let $m = 1$ kg, $k = 1000$ N/m, and $c = 50$ N-s/m.

Solution: The impulse of the input force is 10,000 N $\times$ 0.0002 s = 2 N-s. If we model the force as the impulse function, the right-hand side of Equation 4.21 is

$$F(t) = \frac{2}{m}\delta(t).$$

Then, the response is $2g(t)$, or

$$x(t) = \frac{2}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t u(t).$$

¹¹It is also true that $g(t)$ and $G(i\omega)$ are a Fourier transform pair. This is shown in Section 9.8.

From the given mass, spring stiffness, and the damping coefficient, we find

$$\begin{aligned}\omega_n &= \sqrt{1000} \text{ rad/s} \\ \zeta &= \frac{c}{2\sqrt{mk}} = 0.7906 \\ \omega_d &= \omega_n \sqrt{1 - \zeta^2} = 19.36 \text{ rad/s} \\ \zeta\omega_n &= 25 \text{ rad/s}.\end{aligned}$$

At $t = 0.1$ s, the displacement is

$$\begin{aligned}x(0.1 \text{ s}) &= \frac{2 \text{ (N} \cdot \text{s)}}{(1 \text{ kg})(19.4 \text{ rad/s})} e^{-(25 \text{ 1/s})(0.1 \text{ s})} \\ &\quad \times \sin((19.4 \text{ rad/s})(0.1 \text{ s})) \\ &= 0.00792 \text{ m}.\end{aligned}$$

Note that the units are dimensionally consistent; the arguments of the exponential and sinusoidal function are nondimensional and $x(t)$ is in meters. ■

Example 4.9 Impulse Response of an Overdamped System

Find the specific impulse response function for the system governed by the equation

$$2\ddot{x} + 3\dot{x} + x = F(t).$$

Solution: This system is overdamped, which can be verified by showing that the characteristic roots are real and negative, -1 and -0.5 , or by showing that the damping ratio is greater than 1. The impulse response function is obtained by replacing $F(t)$ by $\delta(t)$ in the governing equation and $x(t)$ by $g(t)$,

$$2\ddot{g} + 3\dot{g} + g = \delta(t).$$

Taking the Laplace transform of both sides and assuming zero initial conditions, we find

$$G(s) = \frac{1}{2s^2 + 3s + 1},$$

which can be rewritten, using the partial fraction expansion technique, as

$$G(s) = \frac{-1}{s+1} + \frac{1}{s+0.5}.$$

The inverse Laplace transform gives the impulse response function for this problem,

$$g(t) = (-e^{-t} + e^{-0.5t})u(t).$$

The response is an exponential decay, as expected for an overdamped system. ■

Example 4.10 Impulse Response of a Critically Damped System

Find the specific impulse response function for a critically damped system governed by the equation

$$m\ddot{x} + c\dot{x} + kx = F(t),$$

where the damping coefficient is such that $c = 2\sqrt{mk}$.

Solution: Since $c = 2\sqrt{mk}$, $\zeta = 1$. Dividing the equation by m , we have

$$\ddot{x} + 2\omega_n\dot{x} + \omega_n^2x = \frac{1}{m}F(t).$$

The characteristic roots are repeated and are equal to $s = -\omega_n$. The impulse response function is the solution to the equation

$$\ddot{g} + 2\omega_n\dot{g} + \omega_n^2g = \frac{1}{m}\delta(t).$$

The Laplace transform of both sides, assuming zero initial conditions, results in

$$\begin{aligned}G(s) &= \frac{1}{m} \frac{1}{s^2 + 2\omega_n s + \omega_n^2} \\ &= \frac{1}{m} \frac{1}{(s + \omega_n)^2}.\end{aligned}$$

The inverse Laplace transform is

$$g(t) = \left(\frac{1}{m} t e^{-\omega_n t} \right) u(t).$$

We note that the response decays since the factor t is quickly overpowered by the decaying exponential. ■

PAUL ADRIEN MAURICE DIRAC

(August 8, 1902 – October 20, 1984)



Contribution: Dirac was a British theoretical physicist

who made fundamental contributions to the development of both quantum mechanics and electrodynamics. His first major contribution to quantum theory was a paper written in 1925. He published *The Principles of Quantum Mechanics* in 1930 and for this work was awarded the Nobel Prize for Physics in 1933. He shared the Nobel Prize with Erwin Schrödinger “for the discovery of new productive forms of atomic theory.”

Among other discoveries, he formulated the Dirac equation, which describes the behavior of fermions and which led to the prediction of the existence of antimatter.

Life Sketch: Dirac was born in Bristol, England. His father, a French teacher, was an immigrant from Switzerland; his mother was originally from Cornwall. He had an elder brother, who committed suicide in 1925, and a younger sister. His early family life appears to have been unhappy due to his father’s unusually strict and authoritarian nature. (Dirac once said “I never knew love or affection when I was a child.”) He was educated at a school that was attached to the University of Bristol that emphasized scientific subjects and modern languages. This was an unusual arrangement since at that time secondary education in Britain was dedicated largely to the classics, and was something for which Dirac later expressed gratitude.

Dirac studied electrical engineering at the University of Bristol, completing his degree in 1921. He then decided that his true calling lay in the mathematical sciences and, after completing a BA in applied mathematics at Bristol in 1923, he received a grant to conduct research at St. John’s College, Cambridge, where he remained for most of his career.

At Cambridge, Dirac pursued his interests in the theory of general relativity and in the nascent field of quantum physics. Dirac noticed an analogy between the Poisson brackets of classical mechanics and the quantization rules in Heisenberg’s formulation of quantum mechanics. This observation allowed Dirac to obtain the quantization rules in a novel and more illuminating manner. For this work, published in 1926, he received a PhD from Cambridge.

Dirac was appointed Lucasian professor of mathematics at the University of Cambridge in 1932, a post he held for 37 years. During World War II, he conducted important theoretical and experimental research on uranium enrichment by gas centrifuge.

Dirac married in 1937 and adopted his wife’s two children. He and his wife had two children together, both daughters.

He relocated to Florida in 1971 to be near his elder daughter. Dirac spent his last years at the University of Miami in Coral Gables, Florida and Florida State University in Tallahassee, where he was a professor of physics.

Dirac died at age 82 in Tallahassee, Florida, where he is buried.

Noteworthy: He was made a fellow of the Royal Society in 1930 and was awarded the Royal Society’s Royal Medal in 1939 and both the Copley Medal and the Max Planck medal in 1952. He was elected a Fellow of the American Physical

Society in 1948. Dirac was appointed to the Order of Merit in 1973.

An amazingly large number of quantities, operators, and phenomena are named after him: Dirac equation, Dirac comb, Dirac delta function, Fermi–Dirac statistics, Dirac sea, Dirac spinor, Dirac measure, Dirac adjoint, Dirac large numbers hypothesis, Dirac fermion, Dirac string, Dirac algebra, Dirac operator, Abraham-Lorentz-Dirac force, Dirac bracket, and Fermi–Dirac integral.

In 1933, following his 1931 paper on magnetic monopoles, Dirac showed that the existence of a single magnetic monopole in the universe would suffice to explain the observed quantization of electrical charge. In 1975 and 1982 intriguing results suggested the possible detection of magnetic monopoles, but there is no convincing evidence for their existence.

Dirac’s quantum electrodynamics made predictions that were infinite and therefore unacceptable. A workaround known as renormalization was developed, but Dirac never accepted this. “I must say that I am very dissatisfied with the situation,” he said in 1975, “because this so-called ‘good theory’ does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small — not neglecting it just because it is infinitely great and you do not want it!”

Dirac was noted for his personal modesty. He called the equation for the time evolution of a quantum-mechanical operator, which he was the first to write down, the “Heisenberg equation of motion.”

Both Dirac and his father may have suffered from autism to differing degrees. In biographies of the Nobel winner this is offered to help explain Dirac’s reticence, literal-mindedness, rigid patterns, concentration, and determination.

Dirac was known among his colleagues for his precise and taciturn nature. When Niels Bohr complained that he did not know how to finish a sentence in a scientific article he was writing, Dirac replied, “I was taught at school never to start a sentence without knowing the end of it.”

He criticized Robert Oppenheimer’s interest in poetry: “The aim of science is to make difficult things understandable in a simpler way; the aim of poetry is to state simple things in an incomprehensible way. The two are incompatible.”

Among his quotes:

“The laws of nature should be expressed in beautiful equations.”

“God used beautiful mathematics in creating the world.”

“I cannot understand why we idle discussing religion. If we are honest – and as scientists honesty is our precise duty – we cannot help but admit that any religion is a pack of false statements, deprived of any real foundation. The very idea of God is a product of human imagination.... I do not recognize any religious myth, at least because they contradict one another....” (Heisenberg recollects this comment by Dirac

in a friendly conversation at a conference in 1927 discussing Einstein's and Planck's views on religion. When Pauli was asked for his opinion, he said: "Well, I'd say that also our friend Dirac has got a religion and the first commandment of this religion is 'God does not exist and Paul Dirac is his prophet.'" Everybody burst into laughter, including Dirac.) Dirac was an atheist.

4.4 Arbitrary Loading: Convolution Integral

An important goal is to evaluate the response of a system to an arbitrary load. We have already considered the response to arbitrary loading via the Laplace transform. Here, we will do so again but with the convolution integral.

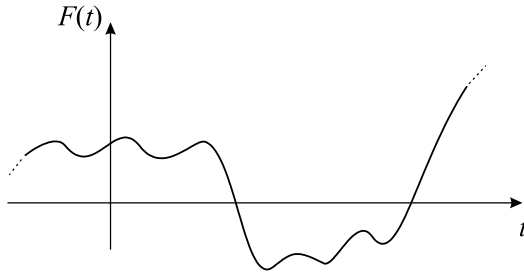


Figure 4.9: Arbitrary deterministic load $F(t)$.

Consider Figure 4.9 representing an arbitrary deterministic load $F(t)$. To derive the response to such a load, our approach is to approximate the arbitrary function by a series of rectangular pulses.¹² The rectangular pulses can be vertical or horizontal, as shown in Figure 4.10. In the limit of zero thickness, both will approach the exact curve, but the approximating equations will be different. We develop both approaches and show how they are related to the impulse response and the unit step response.

Vertical Impulses Consider the case where vertical impulses are used to approximate $F(t)$. As the top sketch in Figure 4.10 shows, each impulse begins at time τ and has an area $F(\tau) \Delta\tau$. Given the unit impulse response $g(t)$, the response to an impulsive force of area $F(\tau) \Delta\tau$ applied at $t = \tau$ is

$$\Delta x_{forced}(t, \tau) = F(\tau) \Delta\tau g(t - \tau). \quad (4.25)$$

The subscript *forced* indicates that it is the forced response due only to $F(t)$ with zero initial conditions. Approximating the full load by a sum of such impulses,

$$F(t) \simeq \sum_{\tau} F(\tau) \Delta\tau \delta(t - \tau)$$

¹²The approach of approximating an arbitrary function by a series of rectangles is a crude method of numerical integration.

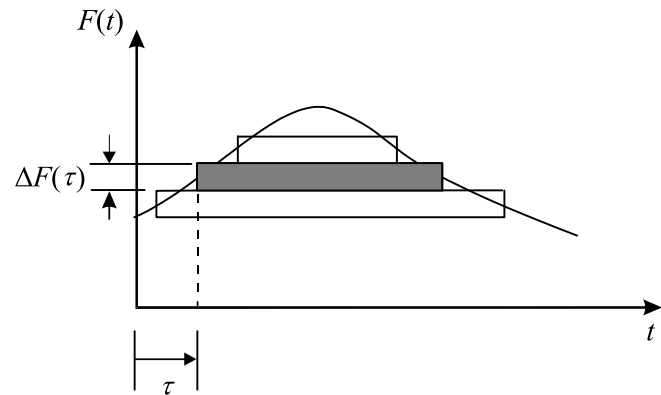
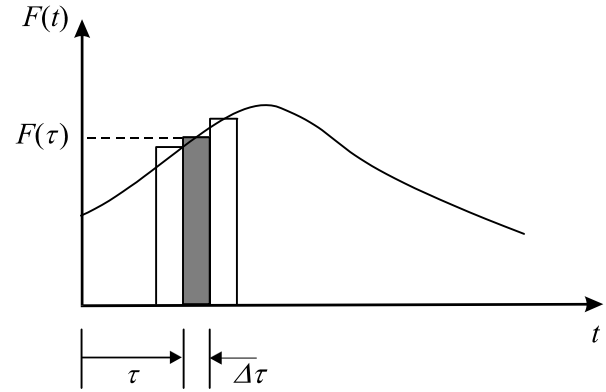


Figure 4.10: Two limiting cases to model the arbitrary load $F(t)$. The vertical and horizontal shaded elements are drawn so that they intersect the curve centrally. In the limit they match the curve exactly.

results in the approximate forced response $x_{forced}(t) \simeq \sum_{\tau} \Delta x_{forced}(t, \tau)$, or

$$x_{forced}(t) \simeq \sum_{\tau} F(\tau) \Delta\tau g(t - \tau).$$

The limit, as $\Delta\tau \rightarrow 0$, is the forced response,

$$x_{forced}(t) = \int_0^t F(\tau) g(t - \tau) d\tau, \quad (4.26)$$

where the right-hand side is the *convolution* integral. It is the principle of superposition for linear systems that validates the idea of the convolution. (We see convolution equations in all the physical sciences, with $g(t)$ expressed in terms of the parameters of that discipline.) Equation 4.26 is called the *Duhamel integral*, when $g(t)$ is given by Equation 4.24.

It should be noted that if the system has non-zero initial conditions, the free response must be added to this forced response to obtain the total solution.

Horizontal Steps An alternate derivation uses a horizontal decomposition of the arbitrary load function.

Each of these are horizontal steps, and the response can be modeled by the unit step response $v(t)$ using [Equation 4.19](#). For a step size $\Delta F(\tau)$ initiated at $t = \tau$, the forced response is

$$\Delta x_{forced}(t, \tau) = \Delta F(\tau) v(t - \tau),$$

which can be written as

$$\Delta x_{forced}(t, \tau) = \frac{\Delta F(\tau)}{\Delta \tau} v(t - \tau) \Delta \tau.$$

Letting the step size at $t = 0$ be $F(0)$ and summing the effects of all the steps up to an arbitrary time t gives an approximation of the exact response,

$$x_{forced}(t) \simeq F(0)v(t) + \sum_{\tau} \frac{\Delta F(\tau)}{\Delta \tau} v(t - \tau) \Delta \tau.$$

In the limit as $\Delta \tau \rightarrow 0$,

$$x_{forced}(t) = F(0)v(t) + \int_0^t \frac{dF(\tau)}{d\tau} v(t - \tau) d\tau. \quad (4.27)$$

Since the goal is to derive the convolution integral, we integrate the second term on the right-hand side by parts to find

$$\begin{aligned} x_{forced}(t) &= F(0)v(t) + v(t - \tau)F(\tau)|_0^t \\ &\quad - \int_0^t \frac{dv(t - \tau)}{d\tau} F(\tau) d\tau \\ &= v(0)F(t) + \int_0^t v'(t - \tau)F(\tau) d\tau, \end{aligned} \quad (4.28)$$

where $v(0) = 0$. Using [Equation 4.23](#),

$$g(t - \tau) = \frac{dv(t - \tau)}{d\tau}$$

leads us again to the convolution integral,

$$\begin{aligned} x_{forced}(t) &= \int_0^t g(t - \tau)F(\tau) d\tau \\ &= \frac{1}{m\omega_d} \int_0^t e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau) F(\tau) d\tau. \end{aligned} \quad (4.29)$$

The complete solution is obtained by adding [Equation 4.29](#) to the free response. [Figure 4.11](#) visually depicts the meaning of the convolution of two functions $g(t)$ and $F(t)$. The time lag can be placed in either function in the integrand. [Equation 4.29](#) is equivalent to

$$\begin{aligned} x_{forced}(t) &= \int_0^t g(\tau)F(t - \tau) d\tau \\ &= \frac{1}{m\omega_d} \int_0^t e^{-\zeta\omega_n\tau} \sin(\omega_d\tau) F(t - \tau) d\tau. \end{aligned}$$

The derivation of the convolution integral above has been formulated for a force starting at $t = 0$. If the force exists for an arbitrary time span, including for $t < 0$, then the convolution integral can be generalized. One way is to define a new time variable t' such that $t' = 0$ when the force begins.

We can also use the fact that $g(t - \tau) = 0$ for $t < \tau$ to show

$$\begin{aligned} x_{forced}(t) &= \int_0^t g(t - \tau)F(\tau) d\tau \\ &= \int_{-\infty}^t g(t - \tau)F(\tau) d\tau. \end{aligned}$$

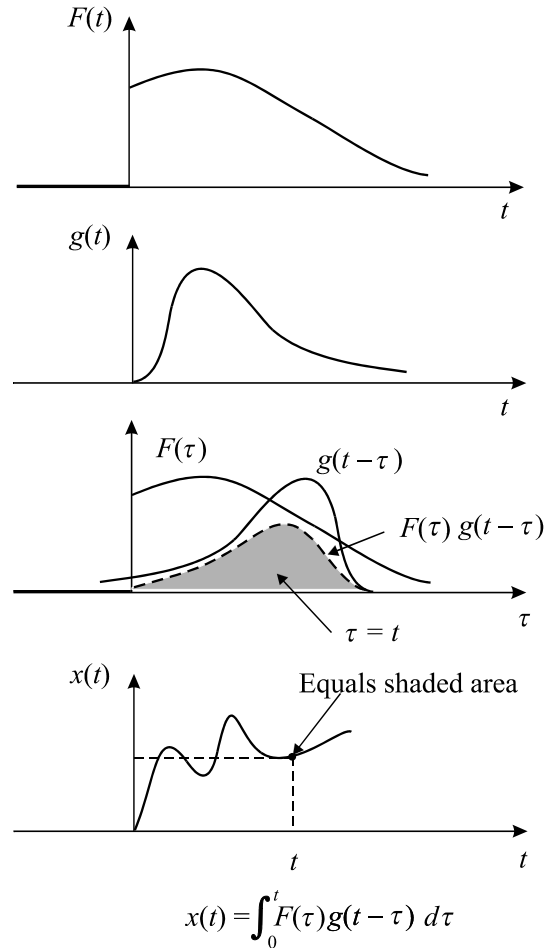


Figure 4.11: The convolution of $g(t)$ and $F(t)$. The shaded area equals the value of x at t .

Example 4.11 One Cycle of Cosine Function - revisited

Find the response of the system in [Example 4.7](#) using the convolution integral. Let $m = 1$ kg and use $x(0) = 0$ and $\dot{x}(0) = 1$ m/s.

Solution: The equation of motion is given by

$$\ddot{x} + 16x = \begin{cases} \cos 4t & \text{if } 0 \leq t < \pi \\ 0 & \text{elsewhere.} \end{cases}$$

The impulse response function given in Equation 4.24 with $\zeta = 0$ is

$$g(t) = \frac{1}{m\omega_n} \sin \omega_n t u(t).$$

For this problem,

$$g(t) = \frac{1}{4} \sin 4t u(t).$$

The forced response is given by the convolution integral

$$\begin{aligned} x_{forced}(t) &= \int_0^t g(t-\tau) F(\tau) d\tau \\ &= \int_0^t \frac{1}{4} \sin 4(t-\tau) F(\tau) d\tau. \end{aligned}$$

For $t < \pi$,

$$\begin{aligned} x_{forced}(t) &= \int_0^t \frac{1}{4} \sin 4(t-\tau) \cos 4\tau d\tau \\ &= \frac{1}{8} t \sin 4t. \end{aligned}$$

For $t > \pi$, the integral needs to be split into two parts,

$$\begin{aligned} x_{forced}(t) &= \int_0^\pi \frac{1}{4} \sin 4(t-\tau) \cos 4\tau d\tau \\ &\quad + \int_\pi^t \frac{1}{4} \sin 4(t-\tau) 0 d\tau \\ &= \frac{\pi}{8} \sin 4t. \end{aligned}$$

The free response is found by solving the same problem with the forcing term set to zero and satisfying the initial conditions,

$$x_{free}(t) = \frac{1}{4} \sin 4t.$$

The total response is

$$\begin{aligned} x(t) &= x_{forced}(t) + x_{free}(t) \\ &= \begin{cases} \frac{1}{4} \sin 4t + \frac{1}{8} t \sin 4t & 0 \leq t < \pi \\ \frac{2+\pi}{8} \sin 4t & t \geq \pi, \end{cases} \end{aligned}$$

which is identical to the solution obtained in Example 4.7. ■

Example 4.12 Convolution for General Loading

An underdamped oscillator is subject to the complicated forcing function shown in Figure 4.12. Determine the equations that must be solved to obtain the response $x(t)$ for all time.

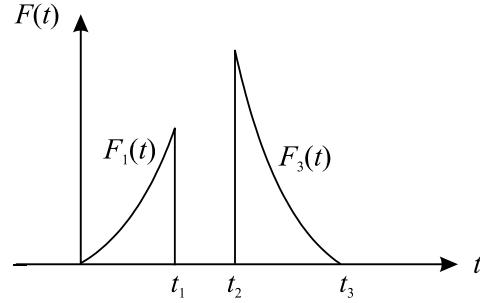


Figure 4.12: Loading history showing positive forces $F_1(t)$ for $0 < t < t_1$ and $F_3(t)$ for $t_2 < t < t_3$ and zero forces $F_2(t) = 0$ for $t_1 < t < t_2$ and $F_4(t) = 0$ for $t \geq t_3$.

Solution: Since the loading function is discontinuous and of a general nature, we choose the convolution integral to represent the response of the oscillator. It is necessary to set up the convolution integral solution over each continuous time interval as follows:

$$x(t) = \int_0^t F_1(\tau) g(t-\tau) d\tau, \quad 0 \leq t \leq t_1$$

$$x(t) = \int_0^{t_1} F_1(\tau) g(t-\tau) d\tau, \quad t_1 \leq t \leq t_2$$

$$\begin{aligned} x(t) &= \int_0^{t_1} F_1(\tau) g(t-\tau) d\tau \\ &\quad + \int_{t_2}^t F_3(\tau) g(t-\tau) d\tau, \quad t_2 \leq t \leq t_3 \end{aligned}$$

$$\begin{aligned} x(t) &= \int_0^{t_1} F_1(\tau) g(t-\tau) d\tau \\ &\quad + \int_{t_2}^{t_3} F_3(\tau) g(t-\tau) d\tau, \quad t \geq t_3, \end{aligned}$$

where $g(t)$ is the oscillator impulse response function, given in Equation 4.24. In the integrals above, we can interchange arguments τ and $t-\tau$ for convenience, depending on the functional forms of the forcing functions.

At each time instant, all the loads from earlier time spans up to the present must be added. Due to the discontinuities of the loading function, the convolution integral is evaluated for each continuous segment.

During zero forcing the system is in free vibration. ■

Example 4.13 Harmonic Forcing for Positive Time¹³

Consider the *causal*¹⁴ system governed by

$$m\ddot{x} + c\dot{x} + kx = F(t). \quad (4.30)$$

Determine the particular solution by solving the system with $x(0) = 0$ and $\dot{x}(0) = 0$, where $F(t) = \sin \omega t \cdot u(t)$ and assuming $c = 0$.

Solution: The particular solution is the general solution with $x(0) = \dot{x}(0) = 0$ only if the excitation $F(t)$ is non-zero for $t \geq 0$. The steady-state solution is

$$x_s(t) = \left[\frac{1}{k} \cdot \frac{1}{1 - (\omega/\omega_n)^2} \right] \sin \omega t.$$

This solution cannot be obtained by the Laplace transform method since the range of integration of the Laplace transform covers positive time only. The function $F(t) = \sin \omega t$ is harmonic over all time and the convolution cannot account for its values for $t < 0$.

The general solution of Equation 4.30 is the sum of the transient and steady-state solutions, given by

$$\begin{aligned} x(t) &= x(0) \cos \omega_n t \\ &+ \left[\frac{\dot{x}(0)}{\omega_n} - \frac{1}{k} \cdot \frac{\omega/\omega_n}{1 - (\omega/\omega_n)^2} \right] \sin \omega_n t \\ &+ \left[\frac{1}{k} \cdot \frac{1}{1 - (\omega/\omega_n)^2} \right] \sin \omega t. \end{aligned}$$

With $x(0) = \dot{x}(0) = 0$, let $x(t) = y(t)$. It is obvious that

$$\begin{aligned} y(t) &= \left[\frac{1}{k} \cdot \frac{1}{1 - (\omega/\omega_n)^2} \right] \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t \right) \\ &\neq x_s(t). \end{aligned}$$

The first two terms of the general solution, defined as $z(t)$,

$$z(t) \equiv x(0) \cos \omega_n t + \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t,$$

coincide with the response in free vibration to which the initial conditions are matched. Thus the term

$$x(t) - x_s(t) - z(t) = - \left[\frac{1}{k} \cdot \frac{1}{1 - (\omega/\omega_n)^2} \right] \sin \omega_n t$$

arises because the general solution is used to satisfy the initial conditions in order to establish the two arbitrary constants.

Suppose that the system is subjected to a positive-time harmonic driving force,

$$F(t) = \begin{cases} \sin \omega t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases}$$

with $c = 0$. Since $F(t)$ is non-zero only for $t \geq 0$, the assumption of a solution of the form $x(t) = A \cos \omega t + B \sin \omega t$ cannot satisfy the governing equation. Instead, the steady-state solution can be found using the convolution equation

$$x_s(t) = \int_0^t F(\tau) g(t - \tau) d\tau,$$

giving

$$x_s(t) = \left[\frac{1}{k} \cdot \frac{1}{1 - (\omega/\omega_n)^2} \right] \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t \right)$$

for $t \geq 0$. Thus, $x(t) = y(t) \neq x_s(t)$ in this case. ■

JEAN MARIE CONSTANT DUHAMEL

(February 5, 1797 – April 29, 1872)



Contribution: Duhamel was a noted French mathematician and physicist who proposed a theory related to the transmission of heat in crystal structures, based on the work of the French mathematicians Jean Baptiste Joseph Fourier and Siméon Denis Poisson. He was primarily engaged in the mathematics of heat, mechanics, and acoustics.

His acoustical studies focused on vibrating strings and the vibration of air in cylindrical and conical pipes, as well as the physics of harmonic overtones. Using partial differential equations, he discovered a solution to the problem of distribution of heat in a solid with a variable boundary temperature, now known as Duhamel's principle.

He also developed methods in calculus using infinitesimals. Duhamel's theorem for infinitesimals states that the sum of a series of infinitesimals is unchanged by replacing each infinitesimal with its principal part.

Life Sketch: Duhamel was born in St. Malo, France. He studied at the Lycée (French high school) in Rennes before becoming a student at the École Polytechnique in Paris in 1814. He attended the École Polytechnique until 1816 and then moved back to Rennes to study jurisprudence. The disruption of his studies was a direct result of the turbulent

¹³ This example was provided by Prof. Fai Ma of the University of California, Berkeley.

¹⁴ A causal system is one where the response at the present time depends on inputs that occurred prior to the present time.

Napoleonic era. In 1804 Napoleon made the École Polytechnique into a military school. Even after Napoleon was defeated at Waterloo, there was severe friction related to the running of the Polytechnique. In 1815 King Louis XVIII, who had been returned to power by the allies, moved against the students and all courses were cancelled until 1817. Duhamel returned to Rennes, but did not go back to Paris after the École Polytechnique was reorganized and reopened in 1817, preferring to remain in Rennes where he studied law.

Duhamel returned to Paris after taking his law degree and taught mathematics and physics at the Institution Massin and at the Lycée Louis le Grand. He then decided to open his own school, which was later called the École Sainte-Barbe. Despite a heavy work load, Duhamel continued his mathematical studies and in 1823 he presented his first paper with Antoine André Louis Reynaud. He began teaching at the École Polytechnique in 1830. He submitted his work on the mathematical theory of heat, written up as a doctoral thesis to the Faculty of Science, and was awarded his doctorate in 1834. Duhamel was named professor of analysis and mechanics in 1836 and was a highly regarded teacher. He remained at the École Polytechnique until his retirement in 1869.

Duhamel also taught at the École Normale Supérieure and at the Sorbonne (they are now part of the University of Paris), and he was a member of the French Academy of Sciences.

Duhamel died in Paris at age 75.

Noteworthy: Duhamel was elected to the Academy of Sciences in 1840.

Duhamel did some experimental work of his own, particularly in the area of vibrating strings. He invented a recording instrument consisting of a pen attached to a vibrating string which left a record on a moving plate. He suggested that the different sounds that one perceives from musical instruments are due to the ear receiving a complex number of harmonics heard as a single sound. (This understanding of sound was made independently by G.S. Ohm.)

Duhamel was a highly regarded teacher of mathematics and was reported to have given very fine lectures.

Duhamel's theoretical predictions relating to the propagation of heat in non-isotropic solids were later verified experimentally by the physicist Henri de Sénarmont.

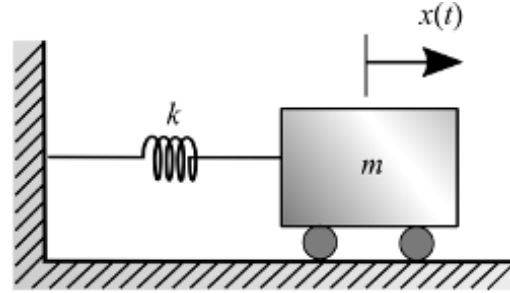


Figure 4.13: Model of a mass-spring system.

vectors. The energy approach can be easier to use to derive the equations of motion for systems that have many degrees of freedom. Here we will only state Lagrange's equation and show an application to demonstrate its use.

Lagrange's equation for a system with a single degree of freedom is

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q(t),$$

where $q(t)$ is used to denote the coordinate, such as x for translational displacement or θ for angular displacement, T equals the system kinetic energy, V equals the system potential and strain energy, and $Q(t)$ includes nonconservative forces associated with $q(t)$. When considering problems with more than one degree of freedom, there will be one Lagrange equation for each degree of freedom.

Example 4.14 A Single Oscillator

Consider the mass m attached to a spring of stiffness k , as shown in Figure 4.13. Derive the equation of motion using Lagrange's equation.

Solution: Lagrange's equation for the derivation of the equation of motion requires the system kinetic and potential energies and derivatives,

$$\begin{aligned} T &= \frac{1}{2} m \dot{x}^2, & \frac{\partial T}{\partial x} &= 0, & V &= \frac{1}{2} k x^2, \\ \frac{\partial T}{\partial \dot{x}} &= m \dot{x}, & \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) &= m \ddot{x}, & \frac{\partial V}{\partial x} &= kx. \end{aligned}$$

Substituting these results into Lagrange's equation yields the governing equation of motion,

$$m \ddot{x} + kx = 0.$$

This approach demonstrates how the energies of the system add to yield the equation of motion. ■

Example 4.15 A Simple Pendulum

Consider the simple pendulum of mass m suspended by an inextensible cable of length l from a fixed point O

4.5 Introduction to Lagrange's Equation

In Chapter 5 we will learn the details of *Lagrange's equation*. It is one of the most important and practical principles of dynamics and useful in the derivation of the equations of motion of discrete and continuous dynamic systems. While giving equations of motion equivalent to Newton's second law of motion, Lagrange's equation is based on energy, a scalar, rather than forces, which are

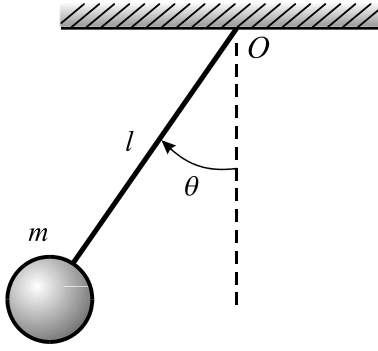


Figure 4.14: A simple pendulum model.

shown in Figure 4.14. Use Lagrange's equation to derive the equation of motion.

Solution: The angular displacement of the cable-mass system from its vertical equilibrium position is given by θ . The kinetic and potential energies are

$$T = \frac{1}{2} I_O \dot{\theta}^2 \text{ and } V = mgh = mg(l - l \cos \theta),$$

and the needed derivatives are

$$\begin{aligned} \frac{\partial T}{\partial \theta} &= 0, & \frac{\partial T}{\partial \dot{\theta}} &= I_O \dot{\theta}, \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) &= I_O \ddot{\theta}, & \frac{\partial V}{\partial \theta} &= mgl \sin \theta. \end{aligned}$$

Substituting these results into Lagrange's equation yields the governing equation of motion,

$$I_O \ddot{\theta} + mgl \sin \theta = 0.$$

Since the moment of inertia about the point of rotation O is $I_O = ml^2$, the governing equation becomes

$$\ddot{\theta} + (g/l) \sin \theta = 0.$$

This equation can be linearized by assuming small oscillations about $\theta = 0$ where $\sin \theta \simeq \theta$. Since this problem is formulated without damping, the pendulum oscillates about the equilibrium position indefinitely. ■

Example 4.16 A Rolling Disk

The disk on the inclined plane of Figure 4.15 is supported by two massless springs. It rolls without slipping, and thus there is no dissipation of energy. The disk oscillates about the static equilibrium position x_0 indefinitely, and at the equilibrium position has a velocity of v_0 . Use Lagrange's equation to derive the equation of motion for arbitrary initial displacement and velocity.

Solution: Consider first the potential energy of the system. Since the disk is assumed rigid, there is no deformation strain energy. However, it does store and give

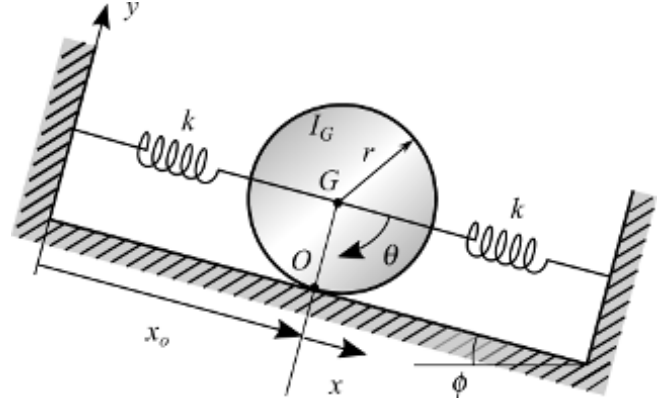


Figure 4.15: Disk on an inclined plane connected to two springs. Since it is assumed that the disk rolls without slipping, there is no energy loss from friction.

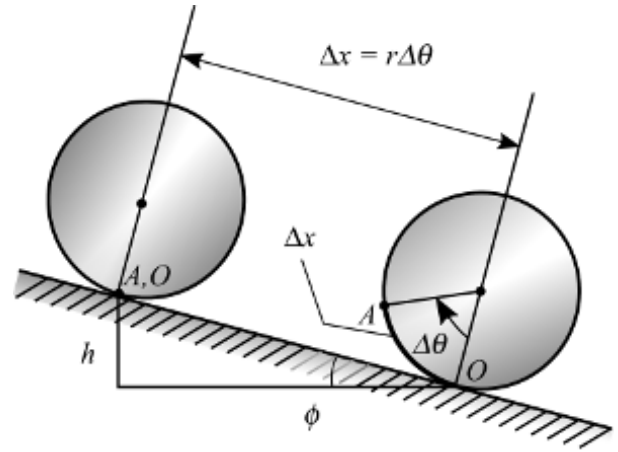


Figure 4.16: Two arbitrary positions of the disk. When the disk rolls, both gravitational and elastic potential energies change.

up gravitational potential energy V_g as it moves up and down the incline. To evaluate the change in potential energy of the disk as it rolls down the ramp, consider Figure 4.16. Assuming the original orientation to be θ_0 , then the change in height is $h = -r(\theta - \theta_0) \sin \phi$ and

$$V_g = mgh = -mgr(\theta - \theta_0) \sin \phi.$$

The two springs store and release potential (strain) energy. Work is done on the springs by the disk as the springs are extended and compressed. The potential energy stored in each spring equals the negative work done on it by the spring force, with the total spring potential given by

$$V_s = 2 \cdot \frac{1}{2} k (\Delta x)^2 = kr^2(\theta - \theta_0)^2.$$

The total potential energy for the system is then $V = V_g + V_s$,

$$V = -mgr(\theta - \theta_0) \sin \phi + kr^2(\theta - \theta_0)^2.$$

Next consider the system kinetic energy, due to the motion of the disk. There are two ways to visualize the kinetic energy. One way is the rectilinear energy plus the rotational energy of the disk,

$$\begin{aligned} T &= \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2 \\ &= \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\dot{\theta}^2 = \frac{3}{4}mr^2\dot{\theta}^2. \end{aligned}$$

The other view is that the disk is in pure rotation about the instantaneous center at the point of contact O ,

$$T = \frac{1}{2}I_O\omega^2 = \frac{1}{2}\left(\frac{1}{2}mr^2 + mr^2\right)\dot{\theta}^2 = \frac{3}{4}mr^2\dot{\theta}^2.$$

To use Lagrange's equation, we need the necessary derivatives of T and V ,

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}} &= \frac{3}{2}mr^2\dot{\theta} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) &= \frac{3}{2}mr^2\ddot{\theta} \\ \frac{\partial V}{\partial \theta} &= -mgr \sin \phi + 2kr^2(\theta - \theta_0). \end{aligned}$$

Substituting into Lagrange's equation yields

$$\frac{3}{2}mr^2\ddot{\theta} - mgr \sin \phi + 2kr^2(\theta - \theta_0) = 0,$$

or

$$\ddot{\theta} + \frac{4}{3}\frac{k}{m}(\theta - \theta_0) = \frac{2}{3}\frac{g}{r} \sin \phi, \quad (4.31)$$

which is the equation of motion of the disk with the natural frequency $\sqrt{4k/3m}$. The "input" is a function of ϕ . The initial conditions are $\theta_0 = \theta(0)$ and $\dot{\theta}_0 = \dot{\theta}(0)$. We could also have used x as the generalized coordinate.

The solution of Equation 4.31 can be used to find the displacement and velocity of point A on the disk via the relations $x = r\theta$ and $\dot{x} = r\dot{\theta}$. ■

JOSEPH-LOUIS LAGRANGE

(January 25, 1736 – April 10, 1813)



Contribution: Lagrange was an Italian mathematician and astronomer, who lived most of his life in Prussia and France, making significant contributions to all fields of analysis, including calculus of variations, the theory of equations, probability theory, number theory, and to classical and celestial mechanics. On the recommendations of Euler and d'Alembert, in 1766 Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, where he stayed for over twenty years, producing a large body of work and winning several prizes of the French Academy of Sciences. Lagrange's treatise on analytical mechanics published in 1788 offered the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the nineteenth century.

Life Sketch: Joseph-Louis Lagrange was born in Turin, Sardinia-Piedmont (now Italy) under the name Giuseppe Lodovico Lagrangia. He was the eldest of 11 children but one of only two to live to adulthood. His father was in charge of the Kingdom of Sardinia's military chest and was of good social position and affluent, but before Lagrange grew up he lost most of his property in speculations, and the family was not wealthy.

A career as a lawyer was planned for Lagrange by his father, and Lagrange seems to have accepted this willingly. He studied at the College of Turin and his favorite subject was classical Latin. At first he had no enthusiasm for mathematics, finding Greek geometry rather dull.

Lagrange's interest in mathematics began when he read a copy of Halley's 1693 work on the use of algebra in optics. He was also attracted to physics at the College of Turin and decided to make a career for himself in mathematics.

He devoted himself to mathematics, and was largely self taught, not having the benefit of studying with leading mathematicians. In July 1754 Lagrange published his first mathematical work which took the form of a letter written in Italian. Perhaps most surprising was the name Luigi De la Grange Tournier under which Lagrange wrote this paper. It drew an analogy between the binomial theorem and the successive derivatives of the product of functions.

Before writing the paper in Italian for publication, Lagrange had sent the results to Euler in a letter written in Latin. The month after the paper was published, he found that the results appeared in correspondence between Johann Bernoulli and Leibniz. Lagrange was greatly upset by this discovery since he feared being branded a plagiarist.

This less than outstanding beginning spurred Lagrange to redouble his efforts to produce results of real merit in mathematics. He began working on the tautochrone, the curve on which a weighted particle will always arrive at a fixed point in the same time, independent of its initial position. By the end of 1754 he had made some important discoveries on the tautochrone, which would contribute substantially to the new subject of the calculus of variations. (It was not called "cal-

culus of variations" until Euler named it that in 1766.)

Lagrange sent Euler his results on the tautochrone containing his method of maxima and minima. Euler replied saying how impressed he was with Lagrange's new ideas. Although he was only 19 years old, Lagrange was appointed professor of mathematics at the Royal Artillery School in Turin in September 1755. It was well deserved; he had already shown the world of mathematics the originality of his thinking and the depth of his great talents.

In 1756 Lagrange sent Euler results that he had obtained on applying the calculus of variations to mechanics. These results generalized those that Euler had himself obtained. Euler consulted the president of the Berlin Academy about Lagrange to try to obtain a position for him in Prussia. However, Lagrange was not after prestige and only wanted to be able to devote his time to mathematics and politely refused the position.

Euler recommended Lagrange for election to the Berlin Academy, and he was elected in September 1756. The following year Lagrange was a founding member of a scientific society in Turin, which was to become the Royal Academy of Sciences of Turin. One of the major roles of this new Society was to publish the scientific journal *Mélanges de Turin*, which published articles in French or Latin. Lagrange was a major contributor: volume 1 appeared in 1759, volume 2 in 1762, and volume 3 in 1766.

Lagrange's articles there covered a variety of topics, including the calculus of variations and the calculus of probabilities. In a work on the foundations of dynamics, Lagrange based his development on the principle of least action and on kinetic energy. Lagrange wrote a major study on the propagation of sound, making important contributions to the theory of vibrating strings. Lagrange used a discrete mass model for his vibrating string, which he took to consist of n masses joined by weightless strings. He solved the resulting system of $n + 1$ differential equations, then let n tend to infinity to obtain the same functional solution as Euler had done. He presented the complete solution of the problem of a string vibrating transversely, arriving at the conclusion that the form of the curve at any time t is given by the equation $y = a \sin(mx) \sin(nt)$.

In papers in the third volume, Lagrange studied the integration of differential equations and topics such as fluid mechanics (where he introduced the Lagrangian function). Also contained are methods to solve systems of linear differential equations, which used the characteristic value of a linear substitution for the first time. Another problem to which he applied his methods was the study of the orbits of Jupiter and Saturn.

In 1763 Lagrange left Turin to make his first long journey to London, but on arriving in Paris he became ill and did not complete the trip. D'Alembert was upset that a mathematician as fine as Lagrange did not receive more honors. D'Alembert, who had visited the Berlin Academy and was friendly with Frederick II of Prussia, arranged for Lagrange to be offered a position in the Berlin Academy. Despite no

improvement in Lagrange's position in Turin, he again turned the offer down.

In 1766 d'Alembert, who knew that Euler was returning to St. Petersburg, wrote to Lagrange to encourage him to accept a post in Berlin. Frederick wrote to Lagrange expressing the wish of "the greatest king in Europe" to have "the greatest mathematician in Europe" resident at his court. Lagrange accepted the offer and succeeded Euler as Director of Mathematics at the Berlin Academy in November 1766.

Lagrange was greeted warmly by most members of the Academy and he soon became close friends with Lambert and Johann (III) Bernoulli. (Not everyone was pleased to see a young man in such a prestigious position. Castillon who was 32 years older than Lagrange thought that he should have been appointed as Director of Mathematics.)

Less than a year after his arrival in Berlin, Lagrange married his cousin. They had no children.

For 20 years Lagrange worked in Berlin, producing a steady stream of top quality papers and regularly winning the prize from the Academy of Sciences in Paris. He shared the 1772 prize on the three body problem with Euler, won the 1774 prize on the motion of the Moon, and he won the 1780 prize on perturbations of the orbits of comets by the planets. His work in Berlin covered many topics: astronomy, the stability of the solar system, mechanics, dynamics, fluid mechanics, probability, and the foundations of the calculus. He also worked on number theory proving in 1770 that every positive integer is the sum of four squares. In 1771 he proved Wilson's theorem that n is prime if and only if $(n - 1)! + 1$ is divisible by n . In 1770 he made a fundamental investigation of why equations of up to 4 degrees could be solved by radicals. The paper is the first to consider the roots of an equation as abstract quantities rather than having numerical values. He studied permutations of the roots and it was a first step in the development of group theory.

During Lagrange's years in Berlin his health was poor on many occasions, and that of his wife was even worse. She died in 1783 after years of illness and Lagrange was very depressed. Three years later Frederick II died and Lagrange's position in Berlin became less attractive.

He was made several offers to return to Italy. The most enticing offer came from Paris and included a clause that Lagrange had no teaching responsibilities. In May 1787 he left Berlin to become a member of the Academy of Sciences in Paris, where he remained for the rest of his career.

Lagrange survived the French Revolution while others did not and this may be due to his attitude which he had expressed many years before when he wrote: "I believe that, in general, one of the first principles of every wise man is to conform strictly to the laws of the country in which he is living, even when they are unreasonable."

Lagrange's *Mécanique Analytique*, which he had written in Berlin, was published in 1788. It had been approved for publication by a committee of the Academy of Sciences, comprising of Laplace, Legendre, and others. Legendre acted as

an editor for the work, doing proofreading and other tasks. The *Mécanique Analytique* summarized all the work done in the field of mechanics since the time of Newton and is notable for its use of the theory of differential equations. With this work Lagrange transformed mechanics into a branch of mathematical analysis. He wrote in the Preface: “One will not find figures in this work. The methods that I expound require neither constructions, nor geometrical or mechanical arguments, but only algebraic operations, subject to a regular and uniform course.”

In May 1790 Lagrange was made a member of the committee of the Academy of Sciences to standardize weights and measures. He worked on the metric system and advocated a decimal base.

In 1792 Lagrange married his second wife, the daughter of one of his astronomer colleagues at the Academy.

He was not unaffected by political events. In 1793 the Reign of Terror commenced and the Academy of Sciences was closed. The weights and measures commission was the only one allowed to continue and Lagrange became its chairman when others such as the chemist Lavoisier, Laplace, Coulomb, and others were thrown off the commission.

In September 1793 a law was passed ordering all foreigners born in enemy countries to be arrested and all their property to be confiscated. Lavoisier intervened on behalf of Lagrange, who fell under the terms of the law, and he was granted an exception. On May 8, 1794, after a trial that lasted less than a day, a revolutionary tribunal condemned Lavoisier, who had saved Lagrange from arrest, and 27 others, to death. Lagrange said on the death of Lavoisier, who was guillotined on the afternoon of the day of his trial: “It took only a moment to cause this head to fall and a hundred years will not suffice to produce its like.”

The École Polytechnique was founded in March 1794 and opened in December 1794. Lagrange was its first professor of analysis. In 1795 the École Normale was founded with the aim of training school teachers. Lagrange taught courses on elementary mathematics there. Although Lagrange had a “no teaching” clause in his contract, he was required to teach after the Revolution. He was not known to be a good lecturer.

Lagrange published two volumes of his calculus lectures. In 1797 he published the first theory of functions of a real variable although he failed to give enough attention to matters of convergence. He stated that the aim of the work was to give the principles of the differential calculus, freed from all consideration of the infinitely small or vanishing quantities, of limits or fluxions, and reduced to the algebraic analysis of finite quantities. The second work of Lagrange on this topic appeared in 1800.

Lagrange died in Paris, France at age 77. He is buried in the Panthéon.

Noteworthy: In 1808 Napoleon named Lagrange to the Legion of Honor and Count of the Empire. In April 1813 he was awarded the Grand Croix of the Order Impérial de la Réunion, a week before his death.

The fundamental equations of analytical mechanics that bear Lagrange’s name employ energy functions and geometrical relations in a viewpoint that emphasizes the way a system behaves as an entity. Lagrange’s equations are considered equivalent to Newton’s laws, although they are actually more general.

In 1810, Lagrange commenced a thorough revision of the *Mécanique Analytique*. He was able to complete only about two-thirds of it before his death in 1813.

Lagrange is usually considered to be a French mathematician, but by birth he was Italian. Lagrange always leaned towards his French ancestry. As a youth he would sign himself Lodovico LaGrange or Luigi Lagrange, using the French form of his family name. Lagrange is considered as French in some books and articles about him and Italian in others.

When he arrived in Paris in 1787 he was received with every mark of distinction. Special apartments in the Louvre were prepared for his reception. However, at the beginning of his residence in Paris he was seized with an attack of melancholy. It is claimed that even the printed copy of his *Mécanique* on which he had worked for a quarter of a century lay for more than two years unopened on his desk. Curiosity as to the results of the French revolution stirred him out of his lethargy and depression, a curiosity which soon turned to alarm as the revolution developed.

His doctoral advisor was Leonhard Euler. His doctoral students included Joseph Fourier and Simeon Poisson.

According to some reports Lagrange always thought out the subject of his papers before he began to compose them, and usually wrote them without a single correction.

Perhaps the world of mathematics has to thank Lagrange’s father for his unsound financial speculation, for Lagrange later claimed: “If I had been rich, I probably would not have devoted myself to mathematics.”

4.6 Notions of Randomness

Sometimes it is not possible to specify exactly the loading on a structure or machine. How then do we proceed to evaluate an expression such as the convolution integral? Consider what it means when the force is random; if an experiment is run many times to measure this force, the results would be different each time. This difference could be small and insignificant, or large and very important.

One way to approach this dilemma is to average the force and use this quantity in the convolution integral. This would at least provide us with a *mean value* response, but it would not provide us with a measure of how scattered the results might be. In analysis and design, we need to know (in a statistical sense) what the largest loads are and what spread of values is possible.

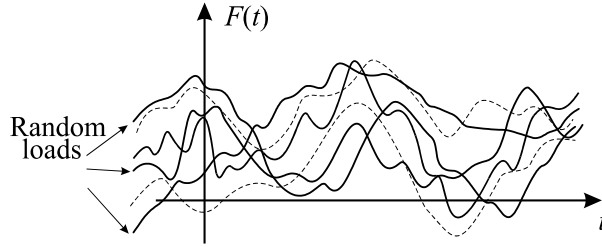


Figure 4.17: Possible random loads: dotted lines represent $\pm$ one standard deviation about the mean.

A measure of the scatter in the data is given by the standard deviation of the loading.

Example 4.17 Random Loading

Figure 4.17 depicts three time histories of a series of tests to estimate the statistics of a force acting on a structure. We see significant variability in the loading. Examples include the aerodynamic loads acting on an airplane, and the ocean wave forces acting on an oil drilling platform. While each force in the figure is similar in the way it varies with time, the magnitudes are far apart and no two are alike. The question the designer will ask when given such data is: *How do we decide which loading history to use when designing the structure?*

Solution: The *average* of the forces is a useful first measure of the general character of the loading, but it does not convey the *spread* in force magnitudes. This is given by the dashed set of curves above and below the average force, the *mean $\pm$ one standard deviation* curves. The larger the band between these two curves, the larger the spread in the original test forces. These standard deviation curves, in conjunction with the mean value response, provide a great deal of useful information on the variability of the force and how to proceed with a design. One possible approach is to design the structure to resist a force that equals the mean plus one standard deviation. ■

4.7 Notions of Control

Suppose the convolution integral is solved for the response of a structural component and the displacement $x(t)$ is too large, that is, $x(t_i) > x_{max}$. Because of this, the vibrating component hits an adjacent structure. Another possibility is that the acceleration $\ddot{x}$ is too rapid, $\ddot{x}(t_j) > \ddot{x}_{max}$, resulting in component damage. What can be done to alleviate these problems? There are two choices. One is to create a new system with different mass, damping, and/or stiffness properties so that the response of the structure is acceptable. In general, this is the preferred approach.

Other times, it is not possible to change the structural configuration or properties, leading to the second possibility – to design a system to *control* the structure. Control implies the introduction of additional forces or moments that, when added to the existing forces, results in a structural response that satisfies user-prescribed criteria. We will study *feedback control* approaches in Chapter 10. Here is a simple example that demonstrates how one can add a force to an existing vibrating system and modify its behavior.

Example 4.18 Control of an Oscillator

A vibrating component within a structure is found to be resonating, leading to excessively large amplitude motion. Occasionally, the component strikes the inside of the structure. Since this is unacceptable, it is concluded that some way must be found to reduce the amplitude of the response. Due to the nature of the component, modifying its *mass* or *stiffness* properties is not feasible. Therefore, a *feedback force* is used to reduce the response. Discuss how we might approach this problem.

Solution: Suppose the feedback force is applied to the structure so that its total response $x(t)$ is modified. The governing equation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{1}{m}[F(t) + F_{control}(t)], \quad (4.32)$$

where $F_{control}(t)$ is the feedback control force. The total response is then

$$x(t) = x_{no\ control}(t) + x_{control}(t).$$

With such a solution, the designer can build a control system to modify $F_{control}(t)$ so that $x(t)$ responds in an acceptable way, within certain amplitude and frequency bounds. ■

4.8 The Inverse Problem

For applications such as *system identification*, *fault detection*, and *nondestructive evaluation*, the system properties are not well known. Usually, the damping of a system is a difficult property to evaluate accurately. Thus, in the convolution integral, $F(t)$ can be prescribed, but not the impulse response $g(t)$, which depends on (m, c, k) . One needs to devise a method to estimate such properties. System identification is a means by which the mass, damping, and stiffness properties of a structure are estimated using a known input force and a measured response. Fault detection and nondestructive evaluation are techniques used to evaluate the health of a structure or to locate damage.

The *inverse* problem focuses on estimating $g(t)$ by loading the structure with a prescribed force and measuring the structural response $x(t)$. Recall that the response

$g(t)$ of an oscillator to the unit impulse $\delta(t)$ is governed by

$$\ddot{g} + 2\zeta\omega_n\dot{g} + \omega_n^2 g = \delta(t),$$

where

$$g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t, \quad t \geq 0, \quad (4.33)$$

for an underdamped system. Equation 4.33 is only a function of system properties ζ and ω_n .

Another way to look at the inverse problem is by considering the simple oscillator governed by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}F(t).$$

In this equation, we know the force $F(t)/m$ and the response function $x(t)$. The system parameters ζ and ω_n are unknown. A series of experiments can be devised to provide enough data (and equations) to estimate these parameters.¹⁵ One type of inverse problem is introduced in Chapter 6.

4.9 A Self-Excited System and Its Stability

Our studies focus on stable system behavior, that is, vibration about a stable equilibrium. However, certain classes of loading can result in unstable behavior, meaning unbounded growth of the response amplitude.

We have already studied the unstable behavior of a structure that is harmonically forced at or near its natural frequency. These *self-excited* systems also have the possibility of becoming unstable. A self-excited system is also called a *parametrically excited* system when the loading occurs through a parameter in the system, usually through a stiffness or a damping term.

For example, assume that the loading on a structure is directly proportional to its velocity with the force given by $F(t) = 2m\beta\omega_n\dot{x}$, where β is a nondimensional parameter. This can be the case for aerodynamically and fluid-loaded structures; the wave force on an offshore structure is related to its structural velocity. This will be examined in Section 9.2.3.

The equation of motion is then

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\beta\omega_n\dot{x},$$

which can be re-written as a “free vibration” problem,

$$\ddot{x} + 2(\zeta - \beta)\omega_n\dot{x} + \omega_n^2 x = 0,$$

with the viscous damping factor being replaced by $\zeta - \beta$. The response, given by Equation 3.9, becomes

$$x(t) = Ce^{-(\zeta-\beta)\omega_n t} \cos(\omega_d t - \phi),$$

where ω_d and ϕ are changed accordingly. If $\zeta > \beta$, the response amplitude decays slower than if $\beta = 0$, but it still decays exponentially. If $\zeta < \beta$, the response amplitude *increases* exponentially without bound and the system is unstable. In terms of system parameters,

$$\zeta = \frac{c}{2\sqrt{km}} < \beta,$$

for this unstable case.

Some applications where self-excited oscillations are possible include engines and governors, mechanical and hydraulic devices, electronic and acoustic devices, rotating shaft phenomena such as whirling, mechanical slip-stick or variable damping phenomena, and aerodynamic phenomena such as the galloping of transmission lines and the flutter of airfoils.¹⁶ Although damping can be added to stabilize such systems, they are generally nonlinear and complicated, and stability becomes a fundamental concern. An introduction is provided in Chapter 11.



MEASURING MOMENT OF INERTIA. The moment of inertia of a complex system, such as a vehicle, or complicated part can be determined experimentally by suspending it from three points to form a *trifilar pendulum*. A trifilar pendulum is a platform supported by three wires designed to oscillate in torsion around its vertical centroidal axis. From the period of oscillation of the trifilar pendulum the moment of inertia around a vertical axis can be calculated.



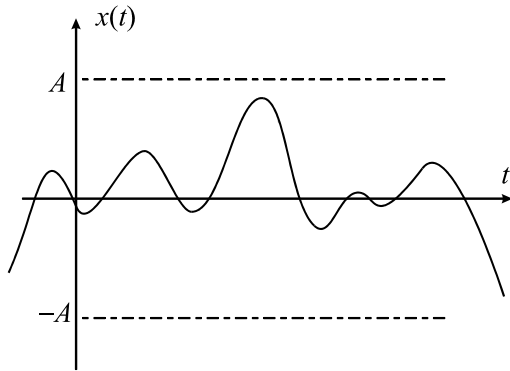
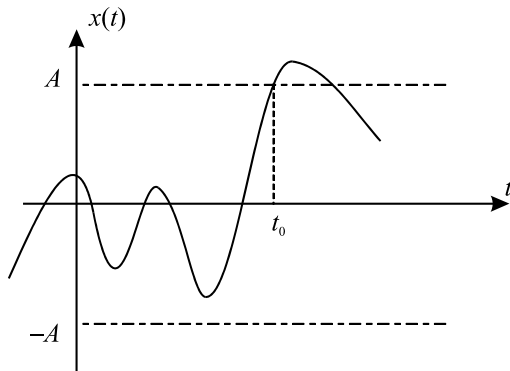
4.10 Solution Analysis and Design Techniques

The design of a structure or machine always has two aspects. The first is to design the system so that it meets its function adequately. For example, a pump must move a certain amount of fluid in a specified time. An engine must develop a certain thrust while maintaining specified temperature and vibration constraints. These vibration specifications can be expressed in a variety of ways. Figure 4.18 shows an example of an amplitude constraint for which the vibration amplitude $|x(t)|$ can never exceed the value $|A|$. Such a requirement exists because there may not be room for larger magnitude deflections or because larger magnitude deflections will result in a structural failure.

The second aspect of design is for structural integrity, which is connected to structural life and reliability. Fatigue life is always a concern when designing vibrating

¹⁵ When $F(t) = 0$, the logarithmic decrement and damped period provide such a way.

¹⁶ **Aircraft Vibration and Flutter**, R.H. Scanlan and R. Rosenbaum, Dover Publications, 1968.

Figure 4.18: Design for $|x(t)| < A$ for all time.Figure 4.19: Design for $|x(t)| < A$ for $t < t_0$.

components since it is related directly to the number of cycles that a component undergoes during its lifetime. The ideal situation is to simultaneously design for function and for structural integrity. They are often addressed separately and involve design trade-offs.

Figure 4.19 has a similar design goal, except that amplitudes greater than A are permissible after time t_0 . Other variations of such design constraints exist.

This chapter closes with examples of how single degree-of-freedom models can provide useful preliminary designs.

Example 4.19 Water Landing of a Space Module

The international space station and its crew were serviced by Space Transportation Systems (STS). Currently the space station is serviced using various Russian rockets. It is necessary to provide the crew with an emergency system that can transport them safely back to Earth in the event of a medical emergency, a space station catastrophe, or if the next rocket launch is not soon enough.¹⁷

Current plans call for a water landing space module, a capsule design based on the Apollo command module

¹⁷This example is based on the paper by D.E. Van Sickle and L.A. Anderson, "Pitch Analysis of a Space Module After Water Landing," *AIAA Journal of Spacecraft and Rockets*, Vol.32, No.4, July-August 1995, pp.601-607.

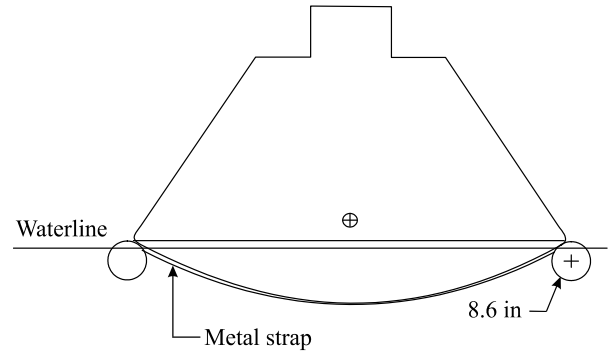


Figure 4.20: Attitude spheres attached to model below the waterline. (After Van Sickle and Anderson)

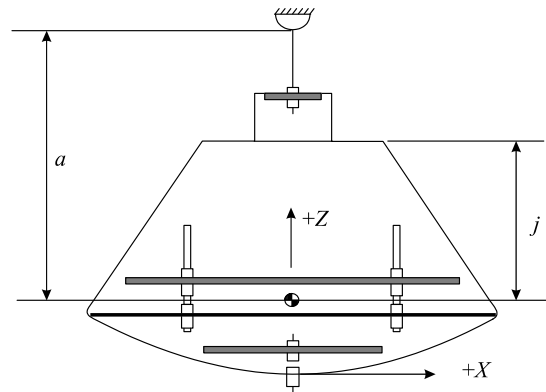


Figure 4.21: Pendulum test setup to determine module dynamic characteristics. (After Van Sickle and Anderson)

that was flown to the Moon. There are several concerns regarding the design of this module. It is expected to be used in an emergency, with little warning. Once it ejects from the space station at some random location in orbit and falls to some body of water, it is likely to be floating for a relatively long time before it can be rescued. This necessitates that the crew be comfortable and safe. The module would parachute to the water and maintain an upright position.

Consider the use of a single degree-of-freedom model for the analysis and preliminary design of such a capsule.

Solution: To assist in the design of such a module, a 1/5 scale model was built and tested. Tests were performed for various center of gravity (CG) locations with and without the use of flotation or stabilization mechanisms. It was found that the pitch oscillation increased as the CG was offset horizontally. Less significantly, pitch oscillations increased with increasing vertical CG offset. To assist in stabilization, *attitude spheres* were attached to the model at various positions just below the water line. Figure 4.20 shows a schematic of the attitude spheres.

To determine the mass moment of inertia for the module and the equation of motion, a pendulum test setup was built, as shown in Figure 4.21. From the figure,

we see that the test module can oscillate as a pendulum about the support point. If the module mass moment of inertia about its CG is I_G , then by the parallel axis theorem, the inertia about the support point distance a from the CG is $(I_G + ma^2)$, where m is the mass of the module.

From our studies, we can find the nonlinear pendulum equation of rotational motion to be

$$(I_G + ma^2)\ddot{\theta} + mga \sin \theta = 0, \quad (4.34)$$

where damping is neglected and θ is measured relative to the horizontal. If only small angles are allowed in the testing, then Equation 4.34 can be simplified by letting $\sin \theta \simeq \theta$. When solved, we have

$$\theta(t) = A \sin \left(\sqrt{\frac{mga}{I_G + ma^2}} t + \gamma \right),$$

where A and γ are determined using the initial conditions. From the natural frequency, the period can be expressed as

$$T = 2\pi \sqrt{\frac{I_G + ma^2}{mga}}.$$

The unknowns are a and I_G . The location of the CG is found by swinging the module from its top and its bottom in two tests. In this manner, two equations for the period are used to determine the location of the CG. The pendulum tests also permit the calculation of the mass moments of inertia about each horizontal axis, I_x and I_y . The tests were performed to establish the geometric and mass properties of the module.

Next, it is necessary to model the oscillation of the module as it floats in a variety of sea states, from calm to severe. A floating body will generally have six degrees of freedom. As an initial study, only *pitch* motion is considered to establish the stability of the module. Figure 4.22 depicts the coordinate system and forces acting on the module as it *pitch*s.

The pitch equation of motion can be obtained via the Newton-Euler equation for the moments acting on the module. From the figure, the restoring force can be seen to be the buoyancy moment due to the angular deflection. The buoyancy force is the weight W and the moment arm is the distance from the CG to the new center of volume of the immersed section. Also taken into consideration is the added inertia term I_w due to the water entrained¹⁸ by the module. Therefore, the total inertia is equal to $I = I_G + I_w$.

Taking the sum of the moments, one finds¹⁹ the nonlinear equation of motion to be

$$I\ddot{\theta} + c\dot{\theta} + k \sin(\theta - \phi) = 0,$$

¹⁸ Entrained means to pull or draw along after itself. Therefore, as the module moves in the water, it pulls along an amount of water with it. Thus, the module behaves as though it is more massive.

¹⁹ For complete details, see the original paper.

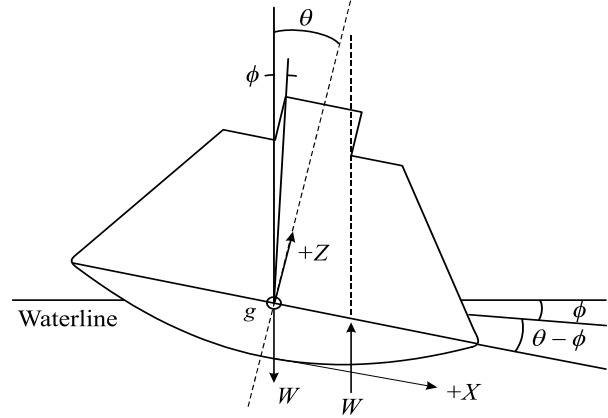


Figure 4.22: Pitch oscillations for module. (After Van Sickle and Anderson)

where ϕ equals the angular orientation of the wave,

$$\phi(t) = \alpha \sin \omega t.$$

For small angles, the usual linearization is made, and the wave forcing term is placed on the right-hand side of the equation of motion, resulting in

$$\ddot{\theta} + \frac{c}{I}\dot{\theta} + \frac{k}{I}\theta = \frac{k\alpha}{I} \sin \omega t.$$

The linearized equation of motion of free vibration is

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0,$$

where

$$c = 2\zeta\omega_n I, \quad k = \omega_n^2 I.$$

The damping factor or ratio ζ is obtained by measuring several successive peak amplitudes of a free oscillation and applying the logarithmic decrement method. Once ζ is evaluated, the natural frequency ω_n can be determined using the damped period measurements, $\omega_n = 2\pi/(T_d\sqrt{1 - \zeta^2})$.

Studies based on this model show that the natural frequency of the module may overlap those of certain wave configurations, resulting in large amplitude pitching. This potential resonance problem can be resolved by altering the module's natural frequency by changing the position of the CG or by adding attitude spheres.

The solution to the simplified problem helps us develop general design criteria for the actual craft. ■

In the next example, we consider the problem of protecting manufactured components and structures during shipment from the point of manufacture to the end user. Here, a spacecraft must be protected when shipped to its launch site over bumpy roads and other obstacles. Vibration loads are difficult to quantify and, depending on the type of transportation used (truck, rail, or air), test

data from the actual route need to be used to design the shipping container's suspension system. Other considerations include the thermal environment, contamination issues, physical compatibility between the spacecraft and the container, as well as between the container and the transportation system being utilized.

Example 4.20 Vibration Analysis of a Spacecraft in a Shipping Container²⁰

A shipping container needs to be designed to transport a spacecraft of weight $W = 4000$ lb from the factory to the launch site. Designing this container is a critical aspect of spacecraft design and manufacture. We are concerned with the vibration design criteria.

In this example an engineer receives the design order with specifications for truck shipment: the load is specified as a “ $3g$ decaying sinusoid with a decay rate (logarithmic decrement) $\delta = 0.5$, with load occurring over the frequency range of 2 to 200 Hz.” Figure 4.23 illustrates this loading case for 5 Hz. The loading characteristics and the frequency range depend on the kinds of bumps in the road as well as the truck speed. How disturbances such as bumps get transmitted to the satellite is a function of the container properties.

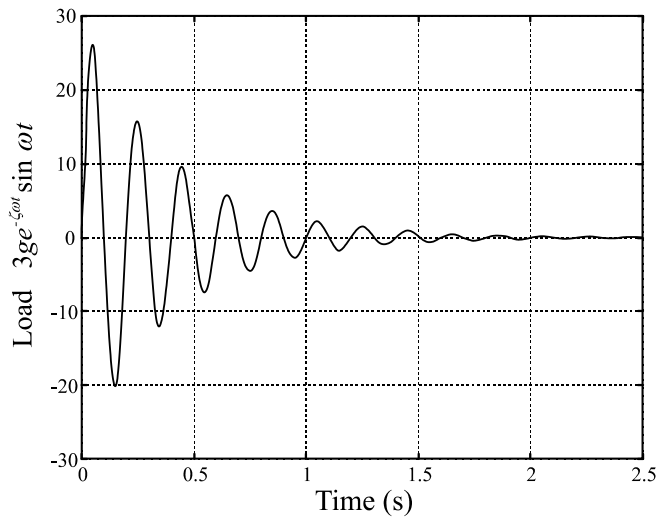


Figure 4.23: The loading is given as a base acceleration in the form of a decaying sinusoid at 5 Hz.

It is common in the study of shock and vibration, for example in earthquake engineering, to specify loading in terms of an acceleration. Here, the forcing is given in terms of the base acceleration, $\ddot{x}$,

$$\ddot{x} = 3ge^{-t/\tau} \sin \omega t, \quad (4.35)$$

where ω is the driving frequency of the load, and $\tau = 1/(\zeta\omega)$ is the relaxation time. We will assume that the

²⁰ From in-house notes, Lockheed Martin Corporation. Copyright 1993, 1996. Used with permission.

maximum load the spacecraft can withstand is $8g$ at the center of gravity (CG) at frequencies below 15 Hz. Above 15 Hz, we assume that the spacecraft cannot resist any significant acceleration. Thus, the suspension system requirements must be derived for maximum attenuation in the vertical direction at frequencies greater than 15 Hz.

Furthermore, we are told that to prevent *coupling* between the container suspension system and the spacecraft, the suspension system must have a natural frequency of 5 Hz or lower.²¹ If coupling occurs, a beating-type behavior will result, leading to large amplitude motion. The satellite in its shipping configuration has a natural frequency of 20 Hz.

The above information is provided to a shipping container subcontractor who then translates the requirements into a container design proposal for eventual approval by the owner of the spacecraft. One such proposal is:

“The spacecraft will be mounted on a frame assembly which weighs 1550 lb. The frame assembly will be mounted on four shock mounts attached to the shipping container base. With the spacecraft installed, 4.0 in of clearance exists between the shipping container insulation and the spacecraft envelope. With a static spring constant of 1275 lb/in, dynamic spring constant²² of 1.3×1275 lb/in, and a damping factor of 11%, this system will satisfy suspension performance requirements.”

The spacecraft owner must find some way to verify that the above container will safeguard the spacecraft on route to the launch site. Typically, shipping containers and their suspension systems are verified through drop tests. In this test, an equivalent mock satellite is dropped in such a container, released from a height resulting in a $3g$ decaying sinusoid vibration input. This equivalent drop height must be estimated analytically to test the container.

Solution Part One: The spacecraft-container system is idealized as a base-excited, single degree-of-freedom system. To establish the safety of the spacecraft in the container, we derive and solve the equation governing the spacecraft oscillation inside the container and show that the maximum response to the $3g$ load is within the allowable space. This will verify the suspension system design. If internal impact occurs, the system must be redesigned. Once the system model is verified, we can

²¹ As we will learn, structures generally vibrate at many frequencies; each frequency is associated with a degree of freedom. Since we have idealized the suspension system as a single degree-of-freedom oscillator, we characterize its behavior with a single natural frequency.

²² The spring constant is generally a fixed value. But when the rate of loading is very rapid, the spring behaves as though it is more stiff. Thus, for average forcing rates, the spring constant is a certain value, but for rapid loading, such as an impact, the value is higher. We see this in materials where the strain-rate alters deformation properties.

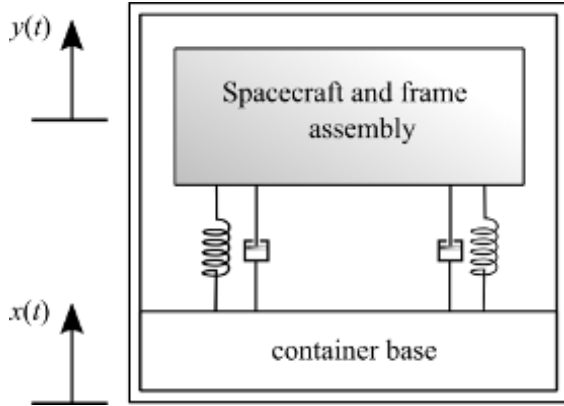


Figure 4.24: Idealized spacecraft in container. The displacement of the base is denoted by x , and of the spacecraft by y , both positive in the up direction.

determine a drop height to test the container's suspension system. The test will verify that the design can be based on the simplified model.

The base-excited system is shown in Figure 4.24. We assume that the satellite and the container CGs are centered. Thus, upon impact there are no eccentricities that will induce rotational motion. Also, we assume that there is no other motion of the container or of the structure (truck) upon which the container sits. All these assumptions can be relaxed if necessary for a more realistic and complicated model.

The spacecraft mass is given by

$$m = \frac{W}{g} = \frac{4000}{386.4} = 10.35 \frac{\text{lb-s}^2}{\text{in}}.$$

The input frequency has a range of $f = 2$ to 200 Hz. We calculate displacements over this range of frequencies in order to be certain that the maxima have been captured.

From the definition of logarithmic decrement, we have

$$\delta = \ln \frac{n^{\text{th}} \text{ peak of } \ddot{x}}{(n+1)^{\text{th}} \text{ peak of } \ddot{x}},$$

into which we can substitute the second derivative of $x(t)$,

$$\delta = \ln \frac{3g \exp(-t_n/\tau)}{3g \exp(-[t_n + T]/\tau)},$$

where t_n is taken when $\sin \omega t_n = 1$ and $T = 2\pi/\omega = 1/f$. Therefore,

$$\delta = \frac{T}{\tau},$$

and

$$\tau = \frac{T}{\delta} = \frac{1}{f\delta}.$$

Thus, the *relaxation time* constant for a frequency of $f = 200$ Hz and for $\delta = 0.5$ is $\tau = 0.01$ s. This can be

interpreted physically as the time duration the structure is at its peak oscillation.

From the shipping container suspension system parameters, the spring constant of all 4 shock mounts is $k = 4 \times 1.3 \times 1275 = 6630$ lb/in, and we are given the damping factor equal to 11%. The natural frequency of the suspension system is then

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = 4.028 \text{ Hz},$$

the damped natural frequency is

$$f_d = \frac{\omega_d}{2\pi} = \frac{\omega_n \sqrt{1 - \zeta^2}}{2\pi} = 4.003 \text{ Hz},$$

and the damping constant is

$$c = 2m\zeta\omega_n = 57.64 \text{ lb s/in}.$$

The governing equation of motion for a base-excited structure is

$$m\ddot{y} + c(\dot{y} - \dot{x}) + k(y - x) = 0, \quad (4.36)$$

where $y(t)$ is the structural response, and $x(t)$ and $\dot{x}(t)$ are the base container input displacement and velocity, respectively. These can be found by integrating Equation 4.35 twice beginning with $\ddot{x} = 3ge^{-t/\tau} \sin \omega t = 3ge^{\rho t} \sin \omega t$, where we define $\rho = -1/\tau$. Assuming zero initial velocity, the first integration results in

$$\dot{x} = \frac{3g}{\rho^2 + \omega^2} [-\omega e^{\rho t} \cos \omega t + \rho e^{\rho t} \sin \omega t + \omega]. \quad (4.37)$$

A second integration, assuming zero initial displacement, yields

$$x(t) = -\frac{3g}{\rho^4 + 2\rho^2\omega^2 + \omega^4} [2e^{\rho t}\omega\rho \cos \omega t + e^{\rho t}(\omega^2 - \rho^2) \sin \omega t - \omega t\rho^2 - \omega^3 t - 2\omega\rho]. \quad (4.38)$$

Substituting Equations 4.37 and 4.38 into the governing Equation 4.36 results in an equation of motion that can be solved numerically,

$$\begin{aligned} m\ddot{y} + c\dot{y} + ky &= \frac{3gc}{\rho^2 + \omega^2} [-\omega e^{\rho t} \cos \omega t + \rho e^{\rho t} \sin \omega t + \omega] \\ &+ \frac{3gk}{\rho^4 + 2\rho^2\omega^2 + \omega^4} [2e^{\rho t}\omega\rho \cos \omega t + e^{\rho t}(\omega^2 - \rho^2) \sin \omega t \\ &- \omega t\rho^2 - \omega^3 t - 2\omega\rho]. \end{aligned} \quad (4.39)$$

Note that there are secular terms (terms that grow with time) in the last line. The ω term in Equation 4.37 led to the secular terms and the term $2\omega\rho$ of Equation 4.38. These arise from the requirement that $x(0) = 0$ and

Table 4.1: Maximum Spacecraft Acceleration and Maximum Relative Displacement ($y - x$) for Various Driving Frequencies f .

f (Hz)	$\max \ddot{y} $ (g)	$\max(y - x)$ (in)
2.000	4.071	1.886
3.600	6.075	3.443
3.700	6.077	3.512
3.900	6.062	3.545
4.000	6.019	3.518
14.00	1.046	0.3550
20.00	0.6200	0.2220
25.00	0.4649	0.1772
100.0	0.1069	0.0434
200.0	0.0524	0.0217

$\dot{x}(0) = 0$. Such terms are signs of potential trouble and instability, but they do not cause difficulties here in the numerical solution of the Equation 4.39. We are interested only in the first cycle of the oscillation, that is, for small time $t \ll 1$ s. This is when the spacecraft is exposed to the peak amplitudes and accelerations.

For a range of frequencies, this equation can be solved for the maximum spacecraft acceleration and for the maximum spacecraft/container relative displacement. This information allows us to determine whether the system requirements are met. Table 4.1 shows how maximum spacecraft acceleration $\ddot{y}$ and maximum relative displacement ($y - x$) vary as a function of driving frequency f . Since $\omega = 2\pi f$, $\rho = -1/\tau = -f\delta$.

The system model meets the requirements that the maximum acceleration is less than $8g$ and the maximum displacement is less than 4 in. From Table 4.1, for the range of frequencies 2 to 4 Hz there is dynamic coupling between the spacecraft container and the truck suspension, resulting in significant satellite motion. At higher frequencies, for example 14 to 200 Hz, the shock loads are isolated as required by the design specifications.

Solution Part Two: The second part of this solution is the determination of the drop height that results in loads and displacements similar to the $3g$ decaying sinusoid. This drop height will be used to design an experiment to verify compliance with design requirements. The height will be determined using the single degree-of-freedom model just developed. From basic physics, when an object is dropped from rest at height h , its gravitational potential energy is converted to kinetic energy, and at the instant of impact, $mgh = mv_0^2/2$, from which

$$v_0 = \sqrt{2gh}. \quad (4.40)$$

Once the container strikes the ground, the internal component will oscillate in free vibration with *initial velocity*

Table 4.2: Maximum Spacecraft Acceleration and Relative Displacement ($y - x$) for Various Heights.

h (in)	$\max \ddot{y} $ (g)	$\max(y - x)$ (in)
12.00	6.568	1.606
13.00	6.758	1.655
14.00	6.945	1.689
15.00	7.128	1.740
16.00	7.306	1.780
17.00	7.480	1.823
18.00	7.650	1.856
19.00	7.816	1.901
20.00	7.979	1.939

v_0 . It is assumed that upon impact, the container remains in contact with the ground rather than bounce up and down. During the free fall, the springs are in a zero or unstretched position, but after impact the oscillation will be about the equilibrium. Therefore, there will also be an *initial displacement* y_0 given by $ky_0 = mg$, and

$$y_0 = \frac{mg}{k} = \frac{4000}{4 \cdot 1275} = 0.784 \text{ in},$$

where k is the static stiffness constant for all four springs. To make the equations above work, we need to *guess* an initial height h , and then use Equation 4.40 to calculate initial velocity $\dot{y}(0) = -v_0$. The initial displacement $y(0) = y_0$ is independent of the initial height. Governing Equation 4.36 becomes

$$m\ddot{y} + c\dot{y} + ky = 0,$$

since $x = 0$ and $\dot{x} = 0$ at impact. This equation can be solved in closed form, as in Section 3.4, where Equation 3.6 with its solution Equation 3.14 is repeated here,

$$y(t) = \frac{y(0)}{\cos \phi} e^{-\zeta \omega_n t} \cos \left[\omega_d t - \tan^{-1} \left(\frac{\dot{y}(0) + y(0)\zeta \omega_n}{y(0)\omega_d} \right) \right],$$

where

$$\cos \phi = \frac{y(0)\omega_d}{\sqrt{[\dot{y}(0) + y(0)\zeta \omega_n]^2 + [y(0)\omega_d]^2}}.$$

We choose $h = 20$ in as the height for a drop test to verify the protective value of the satellite container. This is because of the requirement that maximum relative displacement be less than 4 in and the maximum acceleration be less than $8g$. Both these conditions are met by the drop test height $h = 20$ in. The interior of the box will be instrumented and the drop test results will (or will not) verify the analytical predictions of Table 4.2. ■

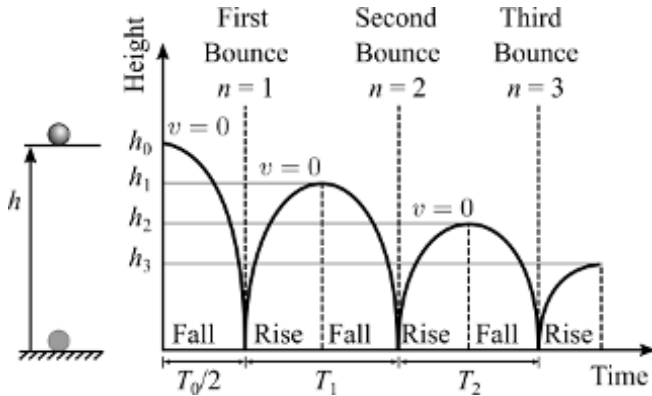


Figure 4.25: Trajectory history of a bouncing ball assuming vertical motion and no aerodynamic drag.

The two examples demonstrate how challenging vibration problems can be using single degree-of-freedom models. The next section presents another example.

4.11 Model of a Bouncing Ball

The bouncing behavior of a dropped ball is a classic problem that links vibration with dynamics. The question is whether one can determine a linear model of a bounced ball based on its impact behavior, essentially replacing the ball by an equivalent mass-spring-damper model.

The trajectory history for the first few bounces is depicted in Figure 4.25. It assumes the ball is dropped from rest from initial height h_0 , the bounces are vertical, and there is no aerodynamic drag, making the rise and fall phases of the trajectory symmetric. At each bounce, the maximum height decreases as energy is lost in the collision.

It is possible to determine an equivalent mass-spring-damper model of a ball.²³ The linear model will be an approximation that only partially captures the true physics. Furthermore, the properties of the model can be related to the coefficient of restitution, a measure of rebound behavior in a collision, and the bounce contact time, the time the ball is in contact with the table or surface, which we will refer to as ground.

The behavior of a dropped ball is studied using the model illustrated in Figure 4.26, where the ball is represented by its mass m , viscous damping c , and linear stiffness k . When the ball is not in contact with the ground, the equation of motion for the case of no drag²⁴

²³The authors are grateful to Shuguang Huang (Marquette University) for his help with this development. For more information about a ball model and its connection to impact-related properties, see M. Nagurka and S. Huang, “A Mass-Spring-Damper Model of a Bouncing Ball,” *International Journal of Engineering Education*, Vol.22, No.2, 2006, pp.393-401.

²⁴For details of a model accounting for aerodynamic drag, the reader is referred to M.L. Nagurka, “Aerodynamic Effects in a Dropped Ping-Pong Ball Experiment,” *International Journal of*

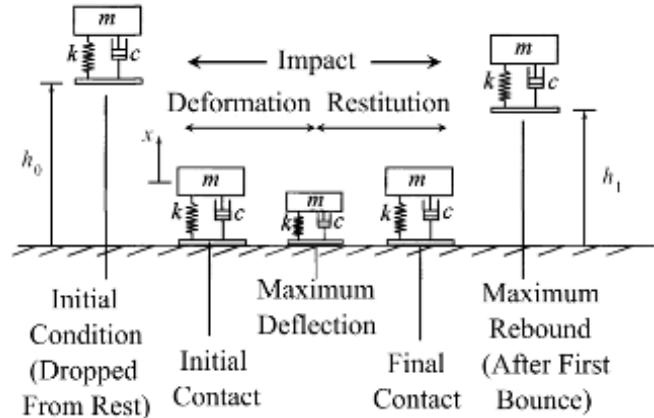


Figure 4.26: A mass-spring-damper model of a ball showing impact phases at the first bounce.

is simply

$$m\ddot{x} = -mg,$$

where x is measured vertically up to the ball's center of mass with $x = 0$ corresponding to initial contact, i.e., when the ball just contacts the ground with no deformation. The initial conditions are $x(0) = h_0$ and $\dot{x}(0) = 0$ for a ball released from rest from height h_0 .

When the ball is in contact with the ground, deformation and restitution occur. The equation of motion is then

$$m\ddot{x} + c\dot{x} + kx = -mg, \quad (4.41)$$

with the initial conditions of $x(0) = 0$ and $\dot{x}(0) = -v_0$ where v_0 is the velocity of the ball just prior to contact with the ground. Integrating Equation 4.41 gives²⁵

$$x(t) = -\frac{mg}{k} + \left[\frac{cg - 2kv_0}{2k\omega_d} \sin \omega_d t + \frac{mg}{k} \cos \omega_d t \right] \exp\left(-\frac{c}{2m}t\right), \quad (4.42)$$

where the damped natural frequency is

$$\omega_d = \frac{1}{2m} \sqrt{4km - c^2}. \quad (4.43)$$

Equation 4.42 gives the motion of the ball during contact with the ground and applies only when $x \leq 0$. Bounce behavior, involving deformation, restitution, and then rebound, requires an underdamped solution for which $\omega_d > 0$ or $(4km - c^2) > 0$. The “steady” or rest solution, applying after the bounces have died out, can be obtained by setting $t \rightarrow \infty$ in Equation 4.42. The equilibrium position is

$$x^* = -\frac{mg}{k},$$

and when $|x| \leq |x^*|$ there will be no further bounces. It follows that the number of bounces is finite.

Engineering Education, Vol.19, No.4, 2003, pp.623-630.

²⁵For clarity, we use the notation here $\exp(\beta) = e^\beta$. Letter e will be used to denote coefficient of restitution.

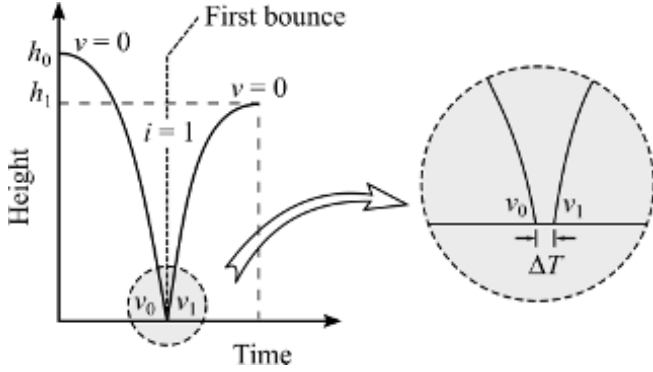


Figure 4.27: Height vs. time and exaggerated view at first bounce.

4.11.1 Time of Contact

The contact time, ΔT , at the first bounce is the time from when the ball reaches $x = 0$ after being dropped to the time it first comes back to $x = 0$. It is shown in exaggerated view in Figure 4.27. Mathematically, the contact time is the first finite solution of the equation $x(\Delta T) = 0$, that is, from Equation 4.42 it is the minimum non-zero solution of

$$\frac{mg}{k} - \left[\frac{cg - 2kv_0}{2k\omega_d} \sin(\omega_d \Delta T) + \frac{mg}{k} \cos(\omega_d \Delta T) \right] \times \exp\left(-\frac{c\Delta T}{2m}\right) = 0, \quad (4.44)$$

which in general has multiple solutions.

Equation 4.44 is difficult to solve analytically. A solution can be found numerically, or alternatively an approximate solution can be obtained by first writing Equation 4.42 in the rearranged form

$$x(t) = -\frac{v_0}{\omega_d} \exp\left(-\frac{c}{2m}t\right) \sin \omega_d t - \frac{mg}{k} \left[1 - \exp\left(-\frac{c}{2m}t\right) \left(\cos \omega_d t + \frac{c}{2m\omega} \sin \omega_d t \right) \right]. \quad (4.45)$$

The maximum magnitude of the first term on the right-hand side, v_0/ω_d , is the dynamic deformation due to the impact for the incoming velocity v_0 ; the maximum of the second term, mg/k , is the static deformation due to the weight. Assuming $mg/k \ll |v_0/\omega_d|$, which is reasonable for a bouncing ball such as a ping-pong ball, the second term on the right-hand side in Equation 4.45 can be neglected and $x(t)$ can be approximated as

$$x(t) = -\frac{v_0}{\omega_d} \exp\left(-\frac{c}{2m}t\right) \sin \omega_d t. \quad (4.46)$$

The contact time, ΔT , can be found as the minimum nonzero solution of Equation 4.46 set equal to zero, giving

$$\Delta T = \frac{\pi}{\omega_d}, \quad (4.47)$$

where ω_d is the damped natural frequency specified by Equation 4.43. Equation 4.47 represents an approximate solution for the contact time at the first bounce.

4.11.2 Stiffness and Damping

The stiffness, k , and damping, c , of the ball can be determined from the contact time, ΔT , and the coefficient of restitution, e , where

$$e = \left| \frac{\dot{x}(\Delta T)}{\dot{x}(0)} \right| \quad (4.48)$$

is the ratio of speed of separation to speed of approach at the first bounce. The coefficient of restitution can be shown to be a measure of the kinetic energy lost in the collision. In a perfectly elastic collision, there is no loss and $e = 1$. In an inelastic collision, $0 < e < 1$. Some kinetic energy is transformed into deformation of the material, heat, sound, and other forms of energy, and is therefore unavailable. This loss is represented in the model by damping, c .

The denominator of Equation 4.48 is simply the speed of the ball prior to the first contact, v_0 , and the numerator is the rebound or post-impact speed of the ball, v_1 . The latter can be found by differentiating Equation 4.45 and imposing the assumption $mg/k \ll |v_0/\omega_d|$, or alternatively, differentiating Equation 4.46 directly to give an expression for the velocity,

$$\dot{x}(t) = \frac{cv_0}{2m\omega_d} \exp\left(-\frac{c}{2m}t\right) \sin \omega_d t - v_0 \exp\left(-\frac{c}{2m}t\right) \cos \omega_d t,$$

and then substituting $t = \Delta T$ with Equation 4.47 to give the rebound velocity,

$$\begin{aligned} v_1 &= \dot{x}(\Delta T) \\ &= v_0 \exp\left(-\frac{c\pi}{2m\omega_d}\right) \\ &= v_0 \exp\left(-\frac{c\Delta T}{2m}\right). \end{aligned}$$

From Equation 4.48, the coefficient of restitution can be written simply as

$$e = \exp\left(-\frac{c\Delta T}{2m}\right). \quad (4.49)$$

For fixed m and c , the shorter the contact time, ΔT , the larger the coefficient of restitution, e , and vice versa.

By manipulating Equations 4.43, 4.47, and 4.49, the stiffness and viscous damping can be written, respectively, as

$$k = m \left(\frac{\pi}{\Delta T} \right)^2 \left[1 + \left(\frac{\ln e}{\pi} \right)^2 \right] \quad (4.50)$$

$$c = -\frac{2m}{\Delta T} \ln e. \quad (4.51)$$

Thus, a shorter contact time, ΔT , corresponds to both a higher stiffness, k , and damping, c . In addition, as e increases, there is a negligible change in k and a reduction in c .

Assuming k , c , and e are constant (independent of the velocity v_0), ΔT will be constant for each contact since ω_d depends only on the system parameters k , c , and m .

4.11.3 Natural Frequency & Damping Ratio

The undamped natural frequency, $\omega_n = \sqrt{k/m}$, can be expressed from Equation 4.50 as

$$\omega_n = \frac{\pi}{\Delta T} \sqrt{1 + \left(\frac{\ln e}{\pi}\right)^2}, \quad (4.52)$$

and is a function of the contact time, and the coefficient of restitution. The damping ratio, ζ ,

$$\zeta = \sqrt{1 - \frac{\omega_d^2}{\omega_n^2}} = \frac{c}{2\sqrt{km}}$$

can be found by substituting Equations 4.47 and 4.52 or Equations 4.50 and 4.51, giving

$$\zeta = \frac{-\ln e}{\pi} \left[1 + \left(\frac{\ln e}{\pi}\right)^2 \right]^{-1/2}. \quad (4.53)$$

The damping ratio depends solely on the coefficient of restitution, e .

4.11.4 Approximations

The approximations developed here are applicable to balls with significant bounce, such as a ping-pong ball. Mathematically, the approximations are valid when $|(\ln e)/\pi| \ll 1$. To see what this means, we can consider the case of $|(\ln e)/\pi| < 0.1$, for which the coefficient of restitution is $0.73 < e < 1$. The high coefficients of restitution indicate that little energy is lost at each collision.

For this case, Equation 4.50 can be approximated as

$$k \simeq m \left(\frac{\pi}{\Delta T} \right)^2, \quad (4.54)$$

which itself is an approximation of Equation 4.47,

$$\Delta T \simeq \frac{\pi}{\omega_n}.$$

From Equation 4.53, it is also possible to write the damping ratio for the case of higher values of e as

$$\zeta \simeq -\frac{\ln e}{\pi},$$

providing a simple direct connection between the damping ratio and the coefficient of restitution.

Figure 4.28 depicts the relationships given in Equation 4.50 and in simplified approximate form in Equation 4.54. The graph shows values of k/m for the range of coefficient of restitution $0.40 \leq e \leq 0.95$ for several values of contact time. The approximate Equation 4.54 provides a highly accurate prediction of the result from Equation 4.50, showing only slight deviation at smaller values of e .

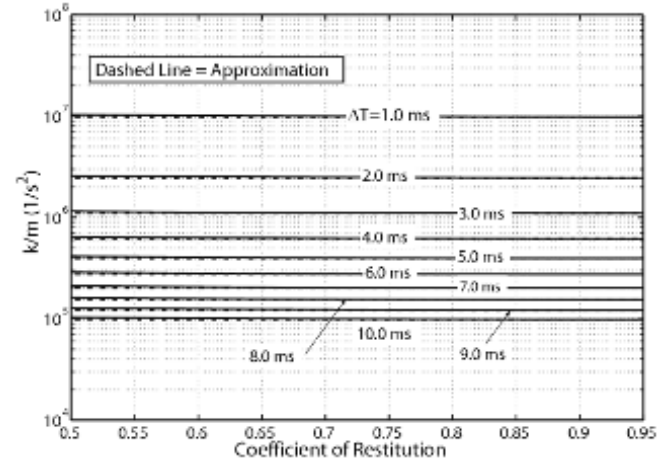


Figure 4.28: Stiffness divided by mass as a function of coefficient of restitution and contact time from Equation 4.50 and for approximation from Equation 4.54.

The damping coefficient c , given by Equation 4.51, is plotted in Figure 4.29 as c/m as a function of both e and ΔT , showing clear dependence on both. Several observations can be made: (a) the larger the contact time ΔT , the smaller the stiffness and the damping, (b) the larger the coefficient of restitution, the smaller the damping, and (c) the coefficient of restitution does not strongly influence the stiffness.

It turns out that the coefficient of restitution, e , and the contact time, ΔT , are parameters that can be found with reasonable ease in an experiment. The results of the experiment can then be used to provide predictions of the equivalent stiffness and damping, as well as natural frequency and damping ratio, for a ball.

4.12 Concepts Summary

This chapter has introduced Laplace transform methods and the convolution integral for systems under arbitrary forcing. In addition, specialized forces such as the impulse and the step loadings were presented. Lagrange's equation has been introduced as a prelude to more powerful energy-based techniques for problem formulation. Some advanced ideas were touched upon: randomness, control, identification, and stability. The chapter closes

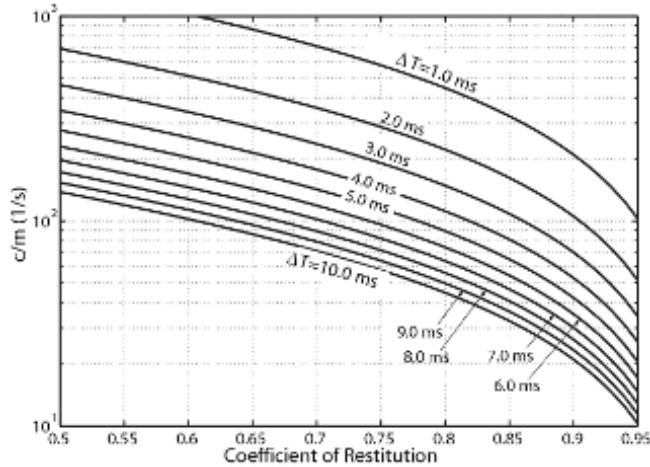


Figure 4.29: Damping coefficient divided by mass as a function of coefficient of restitution and contact time from Equation 4.51.

with two detailed preliminary design problems and a detailed model of a bouncing ball.

4.13 Quotes

- “Look for the answer inside your question.” (Rumi, Mevlana)
- “There is nothing so practical as a good theory.” (Lord Kelvin)
- “I am not getting anything out of the meeting. I am learning nothing. Because there are no experiments, this field is not an active one, so few of the best men are doing work in it. The result is that there are hosts of dopes here ... and it is not good for my blood pressure. Remind me not to come to any more gravity conferences!” (Richard Feynman, 1962, letter to his wife about a meeting in Warsaw where he was presenting his work)
- “We live in a world exquisitely dependent upon science and technology, in which hardly anyone knows anything about science and technology.” (Carl Sagan)
- Mark Nagurka: “I bet the Secretary of Energy doesn’t know what a Joule is.”
Bill Hughes: “I bet the Secretary of Energy thinks it is something you wear.” (Exchange with William F. Hughes)
- “Scientists dream about doing great things. Engineers do them.” (James A. Michener)
- “The engineer has been, and is, a maker of history.” (James Kip Finch)

- “All truth passes through three stages. First, it is ridiculed. Second, it is violently opposed. Third, it is accepted as being self-evident.” (Arthur Schopenhauer)
- “Watch what’s there. And watch what’s not there.” (Halcolm’s Yin and Yang of Observations)
- “We were then satisfied that, with proper lubrication and better adjustments, a little more power could be expected. The completion of the motor according to drawing was, therefore, proceeded with at once.” (Orville Wright)
- “It is not our sexual preferences, the color of our skin, the language we speak, nor the religion we practice that creates friction, hatred and wars amongst in society. It is our words and the words of our leaders that can create that disparity.” (Yehuda Berg)
- “A raised weight can produce work, but in doing so it must necessarily sink from its height, and, when it has fallen as deep as it can fall, its gravity remains as before, but it can no longer do work.” (Hermann von Helmholtz)

4.14 Problems

Problems for Section 4.1 – Arbitrary Loading: Laplace Transform

- Solve the equations of motion using the Laplace transform approach, where $v(t)$ is the speed and $F(t)$ is the force:
 - $\dot{v} + 2v = F(t)$, $v(0) = 1$, $F(t) = 0$
 - $2\dot{v} + 2v = F(t)$, $v(0) = 0$, $F(t) = 10$
 - $\dot{v} + 3v = F(t)$, $v(0) = 1$, $F(t) = 2 \sin(3t)$.
- Solve the equations of motion using the Laplace transform approach:
 - $\ddot{x} + 3\dot{x} + 2x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$
 - $2\ddot{x} + 5\dot{x} + 3x = 0$, $x(0) = 1$, $\dot{x}(0) = 1$
 - $\ddot{x} + 2\dot{x} + 2x = 0$, $x(0) = 2$, $\dot{x}(0) = 4$
 - $\ddot{x} + 2\dot{x} + x = 0$, $x(0) = 1$, $\dot{x}(0) = -1$
 - $\ddot{x} - 2\dot{x} + 3x = 0$, $x(0) = 0$, $\dot{x}(0) = 2$.
- Solve the equations of motion using the Laplace transform approach:
 - $\ddot{x} + 3\dot{x} + 2x = u(t)$, $x(0) = 1$, $\dot{x}(0) = 0$
 - $2\ddot{x} + 5\dot{x} + 3x = e^{-3t}$, $x(0) = 1$, $\dot{x}(0) = 1$
 - $\ddot{x} + 2\dot{x} + 2x = \cos t$, $x(0) = 2$, $\dot{x}(0) = 4$
 - $\ddot{x} + 2\dot{x} + x = e^{-2t} + \sin 2t$, $x(0) = 1$, $\dot{x}(0) = -1$
 - $\ddot{x} - 2\dot{x} + 3x = t$, $x(0) = 0$, $\dot{x}(0) = 2$.

4. Solve the equations of motion using the Laplace transform approach:

- (a) $\ddot{y} + 2\dot{y} + 3y = 5 \cos 3t$, $y(0) = 3$, $\dot{y}(0) = 4$
 (b) $4\ddot{y} + 5\dot{y} + 5y = 4u(t)$, $y(0) = 1$, $\dot{y}(0) = 1$, where $u(t)$ is the unit step function
 (c) $3\ddot{y} + 3\dot{y} + 6y = 3e^{-t} + 2 \cos 3t$, $y(0) = 2$, $\dot{y}(0) = 4$
 (d) $\ddot{y} + \dot{y} + y = F(t)$, $y(0) = 0$, $\dot{y}(0) = 0$, where $F(t)$ is given by a *square wave* function with maximum amplitude 1 and period 1
 (e) $\ddot{y} + 2\dot{y} + 3y = \cos 3t + \cos 5t$, $y(0) = 0$, $\dot{y}(0) = 0$.

5. For the given transfer functions, find the response of the system due to the unit step input force $F(t) = u(t)$ and zero initial conditions:

- (a) $X(s)/\mathcal{F}(s) = 1/(s+2)$
 (b) $X(s)/\mathcal{F}(s) = 1/(s^2 + 3s + 2)$
 (c) $X(s)/\mathcal{F}(s) = (3s+2)/(s^2 + 3s + 2)$.

6. For the given transfer functions, find the response of the system due to the input force $F(t) = 2 \cos(3t)u(t)$ and zero initial conditions:

- (a) $X(s)/\mathcal{F}(s) = 1/(s+2)$
 (b) $X(s)/\mathcal{F}(s) = 1/(s^2 + 3s + 2)$
 (c) $X(s)/\mathcal{F}(s) = 1/(s^2 + 2s + 1)$.

7. Consider a system subjected to two input forces, as shown in Figure 4.30. Find the transfer functions, $X(s)/\mathcal{F}_1(s)$ and $X(s)/\mathcal{F}_2(s)$. Find the response $x(t)$ due to $F_1(t) = t$ and $F_2(t) = u(t)$. Use zero initial conditions, and let $m = 1$ kg, $k_1 = 500$ N/m, and $k_2 = 1000$ N/m.

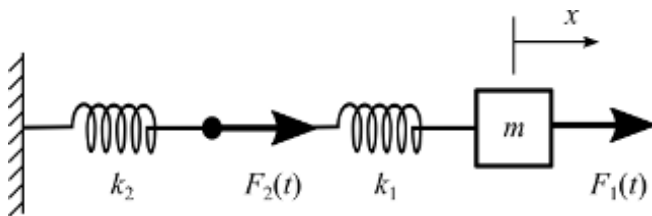


Figure 4.30: Two-input system.

Problems for Section 4.2 – Step Loading

8. A mass of $m = 50$ kg is suspended by a spring attached to a rope that is wound around a drum with radius $r = 0.4$ m and moment of inertia about the point of rotation $I_O = 45$ kg m², as shown in Figure 4.31. The spring of stiffness $k = 3$ kN/m reduces the shock of an applied load. The system is initially at rest when a constant torque $T = 1$ kN-m is applied to the system. (a) Derive an expression for the tension in the rope after a time t . (b) What is the maximum tension in the rope?

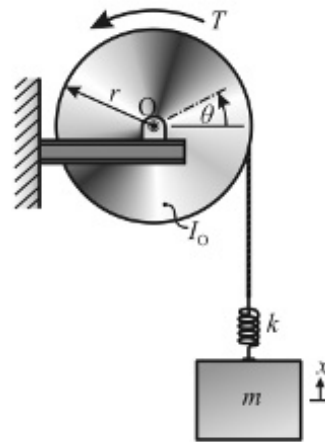


Figure 4.31: A mass suspended via a spring and a rope to a drum.

9. Derive Equation 4.19:

$$v(t) = \frac{1}{\omega_n^2} \left[1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t \right) \right]$$

for $t \geq 0$.

10. Assuming $\zeta < 1$, solve Equation 4.20,

$$v(t) = \mathcal{L}^{-1} \left[\frac{e^{-st_0}}{s} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right].$$

Assume $0 < \zeta < 1$.

11. Using the Laplace transform, solve

$$\ddot{y} + 2\dot{y} + y = F(t),$$

where

$$F(t) = u(t-1) - 2u(t-2) + u(t-3)$$

is in terms of unit step functions, and $y(0) = 0$ and $\dot{y}(0) = 0$.

Problems for Section 4.4 – Arbitrary Loading: Convolution Integral

12. Find the impulse response function of a system whose equation of motion is given by

- (a) $\dot{x} + 4x = f(t)$
 (b) $\ddot{x} + 4x = f(t)$
 (c) $\ddot{x} + 5\dot{x} + 4x = f(t)$
 (d) $\ddot{x} + 2\dot{x} + x = f(t)$
 (e) $\ddot{x} + 5\dot{x} + 4x = 5\dot{f}(t) + 4f(t)$.

13. Find the impulse response function of a system whose transfer function is given by

- (a) $G(s) = 1/(2s + 1)$
- (b) $G(s) = 1/(2s^2 + 3s + 1)$
- (c) $G(s) = 1/(s^2 + 4s + 4)$
- (d) $G(s) = 1/(s^2 + 9)$
- (e) $G(s) = (5s + 4)/(s^2 + 5s + 4)$.

14. Find the response of each of the systems given below to arbitrary inputs, $f(t)$:

- (a) $\dot{x} + 4x = f(t)$, $f(t) = u(t)$
- (b) $\ddot{x} + 4x = f(t)$, $f(t) = t$, $t \geq 0$
- (c) $\ddot{x} + 5\dot{x} + 4x = f(t)$, $f(t) = 1$ for $0 \leq t < 2$, and $f(t) = 0$ for $t \geq 2$
- (d) $\ddot{x} + 2\dot{x} + x = f(t)$, $f(t) = u(t)$.

15. Find the response of the undamped oscillator $\ddot{x} + 4x = F(t)/m$ to each of the following forces per unit mass using the convolution integral:

- (a) $F(t)/m = 1 - e^{-t}$, $t \geq 0$
- (b) $F(t)/m = \cos 2t$, $0 \leq t \leq \pi$
- (c) $F(t)/m = \cos 2t + 3$, $0 \leq t \leq \pi$
- (d) $F(t)/m = \cos 2t + \cos 3t$, $0 \leq t \leq \pi$.

16. Find the response of the damped oscillator $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = F(t)/m$, with $\omega_n^2 = 4$ (rad/s)², to the forces per unit mass listed, and solve for the two underdamped cases of $\zeta = 0.1$ and $\zeta = 0.9$:

- (a) $F(t)/m = 1 - e^{-t}$, $t \geq 0$
- (b) $F(t)/m = \cos 2t$, $0 \leq t \leq \pi$
- (c) $F(t)/m = \cos 2t + 3$, $0 \leq t \leq \pi$
- (d) $F(t)/m = \cos 2t + \cos 3t$, $0 \leq t \leq \pi$.

17. Solve for the damped response beginning with the equation,

$$x(t) = \frac{1}{m\omega_d} \int_0^t \sin \omega(t - \tau) \exp(-\zeta\omega_n\tau) \sin \omega_d\tau d\tau.$$

In the same graph, plot the results for the following cases:

$$\omega = 1.0, 1.5, 2.0, 2.5, 3.0 \text{ rad/s,}$$

where $\zeta = 0.1$, $\omega_n = 2$ rad/s, and $m = 1$ kg. Discuss.

18. The stepped disk in Figure 4.32 is subject to the moment

$$M(t) = 1 - e^{-t}, \quad t \geq 0.$$

The spring is unstretched when $\theta = 0$. Derive the equation of motion and solve for the response $\theta(t)$ assuming zero initial conditions and an underdamped system.

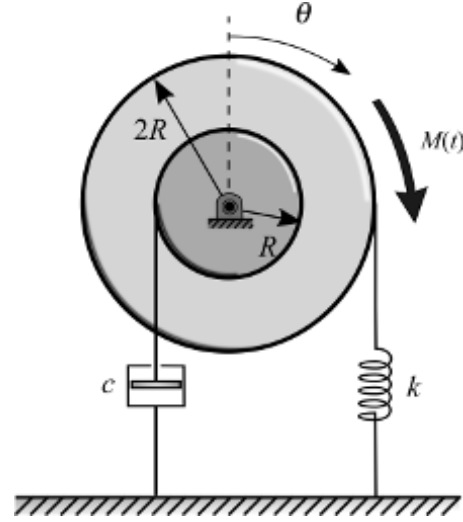


Figure 4.32: A stepped disk undergoing rotation in response to an impulsive moment.

19. Structures are sometimes subjected to very rapidly applied loads of extremely short duration. These types of loads are called *blast* or *explosive* loads. Consider how such a load time history may look. Figure 4.33(a) is a generic blast load. There is a rapid rise time along with an exponential-like decay, and by time t_1 the load is effectively zero, where $t_1 \ll 1$ s. Such a time history is generally a very complicated function that may not be easily determined because of its sensitivity to many factors, such as the medium through which the load passes, be it the atmosphere or the Earth.

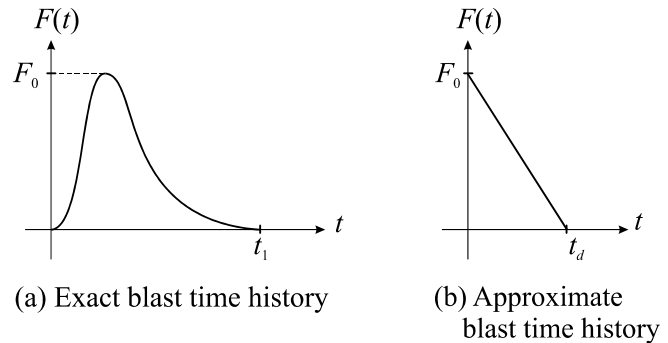


Figure 4.33: Measured (left) and approximate (right) blast time histories.

A common approach is to curve-fit the time history with an appropriate combination of functions. As a first approximation here, we replace the exact curve by the one shown in Figure 4.33(b). To draw such a straight-line approximation, estimate the area under the original curve (this is the impulse) and set it equal to the area under the triangle load, thus fixing the value of t_d . The maximum value F_0 is the

same for both curves. Although not strictly true for such *high strain-rate* loading, we neglect the effects of damping for the early time response. Solve for the structural response for all time; $0 \leq t \leq t_d$ and $t \geq t_d$. Show that:

$$x(t) = \frac{F_0}{m\omega_n^2} \left\{ 1 - \frac{t}{t_d} - \cos \omega_n t + \left(\frac{1}{\omega_n t_d} \right) \sin \omega_n t \right\}, \quad 0 \leq t \leq t_d$$

$$x(t) = \frac{F_0}{m\omega_n^3 t_d} \{ \sin \omega_n t (1 - \cos \omega_n t_d) - \cos \omega_n t (\omega_n t_d - \sin \omega_n t_d) \}, \quad t > t_d.$$

20. Resolve Problem 19 including damping. Discuss the importance of damping by comparing the two results.
21. Solve for the response for all time of an underdamped oscillator that is driven by the forcing function shown in (a) [Figure 4.34](#), and (b) [Figure 4.35](#). Use the convolution integral.

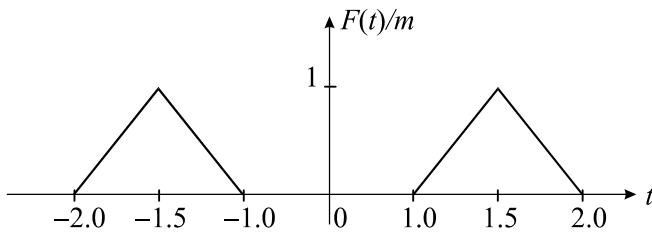


Figure 4.34: Triangular loading function.

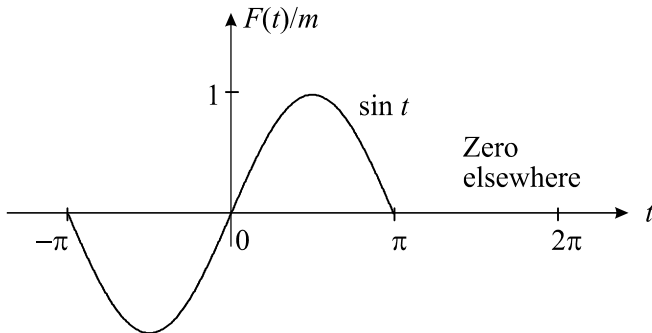


Figure 4.35: Sinusoidal loading function.

Problems for [Section 4.5](#) – Introduction to Lagrange's Equation

22. Use Lagrange's equation to derive the equation of motion for the simple pendulum of [Example 4.15](#), except here assume that the mass m is suspended on a rigid bar that is connected to the support via a torsional spring of stiffness K .

23. For [Example 4.16](#) derive the equation of motion using x as the generalized coordinate. Then solve this equation with initial conditions x_0, v_0 to find

$$x(t) = \frac{v_0}{\omega} \sin \omega t - \frac{mg}{2k} \sin \phi (\cos \omega t - 1) + x_0,$$

where $\omega = \sqrt{4k/3m}$.

24. Derive the equation of motion of [Example 4.16](#) using Newton's second law of motion.

Problems for [Section 4.6](#) – Notions of Randomness

25. For the problem of oscillator control, given by [Equation 4.32](#), consider the specific governing equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{A}{m} \cos \omega t + F_{\text{control}}(t),$$

where ζ , ω_n , and $F_{\text{control}}(t)$ must be determined so that the maximum amplitude of the response is $x_{\text{max}} < \beta A$, where $\beta = 0.5$. Since there is no single answer, describe how to proceed and what considerations must be made during the analysis. Provide one solution.

26. How does randomness of excitation alter the analyst's ability to evaluate structural response?
27. What difficulties arise if a system parameter such as k or m is known only approximately?

Problems for [Section 4.7](#) – Notions of Control

28. If the structure responds in a way that is unacceptable for a particular application, what options exist for the designer?
29. If the analyst decides that a structure requires control in order to fulfill the needs of an application, describe the procedure by which a control force can be derived and then verified that it performs as expected.
30. A simple system is represented by the governing equation

$$m\ddot{x} + kx = F(t),$$

where $m = 1$ kg, $k = 10$ N/m, and $F(t) = \cos t$ N, with zero initial conditions. Find the steady-state response $x_s(t)$. Now suppose that the system is constrained regarding its maximum allowable amplitude, $|x_{\text{max}}| \leq 0.05$ m. Determine the needed $F_{\text{control}}(t)$.

Problems for Section 4.8 – The Inverse Problem

31. Consider the simple system, $m\ddot{x} + kx = F(t)$, where $F(t)$ is known and m and k are unknown. Suppose the initial conditions are both equal to zero. What can we determine about the values of m and k ?
32. Consider the simple system, $m\ddot{x} + kx = F(t)$, where $F(t)$ and steady-state solution $x_s(t)$ are known and m and k are unknown. Suppose the initial conditions are both equal to zero. What can we determine about the values of m and k ?
33. Consider the simple system, $m\ddot{x} + kx = F(t)$, where steady-state solution $x_s(t)$, m , and k are known and $F(t)$ is unknown. Suppose the initial conditions are both equal to zero. What can we determine about $F(t)$?

Problems for Section 4.9 – A Self-Excited System and Its Stability

34. Consider a damped oscillator where the force is a function of velocity,

$$m\ddot{x} + c\dot{x} + kx = F_0\dot{x}.$$

Discuss the stability of this system in terms of the parameters m , c , k , F_0 .

35. Consider a damped oscillator where the force is a function of displacement,

$$m\ddot{x} + c\dot{x} + kx = F_0x.$$

Discuss the stability of this system in terms of the parameters m , c , k , F_0 .

Problems for Section 4.10 – Solution Analysis and Design Techniques

36. For an m - c - k oscillator subjected to harmonic loading, $Ak \cos \omega t$, conceive a design so that the displacement response $|x(t)| < A$ for all time. Assume $A = 10$ cm, $m = 1$ kg, $k = 1$ N/cm, and $\omega = 3$ rad/s. In your design, specify c and k . To do this, find the upper limit for $x(t)$.
37. For an oscillator governed by $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \cos \omega t$, conceive a design so that the displacement response $|x(t)| < A$ for duration t_0 . Assume $A = 10$ cm. In your design, specify c and k assuming $m = 1$.
38. Solve Problem 37 for the load shown in Figure 4.36.

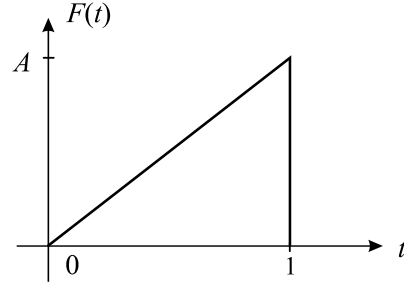


Figure 4.36: Ramp loading.

39. Solve Problem 37 for the load as shown in Figure 4.37.

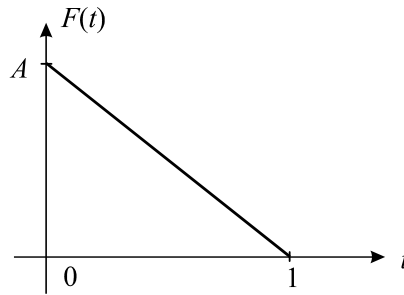


Figure 4.37: Simplified representation of a shock loading.



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Chapter 5

Variational Principles and Analytical Dynamics

“Energy is the basis.”

5.1 Introduction

This chapter presents several fundamental concepts from analytical dynamics. The most important of these concepts is *Lagrange’s equation*, which we were introduced to in [Chapter 4](#). We derive it in this chapter and use it to obtain the equations of motion for discrete systems (and continuous systems later). We will also derive *Hamilton’s principle*, an integral energy formulation applicable to both discrete and continuous systems, and see how it is related to Lagrange’s equation.¹ The basis of this chapter is the *principle of virtual work*.

In comparison to Newton’s second law of motion, the *analytical approach* of Lagrange and Hamilton has advantages. The advantages are especially evident for systems of interacting bodies, where each body exerts a force on the other and where constraints, such as boundaries, also exert forces on the system, limiting motion. Such *auxiliary conditions* can be handled more easily using the analytical approach.

The equations of motion obtained from the analytical approach are independent of the coordinate system. Newton’s second law of motion requires an inertial coordinate system, and the equations derived from the second law depend on the choice of coordinate system.

The analytical approach, derived from *variational principles*, is the unifying basis of the equations that follow. The term variational comes from the *calculus of variations*,² the foundation for such techniques.

¹There are many excellent books on analytical dynamics. Two classics are (1) **Methods of Analytical Dynamics**, L. Meirovitch, McGraw-Hill, 1970, and (2) **The Variational Principles of Mechanics**, C. Lanczos, Dover, 1986. The latter book is rigorous, yet almost a private conversation with the author. Much emphasis is placed on physical understanding. It was originally published in 1949.

²A recommended book on the calculus of variations, especially for those interested in applications, is **Methods of Applied**

To motivate and explain the variational procedure, consider the simple function: $y = f(x)$. The dependent variable y can represent the vertical displacement of a cable or a beam, for example. The variational approach is based on comparing the function $f(x)$ with a slightly modified function $f_\epsilon(x) = f(x) + \epsilon\phi(x)$, where ϵ is small and ϕ is continuous and differentiable. For any value of the independent variable x , the *variation*, or difference, is defined as δy ,

$$\delta y \equiv f_\epsilon(x) - f(x) = \epsilon\phi(x).$$

There are two fundamental conditions for the variation: (a) it is *arbitrary* or *virtual*, and (b) it is an infinitesimal change since ϵ can be made arbitrarily small. Note that while both δy and dy represent infinitesimal changes in the function $f(x)$, dy refers to a change in $f(x)$ caused by an infinitesimal change of the independent variable dx , whereas δy is an infinitesimal change of y that is due to a change in the *function* and results in a *new function* $y + \delta y = f(x) + \epsilon\phi(x)$.

This process of variation applies for each value of x . The value of x is not varied, meaning that $\delta x = 0$. If the two end points of this function are prescribed, they also do not vary. The variation is between definite limits. When time is the independent variable, the beginning and ending times are prescribed and therefore not varied. More advanced aspects of variational problems allow variations of the end points.

As we will discover, in applying the variational procedures to a particular system, we find the governing equation(s) of motion and the necessary number of boundary conditions. *The stationary value conditions imposed by the variational principles result in both the differential equation(s) and the boundary conditions.*

The key topics to be examined in this chapter are

Mathematics, F.B. Hildebrand, 2nd ed., Prentice Hall, 1965. It is currently available from Dover Publications. A more advanced introduction is **Calculus of Variations**, I.M. Gelfand and S.V. Fomin, Prentice Hall, 1963.

(a) the *principle of virtual work* and its relation to the equilibrium of a body,

(b) the principle of virtual work, in conjunction with *d'Alembert's principle*, and

(c) Lagrange's equation, and Hamilton's and Jourdain's variational principles.

This chapter presents many examples showing the application of these principles. Further applications will be demonstrated for multi degree-of-freedom systems in Chapter 6 and continuous systems in Chapters 7 and 8.

5.2 Constraints

All dynamic systems operate under *constraints*.³ These may limit the motion space of the system and are encountered generally as a result of contact between components and contact between the system and its environment. Restrictions resulting from physical constraints can be expressed through constraint equations that define the limitations on system motion and the constraint forces due to contact.

A particle moving on a surface is constrained by the shape of the surface and the particle experiences reaction forces of a magnitude and direction that are functions of the geometry of the surface and its constitutive properties (its elasticity).

A constraint equation representing the shape of a three-dimensional surface can be written in general as

$$f(x, y, z, t) = 0.$$

This constraint equation is also called a *configuration constraint*. The differential of the constraint is given by the relation

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial t} dt = 0.$$

Differentiating this equation with respect to time, or just dividing by dt , we obtain a *velocity or motion constraint*,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} + \frac{\partial f}{\partial t} = 0.$$

In general, the velocity constraint can be written as

$$a_x \dot{x} + a_y \dot{y} + a_z \dot{z} + a_0 = 0,$$

where a_x, a_y, a_z , and a_0 are the partial derivatives of f with respect to x, y, z , and t , respectively, and the dots represent time derivatives. A constraint that can be expressed as both a configuration constraint and a velocity constraint is called *holonomic*. If it cannot be so expressed then the constraint is called *nonholonomic*.

Holonomic constraints are subdivided into two types: *scleronomic* if the constraint equation is not an explicit function of time, and *rheonomic* if it is an explicit function of time. Constraints that are explicit functions of time pose additional challenges.



POWER FOR KEEPING TIME. A self-winding pocket watch was invented by Swiss horologist Abraham-Louis Perrelet before the French revolution (in about 1770). The mainspring of his self-winding watch was wound using the energy generated by the natural movements of the wearer.

Another source of power was used in the clock "Atmos" invented in 1927 by French engineer Jean-Leon Reutter. The clock is wound using expansion or contraction of a sealed gas-filled capsule due to natural day-night changes in ambient temperature and periodic variations in atmospheric pressure. Before making the self-winding "Atmos" clock, Reutter designed the 400-day Anniversary clock, reducing the power consumption to an extremely low level. It was so low that winding its spring once every 400 days was enough for time-keeping.⁴



5.3 Virtual Work

The principle of virtual work is the basis for solving vibration problems in the remainder of this chapter. It forms the foundation for the variational principles of mechanics, and offers important advantages.

5.3.1 Work and Energy

The concepts of work and energy are reviewed before proceeding to the topic of *virtual work*. Consider a particle of mass m moving along a curve C under the action of a force $\mathbf{F}$ as shown in Figure 5.1. In this chapter, vectors are shown as boldface variables, and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors in the x, y , and z directions, respectively.

The position of the particle with respect to an origin O is given by $\mathbf{r}$, which is a function of time. The work necessary to move the particle a differential distance $d\mathbf{r}$ is $dW = \mathbf{F} \cdot d\mathbf{r}$. The *work done* to move the particle from position 1 to position 2 is

$$W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}.$$

Assuming the mass of the particle to be constant, Newton's second law of motion can be written as

$$\mathbf{F} = m \frac{d\dot{\mathbf{r}}}{dt} = m \frac{d}{dt} \left(\frac{d\mathbf{r}}{dt} \right).$$

³ **Analytical Dynamics**, H. Baruh, McGraw-Hill, 1999.

⁴ V. Leonov and R.J. Vullers, "Wearable Electronics Self-Powered by Using Human Body Heat," *Journal of Renewable and Sustainable Energy*, Vol.1 (062701), 2009, pp.2-3.

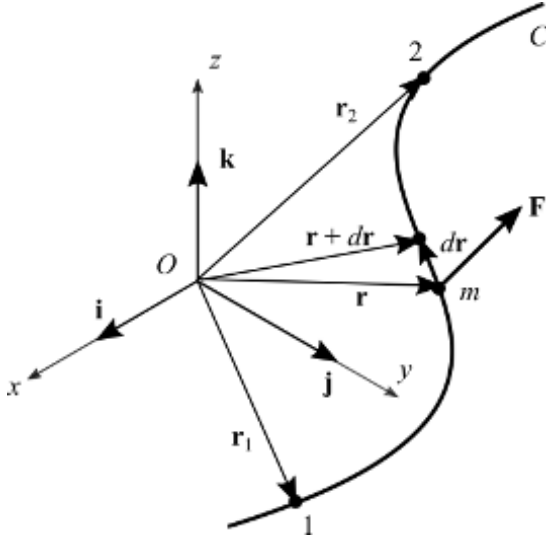


Figure 5.1: Path C of particle of mass m due to force $\mathbf{F}$. The position of the mass is denoted by $\mathbf{r}$.

This equation is equivalent to, but written in a slightly different form than, that used in Chapter 2 to derive the equation of motion for a dynamic system. The goal here is to connect the concepts of force, work, and energy. Using $d\mathbf{r} = \dot{\mathbf{r}}dt$ with the equations above, the work can be rewritten as

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} m \frac{d\dot{\mathbf{r}}}{dt} \cdot \dot{\mathbf{r}} dt \\ &= \frac{1}{2} \int_{t_1}^{t_2} m \frac{d}{dt} (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) dt \\ &= \frac{1}{2} m [(\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2) - (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1)] \\ &= \frac{1}{2} m (\dot{r}_2^2 - \dot{r}_1^2) \\ &= T_2 - T_1, \end{aligned}$$

where the limits of integration have been transformed from $\mathbf{r}$ to t , and $T = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ is the kinetic energy of the mass.

The equation $W_{12} = T_2 - T_1$ is the general *principle of work and energy*. In this equation, both the work and the kinetic energy are scalars. The kinetic energy of a particle is defined as the total work that must be done on the particle to bring it from a state of rest to a velocity $\dot{\mathbf{r}}$. Thus, for $v = |\dot{\mathbf{r}}|$,

$$\begin{aligned} T &= \int_0^v m v dv \\ &= \frac{1}{2} m v^2. \end{aligned}$$

The work done by the conservative force can be related to the change in position of the mass. To develop this relationship, we first introduce the concept of *conservative*

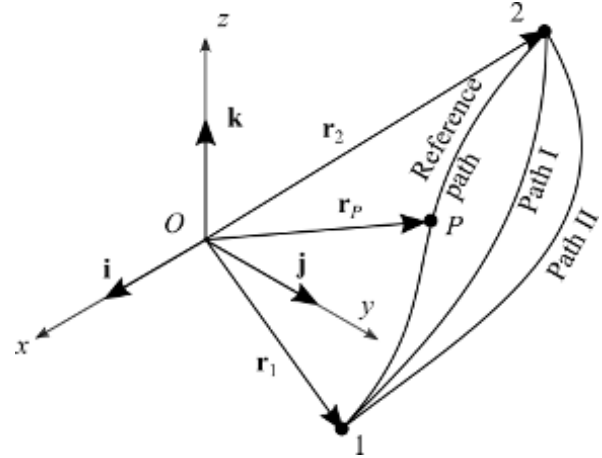


Figure 5.2: Paths in a conservative force field. Any path chosen between points 1 and 2 results in the same work done. Time is not a parameter here.

force field as one where the work done depends only on the *initial* and *final* positions of the particle and is *independent* of the path connecting these positions. An example of a conservative force field is gravity. Nonconservative forces, such as friction – which is *energy-dissipating* – external forces and forces that depend on velocity, are functions that depend on more than position. The work done by nonconservative forces is path-dependent.

From Figure 5.2, any path within the conservative force field that connects points 1 and 2 can be selected, and the work done in changing the particle's position from 1 to 2 will be the same regardless of the path chosen. Denoting this as conservative work, W_{12} , it can be expressed as

$$W_{12c} = \underbrace{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}}_{\text{Path I}} = \underbrace{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}}_{\text{Path II}}.$$

If we denote the *potential energy* associated with position $\mathbf{r}_1$ to be $V(\mathbf{r}_1)$, then we can define the potential energy as the negative of the work done by a conservative force to move an object from its *reference position* $\mathbf{r}_p$ to a current position, $\mathbf{r}_1$,

$$V(\mathbf{r}_1) = - \int_{\mathbf{r}_p}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$

The work done moving a particle in a conservative force field can now be related to the potential energy of the particle. We consider again W_{12c} but choose an arbitrary path through reference position $\mathbf{r}_p$. Then,

$$\begin{aligned} W_{12c} &= \int_{\mathbf{r}_1}^{\mathbf{r}_p} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{r}_p}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{\mathbf{r}_1}^{\mathbf{r}_p} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathbf{r}_2}^{\mathbf{r}_p} \mathbf{F} \cdot d\mathbf{r}, \end{aligned}$$

and

$$\begin{aligned} W_{12c} &= -[V(\mathbf{r}_2) - V(\mathbf{r}_1)] \\ &= -(V_2 - V_1). \end{aligned} \quad (5.1)$$

Equation 5.1 is a statement that *the work done in a conservative force field is the negative of the change in potential energy.*

If W_{12nc} denotes the nonconservative work, then

$$\begin{aligned} W_{12nc} &= W_{12} - W_{12c} \\ &= (T_2 - T_1) + (V_2 - V_1) \\ &= (T_2 + V_2) - (T_1 + V_1) \\ &= E_2 - E_1, \end{aligned}$$

where E_i represents the total energy at point i . Therefore, W_{12nc} is a measure of the change in particle energy due to dissipation. If $W_{12nc} = 0$, then $E_1 = E_2$ and the energy of the particle is constant, indicating that energy is conserved.

Gravitational potential energy is defined as the work done against the gravitational field g to elevate a particle of mass m a distance h above an arbitrary reference plane called the *datum*. It is given by the expression $V_g = mgh$ and is considered positive when the work done is in the direction of gravity.

5.3.2 Principle of Virtual Work

The *principle of virtual work* states that *for a system at equilibrium the virtual work performed by the applied forces undergoing infinitesimal virtual displacements compatible with the system constraints is zero*, that is,

$$\delta W = \sum_{i=1}^N \mathcal{F}_i \cdot \delta \mathbf{r}_i = 0, \quad (5.2)$$

where N is the number of forces and $\mathcal{F}_i$ denotes the resultant force on particle i and includes external forces as well as internal forces between particles. Therefore, if there is only one force acting on the system, the virtual work is given by

$$\delta W = \mathbf{F} \cdot \delta \mathbf{r} = 0. \quad (5.3)$$

A geometric *constraint* is a physical barrier to free motion, for example, a wall, an inextensible link connecting two bodies, or a magnetic field. Such a constraint can be written in terms of position. Equations 5.2 and 5.3 apply to static systems, to quasi-static systems where inertial effects can be ignored, and to discrete as well as continuous systems. The dynamic version of the principle of virtual work, known as *d'Alembert's principle*, is developed in the next section. These two principles form the basis for the variational principles that follow.

For a system of N particles moving in three dimensions, the *virtual displacements* $\delta \mathbf{r}_i$ can be resolved into each dimension for each particle,

$$\delta x_i, \delta y_i, \delta z_i,$$

where $1 \leq i \leq N$. *Virtual displacements may be interpreted as possible alternate configurations of the particles.* These alternate configurations must be consistent with the system constraints. We consider the system at its initial configuration and at its alternate configuration due to the virtual displacements. Time is not a variable here since we are only considering the system in two possible configurations at the same time. Even if time is a parameter in the constraint equation (time-dependent constraint), it is a kinematic, not dynamic, relation. Time will be considered with d'Alembert's principle later.

Consider, for example, a mass on a rigid link of length r . The position of the mass is constrained to be on the circumference of a circle of radius r , that is, $x^2 + y^2 = r^2$. This equation is a *geometrical constraint* relation between coordinates x, y and parameter r . It is not a dynamic relation since time is not a parameter. We do not know the (x, y) position of the mass as a function of time.

More generally, consider the initial configuration of N particles and their constraints expressed in terms of a single *constraint equation*,

$$g(x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_N, y_N, z_N, t) = c. \quad (5.4)$$

An alternate configuration resulting from a virtual displacement can be written as

$$g(x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1, \dots, z_N + \delta z_N, t) = c, \quad (5.5)$$

where in Equations 5.4 and 5.5 parameter t is included to show that it is not varied. The system constraints are embodied within the constant c , since it is equal to the initial and alternate configurations.

In the formulation below, we examine in detail the rules that govern the variations in Equation 5.5. When completed, we will be able to express the virtual work done by forces undergoing a virtual displacement. The result is interpreted as a statement of static equilibrium.

We proceed by expanding Equation 5.5 about the *unvaried* path via a Taylor series representation where only first-order terms are retained,

$$\begin{aligned} &g(x_1, y_1, z_1, \dots, x_N, y_N, z_N, t) \\ &+ \sum_{i=1}^N \left(\frac{\partial g}{\partial x_i} \delta x_i + \frac{\partial g}{\partial y_i} \delta y_i + \frac{\partial g}{\partial z_i} \delta z_i \right) = c. \end{aligned} \quad (5.6)$$

We know from Equation 5.4 that $g(\dots) = c$, and on substituting, Equation 5.6 yields the relations that must

be satisfied to make the virtual displacements compatible with the system constraints,

$$\sum_{i=1}^N \left(\frac{\partial g}{\partial x_i} \delta x_i + \frac{\partial g}{\partial y_i} \delta y_i + \frac{\partial g}{\partial z_i} \delta z_i \right) = 0. \quad (5.7)$$

Each of the N particles can move in three possible coordinate directions. Therefore, in general, Equation 5.7 relates $3N$ unknowns with one equation. Since one variable may be written in terms of the remainder, there are $3N - 1$ variables.

For two particles, there are two virtual displacements each having three components, δx , δy , and δz . Equation 5.7 becomes

$$\begin{aligned} & \left(\frac{\partial g}{\partial x_1} \delta x_1 + \frac{\partial g}{\partial y_1} \delta y_1 + \frac{\partial g}{\partial z_1} \delta z_1 \right) \\ & + \left(\frac{\partial g}{\partial x_2} \delta x_2 + \frac{\partial g}{\partial y_2} \delta y_2 + \frac{\partial g}{\partial z_2} \delta z_2 \right) = 0, \end{aligned} \quad (5.8)$$

where there are six unknown component displacements and one equation. Any five of these unknowns are independent, with the sixth unknown being a function of those five.

Next, assume that the N particles are subject to resultant forces $\mathcal{F}_i = \mathbf{F}_i + \mathbf{f}_i$, where $\mathbf{F}_i$ are applied forces and $\mathbf{f}_i$ are constraint forces of the geometric constraints, for $i = 1, \dots, N$. For the system to be in static equilibrium, every particle must be at rest, and $\mathcal{F}_i = 0$ for all i in any possible configuration. For the virtual displacement configuration, static equilibrium requires that for all i ,

$$\mathcal{F}_i \cdot \delta \mathbf{r}_i = 0$$

$$\mathcal{F}_i \cdot (\delta x_i \mathbf{i} + \delta y_i \mathbf{j} + \delta z_i \mathbf{k}) = 0.$$

For equilibrium, the virtual work for the entire system vanishes according to the relation

$$\begin{aligned} \delta W &= \sum_{i=1}^N \mathcal{F}_i \cdot \delta \mathbf{r}_i = 0 \\ &= \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i + \sum_{i=1}^N \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0. \end{aligned}$$

Before proceeding, we note that various types of constraints may exist. In the case of physical boundaries, the boundary force is perpendicular to the motion of the body and no work is performed. Dissipative forces such as friction will do work if there is motion. These will be introduced later in this chapter when dynamic motion is added. Therefore, for a static system with ideal constraints, $\sum_{i=1}^N \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$. The remaining equation is given the name the *principle of virtual work* for a static system,

$$\delta W = \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0,$$

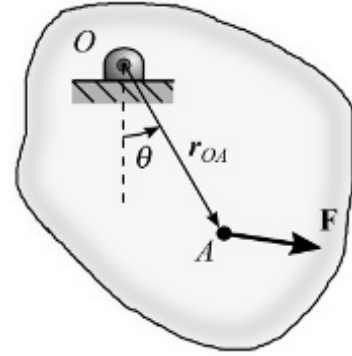


Figure 5.3: A rigid body hinged at O and subjected to an external force, $\mathbf{F}$.

with the applied forces $\mathbf{F}_i$ acting on the system.

For the special case of a conservative system, δW is equal to both the virtual work and the negative of the variation in the potential energy,

$$\begin{aligned} \delta W &= \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \\ \delta W &= -\delta V = - \sum_{i=1}^N \left(\frac{\partial V}{\partial x_i} \delta x_i + \frac{\partial V}{\partial y_i} \delta y_i + \frac{\partial V}{\partial z_i} \delta z_i \right), \end{aligned}$$

where V is the potential energy of the system. Since the variations are independent and arbitrary, the coefficients of the variations must equal zero,

$$\begin{aligned} F_{x_i} &= -\frac{\partial V}{\partial x_i} = 0 \\ F_{y_i} &= -\frac{\partial V}{\partial y_i} = 0 \\ F_{z_i} &= -\frac{\partial V}{\partial z_i} = 0. \end{aligned}$$

These three equations can be used to define the static equilibrium configuration for the system.

For a rigid body hinged at O as shown in Figure 5.3, the virtual work due to the force $\mathbf{F}$ can be written as

$$\delta W = \mathbf{M}_O \cdot \delta \theta,$$

where $\mathbf{M}_O$ is the moment due to $\mathbf{F}$ about O . This can be shown by starting with the definition of work. The work equals the product of the force in the direction of the displacement or

$$\delta W = \mathbf{F} \cdot \delta \mathbf{r}_{OA},$$

where $\delta \mathbf{r}_{OA}$ is the virtual displacement of the point A . Since the system is pivoted about O and the distance between O and A is constant, the virtual displacement can be rewritten as

$$\delta \mathbf{x}_A = \delta \theta \times \mathbf{r}_{OA}.$$

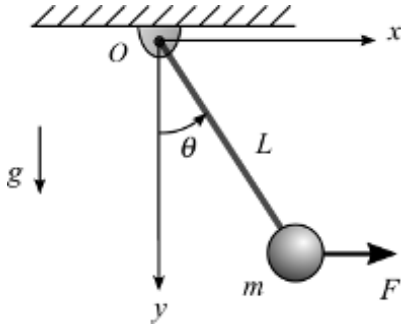


Figure 5.4: A pendulum subject to horizontal force.

Then,

$$\begin{aligned}\delta W &= \mathbf{F} \cdot (\delta \boldsymbol{\theta} \times \mathbf{r}_{OA}) \\ &= (\mathbf{r}_{OA} \times \mathbf{F}) \cdot \delta \boldsymbol{\theta},\end{aligned}$$

where the scalar triple product identity is used. The term, $\mathbf{r}_{OA} \times \mathbf{F}$ is the moment due to the force $\mathbf{F}$. Therefore, we can write the virtual work as

$$\delta W = \mathbf{M}_O \cdot \delta \boldsymbol{\theta}.$$

Example 5.1 Static Equilibrium of a Pendulum

Consider a simple pendulum subject to force F in Figure 5.4. Let the weight of the pendulum be 10 lb and F be 20 lb. Find the equilibrium position θ by (a) Newton's second law and (b) the principle of virtual work.

Solution: This is a one degree-of-freedom system, and we choose θ as our coordinate. (a) To find the equilibrium position via Newton's second law, we draw the free-body diagram as shown, where T_s is the tension in the string or massless rod.

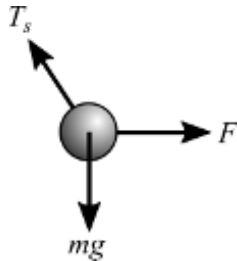


Figure 5.5: Free-body diagram.

Summing the forces in the x and y directions, we obtain

$$\begin{aligned}T_s \sin \theta - F &= 0 \\ T_s \cos \theta - mg &= 0,\end{aligned}$$

from which the equilibrium position θ is

$$\theta = \tan^{-1} (F/mg).$$

(b) The equilibrium position can be found using the principle of virtual work as follows. The virtual work is given by

$$\delta W = F\delta x + mg\delta y = 0,$$

where the virtual displacements, δx and δy , must be determined. Since the horizontal and the vertical coordinates are related to θ by

$$x = L \sin \theta \text{ and } y = L \cos \theta,$$

the virtual displacements are given by

$$\begin{aligned}\delta x &= L \cos \theta \delta \theta \\ \delta y &= -L \sin \theta \delta \theta.\end{aligned}$$

Substituting the virtual displacements into the virtual work expression, we have

$$\begin{aligned}\delta W &= FL \cos \theta \delta \theta - mgL \sin \theta \delta \theta \\ &= (FL \cos \theta - mgL \sin \theta) \delta \theta,\end{aligned}$$

where the tension does not do any work. The virtual work is zero for any choice of $\delta \theta$ if $FL \cos \theta - mgL \sin \theta = 0$. Therefore, the equilibrium position is given by

$$\theta = \tan^{-1} (F/mg),$$

as before. Note that the constraint force – the tension – did not enter into the formulation of virtual work.

The virtual work can be also written in terms of the moment due to the weight and the force F about O , or

$$\delta W = -mgL \sin \theta \delta \theta + FL \cos \theta \delta \theta,$$

from which we obtain the identical equilibrium equation, $-mgL \sin \theta + FL \cos \theta = 0$. ■

Example 5.2 Static Equilibrium of a Bar

Consider the rigid bar shown in Figure 5.6. The bar is pinned at the left and supported by a spring at the right. Assume that the bar is horizontal when the spring is unstretched. Find the equilibrium position of the bar. Assume the bar has a mass of 10 kg and a length of 1 m.

Solution: The free-body diagram of the bar is shown in Figure 5.7. The virtual work done by the individual forces is

$$\delta W = mg\delta \left(\frac{L}{2} \sin \theta \right) - kL \sin \theta \delta (L \sin \theta).$$

Note that the forces at the support O do no work. The virtual work is simplified to

$$\begin{aligned}\delta W &= mg \frac{L}{2} \cos \theta \delta \theta - kL^2 \sin \theta \cos \theta \delta \theta \\ &= \left(mg \frac{L}{2} \cos \theta - kL^2 \sin \theta \cos \theta \right) \delta \theta.\end{aligned}$$

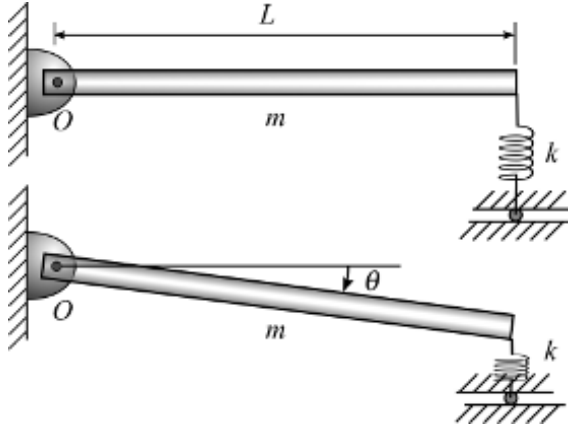


Figure 5.6: Horizontal bar supported by a spring.

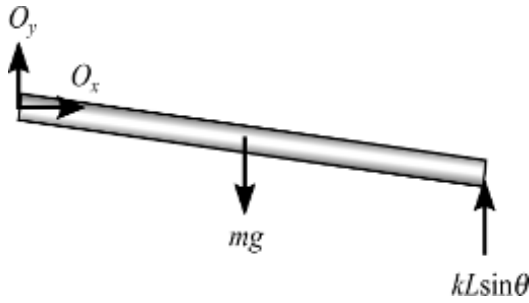


Figure 5.7: Free-body diagram for Example 5.2.

For static equilibrium $\delta W = 0$. Since $\delta\theta$ is arbitrary, the factor in parenthesis must equal zero,

$$\left(mg \frac{L}{2} - kL^2 \sin \theta \right) \cos \theta = 0.$$

Then,

$$\cos \theta = 0 \text{ or } mg \frac{L}{2} - kL^2 \sin \theta = 0.$$

The equilibrium positions are then

$$\theta = -90^\circ, 90^\circ, \text{ and } \sin^{-1} \left(\frac{mg}{2kL} \right).$$

The first two equilibrium positions are unstable, and the third equilibrium position is stable. ■

Example 5.3 Static Equilibrium of a Double Simple Pendulum

Find the equilibrium position of the double simple pendulum in Figure 5.8. An external moment M_O is applied at O and an external force F is applied horizontally to m_2 . Assume the rods are massless and gravity acts in the y direction.

Solution: The system has two degrees of freedom. We will use θ_1 and θ_2 as our coordinates. We can use the coordinates x_1 and y_1 as the horizontal and vertical coordinate of m_1 , and x_2 and y_2 as those of m_2 and then

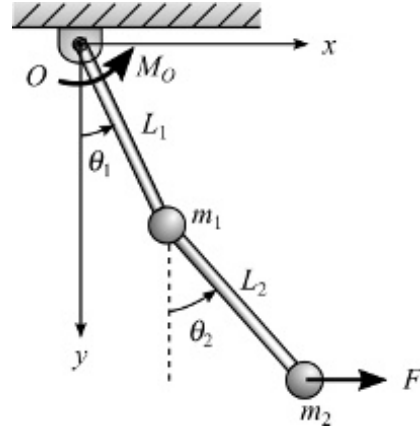


Figure 5.8: Double simple pendulum.

transform these to θ_1 and θ_2 . The virtual work is done by M_O , F , and the weights of each mass, and is given by

$$\delta W = M_O \delta\theta_1 + m_1 g \delta y_1 + m_2 g \delta y_2 + F \delta x_2 = 0.$$

The coordinates, y_1 , y_2 , and x_2 , must now be rewritten in terms of θ_1 and θ_2 . They are related by

$$\begin{aligned} y_1 &= L_1 \cos \theta_1 \\ y_2 &= L_1 \cos \theta_1 + L_2 \cos \theta_2 \\ x_2 &= L_1 \sin \theta_1 + L_2 \sin \theta_2, \end{aligned}$$

giving the virtual displacements

$$\begin{aligned} \delta y_1 &= -L_1 \sin \theta_1 \delta\theta_1 \\ \delta y_2 &= -L_1 \sin \theta_1 \delta\theta_1 - L_2 \sin \theta_2 \delta\theta_2 \\ \delta x_2 &= L_1 \cos \theta_1 \delta\theta_1 + L_2 \cos \theta_2 \delta\theta_2. \end{aligned}$$

Substituting the virtual displacements into the virtual work expression, we obtain

$$\begin{aligned} \delta W &= M_O \delta\theta_1 + m_1 g (-L_1 \sin \theta_1 \delta\theta_1) \\ &\quad + m_2 g (-L_1 \sin \theta_1 \delta\theta_1 - L_2 \sin \theta_2 \delta\theta_2) \\ &\quad + F (L_1 \cos \theta_1 \delta\theta_1 + L_2 \cos \theta_2 \delta\theta_2), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \delta W &= (M_O - m_1 g L_1 \sin \theta_1 - m_2 g L_1 \sin \theta_1 \\ &\quad + F L_1 \cos \theta_1) \delta\theta_1 \\ &\quad + (-m_2 g L_2 \sin \theta_2 + F L_2 \cos \theta_2) \delta\theta_2. \end{aligned}$$

The equilibrium equations are then

$$\begin{aligned} M_O - m_1 g L_1 \sin \theta_1 - m_2 g L_1 \sin \theta_1 + F L_1 \cos \theta_1 &= 0 \\ -m_2 g L_2 \sin \theta_2 + F L_2 \cos \theta_2 &= 0. \end{aligned}$$

These two equations can be solved simultaneously to find the equilibrium angles, θ_1 and θ_2 . ■

We proceed next with d'Alembert's principle, which generalizes the principle of virtual work to time-dependent problems.

5.3.3 D'Alembert's Principle

D'Alembert extended the applicability of the principle of virtual work to dynamic problems. Newton's second law of motion can be rewritten as d'Alembert's principle in the following form for N particles,

$$\mathbf{F}_i + \mathbf{f}_i - m_i \ddot{\mathbf{r}}_i = \mathbf{0}, \quad i = 1, 2, \dots, N. \quad (5.9)$$

The term $-m_i \ddot{\mathbf{r}}_i$ represents an *inertial force*. Each force in Equation 5.9 may be a constant or a function of time. The virtual work performed by the forces acting on the i^{th} particle is given by

$$(\mathbf{F}_i + \mathbf{f}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0,$$

where the virtual displacement $\delta \mathbf{r}_i$ is compatible with the constraints. Assuming the virtual work due to the constraint forces equals zero, the virtual work for the system is given by the relation

$$\sum_{i=1}^N (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0. \quad (5.10)$$

Equation 5.10 is the *generalized principle of d'Alembert* and commonly called d'Alembert's equation. The term $(\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i)$ is sometimes called the *effective force*. D'Alembert's principle will be used in the next section to derive Lagrange's equation.

The importance of d'Alembert's principle was explained by Lanczos:⁵

"The importance of d'Alembert's principle lies in the fact that it is *more* than a reformulation of Newton's equation. It is the expression of a *principle*. We know that the vanishing of a force in Newtonian mechanics means equilibrium. Hence [Equation 5.9] says that the addition of the force of inertia to the other forces produces equilibrium. But this means that if we have any criterion for the equilibrium of a mechanical system, we can immediately extend that criterion to a system which is in motion. All we have to do is add the new 'force of inertia' to the previous forces. By this device *dynamics is reduced to statics*."

The linking of Newton's second law of motion with the principle of virtual work means that the principle is applicable to particles at rest and to particles in motion. The virtual displacement involves a possible but purely mathematical (virtual) experiment that can be applied at any specific time. At that instant, the actual motion of the body is not at issue as the dynamic problem is reduced to a static one.

Another advantage of d'Alembert's principle is that the constraint forces do not appear in the equations.

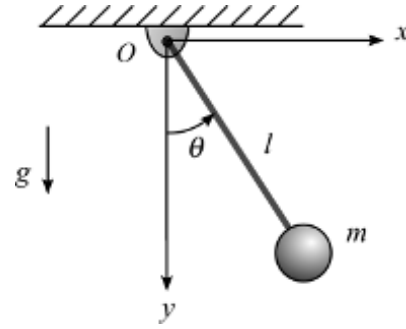


Figure 5.9: A simple pendulum.

D'Alembert's Principle for a Rigid Body

While the above formulation of d'Alembert's principle can be used for more than a particle, there is a version for rigid bodies.⁶ For a single rigid body undergoing planar motion, Equation 5.10 becomes

$$(\mathbf{F} - m\mathbf{a}_G) \cdot \delta \mathbf{r}_G + \left(\mathbf{M}_G - \frac{d}{dt} \mathbf{H}_G \right) \cdot \delta \boldsymbol{\theta} = 0, \quad (5.11)$$

where subscript G stands for the center of gravity and $\mathbf{H}_G$ is the angular momentum vector. For a two-dimensional problem, $\mathbf{H}_G = I_G \boldsymbol{\omega}$. For N rigid bodies, we write

$$\sum_{i=1}^N \left[(\mathbf{F}_i - m\mathbf{a}_{G_i}) \cdot \delta \mathbf{r}_{G_i} + \left(\mathbf{M}_{G_i} - \frac{d}{dt} \mathbf{H}_{G_i} \right) \cdot \delta \boldsymbol{\theta}_i \right] = 0,$$

for $i = 1, \dots, N$. For a rigid body hinged at a fixed O , d'Alembert's principle can be rewritten as

$$\left(\mathbf{M}_O - \frac{d}{dt} \mathbf{H}_O \right) \cdot \delta \boldsymbol{\theta} = 0. \quad (5.12)$$

The two conditions for a point to qualify as O are that it must be part of the rigid body and it must have zero velocity. If no such point O exists, Equation 5.11 must be used instead. This equation can be generalized further for multi-body systems.

Example 5.4 Simple Pendulum Equation from d'Alembert's Principle

Derive the equation of motion for the simple pendulum of Figure 5.9 using d'Alembert's principle. Assume that m is a point mass and the rod is massless.

Solution: This is a one degree-of-freedom system, and we choose θ as our coordinate. There is one point mass in the system, and, therefore, d'Alembert's principle is

$$(\mathbf{F} - m\ddot{\mathbf{r}}) \cdot \delta \mathbf{r} = 0,$$

⁵The Variational Principles of Mechanics, C. Lanczos, 4th ed., Dover Publications, 1970, p.89.

⁶Engineering Dynamics, J. Ginsberg, Cambridge University Press, 2008.

where

$$\begin{aligned}\mathbf{F} &= mg\mathbf{j} \\ \mathbf{r} &= l \sin \theta \mathbf{i} + l \cos \theta \mathbf{j}.\end{aligned}$$

The velocity and the acceleration of the mass are

$$\begin{aligned}\dot{\mathbf{r}} &= (\cos \theta \mathbf{i} - \sin \theta \mathbf{j}) l \dot{\theta} \\ \ddot{\mathbf{r}} &= (\cos \theta \mathbf{i} - \sin \theta \mathbf{j}) l \ddot{\theta} + (-\sin \theta \mathbf{i} - \cos \theta \mathbf{j}) l \dot{\theta}^2 \\ &= (l \ddot{\theta} \cos \theta - l \dot{\theta}^2 \sin \theta) \mathbf{i} + (-l \ddot{\theta} \sin \theta - l \dot{\theta}^2 \cos \theta) \mathbf{j}.\end{aligned}$$

The virtual displacement is given by

$$\delta \mathbf{r} = (\cos \theta \mathbf{i} - \sin \theta \mathbf{j}) l \delta \theta.$$

Substituting into d'Alembert's equation gives

$$\begin{aligned}& \left[(-l \ddot{\theta} \cos \theta + l \dot{\theta}^2 \sin \theta) \mathbf{i} + (g - l \ddot{\theta} \sin \theta - l \dot{\theta}^2 \cos \theta) \mathbf{j} \right] \\ & \quad \cdot (l \delta \theta \cos \theta \mathbf{i} - l \delta \theta \sin \theta \mathbf{j}) = 0 \\ & \quad - (l \ddot{\theta} \cos \theta - l \dot{\theta}^2 \sin \theta) l \cos \theta \delta \theta \\ & \quad + (-l \ddot{\theta} \sin \theta - l \dot{\theta}^2 \cos \theta + g) l \sin \theta \delta \theta = 0 \\ & \quad (l^2 \ddot{\theta} + gl \sin \theta) \delta \theta = 0.\end{aligned}$$

Thus, the equation of motion is

$$l^2 \ddot{\theta} + gl \sin \theta = 0,$$

since $\delta \theta$ is arbitrary. Note that m does not appear in the equation of motion and one l could be canceled. ■

Example 5.5 Redundant Coordinate

Repeat [Example 5.4](#) using the coordinates x and y . Show how [Equation 5.7](#) can be used to eliminate the redundant coordinate and obtain a single equation of motion.

Solution: Although this is a single degree-of-freedom system, for this example we will not express the acceleration and the virtual displacement in terms of a single coordinate but use both x and y . Then,

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{x} \mathbf{i} + \ddot{y} \mathbf{j} \\ \delta \mathbf{r} &= \delta x \mathbf{i} + \delta y \mathbf{j}.\end{aligned}$$

D'Alembert's equation becomes

$$(mg\mathbf{j} - m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j})) \cdot (\delta x\mathbf{i} + \delta y\mathbf{j}) = 0$$

or

$$-m\ddot{x}\delta x + (-m\ddot{y} + mg)\delta y = 0. \quad (5.13)$$

Since δx and δy are not independent of each other, we use the constraint equation, $g(x, y) = x^2 + y^2 - l^2 =$

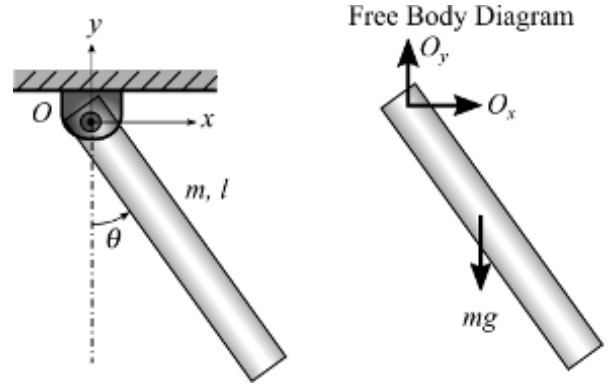


Figure 5.10: A compound pendulum.

0, to relate the two virtual displacements. Taking the variation of the constraint equation, we obtain

$$2x\delta x + 2y\delta y = 0.$$

Replacing δx in [Equation 5.13](#), we have

$$\left[m\ddot{x}\frac{y}{x} + (-m\ddot{y} + mg) \right] \delta y = 0.$$

Thus, the equation of motion for this system is

$$m\ddot{x}\frac{y}{x} + (-m\ddot{y} + mg) = 0.$$

This equation can be converted to an equation in terms of θ by using the transformations $x = l \sin \theta$ and $y = l \cos \theta$ so that the final expression becomes

$$ml^2 \ddot{\theta} + mgl \sin \theta = 0,$$

which is the same result as before. ■

Example 5.6 Compound Pendulum Equation from d'Alembert's Principle

Derive the equation of motion of the compound pendulum of [Figure 5.10](#) using d'Alembert's principle. Let the mass moment of inertia about the center of mass be I_G and the mass moment of inertia about O be I_O .

Solution Method 1: The pendulum is treated as a rigid body instead of a point mass. The pivot at O satisfies the conditions that it is part of the rigid body and it also has zero velocity, and, therefore, we use [Equation 5.12](#). The angular momentum about O for this system is

$$\mathbf{H}_O = I_O \boldsymbol{\omega},$$

where

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{k}.$$

The forces on the system are the reaction forces at O and the weight due to gravity. Only the weight does work, so that

$$\mathbf{M}_O = -mg \frac{l}{2} \sin \theta \mathbf{k}.$$

The virtual displacement is

$$\delta\theta = \delta\theta\mathbf{k}.$$

Then, from Equation 5.12,

$$\left(-mg\frac{l}{2}\sin\theta\mathbf{k} - I_O\ddot{\theta}\mathbf{k}\right) \cdot \delta\theta\mathbf{k} = 0,$$

or

$$\left(mg\frac{l}{2}\sin\theta + I_O\ddot{\theta}\right)\delta\theta = 0.$$

Since $\delta\theta$ is arbitrary, the equation of motion is

$$mg\frac{l}{2}\sin\theta + I_O\ddot{\theta} = 0,$$

where for a rod, $I_O = ml^2/3$.

Solution Method 2: Use the first version of d'Alembert's principle for the rigid body, that is, Equation 5.11,

$$(\mathbf{F} - m\mathbf{a}_G) \cdot \delta\mathbf{r}_G + \left(\mathbf{M}_G - \frac{d}{dt}\mathbf{H}_G\right) \cdot \delta\theta = 0,$$

where $\mathbf{F}$ includes only forces that do virtual work. Since the reaction forces at O do not, we have

$$\mathbf{F} = -mg\mathbf{j}$$

$$\mathbf{v}_G = \frac{l}{2}\dot{\theta}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{a}_G = \frac{l}{2}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\mathbf{i} + \frac{l}{2}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\mathbf{j}.$$

The moment and the angular momentum are

$$\mathbf{M}_G = \mathbf{0} \text{ and } \mathbf{H}_G = I_G\boldsymbol{\omega},$$

where

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k}.$$

The virtual displacements,

$$\delta\mathbf{r}_G = \frac{l}{2}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\delta\theta$$

$$\delta\theta = \delta\theta\mathbf{k},$$

are obtained by replacing d/dt with δ in the velocity expressions, $\mathbf{v}_G$ and $\boldsymbol{\omega}$. After substituting, we obtain

$$-\left(mg\frac{l}{2}\sin\theta + m\left(\frac{l}{2}\right)^2\ddot{\theta} + I_G\ddot{\theta}\right)\delta\theta = 0.$$

Using the parallel axis theorem,

$$I_O = I_G + m\left(\frac{l}{2}\right)^2,$$

we obtain the same equation of motion as in Method 1. ■

JEAN LE ROND D'ALEMBERT

(November 16, 1717 – October 29, 1783)



Contribution: D'Alembert was a French mathematician, mechanician, physicist, and philosopher. In 1743, d'Alembert helped resolve a controversy in mathematical physics by improving Newton's definition of force in his *Traité de dynamique*. In this mathematical masterpiece, d'Alembert established his principle of mechanics: the forces which resist acceleration must be equal and opposite to the forces which produce the acceleration. The result is a condition of kinetic equilibrium. The principle has the advantage of reducing a problem in dynamics to one in statics. It also shows that Newton's third law of motion applies to bodies free to move as well as to stationary bodies.

With the notion of an inertial force, d'Alembert introduced his principle as a way of studying dynamic systems by applying laws of statics. D'Alembert's contribution was crucial to the development by Lagrange of analytical mechanics.

D'Alembert applied his principle to the equilibrium and motion of fluids. In 1744, he published his *Traité de l'équilibre et du mouvement des fluides*, which provided an approach to the study of fluid mechanics that was an alternative to the one published by Daniel Bernoulli.

D'Alembert was the first to use partial differential equations in mathematical physics. In 1747 he published this idea in *Réflexions sur la cause générale des vents* on winds, for which he won the Prussian Academy prize. He applied the concept of partial differential equations to determine the dynamic response of deformable bodies.

In the same year, d'Alembert published an article on the motion of vibrating strings that included the first use of the wave equation in physics. He used mathematically pleasing simplifications of certain boundary conditions that led him to results which were at odds with observation.

In 1754, d'Alembert defined the derivative of a function as the limit of a quotient of increments. His ideas on limits led him to the test for convergence, known today as d'Alembert's ratio test.

D'Alembert wrote 1500 articles for the great French encyclopedia *Encyclopédie*, of which he was co-editor with Denis Diderot. He wrote the introduction and numerous philosophical and mathematical articles, including the famous *Preliminary Discourse*.

Life Sketch: D'Alembert was born in Paris, France. He was the illegitimate child of Louis-Camus Destouches, an artillery officer, and Claudine Guérin de Tencin (from one of her "amorous liaisons"). He was abandoned by his mother on the steps of a church days after his birth. According to custom, he was named after the patron saint of the church. D'Alembert was placed in an orphanage, and adopted by the wife of a glazier. Destouches secretly paid for the education of his son, but did not want his parentage officially recognized.

In 1726, when d'Alembert was only nine years old, his father died. Up until this time, his father paid for him to attend a private school. On his passing, his father's family continued to fund his education. They eventually arranged for him to enter the Jansenist Collège des Quatre Nations. He enrolled in the name of Jean-Baptiste Dairemberg, but soon changed his name to Jean d'Alembert.

The Collège des Quatre Nations was an ideal place for d'Alembert to study mathematics. D'Alembert made great use of the mathematics library at the college and took a course that was based on Varignon's lectures. In addition to mathematical training, d'Alembert studied philosophy, law, and the arts. He learned about Descartes and his views pertaining to physics. He graduated in 1735. In his later life, d'Alembert scorned the Cartesian principles he had been taught by the Jansenists: "physical premotion, innate ideas and the vortices."

In 1739, d'Alembert made his first contribution to the field of mathematics. He presented a paper to the Paris Academy of Science on errors he had found in Reyneau's *Analyse Démontrée*. In 1740, d'Alembert submitted his second paper to the Paris Academy on the mechanics of fluids. In this work he developed the concept of refraction and wrote about what is now called d'Alembert's paradox: the drag on a body immersed in an inviscid, incompressible fluid is zero. In 1741, d'Alembert earned acceptance to the Paris Academy on the strength of these papers and others on integral calculus. However, he left the Academy early due to a rivalry with prominent mathematician Clairaut.

D'Alembert was active in several Parisian salons, including the one founded by the rich and imperious Mme. Geoffrin. Through her and these salons, d'Alembert entered a social life in which he enjoyed great success and popularity. D'Alembert was soon contracted as an editor to write the entries for mathematics and physical astronomy in the *Encyclopédie*. The first volume, which was published in 1751, contained a widely acclaimed preface written by d'Alembert.

Euler and d'Alembert started out as good friends early in their careers, but their relationship soured following a dispute in the Berlin Academy in 1751. The dispute arose when Frederick II, the King of Prussia, suggested d'Alembert's can-

didacy for presidency of the Berlin Academy in 1752. The relationship also suffered because d'Alembert felt Euler owed him more credit for his work.

D'Alembert stopped publishing mathematical articles following his estrangement from the Paris Academy and the Berlin Academy. He collected all his works and published them as *Opuscules Mathématiques*, which appeared in eight volumes between 1761 and 1780.

In the latter part of his life d'Alembert focused on literature and philosophy. His philosophical articles appeared in the five-volume *Mélanges de littérature et de philosophie* between 1753 and 1767. D'Alembert expressed skepticism about metaphysical problems. He accepted the argument in favor of the existence of God based on the belief that intelligence cannot be a product of matter alone.

D'Alembert was elected to the French Academy in November 1754. In 1772 he was elected perpetual secretary of the French Academy and spent substantial time writing obituaries for the academy. He became the academy's most influential member, but despite his efforts the Academy failed to produce anything noteworthy in the way of literature during his pre-eminence.

In 1765, d'Alembert complained that his mind was no longer able to concentrate on mathematics due to illness. In a letter to Lagrange in 1777, he revealed how much he regretted this. D'Alembert suffered from bad health for many years until his death in 1783, the result of a bladder illness. D'Alembert died in Paris, France and, as a known unbeliever, was buried in a common unmarked grave.

Noteworthy: D'Alembert switched fields many times until he discovered his passion for mathematics. He studied theology while in Jansenist Collège des Quatre Nations. He then decided to pursue a career in law and qualified as an advocate in 1738. The following year d'Alembert studied medicine, but he quickly found this even less desirable than theology. Mathematics gave him the most satisfaction. He studied it on his own after graduating from Jansenist Collège and at times when he was supposed to be studying for other qualifications.

D'Alembert had few friends and many enemies. Euler and Bernoulli, once his friends, turned away from him due to his approach to work. Euler admired d'Alembert when he won a prize from the Prussian Academy of Sciences for his essay on wind. Euler saw the power of d'Alembert methods and developed these in further detail. However, Euler turned into an opponent later in life. Daniel Bernoulli also became highly critical of d'Alembert after reading his *Traité de l'équilibre et du mouvement des fluides*. In addition, Clairaut openly attacked d'Alembert's methods in scientific journals.

D'Alembert believed mechanics was based on metaphysical principles and not on experimental evidence. Although his work was mathematically sound, d'Alembert was criticized by many since it was not based on physical evidence. For example, he assumed that wind was generated by tidal effects on the atmosphere and the heating of the atmosphere

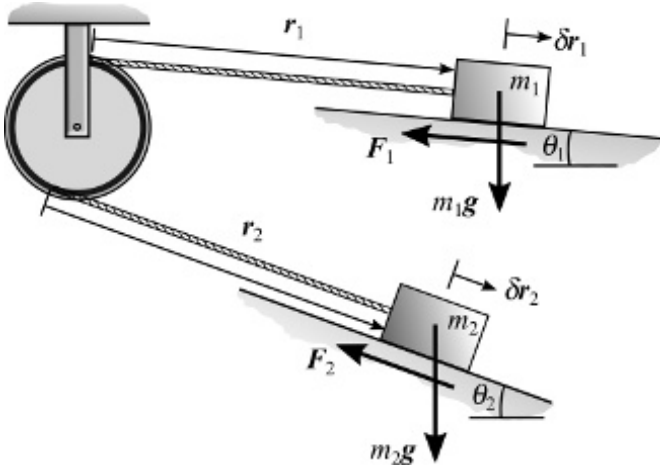


Figure 5.11: Two-body single degree-of-freedom system shown with the virtual displacements δr_1 and δr_2 , and the resisting friction forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

played only a very minor role. D'Alembert had many famous detractors.

While he made great strides in mathematics and physics, d'Alembert is also known for incorrectly arguing that the probability of a coin landing heads increased every time that it came up tails. In gambling, the strategy of decreasing one's bet the more one wins and increasing one's bet the more one loses is called the d'Alembert system, in his honor. (It is a type of martingale, a gambling strategy in which one doubles the stake after each loss.)

D'Alembert was a Latin scholar of some note and worked in the latter part of his life on a superb translation of *Tacitus*.

Example 5.7 Two Mass System Equations from d'Alembert's Principle

Derive the equation of motion for the system in Figure 5.11, consisting of two masses connected by an inextensible string around a frictionless, massless pulley.

Solution: The free-body forces are superimposed in Figure 5.11. To apply d'Alembert's principle, the system must undergo a virtual displacement compatible with the constraints that guide the motion (string and inclines on which the masses move). The assumption that the connecting string is inextensible means that only one coordinate is needed to specify the system configuration. (An elastic string would require two coordinates for the system configuration.) $\mathbf{F}_1$ and $\mathbf{F}_2$ are friction forces acting in the directions opposite to that of the motion.

Assume mass m_2 is displaced in a virtual displacement $\delta \mathbf{r}_2$ as shown. The string constraint equation is its length $l = |\mathbf{r}_1| + |\mathbf{r}_2| + p$, where p is the length of string in contact with the pulley. Since l is constant, $\delta r_1 + \delta r_2 = 0$. Applying d'Alembert's principle, Equation 5.10, for

$i = 1, 2$, we have

$$[(m_1 \mathbf{g} + \mathbf{F}_1) - m_1 \ddot{\mathbf{r}}_1] \cdot \delta \mathbf{r}_1 + [(m_2 \mathbf{g} + \mathbf{F}_2) - m_2 \ddot{\mathbf{r}}_2] \cdot \delta \mathbf{r}_2 = 0.$$

Expanding the dot products we find

$$m_1 g (\sin \theta_1 + \mu \cos \theta_1) \delta r_1 + m_2 g (\sin \theta_2 - \mu \cos \theta_2) \delta r_2 - (m_1 \ddot{r}_1 \delta r_1 + m_2 \ddot{r}_2 \delta r_2) = 0,$$

where μ is the dynamic coefficient of friction between each block and its surface. We use the relations $\delta r_1 = -\delta r_2$ and $\ddot{r}_1 = -\ddot{r}_2$ to find

$$[-m_1 g (\sin \theta_1 + \mu \cos \theta_1) + m_2 g (\sin \theta_2 - \mu \cos \theta_2) - (m_1 + m_2) \ddot{r}_2] \delta r_2 = 0.$$

Since the variation δr_2 is arbitrary, the expression in the square brackets must be equal to zero,

$$(m_1 + m_2) \ddot{r}_2 = -m_1 g (\sin \theta_1 + \mu \cos \theta_1) + m_2 g (\sin \theta_2 - \mu \cos \theta_2). \quad (5.14)$$

This is the equation of motion for the single degree-of-freedom system in terms of coordinate r_2 . If $\ddot{r}_2 = 0$, we recover the equation for static equilibrium,

$$m_1 (\sin \theta_1 + \mu \cos \theta_1) = m_2 (\sin \theta_2 - \mu \cos \theta_2),$$

which is a nonlinear relation linking the masses, the coefficient of friction, and the two angles of inclination. ■

Example 5.8 Double Compound Pendulum Equations from Newton's Second Law

Consider a double compound pendulum attached to a pin at point O as depicted in Figure 5.12. Using the variables defined in the figure, use Newton's second law to derive the two governing equations of motion.

Solution: The links are subjected to two torques, T_1 and T_2 , as shown. Angular displacement θ_2 is measured from the position of link one. We use kinematics to determine the accelerations of the center of mass of each link. Then we apply Newton's second law of motion to obtain the equations of motion for each link. The accelerations derived above are substituted into these equations and then simplified algebraically.

Kinematics: The position vector to the center of mass of link one is

$$\mathbf{r}_{C_1/O} = r_1 \cos \theta_1 \mathbf{i} + r_1 \sin \theta_1 \mathbf{j}.$$

The second derivative of this expression is the acceleration of the center of mass,

$$\mathbf{a}_{C_1/O} = -r_1 \ddot{\theta}_1^2 \cos \theta_1 \mathbf{i} - r_1 \ddot{\theta}_1 \sin \theta_1 \mathbf{i} - r_1 \dot{\theta}_1^2 \sin \theta_1 \mathbf{j} + r_1 \ddot{\theta}_1 \sin \theta_1 \mathbf{j}.$$

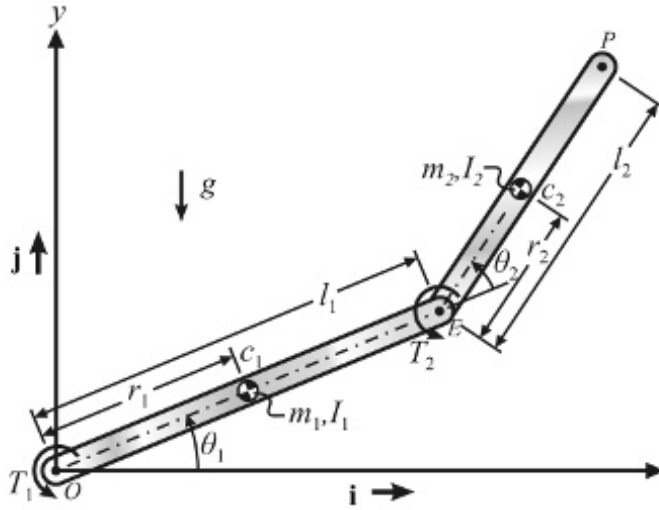


Figure 5.12: Double compound pendulum.

The position vector to the center of mass of link two is

$$\mathbf{r}_{C_2/O} = [r_2 \cos(\theta_1 + \theta_2) + l_1 \cos \theta_1] \mathbf{i} + [r_2 \sin(\theta_1 + \theta_2) + l_1 \sin \theta_1] \mathbf{j},$$

and the acceleration is given by

$$\begin{aligned} \mathbf{a}_{C_2/O} &= [-r_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \cos(\theta_1 + \theta_2) - r_2(\ddot{\theta}_1 + \ddot{\theta}_2) \sin(\theta_1 + \theta_2) \\ &\quad - l_1 \dot{\theta}_1^2 \cos \theta_1 - l_1 \ddot{\theta}_1 \sin \theta_1] \mathbf{i} \\ &\quad + [r_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \sin(\theta_1 + \theta_2) + r_2(\ddot{\theta}_1 + \ddot{\theta}_2) \cos(\theta_1 + \theta_2) \\ &\quad - l_1 \dot{\theta}_1^2 \sin \theta_1 + l_1 \ddot{\theta}_1 \cos \theta_1] \mathbf{j} \\ &\equiv \mathbf{a}_{C_{2x}} \mathbf{i} + \mathbf{a}_{C_{2y}} \mathbf{j}. \end{aligned}$$

Kinetics: We apply Newton's second law of motion to each free-body diagram in the x , y , and θ directions. For link one, the sum of forces and sum of moments are

$$\begin{aligned} \sum F_x &= a_x + E_x \\ \sum F_y &= a_y + E_y + m_1 g \\ \sum M_O &= E_y(l_1 \cos \theta_1) - E_x(l_1 \sin \theta_1) \\ &\quad + T_1 - T_2 - m_1 g r_1 \cos \theta_1, \end{aligned}$$

where E_x and E_y are the reaction forces between the two links at E . For link two, the sum of forces and sum of moments are

$$\begin{aligned} \sum F_x &= -E_x \\ \sum F_y &= -E_y - m_2 g \\ \sum M_{C_2} &= E_y(r_2 \cos(\theta_1 + \theta_2)) - E_x(r_2 \sin(\theta_1 + \theta_2)). \end{aligned}$$

Substituting these forces and moments into Newton's second law for link one we find

$$\begin{aligned} \sum F_x &= m_1 a_{C_{1x}} \\ \sum F_y &= m_1 a_{C_{1y}} \\ \sum M_O &= I_O \alpha_1, \end{aligned}$$

where $I_O = I_1 + m_1 r_1^2$ and $\alpha_1 = \ddot{\theta}_1$. Similarly, for link two, we find

$$\begin{aligned} \sum F_x &= m_2 a_{C_{2x}} \\ \sum F_y &= m_2 a_{C_{2y}} \\ \sum M_O &= I_2 \alpha_2, \end{aligned}$$

where $\alpha_2 = \ddot{\theta}_1 + \ddot{\theta}_2$ since θ_2 is measured from θ_1 . We substitute the forces and moments, respectively, and then substitute the expressions for the accelerations from the kinematics. After considerable algebra, we arrive at the two equations of motion,

$$\begin{aligned} [I_2 + m_2 r_2^2 + m_2 r_2 l_1 \cos \theta_2] \ddot{\theta}_1 + [I_2 + m_2 r_2^2] \ddot{\theta}_2 \\ + [m_2 r_2 l_1 \sin \theta_2] \dot{\theta}_1^2 + m_2 r_2 g \cos(\theta_1 + \theta_2) = T_2, \end{aligned} \quad (5.15)$$

and

$$\begin{aligned} [I_1 + m_1 r_1^2 + m_2(l_1^2 + r_2^2 + 2l_1 r_2 \cos \theta_2) + I_2] \ddot{\theta}_1 \\ + [I_2 + m_2 r_2(r_2 + l_1 \cos \theta_2)] \ddot{\theta}_2 - [2m_2 r_2 l_1 \sin \theta_2] \dot{\theta}_1 \dot{\theta}_2 \\ - [m_2 r_2 l_1 \sin \theta_2] \dot{\theta}_2^2 + g[m_1 r_1 \cos \theta_1 \\ + m_2(l_1 \cos \theta_1 + r_2 \cos(\theta_1 + \theta_2))] = T_1 + T_2. \end{aligned} \quad (5.16)$$

These two equations of motion are highly coupled and nonlinear. ■

Example 5.9 Compound Pendulum Equations from Virtual Work and d'Alembert's Principle

Derive the equations of motion of the double compound pendulum shown in Figure 5.13 using the principle of virtual work in conjunction with the inertial forces of d'Alembert.

Solution: Since the two links of the pendulum are assumed to be rigid, two coordinates are sufficient to describe the motion of the system: ϕ for the upper link of mass m_1 and ψ for the lower link of mass m_2 .

We sketch the free-body diagrams for each link, as shown in Figure 5.14, to help us calculate the virtual work. The unlabeled arrows at the top of each body are reaction forces that are not calculated. Note that inertia terms are included with each free-body diagram. They are included to emphasize that they are part of the force balance as interpreted by d'Alembert. Let I_1

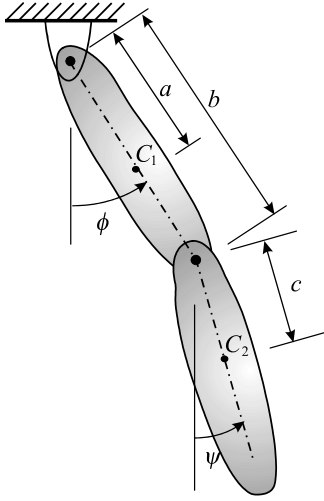


Figure 5.13: Double compound pendulum where C_1 and C_2 are the respective centers of mass.

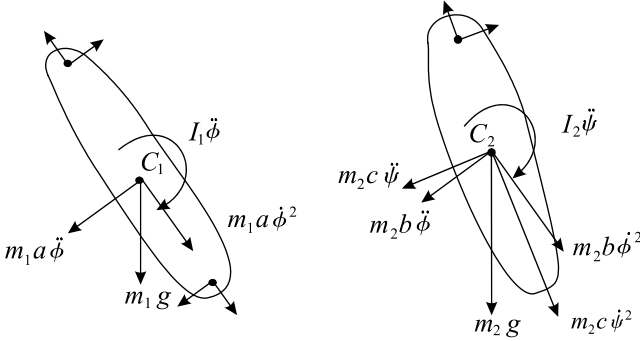


Figure 5.14: Free-body diagrams for each mass of the compound pendulum. Inertia forces and torques have been included.

and I_2 be the mass moments of inertia of the respective bodies about their respective center of mass, C_1 and C_2 .

We derive the virtual work expression for each coordinate and set it equal to zero to find the respective equations of motion. First, we consider the virtual work done by forces and moments corresponding to ψ when it undergoes a small virtual rotation $\delta\psi$, while ϕ is held constant and thus $\delta\phi = 0$. This is permissible since the two generalized coordinates are independent.

We consider one of the components of the virtual work to show how the formulation proceeds. For the lower link, the force equals the product of mass m_2 and the rectilinear acceleration at mass center C_2 , $c\ddot{\psi}$. The virtual displacement is $c\delta\psi$. Therefore, this component of the virtual work is the product $-c\ddot{\psi} c\delta\psi$, where the minus sign indicates a negative rotation according to the *right-hand rule*.

In the next equation, the forces and moments are shown in square brackets. The complete virtual work

is then

$$\begin{aligned}\delta W_\psi &= -[m_2 c \ddot{\psi}] c \delta\psi - [m_2 b \ddot{\phi}] c \cos(\phi - \psi) \delta\psi \\ &\quad - [m_2 g] c \sin \psi \delta\psi - [I_2 \ddot{\psi}] \delta\psi \\ &\quad + [m_2 b \dot{\phi}^2] c \sin(\phi - \psi) \delta\psi \\ &= \left[-(m_2 c^2 + I_2) \ddot{\psi} - m_2 b c (\ddot{\phi} \cos(\phi - \psi) \right. \\ &\quad \left. - \dot{\phi}^2 \sin(\phi - \psi)) - m_2 g c \sin \psi \right] \delta\psi,\end{aligned}$$

where $m_2 c \dot{\psi}^2$, which is perpendicular to the virtual rotation $\delta\psi$, and the torque $I_1 \ddot{\phi}$ do no work.

Similarly, for the virtual rotation $\delta\phi$, with ψ held constant and thus $\delta\psi = 0$, the virtual work in the ϕ coordinate is

$$\begin{aligned}\delta W_\phi &= [-m_1 a^2 \ddot{\phi} - m_1 g a \sin \phi - I_1 \ddot{\phi} \\ &\quad - m_2 b c \ddot{\psi} \cos(\phi - \psi) - m_2 b^2 \ddot{\phi} \\ &\quad - m_2 g b \sin \phi - m_2 b c \dot{\psi}^2 \sin(\phi - \psi)] \delta\phi,\end{aligned}$$

where $m_1 a \dot{\phi}^2$, $m_2 b \dot{\phi}^2$, and $I_2 \ddot{\psi}$ do no work and have been removed from the expression.

Since the virtual work done by all forces, including inertia forces, equals zero by d'Alembert's principle, we can write down the equation of motion for each coordinate system by setting the generalized forces, that is the expression in each of the above square brackets, equal to zero.

If we assume that only small oscillations take place, meaning that ϕ and ψ are small and higher order terms such as $\dot{\phi}^2$ can be neglected, the linearized equations of motion are

$$\begin{aligned}(m_2 c^2 + I_2) \ddot{\psi} + m_2 b c \ddot{\phi} + m_2 c g \psi &= 0 \\ m_2 b c \ddot{\psi} + (m_1 a^2 + I_1 + m_2 b^2) \ddot{\phi} + (m_1 a + m_2 b) g \phi &= 0.\end{aligned}$$

These two coupled equations can be expressed in matrix form,

$$\begin{bmatrix} m_2 c^2 + I_2 & m_2 b c \\ m_2 b c & m_1 a^2 + I_1 + m_2 b^2 \end{bmatrix} \begin{Bmatrix} \ddot{\psi} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} m_2 c g & 0 \\ 0 & (m_1 a + m_2 b) g \end{bmatrix} \begin{Bmatrix} \psi \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

where the coupling is seen in the off-diagonal terms in the inertia matrix. The zero vector on the right-hand side of the equal sign indicates that there is no forcing function; with no external forcing the pendulum oscillations are in free vibration. Matrix methods are introduced to solve such systems in [Chapter 6](#). ■

5.4 Lagrange's Equation

Lagrange's equation is an energy-based formulation for deriving the equations of motion of a dynamic system. The system may be linear or nonlinear. The advantage of the method becomes evident for multi degree-of-freedom systems where it is significantly easier to obtain the equations of motion in comparison to Newton's method. Lagrange's equation is composed of the kinetic and potential energies of the system; unlike Newton's second law of motion, it is not a vector approach.

In the formulation that follows, the equations are written in terms of N *generalized coordinates*. A generalized coordinate, q_i , is a coordinate representing a degree of freedom of the system being modeled. The physical coordinates, $\mathbf{r}_i$, of an n degree-of-freedom system for N particles can be related to the generalized coordinates by an appropriate set of equations,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n), \quad i = 1, 2, \dots, N. \quad (5.17)$$

The purpose of these transformations from physical coordinates, which are vectorial, to generalized coordinates, which are not, is to recast d'Alembert's principle into Lagrange's equation.⁷

In the following derivation, d'Alembert's generalized principle is expanded and rewritten in terms of the potential and kinetic energies, which are functions of the generalized coordinates.⁸ First, we derive the relations between the physical and generalized coordinates.

The total derivative of the position vector $\mathbf{r}_i$ in Equation 5.17 is

$$\begin{aligned} \dot{\mathbf{r}}_i &= \frac{\partial \mathbf{r}_i}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial \mathbf{r}_i}{\partial q_2} \frac{dq_2}{dt} + \dots + \frac{\partial \mathbf{r}_i}{\partial q_n} \frac{dq_n}{dt} \\ &= \frac{\partial \mathbf{r}_i}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{r}_i}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial \mathbf{r}_i}{\partial q_n} \dot{q}_n \\ &= \sum_{k=1}^n \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k, \quad i = 1, 2, \dots, N. \end{aligned} \quad (5.18)$$

Differentiating Equation 5.18 with respect to $\dot{q}_k$ gives

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_i}{\partial q_k}, \quad i = 1, 2, \dots, N, \quad k = 1, 2, \dots, n.$$

Since the variations $\delta \mathbf{r}_i$ follow the same rules as the dif-

⁷Equation 5.17 can include time explicitly,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n, t), \quad i = 1, 2, \dots, N,$$

in which case Equation 5.18 will include the term $\partial \mathbf{r}_i / \partial t$.

⁸We present the details of its derivation because it is important to understand how such an equation is derived and the thought processes that are used. Perhaps there is another Lagrange out there who will be motivated by being exposed to the derivation!

ferentials $d\mathbf{r}_i$, the variations of $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$ are

$$\begin{aligned} \delta \mathbf{r}_i &= \frac{\partial \mathbf{r}_i}{\partial q_1} \delta q_1 + \frac{\partial \mathbf{r}_i}{\partial q_2} \delta q_2 + \dots + \frac{\partial \mathbf{r}_i}{\partial q_n} \delta q_n \\ &= \sum_{k=1}^n \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k, \quad i = 1, 2, \dots, N, \\ \delta \dot{\mathbf{r}}_i &= \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_1} \delta \dot{q}_1 + \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_2} \delta \dot{q}_2 + \dots + \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_n} \delta \dot{q}_n \\ &= \sum_{k=1}^n \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \delta \dot{q}_k, \quad i = 1, 2, \dots, N. \end{aligned}$$

Next, we consider the second term of d'Alembert's principle, Equation 5.10, written in terms of the generalized coordinates,

$$\begin{aligned} \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i &= \sum_{i=1}^N \left(m_i \ddot{\mathbf{r}}_i \cdot \sum_{k=1}^n \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \right) \\ &= \sum_{k=1}^n \left(\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k. \end{aligned} \quad (5.19)$$

In Equation 5.19, for any k , within the interior sum we can perform the following algebraic manipulations,

$$\begin{aligned} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} &= \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \\ &= \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} \\ &= \left[\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_k} \right) - \frac{\partial}{\partial q_k} \right] \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right). \end{aligned} \quad (5.20)$$

We can therefore write Equation 5.19 as

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{k=1}^n \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right] \delta q_k,$$

where the kinetic energy T is defined as

$$T = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = T(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n).$$

The other term in d'Alembert's principle is the virtual work $\delta W = \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i$, and so d'Alembert's equation can be written as

$$\sum_{k=1}^n \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right] \delta q_k = \delta W. \quad (5.21)$$

To explicitly show the virtual work in terms of the forces $\mathbf{F}_i$ we write them in terms of the generalized coordinates.

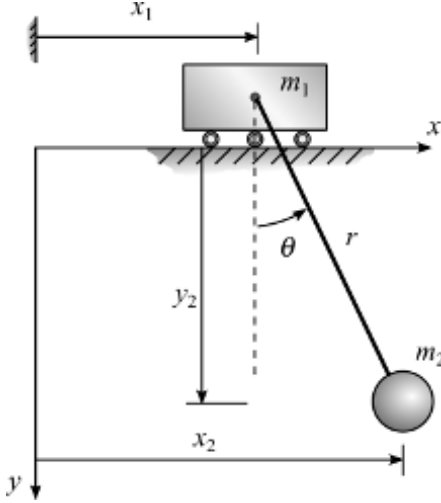


Figure 5.15: Simple pendulum pivoted to a free block that can slide on a smooth (frictionless) surface.

The virtual work done by these forces is

$$\begin{aligned}
 \delta W &= \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i \\
 &= \sum_{i=1}^N \mathbf{F}_i \cdot \sum_{k=1}^n \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \\
 &= \sum_{k=1}^n \left(\sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k \\
 &= \sum_{k=1}^n Q_k \delta q_k,
 \end{aligned} \tag{5.22}$$

where Q_k is called a *generalized force*. The generalized force may be a force or a torque, and is often a complicated expression. Before further discussion of the generalized force, we look at an example that shows how the generalized force can be identified.

Example 5.10 The Generalized Force

Consider the simple pendulum suspended from a block that can translate on a smooth surface, as shown in Figure 5.15. Gravity acts in the y direction. Derive the equations of motion, assuming frictionless contact between the block and the surface, and a frictionless pivot for the pendulum.

Solution: In this example we demonstrate the procedure based on Equations 5.21 and 5.22. The kinetic energy of each mass is derived and appropriate derivatives are set equal to the respective generalized force, resulting in one equation

$$\left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right] = Q_k, \tag{5.23}$$

for each generalized coordinate k . To find the generalized forces, we will evaluate the virtual work done in each generalized coordinate.

The kinetic energy of the two-mass system is given by

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2).$$

The kinematic relations are

$$x_2 = x_1 + r \sin \theta \tag{5.24}$$

$$y_2 = r \cos \theta, \tag{5.25}$$

which are constraint equations. In deriving the equations of motion, we will select x_1 and θ as the two generalized coordinates.

Differentiating Equations 5.24 and 5.25, substituting and eliminating $\dot{x}_2$ and $\dot{y}_2$ from the expression for kinetic energy, we find

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1^2 + 2r\dot{x}_1\dot{\theta} \cos \theta + r^2\dot{\theta}^2).$$

The following derivatives are needed in Equation 5.23. For the equation of motion in generalized coordinate x_1 ,

$$\begin{aligned}
 \frac{\partial T}{\partial \dot{x}_1} &= m_1 \dot{x}_1 + m_2 \dot{x}_1 + m_2 r \dot{\theta} \cos \theta \\
 \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) &= m_1 \ddot{x}_1 + m_2 \ddot{x}_1 + m_2 r (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) \\
 \frac{\partial T}{\partial x_1} &= 0.
 \end{aligned}$$

Substituting these into Equation 5.23 and equating to the generalized force in the x_1 direction,

$$(m_1 + m_2) \ddot{x}_1 + m_2 r (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = F_{x_1}.$$

This is the equation of motion in the x_1 direction.

The expression for F_{x_1} will be evaluated after the equation of motion in the θ direction is derived, for which the following derivatives are needed,

$$\begin{aligned}
 \frac{\partial T}{\partial \dot{\theta}} &= m_2 r \dot{x}_1 \cos \theta + m_2 r^2 \dot{\theta} \\
 \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) &= m_2 r (\ddot{x}_1 \cos \theta - \dot{x}_1 \dot{\theta} \sin \theta) + m_2 r^2 \ddot{\theta} \\
 \frac{\partial T}{\partial \theta} &= -m_2 r \dot{x}_1 \dot{\theta} \sin \theta.
 \end{aligned}$$

Substituting and equating to the generalized force in the θ direction gives the second equation of motion,

$$m_2 r \ddot{x}_1 \cos \theta + m_2 r^2 \ddot{\theta} = F_\theta.$$

Next, we evaluate the virtual work performed by the generalized forces in the x_1 and θ directions. The general virtual work expression is

$$\delta W = F_{x_1} \delta x_1 + F_\theta \delta \theta,$$

where δx_1 and $\delta\theta$ are independent. Since there is no friction between the block and the surface upon which it rests, and there are no other forces in the x_1 direction, $F_{x_1} = 0$.

To evaluate F_θ it is helpful to sketch the virtual displacement $\delta\theta$, as shown in an exaggerated view in Figure 5.16.

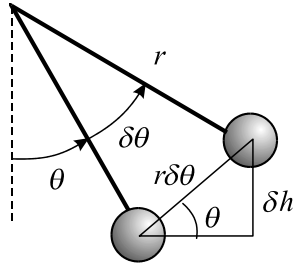


Figure 5.16: Exaggerated virtual displacement $\delta\theta$.

The virtual work is performed in raising the pendulum mass a vertical distance δh , which is related to $\delta\theta$ by $\delta h = r \delta\theta \sin \theta$. The virtual work done by gravity is $F_\theta \delta\theta$, or

$$-m_2 g r \sin \theta \delta\theta,$$

where the negative sign signifies that the work is performed in raising the mass against the gravitational force. Thus, the generalized force in the θ direction, F_θ , is the moment,

$$F_\theta = -m_2 g r \sin \theta.$$

The equation of motion in the θ direction is

$$m_2 r \ddot{x}_1 \cos \theta + m_2 r^2 \ddot{\theta} = -m_2 g r \sin \theta.$$

The coupled equations of motion are then

$$\begin{aligned} (m_1 + m_2) \ddot{x}_1 + m_2 r (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) &= 0 \\ m_2 r \ddot{x}_1 \cos \theta + m_2 r^2 \ddot{\theta} + m_2 g r \sin \theta &= 0. \end{aligned}$$

In Chapter 6 these simultaneous equations will be linearized and solved. Had other external forces existed, they would have been included in F_{x_1} or F_θ and appeared on the right-hand sides of the equations of motion.

Dissipation forces such as damping or friction, if present, are part of the virtual work expression and thus part of the generalized forces in the equations of motion. This is discussed next. ■

D'Alembert's equation can be specialized further by separately considering the *conservative* (derivable from potential energy V) and *nonconservative* forces acting on the system. From Equation 5.22,

$$\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = \delta W = \delta W_c + \delta W_{nc}.$$

The work done in a conservative force field equals the negative of the variation of the potential, $-\delta V$. The total virtual work performed by nonconservative generalized forces $Q_{k_{nc}}$ undergoing virtual displacements δq_k is given by $\sum_{k=1}^n Q_{k_{nc}} \delta q_k$. Therefore,

$$\begin{aligned} \delta W &= - \left(\frac{\partial V}{\partial q_1} \delta q_1 + \dots + \frac{\partial V}{\partial q_n} \delta q_n \right) + \sum_{k=1}^n Q_{k_{nc}} \delta q_k \\ &= - \sum_{k=1}^n \left(\frac{\partial V}{\partial q_k} - Q_{k_{nc}} \right) \delta q_k. \end{aligned}$$

D'Alembert's generalized principle, Equation 5.21, then becomes

$$\sum_{k=1}^n \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} - Q_{k_{nc}} \right] \delta q_k = 0.$$

Since the virtual displacements δq_k are arbitrary, the expression in the square brackets must equal zero for each k . There are n such equations of motion,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k, \quad k = 1, 2, \dots, n, \quad (5.26)$$

where to simplify the notation we use Q_k instead of $Q_{k_{nc}}$. This term includes dissipative forces such as damping and nonconservative external forces. Equations 5.26 are *Lagrange's equations of motion*, with one equation for each of the n degrees of freedom. The addition of damping is considered in a subsequent section.

It is customary to define the *Lagrangian* function as

$$L = T - V.$$

Since potential energy V is a function of position only, it does not vary with velocity. Thus,

$$\frac{\partial T}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{q}_k},$$

resulting in a more compact form for Equation 5.26,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k, \quad k = 1, 2, \dots, n. \quad (5.27)$$

There are several key advantages to Lagrange's equation:

- Lagrange's equation contains only scalar quantities, eliminating the need to express the force and acceleration vectors required in Newton's second law of motion.
- Lagrange's equation is independent of the coordinate system since the energy functions T and V are scalar.

- There is one Lagrange equation for each degree of freedom, and no need to account for constraint forces. In contrast, Newton's formulation leads to extraneous equations associated with the internal forces between coupled bodies. In Newton's approach, such internal forces must be eliminated after the equations of motion are derived. (Of course, in some applications, these internal forces are needed.)

Example 5.11 Generalized Forces due to Nonconservative Forces

Repeat [Example 5.10](#) where the pendulum is subject to horizontal external force $F\mathbf{i}$. What are the generalized forces due to this force? Find the equations of motion.

Solution: The virtual work done by this force is

$$\begin{aligned}\delta W &= \mathbf{F} \cdot \delta \mathbf{r}_2 \\ &= F \delta x_2 \\ &= F \delta (x_1 + r \sin \theta) \\ &= F \delta x_1 + F r \cos \theta \delta \theta.\end{aligned}$$

Then, the generalized forces due to $F\mathbf{i}$ on m_2 are

$$\begin{aligned}F_x &= F \\ F_\theta &= F r \cos \theta.\end{aligned}$$

The equations of motion are then

$$\begin{aligned}(m_1 + m_2)\ddot{x}_1 + m_2 r(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) &= F \\ m_2 r \ddot{x}_1 \cos \theta + m_2 r^2 \ddot{\theta} &= -m_2 g r \sin \theta + F r \cos \theta.\end{aligned}$$

■

Example 5.12 Generalized Forces due to Damping Forces

Repeat [Example 5.10](#) where the pendulum is connected to a damper as shown in [Figure 5.17](#). Find the generalized forces due to the damping force, and then find the equations of motion. The damper is assumed to act only in the vertical direction.

Solution: The damping force is nonconservative and is given by

$$\begin{aligned}\mathbf{F}_{\text{damping}} &= -c\dot{y}_2 \mathbf{j} \\ &= -cr\dot{\theta} \sin \theta \mathbf{j}.\end{aligned}$$

The virtual work done by this force is

$$\begin{aligned}\delta W (\text{due to } \mathbf{F}_{\text{damping}}) &= \mathbf{F}_{\text{damping}} \cdot \delta \mathbf{r}_2 \\ &= -c\dot{y}_2 \delta y_2 \\ &= -cr^2 \dot{\theta} \sin^2 \theta \delta \theta.\end{aligned}$$

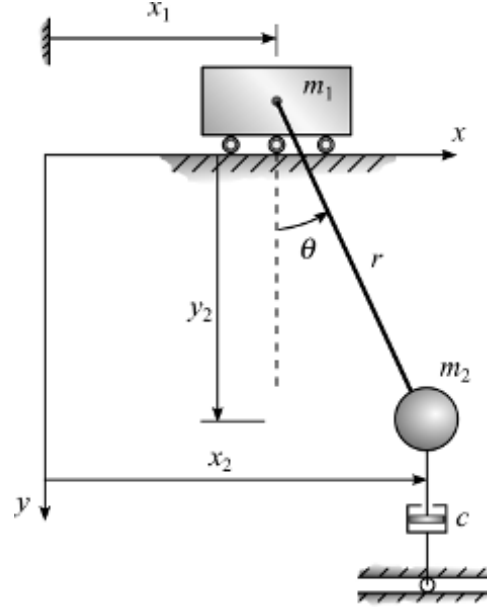


Figure 5.17: Simple pendulum connected to a damper and pivoted to a free block that can roll without friction.

Then, the generalized forces due to the damping force are

$$\begin{aligned}F_x &= 0 \\ F_\theta &= -cr^2 \dot{\theta} \sin^2 \theta.\end{aligned}$$

The equations of motion are then

$$\begin{aligned}(m_1 + m_2)\ddot{x}_1 + m_2 r(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) &= 0 \\ m_2 r \ddot{x}_1 \cos \theta + m_2 r^2 \ddot{\theta} + cr^2 \dot{\theta} \sin^2 \theta + m_2 g r \sin \theta &= 0.\end{aligned}$$

We can compare these equations of motion with those of [Example 5.10](#). ■

Rayleigh's dissipation function is introduced in [Section 5.6](#) as a more formal framework to account for linear dissipation forces.

Example 5.13 Two Mass System Equations from Lagrange's Equation

Formulate the problem of [Example 5.7](#) using Lagrange's equation and derive the equation of motion (a) ignoring friction forces and (b) including friction forces. See [Figure 5.11](#).

Solution: [Figure 5.11](#) depicts the system along with the relevant coordinates. Since this is a one degree-of-freedom system, only one generalized coordinate is needed. However, we will select two coordinates, r_1 and r_2 , and show the link between these coordinates and the string that constrains them. (Other coordinates are possible.) Since there is a constraint condition, $r_1 = l - r_2$,

we will end up with one equation of motion governing one coordinate, which in this case we selected to be r_2 .

(a) The kinetic energy is given by

$$T = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{r}_2^2.$$

The potential energy is given by

$$\begin{aligned} V &= -m_1gh_1 - m_2gh_2 \\ &= -m_1g(l - r_2)\sin\theta_1 - m_2gr_2\sin\theta_2. \end{aligned}$$

The Lagrangian is the difference between the kinetic and potential energies,

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m_1 + m_2)\dot{r}_2^2 + m_1g(l - r_2)\sin\theta_1 + m_2gr_2\sin\theta_2. \end{aligned}$$

Now we can determine the terms in the Lagrangian as necessary to derive the equation of motion,

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}_2} \right) &= (m_1 + m_2)\ddot{r}_2 \\ \frac{\partial L}{\partial r_2} &= -m_1g\sin\theta_1 + m_2g\sin\theta_2. \end{aligned}$$

Substituting these into Equation 5.27, we find the equation of motion for this single degree-of-freedom system to be

$$(m_1 + m_2)\ddot{r}_2 = -m_1g\sin\theta_1 + m_2g\sin\theta_2.$$

This is a nonlinear equation for r_2 that is valid for large θ_1 and θ_2 . Note that the same equation will be obtained if Lagrange's equation is applied to r_1 .

(b) Here friction is included, and therefore the nonconservative forces Q_k from Lagrange's equation are needed. Since this is a two particle one degree-of-freedom system, there is only one dissipative force,

$$Q = -\mu(m_1g\cos\theta_1 + m_2g\cos\theta_2),$$

resulting in the corresponding governing equation

$$\begin{aligned} (m_1 + m_2)\ddot{r}_2 &= -m_1g(\sin\theta_1 + \mu\cos\theta_1) \\ &\quad + m_2g(\sin\theta_2 - \mu\cos\theta_2), \end{aligned}$$

which is the same as Equation 5.14 derived using d'Alembert's approach. ■

5.4.1 Lagrange's Equation for Small Oscillations

We have learned that Lagrange's equation can be used to derive the fully nonlinear equations of motion for a

dynamic system. In many applications, vibration is approximately linear, and we may prefer to take advantage of the simpler linear equations of motion. Therefore, it is of interest to examine how Lagrange's equation simplifies for small amplitude oscillations about equilibrium.

To show this, we expand the expression for the potential energy,

$$V(q_1, q_2, \dots, q_n),$$

in an n variable Taylor series⁹ about an arbitrary equilibrium reference position $V(q_{1eq}, q_{2eq}, \dots, q_{neq})$,

$$\begin{aligned} V(q_1, q_2, \dots, q_n) &= \frac{1}{2} \left(\frac{\partial^2 V}{\partial q_1^2} \Big|_{eq} q_1^2 + \frac{\partial^2 V}{\partial q_2^2} \Big|_{eq} q_2^2 + \dots \right. \\ &\quad \left. + 2 \frac{\partial^2 V}{\partial q_1 \partial q_2} \Big|_{eq} q_1 q_2 + \dots + \dots \right). \end{aligned}$$

Use is made of the equilibrium relations

$$V(q_{1eq}, q_{2eq}, \dots, q_{neq}) = 0$$

and

$$\frac{\partial V}{\partial q_i} \Big|_{eq} = 0.$$

For small amplitudes, q_i to the power three and higher can be ignored, leaving the approximation

$$V \simeq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 V}{\partial q_i \partial q_j} \Big|_{eq} q_i q_j = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n k_{ij} q_i q_j,$$

where k_{ij} are known as the *stiffness coefficients*. The kinetic energy is given by

$$T = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n m_{ij} \dot{q}_i \dot{q}_j,$$

where m_{ij} are known as the *mass coefficients*.

Substituting the above expressions into Lagrange's equation leads to the following n coupled linear differential equations of motion,

$$[m]\{\ddot{q}\} + [k]\{q\} = \{0\}. \quad (5.28)$$

In the expressions for V and T , the property elements k_{ij} and m_{ij} are constants. Equation 5.28 will generally have coupled terms of the form $q_i q_j$.

⁹The Taylor series about $(0, 0, 0)$ is

$$\begin{aligned} f(x, y, z) &= f(0, 0, 0) + f_x(0, 0, 0) \cdot (x - x(0)) \\ &\quad + f_y(0, 0, 0) \cdot (y - y(0)) + f_z(0, 0, 0) \cdot (z - z(0)) \\ &\quad + \frac{1}{2} (f_{xx}(0, 0, 0) \cdot (x - x(0))^2 + \dots \\ &\quad + 2f_{xy}(0, 0, 0) \cdot (x - x(0))(y - y(0)) + \dots) + \dots \end{aligned}$$

This matrix-vector equation of motion will be derived again (and solved) in [Chapter 6](#) via Newton's second law of motion.

WILLIAM ROWAN HAMILTON

(August 4, 1805 – September 2, 1865)



Contribution: Hamilton was an Irish physicist, astronomer, and mathematician who made important contributions to classical mechanics, optics, and algebra. His studies of mechanical and optical systems led him to discover far-reaching mathematical concepts and techniques.

His greatest contribution is perhaps the reformulation of Newtonian mechanics, now called Hamiltonian mechanics, through the development of action principles. He developed a unified formulation for classical mechanics. Hamilton's principle, which draws on concepts from the calculus of variations, contains both the Newtonian and Lagrangian forms of the equations of motion.

The generality of his principle is evidenced by the fact that Hamilton reported it in a work on optics. His principle has been extended to relativistic and quantum mechanics through appropriate redefinitions of the energy functions. Since Newton's laws are axioms, some researchers have argued that Hamilton's principle, rather than Newton's laws, is the foundation for classical mechanics.

In mathematics, he is known for discovering quaternions.

Life Sketch: Hamilton was born in Dublin, Ireland. He was the fourth of nine children. Before his third birthday, Hamilton was sent to live with his uncle, a clergyman of the Church of Ireland and curate of Trim, County Meath.

Hamilton's genius was evident early in life. By age five, he had learned Latin, Greek, and Hebrew through home tutoring. He mastered other languages as he grew. A turning point in his life occurred at age 12 when he met the American Zerah Colburn, who could perform amazing mental arithmetical feats. Hamilton joined in competitions of arithmetical ability with Colburn. Some say that losing to Colburn sparked Hamilton's interest in mathematics.

At age 13, Hamilton studied Clairaut's *Algebra* and later the works of Newton and Laplace. At age 15, he started

studying the works of Newton and Laplace. In 1822, Hamilton found an error in Laplace's *Mécanique Céleste*, which he corrected with a proof of his own. His work brought him to the attention of John Brinkley, the Royal Astronomer of Ireland, who encouraged Hamilton in his studies.

Hamilton entered Trinity College in Dublin at age 18. His academic progress as an undergraduate was exceptional. In his first year he obtained an "optime," a distinction awarded only once in 20 years, in Classics. Two years later, he received another "optime" in mathematical physics. And in 1826, his final year as an undergraduate, he presented the memoir *Theory of Systems of Rays* to the Royal Irish Academy in which he introduced the characteristic function for optics.

Hamilton became a professor of astronomy at Trinity College at the age of 22 when he completed his undergraduate work there. Thus, in 1827 he became the Andrews' Professor of Astronomy. The post carried the honorary title of Royal Astronomer of Ireland and the benefit of residing at the Dunsink Observatory. However, Hamilton lost interest in astronomy and spent all his time on mathematics.

In 1832, Hamilton published the third supplement to his *Theory of Systems of Rays* and predicted the phenomenon of conical refraction under certain conditions in biaxial crystals. Humphrey Lloyd, a Professor of Physics at Trinity College, verified this theoretical prediction and brought Hamilton great fame. (Hamilton encountered controversy with MacCullagh, who had come very close to the theoretical discovery himself.)

Hamilton developed a theory of conjugate functions, or algebraic couples, in which complex numbers are constructed as ordered pairs of real numbers. In 1834, Hamilton published *On a General Method in Dynamics* in which he applied algebra to dynamics. His papers were difficult to read and entirely different from common practice.

In 1835, Hamilton published *Algebra as the Science of Pure Time*. Hamilton presented his paper at a meeting of the British Association for the Advancement of Science. This second paper on algebraic couples identified them with steps in time and he referred to the couples as "time steps." After the discovery of algebraic couples, he tried to extend the theory to triplets, and this became an obsession that troubled him for many years.

Hamilton proposed an energy conservation law. This law is commonly known by his name, Hamilton's Law, and the operator known as the Hamiltonian H . He made the initial and final coordinates the independent variables of the characteristic function. For conservative systems, the total energy H was constant along any real path but varied if the initial and final points were varied, and so the characteristic function in dynamics became a function of the $6n$ coordinates of initial and final position (for n particles) and the Hamiltonian H .

In 1843, Hamilton discovered the algebraic system of quaternions. This played an important role in the development of abstract algebra. In 1853, Hamilton published *Lectures on Quaternions*.

Hamilton died in Dublin, Ireland from a severe attack of gout. His death came shortly after receiving the news that he had been elected the first foreign member of the National Academy of Sciences of the U.S.

Noteworthy: Hamilton's life was influenced deeply by his love for, and inability to marry, Catherine Disney, who he first met in August 1824. Despite their great love, Catherine's family arranged for her to marry a clergyman, Rev. William Barlow, who was fifteen years her senior. He was affluent and they decided he could offer more. Hamilton was distraught, considered suicide, and became dependent on alcohol, which played a critical role in his life.

Hamilton became ill due to overwork and decided to take a vacation to Armagh to visit another astronomer. There he met Lady Campbell, who was to become one of his favorite confidants. William also took the opportunity to visit Catherine, as she was living nearby, and she reciprocated by coming to the observatory. Hamilton was so nervous in her presence that he broke the eyepiece of the telescope while trying to give her a demonstration. This episode inspired another interval of misery and poem writing.

Catherine aside, Hamilton seemed quite fickle when it came to relationships with women. Perhaps this was because he thought that he ought to marry and that if he could not have Catherine, then it did not really matter whom he married. He eventually married Helen Maria Bayly, who lived just across the fields from the observatory. The marriage was ill-fated from the start; Hamilton spent the duration of their honeymoon working on his third supplement to his *Theory of Systems of Rays*. They had two sons.

After discovering quaternions in 1843, Hamilton became obsessed with trying to apply the theory to problems in applied mathematics. Hamilton felt his discovery of quaternions would revolutionize mathematical physics and he spent the rest of his life working on them. He wrote: "I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions [the calculus] was for the close of the seventeenth."

Shortly after Hamilton's discovery of the quaternions, his personal life became troubled again. In 1845, Catherine visited Hamilton at the observatory, after which his alcohol dependency took a turn for the worse. At a meeting of the Geological Society the following February, Hamilton made an exhibition of himself through his intoxication.

In 1847 the deaths of his uncles James and Willey and the suicide of his colleague James MacCullagh at Trinity College greatly disturbed him. The following year Catherine began writing to him, adding to his depression. The correspondence continued for six weeks and became more informal and personal until Catherine felt so guilty that she confessed to her husband, Barlow. Hamilton wrote to Barlow that they would never hear from him again. However, Catherine wrote once more and this time attempted suicide (unsuccessfully) as her remorse was so great. She then spent the rest of her life living with her mother and siblings, although there was no official

separation from Barlow. Hamilton persisted in his correspondence to Catherine, which he sent through her relatives.

Hamilton turned again to alcohol. He threw himself into his work and began writing his *Lectures on Quaternions*. Hamilton helped Catherine's son James prepare for his Fellowship examinations, which focused on quaternions. He saw this as revenge towards Barlow as he was able to help Catherine's son in a way that his father could not. Later that year Hamilton received a pencil case from Catherine with an inscription that read: "From one who you must never forget, nor think unkindly of, and who would have died more contented if we had once more met."

Hamilton went straight to Catherine and gave her a copy of *Lectures on Quaternions*. She died two weeks later. As a way of dealing with his grief, Hamilton overwhelmed the family with incessant correspondence, sometimes writing two letters a day.

Determined to produce a work of lasting quality, Hamilton began to write another book, *Elements of Quaternions*, which he estimated would be 400 pages long and take 2 years to write. The book ended up being double its intended length and taking seven years to write. The final chapter was incomplete when he died and the book was finally published with a preface by his son William Edwin Hamilton. The title suggests that Hamilton modeled his work on Euclid's *Elements* and indeed this was the case.

Not everyone found Hamilton's quaternions a major advance. Cayley compared the quaternions with a pocket map: "... which contained everything but had to be unfolded into another form before it could be understood."

Hamilton had amazing facility with languages. At a young age, Hamilton displayed an uncanny ability to acquire languages. At the age of seven he had already made very considerable progress in Hebrew, and before he was thirteen he had acquired, under the care of his uncle (a linguist), almost as many languages as he had years of age. These included the classical and modern European languages, as well as Persian, Arabic, Hindustani, Sanskrit, and even Malay. He retained much of his knowledge of languages to the end of his life, often reading Persian and Arabic in his spare time, although he had long stopped studying languages, and used them just for relaxation.

Hamilton was friends with the prominent poet Wordsworth. The two men had long debates over science versus poetry. Hamilton liked to compare the two, suggesting that mathematical language was as artistic as poetry. Wordsworth disagreed, saying that "science applied only to material uses of life, waged war with, and wished to extinguish, imagination." Wordsworth eventually told Hamilton quite forcibly that his talents were in science rather than poetry.

His name lives on through his Hamilton's equations, Hamilton's principle, Hamilton's principal function, and the Hamilton-Jacobi equation. The Hamiltonian is the name of both a function (classical) and an operator (quantum) in

physics, and a term from graph theory. The algebra of quaternions is usually denoted by H , in honor of Hamilton.

The year 2005 was the 200th anniversary of Hamilton's birth and the Irish government designated it the Hamilton Year, celebrating Irish science. A commemorative coin was issued by the Central Bank of Ireland in his honor.

5.5 Hamilton's Principle

We now offer an alternate approach to the derivation of Lagrange's equation. Along the way we derive *Hamilton's principle*, a very powerful integral variational principle.¹⁰ It has the advantage that the boundary conditions are a result as well as the equations of motion.

We begin with Equation 5.19, and rewrite the left-hand side of that equation as

$$\begin{aligned} \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i &= \sum_{i=1}^N m_i \frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) - \delta \sum_{i=1}^N \frac{1}{2} m_i (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \\ &= \sum_{i=1}^N m_i \frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) - \delta T, \end{aligned} \quad (5.29)$$

where use has been made of the relation

$$\frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) = \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i + \dot{\mathbf{r}}_i \cdot \delta \dot{\mathbf{r}}_i = \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i + \delta \left(\frac{1}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right).$$

Substituting Equation 5.29 and $\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = \delta W$ into d'Alembert's principle, we find

$$\delta T + \delta W = \sum_{i=1}^N m_i \frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i).$$

Consider a *varied path*, as shown in Figure 5.18, where the paths coincide at the initial and final times. Integrating between t_1 and t_2 ,

$$\begin{aligned} \int_{t_1}^{t_2} (\delta T + \delta W) dt &= \int_{t_1}^{t_2} \sum_{i=1}^N m_i \frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) dt \\ &= \sum_{i=1}^N \int_{t_1}^{t_2} m_i \frac{d}{dt} (\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i) dt \\ &= \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \Big|_{t_1}^{t_2} \\ &= 0, \end{aligned}$$

since $\delta \mathbf{r}_i = 0$ at t_1 and t_2 because there are no variations at t_1 and t_2 . Therefore,

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0. \quad (5.30)$$

¹⁰In general, elementary problems in mechanics are more easily solved with Lagrange's equations than with Hamilton's principle, which plays an important role in quantum mechanics.

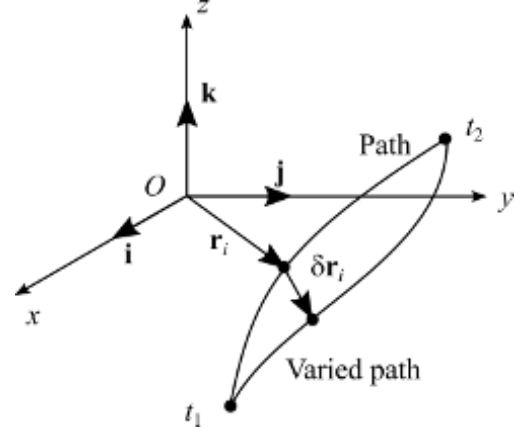


Figure 5.18: Path and varied path for Hamilton's principle. Note that here, unlike in Figure 5.2, the paths are functions of time.

This is called *the extended Hamilton's principle* with the term *extended* implying that δW includes both conservative and nonconservative work. If the forces are only conservative, then $\delta W = -\delta V$, and we find Hamilton's principle,

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0.$$

This equation may be interpreted as nature trying to equalize the kinetic and potential energies of a system, in absence of dissipation.

Lagrange's equation can also be derived beginning with Hamilton's principle. To see this, for each generalized coordinate in Equation 5.30, we vary $T(q_i, \dot{q}_i)$,

$$\delta T = \sum_{i=1}^N \frac{\partial T}{\partial q_i} \delta q_i + \sum_{i=1}^N \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i. \quad (5.31)$$

Letting

$$\delta \dot{q}_i = \frac{d(\delta q_i)}{dt}$$

and integrating by parts, the i^{th} component of the second term in Equation 5.31,

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i} \frac{d(\delta q_i)}{dt} dt = \left. \frac{\partial T}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_i \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) dt,$$

where

$$\left. \frac{\partial T}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} = 0$$

at the end times. Equation 5.30 then becomes

$$\int_{t_1}^{t_2} \sum_{i=1}^N \left[\left\{ \frac{\partial T}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) + Q_i \right\} \delta q_i \right] dt = 0, \quad (5.32)$$

where $\delta W = -\sum_i (\partial V / \partial q_i - Q_{i_{nc}}) \delta q_i \equiv \sum_i Q_i \delta q_i$.

Since all the δq_i are arbitrary except at the end times, for each i in Equation 5.32 the expression within the braces equals zero. This gives us Lagrange's equation, one for each generalized coordinate, as in Equation 5.26.

Example 5.14 Hamilton's Principle for the Derivation of the Equations of Motion of an Elastic Pendulum

Derive the equation of motion of the spring-pendulum system depicted in Figure 5.19. Demonstrate the utility of Hamilton's principle for the derivation of the equations of motion of complex dynamic systems. The system is assumed to have no dissipation.

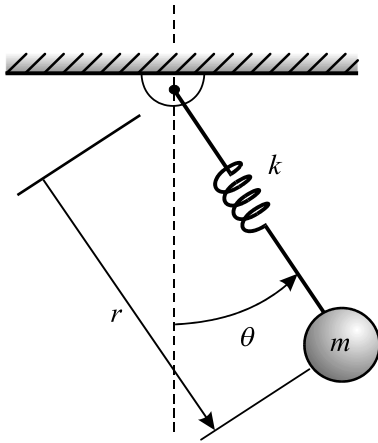


Figure 5.19: A spring pendulum.

Solution: The Lagrangian is

$$L = T - V$$

$$= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2] - [-mgh + \frac{1}{2}k(r - r_0)^2],$$

where $h = r \cos \theta$ and r_0 is the undeformed spring length. Substituting the variation of the Lagrangian into Hamilton's principle yields the following expression,

$$\int_{t_1}^{t_2} \delta L dt$$

$$= \int_{t_1}^{t_2} [m\dot{r} \delta \dot{r} + m(r\dot{\theta}^2 \delta r + r^2 \dot{\theta} \delta \dot{\theta})$$

$$+ mg \delta r \cos \theta - mgr \delta \theta \sin \theta - k(r - r_0) \delta r] dt$$

$$= \int_{t_1}^{t_2} \left\{ m\dot{r} \delta \dot{r} + mr^2 \dot{\theta} \delta \dot{\theta} \right.$$

$$+ [m(r\dot{\theta}^2 + mg \cos \theta - k(r - r_0)) \delta r$$

$$- mgr \sin \theta \delta \theta] dt. \quad (5.33)$$

To proceed we must convert variations of time derivatives of parameters into variations of the parameters themselves. There are two such terms in Equation 5.33,

$$m\dot{r} \delta \dot{r} dt, \quad mr^2 \dot{\theta} \delta \dot{\theta} dt,$$

that need to be converted.

The approach is relatively straightforward and involves an “inverse” chain rule or “integration by parts.” We show the procedure here. For the first case above, we transform a term such as $(\dots) \delta \dot{r} dt$ to one such as $(\dots) \delta r dt$. We begin with the time derivative

$$\frac{d}{dt}(m\dot{r} \delta r) = m\ddot{r} \delta r + m\dot{r} \delta \dot{r},$$

from which we obtain

$$m\dot{r} \delta \dot{r} dt = d(m\dot{r} \delta r) - m\ddot{r} \delta r dt.$$

For the expression $mr^2 \dot{\theta} \delta \dot{\theta} dt$, which is a product of three functions of time, we begin with the time derivative

$$\frac{d}{dt}(mr^2 \dot{\theta} \delta \theta) = m(2r\dot{r} \dot{\theta} \delta \theta) + m(r^2 \ddot{\theta} \delta \theta) + mr^2 \dot{\theta} \delta \dot{\theta},$$

and find

$$mr^2 \dot{\theta} \delta \dot{\theta} dt = d(mr^2 \dot{\theta} \delta \theta) - (mr^2 \ddot{\theta} + 2mr\dot{r}\dot{\theta}) \delta \theta dt.$$

Substituting these expressions into Equation 5.33, we have

$$- \int_{t_1}^{t_2} \delta L dt$$

$$= \int_{t_1}^{t_2} \left[\left\{ m\ddot{r} - mr\dot{\theta}^2 - mg \cos \theta + k(r - r_0) \right\} \delta r \right.$$

$$+ \left\{ mr^2 \ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr \sin \theta \right\} \delta \theta \Big] dt$$

$$- \int_{t_1}^{t_2} \left[d(m\dot{r} \delta r) + d(mr^2 \dot{\theta} \delta \theta) \right] = 0. \quad (5.34)$$

Since $\delta r = 0$ and $\delta \theta = 0$ at t_1 and t_2 , the second integral on the right-hand side of Equation 5.34 equals zero. Further, since δr is independent of $\delta \theta$ and both are arbitrary, it must be that

$$m\ddot{r} - mr\dot{\theta}^2 - mg \cos \theta + k(r - r_0) = 0$$

$$mr^2 \ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr \sin \theta = 0,$$

which are the nonlinear equations of motion for the “elastic pendulum.” It is assumed that the spring behaves linearly. Even if we assume small angles, that is $\cos \theta \simeq 1$, $\sin \theta \simeq \theta$, we cannot remove the nonlinearity $\dot{\theta}^2$ without additional assumptions. ■

5.6 Lagrange's Equation with Damping

Up to now damping was not accounted for in the variational formulation. Here, damping is included. Rather

than proceed with a full derivation, as we have for Lagrange's equation and Hamilton's principle, we will state the final result, and refer the reader to a source¹¹ that includes many relevant details and examples.

As discussed in Chapter 3 and Appendix B, there are many types of damping, and the particular application will determine which is most suitable. We will assume here that viscous damping, which is proportional to the first power of the speed and opposite in direction to its motion, is an adequate representation. Depending on the application, at higher speeds the damping may be proportional to the speed taken to a power greater than one.

For viscous damping where the damping force is proportional to the velocity, the special form of the generalized force is

$$Q_D = -\frac{\partial R}{\partial \dot{q}}$$

for each generalized coordinate, where R is known as the *Rayleigh dissipation function* and is given by

$$R = \frac{1}{2} \sum_k \sum_l c_{kl} \dot{q}_k \dot{q}_l,$$

where the c_{kl} are damping coefficients. For the k^{th} generalized coordinate then

$$Q_{D_k} = -\frac{\partial R}{\partial \dot{q}_k} = -\sum_l c_{kl} \dot{q}_l,$$

with the resulting Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = Q_k, \quad k = 1, 2, \dots, n.$$

We will apply Lagrange's equation and Hamilton's principle in Chapters 6 to 8 for systems modeled as multi degree-of-freedom and continuous systems, respectively.

5.7 Jourdain's Principle

In 1909, Jourdain published his variational principle to consider the "gap" between d'Alembert's principle and Gauss's principle of least constraint and the respective differences in the imposed variational constraints.^{12,13} D'Alembert's principle is formulated in terms of virtual displacements and Gauss's principle¹⁴ in terms of a virtual acceleration. Jourdain was looking for a variation

¹¹Lagrangian Dynamics, D.A. Wells, Schaum's Outline Series, McGraw-Hill, 1967.

¹²P.E.B. Jourdain, "Note on an Analogue of Gauss's Principle of Least Constraint," *Quarterly Journal of Pure and Applied Mathematics*, Vol.40, 1909, pp.153-157.

¹³S. Mottaghi, **Modeling vortex-induced fluid-structure interaction using an extension of Jourdain's principle**, PhD Dissertation, Rutgers University, January 2015.

¹⁴An advantage of Gauss's principle over the principle of virtual work is that it offers a more intuitive formulation of the motion of a constrained system.

in the velocity and considered the variational constraints to be

$$\delta \mathbf{r} = \delta t = 0. \quad (5.35)$$

Analogous to the other variational methods, Jourdain's principle is based on a *dynamic equilibrium* relation, and, for a system of N particles, is given by the relation

$$\sum_{i=1}^N (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \dot{\mathbf{r}}_i = 0, \quad (5.36)$$

$$\delta \mathbf{r} = 0, \quad \delta t = 0,$$

where m_i is the mass of particle i , vector $\mathbf{F}_i$ is the sum of the forces acting on particle i , and $\delta \dot{\mathbf{r}}_i$ is the variation of the velocity of the particle i , called the *virtual velocity*.

Jourdain showed that his method demands fewer derivations for nonholonomic systems when compared with d'Alembert's and Gauss's principles. Jourdain concluded that his constraints lead to a new principle where the virtual velocities do not correspond to the virtual displacements of d'Alembert's principle.

Since the terms in Equation 5.36 are power relations, the equation has also been referred to as the *principle of virtual power*, and is also known as *Kane's equation*.

5.7.1 Jourdain's Principle from d'Alembert's Principle

Jourdain's principle can be derived from d'Alembert's principle by direct differentiation, beginning with d'Alembert's principle,

$$\sum_{i=1}^N (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \mathbf{r}_i = 0, \quad (5.37)$$

$$\delta t = 0.$$

Differentiating with respect to time yields

$$\sum_{i=1}^N \left\{ \frac{d}{dt} (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \mathbf{r}_i + (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \frac{d(\delta \mathbf{r}_i)}{dt} \right\} = 0,$$

$$\delta t = 0.$$

Using the *commutation rule*,

$$\frac{d}{dt} (\delta \mathbf{r}) - \delta \left(\frac{d}{dt} \mathbf{r} \right) = 0, \quad (5.38)$$

we have

$$\sum_{i=1}^N \left\{ \frac{d}{dt} (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \mathbf{r}_i + (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \dot{\mathbf{r}}_i \right\} = 0,$$

$$\delta t = 0.$$

Now, we can impose Jourdain's constraints,

$$\delta t = 0, \quad \delta \mathbf{r} = 0, \quad d(\delta \mathbf{r}) \neq 0, \quad (5.39)$$

and obtain Jourdain's principle,

$$\sum_{i=1}^N (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \dot{\mathbf{r}}_i = 0, \quad (5.40)$$

where $d(\delta \mathbf{r})$ is the differential of the virtual displacement $\delta \mathbf{r}$. Jourdain's constraints, expressed by Equation 5.39, assume that a system can have an alternate possible velocity field at a time instant with the same corresponding displacement field.

It has been suggested that the position vector, $\mathbf{r}$, can be represented by using trajectories in both the Lagrangian and Eulerian frames of reference. Unlike the virtual displacements and velocities utilized in Lagrange's equation, d'Alembert principle, and Hamilton's principle, the virtual velocities in Jourdain's principle are not necessarily infinitesimal quantities and result in an alternative possible state.

While the limited literature on Jourdain's principle is focused primarily on its application to nonholonomic systems, an important feature of this principle was revealed by Papastavridis.¹⁵ While examining the principle and comparing it with Lagrange's equations, he showed how Jourdain's principle can be derived without the assumption of the commutation rule, Equation 5.38. He showed that if the commutation rule is not valid, then there exists a vector $\delta^* \mathbf{r}$ such that

$$\delta^* \mathbf{r} = \frac{d}{dt}(\delta \mathbf{r}) - \delta \left(\frac{d}{dt} \mathbf{r} \right) \neq 0$$

where the correct equation of motion is still obtained using Jourdain's variational principle.

Vujanovic¹⁶ demonstrates the use of Jourdain's principle to find approximate solutions for nonlinear oscillators and other nonlinear phenomena. Kövecses¹⁷ discusses the application of Jourdain's principle to the modeling of finite and impulsive motion of constrained mechanical systems, such as a robotic arm undergoing prescribed motion. Bahar¹⁸ offers a unified approach to nonholonomic dynamics. Such problems arise where the constraints are actively imposed.

¹⁵J.G. Papastavridis, "On Jourdain's Principle," *International Journal of Engineering Science*, Vol.30, No.2, pp.135-140, 1992.

¹⁶B. Vujanovic, T. Atanackovic, "On the Use of Jourdain's Variational Principle in Nonlinear Mechanics and Transport Phenomena," *Acta Mechanica*, Vol.29, 1978, pp.229-238.

¹⁷J. Kövecses, W.L. Cleghorn, "Finite and impulsive motion of constrained mechanical systems via Jourdain's principle: discrete and hybrid parameter models," *International Journal of Nonlinear Mechanics*, Vol.38, 2003, pp.935-956.

¹⁸L.Y. Bahar, "A unified approach to non-holonomic dynamics," *International Journal of Non-Linear Mechanics*, Vol.35, 2000, pp.613-625.

Example 5.15 Applying Jourdain's Principle

Consider the two degree-of-freedom vertical system in Figure 5.20 of two masses connected to each other by a spring of stiffness constant k_2 and both masses connected to a fixed support at top by another spring of constant k_1 . The top mass is m_1 and its vertical displacement is given by r_1 . The bottom mass m_2 has an external force F acting on it, and its vertical displacement is given by r_2 . Assume that r_1 and r_2 are measured from the respective equilibrium positions. Derive the equations of motion for the masses using Jourdain's principle.

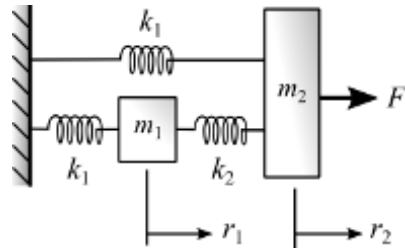


Figure 5.20: A two degree-of-freedom system for Example 5.15.

Solution From free-body diagrams for the masses, using Jourdain's Equation 5.40, we have

$$[m_1 \ddot{r}_1 - k_1 r_1 + k_2(r_2 - r_1)]\delta \dot{r}_1 + [m_2 \ddot{r}_2 - k_2(r_2 - r_1) - F]\delta \dot{r}_2 = 0.$$

Since the velocity variations are not equal to zero, then it must be that each factor is independently equal to zero,

$$m_1 \ddot{r}_1 - k_1 r_1 + k_2(r_2 - r_1) = 0$$

$$m_2 \ddot{r}_2 - k_2(r_2 - r_1) - F = 0.$$

Note that if the external force F is set equal to zero and that the accelerations are both set equal to zero, we find $r_1 = r_2 = 0$, which are the equilibrium positions. ■

5.8 Concepts Summary

This chapter has introduced the fundamental concepts underlying the variational approaches to problems of vibration. The basis for these concepts is the principle of virtual work. D'Alembert's principle extends virtual work to dynamic problems. From these, Lagrange's equation and Hamilton's principle have been derived. They are tools for deriving the governing equations of motion.

Chapter 6 formulates idealized multi degree-of-freedom models of vibration. Chapters 7 and 8 develop idealized continuous system models of vibration. In each of these chapters, the equations of motion are derived by the application of Newton's second law of motion, by Lagrange's equation, and by Hamilton's principle.

5.9 Quotes

- “A man will turn over half a library to make one book.” (Samuel Johnson)
- “We all agree that your theory is crazy, but is it crazy enough?” (Niels Bohr)
- “A professor on the first day of class goes up to the board and draws two circles, a very small one and a large one. He then writes ‘You’ next to the small circle and ‘Me’ near the large one. The students rumble. He then says, ‘You see the small circle. This is what you think you don’t know about this subject. The large circle. This is what I know I don’t know about the subject.’”
- “That’s one small step for a man, one giant leap for mankind.” (Neal Armstrong)
- “Experience serves not only to confirm theory, but differs from it without disturbing it, it leads to new truths which theory only has not been able to reach.” (D’Alembert, quoted in introduction to *PS Girard Traite Analytique de la Resistance des Solides*)
- “I am an old man and I have known a great many troubles, but most of them never happened.” (Mark Twain)
- “Logic will take you from A to B. Imagination will take you everywhere.” (Albert Einstein)
- “I’m great at multitasking. I can waste time, be unproductive, and procrastinate all at once.”
- “The saddest aspect of life right now is that science gathers knowledge faster than society gathers wisdom.” (Isaac Asimov)
- “I’m not interested in my legacy. I made up a word: ‘live-acy.’ I’m more interested in living.” (John Glenn)

5.10 Problems

Many real vibration problems can be interpreted as pendulum problems and we have included an assortment of such problems here. Many of the problems here can be solved using Lagrange’s equation, the most widely used approach.

Problems for Section 5.3 – Virtual Work and D’Alembert’s Equation

1. Consider the system shown in Figure 5.21. Let x_1 and θ be the displacement and angle of rotation

of the pulley measured from their original locations when the spring is undeformed. Let x_2 be the displacement of the cylinder, also measured from its original location when the spring is undeformed. The pulley has mass M , radius R , and its mass moment of inertia about its centroidal axis is I_G . The cylinder has mass m . Find (a) the static equilibrium position of the system using the principle of virtual work, and (b) the equation of motion of the system for generalized coordinate x_2 using d’Alembert’s principle.

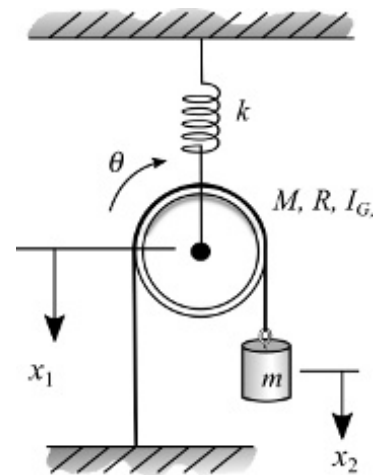


Figure 5.21: Pulley-cylinder coupled system.

2. Consider the point mass supported by two linear springs shown in Figure 5.22. The springs are undeformed when the mass and the springs are along a straight line, where the coordinate y defines the displacement of the mass. Do not assume small angles. Derive (a) the equilibrium equation using the principle of virtual work, and (b) the equation of motion of the system using d’Alembert’s principle.

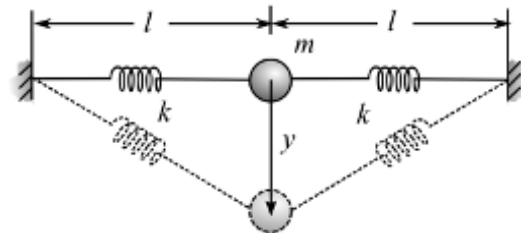


Figure 5.22: Point mass supported by two identical springs.

3. Consider again the system shown in Figure 2.58, reproduced in Figure 5.23. The system consists of massless pulleys, a block of mass m , and two linear springs. The coordinate, θ , is the angle of rotation of the moving pulley, x_1 is the displacement of the pulley center of mass, and x_2 is the displacement of the

mass. The displacements are measured from the positions where the springs are undeformed. (a) Find the static equilibrium position of the system using the principle of virtual work, and (b) derive the equation of motion of the system using d'Alembert's principle for generalized coordinate x_2 .

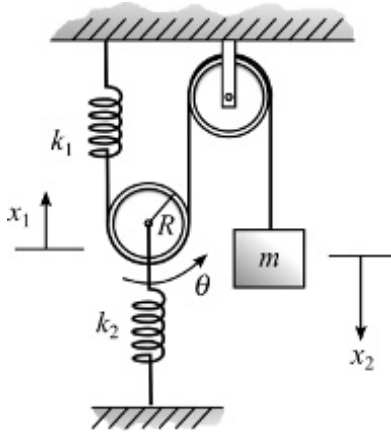


Figure 5.23: A mass suspended by springs and pulleys.

4. Consider the system shown in Figure 5.24. Let x be the displacement of the mass measured from its original location where the springs are undeformed. Assume that the pulleys are massless and the springs are linear. (a) Find the static equilibrium position of the system using the principle of virtual work, and (b) derive the equation of motion using d'Alembert's principle.

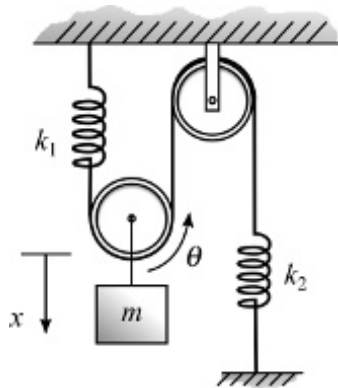


Figure 5.24: A mass suspended by springs and pulleys.

5. Consider again the system shown in Figure 2.46, reproduced in Figure 5.25. The bar is horizontal when the massless springs are undeformed. Assume that the rod mass is m , and the mass moment of inertia about the torsional spring is I_O . (a) Find the static equilibrium position of the system using the principle of virtual work, and (b) derive the equation of motion of the system using d'Alembert's principle.

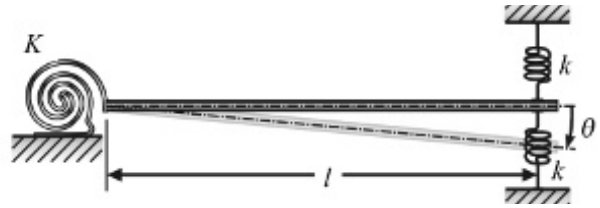


Figure 5.25: Restrained rigid rod.

6. Consider again the system shown in Figure 2.56, reproduced in Figure 5.26. A disk is on an inclined plane supported by a spring. The coordinates x and θ are zero when the spring is undeformed. The disk has mass moment of inertia I_G about its mass center and has radius r . Assume that the disk rolls without slip. (a) Find the static equilibrium position of the system using the principle of virtual work, and (b) derive the equation of motion of the system using d'Alembert's principle.

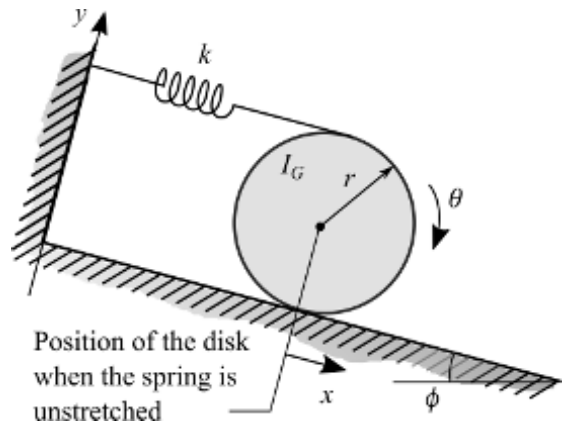


Figure 5.26: A rolling disk on an incline restrained by a spring.

7. Consider the system comprised of the disk, pulley, and block shown in Figure 5.27. The spring is connected to the disk of mass M at its mass center. The disk has radius r and mass moment of inertia I_G about its mass center. A cable is connected to and wrapped around the disk and connected to the block of mass m . Assume that the pulley is massless and that the disk rolls without slip. (a) Find the static equilibrium position of the system using the principle of virtual work, and (b) derive the equation of motion of the system using d'Alembert's principle.

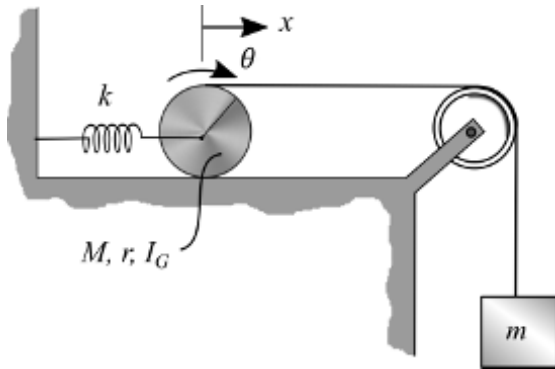


Figure 5.27: Spring-disk-pulley mass system.

8. Consider the two degree-of-freedom system shown in Figure 5.28. The springs are undeformed when the rods are horizontal. The mass of each rod is m , with length L , and mass moment of inertia about each center of gravity is I_G . (a) Find the static equilibrium position of the system using the principle of virtual work, and (b) derive the equation of motion of the system using d'Alembert's principle.

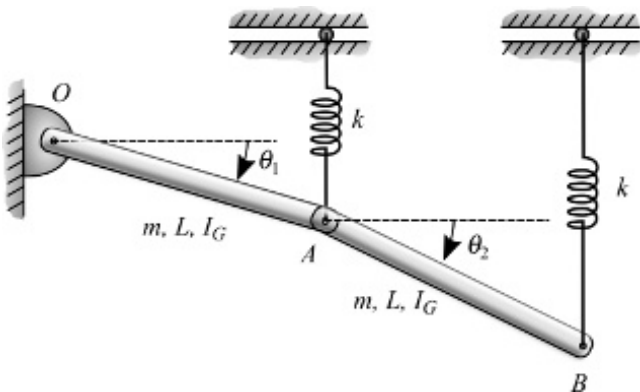


Figure 5.28: Spring-constrained double pendulum.

Problems for Sections 5.4 and 5.5 – Lagrange's Equation, and Hamilton's Principle

9. Derive the equation of motion for the system in Figure 5.21 for generalized coordinate x_2 using (a) Lagrange's equation, and (b) Hamilton's principle.
10. Derive the equation of motion for the system in Figure 5.22 for generalized coordinate y using (a) Lagrange's equation, and (b) Hamilton's principle.
11. Derive the equation of motion for the system in Figure 5.23 for generalized coordinate x_2 using (a) Lagrange's equation, and (b) Hamilton's principle.
12. Derive the equation of motion for the system in Figure 5.24 for generalized coordinate x using (a) Lagrange's equation, and (b) Hamilton's principle.
13. Derive the equation of motion for the system in Figure 5.25 for generalized coordinate θ using (a) Lagrange's equation, and (b) Hamilton's principle.
14. Derive the equation of motion for the system in Figure 5.26 for generalized coordinate x using (a) Lagrange's equation, and (b) Hamilton's principle.
15. Derive the equation of motion for the system in Figure 5.27 for generalized coordinate x using (a) Lagrange's equation, and (b) Hamilton's principle.
16. Derive the equations of motion for the system in Figure 5.28 for generalized coordinates θ_1 and θ_2 using (a) Lagrange's equation and (b) Hamilton's principle.
17. Consider again the system of Figure 2.52, reproduced in Figure 5.29. A disk of mass m and mass moment of inertia I_G is supported by a massless spring and a rope. Derive the equation of motion of the system using (a) Lagrange's equation, and (b) Hamilton's principle, for generalized coordinate x .
18. Consider again the system of Figure 2.53, reproduced in Figure 5.30. A mass m is suspended by a pulley system with a spring. Derive the equation of motion of the system using (a) Lagrange's equation, and (b) Hamilton's principle, for generalized coordinate x .

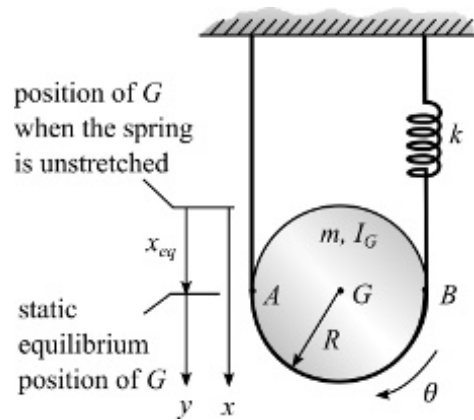


Figure 5.29: A disk suspended by an inextensible cable and a spring.

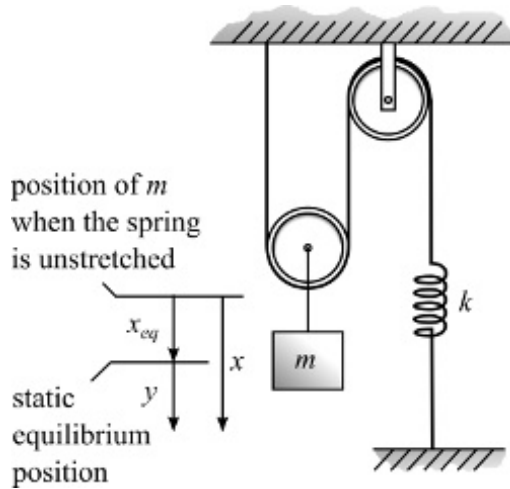


Figure 5.30: A mass suspended in a pulley system.

19. Consider again the system of Figure 2.54, reproduced in Figure 5.31. Derive the equation of motion of the system using (a) Lagrange's equation, and (b) Hamilton's principle, for generalized coordinate x_1 .

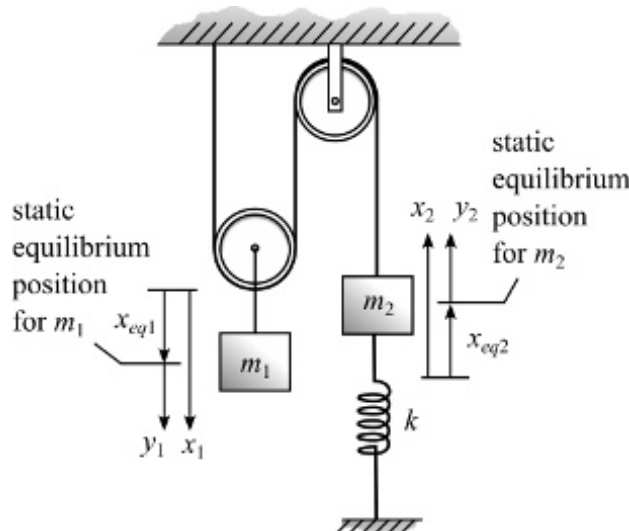


Figure 5.31: Two masses suspended in a pulley system.

20. Consider again the system of Figure 2.57, reproduced in Figure 5.32. Derive the equation of motion using (a) Lagrange's equation, and (b) Hamilton's principle, for generalized coordinate x .

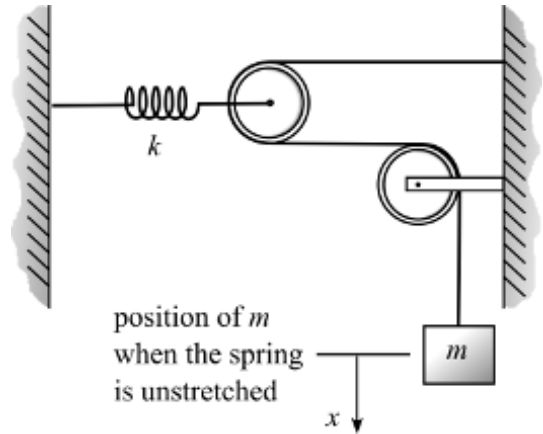


Figure 5.32: Two masses suspended in a pulley system.

21. For the simple pendulum of Figure 5.33, derive the governing equation of motion assuming (a) that m is a point mass, and (b) that the mass is a sphere with small but finite mass moment of inertia. Given $l = 10$ cm and $m = 5$ kg, how significant is the effect of the mass moment of inertia if $r = 0.5$ cm? Derive the governing equation of motion using (i) Newton's second law, (ii) Lagrange's equation, and (iii) Hamilton's principle.

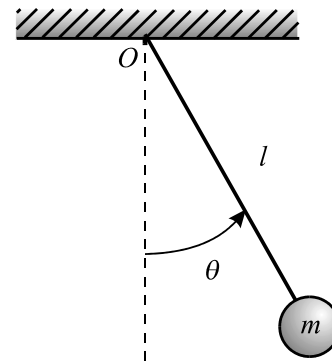


Figure 5.33: Simple pendulum.

22. The lightweight bar in Figure 5.34 is released from rest when the spring is undeformed at $\theta = 0$. The two masses at A and B slide in frictionless guides that are horizontal at the left and vertical at the right. Formulate the equation of motion using (a) Newton's second law, (b) Lagrange's equation, and (c) Hamilton's principle.

Assume the bar is massless. Show that the equation of motion is

$$\begin{aligned} & (a^2 + 4l^2 \cos^2 \theta + 4b^2 \sin^2 \theta) \ddot{\theta} \\ & + 2(b^2 - l^2) \dot{\theta}^2 \sin 2\theta \\ & + g(5a + 4b) \cos \theta + \frac{1}{2} \frac{k}{m} a^2 \sin 2\theta = 0. \end{aligned}$$

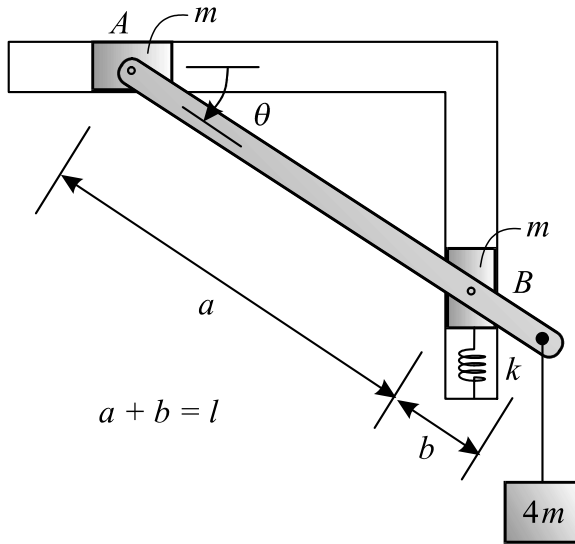


Figure 5.34: Oscillation of constrained bar.

23. The spring supporting the rod-sphere system in Figure 5.35 is undeformed when the rod is horizontal. Assume the rod mass is negligible. The roller from which the spring is suspended permits the spring to maintain a vertical configuration. If the system is in this position when it is released from rest, derive the equation of motion using (a) the principle of virtual work along with d'Alembert's principle, and (b) using Lagrange's equation. Show that the equation of motion is

$$\left(m(l+r)^2 + \frac{2}{5}mr^2 \right) \ddot{\theta} + ka^2 \sin \theta \cos \theta - mg(l+r) \cos \theta = 0.$$

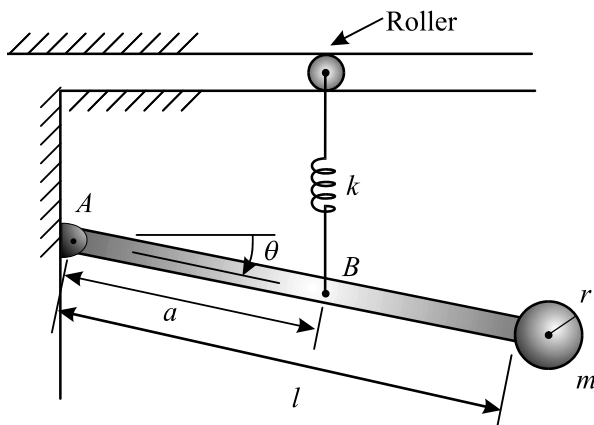


Figure 5.35: Oscillation of rod-sphere system.

24. Verify Equation 5.20,

$$m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_k} = \left[\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_k} \right) - \frac{\partial}{\partial q_k} \right] \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right).$$

25. For Example 5.10, use Newton's second law of motion to derive the equations of motion.

26. Derive Equation 5.28, assuming a linear model with $m_{ij} = \text{constant}$ and $k_{ij} = \text{constant}$,

$$[m]\{\ddot{q}\} + [k]\{q\} = \{0\}.$$

27. Formulate the problem of Example 5.13 where the string is elastic with stiffness k and can stretch. Derive the equations of motion.

28. For the elastic pendulum discussed in Example 5.14, and shown in Figure 5.19, derive the governing equations of motion using Lagrange's equation.

29. In the pendulum shown in Figure 5.36 the length r is variable. Derive the equations of motion. Discuss the possibility and implications if $\omega = \dot{\theta}$.

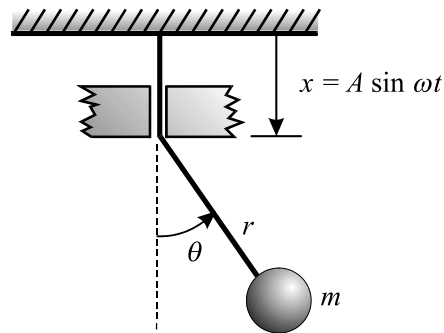


Figure 5.36: Variable length pendulum.

30. (a) For the elastically supported mass on a vertically moving base shown in Figure 5.37 derive the governing equation of motion first using d'Alembert's principle, and then using Lagrange's equation. Discuss the effect of base motion and show that it is equivalent to a force on mass m equal to $kA \sin \omega t$. (b) Repeat the problem where viscous damping exists due to the motion of the mass through the surrounding medium. Assume a damping constant denoted by c .

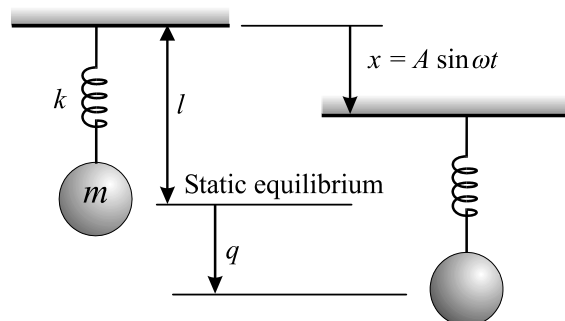


Figure 5.37: Elastically supported mass on a moving base.

31. Consider the motion of a pendulum that is supported by springs that are elastically restrained to horizontal motion, as depicted in Figure 5.38. Assume that the springs are massless and remain horizontal, that θ is small, and r is a constant. Formulate the equations of motion using (a) Newton's second law, (b) Lagrange's equation, and (c) Hamilton's principle. Show that the period τ is given by

$$\tau = 2\pi \sqrt{\frac{mg + 2kr}{2kg}}.$$

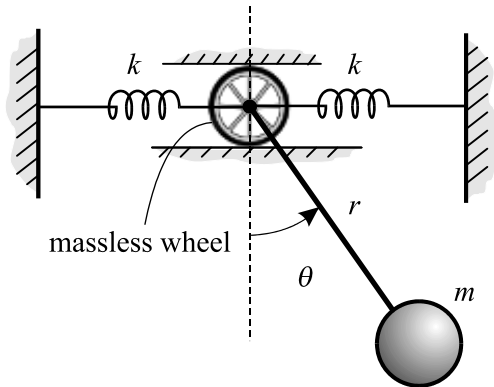


Figure 5.38: Pendulum supported by horizontal springs.

32. For a pendulum supported from a horizontally restrained mass, as shown in Figure 5.39, derive the governing equations of motion using (a) Lagrange's equation and (b) Hamilton's principle. Identify the constraint equations. Discuss the effect of setting $m_1 = 0$. Finally, simplify the governing equations if small motions are assumed.

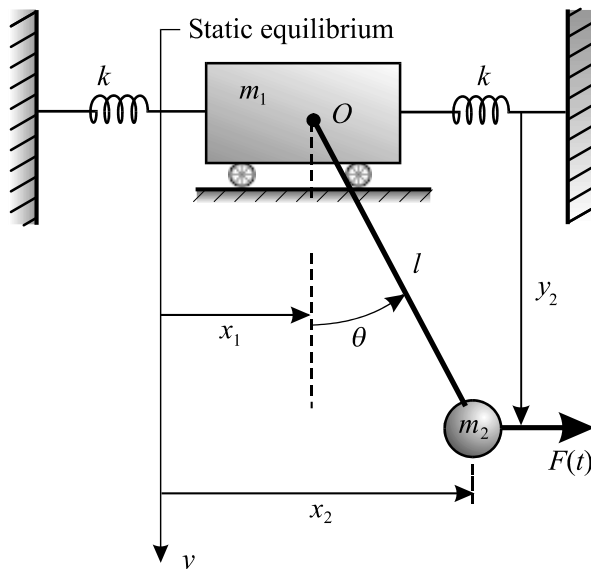


Figure 5.39: Pendulum suspended from a horizontally restrained mass.

33. Fill in the missing steps in the derivation of the equations of motion for Example 5.8, that is, Equations 5.15 and 5.16.
34. Use Lagrange's equation to derive the equations of motion governing the displacements x_1 and x_2 for the two degree-of-freedom system depicted in Figure 5.40.

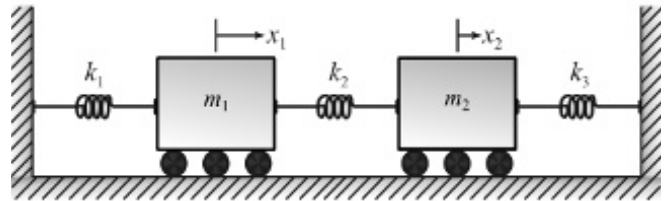


Figure 5.40: Undamped two degree-of-freedom system.

35. Use Lagrange's equation to derive the equations governing the rotations θ_1 , θ_2 , and θ_3 for the spring-connected triple pendulum system shown in Figure 5.41.

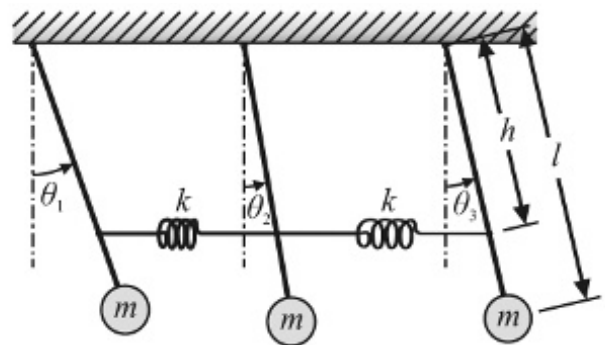


Figure 5.41: Triple pendulum coupled by springs.

36. A pendulum is suspended from a torsionally restrained disk, as in Figure 5.42, where α is the angle of rotation of the disk, and θ is the relative angle of the pendulum measured from the disk. Derive the governing equation of motion using (a) Lagrange's equation, and (b) Hamilton's principle. Compare the equation of motion to the equation of the simple pendulum.

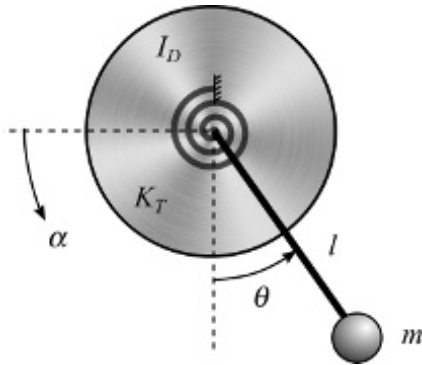


Figure 5.42: Pendulum supported from a torsionally restrained disk.

37. A rigid beam acts as a compound pendulum, suspended from an elastically restrained block that can undergo horizontal motion, as drawn in Figure 5.43. Derive the equations of motion using Lagrange's equation. Consider the following cases: (a) no friction between the block and surface, and (b) viscous damping between the block and surface.

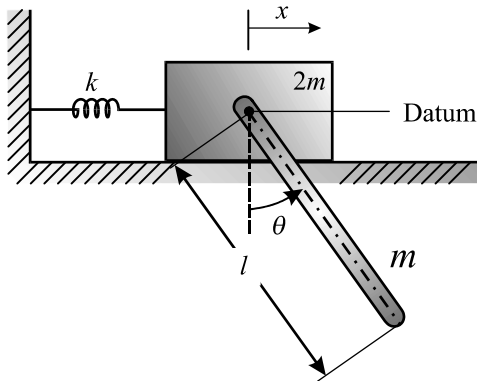


Figure 5.43: Compound pendulum pivoted to a block.

38. For the inverted simple pendulum of length l and mass m_2 , supported on a cart of mass m_1 , as shown in Figure 5.44, derive the equations of motion if the cart is forced by $F(t)$.

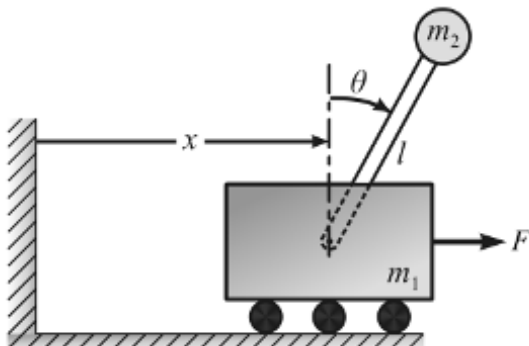


Figure 5.44: Inverted pendulum on a moving base.

39. The system in Figure 5.45 oscillates about the equilibrium position denoted by the horizontal lines. The coil torsional springs are each undeformed when $\theta_1 = \theta_2 = 0$. If the link masses are neglected, derive the equations of motion to be

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2l_1l_2\ddot{\theta}_2 \cos(\theta_2 - \theta_1) - m_2l_1l_2\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) + (K_1 + K_2)\theta_1 - K_2\theta_2 - (m_1 + m_2)gl_1 \cos \theta_1 = 0$$

$$m_2l_2^2\ddot{\theta}_2 + m_2l_1l_2\ddot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2l_1l_2\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + K_2(\theta_2 - \theta_1) - m_2gl_2 \cos \theta_2 = 0.$$

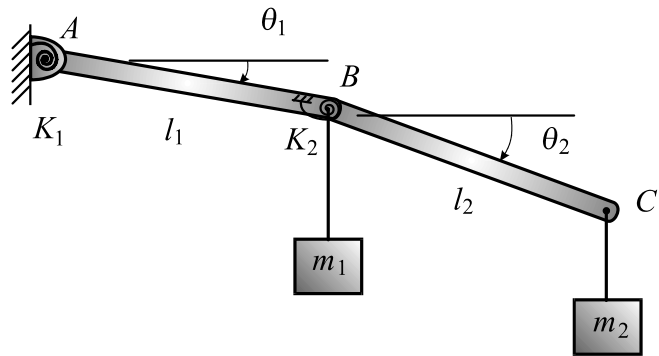


Figure 5.45: Torsionally restrained beams carrying supporting masses.

40. Figure 5.46 shows a pulley system that is acted on by an applied moment $M(t)$, resulting in oscillatory motion. Use Lagrange's equation to derive the equations of motion. Identify the generalized forces Q_θ and Q_x .

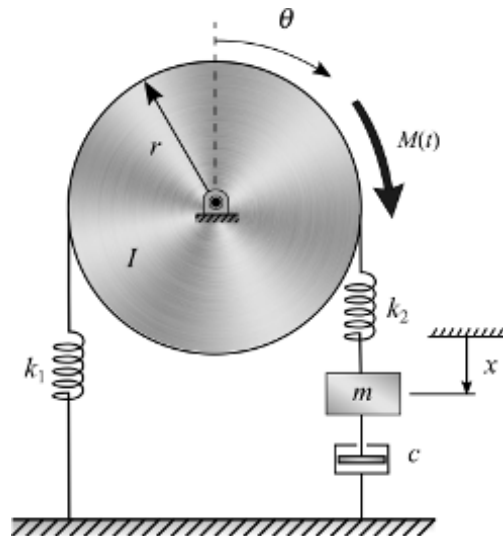


Figure 5.46: A pulley system subjected to a moment.

41. The system sketched in Figure 5.47 depicts two shafts coupled through meshing gears. Assume that gears 2 and 4 have gear ratio $n = r_2/r_4$, and that external moment $M(t)$ is applied to gear 1. Identify the constraint equation and the generalized coordinates, and derive the governing equation(s) of motion. Discuss how the value of n affects the resulting behavior.

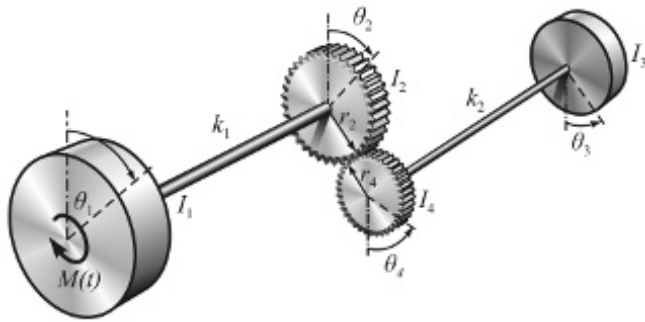


Figure 5.47: A gear-shaft system.



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Chapter 6

Multi Degree-of-Freedom Vibration

“And now we no longer operate in isolation since we are aware of others.”

Many engineering systems, such as turbomachines, bridges, and aircraft, are modeled with hundreds if not thousands of degrees of freedom. Each degree of freedom has a frequency of vibration, much in the same way as a single degree-of-freedom oscillator has a natural frequency. However, systems with multiple degrees of freedom are more complicated due to the physical *coupling* that exists between pairs of degrees of freedom.

In this chapter we introduce a new concept – called the *mode* of vibration – which exists in systems of two or more degrees of freedom. A vibrating linear system, moving in very complicated patterns, can be modeled as n uncoupled vibrating oscillators if special coordinates – called principal coordinates – are used. We will demonstrate that the single degree-of-freedom oscillator is the basis for solving many linear multi degree-of-freedom vibrating systems.

Later in this chapter, we extend the study of multi degree-of-freedom vibration to systems of a more general nature and those with unique and complicating factors, such as imperfections and uncertainties. A deeper understanding of the eigenvalue problem is gained by considering its meaning geometrically. The inverse vibration problem is considered – that is, given the response and the forcing function, we wish to find the system properties of mass, damping, and stiffness.

6.1 Motivating Examples

Most engineering systems can be modeled as multi degree-of-freedom systems. Three interesting applications motivate the study of multi degree-of-freedom discrete models. Additional applications are introduced throughout the chapter.

6.1.1 Periodic Structures

Periodic structures have geometry and material properties that are repetitive with a certain pattern. Such structures are very important in applications such as the solar panel sections of Figure 6.1, the truss lattice of a space station, or the reticulated pattern of stiffeners on the inside shell of an aircraft fuselage. These are well known applications. A less known example is a dynamic model of DNA,¹ a biological polymer that plays an essential role in the conservation and transportation of genetic information. A common example from nature is the spider web shown in Figure 6.2.

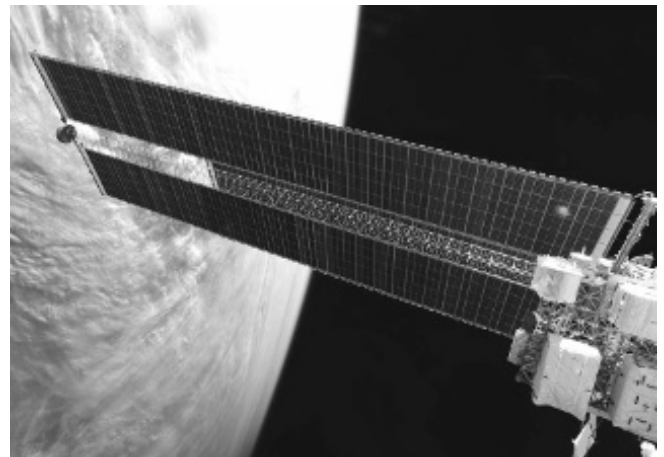


Figure 6.1: Giant solar arrays on the International Space Station are periodic structures. (Photograph taken February 12, 2015. European Space Agency/NASA)

6.1.2 Inverse Problems

In *inverse problems* the output, such as the displacement response, is known and the analysis leads to estimates of parameters such as the mass and stiffness. Such prob-

¹L.V. Yakushevich, “Nonlinear DNA dynamics: hierarchy of the models,” *Physica D*, Vol.79, 1994, pp.77-86. In this paper five increasingly complex models of the DNA molecule are used to study its dynamics.



Figure 6.2: A spider web is an approximation of a circular periodic structure. (Photograph by M. Nagurka)

lems are sometimes referred to as *identification problems*. For example, an engineer applies a known force, such as a step or sinusoid, to a structure – with unknown mass and stiffness properties – and measures the resulting response. In an inverse analysis the response data, such as the frequency response data, along with the known input force are used to estimate the structural parameters. We return to this problem after studying the basic tools of vibration analysis for multi degree-of-freedom systems.



SPIDER WEBS AS VIBRATING STRUCTURE. Spider webs are superbly tuned instruments for vibration transmission. When plucked like a guitar string, spider silk transmits vibrations across a wide range of frequencies.

Web-dwelling spiders have poor vision and rely almost exclusively on web vibrations for their “view” of the world. The vibration patterns provide spiders with crucial information on the type of prey caught in the web and of predators approaching, as well as the quality of prospective mates and even the structural integrity of the web. Spiders carefully engineer their webs out of a range of silks to control web architecture, tension, and stiffness, analogous to constructing and tuning a musical instrument. It is believed that the spider can use behavior and silk properties to control the function of its web. These control mechanisms could alter vibration filtering, as well as orientation to and discrimination of vibration sources in the web.²



²B. Mortimer, A. Soler, C.R. Siviour, R. Zaera, F. Vollrath, “Tuning the instrument: sonic properties in the spider’s web,” *Journal of the Royal Society Interface*, Vol.13, No.122, September 2016.

6.1.3 Vehicle Vibration Testing

In general, automobile and truck suspension systems are tuned to one specific performance target. A suspension system may be optimized for handling performance or it may be optimized to isolate the occupants from road disturbances, but in a passive system it cannot excel at both. In practice, the performance of conventional vehicle suspensions is a compromise between isolation (offering occupants comfort from road inputs) and handling (how well the vehicle holds the road, for example, drivability and maneuverability).³



Figure 6.3: A Mazda CX-7 being tested on an MTS Model 320 7-Post Tire Coupled Road Simulator. (Courtesy Thomas R. Kurfess, May 2008)

To test suspension systems, vehicles can be “shaken” in a laboratory simulating different road profiles. Figure 6.3 shows such a test system in a fully equipped laboratory for dynamically analyzing a vehicle under simulated road conditions.⁴ The system includes an actuator at each wheel to move the vehicle in response to computer-generated signals. The actuators, which are hydraulic, can vibrate with an amplitude of ± 1.5 mm at 30 Hz and ± 0.5 mm at 50 Hz. (The system is rated to 100 Hz, but at that frequency the amplitudes would be so small that the actuators would just buzz.)

The photograph taken in the test-cell shows the two right-side posts, below which are the actuators. In the actual test, one post excites each tire. The shaker is said to be “tire coupled” – meaning that the shaker is coupled to the vehicle via the tires – as opposed to hub-coupled. Not seen in the photograph are the other posts (seven total) and three stingers that come from below

³Some vehicles offer driver selectable suspension systems. Some offer adjustable ride height to improve handling or to increase ground clearance – when driving on poorly surfaced roads. The vehicle can be “tuned” to meet the driver’s desired handling or ride comfort level, that is, the suspension stiffness may be lowered to improve isolation or raised to improve handling performance.

⁴The vehicle is in an environmental chamber capable of a temperature between -40°C and $+85^{\circ}\text{C}$ and relative humidity between 15% and 95%, covering virtually all possible ambient conditions.

and can load and unload the overall vehicle. These are used to simulate aerodynamic drag, cornering forces, and banking.

The system can be used to conduct a wide range of tests of vehicles of different size and weight. These tests include (a) performance, reliability, and design life assessment of suspension systems, (b) vehicle ride quality assessment, (c) active suspension control system development, (d) durability (shake) testing of vehicles, (e) tire and suspension dynamic characterization at various temperature and humidity levels, (f) evaluation of various tire and/or suspension designs for vehicles, (g) squeak and rattle testing at various temperatures and humidity levels, (h) validation of simulation models, (i) evaluation of vehicle testing methods, and (j) road and terrain profile development.

6.1.4 Scope

This chapter introduces the basics of multi degree-of-freedom systems, with applications to simple models that are introduced to develop our understanding. Much of our efforts are directed to the derivation of the equations of motion. In the next section, the concepts of stiffness and flexibility are introduced as a basis for deriving the equations of motion.

6.2 The Concepts of Stiffness and Flexibility

The modeling of multi degree-of-freedom systems requires one governing equation of motion for each degree of freedom. These *simultaneous* equations can be written in matrix form, leading to the matrix counterparts of mass, damping, and stiffness.

Different methods may be used to derive the equations of motion and the property matrices. These include Newton's second law of motion, Lagrange's equation, Hamilton's principle, and *influence coefficients*. The method of influence coefficients provides additional insights into the concepts of stiffness and flexibility that arise naturally in the derivation of the equations of motion.

In this chapter we adopt braces $\{ \}$ for vectors and brackets $[]$ for matrices rather than the bold notation of previous chapters.

6.2.1 Influence Coefficients

We begin with the concepts of *stiffness* and *flexibility* since these properties of a dynamic system are fundamental to its behavior. In this section these ideas are introduced for a very simple system. In the following sections, more realistic and complex systems are tackled.

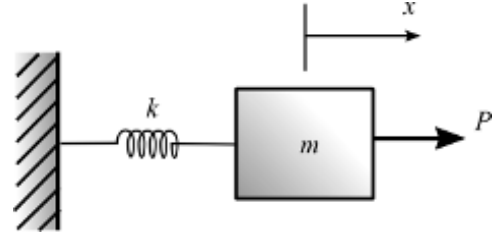


Figure 6.4: The concepts of stiffness and flexibility are introduced by considering a mass-spring system with static force P .

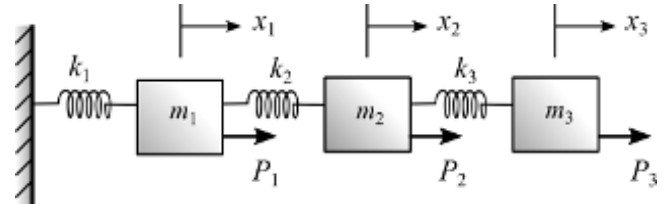


Figure 6.5: Three-mass system coupled by stiffness components subjected to static forces P_1 , P_2 , and P_3 .

In systems with many degrees of freedom, a force acts on mass m_i resulting in the motion of mass m_j . The concept of flexibility and stiffness leads to relations between a force and a displacement. The following is a static analysis, but d'Alembert's principle⁵ can be used to extend the analysis to dynamic problems.

We begin with a simple single degree-of-freedom mass-spring system shown in Figure 6.4. Let the mass be subjected to static force P , resulting in displacement x of mass m . Hooke's law, $P = kx$, applies along with the inverse relation $x = k^{-1}P = fP$, where $k = \text{stiffness influence coefficient}$ and $f = \text{flexibility influence coefficient}$. If one applies a *unit force*, $P = 1$, and measures the displacement x , then the flexibility coefficient can be evaluated directly since $f = x$. Similarly, if a *unit displacement* is applied, $x = 1$, then the stiffness coefficient is given by $k = P$. These ideas can be generalized to systems of more than one mass.

Consider now the system of three masses and three springs shown in Figure 6.5. The system is a fixed-free system because of the way it is configured.

Displacement-force relations can be written in the form

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}, \quad (6.1)$$

where the *flexibility influence coefficient* f_{ij} is defined as the displacement at location i due to a unit force at location j , with all other forces set equal to zero. If we set the values of forces P_j and measure displacements x_i , the coefficients f_{ij} can be evaluated. For example, if we

⁵D'Alembert's principle was discussed in Chapter 5.

set $P_1 = 1$ and $P_2 = P_3 = 0$, then the first relation in the above matrix equation,

$$x_1 = f_{11}P_1 + f_{12}P_2 + f_{13}P_3,$$

becomes $x_1 = f_{11}$, providing the value of flexibility coefficient f_{11} given the measurement x_1 . In a similar manner, the other influence coefficients can be determined. The matrix in Equation 6.1 is called the *flexibility matrix*.

Alternatively, we can write the reciprocal force-displacement relations,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}. \quad (6.2)$$

If we fix displacement x_i and measure force P_j , then k_{ij} can be found from the first equation in Equation 6.2,

$$P_1 = k_{11}x_1 + k_{12}x_2 + k_{13}x_3.$$

Setting $x_1 = 1$, and $x_2 = x_3 = 0$, yields $P_1 = k_{11}$, the value of stiffness coefficient k_{11} . The *stiffness influence coefficient* k_{ij} is the force at location i producing unit displacement at location j , with all other displacements set equal to zero. The matrix in Equation 6.2 is called the *stiffness matrix*, and it arises naturally when the equations of motion for the system are derived, as we will see in the next section.

In this way, we see the beginnings of a procedure by which the flexibility and stiffness coefficients can be determined, as is shown in the two examples that follow.

Example 6.1 Flexibility Matrix

Determine the flexibility matrix $[f]$ of the three-mass, three-spring system shown in Figure 6.5.

Solution: We apply a unit force, $P_1 = 1$, on mass 1 and set $P_2 = P_3 = 0$. Since k_2 and k_3 are unstretched, the springs displace as a rigid body with displacement

$$\begin{aligned} x_1 &= P_1/k_1 = 1/k_1 \\ &= x_2 = x_3. \end{aligned}$$

Thus, the three displacements are related to $P_1 = 1$ by the equation

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1/k_1 & 0 & 0 \\ 1/k_1 & 0 & 0 \\ 1/k_1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}.$$

Next, we set $P_1 = P_3 = 0$ and $P_2 = 1$. From the figure, we see that $x_2 = x_3$. The force-displacement relation for mass 2 is $P_2 = (x_2 - x_1)k_2$ and for mass 1 is $P_2 = x_1k_1$. Combining these two equations, we have

$$x_2 = \left(\frac{1}{k_1} + \frac{1}{k_2} \right).$$

Therefore, the three displacements are related to $P_2 = 1$ by the equation

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1/k_1 & 0 \\ 0 & (1/k_1 + 1/k_2) & 0 \\ 0 & (1/k_1 + 1/k_2) & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}.$$

Finally, we set $P_1 = P_2 = 0$ and $P_3 = 1$. Then, the displacement of mass 1 is governed by $x_1 = P_3/k_1$, of mass 2 by $x_2 = P_3(1/k_1 + 1/k_2)$, and of mass 3 by $x_3 = P_3(1/k_1 + 1/k_2 + 1/k_3)$. The three displacements are related to $P_3 = 1$ by the equation

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1/k_1 \\ 0 & 0 & \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \\ 0 & 0 & \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \right) \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

By combining the three cases above, the complete relation between displacement and forcing is

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \left(\frac{1}{k_1} + \frac{1}{k_2} \right) & \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \\ \frac{1}{k_1} & \left(\frac{1}{k_1} + \frac{1}{k_2} \right) & \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \right) \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix},$$

with the matrix of flexibility coefficients called the *flexibility matrix*. ■

Example 6.2 Stiffness Matrix

For the model in Figure 6.5 mass 2 is now also connected to a wall via a spring of constant k_4 . Derive the stiffness influence coefficients and thus the stiffness matrix $[K]$. Note that by setting $k_4 = 0$, the system of the previous example is regained.

Solution: We proceed according to the definition of stiffness influence coefficients, setting $x_1 = 1$ with $x_2 = x_3 = 0$. The forces required at masses 1, 2, and 3 to enforce the required displacements are

$$\begin{aligned} P_1 &= (k_1 + k_2)x_1 = k_1 + k_2 \\ P_2 &= -k_2x_1 = -k_2 \\ P_3 &= 0. \end{aligned}$$

Repeating this procedure with $x_2 = 1$ and $x_1 = x_3 = 0$, the required forces are

$$\begin{aligned} P_1 &= -k_2x_2 = -k_2 \\ P_2 &= (k_2 + k_3)x_2 \\ P_3 &= -k_3x_2 = -k_3. \end{aligned}$$

Finally, letting $x_3 = 1$ with $x_1 = x_2 = 0$, the forces are

$$\begin{aligned} P_1 &= 0 \\ P_2 &= -k_3 x_3 = -k_3 \\ P_3 &= (k_3 + k_4)x_3 = k_3 + k_4. \end{aligned}$$

Combining these force-displacement relations, we have the relation between an arbitrary displacement vector and the force vector,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}. \quad (6.3)$$

The matrix of stiffness coefficients is called the *stiffness matrix*. It will be derived again using Newton's second law of motion and Lagrange's equation. ■

We note that the influence coefficient matrices are symmetric about their diagonals. Therefore, $f_{ij} = f_{ji}$ and $k_{ij} = k_{ji}$. Furthermore, $[f] = [K]^{-1}$ and $[K] = [f]^{-1}$ and therefore $[f][K] = [I]$. This is known as the *Maxwell-Betti reciprocity theorem*,⁶ which is derived next.

Reciprocity in a linear system means that load and deflection can be interchanged in the following way. Consider a simple but arbitrary elastic structure, say a disk where we focus on two locations, 1 and 2. It turns out that the deflection at location 2 due to a force at location 1 is equal to the deflection at location 1 due to the identical force at location 2. This can be shown using work-energy considerations.

We recall that *work* = *force* × *displacement* where the displacement is in the direction of the force. In terms of the influence coefficients, we can integrate the product of the force and the displacement,

$$W = \int_0^x kx \, dx = \int_0^P fP \, dP,$$

to find $W = kx^2/2 = fP^2/2$ where x is the maximum displacement and P is the static force at the maximum displacement. The second equality is obtained by using the relation $P = kx$ and replacing the limits of integration appropriately.

Next, we consider the work done by two loads P_i and P_j acting in this order at locations i and j , respectively. In terms of the flexibility coefficients, the work due to

the first force P_i is $W_i = P_i^2 f_{ii}/2$ and due to the second force P_j is $W_j = P_j^2 f_{jj}/2$. Note that there is an additional displacement at location i due to the second force. Therefore, the additional work by P_j at location i is $W_{ij} = (f_{ij}P_j)P_i$ and the total work is

$$W_{tot} = \frac{P_i^2 f_{ii}}{2} + \frac{P_j^2 f_{jj}}{2} + f_{ij}P_jP_i.$$

Repeating these steps for the case where load P_j is applied first results in the total work,

$$W_{tot} = \frac{P_j^2 f_{jj}}{2} + \frac{P_i^2 f_{ii}}{2} + f_{ji}P_iP_j.$$

The total work is a scalar and does not depend on the order of loading, and thus these two work values must be equal. This leads us to the conclusion that $f_{ij} = f_{ji}$ and confirms the previous observation of symmetry for a linear system.

Example 6.3 Equations of Motion via Flexibility

Offshore structures are designed for a variety of applications, such as oil drilling, communications, and national defense. Generally, they are very complex structures. Simple models provide the engineer with an approximation of how the actual structure will behave. From this basic understanding, the engineer can proceed with more advanced and realistic mathematical models.

Formulate a two degree-of-freedom model of an offshore structure and use the stiffness-flexibility approach to derive the equations of motion.

Solution: Consider the two degree-of-freedom model, shown in Figure 6.6, of a tower that oscillates transversely.⁷ Coordinate x is in the direction along the axis of the tower and coordinate y is in the transverse direction. In this model, half of the mass of the tower is lumped at the top. (The other half of the mass is lumped at the bottom and is incorporated into the foundation.) For the purposes of this example, the tower is fixed at the bottom. The lumped mass at the top of the tower can translate and rotate as the oscillation progresses. We neglect the axial extension of the tower and assume that the lumped mass remains essentially at height L from the base. In reality, the lumped mass decreases in height as θ increases. However, for linear oscillation, θ is small and the position of m remains along the horizontal line.

If the tower oscillates in a fluid such as water, then the *added mass* of the water that oscillates with the structure is similarly lumped at the ends. (The concept of added mass was discussed in Chapter 2.) Two coordinates are needed to delineate this oscillation, a translation $y(t)$ and a rotation $\theta(t)$. From the free-body diagram for the lumped mass in Figure 6.6, the forces acting on the

⁶Betti's theorem, which was discovered by Enrico Betti in 1872, states that for a linear elastic structure subject to two sets of forces $\{P_i\}$, $i = 1, \dots, m$ and $\{Q_j\}$, $j = 1, 2, \dots, n$, the work done by the set P through the displacements produced by the set Q is equal to the work done by the set Q through the displacements produced by the set P . This result is also known as the Maxwell-Betti reciprocal work (or reciprocity) theorem. It has applications in structural engineering.

⁷This example is adapted from **Dynamics of Offshore Structures**, J.F. Wilson, Wiley-Interscience, 1984, p.208.

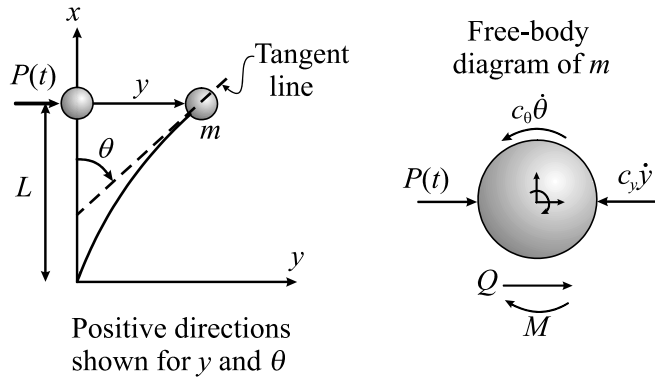


Figure 6.6: Idealization of a tower using a mass on top of an elastic beam. The mass of the structure $2m$ is lumped at the ends (half on top and half at the fixed base), and the stiffness and damping properties of the beam are distributed along the beam. The free-body diagram is on the right.

mass can be summed to obtain the rectilinear equation of motion using Newton's second law of motion,

$$m\ddot{y} + c_y\dot{y} + Q(t) = P(t),$$

and the moments acting on the mass about the base can be summed to obtain the rotational equation of motion,

$$J\ddot{\theta} + c_\theta\dot{\theta} + M(t) = 0,$$

where c_y and c_θ are the viscous damping coefficients in the two directions, J is the mass moment of inertia for the lumped mass about the base, $Q(t)$ is the shear force between the lumped mass and the tower, $M(t)$ is the internal moment between the lumped mass and the tower, and $P(t)$ is the externally applied force on the tower. If there is an *external* moment acting on the tower, it would appear on the right-hand side of the moment equation of motion.

The equations of motion can be written as

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_y & 0 \\ 0 & c_\theta \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} P(t) \\ 0 \end{Bmatrix}.$$

These equations cannot be solved as we do not know the stiffness coefficients k_{ij} . We next derive the force-displacement expressions relating Q to y and M to θ .

Even though (y, θ) and (Q, M) are time-dependent, we are only looking for the stiffness properties which are static relations. There are two options for deriving these relations. The force-displacement relation can be written in terms of a stiffness matrix,

$$\begin{Bmatrix} Q \\ M \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix}. \quad (6.4)$$

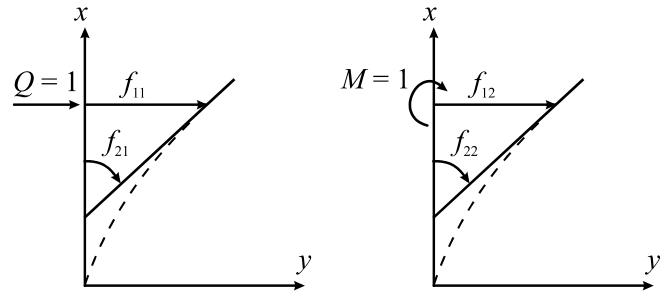


Figure 6.7: Meaning of flexibility coefficients for Example 6.3.

Alternatively, the displacement-force relationship (the inverse of Equation 6.4) can be written in terms of a flexibility matrix,

$$\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} Q \\ M \end{Bmatrix}. \quad (6.5)$$

The relation between the stiffness and flexibility matrices is

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{-1} = \frac{1}{f_{11}f_{22} - f_{12}f_{21}} \begin{bmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{bmatrix}. \quad (6.6)$$

It turns out to be simpler to derive the flexibility matrix and then invert it to obtain the stiffness matrix.⁸

For the flexibility approach, $y = f_{11}Q + f_{12}M$, with $Q = 1$ and $M = 0$. Let y be the deflection of a cantilever beam due to a unit load at the end, as shown in Figure 6.7. From strength of materials, the deflection at the end of a cantilever beam due to a unit force at the free end is $L^3/3EI$. Therefore,

$$y = f_{11}Q = f_{11} = \frac{L^3}{3EI},$$

where I is the area moment of inertia about the axis perpendicular to bending.

Similarly, the rotation at the end of a cantilever beam is given by $\theta = f_{21}Q + f_{22}M$. For $Q = 1$ at the free end and $M = 0$, we have

$$\theta = f_{21}Q = f_{21} = \frac{L^2}{2EI}.$$

To determine the second column of the flexibility matrix, we set $Q = 0$ and $M = 1$, and again use results from strength of materials. The deflection of the free end of

⁸Experimentally, it is easier to apply a specified force and measure displacements than to apply a specified displacement and measure forces.

the cantilever beam due to a unit moment applied at the free end is

$$y = f_{12}M = f_{12} = \frac{L^2}{2EI}.$$

Similarly, the rotation of the free end of the cantilever beam due to unit moment applied at the free end is

$$\theta = f_{22}M = f_{22} = \frac{L}{EI}.$$

As predicted by the Maxwell-Betti reciprocity theorem, $f_{12} = f_{21}$, and the flexibility matrix is

$$[f] = \frac{1}{EI} \begin{bmatrix} L^3/3 & L^2/2 \\ L^2/2 & L \end{bmatrix},$$

with Equation 6.6 yielding the stiffness matrix,

$$[K] = \frac{EI}{L} \begin{bmatrix} 12/L^2 & -6/L \\ -6/L & 4 \end{bmatrix}.$$

The equations of motion are then

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_y & 0 \\ 0 & c_\theta \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix} + \frac{EI}{L} \begin{bmatrix} 12/L^2 & -6/L \\ -6/L & 4 \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} P(t) \\ 0 \end{Bmatrix}. \quad (6.7)$$

We will learn how to solve such equations later in this chapter. ■

The matrix framework is useful because it is compact and allows us to examine key properties of multi degree-of-freedom systems. The object of this chapter is twofold: first, to derive matrix equations of motion such as Equation 6.7, and second, to solve these equations. The example just completed showed a formulation based on Newton's second law of motion in conjunction with the method of influence coefficients. Next we derive the equations of motion for a two degree-of-freedom system using Newton's second law applied directly to the free-body diagrams.

LILLIAN MOLLER GILBRETH

(May 24, 1878 – January 2, 1972)



Contribution: Lillian Evelyn Moller Gilbreth was an American industrial engineer and psychologist. She was one of the first working female engineers holding a PhD. She and her husband Frank Bunker Gilbreth Sr. were efficiency experts who contributed to the study of industrial engineering in fields such as motion study and human factors.

Life Sketch: Gilbreth was born in Oakland, California. She was the second of eleven children of William Moller, a builder's supply merchant, and Annie Delger. Both parents were of German descent. She was educated at home until she was nine years old, when her formal schooling began at a public elementary school, where she was required to start from the first grade (although she was rapidly promoted through the grades). She attended Oakland High School, where she was elected vice president of her senior class. She graduated with exemplary grades in May 1896.

Gilbreth started college at the University of California, Berkeley shortly after, and graduated in 1900 with a bachelor's degree in English literature. She originally pursued her master's degree at Columbia University, where she was exposed to the subject of psychology. However, she became ill and returned home, finishing her master's degree in literature at UC Berkeley in 1902.

She met her husband Frank Bunker Gilbreth Sr. in June 1903 in Boston, Massachusetts, en route to Europe with her chaperone, who was Frank's cousin. The couple married on October 19, 1904, in Oakland, California. They became the parents of thirteen children (one was stillborn in 1915), eleven of whom lived to adulthood.

Gilbreth completed a dissertation and attempted to obtain a doctorate from the University of California in 1911, but was not awarded the degree due to noncompliance with residency requirements; this dissertation was later published as *The Psychology of Management*. Since her immediate family had relocated to New England, she attended Brown University and earned a PhD in 1915, having written a second dissertation on efficient teaching methods called *Some Aspects of Eliminating Waste in Teaching*. It was the first degree granted in industrial psychology.

She and her husband were partners in the management consulting firm of Gilbreth, Inc., which performed time and motion studies. The Gilbreths also conducted research on fatigue, the forerunner to ergonomics.

Gilbreth had always been interested in teaching and education. She began lecturing at Purdue University in 1925, where her husband had previously given annual lectures. This led to a visiting professorship in 1935, when she became the first female engineering professor at Purdue; she was granted full professorship in 1940, dividing her time among the departments of industrial engineering, industrial psychology, home economics, and the dean's office where she consulted on careers for women. In the School of Industrial Engineering, she helped establish a time and motion study laboratory, and transferred motion study techniques to the home economics

department under the banner of “work simplification.” She retired from Purdue in 1948.

Besides teaching at Purdue, she was also appointed Knapp Visiting Professor at the University of Wisconsin’s School of Engineering, and taught at Newark College of Engineering (1941–43), Bryn Mawr College, and Rutgers University. She became resident lecturer at M.I.T. in 1964, at the age of 86.

She died in Phoenix, Arizona at the age of 93.

Noteworthy: Gilbreth was instrumental in the development of the modern kitchen, creating the “work triangle” and linear kitchen layouts that are used today. She is credited with the invention of both the foot pedal trash can and the shelves on the inside of refrigerator doors, including the butter tray and egg keeper. She filed numerous patents, including ones on improving the electric can opener and the wastewater hose for washers. When she was an industrial engineer working at General Electric, she “interviewed over 4,000 women to design the proper height for stoves, sinks, and other kitchen fixtures as she worked on improving kitchen designs.”

Gilbreth combined the perspectives of an engineer and a psychologist; she helped industrial engineers see the importance of the psychological dimensions of work. By applying the principles of scientific management to household tasks, Gilbreth “sought to provide women with shorter, simpler, and easier ways of doing housework to enable them to seek paid employment outside the home.”

She coauthored multiple books with her husband. However, due to publishers’ concerns about the books, credibility with a female author (despite the fact that Lillian had earned a doctorate while her husband never attended college), she was not named as such.

During her career, Gilbreth received numerous awards and honors, including 23 honorary degrees from schools such as Princeton, Brown, and the University of Michigan. She was named 1954 Alumna of the Year by the University of California’s alumni association. She was accepted to the membership of the American Society of Mechanical Engineers in 1926, becoming its second female member; the society later awarded her and her husband (posthumously) the Henry Laurence Gantt Medal in 1944 for her contributions to industrial engineering. In 1950, she was the first honorary member of the newly created Society of Women Engineers.

In 1965, Gilbreth became the first woman elected to the National Academy of Engineering. The next year, she received the Hoover Medal, an engineering prize awarded jointly by five engineering societies, for her “contributions to motion study and to recognition of the principle that management engineering and human relations are intertwined.... Additionally, her unselfish application of energy and creative efforts in modifying industrial and home environments for the handicapped has resulted in full employment of their capabilities and elevation of their self-esteem.”

Multiple engineering awards have been named in her honor. The Lillian M. Gilbreth Lectureships were established in 2001 by the National Academy of Engineering, to recognize out-

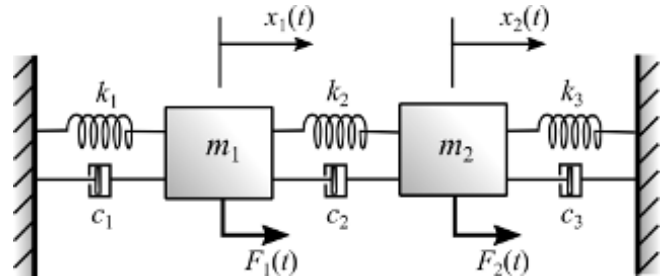


Figure 6.8: Two degree-of-freedom system subjected to external forces.

standing young American engineers. The highest honor bestowed by the Institute of Industrial Engineers is the Frank and Lillian Gilbreth Industrial Engineering Award, for “those who have distinguished themselves through contributions to the welfare of mankind in the field of industrial engineering.” At Purdue University, the Lillian M. Gilbreth Distinguished Professor is an honor bestowed on a member of the industrial engineering department. Additionally, the Society of Women Engineers awards the Lillian Moller Gilbreth Memorial Scholarship to deserving female engineering undergraduates.

In 1984, the United States Postal Service issued a 40¢ Great Americans series postage stamp in Gilbreth’s honor, and she was lauded by the American Psychological Association as the first psychologist to be so commemorated.

The books *Cheaper by the Dozen* and *Belles on Their Toes* (written by their children Ernestine and Frank Jr.) tell the story of their family life with their twelve children, and describe how they applied their interest in time and motion study to the organization and daily activities of such a large family.

6.3 Equations of Motion

Systems with many degrees of freedom are more complicated than systems modeled with two degrees of freedom. However, the concepts used to solve multi degree-of-freedom systems can be introduced by working through the solution of two degree-of-freedom models. Key ideas can be demonstrated more easily with the two degree-of-freedom model without the major algebraic and numeric demands imposed by larger systems. All the ideas transfer to larger systems.

Consider the two degree-of-freedom linear model shown in Figure 6.8 that includes external forcing and mass, damping, and stiffness elements. The sketch shows the assumed positive directions of horizontal forces and displacements. To derive the governing equations of motion using Newton’s second law of motion, a free-body diagram is needed for each mass, as shown in Figure 6.9.

Applying Newton’s second law of motion to each mass results in the following differential equations in terms of

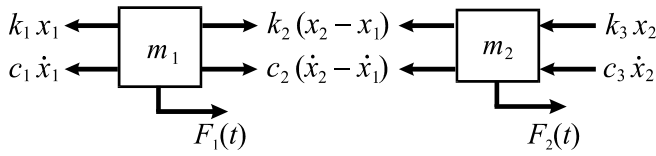


Figure 6.9: Free-body diagrams of the two degree-of-freedom system.

the unknown displacement functions $x_1(t)$ and $x_2(t)$,

$$\begin{aligned} m_1 \ddot{x}_1 &= F_1(t) - c_1 \dot{x}_1 - k_1 x_1 \\ &\quad + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) \\ m_2 \ddot{x}_2 &= F_2(t) - c_3 \dot{x}_2 - k_3 x_2 \\ &\quad - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1), \end{aligned}$$

which can be written in standard form as

$$\begin{aligned} m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 &= F_1(t) \\ m_2 \ddot{x}_2 + (c_2 + c_3) \dot{x}_2 - c_2 \dot{x}_1 + (k_2 + k_3) x_2 - k_2 x_1 &= F_2(t). \end{aligned}$$

These two equations of motion are *coupled*, meaning that the independent variables appear in both equations, and they must be solved *simultaneously*. The coupling terms are $-c_2 \dot{x}_2 - k_2 x_2$ in the first equation and $-c_2 \dot{x}_1 - k_2 x_1$ in the second equation. This mathematical coupling is due to the *physical* coupling between the masses, that is, the damping and stiffness elements, c_2 and k_2 , respectively, that connect the two masses to each other.

It is customary and convenient to recast the equations in matrix-vector form,

$$\begin{aligned} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \\ + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} &= \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}. \quad (6.8) \end{aligned}$$

Equation 6.8, or any n degree-of-freedom system of equations, can be written in general matrix-vector form as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}, \quad (6.9)$$

where the mass $[M]$, damping $[C]$, and stiffness $[K]$ matrices are symmetric: $[M] = [M]^T$, $[C] = [C]^T$, $[K] = [K]^T$ with superscript T denoting the matrix transpose.⁹ We can see that the coupling occurs in the off-diagonal terms of the property matrices. Without coupling, the equations of motion can be solved independently as single degree-of-freedom oscillators.

⁹This is not true for systems where electromagnetic forces or couples exist. Systems with symmetric matrices are known as *self-adjoint*. In addition, for a self-adjoint linear system with a characteristic matrix that is free of zero eigenvalues, a unique solution is always possible. These terms are defined later in the chapter.

The matrix-vector equation of motion, Equation 6.9, is identical to Lagrange's equation for *small oscillations*, from Section 5.4.1 for the undamped, unforced system.

Example 6.4 Derivation of Equation 6.8 by Lagrange's Equation and by Hamilton's Principle Use Lagrange's equation and Hamilton's principle to derive the equations of motion for the model depicted in Figure 6.8.

Solution: We need the kinetic, potential, and dissipation energies, respectively,

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \\ V &= \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 x_2^2 \\ R &= \frac{1}{2} c_1 \dot{x}_1^2 + \frac{1}{2} c_2 (\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2} c_3 \dot{x}_2^2. \end{aligned}$$

Lagrange's equation for generalized coordinate x_1 is

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) + \frac{\partial V}{\partial x_1} + \frac{\partial R}{\partial \dot{x}_1} = F_1(t).$$

Taking the appropriate derivatives and substituting, the equation of motion for mass m_1 is

$$m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 = F_1(t).$$

Following the same procedure for mass m_2 , the second governing equation is

$$m_2 \ddot{x}_2 + (c_2 + c_3) \dot{x}_2 - c_2 \dot{x}_1 + (k_2 + k_3) x_2 - k_2 x_1 = F_2(t).$$

Hamilton's principle requires that the following variational equation must be satisfied,

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0. \quad (6.10)$$

The virtual work can be written as the sum of the conservative and nonconservative components. The conservative portion is equal to the negative of the potential energy. Nonconservative work W_{nc} is performed by dissipative forces, such as friction, and by applied forces external to the system. Therefore, $\delta W = -\delta V + \delta W_{nc}$. Equation 6.10 can then be written as

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc}) dt = 0,$$

where δL is the variation of the Lagrangian, given by

$$\begin{aligned} \delta L &= \delta(T - V) \\ &= \delta \left(\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 \right. \\ &\quad \left. - \frac{1}{2} k_2 (x_2 - x_1)^2 - \frac{1}{2} k_3 x_2^2 \right) \\ &= m_1 \dot{x}_1 \delta \dot{x}_1 + m_2 \dot{x}_2 \delta \dot{x}_2 - k_1 x_1 \delta x_1 \\ &\quad - k_2 (x_2 - x_1) (\delta x_2 - \delta x_1) - k_3 x_2 \delta x_2. \end{aligned}$$

As indicated in the last chapter, we need to rewrite terms that are variations of derivatives as variations of the parameters themselves. For example,

$$\frac{d}{dt}(m_1 \dot{x}_1 \delta x_1) = m_1 \ddot{x}_1 \delta x_1 + m_1 \dot{x}_1 \delta \dot{x}_1,$$

and, therefore,

$$m_1 \dot{x}_1 \delta \dot{x}_1 dt = d(m_1 \dot{x}_1 \delta x_1) - m_1 \ddot{x}_1 \delta x_1 dt.$$

We proceed for the case with no damping or external forces. Hamilton's principle becomes

$$\begin{aligned} \int_{t_1}^{t_2} \{ & [-m_1 \ddot{x}_1 - k_1 x_1 + k_2(x_2 - x_1)] \delta x_1 \\ & + [-m_2 \ddot{x}_2 - k_3 x_2 - k_2(x_2 - x_1)] \delta x_2 \} dt \\ & + \int_{t_1}^{t_2} [d(m \dot{x}_1 \delta x_1) + d(m \dot{x}_2 \delta x_2)] = 0. \end{aligned} \quad (6.11)$$

The second integral can be integrated directly resulting in

$$[m \dot{x}_1 \delta x_1 + m \dot{x}_2 \delta x_2]_{t_1}^{t_2}.$$

Since variations δx_1 and δx_2 equal zero at t_1 and t_2 , both these expressions equal zero as well. Furthermore, in the first integral above, variations δx_1 and δx_2 are independent of each other and are arbitrary; the equality can then only hold if the expressions in each set of square brackets add to zero, resulting in the equations of motion,

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 &= 0. \end{aligned}$$

Next, we add the effects of damping. The virtual work done by viscous damping equals $F_d \delta x = (-c\dot{x})\delta x$. For the system considered here,

$$\delta W_{nc} = -c_1 \dot{x}_1 \delta x_1 - c_2 (\dot{x}_2 - \dot{x}_1)(\delta x_2 - \delta x_1) - c_3 \dot{x}_2 \delta x_2.$$

We insert these terms into Equation 6.11 and group them according to like variations, resulting in the damped equations of free motion,

$$\begin{aligned} m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2 \dot{x}_2 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + (c_2 + c_3)\dot{x}_2 + (k_2 + k_3)x_2 - c_2 \dot{x}_1 - k_2 x_1 &= 0. \end{aligned}$$

Finally, we add the virtual work done by the externally applied nonconservative forces,

$$\delta W_{nc} = F_1 \delta x_1 + F_2 \delta x_2.$$

These are substituted into Equation 6.11 resulting in the same equations derived earlier via Lagrange's equation.

While Hamilton's principle appears more involved than Lagrange's equation, there is little difference between the two approaches. However, depending on the application, one method may have an advantage over

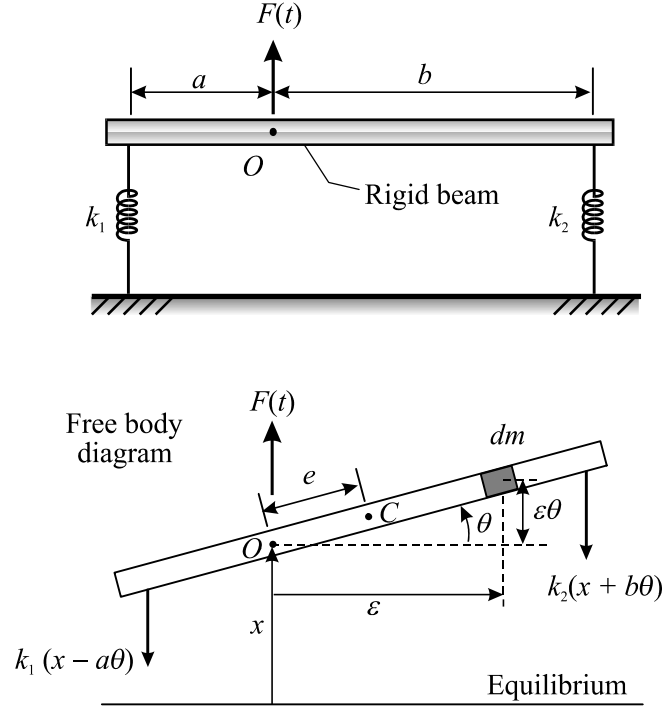


Figure 6.10: Two degree-of-freedom model of a rigid beam, resulting in mass and stiffness coupling. The external force is applied at O and C is the center of mass. The system configuration is shown on top, and the free-body diagram on bottom.

the other. The advantage of energy methods is most evident for systems with more degrees of freedom where Newton's second law of motion becomes cumbersome to apply. ■

In the last example, the mass matrix was diagonal and the stiffness matrix was coupled. The next example examines a system with mass and stiffness coupling.

Example 6.5 Equations of Motion with Mass and Stiffness Coupling

Consider the two degree-of-freedom rigid-body model of Figure 6.10. For the rigid beam, modeled as a lumped mass, two coordinates are required to specify its translation and rotation. If the force is applied at the center of mass C and if $k_1 = k_2$, then the beam will only translate. The location of C depends on the distribution of mass in the beam. If the mass is uniformly distributed, then C is at the geometric center.

Derive the governing equations of motion, assuming small motion, for the case in which a force $F(t)$ is applied at an arbitrary point O and the mass is distributed in an arbitrary manner.

Solution: From the problem statement, we assume that the beam of total mass m undergoes small rotations and translations. In the free-body diagram, we consider an

element of the rigid beam dm at a horizontal distance ε from O . Due to the small motion, we can reasonably say that the acceleration of the mass element is primarily in the vertical direction and has a magnitude of

$$\frac{d^2}{dt^2}(x + \varepsilon\theta) = \ddot{x} + \varepsilon\ddot{\theta}.$$

By Newton's second law of motion, the sum of the forces in the vertical direction due to the deformation of the springs (from the static equilibrium position) is equal to the mass times acceleration,

$$\begin{aligned} F(t) - k_1(x - a\theta) - k_2(x + b\theta) \\ &= \int_{beam} (\ddot{x} + \varepsilon\ddot{\theta}) dm \\ &= \ddot{x} \int_{beam} dm + \ddot{\theta} \int_{beam} \varepsilon dm \\ &= \ddot{x}m + \ddot{\theta}me, \end{aligned}$$

where the location of the mass center from O is $e = (\int \varepsilon dm)/m$.

Similarly, the moment equation about O is

$$\begin{aligned} k_1(x - a\theta)a - k_2(x + b\theta)b \\ &= \int_{beam} (\ddot{x} + \varepsilon\ddot{\theta}) \varepsilon dm \\ &= \ddot{x} \int_{beam} \varepsilon dm + \ddot{\theta} \int_{beam} \varepsilon^2 dm \\ &= \ddot{x}me + \ddot{\theta}I_O, \end{aligned}$$

where I_O is the mass moment of inertia of the body about O and the sign of the moment is positive in the positive θ direction.

The two equations of motion can be written in standard form as

$$\begin{aligned} m\ddot{x} + me\ddot{\theta} + (k_1 + k_2)x + (k_2b - k_1a)\theta &= F(t) \\ me\ddot{x} + I_O\ddot{\theta} + (k_2b - k_1a)x + (k_1a^2 + k_2b^2)\theta &= 0, \end{aligned}$$

or in matrix-vector form,

$$\begin{bmatrix} m & me \\ me & I_O \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2b - k_1a \\ k_2b - k_1a & k_1a^2 + k_2b^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}.$$

Both the mass and stiffness matrices are coupled. The mass matrix decouples if C and O coincide, that is, $e = 0$; the stiffness matrix decouples only if $k_2b = k_1a$. Later in this chapter we will show how we can decouple both the mass and stiffness matrices by transforming the problem to a new coordinate system using a procedure known as modal analysis. ■

The mass and stiffness matrices are related to the energy of the system: the stiffness elements store strain energy and the mass stores kinetic and potential energies. In the next section we examine the energy properties of systems in order to better understand system behavior.

6.3.1 Mass and Stiffness Matrices

We know that system energies can be used to derive the equation of motion via Lagrange's equation and Hamilton's principle. We expect that the kinetic T is related to the mass matrix, and the strain energy V is related to the stiffness or flexibility matrix. For a single linear spring element with constant k , the area under the force-displacement curve is its strain energy, which can be written as $V = \frac{1}{2}kx^2$ or $V = \frac{1}{2}Fx$ from Hooke's law $F = kx$. Generalizing to an n degree-of-freedom vibrating system, the total strain energy can be written in two ways as well,

$$\begin{aligned} V &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n k_{ij}x_i x_j = \frac{1}{2} \{x\}^T [K] \{x\} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n f_{ij}F_i F_j = \frac{1}{2} \{F\}^T [f] \{F\}. \end{aligned}$$

The kinetic energy for a single mass is $T = \frac{1}{2}m\dot{x}^2$. For n masses the kinetic energy is

$$T = \frac{1}{2} \sum_{i=1}^n m_i \dot{x}_i^2 = \frac{1}{2} \{\dot{x}\}^T [M] \{\dot{x}\}.$$

Since kinetic energy is always positive, $T > 0$ and $[M]$ is a *positive definite* matrix.¹⁰ The potential energy will always be positive for a restrained system undergoing oscillation. For the case of an unrestrained system, the potential energy may take a minimum value of $V = 0$. That is, there may not be any relative motion between the masses and the springs and the elements remain undeformed. In this case $[K]$ is called *positive semi-definite*.¹¹ We will look more closely at unrestrained systems in Section 6.9.1.

Example 6.6 Strain Energy and Stiffness Matrix

Consider the three degree-of-freedom system of Figure 6.11 with the masses oscillating along the horizontal axis. The system strain energy is a function of the

¹⁰ An $n \times n$ real symmetric matrix $[M]$ is positive definite if $\{\dot{x}\}^T [M] \{\dot{x}\} > 0$ for all non-zero vectors $\{\dot{x}\}$ with real entries.

¹¹ Positive semi-definite has the same meaning as positive definite with the exception that we use $\geq$ rather than $>$.

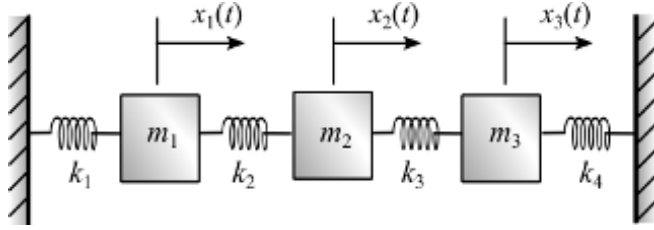


Figure 6.11: Three degree-of-freedom system.

positions of the masses,

$$\begin{aligned} V &= \frac{1}{2} [k_1 x_1^2 + k_2 (x_2 - x_1)^2 + k_3 (x_3 - x_2)^2 + k_4 x_3^2] \\ &= \frac{1}{2} [(k_1 + k_2)x_1^2 + (k_2 + k_3)x_2^2 \\ &\quad + (k_3 + k_4)x_3^2 - 2k_2 x_1 x_2 - 2k_3 x_2 x_3]. \end{aligned}$$

In matrix form, the strain energy can be written as the triple product $V = \frac{1}{2} \{x\}^T [K] \{x\}$, where

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix},$$

which is the same stiffness matrix as in Equation 6.3. (Performing the triple product will verify the previous expression for V .) If the system above is unrestrained, that is, $k_1 = k_4 = 0$, then

$$[K] = \begin{bmatrix} k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix},$$

which is a *singular matrix*. This means that the determinant of the stiffness matrix equals zero, $\det [K] = 0$, and its inverse does not exist. The flexibility matrix $[f]$ then does not exist, which is expected since the concept of flexibility is not applicable to an unrestrained system. Such systems are important in practice and will be discussed in Section 6.9.1. ■

In the next section, we will solve the free vibration problem without damping, $[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$. In solving the coupled equations, we will introduce a new concept, namely, *mode of vibration*, that is useful for modeling multi degree-of-freedom vibrating systems.

6.4 Undamped Vibration

There are two analytical approaches to solving linear vibration problems: (1) the direct method and (2) modal analysis. The direct method is general in that no assumptions are needed regarding the damping properties

of the system. Modal analysis is a more efficient method but requires that we make assumptions about the nature of the damping. In this section we begin with the direct method for systems without damping. We will study the application of both methods to undamped and damped systems.

6.4.1 Two Degree-of-Freedom Vibration: Direct Method

The free vibration problem for an n degree-of-freedom undamped system is governed by the equation of motion

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}, \quad (6.12)$$

where the matrices $[M]$ and $[K]$ are positive definite of dimension $n \times n$ and the vector of displacements $\{x\}$ is of dimension $n \times 1$.

Generalizing from single degree-of-freedom systems, we anticipate harmonic solutions. We will find that an n degree-of-freedom system generally has n natural frequencies and for the undamped, unforced system, the complete response of the masses involves the sum of n harmonic functions.

We start by considering a two degree-of-freedom system to introduce the ideas. Referring to Figures 6.8 and 6.9, with $c_1 = 0$, $c_2 = 0$, $F_1(t) = 0$, and $F_2(t) = 0$, the equations for free vibration are

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \quad (6.13)$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0, \quad (6.14)$$

or in matrix-vector form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.15)$$

For simplicity and generality, we rewrite the stiffness matrix as

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}.$$

Two points are important to help understand the solution of Equation 6.15. First, the individual governing equations are homogeneous and therefore it is only possible to obtain a solution of the displacement of each mass to within an arbitrary constant multiplier. Thus, if $x_1(t)$ and $x_2(t)$ are solutions that satisfy the governing differential equations, then so are $Ax_1(t)$ and $Ax_2(t)$ (the constant A can be canceled since the right-hand sides equal zero). This was also the case for the single degree-of-freedom free vibration solution. Second, a harmonic

solution can be assumed based on physical considerations in a general solution.

The governing equations for the displacement coordinates $x_1(t)$ and $x_2(t)$ are of the same mathematical form, implying a *synchronicity of motion*. This means that $x_1(t)$ and $x_2(t)$ vary in a constant proportion to each other as a function of time. This motion of the two masses is either *in phase* or 180 degrees out of phase. As second-order differential equations, we expect two solutions each for Equations 6.13 and 6.14. (We will see this below in Equation 6.22.)

Based on the above discussion, $x_1(t)$ and $x_2(t)$ are assumed to be, respectively, the product of constants u_1 and u_2 and a function of time $Y(t)$,

$$x_1(t) = u_1 Y(t) \quad (6.16)$$

$$x_2(t) = u_2 Y(t), \quad (6.17)$$

where it is noted that

$$\frac{x_1(t)}{x_2(t)} = \frac{u_1}{u_2}.$$

The ratio between the two displacements is called a *mode of vibration*.

Equations 6.16 and 6.17 are differentiated twice and substituted into governing Equations 6.13 and 6.14 that they must satisfy, resulting in

$$m_1 u_1 \ddot{Y} + (k_{11} u_1 + k_{12} u_2) Y = 0$$

$$m_2 u_2 \ddot{Y} + (k_{12} u_1 + k_{22} u_2) Y = 0.$$

Rearranging, the following ratios must hold,

$$\frac{k_{11} u_1 + k_{12} u_2}{m_1 u_1} = \frac{k_{12} u_1 + k_{22} u_2}{m_2 u_2} = -\frac{\ddot{Y}}{Y} \equiv \lambda. \quad (6.18)$$

Since the first two ratios are constants, they must be equal to some constant λ , evaluated below.

Based on Equation 6.18, we have the following three equations, one differential and two algebraic, to solve

$$\ddot{Y} + \lambda Y = 0 \quad (6.19)$$

$$(k_{11} - \lambda m_1) u_1 + k_{12} u_2 = 0 \quad (6.20)$$

$$k_{12} u_1 + (k_{22} - \lambda m_2) u_2 = 0. \quad (6.21)$$

Equations 6.19 to 6.21 have four unknowns, λ , Y , u_1 , and u_2 .

We first solve Equation 6.19, which is decoupled from the other two for Y , to establish the physical meaning of λ . From Chapters 2 and 3, we know the solution is of the form $Y(t) = A \exp(rt)$ or $A \cos(rt)$, with characteristic roots of the form $r = \pm\sqrt{-\lambda}$. The complete solution for $Y(t)$ is composed of two linearly independent solutions,

$$Y(t) = A_1 e^{\sqrt{-\lambda}t} + A_2 e^{-\sqrt{-\lambda}t}. \quad (6.22)$$

If λ is negative, the first solution grows to infinity with time, contradicting the assumption of stable motion. Therefore, λ must be positive, resulting in the solution for the harmonic oscillator. Letting $\lambda = \omega^2$, Equation 6.22 can be written as $Y(t) = A_1 \exp(i\omega t) + A_2 \exp(-i\omega t)$, or as

$$\begin{aligned} Y(t) &= B_1 \cos(\omega t) + B_2 \sin(\omega t) \\ &= C \cos(\omega t - \phi). \end{aligned} \quad (6.23)$$

Equations 6.20 and 6.21 can be written in matrix-vector form,

$$\begin{bmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{12} & k_{22} - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.24)$$

The solution of Equation 6.24 for nontrivial¹² values of u_1 and u_2 leads to parameter values of ω called the *characteristic values* or *eigenvalues* of the problem. The nontrivial solution can be found only if the *determinant* of the characteristic matrix in Equation 6.24 equals zero.

This requirement is due to the way in which a matrix equation is solved, as we see next. Let Equation 6.24 be written in shorthand as $[K]\{u\} = \{0\}$. A trivial solution can be found by multiplying both sides by the inverse of $[K]$, that is, $[K]^{-1}[K]\{u\} = \{0\}$. Then $\{u\} = \{0\}$. A nontrivial solution cannot be obtained if the inverse does not exist. The inverse is given by $[K]^{-1} = \text{adj}[K] / \det[K]$, which exists only if $\det[K] \neq 0$. Requiring that $\det[K] = 0$ implies that there is no inverse and no trivial solution.

For nontrivial solutions, we require that $\det[\] = 0$ in Equation 6.24,

$$\det \begin{bmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{12} & k_{22} - \omega^2 m_2 \end{bmatrix} = 0. \quad (6.25)$$

The left-hand side of Equation 6.25 is called the *characteristic determinant*. We can expand the determinant to obtain the *characteristic equation* (also called *eigenequation* and *frequency equation*),

$$m_1 m_2 \omega^4 - (m_1 k_{22} + m_2 k_{11}) \omega^2 + k_{11} k_{22} - k_{12}^2 = 0,$$

with roots given by the quadratic formula,

$$\begin{aligned} \omega_{1,2}^2 &= \frac{m_1 k_{22} + m_2 k_{11}}{2m_1 m_2} \\ &\mp \sqrt{\left(\frac{m_1 k_{22} + m_2 k_{11}}{2m_1 m_2} \right)^2 - \frac{k_{11} k_{22} - k_{12}^2}{m_1 m_2}}. \end{aligned} \quad (6.26)$$

The $\mp$ ordering of the roots follows the convention that the frequencies are numbered from lowest to highest,

¹²The trivial solutions are $u_1 = u_2 = 0$, for which there is no motion.

with ω_1 called the *fundamental frequency*. The roots $\omega_{1,2}^2$ are known as the *eigenvalues* or *characteristic values* of the system.

The two degree-of-freedom system has two (positive) natural frequencies of oscillation. Once ω_1 and ω_2 have been calculated, either Equation 6.20 or 6.21 can be used to find the *ratios* u_1/u_2 for each ω that solves the two equations. Since the equations are homogeneous, only relative values of u_1 and u_2 can be found.

There is one ratio for each ω , and we will use the notation $(u_2/u_1)_1$ or u_{21}/u_{11} for ω_1 and $(u_2/u_1)_2$ or u_{22}/u_{12} for ω_2 , where the first subscript denotes the mass number and the second denotes the frequency number. These ratios are known as the *eigenvectors* or *characteristic vectors* of the system.

For the first eigenvalue ω_1^2 , Equations 6.20 and 6.21 become

$$(k_{11} - \omega_1^2 m_1)u_{11} + k_{12}u_{21} = 0 \quad (6.27)$$

$$k_{12}u_{11} + (k_{22} - \omega_1^2 m_2)u_{21} = 0. \quad (6.28)$$

Equivalently, these equations can be written as

$$k_{11}u_{11} + k_{12}u_{21} = \omega_1^2 m_1 u_{11}$$

$$k_{12}u_{11} + k_{22}u_{21} = \omega_1^2 m_2 u_{21},$$

or, in matrix-vector form,

$$[K]\{u\}_1 = \omega_1^2 [M]\{u\}_1, \quad (6.29)$$

where $\{u\}_1 = [u_{11} \ u_{21}]^T$. Either Equation 6.27 or 6.28 can be solved for the *eigenvectors*,

$$\frac{u_{21}}{u_{11}} = -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_1^2 m_2}, \quad (6.30)$$

and similarly for the second eigenvalue ω_2^2 ; $[K]\{u\}_2 = \omega_2^2 [M]\{u\}_2$ with the eigenvectors

$$\frac{u_{22}}{u_{12}} = -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_2^2 m_2}, \quad (6.31)$$

where ω_1^2 and ω_2^2 are substituted from Equation 6.26.

Equations 6.30 and 6.31 represent the relative displacement magnitudes of each mass for the frequencies ω_1 and ω_2 , respectively, and are known as the *modal ratios*. An equation exists for each mode and the modal characteristics of a system include its frequency and mode shape, denoted by $[\omega_i, \{u\}_i]$, $i = 1, 2$ for this two degree-of-freedom case. The solution is incomplete until the initial conditions are satisfied as we see next.

The motion of the system is the linear superposition of the two solutions (modes). From Equations 6.16, 6.17, and 6.23, we have

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= C_1 \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} \cos(\omega_1 t - \phi_1) \\ &+ C_2 \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} \cos(\omega_2 t - \phi_2), \end{aligned} \quad (6.32)$$

where four *initial conditions*, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, and $\dot{x}_2(0)$, are needed to evaluate the constants, C_1 , C_2 , ϕ_1 , and ϕ_2 . In order to substitute the modal ratios of Equations 6.30 and 6.31 into Equation 6.32, we can rewrite Equation 6.32 in the following equivalent form,

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \bar{C}_1 \begin{Bmatrix} 1 \\ u_{21}/u_{11} \end{Bmatrix} \cos(\omega_1 t - \phi_1) \\ &+ \bar{C}_2 \begin{Bmatrix} 1 \\ u_{22}/u_{12} \end{Bmatrix} \cos(\omega_2 t - \phi_2), \end{aligned} \quad (6.33)$$

where $\bar{C}_1 = C_1 u_{11}$ and $\bar{C}_2 = C_2 u_{12}$ are new integration constants.

Equation 6.33 indicates that the response of each mass is a sum of natural harmonics, where the proportion contributed by each harmonic to the total response is given by the modal ratio. The numerical values of $\bar{C}_1$, $\bar{C}_2$, ϕ_1 , and ϕ_2 are determined by satisfying the four initial conditions, that is, by simultaneously solving the following four equations,

$$\begin{aligned} \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} &= \bar{C}_1 \begin{Bmatrix} 1 \\ u_{21}/u_{11} \end{Bmatrix} \cos(-\phi_1) \\ &+ \bar{C}_2 \begin{Bmatrix} 1 \\ u_{22}/u_{12} \end{Bmatrix} \cos(-\phi_2) \end{aligned} \quad (6.34)$$

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} &= -\bar{C}_1 \omega_1 \begin{Bmatrix} 1 \\ u_{21}/u_{11} \end{Bmatrix} \sin(-\phi_1) \\ &- \bar{C}_2 \omega_2 \begin{Bmatrix} 1 \\ u_{22}/u_{12} \end{Bmatrix} \sin(-\phi_2). \end{aligned} \quad (6.35)$$

If for some particular initial conditions $\bar{C}_2 = 0$, then the two degree-of-freedom structure will oscillate only in its first mode at its first natural frequency. Similarly, if $\bar{C}_1 = 0$, the structure will oscillate solely in its second mode at its second natural frequency.

To emphasize, a mode of oscillation is one in which all points on a system execute simple harmonic motion at the same frequency and phase. This is known as synchronous motion. The amplitude generally varies from point to point. Figure 6.12 shows the two modes of a two degree-of-freedom model in synchronous motion.

Next, we consider an example of the above procedure.

Example 6.7 Two Degree of Freedom, Undamped

Consider the system of Figure 6.8, simplified to the undamped, unforced system in Figure 6.13. Let $m_1 = m$, $m_2 = 2m$, $k_1 = k$, $k_2 = 2k$, $k_3 = 3k$, and obtain the natural frequencies and modes of vibration.

Solution: The equations of motion are

$$\begin{aligned} m\ddot{x}_1 + 3kx_1 - 2kx_2 &= 0 \\ 2m\ddot{x}_2 + 5kx_2 - 2kx_1 &= 0. \end{aligned}$$

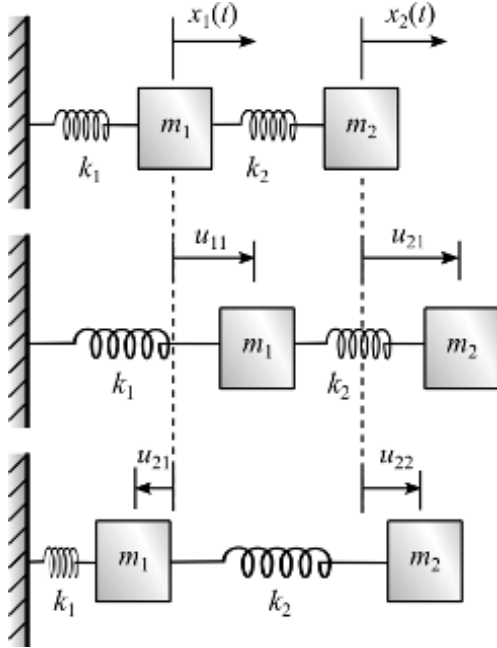


Figure 6.12: The synchronous motion of the two modes of a two degree-of-freedom system. The actual motion is a combination of the two modes of vibration.

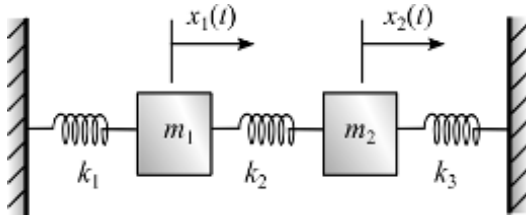


Figure 6.13: Two degree-of-freedom system.

In matrix-vector form, the symmetries of the mass and stiffness matrices are evident,

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 5k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.36)$$

Assuming harmonic responses of the form

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \exp(i\omega t),$$

differentiating twice, and substituting into the governing

Equation 6.36 gives

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} u_1 e^{i\omega t} \\ u_2 e^{i\omega t} \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 5k \end{bmatrix} \begin{Bmatrix} u_1 e^{i\omega t} \\ u_2 e^{i\omega t} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Combining terms, we have

$$\begin{bmatrix} -\omega^2 m + 3k & -2k \\ -2k & -2\omega^2 m + 5k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.37)$$

Setting the determinant of the coefficient matrix to zero leads to the characteristic equation

$$2m^2\omega^4 - 11mk\omega^2 + 11k^2 = 0,$$

with roots

$$\begin{aligned} \omega_{1,2}^2 &= \frac{11mk}{4m^2} \mp \sqrt{\frac{(11mk)^2}{16m^4} - 4 \frac{22(mk)^2}{16m^4}} \\ &= [2.75 \mp 1.44] \frac{k}{m} = 1.31 \frac{k}{m} \text{ and } 4.19 \frac{k}{m}. \end{aligned}$$

The two natural frequencies are then

$$\omega_1 = 1.14 \sqrt{\frac{k}{m}} \text{ rad/s}, \quad \omega_2 = 2.05 \sqrt{\frac{k}{m}} \text{ rad/s}.$$

With the natural frequencies known, Equation 6.30 (or Equation 6.37) can be used to obtain the respective modes of vibration,

$$\begin{aligned} \frac{u_{21}}{u_{11}} &= -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} \\ &= -\frac{3k - 1.31(k/m)m}{-2k} = 0.845 \\ \frac{u_{22}}{u_{12}} &= -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} \\ &= -\frac{3k - 4.19(k/m)m}{-2k} = -0.595. \end{aligned}$$

The denominators cannot equal zero since the stiffness must be positive. Physically, zero stiffness would imply that one mass is motionless while the other is vibrating. This can occur only if the masses are not coupled. The modes are sketched in Figure 6.14. Modes generally have amplitudes of alternating signs and, therefore, the modal amplitude lines¹³ cross the zero $(n-1)$ times where n is the mode number. The crossing points are called *nodes*, indicative of no motion.

¹³The lines are drawn for better visualization of the eigenvectors.

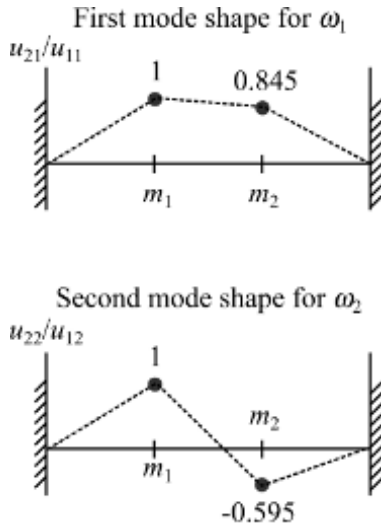


Figure 6.14: Two modes of vibration: the first mode (top) and the second mode (bottom). Even though the masses are only at locations 1 and 2, dotted lines are drawn connecting these displacements.

The complete response, following Equation 6.33, is

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \bar{C}_1 \begin{Bmatrix} 1 \\ 0.845 \end{Bmatrix} \cos \left(1.14 \sqrt{\frac{k}{m}} t - \phi_1 \right) + \bar{C}_2 \begin{Bmatrix} 1 \\ -0.595 \end{Bmatrix} \cos \left(2.05 \sqrt{\frac{k}{m}} t - \phi_2 \right).$$

The second mode in the figure has one node. The response of each mass is a combination of two harmonics that are weighted by the respective modal ratios.

Given the initial conditions $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, and $\dot{x}_2(0)$, the constants of integration $\bar{C}_1$, $\bar{C}_2$, ϕ_1 , and ϕ_2 can be found by solving the four simultaneous algebraic Equations 6.34 and 6.35. ■

Example 6.8 Elevator Cable System

Develop three possible idealized models of an elevator cable system for the purpose of estimating its natural frequency or frequencies. The models considered are a one-mass, a one-mass with spring correction, and a three-mass system. The system parameters are given:

weight of car	$W_{car} = 1000 \text{ lb}$
length of cable	$l = 750 \text{ ft}$
cable weight/length	$\gamma = 1 \text{ lb/ft}$
cable cross-sectional area	$A = 1 \text{ in}^2$
cable modulus of elasticity	$E = 4.5 \times 10^7 \text{ psi}$.

Solution: For the one-mass model, shown in Figure 6.15,

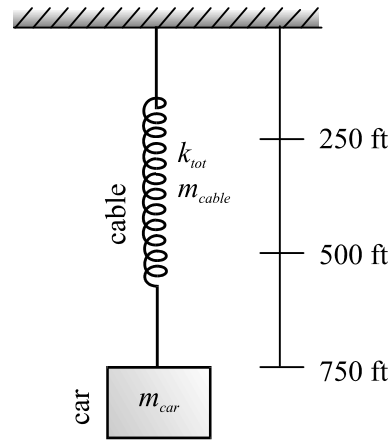


Figure 6.15: One-mass model of elevator plus cable.

the total spring stiffness is

$$k_{tot} = \frac{EA}{l} = \frac{4.5 \times 10^7 \text{ psi} \cdot 1 \text{ in}^2}{750 \text{ ft}} = 6.0 \times 10^4 \frac{\text{lb}}{\text{ft}}.$$

The weight of the cable is

$$W_{cable} = \gamma l = 1 \frac{\text{lb}}{\text{ft}} \cdot 750 \text{ ft} = 750 \text{ lb}.$$

For the one-mass model, the cable weight is added to the car weight in the calculation of the natural frequency of oscillation,

$$\begin{aligned} W_{total} &= W_{car} + W_{cable} \\ &= 1000 \text{ lb} + 750 \text{ lb} = 1750 \text{ lb}. \end{aligned}$$

The natural frequency for the one-mass model is then

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_{tot}}{m_{tot}}} = \sqrt{\frac{k g}{W_{total}}} \\ &= \sqrt{\frac{\left(6.0 \times 10^4 \frac{\text{lb}}{\text{ft}} \right) \left(32.2 \frac{\text{ft}}{\text{s}^2} \right)}{(1750 \text{ lb})}} \\ &= 33.3 \text{ rad/s} \end{aligned}$$

Since $f_n = \omega_n / 2\pi$, the cyclic natural frequency is $f_n = 5.25 \text{ Hz}$.

Next, in the interest of improving the model we account for the mass of the spring and use an equivalent weight for the mass, as shown in Figure 6.16. Following Example 2.4,

$$\begin{aligned} W_{equiv} &= W_{car} + \frac{1}{3} W_{cable} \\ &= 1000 \text{ lb} + \frac{1}{3} (750 \text{ lb}) = 1250 \text{ lb}. \end{aligned}$$

For this second model, the natural frequency increases

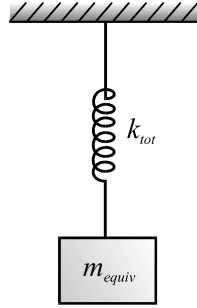


Figure 6.16: One-mass model with spring correction.

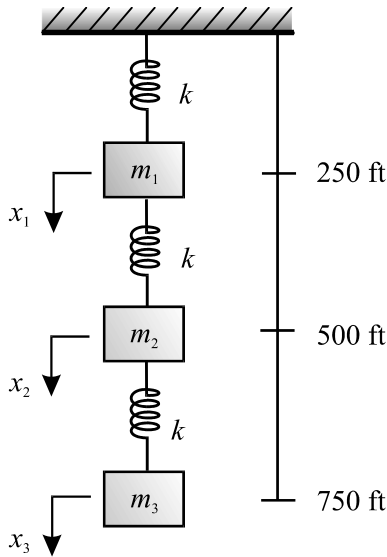


Figure 6.17: Three-mass model of elevator system.

as follows,

$$\begin{aligned}\omega_n &= \sqrt{\frac{k_{tot} g}{W_{equiv}}} = \sqrt{\frac{6.0 \times 10^4 \frac{\text{lb}}{\text{ft}} \cdot 32.2 \frac{\text{ft}}{\text{s}^2}}{1250 \text{ lb}}} \\ &= 39.3 \text{ rad/s} \quad \text{or} \quad f_n = 6.26 \text{ Hz}.\end{aligned}$$

This is a more accurate representation of the system.

Finally, a third model is considered which distributes the cable weight at two discrete locations, at 250 ft and 500 ft from the top, as shown in Figure 6.17. The mass of the cable is distributed in thirds, one third to each mass and the other third to the mass of the car, as before,

$$\begin{aligned}m_1 &= m_2 = \frac{1}{3} \frac{W_{cable}}{g} = \frac{250 \text{ lb}}{g} \\ m_3 &= \frac{W_{car} + W_{cable}}{g} = \frac{1250 \text{ lb}}{g}.\end{aligned}$$

The spring stiffness changes due to the shorter lengths,

$$k = \frac{EA}{(l/3)} = \frac{4.5 \times 10^7 \text{ psi} \cdot 1 \text{ in}^2}{250 \text{ ft}} = 1.8 \times 10^5 \frac{\text{lb}}{\text{ft}}.$$

To calculate the three natural frequencies, we need the equations of motion,

$$\begin{aligned}m_1 \ddot{x}_1 + 2kx_1 - kx_2 &= 0 \\ m_2 \ddot{x}_2 - kx_1 + 2kx_2 - kx_3 &= 0 \\ m_3 \ddot{x}_3 - kx_2 + kx_3 &= 0.\end{aligned}$$

Rewriting these equations of motion in matrix-vector form, differentiating and substituting the assumed harmonic responses for each displacement $A_i \exp(i\omega t)$ into the equation of motion, we have

$$\begin{aligned}-\omega^2 \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} A_1 \exp(i\omega t) \\ A_2 \exp(i\omega t) \\ A_3 \exp(i\omega t) \end{Bmatrix} \\ + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} A_1 \exp(i\omega t) \\ A_2 \exp(i\omega t) \\ A_3 \exp(i\omega t) \end{Bmatrix} &= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.\end{aligned}$$

Using the fact that $m_1 = m_2$, combining like terms, and multiplying all terms by $\exp(-i\omega t)$, we find

$$\begin{aligned}\begin{bmatrix} -\omega^2 m_1 + 2k & -k & 0 \\ -k & -\omega^2 m_2 + 2k & -k \\ 0 & -k & -\omega^2 m_3 + k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \\ = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},\end{aligned}$$

which is the eigenvalue problem. Setting the determinant of the coefficient matrix equal to zero, we obtain the characteristic equation,

$$\begin{aligned}\omega^6 (-m_1^2 m_3) + \omega^4 [k m_1 (m_1 + 4m_3)] \\ + \omega^2 [-3k^2 (m_1 + m_3)] + k^3 = 0.\end{aligned}$$

Substituting the parameter values and solving the equivalent cubic equation for ω^2 gives the frequencies of oscillation,

$$\omega_{1,2,3} = 46.3, 165, 266 \text{ rad/s}$$

or

$$f_{1,2,3} = 7.37, 26.26, 42.34 \text{ Hz}.$$

Here, the fundamental frequency (7.37 Hz) is higher than those of the two simpler models and most likely more accurate. The next level of realism would be to use a string model for the cable that represents its physical properties as a continuum. A conclusion from this example is that a range of models can be used to represent a complex system, and the accuracy of numerical results depends on the reasonableness of the underlying assumptions. ■

6.4.2 Harmonically Forced Vibration: Direct Method

Next, we consider the undamped vibration of a two degree-of-freedom system acted on by external forces. We begin this case of forced vibration with the generic two degree-of-freedom matrix-vector equation of motion, Equation 6.15, with $F_1(t)$ and $F_2(t)$ applied to the respective masses,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}, \quad (6.38)$$

where, due to the symmetry of the property matrices, $k_{12} = k_{21}$. First, we examine the response due to harmonic $F_1(t)$ only,

$$\begin{aligned} F_1(t) &= f_1 \cos \omega t \\ F_2(t) &= 0, \end{aligned}$$

where ω is the forcing frequency. Since there is no damping, the response will either be in phase or 180° out of phase with the forcing.

The particular solution also oscillates at the forcing frequency,

$$\begin{aligned} x_1(t) &= X_1 \cos \omega t \\ x_2(t) &= X_2 \cos \omega t, \end{aligned}$$

where amplitudes X_1 and X_2 need to be determined. Here, we look for the particular solutions only; the total solution must include both the homogeneous and the particular solutions.

Substituting $x_1(t)$ and $x_2(t)$ and their second derivatives into Equation 6.38 and canceling the common cosine term results in

$$\begin{aligned} -\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} &= \begin{Bmatrix} f_1 \\ 0 \end{Bmatrix}. \quad (6.39) \end{aligned}$$

Combining terms, we find an expression that looks very similar to the eigenvalue problem, except that the algebraic equations are nonhomogeneous,

$$\begin{bmatrix} k_{11} - m_1\omega^2 & k_{12} \\ k_{12} & k_{22} - m_2\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \end{Bmatrix}.$$

Denoting the square matrix, which is the system characteristic matrix, by $[S]$ or $[S(i\omega)]$, we can solve for X_1 and X_2 by taking the inverse of $[S]$,

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [S]^{-1} \begin{Bmatrix} f_1 \\ 0 \end{Bmatrix}.$$

From matrix algebra, the inverse equals the adjoint matrix divided by the determinant,

$$[S]^{-1} = \frac{\text{adj}[S]}{\det[S]},$$

and thus we can write

$$\begin{aligned} \begin{Bmatrix} X_1(\omega) \\ X_2(\omega) \end{Bmatrix} &= \frac{1}{\det[S]} \begin{bmatrix} k_{22} - m_2\omega^2 & -k_{12} \\ -k_{12} & k_{11} - m_1\omega^2 \end{bmatrix} \begin{Bmatrix} f_1 \\ 0 \end{Bmatrix}, \end{aligned} \quad (6.40)$$

where the response amplitudes, X_1 and X_2 , are functions of the forcing frequency ω . Expanding the matrix equation, X_1 and X_2 are given by

$$\begin{aligned} X_1(\omega) &= \frac{(k_{22} - m_2\omega^2)}{\det[S]} f_1 \\ X_2(\omega) &= \frac{-k_{12}}{\det[S]} f_1, \end{aligned} \quad (6.41)$$

where

$$\det[S] = (k_{22} - m_2\omega^2)(k_{11} - m_1\omega^2) - k_{12}^2.$$

The determinant of $[S]$, $\det[S]$, is the characteristic polynomial with the squares of natural frequencies as roots, equivalent to the characteristic polynomial obtained in Equation 6.25 and roots in Equation 6.26. Therefore, at the natural frequencies, the equations for X_1 and X_2 will have a zero in the denominator, resulting in an unbounded response. From the single degree-of-freedom case, Equation 2.35, we already know that if the forcing frequency equals the natural frequency, the response grows without bound for an undamped system. For a multi degree-of-freedom system, such an instability can occur for each natural frequency.

Figures 6.18 and 6.19 show the response amplitudes of the two degree-of-freedom system in Figure 6.13 for parameter values $m_1 = m_2 = 1$ kg, $k_1 = k_2 = k_3 = 1$ N/m, and $f_1 = 1$ N and $f_2 = 0$ N. The corresponding mass and stiffness matrices are

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The natural frequencies are 1.00 and $\sqrt{3} = 1.73$ rad/s. These figures are analogous to Figure 2.26 for the single degree-of-freedom system.

Example 6.9 Multiple Input Multiple Outputs

Consider the system whose equation of motion is given by Equation 6.38. Here, let m_1 be subject to $F_1(t) = f_1 \cos \omega_1 t$ and m_2 be subject to $F_2(t) = f_2 \cos \omega_2 t$. Find

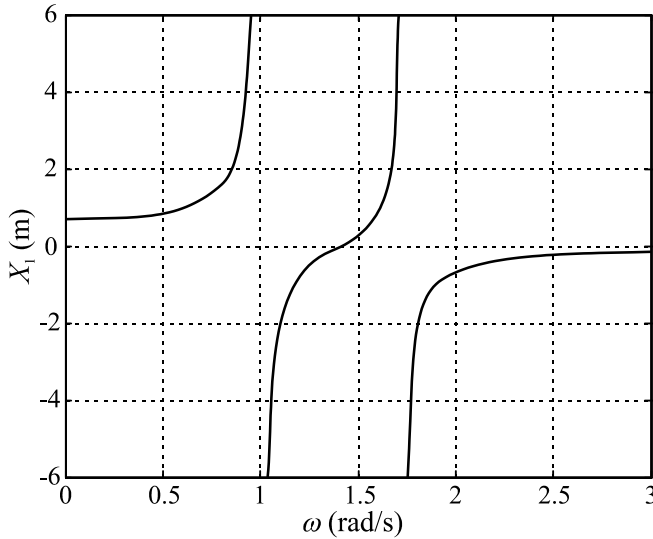


Figure 6.18: Response amplitude X_1 as a function of driving frequency ω for $f_1 = 1$ and $f_2 = 0$. The magnitude grows to infinity at the two natural frequencies: $\omega = 1.00$ and 1.73 rad/s. Note that there is one (finite) frequency corresponding to zero amplitude.

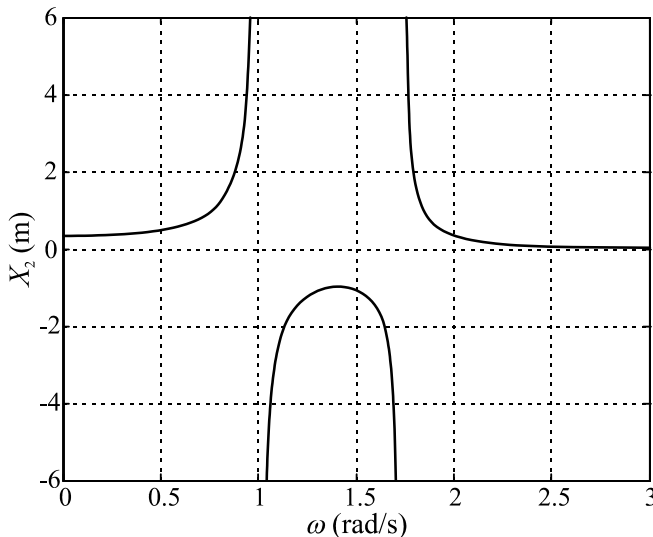


Figure 6.19: Response amplitude X_2 as a function of driving frequency ω for $f_1 = 1$ and $f_2 = 0$. The magnitude grows to infinity at the two natural frequencies: $\omega = 1.00$ and 1.73 rad/s. For the second mass there is no frequency corresponding to zero amplitude.

the displacement amplitudes, X_1 and X_2 , of the particular solution as functions of input frequencies ω_1 and ω_2 .

Solution: In order to obtain the particular solutions, we use the principle of superposition. We find the particular solutions to $F_1(t)$ and $F_2(t)$ separately and add them. The response amplitudes due to $F_1(t)$ are given by Equation 6.41. We now turn to the second input, $F_2(t)$, using Equation 6.40,

$$\begin{aligned} & \begin{Bmatrix} X_1(\omega_2) \\ X_2(\omega_2) \end{Bmatrix} \\ &= \frac{1}{\det[S(\omega_2)]} \begin{bmatrix} k_{22} - m_2\omega_2^2 & -k_{12} \\ -k_{12} & k_{11} - m_1\omega_2^2 \end{bmatrix} \begin{Bmatrix} 0 \\ f_2 \end{Bmatrix}. \end{aligned} \quad (6.42)$$

Then,

$$\begin{aligned} X_1(\omega_2) &= \frac{-k_{12}}{\det[S(\omega_2)]} f_2 \\ X_2(\omega_2) &= \frac{k_{11} - m_1\omega_2^2}{\det[S(\omega_2)]} f_2. \end{aligned} \quad (6.43)$$

Combining the two response amplitudes, we obtain

$$\begin{aligned} X_1 &= \frac{(k_{22} - m_2\omega_1^2)}{\det[S(\omega_1)]} f_1 + \frac{-k_{12}}{\det[S(\omega_2)]} f_2 \\ X_2 &= \frac{-k_{12}}{\det[S(\omega_1)]} f_1 + \frac{k_{11} - m_1\omega_2^2}{\det[S(\omega_2)]} f_2. \end{aligned}$$

■

Example 6.10 Base-Excited Motion of a Two Degree-of-Freedom System

Consider the two degree-of-freedom model of a building subject to base motion, as shown in Figure 6.20. Let $y(t) = Y \sin \omega t$. Plot the displacement transmissibilities and phase angles. Use $m_1 = m_2 = m$ and $k_1 = k_2 = k$.

Solution: The free-body diagrams for the two floors are shown in Figure 6.21. Using Newton's second law, the equations of motion are given by

$$\begin{aligned} k(x_2 - x_1) - k(x_1 - y) &= m\ddot{x}_1 \\ -k(x_1 - x_2) &= m\ddot{x}_2. \end{aligned}$$

Since the base motion is harmonic with amplitude Y ,

$$y(t) = Y \sin \omega t,$$

the responses are also harmonic with

$$\begin{aligned} x_1(t) &= X_1(\omega) \sin \omega t \\ x_2(t) &= X_2(\omega) \sin \omega t. \end{aligned}$$

Differentiating $x_1(t)$ and $x_2(t)$ twice and substituting into the equation of motion, we obtain

$$\begin{aligned} k(X_2 - X_1) - k(X_1 - Y) &= -m\omega^2 X_1 \\ -k(X_1 - X_2) &= -m\omega^2 X_2. \end{aligned}$$

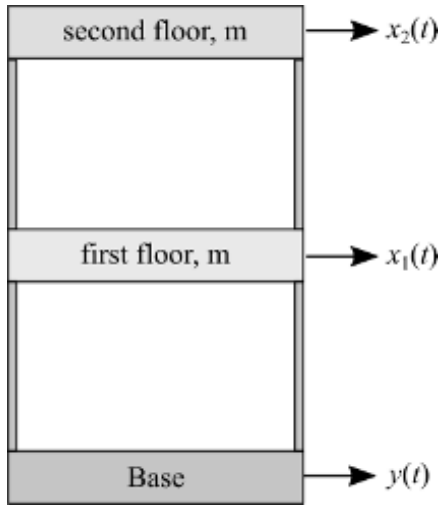


Figure 6.20: Base-excited two-story building model.

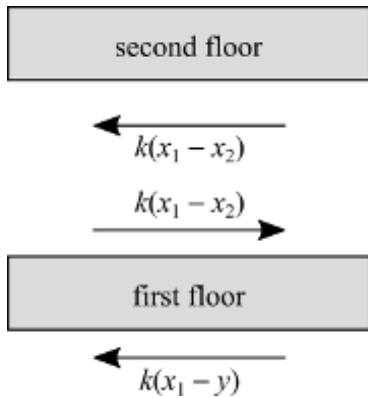


Figure 6.21: The free-body diagrams of two-story model.

In matrix form, we have

$$\begin{bmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} kY \\ 0 \end{bmatrix}.$$

The response amplitudes are given by

$$\begin{aligned} X_1(\omega) &= \frac{-mk\omega^2 + k^2}{(-m\omega^2 + 2k)(-m\omega^2 + k) - k^2} Y \\ &= \frac{-\frac{m}{k}\omega^2 + 1}{\left(-\frac{m}{k}\omega^2 + 2\right)\left(-\frac{m}{k}\omega^2 + 1\right) - 1} Y \\ X_2(\omega) &= \frac{k^2}{(-m\omega^2 + 2k)(-m\omega^2 + k) - k^2} Y \\ &= \frac{1}{\left(-\frac{m}{k}\omega^2 + 2\right)\left(-\frac{m}{k}\omega^2 + 1\right) - 1} Y, \end{aligned}$$

where the response amplitudes approach infinity at the

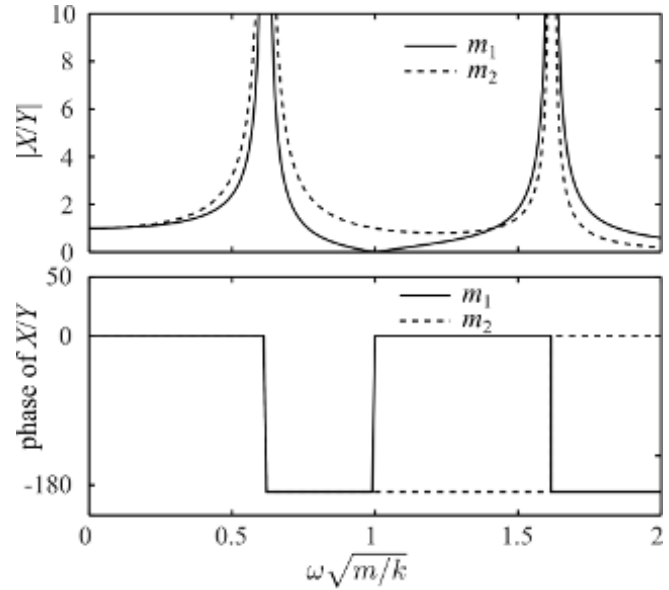


Figure 6.22: The amplitude and phase of displacement transmissibilities used in Equation 6.44. The phase for X_1 is 0° for $\omega\sqrt{m/k} < 0.618$ or $1 < \omega\sqrt{m/k} < 1.618$. The phase of X_1 is -180° elsewhere. Similarly, the phase of X_2 is 0° for $\omega\sqrt{m/k} < 0.618$ or $\omega\sqrt{m/k} > 1.618$. The phase of X_2 is -180° elsewhere.

two natural frequencies given by

$$\begin{aligned} \omega_1 &= 0.618\sqrt{\frac{k}{m}} \\ \omega_2 &= 1.618\sqrt{\frac{k}{m}}. \end{aligned}$$

The displacement transmissibilities are then

$$\begin{aligned} \frac{X_1(\omega)}{Y} &= \frac{-\frac{m}{k}\omega^2 + 1}{\left(-\frac{m}{k}\omega^2 + 2\right)\left(-\frac{m}{k}\omega^2 + 1\right) - 1} \\ \frac{X_2(\omega)}{Y} &= \frac{1}{\left(-\frac{m}{k}\omega^2 + 2\right)\left(-\frac{m}{k}\omega^2 + 1\right) - 1}, \end{aligned} \quad (6.44)$$

which are plotted in Figure 6.22. In addition, we note that the displacement transmissibility of mass 1 becomes zero when $\omega = \sqrt{k/m}$. This frequency is referred to as “zero frequency” since the first floor does not move at this frequency.

It is interesting to observe the amplitude of each floor as ω increases. At very low base oscillation frequencies, the displacement transmissibility is nearly unity and the phases equal 0. That is, the displacement amplitudes of the two floors are nearly identical to that of the base and have the same sign as the base. Figure 6.23 depicts the responses for different ranges of values of ω .

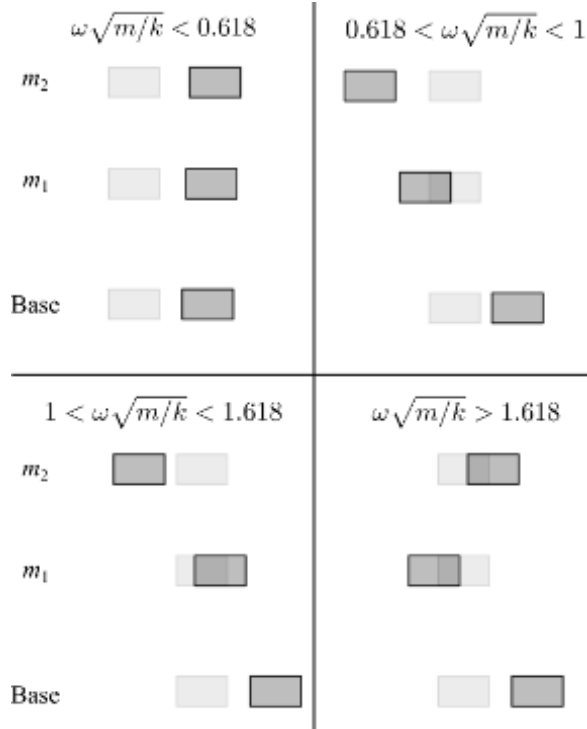


Figure 6.23: The response of a two-story building due to base excitation at various input frequencies. The light gray boxes are the static locations of the base, m_1 , and m_2 . The dark gray boxes are the dynamic amplitudes of the base, m_1 , and m_2 .

The shapes of vibration resemble the two modes of vibration. ■

6.4.3 Undamped Vibration Absorber

A practical application of forced vibration theory is the *vibration absorber*.¹⁴ Figure 6.24 shows an undamped system subjected to a harmonic force $F_1(t) = f_1 \sin \omega t$, and the system modified by the addition of a mass-spring system. This added mass is a vibration absorber. The equations of motion of the modified two degree-of-freedom system are¹⁵

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1 \sin \omega t \quad (6.45)$$

$$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0. \quad (6.46)$$

¹⁴There are numerous ways to modify the character of system oscillations, including system control. Vibration absorbers are devices that are attached to a primary structure with the intent to change its behavior. Such absorbers may be passive, active, or hybrid devices. One reference is **Dynamic Vibration Absorbers**, B.G. Korenev and L.M. Reznikov, Wiley, 1993.

¹⁵An alternative modification would be to subject the absorber mass m_2 to a force $F_2(t)$ that can be *controlled*. In this case, the control parameters that can be specified by the designer are m_2 , k_2 , and $F_2(t)$. By including more parameters that can be modified in the model, finer tuning is possible. Here we proceed with the simpler case where $F_2(t) = 0$.

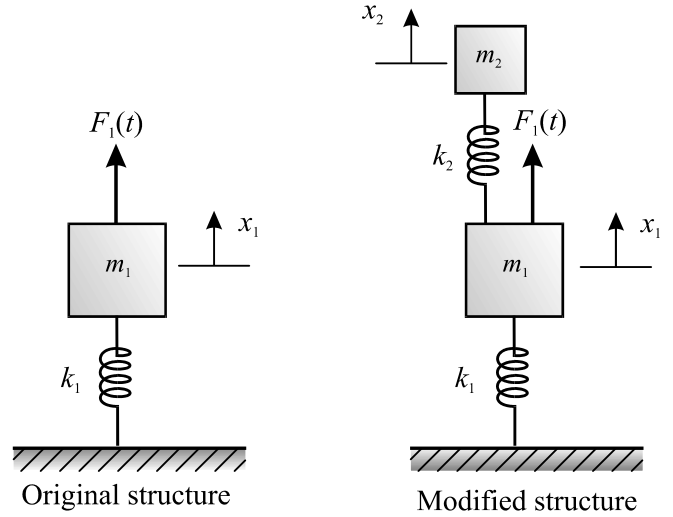


Figure 6.24: Mass-spring system without absorber on the left and with absorber on the right.

We rewrite each equation of motion in standard form,

$$\begin{aligned} \ddot{x}_1 + \frac{(k_1 + k_2)}{m_1} x_1 - \frac{k_2}{m_1} x_2 &= \frac{f_1}{m_1} \sin \omega t \\ \ddot{x}_2 - \frac{k_2}{m_2} x_1 + \frac{k_2}{m_2} x_2 &= 0, \end{aligned}$$

and define the following parameters,

$$\mu = \frac{m_2}{m_1}, \quad \omega_n^2 = \frac{k_1}{m_1}, \quad \omega_a^2 = \frac{k_2}{m_2},$$

where μ is the mass ratio, ω_n is the natural frequency of the primary system, and ω_a is the natural frequency of the absorber. Substituting these, along with the derived relation $k_2/m_1 = \mu\omega_a^2$, into the equations of motion results in the equations

$$\begin{aligned} \ddot{x}_1 + (\omega_n^2 + \mu\omega_a^2)x_1 - \mu\omega_a^2 x_2 &= \frac{f_1}{m_1} \sin \omega t \\ \ddot{x}_2 - \omega_a^2 x_1 + \omega_a^2 x_2 &= 0. \end{aligned}$$

The direct method of the previous section can be used here, where harmonic forcing on mass m_1 results in harmonic responses,

$$x_1(t) = X_1(\omega) \sin \omega t, \quad x_2(t) = X_2(\omega) \sin \omega t.$$

The anticipated dependence of the displacement response amplitudes $X_1(\omega)$ and $X_2(\omega)$ on driving frequency ω is shown explicitly. Differentiating $x_1(t)$ and $x_2(t)$ twice and substituting the resulting expressions into the equations of motion, simplifying, and writing in matrix-vector form leads to the equation

$$\begin{bmatrix} \omega_n^2 + \mu\omega_a^2 - \omega^2 & -\mu\omega_a^2 \\ -\omega_a^2 & \omega_a^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} \frac{f_1}{m_1} \\ 0 \end{Bmatrix}. \quad (6.47)$$

The displacement response magnitudes can be solved by taking the inverse of the matrix in brackets, the characteristic matrix $[S]$, resulting in

$$\begin{Bmatrix} X_1(\omega) \\ X_2(\omega) \end{Bmatrix} = \frac{1}{\det[S]} \begin{bmatrix} \omega_a^2 - \omega^2 & \mu\omega_a^2 \\ \omega_a^2 & \omega_n^2 + \mu\omega_a^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} f_1/m_1 \\ 0 \end{Bmatrix}.$$

Then,

$$X_1(\omega) = \frac{(\omega_a^2 - \omega^2)f_1/m_1}{\det[S]} \quad (6.48)$$

$$X_2(\omega) = \frac{\omega_a^2 f_1/m_1}{\det[S]}. \quad (6.49)$$

If another harmonic force at the same frequency ω is applied to the added mass m_2 its effect would appear as a second term in the numerator of each expression. The force vector in Equation 6.47 would be $\{f_1/m_1 \ f_2/m_2\}^T$. If there are two different forcing frequencies, we solve for each forcing separately and add the results.

Equations 6.48 and 6.49 can be simplified and written in a more useful form where dimensionless frequency ratios are introduced. In these equations, we replace f_1/m_1 by its equivalent $f_1\omega_n^2/k_1$ and multiply numerator and denominator by $1/\omega_n^2\omega_a^2$. The ratio f_1/k_1 equals x_{static} , the static displacement of mass m_1 . Performing these operations on both equations results in the relations

$$X_1(\omega) = \frac{\left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] x_{static}}{\det[S]/(\omega_a^2 k_1/m_1)}$$

$$X_2(\omega) = \frac{x_{static}}{\det[S]/(\omega_a^2 k_1/m_1)},$$

where

$$\begin{aligned} \frac{\det[S]}{\omega_a^2 \frac{k_1}{m_1}} &= \left[1 + \mu \left(\frac{\omega}{\omega_a}\right)^2 - \left(\frac{\omega}{\omega_n}\right)^2\right] \\ &\times \left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] - \mu \left(\frac{\omega_a}{\omega_n}\right)^2. \end{aligned}$$

If the driving frequency ω equals ω_a , then the amplitude $X_1(\omega_a) = 0$, that is, the primary mass does not move despite being subjected to a harmonic external force. Let us see if this makes physical sense. For $\omega = \omega_a$, the secondary amplitude $X_2(\omega_a) = -f_1/k_2$, and the response of the absorber structure is

$$x_2(t) = -\frac{f_1}{k_2} \sin \omega t,$$

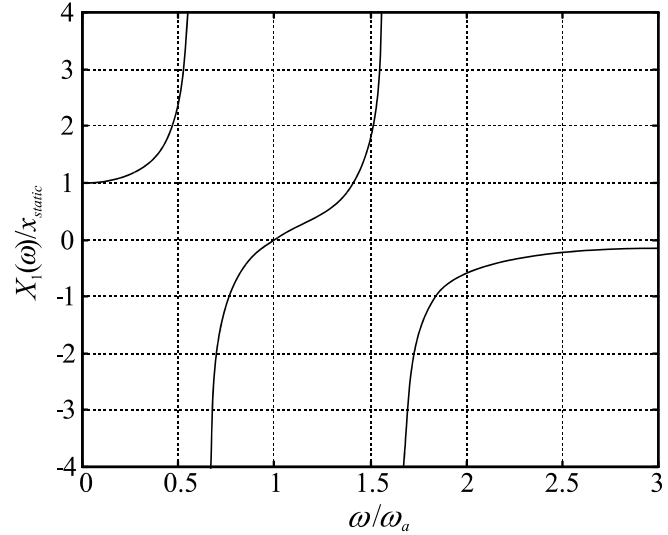


Figure 6.25: Nondimensional displacement X_1/x_{static} of primary mass vs. ratio ω/ω_a , where $\omega_n = \omega_a$ and the natural frequencies are $\omega_{1,2} = 0.618, 1.618$ rad/s.

or $k_2 x_2 = -f_1 \sin \omega t$. The left-hand side of this equation is the force exerted by m_2 on m_1 . This is equal and opposite to the force applied to m_1 . Since the force due to the absorber is exactly opposite to the applied force, the primary mass does not move.

Plotting $X_1(\omega)/x_{static}$ vs. the ratio ω/ω_a , as in Figure 6.25, we see that a slight shift away from $\omega/\omega_a = 1$ results in a large increase in amplitude. In this figure, $\mu = 1$, $\omega_a = \omega_n$, and the natural frequencies are $\omega_{1,2} = 0.618, 1.618$ rad/s, where the magnitudes grow to infinity. Even though this is a conceptual example, it has significant practical implications.

Stockbridge Vibration Absorber

A *Stockbridge damper* is the name given to a passive vibration absorber used to suppress wind-induced vibrations on cables, such as overhead power lines.¹⁶ Figure 6.26 shows an example of the dumbbell-shaped device in use. It consists of two masses at the ends of a short length of cable or flexible rod, which is clamped at its middle to the main cable. The Stockbridge damper – really a tuned mass-spring-damper – is designed to dissipate the energy of oscillations in the main cable to an acceptable level. Due to its distinctive shape it is sometimes called a “dog-bone damper” as a nickname.

Wind can generate three major modes of oscillation¹⁷ in suspended cables: (a) *gallop*, which has amplitudes in

¹⁶M. Markiewicz, “Optimum dynamic characteristics of Stockbridge dampers for dead-end spans,” *Journal of Sound and Vibration*, Vol.188, No.2, 1995, pp.243-256.

¹⁷Electric Power Research Institute Transmission Line Reference Book, **Wind-Induced Conductor Motion**, EPRI 1012317, 2008.



Figure 6.26: A Stockbridge damper on a power line is shown at the center. The damper is placed near a suspension clamp, at a distance less than one-half wavelength of the highest frequency mode to be absorbed. (Glendale, WI; photograph by M. Nagurka)

meters and a frequency range of 0.08 to 3 Hz,¹⁸ (b) *aerolian vibration*, or simply *flutter*, which has amplitudes of millimeters to centimeters and a frequency of 3 to 150 Hz,¹⁹ and (c) *wake-induced vibration*, which has amplitudes of centimeters and a frequency of 0.15 to 10 Hz.

The Stockbridge damper is designed to mitigate oscillations due to flutter.²⁰ Flutter occurs in the vertical plane and is caused by alternating shedding of vortices on the downwind side of the cable. A steady but moderate wind can induce a standing wave pattern on the line consisting of several wavelengths per span. Flutter causes damaging stress fatigue²¹ to the cable and represents the principal cause of failure of conductor strands. The ends of a power line span, where it is clamped to the transmission towers, are at most risk.

The Stockbridge damper was invented in the 1920s by George H. Stockbridge, an engineer working for Southern California Edison.²² His patent described three means of damping vibrations on overhead lines: (a) a sack of metal components (punchings) tied to the line, (b) a short length of cable clamped parallel to the main cable, and (c) a short (75 cm $\approx$ 30 in) cable with a concrete

mass fixed at each end. A sketch of this last device from Stockbridge's patent is shown in Figure 6.27. This device developed into the widely used Stockbridge damper.

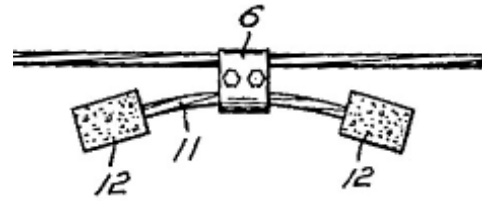


Figure 6.27: Drawing from Stockbridge's patent showing a short cable with concrete masses fixed at the ends (6 denotes the clamp, 11 the damper cable, and 12 the concrete blocks).

In his design, vibration in the main cable would be passed down through the clamp and into the damper. This shorter system would flex and cause the symmetrically placed concrete blocks at its ends to oscillate. Through careful choices of the mass of the blocks and the stiffness and length of the damper cable the damper's impedance would match that of the line. If tuned properly, it would greatly attenuate oscillation of the main cable. Since Stockbridge dampers were economic, effective, and very easy to fit, they were commonly installed on overhead lines.

Modern designs use metal bell-shaped weights rather than Stockbridge's concrete blocks. The bell is hollow and the damper cable is fixed internally to the distal end, which permits relative motion between the cable and damping weights.²³ To provide for greater freedom of motion, the weights may be partially slotted in the vertical plane, allowing the cable to travel outside the confines of the bell. More complex designs use weights with asymmetric mass distribution, which enables the damper to oscillate in several different frequency modes.

Overhead transmission lines form a catenary²⁴ for which vibration is predominately in the vertical plane. When more than one plane of vibration is anticipated, Stockbridge dampers may be mounted at right angles to each other. This is common when the cable runs in a vertical or off-horizontal plane, for example in cable-stayed bridges or radio mast guy-wires.

¹⁸Galloping vibrations typically occur in one of the first eigenfrequencies of the conductor, corresponding to a frequency below 1 Hz, with very large amplitudes. They occur sporadically, and are more common in winter when ice deposits on the cable making it asymmetric and causing an aero-elastic instability.

¹⁹Some sources indicate a narrower bandwidth, that is, a frequency range of 10 Hz to 50 Hz. Flutter is almost always present in a transmission line, usually with amplitudes well below one conductor diameter.

²⁰For a detailed dynamic model of the Stockbridge damper see P. Hagedorn, "Mechanical Vibrations and Vibration Control," *Passive and Active Structural Vibration Control in Civil Engineering*, ed. T.T. Soong and M.C. Costantino, Springer-Verlag, 1994, pp.51-62. (The Stockbridge damper is less effective for the other two modes. Hagedorn develops a simple one degree-of-freedom model for galloping vibration on pp.62-64.)

²¹The damage becomes more pronounced with increased cable tension, as the cable's natural self-damping is reduced.

²²Stockbridge obtained U.S. patent 1675391 on July 3, 1928 for a "vibration damper."

²³*Electrical Power Systems*, C.L.Wadhwa, New Age Publishers, 2006, pp.169-170.

²⁴A catenary is the shape of a hanging flexible cable when supported at its ends and acted upon by gravity. Due to its own weight, in equilibrium the curve has a U-shape that is similar in appearance to a parabola. It turns out that it is not actually a parabola. The equation of a catenary has the form $y = a \cosh(x/a)$, where $\cosh$ is the hyperbolic cosine function and a is a scaling factor that can be interpreted as the ratio between the horizontal component of the tension (assumed constant) and the weight of the cable per unit of length.

Vibration Absorbers on Pipelines

Passive vibration absorbers²⁵ are used to mitigate vibrations of long pipes. Figures 6.28 and 6.29 show designs of vibration absorbers installed on pipelines in Alaska.



Figure 6.28: A Tuned Vibration Absorber (TVA) used on a pipeline in Alaska. The photograph shows a “hammer” type design in which a torsional spring in the base supports the arm and weight. The length of the arm and number of weights is variable. The photograph was taken at a BP-operated facility in the Prudhoe Bay area of Alaska, north of the Arctic circle, where the Trans-Alaska pipeline begins. (Courtesy of Evan D. Kane, February 2009)

6.4.4 Beating Oscillations

An important phenomenon that can occur in coupled systems with *close* natural frequencies is called *beating*. Because of this closeness, the system undergoes a high frequency oscillation that is modulated (enveloped) by a lower frequency oscillation.

Beating Phenomena

An example of the *beating phenomenon* is the periodically changing sound of a twin engine propeller-driven aircraft. Whenever the speed of one engine varies slightly from the other, one can hear the vibration acoustically – and often feel the beating in the aircraft’s structure. Another example is when two adjacent notes on the piano are struck simultaneously. Their sound rises and falls periodically. The beating phenomenon can be exploited to provide a simple, accurate way to detect phase differences between two waves with nearly identical frequencies.²⁶

²⁵The authors are grateful to Evan D. Kane for supplying the photographs and information about vibration absorbers used on the pipeline.

²⁶The beating phenomenon allows the difference in two very fast, otherwise nearly immeasurable frequencies – for example, interfering light waves – to be measured fairly easily by superimposing the waves and measuring the occurrence of intensity peaks. The use of the beating phenomenon to assist in the measurement of the frequency difference in two electromagnetic waves is known as *heterodyne detection*.



Figure 6.29: Two designs of Tuned Vibration Absorbers (TVAs) used on pipelines in Alaska. The photograph shows (1) a hanging spring-mass design (classic vibration absorber) and (2) a “hammer” type design in which a torsional spring in the base supports an arm and weight. The absorbers are tuned to absorb three main frequencies. They are used for most of the pipeline, starting at an anchored support in the middle and alternating through the three frequencies on spans between sliding supports until the line reaches an expansion loop. The photograph was taken at a BP-operated facility in the Prudhoe Bay area of Alaska, north of the Arctic circle, where the Trans-Alaska pipeline begins. (Courtesy of Evan D. Kane, February 2009)

Such behavior (shown in Figure 6.32) is the topic of this section. We refer the reader to [Section 2.2](#) to review the formulation of the equation of motion for the simple pendulum as a basis for understanding the coupled pendulum discussed next.

Coupled Pendulums

Consider the coupled pendulums each of length L of Figure 6.30 that includes free-body diagrams of the masses. In the problem considered here, we assume that $k_2 = k_1$. Taking the sum of the moments about O for the right mass leads to the equation of motion,

$$\begin{aligned} I_O \alpha &= \sum M_O \\ mL^2 \ddot{\theta}_1 &= -mgL \sin \theta_1 \\ &\quad - \{ka(\tan \theta_1 - \tan \theta_2) + k_1 a \tan \theta_1\} \\ &\quad \times \cos \theta_1] a(1 + \tan^2 \theta_1)^{1/2}. \end{aligned} \quad (6.50)$$

Making the assumption of small θ_1 yields the simplifications $\cos \theta_1 \simeq 1$, $\tan \theta_1 \simeq \sin \theta_1 \simeq \theta_1$, and $\tan^2 \theta_1 \ll 1$, and the equation of motion becomes

$$mL^2 \ddot{\theta}_1 + mgL \theta_1 + (k + k_1) a^2 \theta_1 - k a^2 \theta_2 = 0.$$

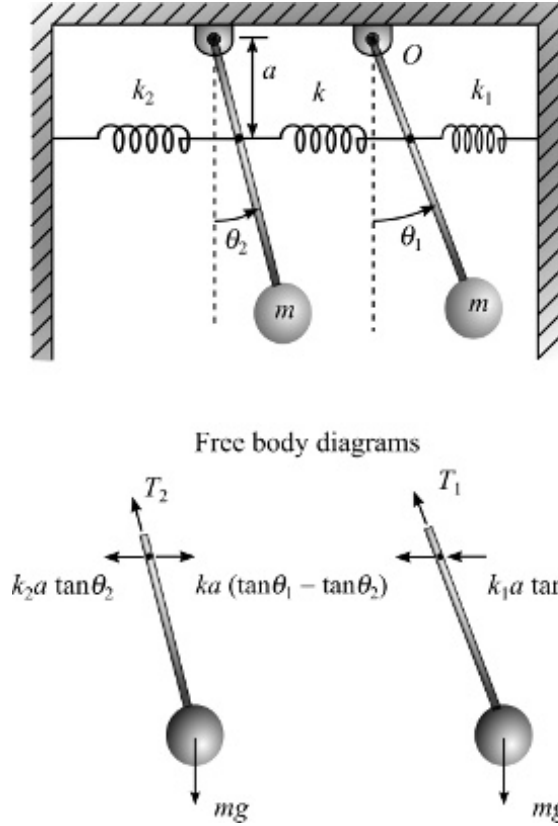


Figure 6.30: Exaggerated view of simple coupled pendulums each of length L (top) and free-body diagrams (bottom). The angular displacements are assumed small.

Similarly for the left mass, with the small angle assumption, the equation of motion is

$$mL^2\ddot{\theta}_2 + mgL\theta_2 + (k + k_1)a^2\theta_2 - ka^2\theta_1 = 0.$$

In matrix-vector form, the equations of motion become

$$[M] \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + [K] \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (6.51)$$

where

$$[M] = \begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix}$$

$$[K] = \begin{bmatrix} mgL + (k + k_1)a^2 & -ka^2 \\ -ka^2 & mgL + (k + k_1)a^2 \end{bmatrix}.$$

As expected, if $k = 0$, there is no coupling between the pendula, and the simple pendulum equation is recovered for each uncoupled pendulum. To solve coupled [Equations 6.51](#), we assume a harmonic solution²⁷ for each de-

²⁷We could have just as easily assumed solutions of the form $\theta(t) = \Theta \cos(\omega t - \phi)$.

gree of freedom,

$$\theta_1(t) = \Theta_1 e^{i\omega t}$$

$$\theta_2(t) = \Theta_2 e^{i\omega t},$$

differentiate each twice, and substitute the functions and their derivatives into [Equation 6.51](#). Canceling the exponential function yields

$$-\omega^2 [M] \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} + [K] \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Combining terms as before we obtain the equation,

$$[A] \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (6.52)$$

where

$$[A] = [K] - \omega^2 [M]$$

$$= \begin{bmatrix} \begin{pmatrix} -\omega^2 mL^2 \\ +mgL \\ +(k + k_1)a^2 \end{pmatrix} & -ka^2 \\ -ka^2 & \begin{pmatrix} -\omega^2 mL^2 \\ +mgL \\ +(k + k_1)a^2 \end{pmatrix} \end{bmatrix}.$$

Setting the determinant of the characteristic matrix $[A]$ to zero, we find the characteristic equation for the system to be

$$(-\omega^2 mL^2 + mgL + (k + k_1)a^2)^2 - (ka^2)^2 = 0,$$

or

$$-\omega^2 mL^2 + mgL + (k + k_1)a^2 = \pm ka^2,$$

which has two solutions for ω (as required for a two degree-of-freedom system). For $+ka^2$,

$$\omega_1 = \sqrt{\frac{g}{L} + \frac{k_1 a^2}{m L^2}}, \quad (6.53)$$

and for $-ka^2$,

$$\omega_2 = \sqrt{\frac{g}{L} + \frac{2k + k_1 a^2}{m L^2}}. \quad (6.54)$$

To evaluate the natural mode for each natural frequency, substitute each ω_i into [Equation 6.52](#) as follows,

$$\begin{bmatrix} \begin{pmatrix} -\omega_i^2 mL^2 + mgL \\ +(k + k_1)a^2 \end{pmatrix} & -ka^2 \\ -ka^2 & \begin{pmatrix} -\omega_i^2 mL^2 + mgL \\ +(k + k_1)a^2 \end{pmatrix} \end{bmatrix}$$

$$\times \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}_i = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

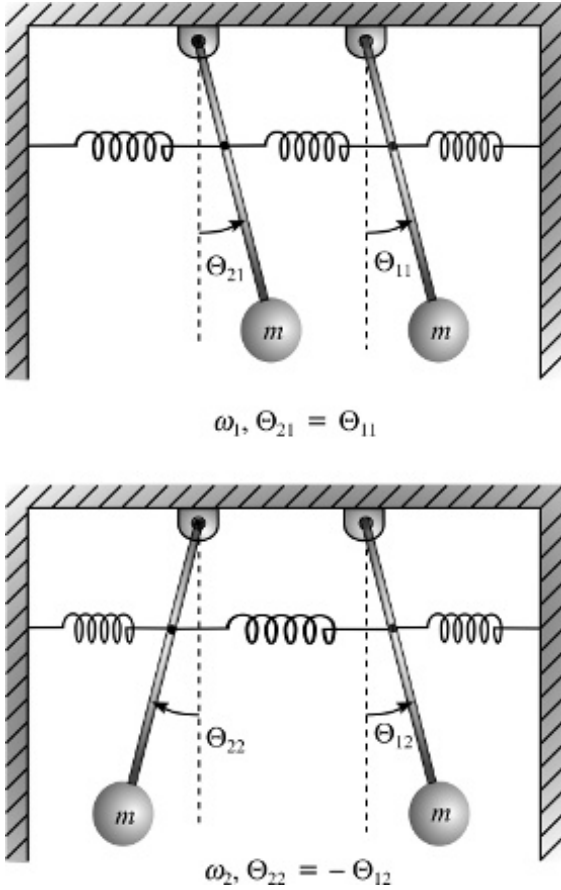


Figure 6.31: Modes for two coupled pendulums. Mode 1 (top) is in-phase. Mode 2 (bottom) is out-of-phase.

Solving for the modal ratios, we find

$$\begin{aligned}\frac{\Theta_{21}}{\Theta_{11}} &= 1 \\ \frac{\Theta_{22}}{\Theta_{12}} &= -1.\end{aligned}$$

These modes are shown in Figure 6.31.

The coupling parameter k affects ω_2 but not ω_1 as can be seen from Equations 6.53 and 6.54. The physical reason is that in the mode associated with ω_1 , the masses move in phase while for the second mode the masses move in opposite directions, with k being engaged.

The general motion of the two coupled pendula is given by the superposition of the two natural modes of vibra-

tion,

$$\begin{aligned}\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix} &= \bar{C}_1 \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}_1 \cos(\omega_1 t - \phi_1) \\ &+ \bar{C}_2 \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}_2 \cos(\omega_2 t - \phi_2) \\ &= C_1 \begin{Bmatrix} 1 \\ \Theta_{21}/\Theta_{11} \end{Bmatrix} \cos(\omega_1 t - \phi_1) \\ &+ C_2 \begin{Bmatrix} 1 \\ \Theta_{22}/\Theta_{12} \end{Bmatrix} \cos(\omega_2 t - \phi_2),\end{aligned}$$

where the factors Θ_{11} and Θ_{12} have been incorporated into coefficients C_1 and C_2 , respectively. The modification of the integration constants does not change the final result since these constants are determined by satisfying the initial conditions. Substituting the modal ratios leads to the response

$$\begin{aligned}\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix} &= C_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t - \phi_1) \\ &+ C_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t - \phi_2).\end{aligned}\quad (6.55)$$

All that remains is the evaluation of C_1, C_2, ϕ_1 , and ϕ_2 using the initial conditions.

In order to demonstrate the beating phenomenon, we assume the following initial conditions,²⁸

$$\begin{aligned}\theta_1(0) &= \theta_0, \quad \theta_2(0) = 0 \\ \dot{\theta}_1(0) &= 0, \quad \dot{\theta}_2(0) = 0.\end{aligned}$$

From Equation 6.55 and its derivative, satisfying the initial conditions results in the values

$$C_1 = C_2 = \frac{1}{2}\theta_0, \quad \phi_1 = \phi_2 = 0.$$

Before proceeding with the analysis, let us try to anticipate the behavior of the coupled pendula, which depends entirely on the coupling force in the middle spring. If the coupling is “strong” (high stiffness), then we expect the two masses to oscillate almost in phase. The strong coupling will not allow one mass to lag far behind the motion of the other. If the coupling is “weak” (defined below), then we may expect that the two masses will oscillate, with one sometimes reinforcing the motion of the other (larger amplitudes) and at other times canceling out the motion of the other (smaller amplitudes). The

²⁸These initial conditions imply that one pendulum is pulled to one side θ_0 while holding the other pendulum fixed. Assuming zero initial velocities does not change the character of the beating except to provide a zero slope at the initial time of the displacements.

oscillations do not look like those of the separate pendula because energy is being transferred continuously between the two masses.

In order to better observe this behavior, we can rewrite Equation 6.55 by introducing

$$\alpha = \frac{(\omega_2 - \omega_1)t}{2}, \quad \beta = \frac{(\omega_2 + \omega_1)t}{2},$$

and using a trigonometric identity,²⁹ to find

$$\theta_1(t) = \theta_0 \cos \frac{(\omega_2 - \omega_1)t}{2} \cos \frac{(\omega_2 + \omega_1)t}{2} \quad (6.56)$$

$$\theta_2(t) = \theta_0 \sin \frac{(\omega_2 - \omega_1)t}{2} \sin \frac{(\omega_2 + \omega_1)t}{2}. \quad (6.57)$$

From Equations 6.53 and 6.54, the difference and sum of the natural frequencies can be written as

$$\frac{(\omega_2 \mp \omega_1)}{2} = \frac{1}{2} \left(\sqrt{\frac{g}{L} + \frac{2k + k_1}{m} \frac{a^2}{L^2}} \mp \sqrt{\frac{g}{L} + \frac{k_1}{m} \frac{a^2}{L^2}} \right).$$

Furthermore, using the binomial series representation³⁰ for the square root, the following approximation for small $(2k + k_1)a^2/mgL$ can be used,

$$\begin{aligned} \sqrt{\frac{g}{L} + \frac{2k + k_1}{m} \frac{a^2}{L^2}} &= \sqrt{\frac{g}{L}} \sqrt{1 + \frac{2k + k_1}{mg} \frac{a^2}{L}} \\ &\simeq \sqrt{\frac{g}{L}} \left[1 + \frac{1}{2} \frac{(2k + k_1)a^2}{mgL} \right]. \end{aligned}$$

Weak coupling here implies $(2k + k_1)a^2 \ll mgL$, and therefore

$$\begin{aligned} \frac{(\omega_2 - \omega_1)}{2} &\simeq \frac{1}{2} \frac{k}{m} \frac{a^2}{\sqrt{gL^3}} \equiv \frac{\omega_b}{2} \\ \frac{(\omega_2 + \omega_1)}{2} &\simeq \sqrt{\frac{g}{L}} + \frac{1}{2} \frac{k + k_1}{m} \frac{a^2}{\sqrt{gL^3}} \\ &\equiv \omega_{avg}, \end{aligned} \quad (6.58)$$

where ω_{avg} is the frequency of oscillation and ω_b is the beating frequency – the frequency of the envelope around the high frequency oscillation. With weak coupling, Equations 6.56 and 6.57 can be written as

$$\begin{aligned} \theta_1(t) &\simeq \theta_0 \cos \frac{\omega_b t}{2} \cos \frac{\omega_{avg} t}{2} \\ \theta_2(t) &\simeq \theta_0 \sin \frac{\omega_b t}{2} \sin \frac{\omega_{avg} t}{2}, \end{aligned}$$

representing *amplitude modulated harmonic functions*. The beat frequency is $\omega_b = ka^2/m\sqrt{gL^3}$ from Equation 6.58. The maximum time between two peaks of the

²⁹ $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$.

³⁰ $(1 + x)^{\pm \frac{1}{2}} = 1 \pm \frac{1}{2}x \mp \frac{1}{8}x^2 \pm \dots$ for $|x| < 1$. The smaller the value of x , the more accurate is the two-term approximation $1 \pm \frac{1}{2}x$.

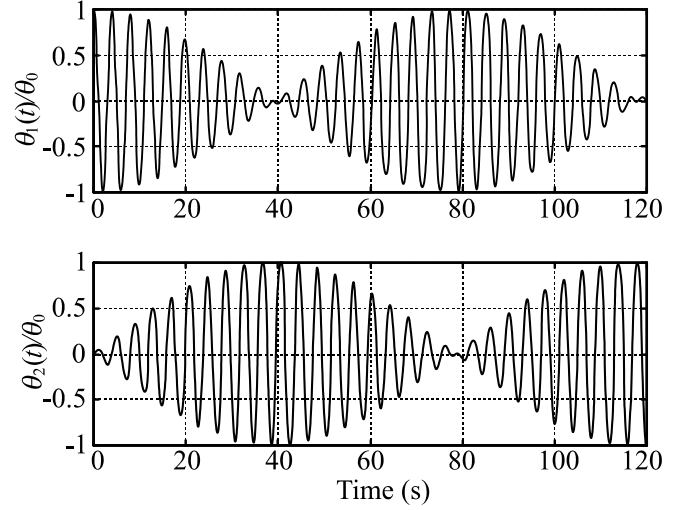


Figure 6.32: Beating behavior for weak coupling for the right pendulum (top) and the left pendulum (bottom).

beat is $T/2 = 2\pi/\omega_b$, where the period of the amplitude modulated envelope is T .

For parameter values $k = 10$ N/m, $k_1 = 10$ N/m, $a = 0.5$ m, $m = 10$ kg, $g = 9.81$ m/s², and $L = 1$ m, the condition $(2k + k_1)a^2 \ll mgL$ is 7.5 N-m $\ll 98.0$ N-m, which satisfies the weak coupling approximation. The two characteristic frequencies are

$$\omega_b = 0.080 \text{ rad/s}, \quad \omega_{avg} = 3.21 \text{ rad/s}.$$

Figure 6.32 shows the time histories for $\theta_1(t)/\theta_0$ and $\theta_2(t)/\theta_0$, given by

$$\begin{aligned} \frac{\theta_1(t)}{\theta_0} &\simeq \cos \frac{0.080t}{2} \cos 3.21t \\ \frac{\theta_2(t)}{\theta_0} &\simeq \sin \frac{0.080t}{2} \sin 3.21t. \end{aligned}$$

The beating response is reminiscent of amplitude modulated (AM) radio signals.

Next, we consider the damped, free vibration problem.

6.5 Free Vibration with Damping: Direct Method

There are two general approaches for analyzing damped oscillations: the direct method of this section and the modal method of the following section. The direct method is valid for any damping matrix $[C]$ whereas the modal method is generally limited to $[C]$ matrices of certain forms.

The equation of free motion with viscous damping,

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\},$$

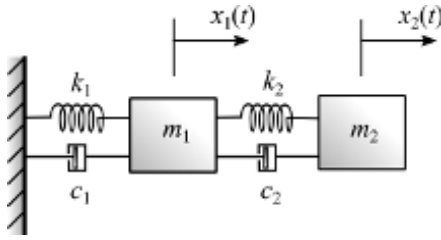


Figure 6.33: Two degree-of-freedom system.

can be solved in a direct manner by assuming a harmonic form for the response. Due to the presence of damping, the characteristic equation will be a polynomial that has complex conjugate roots. Given complex conjugate eigenvalues, there are complex conjugate eigenvectors. We explore this approach by way of a specific system.³¹

For the two degree-of-freedom system in Figure 6.33, the equations of motion are

$$\begin{aligned} m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 + (k_1 + k_2)x_1 - c_2 \dot{x}_2 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 - c_2 \dot{x}_1 - k_2 x_1 &= 0, \end{aligned}$$

or in matrix-vector form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

We assume solutions of the form

$$x_1(t) = Ae^{\lambda t} \quad (6.59)$$

$$x_2(t) = Be^{\lambda t}, \quad (6.60)$$

and substitute these into the governing equations. Eliminating the common factor $\exp\{\lambda t\}$ leads to two simultaneous algebraic equations,

$$[m_1 \lambda^2 + (c_1 + c_2)\lambda + (k_1 + k_2)]A - (c_2 \lambda + k_2)B = 0 \quad (6.61)$$

$$-(c_2 \lambda + k_2)A + (m_2 \lambda^2 + c_2 \lambda + k_2)B = 0. \quad (6.62)$$

In matrix-vector form, these equations can be written as

$$\begin{bmatrix} m_1 \lambda^2 + (c_1 + c_2)\lambda + (k_1 + k_2) & -(c_2 \lambda + k_2) \\ -(c_2 \lambda + k_2) & m_2 \lambda^2 + c_2 \lambda + k_2 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.63)$$

³¹This development follows **Vibration Problems in Engineering**, W. Weaver, S.P. Timoshenko, D.H. Young, 5th ed., Wiley-Interscience, 1990, p.260.

For a nontrivial solution to exist, the determinant of the matrix must equal zero,

$$\begin{vmatrix} m_1 \lambda^2 + (c_1 + c_2)\lambda + (k_1 + k_2) & -(c_2 \lambda + k_2) \\ -(c_2 \lambda + k_2) & m_2 \lambda^2 + c_2 \lambda + k_2 \end{vmatrix} = 0,$$

with the resulting characteristic equation

$$[m_1 \lambda^2 + (c_1 + c_2)\lambda + (k_1 + k_2)](m_2 \lambda^2 + c_2 \lambda + k_2) - (c_2 \lambda + k_2)^2 = 0,$$

which can be expanded to

$$\begin{aligned} m_1 m_2 \lambda^4 + (m_1 c_2 + [c_1 + c_2]m_2) \lambda^3 \\ + (m_1 k_2 + c_1 c_2 + [k_1 + k_2]m_2) \lambda^2 \\ + (c_1 k_2 + k_1 c_2) \lambda + k_1 k_2 = 0. \end{aligned}$$

This quartic characteristic equation will generally have four distinct roots, the eigenvalues λ_1 , λ_2 , λ_3 , and λ_4 that must be evaluated numerically.

From the theory of algebraic equations, since all the coefficients are positive, the nonzero roots can be *neither* real and positive *nor* complex with positive real parts. The roots must be either real and negative, or complex with negative real parts.³²

Physically, we know that most engineering structures are lightly damped and will vibrate. From our studies of single degree-of-freedom systems, we expect complex conjugate roots that occur in conjugate pairs.³³ For this two degree-of-freedom system, there are two complex conjugate pairs of the form

$$\lambda_{11} = -a_1 + i \omega_{d1}$$

$$\lambda_{12} = -a_1 - i \omega_{d1}$$

$$\lambda_{21} = -a_2 + i \omega_{d2}$$

$$\lambda_{22} = -a_2 - i \omega_{d2},$$

where we have relabeled the four roots with two subscripts to signify their complex conjugate pairings. The damped natural frequencies are given by $\omega_{di} = \omega_{ni} \sqrt{1 - \zeta_i^2}$, and a_i are positive numbers (for a stable system) that define the rate of decay of the exponential envelope. These expressions correspond to similar ones for single degree-of-freedom structures.

Substituting each root in turn into Equation 6.63 we can derive the amplitude ratios, or modes, A_{ij}/B_{ij} , for

³²A positive real root or real part of a complex root results in a solution with a growing exponential factor; this is an unstable system.

³³Recall that the real part appears as the coefficient of time in the exponential envelope, and the imaginary part becomes the damped frequency of oscillation.

each frequency,

$$\begin{bmatrix} \begin{pmatrix} m_1\lambda_{ij}^2 + (c_1 + c_2)\lambda_{ij} \\ + (k_1 + k_2) \end{pmatrix} & -c_2\lambda_{ij} - k_2 \\ -c_2\lambda_{ij} - k_2 & \begin{pmatrix} m_2\lambda_{ij}^2 + c_2\lambda_{ij} \\ + k_2 \end{pmatrix} \end{bmatrix} \times \begin{Bmatrix} A_{ij} \\ B_{ij} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.64)$$

Substituting each root λ_{ij} into Equation 6.64, we can find four possible ratios of the form

$$\begin{aligned} r_{ij} &\equiv \frac{A_{ij}}{B_{ij}} \\ &= \frac{c_2\lambda_{ij} + k_2}{m_1\lambda_{ij}^2 + (c_1 + c_2)\lambda_{ij} + (k_1 + k_2)} \\ &= \frac{m_2\lambda_{ij}^2 + c_2\lambda_{ij} + k_2}{c_2\lambda_{ij} + k_2}, \end{aligned}$$

where $i = 1, 2$ and $j = 1, 2$. The ratios r_{11} , r_{12} , r_{21} , and r_{22} are complex conjugate pairs that relate the modal amplitudes. These are the modes of vibration and are generally *complex modes*. Using the relations $A_{ij} = r_{ij}B_{ij}$ and Equations 6.59 and 6.60, the complete solution may be written as

$$x_1(t) = r_{11}B_{11}e^{\lambda_{11}t} + r_{12}B_{12}e^{\lambda_{12}t} + r_{21}B_{21}e^{\lambda_{21}t} + r_{22}B_{22}e^{\lambda_{22}t} \quad (6.65)$$

$$x_2(t) = B_{11}e^{\lambda_{11}t} + B_{12}e^{\lambda_{12}t} + B_{21}e^{\lambda_{21}t} + B_{22}e^{\lambda_{22}t}, \quad (6.66)$$

where (B_{11}, B_{12}) and (B_{21}, B_{22}) are complex conjugate pairs that are determined using the initial conditions.

We proceed as for a single mass system by converting Equations 6.65 and 6.66 into their equivalent trigonometric expressions by combining complex conjugate pairs. From Equation 6.66 for $x_2(t)$, we can rewrite

$$B_{11}e^{\lambda_{11}t} + B_{12}e^{\lambda_{12}t} = e^{-a_1t}(C_1 \cos \omega_{d1}t + C_2 \sin \omega_{d1}t),$$

where we used Euler's equation for the complex exponential, and where $C_1 = B_{11} + B_{12}$ and $C_2 = i(B_{11} - B_{12})$ where C_1 and C_2 are real numbers. Similarly, we can rewrite the last two terms in Equation 6.66 by following the same procedure for the complex conjugate pairs. For $x_1(t)$, the ratios r_{11} and r_{12} are a complex conjugate pair of the form

$$r_{11} = \alpha_1 + i\beta_1, \quad r_{12} = \alpha_1 - i\beta_1. \quad (6.67)$$

Since a complex number represents a magnitude and a phase, in Equation 6.65 we can rewrite the first two terms

as

$$\begin{aligned} r_{11}B_{11}e^{\lambda_{11}t} + r_{12}B_{12}e^{\lambda_{12}t} \\ = e^{-a_1t}[(C_1\alpha_1 + C_2\beta_1) \cos \omega_{d1}t \\ + (-C_1\beta_1 + C_2\alpha_1) \sin \omega_{d1}t]. \end{aligned}$$

The second pair of complex conjugates can be converted using the real constants $C_3 = B_{21} + B_{22}$ and $C_4 = i(B_{21} - B_{22})$. Introducing the notation

$$r_{21} = \alpha_2 + i\beta_2, \quad r_{22} = \alpha_2 - i\beta_2, \quad (6.68)$$

the displacement response equations then become

$$\begin{aligned} x_1(t) &= e^{-a_1t}(r_1C_1 \cos \omega_{d1}t + r'_1C_2 \sin \omega_{d1}t) \\ &\quad + e^{-a_2t}(r_2C_3 \cos \omega_{d2}t + r'_2C_4 \sin \omega_{d2}t) \end{aligned} \quad (6.69)$$

$$\begin{aligned} x_2(t) &= e^{-a_1t}(C_1 \cos \omega_{d1}t + C_2 \sin \omega_{d1}t) \\ &\quad + e^{-a_2t}(C_3 \cos \omega_{d2}t + C_4 \sin \omega_{d2}t), \end{aligned} \quad (6.70)$$

where

$$\begin{aligned} r_1C_1 &= C_1\alpha_1 + C_2\beta_1 \\ r'_1C_2 &= -C_1\beta_1 + C_2\alpha_1 \\ r_2C_3 &= C_3\alpha_2 + C_4\beta_2 \\ r'_2C_4 &= -C_3\beta_2 + C_4\alpha_2 \end{aligned}$$

are real numbers. Equations 6.69 and 6.70 can be compared to their counterparts for undamped vibration in Equation 6.33. They differ in the same way that a damped oscillator differs from the undamped one, that is, a phasing term is introduced. In matrix-vector form we have

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \\ = e^{-a_1t} \left[\begin{Bmatrix} r_1C_1 \\ C_1 \end{Bmatrix} \cos \omega_{d1}t + \begin{Bmatrix} r'_1C_2 \\ C_2 \end{Bmatrix} \sin \omega_{d1}t \right] \\ + e^{-a_2t} \left[\begin{Bmatrix} r_2C_3 \\ C_3 \end{Bmatrix} \cos \omega_{d2}t + \begin{Bmatrix} r'_2C_4 \\ C_4 \end{Bmatrix} \sin \omega_{d2}t \right]. \end{aligned}$$

We also have four amplitude ratios that characterize the damped case, whereas only two ratios were needed in Equation 6.33 for the undamped modes of vibration. The two additional ratios permit us to evaluate the *phasing difference* between respective complex conjugate pairs in the solutions for $x_1(t)$ and $x_2(t)$.

The first complex conjugate pair in the solution for $x_1(t)$ is not in phase with the respective complex conjugate pair in the solution for $x_2(t)$. This can be observed by using the phase angle form for Equations 6.69 and 6.70:

$$\begin{aligned} x_1(t) &= B'_1e^{-a_1t} \cos(\omega_{d1}t - \phi'_{d1}) \\ &\quad + B'_2e^{-a_2t} \cos(\omega_{d2}t - \phi'_{d2}) \end{aligned} \quad (6.71)$$

$$\begin{aligned} x_2(t) &= B_1e^{-a_1t} \cos(\omega_{d1}t - \phi_{d1}) \\ &\quad + B_2e^{-a_2t} \cos(\omega_{d2}t - \phi_{d2}), \end{aligned} \quad (6.72)$$

where

$$\begin{aligned} B_1 &= \sqrt{C_1^2 + C_2^2}, & B_2 &= \sqrt{C_3^2 + C_4^2} \\ B'_1 &= B_1 \sqrt{\alpha_1^2 + \beta_1^2}, & B'_2 &= B_2 \sqrt{\alpha_2^2 + \beta_2^2} \\ \phi_{d1} &= \arctan\left(\frac{C_2}{C_1}\right), & \phi_{d2} &= \arctan\left(\frac{C_4}{C_3}\right) \\ \phi'_{d1} &= \arctan\left(\frac{r'_1 C_2}{r_1 C_1}\right), & \phi'_{d2} &= \arctan\left(\frac{r'_2 C_4}{r_2 C_3}\right). \end{aligned}$$

Thus, principal modes that are in phase do not generally exist for a damped system of the type considered here. The natural modes that do exist have a phase relationship, that is *out of phase* as shown above.

Very Low Viscous Damping

Some simplifications are possible if certain assumptions can be made regarding the damping. Suppose that a structure has *very low viscous damping*. Then the characteristic equation will be very close to that for the undamped case, and the following approximations seem reasonable,

$$\begin{aligned} \omega_{d1} &\simeq \omega_1, & r'_1 &\simeq r_1 \\ \omega_{d2} &\simeq \omega_2, & r'_2 &\simeq r_2, \end{aligned}$$

with the displacement responses approximated by

$$\begin{aligned} x_1(t) &\simeq r_1 e^{-a_1 t} (C_1 \cos \omega_1 t + C_2 \sin \omega_1 t) \\ &\quad + r_2 e^{-a_2 t} (C_3 \cos \omega_2 t + C_4 \sin \omega_2 t) \\ x_2(t) &\simeq e^{-a_1 t} (C_1 \cos \omega_1 t + C_2 \sin \omega_1 t) \\ &\quad + e^{-a_2 t} (C_3 \cos \omega_2 t + C_4 \sin \omega_2 t), \end{aligned}$$

where C_1 through C_4 are constants that depend on the initial conditions. In this case, there will be negligible phase difference between the modes.

Very High Viscous Damping

For a structure with *very high viscous damping*, there are two possibilities. The first is that all the roots, λ_1 , λ_2 , λ_3 , and λ_4 , of the characteristic equation are real and negative. As in the single degree-of-freedom case, the response is *not oscillatory* but a decaying exponential,

$$\begin{aligned} x_1(t) &\simeq r_1 D_1 e^{-\lambda_1 t} + r_2 D_2 e^{-\lambda_2 t} \\ &\quad + r_3 D_3 e^{-\lambda_3 t} + r_4 D_4 e^{-\lambda_4 t} \\ x_2(t) &\simeq D_1 e^{-\lambda_1 t} + D_2 e^{-\lambda_2 t} \\ &\quad + D_3 e^{-\lambda_3 t} + D_4 e^{-\lambda_4 t}, \end{aligned}$$

where $D_1, \dots, D_4$ and $r_1, \dots, r_4$ are real.

The second possibility is that two of the four roots are real and negative, and the other two roots are complex

conjugates with negative real parts. This solution has the form

$$\begin{aligned} x_1(t) &\simeq e^{-at} (r_1 C_1 \cos \omega_d t + r'_1 C_2 \sin \omega_d t) \\ &\quad + r_3 C_3 e^{-\lambda_3 t} + r_4 C_4 e^{-\lambda_4 t} \\ x_2(t) &\simeq e^{-at} (C_1 \cos \omega_d t + C_2 \sin \omega_d t) \\ &\quad + C_3 e^{-\lambda_3 t} + C_4 e^{-\lambda_4 t}. \end{aligned}$$

These solutions show an exponentially decaying oscillation superimposed with a purely exponentially decaying response.

Two numerical examples are studied next to better understand the solution procedure, the meaning of phasing, and how it comes into a response.

Example 6.11 Direct Method: Damped, Free Vibration, Case I

For the system of Figure 6.33, the system parameters are given as $m_1 = m_2 = 1$ kg, $c_1 = c_2 = 1$ N-s/m, and $k_1 = k_2 = 1$ N/m. The equation of motion in matrix-vector form is then

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \\ + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \end{aligned}$$

Solve for the displacement response using the direct method.

Solution: Assuming the following solution,

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A \\ B \end{Bmatrix} e^{\lambda t}$$

and taking the first and second derivatives and substituting these into the equation of motion, we find

$$\begin{aligned} \lambda^2 [M] \begin{Bmatrix} A \\ B \end{Bmatrix} e^{\lambda t} + \lambda [C] \begin{Bmatrix} A \\ B \end{Bmatrix} e^{\lambda t} \\ + [K] \begin{Bmatrix} A \\ B \end{Bmatrix} e^{\lambda t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ [\lambda^2 [M] + \lambda [C] + [K]] \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \end{aligned}$$

Substituting the mass, damping, and stiffness matrices leads to the characteristic matrix equation,

$$\begin{bmatrix} \lambda^2 + 2\lambda + 2 & -(\lambda + 1) \\ -(\lambda + 1) & \lambda^2 + \lambda + 1 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.73)$$

The determinant of the characteristic matrix is set equal to zero, leading to the characteristic equation,

$$\lambda^4 + 3\lambda^3 + 4\lambda^2 + 2\lambda + 1 = 0,$$

that must be solved numerically.³⁴ The roots are

$$\begin{aligned}\lambda_{11} &= -0.191 + 0.588i \\ \lambda_{12} &= -0.191 - 0.588i \\ \lambda_{21} &= -1.309 + 0.951i \\ \lambda_{22} &= -1.309 - 0.951i.\end{aligned}$$

Let us go through the steps used in proceeding from the value of λ_{11} to that of r_{11} . The same steps would need to be followed for each root. Substituting λ_{11} into the first equation of matrix Equation 6.73 we find

$$(\lambda_{11}^2 + 2\lambda_{11} + 2)A_{11} - (\lambda_{11} + 1)B_{11} = 0,$$

or

$$\begin{aligned}\frac{A_{11}}{B_{11}} &= r_{11} = \frac{\lambda_{11} + 1}{\lambda_{11}^2 + 2\lambda_{11} + 2} \\ &= \frac{0.809 + 0.588i}{1.309 + 0.951i},\end{aligned}\quad (6.74)$$

where A_{11} , B_{11} are A , B , respectively, for λ_{11} .

Multiplying the top and bottom of Equation 6.74 by the complex conjugate of its denominator results in

$$r_{11} = 0.618,$$

where a small imaginary part may remain because of round-off error. The eigenvector in this case (not needed in this solution) can be written as $\{0.618 \ 1\}^T$. Similarly, we have

$$\begin{aligned}\frac{A_{22}}{B_{22}} &= r_{22} = \frac{\lambda_{22} + 1}{\lambda_{22}^2 + 2\lambda_{22} + 2} \\ &= -1.618.\end{aligned}$$

The eigenvector in this case is $\{-1.618 \ 1\}^T$. Figure 6.34 is a sketch of these non-complex modes. There are two distinct eigenvalues.³⁵

What does it mean physically that the imaginary parts drop out completely? It means that there is no phasing in the modes, and the system behaves as though there is no damping. Since we do have damping, the damping matrix is linearly related to the $[M]$ and $[K]$ matrices. This is a special type of damping known as *proportional damping*. We will learn more about it in Section 6.6.3.

³⁴Using a program such as MATLAB makes it easy to find the roots in the general form $\lambda_{jk} = -a_j \pm i\omega_{d_j}$.

³⁵Had these ratios included imaginary parts, then r_{11} and r_{12} would have been complex conjugates, as would have been r_{21} and r_{22} . In general, the ratios r_{ij} are complex numbers. See Equations 6.67 and 6.68. The next example will show such a case.

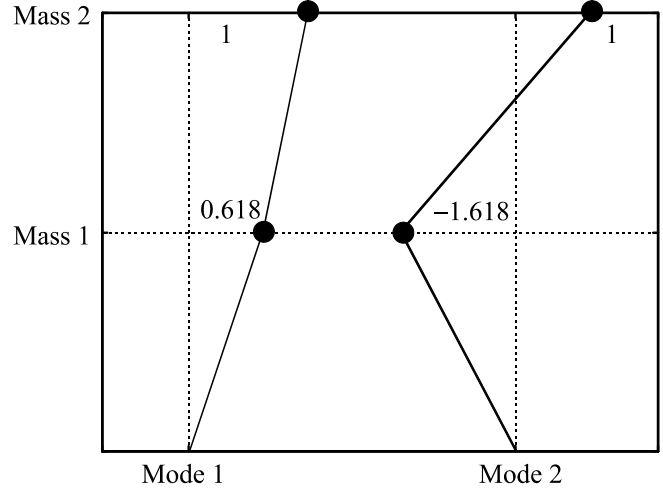


Figure 6.34: Case I modes. While we might have expected complex modes, the imaginary parts of the modal vectors equal zero.

Using the notation introduced earlier,

$$a_1 = 1.309, \quad \omega_{d1} = 0.951; \quad a_2 = 0.191, \quad \omega_{d2} = 0.588,$$

and

$$\alpha_1 = -1.618, \quad \beta_1 = 0; \quad \alpha_2 = 0.618, \quad \beta_2 = 0.$$

These parameters are substituted into the general Equations 6.69 and 6.70. In order to ensure that the procedure is clear, β_1 and β_2 will be retained even though they are equal to zero,

$$\begin{aligned}x_1(t) &= e^{-1.309t} ([C_1(-1.618) + C_2(\beta_1)] \cos 0.951t \\ &\quad + [-C_1(\beta_1) + C_2(-1.618)] \sin 0.951t) \\ &\quad + e^{-0.191t} ([C_3(0.618) + C_4(\beta_2)] \cos 0.588t \\ &\quad + [-C_3(\beta_2) + C_4(0.618)] \sin 0.588t).\end{aligned}$$

Simplifying and repeating similarly for $x_2(t)$, we obtain the equations governing the motion of the two masses,

$$\begin{aligned}x_1(t) &= -1.618e^{-1.309t}(C_1 \cos 0.951t + C_2 \sin 0.951t) \\ &\quad + 0.618e^{-0.191t}(C_3 \cos 0.588t + C_4 \sin 0.588t)\end{aligned}\quad (6.75)$$

$$\begin{aligned}x_2(t) &= e^{-1.309t}(C_1 \cos 0.951t + C_2 \sin 0.951t) \\ &\quad + e^{-0.191t}(C_3 \cos 0.588t + C_4 \sin 0.588t).\end{aligned}\quad (6.76)$$

The constants of integration, C_i , are obtained by satisfying the initial conditions. For initial conditions $x_1(0) = 1$, $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$, we find $C_1 = -0.447$ and $C_3 = 0.447$. The initial velocities can be used to find the other two constants. The responses of Equations 6.75 and 6.76 are plotted in Figure 6.35 for these initial conditions.

It turns out in this example that the imaginary parts of the modal vectors equal zero. In the next example,

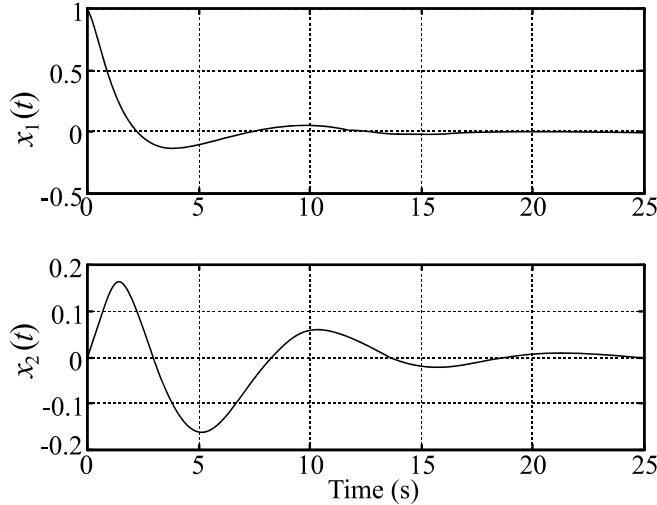


Figure 6.35: Case I. Displacement response time histories.

a more general damping case is considered where the ratios r_{ij} are fully complex conjugate. ■

Example 6.12 Direct Method: Damped, Free Vibration, Case II

We now re-examine the two degree-of-freedom system of [Example 6.11](#) with different property matrices $[M]$, $[C]$, and $[K]$. Here the matrix-vector equation of motion is

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 7 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Assuming the following solution,

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{\lambda t},$$

taking the first and second derivatives of this vector equation and substituting these terms into the equation of motion, we find

$$\begin{aligned} \lambda^2 [M] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{\lambda t} + \lambda [C] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{\lambda t} + [K] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{\lambda t} &= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ [\lambda^2 [M] + \lambda [C] + [K]] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} &= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \end{aligned}$$

To derive the characteristic equation, we set the determinant of the resulting matrix in the square brackets to

zero,

$$\left| \lambda^2 \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} + \lambda \begin{bmatrix} 4 & -3 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 7 & -3 \\ -3 & 3 \end{bmatrix} \right| = 0$$

$$\left| \begin{array}{cc} 2\lambda^2 + 4\lambda + 7 & -3\lambda - 3 \\ -3\lambda - 3 & 4\lambda^2 + 3\lambda + 3 \end{array} \right| = 0.$$

The characteristic equation is found to be

$$8\lambda^4 + 22\lambda^3 + 37\lambda^2 + 15\lambda + 12 = 0,$$

which gives the following four roots,

$$\lambda_{11} = -1.260 + 1.430i$$

$$\lambda_{12} = -1.260 - 1.430i$$

$$\lambda_{21} = -0.115 + 0.632i$$

$$\lambda_{22} = -0.115 - 0.632i.$$

These roots can be used to generate the complex modes by substituting each value of λ_{ij} into the equation

$$\begin{bmatrix} 2\lambda^2 + 4\lambda + 7 & -3\lambda - 3 \\ -3\lambda - 3 & 4\lambda^2 + 3\lambda + 3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

For example, substituting λ_{11} into the top equation we find

$$(2\lambda_{11}^2 + 4\lambda_{11} + 7) X_1 + (-3\lambda_{11} - 3) X_2 = 0.$$

Then,

$$\frac{X_{11}}{X_{21}} = r_{11} = \frac{3\lambda_{11} + 3}{2\lambda_{11}^2 + 4\lambda_{11} + 7}.$$

At this point we make the substitution $\lambda_{11} = -1.260 + 1.430i$ and then multiply the numerator and denominator by the complex conjugate of the denominator in order to eliminate the fraction. Doing this we find

$$r_{11} = -2.178 + 1.004i.$$

Following the same procedure for the other roots, we can solve for the other three complex modes,

$$r_{12} = -2.178 - 1.004i$$

$$r_{21} = 0.511 + 0.131i$$

$$r_{22} = 0.511 - 0.131i,$$

giving two pairs of complex conjugate eigenvectors. Given the values of λ_{ij} and r_{ij} , the following parameters can be calculated,

$$a_1 = 1.260, \quad \omega_{d1} = 1.430; \quad a_2 = 0.115, \quad \omega_{d2} = 0.632,$$

and

$$\alpha_1 = -2.178, \quad \beta_1 = 1.004; \quad \alpha_2 = 0.511, \quad \beta_2 = 0.131.$$

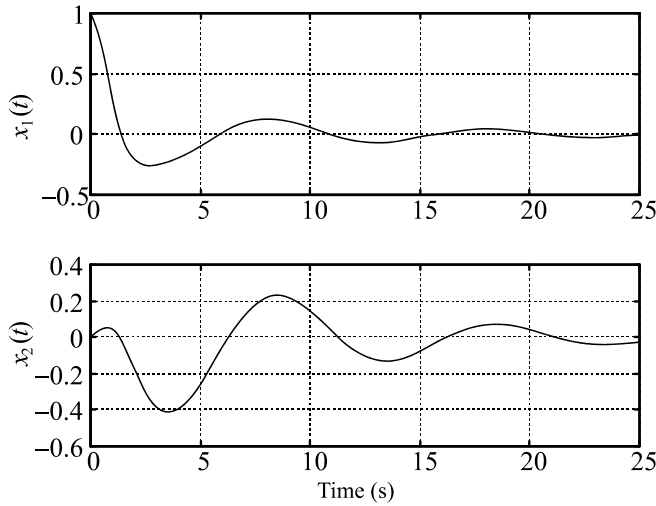


Figure 6.36: Case II. Displacement response time histories.

Assuming $x_1(0) = 1$ with all other initial conditions set equal to zero, the solutions can be found using Equations 6.69 and 6.70,

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = e^{-1.260t} \left[\begin{Bmatrix} -2.178C_1 + 1.004C_2 \\ C_1 \end{Bmatrix} \cos 1.430t + \begin{Bmatrix} -2.178C_2 - 1.004C_1 \\ C_2 \end{Bmatrix} \sin 1.430t \right] + e^{-0.115t} \left[\begin{Bmatrix} 0.511C_3 + 0.131C_4 \\ C_3 \end{Bmatrix} \cos 0.632t + \begin{Bmatrix} 0.511C_4 - 0.131C_3 \\ C_4 \end{Bmatrix} \sin 0.632t \right],$$

or, with the initial conditions,

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = e^{-1.260t} \left[\begin{Bmatrix} 0.820 \\ -0.460 \end{Bmatrix} \cos 1.430t + \begin{Bmatrix} 0.860 \\ -0.182 \end{Bmatrix} \sin 1.430t \right] \quad (6.77)$$

$$+ e^{-0.115t} \left[\begin{Bmatrix} 0.180 \\ 0.460 \end{Bmatrix} \cos 0.632t + \begin{Bmatrix} -0.276 \\ -0.422 \end{Bmatrix} \sin 0.632t \right]. \quad (6.78)$$

The response time histories are shown in Figure 6.36. The complex modes r_{ij} are plotted in the complex plane shown in Figure 6.37, which shows the complex modes as vectors in the complex plane. The complex

mode represents a relative amplitude and a phase. The introduction of a phase implies that there is relative motion between the modes. They are not synchronous. If the modes are real, as in the last example, they appear as vectors along the real axis only and undergo synchronous motion. The angle the complex vector makes with the real axis is representative of the phase. ■

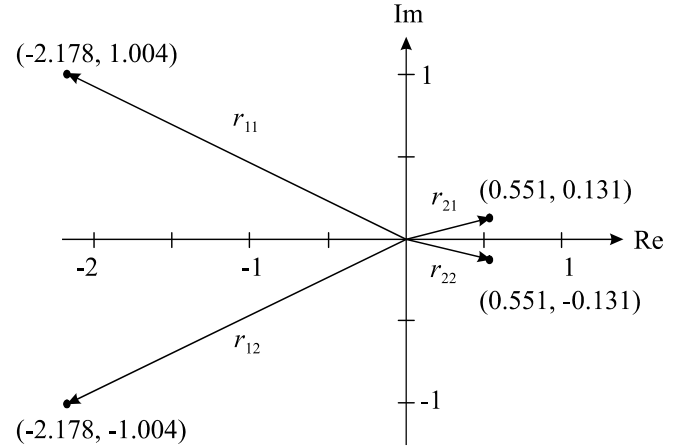


Figure 6.37: Case II. The complex modes in the complex plane.

6.6 Modal Analysis

Modal analysis is a procedure by which the orthogonality properties of the modes of vibration are utilized to transform the equations of motion from a coupled physical coordinate system to a *principal coordinate* system where the equations of motion are decoupled. Rather than solving the n original simultaneous ordinary differential equations, each of the decoupled equations representing single oscillators can be solved independently. This procedure is valid for undamped systems as well as *proportionally damped* systems.

The reason such a principal coordinate system exists is because the modes have certain orthogonality properties with respect to the mass and stiffness matrices. We examine these orthogonality properties first.

6.6.1 Modal Orthogonality

Orthogonality is a property that indicates that two vectors are perpendicular. The modes possess special orthogonality relations with respect to the mass and stiffness matrices.

Rather than prove the orthogonality properties in general, we demonstrate these properties for the two degree-of-freedom system of Section 6.4.1 using the modal ratios

from Equations 6.30 and 6.31. We recall the modal vectors

$$\begin{aligned}\{u\}_1 &= \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = u_{11} \begin{Bmatrix} 1 \\ u_{21}/u_{11} \end{Bmatrix} \\ &= u_{11} \begin{Bmatrix} 1 \\ -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} \end{Bmatrix} \\ \{u\}_2 &= \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = u_{12} \begin{Bmatrix} 1 \\ u_{22}/u_{12} \end{Bmatrix} \\ &= u_{12} \begin{Bmatrix} 1 \\ -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} \end{Bmatrix},\end{aligned}$$

where ω_1 and ω_2 are the natural frequencies, and only elastic (stiffness) coupling is included to simplify the demonstration. We first form the matrix product

$$\begin{aligned}\{u\}_2^T [M] \{u\}_1 &= u_{11} u_{12} \begin{Bmatrix} 1 \\ -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} \end{Bmatrix}^T \\ &\quad \times \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} \end{Bmatrix} \\ &= u_{11} u_{12} [m_1 + \frac{m_2}{k_{12}^2} (k_{11} - \omega_1^2 m_1)(k_{11} - \omega_2^2 m_1)],\end{aligned}$$

into which Equations 6.26 for ω_1 and ω_2 are substituted. After some matrix algebra³⁶ we find

$$\{u\}_2^T [M] \{u\}_1 = u_{11} u_{12} \cdot 0 = 0,$$

which shows the *orthogonality of the modes of vibration with respect to the mass matrix*. Using Equation 6.29, $\omega_1^2 [M] \{u\}_1 = [K] \{u\}_1$, and pre-multiplying by $\{u\}_2^T$,

$$\omega_1^2 \{u\}_2^T [M] \{u\}_1 = \{u\}_2^T [K] \{u\}_1,$$

we find

$$\{u\}_2^T [K] \{u\}_1 = 0, \quad (6.79)$$

that is, *the modes are orthogonal with respect to the stiffness matrix* as well.

Consider the matrix product, $\{u\}_1^T [M] \{u\}_1$, which will equal a constant when evaluated. It is customary to define a normalized mode $\{\hat{u}\}$ so that the matrix product equals one. To do this, we multiply the mode by a constant, say a , such that $\{\hat{u}\} = a\{u\}$. Then, for mode i , we force the equality to one,

$$a_i^2 \{u\}_i^T [M] \{u\}_i = 1,$$

³⁶It is instructive to proceed through the matrix manipulations of the type encountered in modal analysis. Also, the end result will be the same for a fully populated mass matrix rather than the diagonal mass matrix.

from which the necessary value of a_i can be found, resulting in

$$\{\hat{u}\}_i^T [M] \{\hat{u}\}_i = 1, \quad i = 1, 2, \quad (6.80)$$

and using Equation 6.29 for normal mode i , and pre-multiplying by $\{\hat{u}\}_i^T$, we find

$$\omega_i^2 \{\hat{u}\}_i^T [M] \{\hat{u}\}_i = \{\hat{u}\}_i^T [K] \{\hat{u}\}_i,$$

or

$$\{\hat{u}\}_i^T [K] \{\hat{u}\}_i = \omega_i^2, \quad i = 1, 2. \quad (6.81)$$

These modal orthogonality and normalization properties are needed to decouple the equations of motion as part of a modal analysis. The physical coordinates, $x_1(t)$ and $x_2(t)$, are transformed into uncoupled *modal coordinates* $q_1(t)$ and $q_2(t)$ that are yet to be defined. The notation $q_1(t)$ and $q_2(t)$ is customary for the modal coordinates, and also matches the notation for the generalized coordinates of Chapter 5. These coordinates are also known as *natural* or *principal* coordinates and are discussed in a physical sense subsequently.

Consider a simple problem that demonstrates the procedure hinted at above. We begin with the general matrix equation of free, undamped motion,

$$[M] \{\ddot{x}\} + [K] \{x\} = \{0\}, \quad (6.82)$$

for a system with only two degrees of freedom. We write the response $\{x(t)\}$ in terms of the known normal modes,

$$\{x(t)\} = \{\hat{u}\}_1 q_1(t) + \{\hat{u}\}_2 q_2(t) = \sum_{i=1}^2 \{\hat{u}\}_i q_i(t), \quad (6.83)$$

where the modal or normal coordinates $q_1(t)$ and $q_2(t)$ are unknown modal displacements that are found in the following modal analysis solution. We note $q(t)$ and $x(t)$ have the same units. The frequencies and modes have already been evaluated in an eigenvalue analysis using the property matrices $[M]$ and $[K]$ in the characteristic matrix equation $[k - \omega^2 m] \{u\} = \{0\}$.

Equation 6.83, which relates the physical coordinates $x_i(t)$ to the modal coordinates $q_i(t)$, is a reduced version of the general expression known as the *expansion theorem*,

$$\{x(t)\} = \sum_{i=1}^n \{\hat{u}\}_i q_i(t).$$

From matrix theory, this is the expansion of function $\{x(t)\}$ in terms of orthogonal functions $\{\hat{u}\}_i$. Equation 6.83 can be written in matrix form to show the generality of the procedure for n degrees of freedom,

$$\{x(t)\} = [P] \{q(t)\}, \quad (6.84)$$

where $[P]$ is called the normalized *modal matrix*. For a two degree-of-freedom system, $[P]$ is the 2×2 *modal matrix*

$$[P] = [\{\hat{u}\}_1 \ \{\hat{u}\}_2]. \quad (6.85)$$

The braces are included to emphasize that $[P]$ is built of modal vectors.

We differentiate and substitute Equation 6.83 into governing Equation 6.82 to find

$$[M](\{\hat{u}\}_1 \ddot{q}_1 + \{\hat{u}\}_2 \ddot{q}_2) + [K](\{\hat{u}\}_1 q_1 + \{\hat{u}\}_2 q_2) = \{0\}. \quad (6.86)$$

Pre-multiplying by the transpose of the first modal vector $\{\hat{u}\}_1^T$ creates the matrix triple products that possess the orthogonality properties needed to decouple the equations of motion,

$$\begin{aligned} \{\hat{u}\}_1^T [M](\{\hat{u}\}_1 \ddot{q}_1 + \{\hat{u}\}_2 \ddot{q}_2) \\ + \{\hat{u}\}_1^T [K](\{\hat{u}\}_1 q_1 + \{\hat{u}\}_2 q_2) = 0. \end{aligned}$$

Orthogonality of the modes with respect to the mass and stiffness matrices reduces this equation to

$$\{\hat{u}\}_1^T [M]\{\hat{u}\}_1 \ddot{q}_1 + \{\hat{u}\}_1^T [K]\{\hat{u}\}_1 q_1 = 0. \quad (6.87)$$

The normalization relations $\{\hat{u}\}_r^T [M]\{\hat{u}\}_r = 1$ and $\{\hat{u}\}_r^T [K]\{\hat{u}\}_r = \omega_r^2$ further reduce Equation 6.87 to the equation for the first modal coordinate,

$$\ddot{q}_1 + \omega_1^2 q_1 = 0. \quad (6.88)$$

Similarly, pre-multiplying Equation 6.86 by the transpose of the second modal vector $\{\hat{u}\}_2^T$ leads to the decoupled equation of motion for the second modal coordinate $q_2(t)$,

$$\ddot{q}_2 + \omega_2^2 q_2 = 0. \quad (6.89)$$

Thus, the orthogonality properties of the modes transform the coupled equations of motion in physical space into uncoupled equations in modal space.

Equations 6.88 and 6.89 are then solved for $q_1(t)$ and $q_2(t)$, respectively, obtaining

$$q_i(t) = C_i \cos(\omega_i t - \phi_i), \quad i = 1, 2. \quad (6.90)$$

Inserting Equation 6.90 into either of Equations 6.83 or 6.84 we recover the physical response,

$$\begin{aligned} \{x(t)\} &= \sum_{i=1}^2 \{\hat{u}\}_i q_i(t) = [P]\{q(t)\} \\ &= [\{\hat{u}\}_1 \ \{\hat{u}\}_2] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \\ &= \sum_{i=1}^2 \{\hat{u}\}_i C_i \cos(\omega_i t - \phi_i) \\ &= \sum_{i=1}^2 C_i \{\hat{u}\}_i [\cos \omega_i t \cos \phi_i + \sin \omega_i t \sin \phi_i]. \end{aligned} \quad (6.91)$$

Given the initial conditions $\{x(0)\}$ and $\{\dot{x}(0)\}$, Equation 6.91 can be used to evaluate the arbitrary constants C_i and ϕ_i , as follows,

$$\{x(0)\} = \sum_{i=1}^2 C_i \{\hat{u}\}_i \cos \phi_i \quad (6.92)$$

$$\{\dot{x}(0)\} = \sum_{i=1}^2 C_i \omega_i \{\hat{u}\}_i \sin \phi_i. \quad (6.93)$$

Pre-multiplying Equations 6.92 and 6.93 by $\{\hat{u}\}_j^T [M]$, $j = 1, 2$, we find

$$\begin{aligned} \{\hat{u}\}_j^T [M]\{x(0)\} &= \sum_{i=1}^2 C_i \{\hat{u}\}_j^T [M]\{\hat{u}\}_i \cos \phi_i \\ \{\hat{u}\}_j^T [M]\{\dot{x}(0)\} &= \sum_{i=1}^2 C_i \omega_i \{\hat{u}\}_j^T [M]\{\hat{u}\}_i \sin \phi_i, \end{aligned}$$

from which, by orthogonality, the only nonzero terms on the right-hand sides are for $j = i$. To evaluate C and ϕ for each degree of freedom the two equations

$$\begin{aligned} \{\hat{u}\}_i^T [M]\{x(0)\} &= C_i \cos \phi_i, \quad i = 1, 2, \\ \frac{\{\hat{u}\}_i^T [M]\{\dot{x}(0)\}}{\omega_i} &= C_i \sin \phi_i, \quad i = 1, 2, \end{aligned}$$

must be solved. Substituting these two expressions into Equation 6.91 we obtain the complete free vibration response,

$$\begin{aligned} \{x(t)\} &= \sum_{i=1}^2 \{\hat{u}\}_i (\{\hat{u}\}_i^T [M]\{x(0)\} \cos \omega_i t \\ &\quad + \frac{1}{\omega_i} \{\hat{u}\}_i^T [M]\{\dot{x}(0)\} \sin \omega_i t), \end{aligned} \quad (6.94)$$

which has been found using a modal analysis approach.³⁷

It is informative to consider the solution Equation 6.94 with the following initial conditions:

$$\begin{aligned} \{\dot{x}(0)\} &= \{0\} \\ \{x(0)\} &= x_0 \{\hat{u}\}_s, \end{aligned}$$

where the initial displacement is related linearly to the s^{th} normal mode. Then,

$$\begin{aligned} \{x(t)\} &= \sum_{i=1}^2 x_0 \{\hat{u}\}_i [\{\hat{u}\}_i^T [M]\{\hat{u}\}_s] \cos \omega_i t \\ &= \sum_{i=1}^2 x_0 \{\hat{u}\}_i [\delta_{is}] \cos \omega_i t \\ &= x_0 \{\hat{u}\}_s \cos \omega_s t, \end{aligned}$$

³⁷From linear algebra vector-matrix multiplication is not associative and therefore in Equation 6.94 we cannot move the factor $\{\hat{u}\}_i$ arbitrarily inside the parenthesis within the matrix products to obtain triple products such as $\{\hat{u}\}_i^T [m]\{\hat{u}\}_i$.

where the Kronecker delta $\delta_{is} = 1$ for $i = s$ and is zero otherwise. The solution represents synchronous harmonic oscillation at the natural frequency ω_s ; the system configuration resembles the s^{th} mode at all times. Thus, any of the normal modes can be excited independently. (This can occur only in a linear system.)

Modal analysis works because *the mass and stiffness matrices* can be *diagonalized simultaneously*. Rather than solve n simultaneous equations of motion, modal analysis involves a series of matrix multiplications and additions, as we see by Equation 6.94, that decouple the governing equations.

Example 6.13 Free Vibration via Modal Analysis

Solve Example 6.7 using modal analysis assuming the initial conditions: $x_1(0) = 1$ cm, $x_2(0) = -1$ cm, $\dot{x}_1(0) = 3$ cm/s, and $\dot{x}_2(0) = -2$ cm/s.

Solution: The matrix-vector equation of motion is

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 5k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

with natural frequencies

$$\omega_1 = 1.14\sqrt{\frac{k}{m}}, \quad \omega_2 = 2.05\sqrt{\frac{k}{m}},$$

and respective modal ratios

$$\frac{u_{21}}{u_{11}} = 0.845$$

$$\frac{u_{22}}{u_{12}} = -0.595.$$

First, normalize the modes with respect to the mass matrix as follows. Let the normalized modes be

$$\{\hat{u}\}_1 = a_1\{u\}_1 = a_1 \begin{Bmatrix} 1.000 \\ 0.845 \end{Bmatrix}$$

$$\{\hat{u}\}_2 = a_2\{u\}_2 = a_2 \begin{Bmatrix} 1.000 \\ -0.595 \end{Bmatrix},$$

where a_1 and a_2 are the normalizing factors. Then

$$\{u\}_1^T [M] \{u\}_1 = \frac{1}{a_1^2},$$

and substituting the appropriate vectors and matrix,

$$\{1.000 \quad 0.845\} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1.000 \\ 0.845 \end{Bmatrix} = \frac{1}{a_1^2},$$

we can solve for a_1 , to find

$$a_1 = \frac{0.642}{\sqrt{m}}.$$

Following the same procedure for the second modal vector,

$$\{1.000 \quad -0.595\} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1.000 \\ -0.595 \end{Bmatrix} = \frac{1}{a_2^2},$$

results in the value

$$a_2 = \frac{0.765}{\sqrt{m}}.$$

The normalized modes are then

$$\{\hat{u}\}_1 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.642 \\ 0.542 \end{Bmatrix}, \quad \{\hat{u}\}_2 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.765 \\ -0.455 \end{Bmatrix}.$$

Now we have what we need to evaluate Equation 6.94:

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.642 \\ 0.542 \end{Bmatrix} \\ &\times \left(\frac{\{0.642 \quad 0.542\}}{\sqrt{m}} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos \omega_1 t \right. \\ &+ \frac{\{0.642 \quad 0.542\}}{\omega_1 \sqrt{m}} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 3 \\ -2 \end{Bmatrix} \sin \omega_1 t \Big) \\ &+ \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.765 \\ -0.455 \end{Bmatrix} \\ &\times \left(\frac{\{0.765 \quad -0.455\}}{\sqrt{m}} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos \omega_2 t \right. \\ &+ \frac{\{0.765 \quad -0.455\}}{\omega_2 \sqrt{m}} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 3 \\ -2 \end{Bmatrix} \sin \omega_2 t \Big). \end{aligned}$$

For a numerical solution, we need the values of k and m . If we take $k = 1$ N/m and $m = 1$ kg, then $\omega_1 = 1.14$ rad/s and $\omega_2 = 2.05$ rad/s, with the result,

$$\begin{aligned} x_1(t) &= -0.284 \cos 1.14t - 0.136 \sin 1.14t \\ &\quad + 1.281 \cos 2.05t + 1.536 \sin 2.05t \text{ cm} \\ x_2(t) &= -0.240 \cos 1.14t - 0.115 \sin 1.14t \\ &\quad - 0.762 \cos 2.05t - 0.914 \sin 2.05t \text{ cm}. \end{aligned}$$

The displacements are plotted in Figure 6.38. ■

The example above has been carried out to demonstrate a procedure, and may not be a realistic application with all its inherent difficulties. Modal analysis begins to be advantageous for a forced system, as will be shown in the next section. A discussion of the advantages of the modal approach follows in Section 6.8.1.

6.6.2 Modal Analysis with Forcing

We begin with the general matrix equation of forced, undamped motion,

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F(t)\}, \quad (6.95)$$

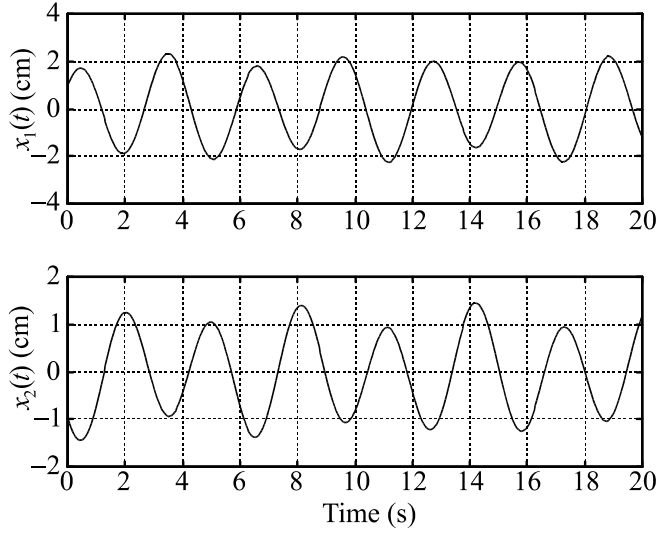


Figure 6.38: Response of two coupled masses.

and again consider a two degree-of-freedom system. The expansion of $\{x(t)\}$ as a function of the normal modes yields

$$\begin{aligned}\{x(t)\} &= \{\hat{u}\}_1 q_1(t) + \{\hat{u}\}_2 q_2(t) = \sum_{i=1}^2 \{\hat{u}\}_i q_i(t) \quad (6.96) \\ &= [P]\{q(t)\}.\end{aligned}$$

Substituting Equation 6.96 and its second derivative into Equation 6.95 leads to

$$[M](\{\hat{u}\}_1 \ddot{q}_1 + \{\hat{u}\}_2 \ddot{q}_2) + [K](\{\hat{u}\}_1 q_1 + \{\hat{u}\}_2 q_2) = \{F(t)\}. \quad (6.97)$$

Pre-multiplying each term by $\{\hat{u}\}_1^T$ we obtain

$$\begin{aligned}\{\hat{u}\}_1^T [M](\{\hat{u}\}_1 \ddot{q}_1 + \{\hat{u}\}_2 \ddot{q}_2) \\ + \{\hat{u}\}_1^T [K](\{\hat{u}\}_1 q_1 + \{\hat{u}\}_2 q_2) = \{\hat{u}\}_1^T \{F(t)\},\end{aligned}$$

where the two matrix triple products possess the orthogonality properties. Orthogonality of the modes with respect to the mass and stiffness matrices reduces this equation to

$$\{\hat{u}\}_1^T [M]\{\hat{u}\}_1 \ddot{q}_1 + \{\hat{u}\}_1^T [K]\{\hat{u}\}_1 q_1 = \{\hat{u}\}_1^T \{F(t)\},$$

and the use of normalization Equations 6.80 and 6.81 reduces this further to the equation governing the first modal coordinate,

$$\ddot{q}_1 + \omega_1^2 q_1 = N_1(t), \quad (6.98)$$

where $N_1(t) = \{\hat{u}\}_1^T \{F(t)\}$. Similarly, multiplying Equation 6.97 by $\{\hat{u}\}_2^T$ leads to the decoupled equation for the second modal coordinate,

$$\ddot{q}_2 + \omega_2^2 q_2 = N_2(t), \quad (6.99)$$

where $N_2(t) = \{\hat{u}\}_2^T \{F(t)\}$. Equations 6.98 and 6.99 are solved for $q_1(t)$ and $q_2(t)$, respectively,

$$\begin{aligned}q_i(t) &= \frac{1}{\omega_i} \int_0^t N_i(\tau) \sin \omega_i(t - \tau) d\tau \\ &+ q_i(0) \cos \omega_i t + \frac{\dot{q}_i(0)}{\omega_i} \sin \omega_i t, \quad i = 1, 2. \quad (6.100)\end{aligned}$$

The first term in the solution is the Duhamel integral. The second and third terms are the free responses due to the initial displacement and velocity.

The initial conditions in modal coordinates can be related to the physical initial conditions using Equation 6.96, and setting $t = 0$,

$$\begin{aligned}\{x(0)\} &= [P]\{q(0)\} = \sum_{i=1}^2 \{\hat{u}\}_i q_i(0) \quad (6.101) \\ \{\dot{x}(0)\} &= [P]\{\dot{q}(0)\} = \sum_{i=1}^2 \{\hat{u}\}_i \dot{q}_i(0).\end{aligned}$$

The relation for the initial displacements is

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}_1 q_1(0) + \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}_2 q_2(0).$$

Similarly, for the initial velocities,

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}_1 \dot{q}_1(0) + \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}_2 \dot{q}_2(0).$$

We now have four equations that can be solved for four unknowns: $q_1(0)$, $q_2(0)$, $\dot{q}_1(0)$, and $\dot{q}_2(0)$.

Rather than solve four simultaneous equations (or $2n$ simultaneous equations for an n degree-of-freedom system), the initial conditions in modal space can be found directly by pre-multiplying Equation 6.101 by $\{\hat{u}\}_j^T [M]$. Then,

$$\{\hat{u}\}_j^T [M]\{x(0)\} = \{\hat{u}\}_j^T [M] \sum_{i=1}^2 \{\hat{u}\}_i q_i(0), \quad j = 1, 2.$$

By orthogonality, only one term on the right-hand side will remain ($j = i$),

$$\{\hat{u}\}_i^T [M]\{x(0)\} = q_i(0).$$

Similarly, the result for the velocity equation is

$$\{\hat{u}\}_i^T [M]\{\dot{x}(0)\} = \dot{q}_i(0).$$

In this way all the modal initial conditions can be found. If we do not need to use all the modes in our analysis, we can find just the initial conditions we need, for example, $q_i(0)$ and $\dot{q}_i(0)$ for $i = 1, 2, 3$ if we use only the first three

modes to describe the behavior. (See the discussion on modal participation factor below for further discussion.)

Equation 6.100 can be solved for each modal degree of freedom and then substituted into Equation 6.96 to obtain the complete response in the physical coordinates, $\{x(t)\} = \{\hat{u}\}_1 q_1(t) + \{\hat{u}\}_2 q_2(t)$,

$$\begin{aligned} \{x(t)\} &= \sum_i^2 \{\hat{u}\}_i \left(\frac{1}{\omega_i} \int_0^t N_i(\tau) \sin \omega_i(t - \tau) d\tau \right. \\ &\quad \left. + q_i(0) \cos \omega_i t + \frac{\dot{q}_i(0)}{\omega_i} \sin \omega_i t \right) \\ &= \sum_i^2 \{\hat{u}\}_i \left(\frac{1}{\omega_i} \int_0^t N_i(\tau) \sin \omega_i(t - \tau) d\tau \right. \\ &\quad \left. + \{\hat{u}\}_i^T [M] \{x(0)\} \cos \omega_i t \right. \\ &\quad \left. + \{\hat{u}\}_i^T [M] \{\dot{x}(0)\} \frac{1}{\omega_i} \sin \omega_i t \right), \quad (6.102) \\ N_i(t) &= \sum_{j=1}^2 \hat{u}_{ji} F_j(t), \quad i = 1, 2. \end{aligned}$$

In Equation 6.102, the first term of the response is the steady-state component given by a Duhamel integral. The next two terms are the response components due to initial displacements and initial velocities, respectively. Since there is no damping in this model, there are no decaying terms.

Modal Participation Factor

When a structure is vibrating, each mode “participates” in the vibration to some extent. The *modal participation factor* indicates how much each mode contributes to the response. It is calculated for each translation direction as well as for rotation around each of those axes. For instance, a given mode may have significant participation in the x direction, but have very little participation in the other directions. This knowledge can be valuable in a numerical calculation involving many modes since those that have very low participation can be ignored, reducing computation time.

Figure 6.39 shows schematics of two possible loading combinations and the two modes of a two degree-of-freedom structure. For the first loading case, the two forces are acting in the same direction and will primarily excite the first mode of vibration. The second case, where the two forces act in opposite directions, will excite the second mode more than the first mode. The response to loading case one would be adequately modeled using only mode one. Similarly, for loading case two, the response would be primarily in the second mode.

Generalizing to a structure with hundreds of degrees of freedom, we can see that a structure will generally

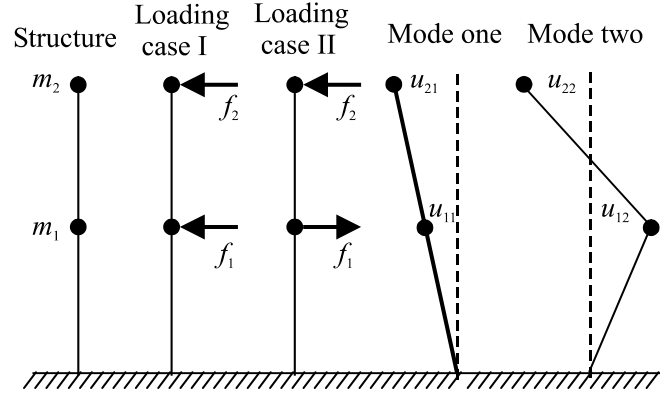


Figure 6.39: Modal participation: two loading cases and the two modes.

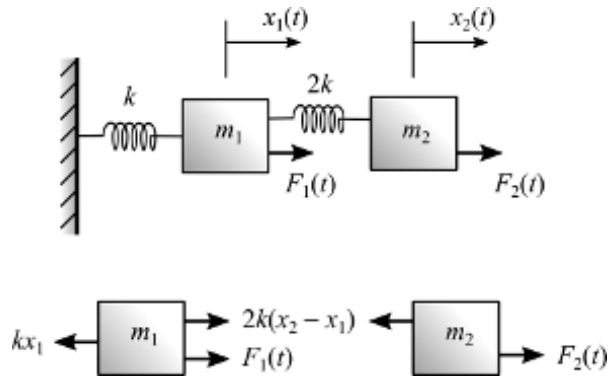


Figure 6.40: Two degree-of-freedom undamped system and free-body diagrams.

respond in the modes that resemble the loading. Such understanding can facilitate the development of more realistic models for particular loading cases. This concept is also applicable to continuous systems studied in Chapters 7 and 8.

Example 6.14 Forced Vibration Modal Analysis

Using modal analysis solve for the displacements in the two degree-of-freedom system without damping, shown in Figure 6.40.

Solution: The free-body diagrams, in conjunction with Newton’s second law of motion, lead to the two governing equations of motion,

$$\begin{aligned} F_1(t) - kx_1 + 2k(x_2 - x_1) &= m\ddot{x}_1 \\ F_2(t) - 2k(x_2 - x_1) &= m\ddot{x}_2, \end{aligned}$$

or, in matrix-vector form,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}.$$

First, we solve the free vibration problem for the natural frequencies and modes. For free vibration, we assume harmonic behavior,

$$x_1(t) = X_1 e^{-i\omega t}, \quad x_2(t) = X_2 e^{-i\omega t}.$$

Taking the second derivatives of these functions

$$\ddot{x}_1(t) = -X_1 \omega^2 e^{-i\omega t}, \quad \ddot{x}_2(t) = -X_2 \omega^2 e^{-i\omega t},$$

and substituting them into the matrix equation of motion, combining, and canceling the exponential function which appears in all terms, leads to

$$\begin{bmatrix} 3k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (6.103)$$

Setting the determinant of the characteristic matrix equal to zero leads to the characteristic equation,

$$\omega^4 - 5\frac{k}{m}\omega^2 + 2\left(\frac{k}{m}\right)^2 = 0.$$

This equation is a quadratic in ω^2 , with solutions

$$\omega_1^2 = 0.44\frac{k}{m}, \quad \omega_2^2 = 4.56\frac{k}{m}.$$

The natural frequencies are then

$$\omega_1 = 0.66\sqrt{k/m}, \quad \omega_2 = 2.14\sqrt{k/m}.$$

To obtain the modes, we substitute the squares of the natural frequencies into one row of matrix [Equation 6.103](#). From the first row, we can solve for the modal ratios, as follows,

$$\begin{aligned} \left(3k - m \cdot 0.44\frac{k}{m}\right) X_1 - 2kX_2 &= 0 \\ \left(\frac{X_2}{X_1}\right)_1 &= 1.28, \end{aligned}$$

and

$$\begin{aligned} \left(3k - m \cdot 4.56\frac{k}{m}\right) X_1 - 2kX_2 &= 0 \\ \left(\frac{X_2}{X_1}\right)_2 &= -0.78. \end{aligned}$$

The two modes, in standard notation, are

$$\{u\}_1 = \begin{Bmatrix} 1.00 \\ 1.28 \end{Bmatrix}, \quad \{u\}_2 = \begin{Bmatrix} 1.00 \\ -0.78 \end{Bmatrix}.$$

In order to proceed with a modal analysis and decouple the equations of motion, we must first normalize the

modes. For this we introduce constant multiples a_1 and a_2 in the modal vectors,

$$\{\hat{u}\}_1 = a_1 \begin{Bmatrix} 1.00 \\ 1.28 \end{Bmatrix}, \quad \{\hat{u}\}_2 = a_2 \begin{Bmatrix} 1.00 \\ -0.78 \end{Bmatrix}.$$

Normalizing the first mode requires that $\{\hat{u}\}_1^T [M] \{\hat{u}\}_1 = 1$. The procedure is applied for the first mode,

$$\begin{aligned} 1 &= a_1^2 \begin{Bmatrix} 1.00 \\ 1.28 \end{Bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.28 \end{Bmatrix} \\ 1 &= 2.64 a_1^2 m \\ a_1 &= \frac{0.62}{\sqrt{m}} \\ \{\hat{u}\}_1 &= \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.62 \\ 0.79 \end{Bmatrix}. \end{aligned}$$

Normalizing the second mode requires that $\{\hat{u}\}_2^T [M] \{\hat{u}\}_2 = 1$. The following procedure is used,

$$\begin{aligned} 1 &= a_2^2 \begin{Bmatrix} 1.00 \\ -0.78 \end{Bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1.00 \\ -0.78 \end{Bmatrix} \\ 1 &= 1.61 a_2^2 m \\ a_2 &= \frac{0.79}{\sqrt{m}} \\ \{\hat{u}\}_2 &= \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.79 \\ -0.62 \end{Bmatrix}. \end{aligned}$$

Next we take the external forcing to be of the form

$$\begin{aligned} F_1(t) &= A_1 u(t) \\ F_2(t) &= A_2 u(t), \end{aligned}$$

where $u(t)$ is the unit step function and A_1 and A_2 are known constants. To decouple the matrix equations of motion, we use the modal expansion [Equation 6.84](#), $\{x(t)\} = [P]\{q(t)\}$, where

$$[P] = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.62 & 0.79 \\ 0.79 & -0.62 \end{bmatrix}$$

is the normalized modal matrix. Substituting the modal expansion into the matrix equation of motion and pre-multiplying by $[P]^T$ leads to

$$[P]^T [M] [P] \{\ddot{q}\} + [P]^T [K] [P] \{q\} = [P]^T \{F(t)\}.$$

The equations are now decoupled since the matrix triple products are diagonal. In using the modal matrix $[P]$, all the equations are decoupled simultaneously rather than

one equation at a time as was done earlier. The right-hand side of the above equation can be expanded,

$$\begin{aligned}\{N(t)\} &\equiv [P]^T \{F(t)\} = [P]^T \begin{Bmatrix} A_1 u(t) \\ A_2 u(t) \end{Bmatrix} \\ &= \frac{1}{\sqrt{m}} \begin{bmatrix} 0.62 & 0.79 \\ 0.79 & -0.62 \end{bmatrix} \begin{Bmatrix} A_1 u(t) \\ A_2 u(t) \end{Bmatrix} \\ \begin{Bmatrix} N_1(t) \\ N_2(t) \end{Bmatrix} &= \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.62A_1 + 0.79A_2 \\ 0.79A_1 - 0.62A_2 \end{Bmatrix} u(t).\end{aligned}$$

The decoupled modal equations of motion are then

$$\begin{aligned}\ddot{q}_1 + \omega_1^2 q_1 &= N_1(t) \\ \ddot{q}_2 + \omega_2^2 q_2 &= N_2(t).\end{aligned}$$

The solution of each is the undamped Duhamel integral plus terms due to the initial conditions. Assuming zero initial conditions,

$$q_i(t) = \frac{1}{\omega_i} \int_0^t N_i(\tau) \sin \omega_i(t - \tau) d\tau, \quad i = 1, 2.$$

Therefore, the modal solution is

$$\begin{aligned}q_1(t) &= \frac{1}{\omega_1} \frac{1}{\sqrt{m}} (0.62A_1 + 0.79A_2) \int_0^t u(\tau) \sin \omega_1(t - \tau) d\tau \\ &= \frac{(0.62A_1 + 0.79A_2)}{\omega_1 \sqrt{m}} \frac{(1 - \cos \omega_1 t)}{\omega_1},\end{aligned}\quad (6.104)$$

$$\begin{aligned}q_2(t) &= \frac{1}{\omega_2} \frac{1}{\sqrt{m}} (0.79A_1 - 0.62A_2) \int_0^t u(\tau) \sin \omega_2(t - \tau) d\tau \\ &= \frac{(0.79A_1 - 0.62A_2)}{\omega_2 \sqrt{m}} \frac{(1 - \cos \omega_2 t)}{\omega_2}.\end{aligned}\quad (6.105)$$

To complete the solution, we need to transform the modal solution back to physical coordinates using $\{x(t)\} = [P]\{q(t)\}$,

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.62 & 0.79 \\ 0.79 & -0.62 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}.$$

The forced response for each mass is

$$x_1(t) = \frac{1}{\sqrt{m}} [0.62q_1(t) + 0.79q_2(t)] \quad (6.106)$$

$$x_2(t) = \frac{1}{\sqrt{m}} [0.79q_1(t) - 0.62q_2(t)], \quad (6.107)$$

where $q_1(t)$ and $q_2(t)$ are given by Equations 6.104 and 6.105. We can easily identify the contribution of each mode to the total motion of each mass. If the initial conditions above are not zero, then Equations 6.106 and

6.107 would be added to the free vibration response as was done in Equation 6.100, resulting in the complete response. ■

We next examine the possibility of using modal analysis for a system with damping.

6.6.3 Modal Analysis with Proportional Damping

Viscous damping can be incorporated directly into modal analysis if it is *proportional damping* – that is, if the damping matrix is proportional to a linear combination of the mass and stiffness matrices, $[C] \sim [M] + [K]$. In this case, modal analysis of a damped system leads to a decoupled set of governing equations as well. Otherwise, the first derivative damping related terms prevent such a decoupling and the direct method of Section 6.5 or other related procedures become necessary. The physical meaning of this damping model is discussed at the end of this section.

We begin with the general matrix equation of forced, damped motion,

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}, \quad (6.108)$$

and consider a two degree-of-freedom system with a two-term solution,

$$\{x(t)\} = [P]\{q(t)\} = \sum_{i=1}^2 \{\hat{u}\}_i q_i(t). \quad (6.109)$$

Differentiating and substituting Equation 6.109 into governing Equation 6.108, we find

$$\begin{aligned}[M](\{\hat{u}\}_1 \ddot{q}_1 + \{\hat{u}\}_2 \ddot{q}_2) + [C](\{\hat{u}\}_1 \dot{q}_1 + \{\hat{u}\}_2 \dot{q}_2) \\ + [K](\{\hat{u}\}_1 q_1 + \{\hat{u}\}_2 q_2) = \{F(t)\}.\end{aligned}\quad (6.110)$$

In terms of the modal matrix $[P] = [\{\hat{u}\}_1 \ \{\hat{u}\}_2]$, Equation 6.110 can be written as

$$[M][P]\{\ddot{q}\} + [C][P]\{\dot{q}\} + [K][P]\{q\} = \{F(t)\}.$$

Pre-multiplying by $[P]^T$ in order to obtain the matrix triple products that possess the needed orthogonality properties,³⁸ we have

$$\begin{aligned}[P]^T [M] [P] \{\ddot{q}\} + [P]^T [C] [P] \{\dot{q}\} \\ + [P]^T [K] [P] \{q\} = [P]^T \{F(t)\}.\end{aligned}$$

³⁸We could have premultiplied by $\{\hat{u}\}_1^T$ and $\{\hat{u}\}_2^T$ in sequence as in Section 6.6.1, but by showing this procedure we see how to decouple the equations of motion in one step.

We know that orthogonality of the modes with respect to the mass and stiffness matrices permits the substitutions

$$\begin{aligned}[P]^T[M][P] &= [I] \\ [P]^T[K][P] &= [\text{diag } \omega^2] \equiv [\Omega],\end{aligned}$$

where $[\text{diag } \omega^2]$ is a square matrix with elements ω_i^2 on the diagonal. But what about the triple product $[P]^T[C][P]$? In general, there is no decoupling and it is not possible to *simultaneously* diagonalize the mass, stiffness, and damping matrices. However, suppose the viscous damping can be represented as a numerical proportion of mass and stiffness in the following way,

$$[C] = C_m[M] + C_k[K],$$

where C_m and C_k are constants. Then,

$$\begin{aligned}[P]^T[C][P]\{\dot{q}\} &= [P]^T(C_m[M] + C_k[K])[P]\{\dot{q}\} \\ &= C_m[P]^T[M][P]\{\dot{q}\} + C_k[P]^T[K][P]\{\dot{q}\} \\ &= (C_m[I] + C_k[\Omega])\{\dot{q}\} \\ &= (C_m + C_k\omega_i^2)\dot{q}_i.\end{aligned}$$

Such damping is known as proportional³⁹ or *Rayleigh damping*.

The equations can now be decoupled giving a governing equation for each degree of freedom,

$$\ddot{q}_i + (C_m + C_k\omega_i^2)\dot{q}_i + \omega_i^2 q_i = N_i(t), \quad (6.111)$$

where it is customary to define

$$2\zeta_i\omega_i \equiv C_m + C_k\omega_i^2,$$

or

$$\zeta_i = \frac{1}{2} \left(\frac{C_m}{\omega_i} + C_k\omega_i \right), \quad (6.112)$$

representing the i^{th} mode damping ratio, and $N_i(t) = \sum_{j=1}^n u_{ji} F_j(t)$. The constants C_m and C_k have units that make both sides of Equation 6.112 dimensionless. C_m has dimensions of s^{-1} and C_k has dimensions of s .

Equation 6.111 is solved by the Duhamel integral,

$$\begin{aligned}q_i(t) &= \frac{1}{\omega_{d_i}} \int_0^t N_i(\tau) e^{-\zeta_i\omega_{d_i}(t-\tau)} \sin \omega_{d_i}(t-\tau) d\tau \\ &\quad + e^{-\zeta_i\omega_{d_i}t} \left[\frac{q_i(0)}{(1-\zeta_i^2)^{1/2}} \cos(\omega_{d_i}t - \phi_i) + \frac{\dot{q}_i(0)}{\omega_{d_i}} \sin \omega_{d_i}t \right],\end{aligned}$$

³⁹ Example 6.11 can be reexamined to see that the damping was indeed proportional and therefore that problem can be solved by modal analysis. Example 6.12, on the other hand, cannot be solved by modal analysis.

where

$$\omega_{d_i} = \omega_i \sqrt{1 - \zeta_i^2}, \quad \phi_i = \arctan \frac{\zeta_i}{\sqrt{1 - \zeta_i^2}},$$

and

$$\begin{aligned}q_i(0) &= \{\hat{u}\}_i^T [M] \{x(0)\} \\ \dot{q}_i(0) &= \{\hat{u}\}_i^T [M] \{\dot{x}(0)\}.\end{aligned}$$

Once each $q_i(t)$ is evaluated, the response $\{x(t)\}$ is found by performing the matrix-vector multiplication $[P]\{q(t)\}$ of Equation 6.109, as was done in earlier examples.

Example 6.15 Forced Vibration with Proportional Damping via Modal Analysis

Solve Example 6.14 with damping added to the equations of motion using a modal analysis procedure.

Solution: We assume proportional damping of the form

$$\begin{aligned}[C] &= 2[M] + 3[K] \\ &= \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} + \begin{bmatrix} 9k & -6k \\ -6k & 6k \end{bmatrix}.\end{aligned}$$

Since we have proportional damping we can decouple the equations of motion. In modal coordinates, the equations are

$$\begin{aligned}\ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 &= N_1(t) \\ \ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 &= N_2(t),\end{aligned}$$

where $N_1(t)$ and $N_2(t)$ are the same as in Example 6.14, and

$$\begin{aligned}\zeta_1 &= \frac{1}{2} \left(\frac{2}{\omega_1} + 3\omega_1 \right), \\ \zeta_2 &= \frac{1}{2} \left(\frac{2}{\omega_2} + 3\omega_2 \right).\end{aligned}$$

The remainder of the solution follows the procedure after Equation 6.112. ■

We have learned how to use the direct method in Section 6.5 to find the free response of a damped multi degree-of-freedom system. We also learned how to use modal analysis in Section 6.6.2 to find the response of a proportionally damped multi degree-of-freedom system. Modal analysis cannot be used if the damping is not proportional.

Example 6.16 Response of a MDOF System via Laplace Transform Method

Solve the system shown in Figure 6.33 and Example 6.12 via Laplace transform. In addition, m_1 is subject to a constant force of 10 N, and m_2 is subject to an impulsive

force with an impulse of 2 N/s at $t = 0$. Compare the solution to that of [Example 6.12](#).

Solution: In this example, we find the response for a general linear system using the Laplace transform method. The equation of motion is

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 7 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 2\delta(t) \end{Bmatrix}.$$

The initial conditions are given by

$$\{x(0)\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ and } \{\dot{x}(0)\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Taking the Laplace transform of the equation of motion we obtain

$$\begin{aligned} [M] \left(s^2 \{X(s)\} - s \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right) \\ + [C] \left(s \{X(s)\} - \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right) \\ + [K] \{X(s)\} = \begin{Bmatrix} 10/s \\ 2 \end{Bmatrix}. \end{aligned}$$

Solving for $\{X(s)\}$ we find

$$\{X(s)\} = (s^2 [M] + s [C] + [K])^{-1} \begin{Bmatrix} 10/s + 2s + 4 \\ -1 \end{Bmatrix},$$

or

$$X_1(s) = \frac{55s^4 + 8s^3 + 22s^2 + 39s + 30}{(8s^4 + 22s^3 + 37s^2 + 15s + 12)s}$$

and

$$X_2(s) = \frac{4s^3 + 14s^2 + 35s + 30}{(8s^4 + 22s^3 + 37s^2 + 15s + 12)s}.$$

Taking the inverse Laplace transform we can obtain $x_1(t)$ and $x_2(t)$. The denominators have roots $s = 0, -0.115 \pm 0.632i, -1.26 \pm 1.43i$, and, therefore, $x_1(t)$ and $x_2(t)$ have the form

$$\begin{aligned} x(t) = A + e^{-0.115t} (B \cos 0.632t + C \sin 0.632t) \\ + e^{-1.26t} (D \cos 1.43t + E \sin 1.43t). \end{aligned}$$

The constant coefficients are found by matching coefficients, yielding the solution

$$\begin{aligned} x_1(t) &= 2.5 + e^{-0.115t} (5.24 \cos 0.632t + 15.5 \sin 0.632t) \\ &\quad + e^{-1.26t} (0.259 \cos 1.43t - 6.21 \sin 1.43t) \\ x_2(t) &= 2.5 + e^{-0.115t} (-5.42 \cos 0.632t + 13.8 \sin 0.632t) \\ &\quad + e^{-1.26t} (2.92 \cos 1.43t - 1.18 \sin 1.43t). \end{aligned}$$

An advantage of the Laplace transform method is that damping does not need to be proportional. However, the disadvantages are in taking the inverse matrix and in finding the inverse Laplace transform. If there are 100 degrees of freedom, for example, we would need to invert a 100×100 matrix. In addition, roots to a 100^{th} -order polynomial must be found and sorted in ascending order. ■

Damping in Multi Degree-of-Freedom Systems

The normal mode method, or modal analysis, applies only to undamped systems or systems for which the damping matrix can be made mathematically equivalent to a combination of the mass and stiffness matrices.

Sometimes, damping can be ignored in the forced response of a vibratory system. For example, if the force acts for a relatively short period of time, a small amount of damping will not significantly affect the response, especially during the time immediately after the force stops.

If the forcing frequency is not near one of the system resonances, damping plays an insignificant role in the steady-state response of a periodically forced system. (Recall that in the magnification factor curves of Figure 3.29, except for the region about resonance $\omega/\omega_n = 1$, the curves are quite close to each other for all ζ values.) If the periodic forcing frequency is near or at a resonant frequency, damping plays a crucial role in the response amplitude.

Since the analyst generally does not know in advance the importance of damping in a particular problem, it is included in a vibration analysis unless it can be shown to be insignificant to the response.

For special systems where the damping matrix is linearly related to the mass and stiffness matrices, that is the proportional damping case, the simultaneous diagonalization of the stiffness and mass matrices can be accomplished along with that of the damping matrix. This results in the equation we have found above,

$$\zeta_i = \frac{1}{2} \left(\frac{C_m}{\omega_i} + C_k \omega_i \right).$$

For such systems, two cases are of particular interest. If C_m is very small or $C_m = 0$, then the damping matrix is proportional only to the stiffness matrix, that is $\zeta_i = C_k \omega_i / 2$, which means that the damping ratio for each normal mode is proportional to the undamped resonant frequency of that mode. Therefore, the response of the higher modes of such a system will be damped out more rapidly than those of the lower modes.

The other case is when C_k is very small or $C_k = 0$. Here the damping matrix is proportional to the mass matrix, and $\zeta_i = C_m / 2\omega_i$, meaning that the damping

ratio in each mode is inversely proportional to the undamped resonant frequency. The lower modes for this type of system will damp quicker than the higher modes.

For systems that include both C_m and C_k components, a designer can develop a physical sense of how the system model will behave, given the relative values of C_m and C_k . In the same vein, an analyst working with damping data can create a proportional damping model for a structure using characteristic vibration data by choosing appropriate values of C_m and C_k so that the damping of the modes resembles that found in the data.

If damping is light, the assumption is sometimes made that the damping matrix satisfies the same modal orthogonality properties as do the mass and stiffness matrices,

$$\{u\}_i^T [C] \{u\}_j = 0, \quad i \neq j,$$

which is exactly valid only if the off-diagonal terms are zero. The expression is approximately valid when the off-diagonal terms are very small in comparison to those of the diagonal. Light damping is a qualitative description, but is common in many engineering structures and machines where $0.0 < \zeta_i \leq 0.2$ for most modes.

As we have seen, when the damping cannot be considered proportional the eigenvectors, as well as the eigenvalues, are complex. Physically, the components of each mode (eigenvector) have different phase relationships and therefore do not reach their maximum amplitudes at the same time. Another way of visualizing this is that the *nodes* for each mode are not fixed but travel along the vibrating structure. For structures with proportional or no damping, the components of each mode are either in phase or 180° out of phase. If both components have the same sign they are in phase; if they are of opposite sign they are completely out of phase. There is no “in between,” except for nonproportional damping.

Example 6.17 Wind-Induced Vibration⁴⁰

When a high-rise building is designed, wind-induced oscillations are a critical consideration. The structure needs to withstand wind forces, but in order to resist all possible forces, the structure would have to be designed to be enormous and would be very expensive. Today, buildings are constructed of lightweight and high-strength materials that tend to be more flexible and have less damping.

Many modern high-rise buildings use a vibration absorber – a *tuned mass-spring-damper* system – to reduce excessive oscillation amplitudes without resorting to over-designing. (See Section 6.4.3 for discussion of a passive vibration absorber.) The vibration absorber is an *mck* system that is attached to the primary system.

⁴⁰M-Y. Liu, W-L. Chiang, J-H. Hwang, and C-R. Chu, “Wind-induced vibration of high-rise building with tuned mass damper including soil-structure interaction,” *Journal of Wind Engineering and Industrial Aerodynamics*, Vol.96, 2008, pp.1092-1102.

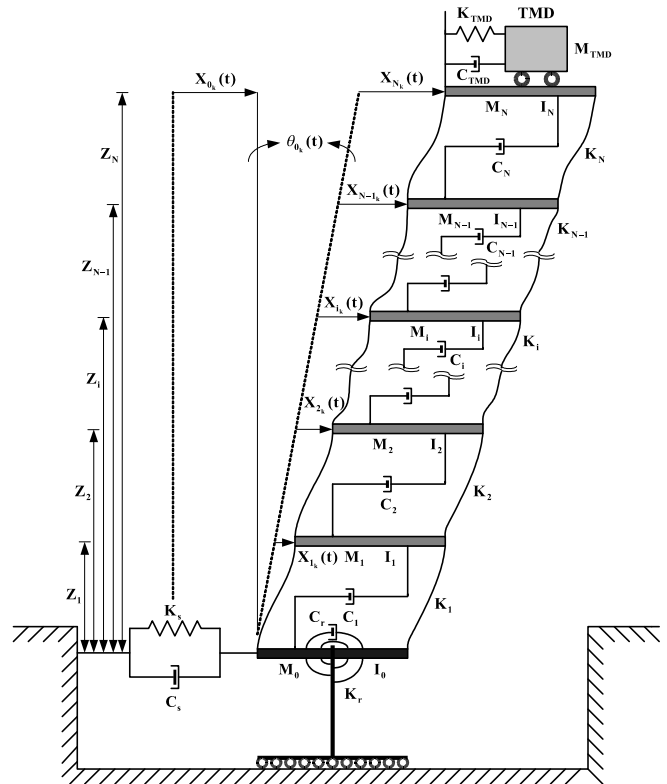


Figure 6.41: Shear building model. (By permission of Ming-Yi Liu)

As the high-rise structure oscillates, it causes the tuned mass-damper to oscillate as well, dissipating energy. Interactions between the building and the soil also affect the structural response.

Figure 6.41 is a multi degree-of-freedom model of the building, the tuned mass-damper (labeled TMD at the top), and the coupling to the soil, denoted by K_s and C_s at the bottom left of the figure.

The numerical studies concluded that soil-structure interaction cannot be ignored for small soil stiffness, since then the results underestimate the effectiveness of the tuned mass-damper. Also, such dampers with more mass are useful for minimizing structural response to wind forces. The Citicorp building in Manhattan has a tuned mass-damper in its tower. ■

6.7 Nonproportional Damping

The popularity of classical modal analysis is due to the possibility of decoupling the equations of motion. For non-classically damped systems, described below, other approaches are needed. These include numerical methods for direct integration, conversion of the equations of motion into state-space (with double the number of equations and possibly less intuitive parameters), or to work

with complex modes. We will not examine or compare these alternatives. Rather, in this section, we present a method due to Ma et al.⁴¹ for the decoupling of viscously damped linear systems in oscillatory free vibration. The free vibration case presented here is a suitable way to introduce the procedure, although more complex cases can be analyzed using this approach.

This method is an extension of classical modal analysis. For classically damped modes, where the equations of motion have been decoupled, the motion is synchronous, as we have discussed elsewhere in this chapter. Each system degree of freedom harmonically oscillates at its respective damped oscillation frequency and passes through its equilibrium position at the same time instant as all the other degrees of freedom. The modes are either in phase or out of phase, and they decay in time.

Non-classically damped modes have complex eigenvectors that perform decaying harmonic oscillations at their respective damped natural frequencies, but they pass through their respective equilibria at different times. The phase difference between any two modes, however, is constant, and the order in which the modes pass through their equilibria is a particular sequence that repeats exactly. We can look back at the decaying oscillation of a single degree-of-freedom system and recall that its damped period of oscillation is a constant. Similarly, for the multi degree-of-freedom system under discussion, we may expect that the phasing between modes is a constant as the amplitudes decay.

We begin with the matrix equation of motion of an n degree-of-freedom viscously damped system undergoing free vibration, where the generalized coordinates $\{q(t)\}$ are governed by

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{0\}, \quad (6.113)$$

where we follow the notation adopted by Ma. The free vibration admits an oscillatory solution of the form

$$\{q(t)\} = \{v\}e^{\lambda t}, \quad (6.114)$$

where $\{v\}$ is an n -dimensional column vector of constants and λ is a scalar parameter to be determined. Substitution of Equation 6.114 into 6.113 results in the quadratic eigenvalue problem,

$$([M]\lambda^2 + [C]\lambda + [K])\{v\} = \{0\}. \quad (6.115)$$

The determinant,

$$\det([M]\lambda^2 + [C]\lambda + [K]) = 0,$$

is the characteristic polynomial that is solved for $2n$ roots, the eigenvalues λ_i , and, in general, n linearly independent eigenvectors $\{v\}_i$, $i = 1, \dots, 2n$. (The number of eigenvectors can be less than n .) The coefficients of the characteristic polynomial are real, and any complex roots occur as complex conjugate pairs. The corresponding eigenvectors are also complex conjugate pairs. The real parts of all roots must be negative because of the dissipation of energy by the system viscous damping.

Assuming distinct eigenvalues, the solution for $\{q(t)\}$ is the linear combination

$$\{q(t)\} = \sum_{i=1}^{2n} c_i \{v\}_i e^{\lambda_i t},$$

where c_i are $2n$ constants that are functions of the initial conditions. Let

$$\begin{aligned} \lambda_j &= \alpha_j + i\omega_j \\ \{v\}_j &= \{r_{j1}e^{-i\phi_{j1}} \quad r_{j2}e^{-i\phi_{j2}} \quad \dots \quad r_{jn}e^{-i\phi_{jn}}\}^T \end{aligned}$$

comprise eigensolution j , where r_{jk} , ϕ_{jk} are real parameters and $j, k = 1, \dots, n$, and $\alpha_j < 0$. Another eigensolution is given by the complex conjugates of eigensolution j : λ_j^* , $\{v\}_j^*$. The damped mode is given by the sum of the two eigensolutions,

$$\{s(t)\}_j = a_j \{v\}_j e^{(\alpha_j + i\omega_j)t} + b_j \{v\}_j^* e^{(\alpha_j - i\omega_j)t},$$

$j = 1, \dots, n$. Physical modes $\{s(t)\}_j$ must be real, implying that $b_j = a_j^*$. Then,

$$\begin{aligned} \{s(t)\}_j &= 2 \operatorname{Re}[a_j \{v\}_j e^{(\alpha_j + i\omega_j)t}] \\ &= 2e^{\alpha_j t} \operatorname{Re}[a_j \{v\}_j e^{i\omega_j t}]. \end{aligned}$$

By writing $2a_j$ in polar form,

$$2a_j = C_j e^{-i\theta_j},$$

where C_j and θ_j are real, it follows that

$$\{s(t)\}_j = C_j e^{\alpha_j t} \begin{Bmatrix} r_{j1} \cos(\omega_j t - \theta_j - \phi_{j1}) \\ r_{j2} \cos(\omega_j t - \theta_j - \phi_{j2}) \\ \vdots \\ r_{jn} \cos(\omega_j t - \theta_j - \phi_{jn}) \end{Bmatrix}. \quad (6.116)$$

The general solution of the governing system is a superposition of the n damped vibration modes,

$$\begin{aligned} \{q(t)\} &= \sum_{j=1}^n \left(a_j \{v\}_j e^{\lambda_j t} + a_j^* \{v\}_j^* e^{\lambda_j^* t} \right) \\ &= \sum_{j=1}^n \{s(t)\}_j. \end{aligned} \quad (6.117)$$

⁴¹F. Ma, A. Imam, M. Morzfeld, "The decoupling of damped linear systems in oscillatory free vibration," *Journal of Sound and Vibration*, Vol.324, 2009, pp.408-428.

Since the phase angles ϕ_{jk} are generally different, the system degrees of freedom do not pass through their equilibria simultaneously. But, the phase difference between any two degrees of freedom in $\{s(t)\}_j$ is constant, as suggested at the beginning of this section.

The eigenvectors of Equation 6.115, $\{v\}_j$, can be normalized in a variety of ways, with one possibility being

$$2\lambda_j\{v\}_j^T[M]\{v\}_j + \{v\}_j^T[C]\{v\}_j = 2i\omega_j. \quad (6.118)$$

For classically damped systems, $\{v\}_j = \{v\}_j^*$ ($j = 1, \dots, n$) is a real eigenvector, denoted by the natural or normal mode, say, $\{u\}_j$, where the modes are synchronous and thus $\phi_{jk} = 0$, so that

$$\{u\}_j = \{v\}_j = \{r_{j1} \ r_{j2} \ \cdots \ r_{jn}\}^T,$$

resulting in the classically damped mode of vibration,

$$\{s(t)\}_j = C_j e^{\alpha_j t} \cos(\omega_j t - \theta_j) \{u\}_j.$$

For an undamped system, $\alpha_j = 0$ and $\lambda_j = i\omega_j$, where there is no decay in the amplitude of vibration and, therefore, ω_j is a natural frequency.

Equation 6.118 reduces to normalization with respect to the mass matrix $[M]$ for undamped or classically damped systems.

6.7.1 Phase Synchronization

For the non-classically damped system governed by Equations 6.116 and 6.117, we introduce and discuss the approach taken to synchronize all the modes so that a standard decoupling can be performed. As per Ma,⁴² “If suitable phase shifts are introduced into each non-classically damped mode so that all components are either in phase or out of phase, it is possible to transform a non-classically damped system into one with classical damping. A basic objective of a process termed *phase synchronization* is to do just that.”

In Equation 6.116, we introduce a phase shift by replacing t with $t + \phi_{jk}/\omega_j$,

$$\{y(t)\}_j = \begin{Bmatrix} y_{j1}(t) \\ y_{j2}(t) \\ \vdots \\ y_{jn}(t) \end{Bmatrix} = \begin{Bmatrix} s_{j1}(t + \phi_{j1}/\omega_j) \\ s_{j2}(t + \phi_{j2}/\omega_j) \\ \vdots \\ s_{jn}(t + \phi_{jn}/\omega_j) \end{Bmatrix}. \quad (6.119)$$

Then, Equation 6.116 becomes

$$\{y(t)\}_j = C_j e^{\alpha_j t} \cos(\omega_j t - \theta_j) \begin{Bmatrix} r_{j1} e^{\alpha_j \phi_{j1}/\omega_j} \\ r_{j2} e^{\alpha_j \phi_{j2}/\omega_j} \\ \vdots \\ r_{jn} e^{\alpha_j \phi_{jn}/\omega_j} \end{Bmatrix}. \quad (6.120)$$

A system that possesses the damped modes of vibration $\{y(t)\}_j$ shown in Equation 6.120 is classically damped. We see the difference between the classically damped Equation 6.120 and the non-classically damped Equation 6.116. The difference is in the cosine term. The classically damped case has cosine terms that are synchronized.

To find the equation of motion for a system with the modes of Equation 6.120 we define the modal vectors $\{z\}_j$,

$$\{z\}_j = \begin{Bmatrix} r_{j1} e^{\alpha_j \phi_{j1}/\omega_j} \\ r_{j2} e^{\alpha_j \phi_{j2}/\omega_j} \\ \vdots \\ r_{jn} e^{\alpha_j \phi_{jn}/\omega_j} \end{Bmatrix}, \quad j = 1, \dots, n. \quad (6.121)$$

Assuming that the n modes are linearly independent (if they are not, they can be orthogonalized), a modal matrix $[Z]$ containing the n modal vectors can be defined,

$$[Z] = [\{z\}_1 \ \{z\}_2 \ \cdots \ \{z\}_n].$$

Then, let the desired governing equation with damped modes $\{y(t)\}_j$ be given by

$$[M]_1 \{\ddot{q}\}_1 + [C]_1 \{\dot{q}\}_1 + [K]_1 \{q\}_1 = \{0\}. \quad (6.122)$$

For this system, the following matrix products must hold,

$$\begin{aligned} [Z]^T [M]_1 [Z] &= [I] \\ [Z]^T [C]_1 [Z] &= [D]_1 \\ [Z]^T [K]_1 [Z] &= [\Omega]_1, \end{aligned}$$

where $[I]$ is the identity matrix,

$$\begin{aligned} [D]_1 &= -\text{diag}[\lambda_j + \lambda_j^*] \\ &= -\text{diag}[2\alpha_1, 2\alpha_2, \dots, 2\alpha_n], \end{aligned}$$

and

$$\begin{aligned} [\Omega]_1 &= \text{diag}[\lambda_j \lambda_j^*] \\ &= \text{diag}[\alpha_1^2 + \omega_1^2, \dots, \alpha_n^2 + \omega_n^2]. \end{aligned}$$

Therefore,

$$\begin{aligned} [M]_1 &= [Z]^{-T} [Z]^{-1} \\ [C]_1 &= [Z]^{-T} [D]_1 [Z]^{-1} \\ [K]_1 &= [Z]^{-T} [\Omega]_1 [Z]^{-1} \end{aligned}$$

are symmetric and positive definite and Equation 6.122 is a classically damped system derived by phase synchronization of its damped modes of vibration.

Equation 6.122 can be decoupled using standard modal analysis procedures, beginning with

$$\{q(t)\}_1 = [Z] \{p(t)\},$$

⁴² Ibid.

resulting in the decoupled equation of motion,

$$[I]\{\ddot{p}\} + [D]_1\{\dot{p}\} + [\Omega]_1\{p\} = \{0\}. \quad (6.123)$$

The solution to this equation is

$$\{p(t)\} = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{Bmatrix} = \begin{Bmatrix} C_1 e^{\alpha_1 t} \cos(\omega_1 t - \theta_1) \\ C_2 e^{\alpha_2 t} \cos(\omega_2 t - \theta_2) \\ \vdots \\ C_n e^{\alpha_n t} \cos(\omega_n t - \theta_n) \end{Bmatrix}. \quad (6.124)$$

Noting that Equation 6.119 has the following inverse,

$$\{s(t)\}_j = \begin{Bmatrix} s_{j1}(t) \\ s_{j2}(t) \\ \vdots \\ s_{jn}(t) \end{Bmatrix} = \begin{Bmatrix} y_{j1}(t - \phi_{j1}/\omega_j) \\ y_{j2}(t - \phi_{j2}/\omega_j) \\ \vdots \\ y_{jn}(t - \phi_{jn}/\omega_j) \end{Bmatrix}, \quad (6.125)$$

and combining Equations 6.120, 6.121, and 6.124, we obtain

$$\{y(t)\}_j = C_j e^{\alpha_j t} \cos(\omega_j t - \theta_j) \{z\}_j \quad (6.126)$$

$$= p_j(t) \{z\}_j.$$

From Equations 6.125 and 6.126, it follows that

$$\{s(t)\}_j = \begin{Bmatrix} p_j(t - \phi_{j1}/\omega_j) r_{j1} e^{\alpha_j \phi_{j1}/\omega_j} \\ p_j(t - \phi_{j2}/\omega_j) r_{j2} e^{\alpha_j \phi_{j2}/\omega_j} \\ \vdots \\ p_j(t - \phi_{jn}/\omega_j) r_{jn} e^{\alpha_j \phi_{jn}/\omega_j} \end{Bmatrix}.$$

Then, since the displacement response $\{q(t)\}$ is the superposition of the damped modes $\{s(t)\}_j$, we have

$$\{q(t)\} = \sum_{j=1}^n \begin{bmatrix} p_j(t - \phi_{j1}/\omega_j) & & \\ & \ddots & \\ & & p_j(t - \phi_{jn}/\omega_j) \end{bmatrix} \times \begin{Bmatrix} r_{j1} e^{\alpha_j \phi_{j1}/\omega_j} \\ \vdots \\ r_{jn} e^{\alpha_j \phi_{jn}/\omega_j} \end{Bmatrix}. \quad (6.127)$$

The method introduced here is in two parts. The first part is to apply phase synchronization to the non-classically damped system $[M], [C], [K]$, transforming it into a classically damped system $[M]_1, [C]_1, [K]_1$. The second part is to apply standard modal analysis to the resulting classically damped system, resulting in a decoupled system $[I], [D]_1, [\Omega]_1$. The eigenvalues of all three systems are identical, but their eigenvectors are different. That is, the original non-classically damped, the

phase synchronized, and the final transformed classically damped systems all have the same eigenvalues. The initial conditions need to go through a similar transformation, but we omit this procedure here and refer the reader to the original paper.

Example 6.18 Two Degrees of Freedom with Viscous Damping

For the two degree-of-freedom mass-spring-damper system governed by Equation 6.113, compare the responses for the (a) undamped, (b) classically damped, and (c) non-classically damped systems.

Solution: This system has the following mass and stiffness property matrices,

$$[M] = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [K] = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

where for convenience we assume $m = 1$ and $k = 1$ with appropriate units, and the damping matrix is

$$[C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}.$$

The initial conditions are given by the vectors

$$\{q(0)\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad \{\dot{q}(0)\} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

For purposes of comparison, we provide the results for an undamped system, a classically damped system, and a non-classically damped system. The natural frequencies are obtained by solving $([M]\lambda^2 + [K])\{v\} = \{0\}$ while the damped frequencies are obtained from $([M]\lambda^2 + [C]\lambda + [K])\{v\} = \{0\}$. The damped frequencies are slightly smaller than the natural frequencies, and so we retain four digits after the decimal point in order to highlight the slight differences.

(a) For an undamped system, $c_1 = c_2 = c_3 = 0$, with the solution of the eigenvalue problem, and subsequent normalization with respect to the mass matrix, yielding the following natural frequencies and modal ratios,

$$[\Omega] = \text{diag}[1, 3]$$

$$[U] = [\{u\}_1 \quad \{u\}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The complete solution is the superposition of the natural modes,

$$\begin{aligned} \{q(t)\} &= \sum_{j=1}^2 \{s(t)\}_j \\ &= C_1 \cos(t - \theta_1) \{u\}_1 \\ &\quad + C_2 \cos(\sqrt{3}t - \theta_2) \{u\}_2. \end{aligned}$$

The constants C_1, C_2, θ_1 , and θ_2 are determined by satisfying the initial conditions. From [U] we see that the modes are either in phase or out of phase.

(b) For a damped system, we assume that $c_1 = c_2 = c_3 = 0.1$. Since $[C] = 0.1[K]$, the system is proportionately damped, with the general solution

$$\begin{aligned}\{q(t)\} &= \sum_{j=1}^2 \{s(t)\}_j \\ &= C_1 e^{-0.05t} \cos(0.9987t - \theta_1) \{u\}_1 \\ &\quad + C_2 e^{-0.15t} \cos(1.7255t - \theta_2) \{u\}_2.\end{aligned}$$

In this case there is a decaying oscillation. The coefficients in the decaying exponentials, -0.05 and -0.15 , are the real parts of the complex eigenvalues.

(c) For the non-classically damped system, we assume that $c_1 = 0.6$, and $c_2 = c_3 = 0.1$. Solving the quadratic eigenvalue problem Equation 6.115 yields

$$\begin{aligned}\lambda_1 &= \lambda_3^* = -0.1792 + 1.0008i \\ \{v\}_1 &= \{v\}_3^* = \begin{Bmatrix} 0.7389e^{-i0.1288} \\ -0.7242e^{-i3.0108} \end{Bmatrix} \\ &= \begin{Bmatrix} 0.7389e^{-i7.38^\circ} \\ -0.7242e^{-i172.51^\circ} \end{Bmatrix},\end{aligned}$$

and

$$\begin{aligned}\lambda_2 &= \lambda_4^* = -0.2708 + 1.6819i \\ \{v\}_2 &= \{v\}_4^* = \begin{Bmatrix} -0.7337e^{-i2.9169} \\ -0.7296e^{-i0.2213} \end{Bmatrix} \\ &= \begin{Bmatrix} -0.7337e^{-i167.13^\circ} \\ -0.7296e^{-i12.68^\circ} \end{Bmatrix},\end{aligned}$$

where the eigenvectors were normalized in accordance with Equation 6.118,

$$2\lambda_j \{v\}_j^T [M] \{v\}_j + \{v\}_j^T [C] \{v\}_j = 2i\omega_j.$$

From Equation 6.116, the two non-classically damped modes of vibration are given by

$$\begin{aligned}\{s(t)\}_1 &= C_1 e^{-0.1792t} \begin{Bmatrix} 0.7389 \cos(1.0008t - \theta_1 - 0.1288) \\ -0.7242 \cos(1.0008t - \theta_1 - 3.0108) \end{Bmatrix}\end{aligned}$$

and

$$\begin{aligned}\{s(t)\}_2 &= C_2 e^{-0.2708t} \begin{Bmatrix} -0.7337 \cos(1.6819t - \theta_2 - 2.9169) \\ -0.7296 \cos(1.6819t - \theta_2 - 0.2213) \end{Bmatrix}.\end{aligned}$$

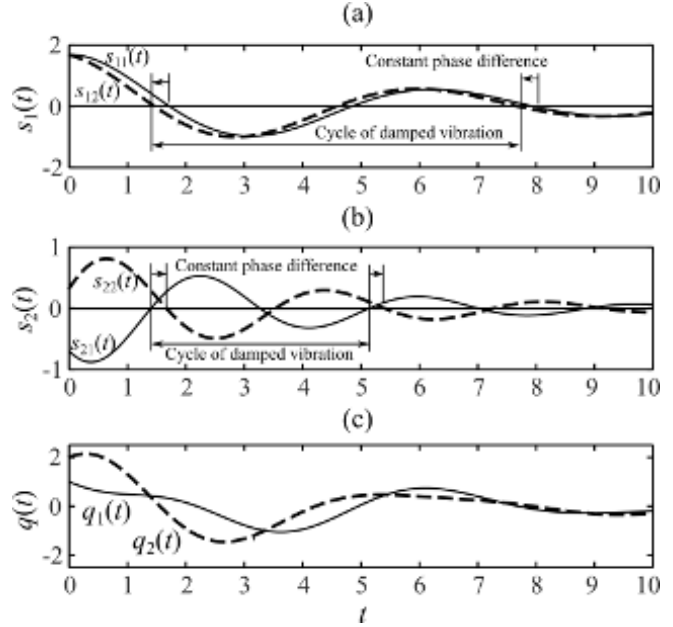


Figure 6.42: Non-classically damped modes and free response shown in three parts. (a) First non-classically damped mode $\{s\}_1$ with first element s_{11} (—) and second element s_{12} (- - -). (b) Second non-classically damped mode $\{s\}_2$ with first element s_{21} (—) and second element s_{22} (- - -). (c) Free response $\{q\}$ with first element q_1 (—) and second element q_2 (- - -). (Reprinted with permission Fai Ma)

The general solution, given by Equation 6.117, is the superposition of these two real damped modes. As shown in Figure 6.42, there is a constant phase difference between the two components in each mode, where $\{s(t)\}_1$ and $\{s(t)\}_2$ are matched with the specified initial conditions $\{q(0)\}$ and $\{\dot{q}(0)\}$. The constants of integration that match the initial conditions given in the problem statement are

$$C_1 = -2.324, \quad C_2 = 1.338, \quad \theta_1 = -9.389, \quad \theta_2 = 4.152.$$

If $\{s(t)\}_1$ and $\{s(t)\}_2$ are graphed simply with $C_1 = C_2 = 1$ and $\theta_1 = \theta_2 = 0$, the plots in Figure 6.43 are obtained.

Equation 6.120 provides the transformation from $\{s\}_j$ to $\{y\}_j$ to create phase synchronization, resulting in

$$\{y(t)\}_1 = C_1 e^{-0.1792t} \cos(1.0008t - \theta_1) \{z\}_1$$

and

$$\{y(t)\}_2 = C_2 e^{-0.2708t} \cos(1.6819t - \theta_2) \{z\}_2,$$

where

$$\begin{aligned}[Z] &= [\{z\}_1 \quad \{z\}_2] \\ &= \begin{bmatrix} 0.7221 & -0.4587 \\ -0.4224 & -0.7040 \end{bmatrix},\end{aligned}$$

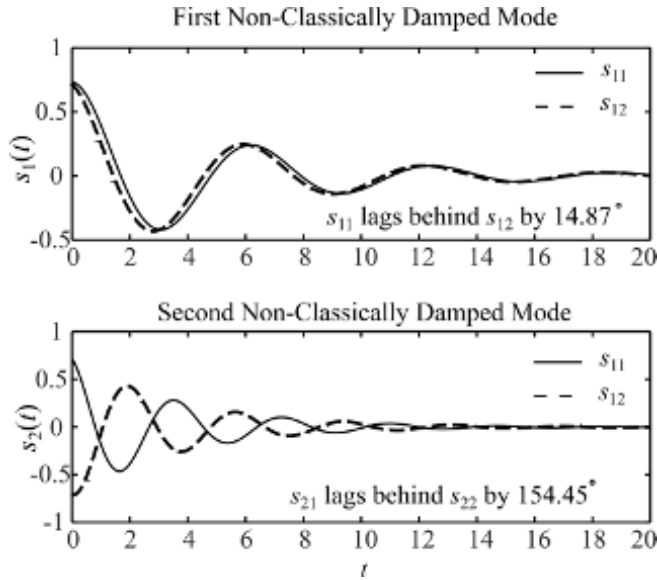


Figure 6.43: Two non-classically damped modes using assumed values $C_1 = C_2 = 1$ and $\theta_1 = \theta_2 = 0$. (Reprinted with permission Fai Ma)

$$[D]_1 = -\text{diag}[\lambda_j + \lambda_j^*] \\ = \begin{bmatrix} 0.3584 & 0 \\ 0 & 0.5416 \end{bmatrix},$$

and

$$[\Omega]_1 = \text{diag}[\lambda_j \lambda_j^*] \\ = \begin{bmatrix} 1.0337 & 0 \\ 0 & 2.9022 \end{bmatrix}.$$

Upon decoupling, the equation of motion becomes

$$[I]\{\ddot{p}\} + [D]_1\{\dot{p}\} + [\Omega]_1\{p\} = \{0\},$$

which can be easily solved for $\{p(t)\}$. The original response $\{q(t)\}$ is then obtained using Equation 6.127. The decoupling transformation, as a linear time-shifting transformation, is given by

$$\{q(t)\} = \begin{bmatrix} p_1(t - \phi_{11}/\omega_1) & 0 \\ 0 & p_1(t - \phi_{12}/\omega_1) \end{bmatrix} \{z\}_1 \\ + \begin{bmatrix} p_2(t - \phi_{21}/\omega_2) & 0 \\ 0 & p_2(t - \phi_{22}/\omega_2) \end{bmatrix} \{z\}_2,$$

which results in

$$\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \\ = \begin{bmatrix} p_1(t - 0.1287) & 0 \\ 0 & p_1(t - 3.0084) \end{bmatrix} \begin{Bmatrix} 0.7221 \\ -0.4224 \end{Bmatrix} \\ + \begin{bmatrix} p_2(t - 1.7343) & 0 \\ 0 & p_2(t - 0.1316) \end{bmatrix} \begin{Bmatrix} -0.4587 \\ -0.7040 \end{Bmatrix} \\ = \begin{bmatrix} 0.7221p_1(t - 0.1287) - 0.4587p_2(t - 1.7343) \\ -0.4224p_1(t - 3.0084) - 0.7040p_2(t - 0.1316) \end{bmatrix}.$$

It can be checked that $\{q(t)\}$, whether generated by decoupling or by direct numerical solution of the original equation of motion, is the same. ■

6.8 Real and Complex Modes

We summarize⁴³ the key aspects of real and complex normal modes, and show some of these characteristics in Figure 6.44. In this figure, on the left are two time histories – the top for real modes and the bottom for complex modes – and on the right are arrows representing the relative location of three points on the respective time history. For the real modes the arrows are along a line, indicating that they are completely in phase or 180° out of phase. The complex modes are out of phase.

Real normal modes have the following characteristics:

1. The mode shape is described by a standing wave (stationary wave). Nodes (where the zero axis is crossed) of standing waves are fixed.
2. All points on the system undergoing oscillation pass through their maxima and minima at the same instant of time.
3. All points on the system undergoing oscillation pass through the zero at the same instant of time.
4. The mode shape can be described by signed real numbers.
5. All points are either totally in phase or 180° out of phase with any other point on the structure.

The mode shapes of the undamped case are the same as the mode shapes of the proportionally damped case. These modes decouple property matrices $[M]$, $[C]$, $[K]$.

The complex modes have the following characteristics:

⁴³Based on P. Avitabile, “Modal Space,” *Experimental Techniques*, May/June 2002, p.17-18.

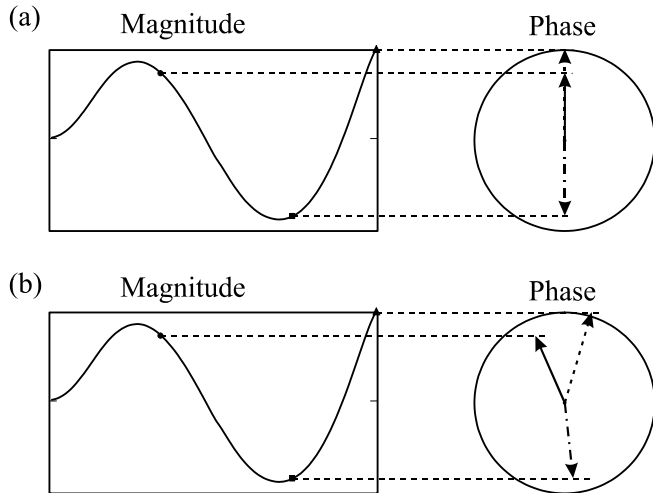


Figure 6.44: Mode Schematic: (a) Proportional (real normal) mode, and (b) Nonproportional (complex) mode. Real normal modes are either in phase or 180° out of phase. Complex modes have a wavelike quality such that different points can be out of phase generally.

1. The mode shape is described by a traveling wave and appears to have moving node points on the structure.
2. All points do not pass through their maxima at the same time. Some points lag.
3. All points do not pass through zero at the same time.
4. The mode shapes are complex valued.
5. The different degrees of freedom will have phase differences between them.
6. The mode shapes from the undamped case will not uncouple the damping matrix.

6.8.1 Modal Analysis vs. Direct Method

It is helpful to compare the direct method with modal analysis. The direct method has the advantage that a solution for the response does not require a special form for the damping. However, in general, the solution for the roots of the characteristic determinant becomes computationally intensive for systems with more than a few degrees of freedom. Since there are no exact solutions for the roots of a characteristic polynomial of order greater than four, in these cases numerical techniques for determining the eigenvalues are used.

Various techniques have been developed for solving eigenvalue problems because of their importance in technical and scientific disciplines. The most suitable method for a specific eigenvalue problem depends to some extent on the size of the matrix (number of degrees of freedom) and on how many eigenvalues and eigenvectors are

needed to model the response. Matrix iteration techniques are the basis for many eigenvalue problem solution techniques used in practice.⁴⁴ It is because the numerical methods for estimating frequencies and vectors are iterative – that is, the estimates are sequential – that the modal approach permits a designer to retain only as many degrees of freedom as needed for an accurate result. Some techniques⁴⁵ for the solution of eigenvalue problems include the *power method*, *Hotelling's deflation method*, *Jacobi's method*, and *Householder's method*, generally the most efficient for large systems (20 and more degrees of freedom) that can be used in conjunction with the *QL* algorithm.

In modal analysis, the symmetries of the stiffness and mass matrices are key to the orthogonality of the natural modes. The orthogonality relations are also useful as a check on the accuracy of a numerical computation of the modes. The physical meaning of the normal coordinates is discussed in Section 6.10. The normal coordinates are the principal coordinates, in the same sense that the term is used in solid mechanics. In a single degree-of-freedom system, by considering the vibration to be in only one direction, we imply that direction to be a normal or principal coordinate.

Modal analysis is especially useful where the structure is forced. A forced response by the direct method requires the *simultaneous solution* of as many equations as there are degrees of freedom. The modal analysis approach proceeds with decoupled oscillator equations that are solved independently in modal space. Modal analysis, even with forcing, transforms the solution of n simultaneous equations of motion to a procedure that only involves a series of matrix multiplications and additions.

Modal analysis is also useful as an approximate method for systems with a large number of degrees of freedom (for example, hundreds to thousands of degrees of freedom). If the forcing has only lower frequency components, modal analysis with only a small number of lower modes can be used. It is numerically possible to selectively pick particular eigenvalue ranges within which the response is needed. (Generally, this is due to loading that has most of its energy in a particular frequency band.) In this way, the critical modes are determined without wasting computational time on those modes that have little or no contribution.

For clarification, suppose that only the first *three* modes of a *one thousand* degree-of-freedom model are the primary contributors to the response of the struc-

⁴⁴**Dynamics of Structures**, R.W. Clough and J. Penzien, 2nd ed., McGraw-Hill, 1993.

⁴⁵**Vibration of Mechanical and Structural Systems**, M.L. James, G.M. Smith, J.C. Wolford, and P.W. Whaley, 2nd ed., Harper Collins, 1994. An introductory discussion is provided in **Applied Numerical Analysis**, C.F. Gerald and P.O. Wheatley, 5th ed., Addison-Wesley, 1994.

ture. The response can then be approximated by

$$\underbrace{\{x(t)\}}_{1000 \times 1} \simeq \underbrace{[P]}_{1000 \times 3} \underbrace{\{q(t)\}}_{3 \times 1},$$

where the modal matrix $[P]$ has been determined previously.

The question of how many and which modes to retain generally depends on the problem. But, if the forcing spectrum is limited to a frequency band of $\omega_i \rightarrow \omega_j$, then the analyst would retain modes corresponding to frequencies as high as two to four times the highest frequency ω_j . This is only a rule of thumb and exceptions exist. Our previous discussion of the modal participation factor is relevant here.

6.9 Special Cases

6.9.1 Unrestrained Systems

So far we have considered systems that are fixed at one location, such as skyscrapers, machines, and bridges. In fixed structures, vibration is constrained and oscillations occur about a stationary equilibrium. However, there are many cases in which the vibration occurs about a moving equilibrium. In this situation, the body is not constrained and its *vibration is superimposed on the larger dynamic behavior*, which is known as *elastic-body unrestrained motion*. Examples of such systems are vehicles (airplanes, rockets, trains, and cars) and rotating machinery that can undergo unlimited rotation. All of these systems vibrate while undergoing large dynamic motion.

Two representative schematics are shown in Figure 6.45. We are interested in examining such systems to determine how the techniques of vibration analysis can be applied to systems that are unrestrained.

Physically, an *unrestrained system* is one which can translate or rotate without bounds. An airplane, modeled as a multi-body system, is an unrestrained system; it moves arbitrary distances relative to any fixed point and it vibrates. A turbine rotates without any restraint except friction. What is the common element in all such systems? It is that *translational and rotational motions have unlimited magnitudes and that the “first mode” is at zero frequency*. There is no relative motion between any two points on the system for this mode. This is known as a *rigid-body mode*.

Example 6.19 Energy of a Rigid-Body Mode

Derive the expressions for the strain energy and the kinetic energy for the translating system of Figure 6.45.

Solution: The strain energy V stored in the springs is a function of the squares of the differences of the relative displacements,

$$V = \frac{1}{2}[k_1(x_2 - x_1)^2 + k_2(x_3 - x_2)^2].$$

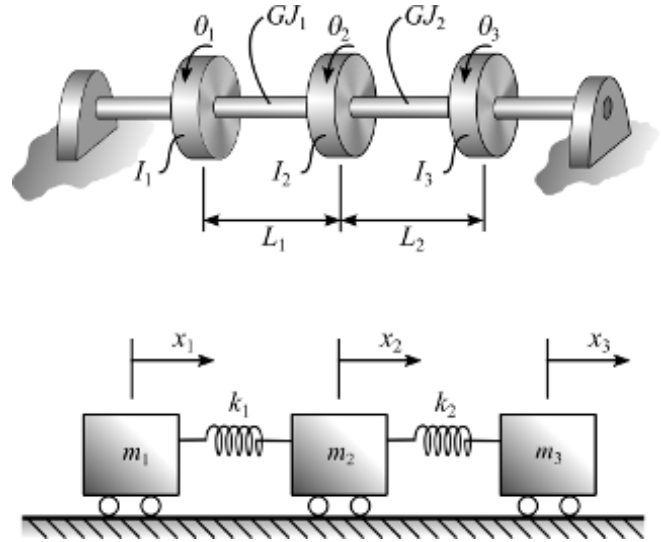


Figure 6.45: Examples of unrestrained rotating and translating systems. In the top figure three rotors with mass moments of inertia I_i are connected via shafts of torsional rigidity GJ_i and supported by frictionless bearings at each end. (G is the shear modulus and J is the area polar moment of inertia.) In the lower figure three masses m_i are connected via springs of stiffnesses k_i .

The kinetic energy T is

$$T = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2),$$

which is always positive except when the system is static for which $T = 0$. The strain energy is positive except when all the displacements are equal, that is, when $x_1 = x_2 = x_3$, then $V = 0$. In this case, the three masses translate as one unit and they behave like a rigid-body with no resulting oscillations. The rigid-body mode has a zero natural frequency. ■

6.9.2 Rigid-Body Mode

We will show that for the rigid-body mode, $\omega_0 = 0$ and the mode shape is $\{u\}_0 = C\{1 \ 1 \ 1\}^T$, where C is an arbitrary constant of integration. The eigenvalue problem for any mode i is given by

$$\omega_i^2[M]\{u\}_i = [K]\{u\}_i.$$

Premultiplying both sides of this equation by $\{u\}_i^T$ we find

$$\omega_i^2\{u\}_i^T[M]\{u\}_i = \{u\}_i^T[K]\{u\}_i. \quad (6.128)$$

Rigid-body motion is given by

$$\{u\}_0 = C\{1 \ 1 \ 1\}^T.$$

We recall from Section 6.3.1 that $[M]$ is a positive definite matrix while $[K]$ is a positive semi-definite matrix. For

an unrestrained system $\det[K] = 0$ and $\{u\}_0^T [K] \{u\}_0 = 0$. In Equation 6.128, for $i = 0$, the left-hand side is positive definite while the right-hand side is zero for the rigid-body mode. This is only possible if $\omega_0 = 0$.

Since the rigid-body mode is a solution of the eigenvalue problem, it must be orthogonal to any other elastic mode. For example, $\{u\}_0^T [M] \{u\}_1 = 0$. Then

$$\{1 \ 1 \ 1\} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_1 = 0,$$

or $m_1 u_1 + m_2 u_2 + m_3 u_3 = 0$. We can replace the modal coordinate by the physical coordinate using $x_i = u_i \exp(i\omega_0 t)$, where, for $\omega_0 = 0$, $x_i = u_i$. Then,

$$\{x_0 \ x_0 \ x_0\} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0,$$

which can be expanded and simplified as follows,

$$\begin{aligned} x_0(m_1 x_1 + m_2 x_2 + m_3 x_3) &= 0 \\ m_1 x_1 + m_2 x_2 + m_3 x_3 &= 0. \end{aligned} \quad (6.129)$$

Note that by differentiating Equation 6.129 with respect to time, the resulting equation is a statement that the linear momentum is conserved for the elastic mode,

$$m_1 \dot{x}_1 + m_2 \dot{x}_2 + m_3 \dot{x}_3 = 0.$$

Equation 6.129 can also be used to reduce the order of the system by one, since one of the displacements can be expressed as a function of the other two, for example,

$$x_3 = -\frac{m_1}{m_3} x_1 - \frac{m_2}{m_3} x_2.$$

This equation is a *constraint* and it can be written as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m_1/m_3 & -m_2/m_3 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}.$$

Since x_3 can be written in terms of the other two coordinates, this matrix equation can be reduced in dimension to

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -m_1/m_3 & -m_2/m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_{\mathcal{C}},$$

where the subscript $\mathcal{C}$ denotes a constrained system.

In matrix-vector notation, the displacement vector and the velocity vector can be written in terms of constrained coordinates,

$$\begin{aligned} \{x\} &= [\mathcal{C}] \{x\}_{\mathcal{C}} \\ \{\dot{x}\} &= [\mathcal{C}] \{\dot{x}\}_{\mathcal{C}}, \end{aligned}$$

where $[\mathcal{C}]$ is the *constraint matrix*,

$$[\mathcal{C}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -m_1/m_3 & -m_2/m_3 \end{bmatrix},$$

that relates the original system of coordinates to the reduced system.

By interpreting the three degree-of-freedom unrestrained system as a constrained system, the order has been reduced by one and the system has a new stiffness matrix that is positive definite. Thus,

$$\begin{aligned} T &= \frac{1}{2} \{\dot{x}\}^T [M] \{\dot{x}\} \\ &= \frac{1}{2} \{\dot{x}\}_{\mathcal{C}}^T [\mathcal{C}]^T [M] [\mathcal{C}] \{\dot{x}\}_{\mathcal{C}} = \frac{1}{2} \{\dot{x}\}_{\mathcal{C}}^T [\Lambda] \{\dot{x}\}_{\mathcal{C}} \end{aligned} \quad (6.130)$$

$$\begin{aligned} V &= \frac{1}{2} \{x\}^T [K] \{x\} \\ &= \frac{1}{2} \{x\}_{\mathcal{C}}^T [\mathcal{C}]^T [K] [\mathcal{C}] \{x\}_{\mathcal{C}} = \frac{1}{2} \{x\}_{\mathcal{C}}^T [\Psi] \{x\}_{\mathcal{C}}, \end{aligned} \quad (6.131)$$

where the newly defined matrices $[\Lambda] = [\mathcal{C}]^T [M] [\mathcal{C}]$ and $[\Psi] = [\mathcal{C}]^T [K] [\mathcal{C}]$ are associated with the constrained system.

At this point we can work with the constrained system,

$$[\Lambda] \{\ddot{x}\}_{\mathcal{C}} + [\Psi] \{x\}_{\mathcal{C}} = \{0\},$$

where we find the eigenvalues (ω_1, ω_2) and eigenvectors ($\{\hat{u}\}_{1\mathcal{C}}, \{\hat{u}\}_{2\mathcal{C}}$) of the reduced order model. Then, to recover the modes of the physical system we need to perform the following operations,

$$\begin{aligned} \{\hat{u}\}_1 &= [\mathcal{C}] \{\hat{u}\}_{1\mathcal{C}} \\ \{\hat{u}\}_2 &= [\mathcal{C}] \{\hat{u}\}_{2\mathcal{C}}, \end{aligned}$$

where $\{\hat{u}\}_1$ and $\{\hat{u}\}_2$ are the 3×1 modal vectors of the original three degree-of-freedom system.

The question is how to combine the rigid-body motion with the underlying modal vibration. We recognize that rigid-body behavior is different than modal behavior. It is necessary to go back to Equation 6.19, which is repeated here, to determine the rigid-body solution,

$$\ddot{Y} + \omega^2 Y = 0.$$

To solve this equation for the rigid-body mode, we set $\omega = 0$ and find that the rigid-body mode is governed by $\ddot{Y} = 0$. This equation can be integrated twice, resulting in

$$Y(t) = C_1 t + C_2,$$

where C_1 and C_2 are determined from the initial displacement and velocity.

The complete solution is then the sum of the rigid-body and the modal displacements,

$$\{x(t)\} = (C_1 t + C_2)\{\hat{u}\}_0 + \sum_{i=1}^2 A_i \{\hat{u}\}_i \cos(\omega_i t - \phi_i).$$

The modes $\{\hat{u}\}_1$ and $\{\hat{u}\}_2$ are superimposed on the rigid-body dynamics of the first mode.

Consider two examples, one which follows the procedure just discussed, and the other which proceeds by a direct analysis.

Example 6.20 Unrestrained Rotation

Solve the three degree-of-freedom rotational model of Figure 6.45 and show how unrestrained rotation has applications to rotating machinery.

Solution: From a free-body diagram of each rotor and the Newton-Euler equation, we can derive three equations of motion for the system. Alternatively, using the approach discussed in Section 6.3.1, the inertia and stiffness matrices can be derived via the expressions of kinetic and strain energies, respectively. These are

$$\begin{aligned} T &= \frac{1}{2}(I_1 \dot{\theta}_1^2 + I_2 \dot{\theta}_2^2 + I_3 \dot{\theta}_3^2) = \frac{1}{2}\{\dot{\theta}\}^T [I] \{\dot{\theta}\} \\ V &= \frac{1}{2} \left[\frac{GJ_1}{L_1} (\theta_2 - \theta_1)^2 + \frac{GJ_2}{L_2} (\theta_3 - \theta_2)^2 \right] \\ &= \frac{1}{2}\{\theta\}^T [K] \{\theta\}, \end{aligned}$$

where the property matrices $[I]$ and $[K]$ can be written as

$$\begin{aligned} [I] &= \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \\ [K] &= \begin{bmatrix} GJ_1/L_1 & -GJ_1/L_1 & 0 \\ -GJ_1/L_1 & GJ_1/L_1 + GJ_2/L_2 & -GJ_2/L_2 \\ 0 & -GJ_2/L_2 & GJ_2/L_2 \end{bmatrix}. \end{aligned} \quad (6.132)$$

We assume that the rotation can be written as $\theta_j(t) = \Theta_j \exp\{i\omega t\}$, and by substituting $\theta_j(t)$ and $\ddot{\theta}_j(t)$ into the equations of motion, the eigenvalue problem is found, $\omega^2 [I] \{\Theta\} = [K] \{\Theta\}$.

In this case the stiffness matrix $[K]$ is singular – that is, $\det [K] = 0$. The conclusion is that the first mode is a rigid-body mode with frequency $\omega_0 = 0$ and mode shape $\{\Theta\}_0 = \Theta_0 \{1 \ 1 \ 1\}^T$.

In order to generate the constraint equation, we make use of the fact that the eigenvectors are orthogonal to each other. Therefore,

$$\{\Theta\}_0 [I] \{\Theta\} = \Theta_0 (I_1 \Theta_1 + I_2 \Theta_2 + I_3 \Theta_3) = 0,$$

and since $\Theta_0 \neq 0$, we set the expression in parenthesis to zero,

$$I_1 \Theta_1 + I_2 \Theta_2 + I_3 \Theta_3 = 0. \quad (6.133)$$

Using the relation between $\theta_j(t)$ and Θ_j , Equation 6.133 becomes

$$\Theta_j = \frac{\theta_j(t)}{\exp\{i\omega t\}}.$$

Therefore,

$$I_1 \theta_1(t) + I_2 \theta_2(t) + I_3 \theta_3(t) = 0,$$

which indicates that the three degree-of-freedom system can be specified in terms of two coordinates,

$$\theta_3(t) = -\frac{I_1}{I_3} \theta_1(t) - \frac{I_2}{I_3} \theta_2(t).$$

The relation between the constrained motion and the arbitrary motion becomes

$$\begin{aligned} \begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{Bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -I_1/I_3 & -I_2/I_3 & 0 \end{bmatrix} \begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{Bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -I_1/I_3 & -I_2/I_3 \end{bmatrix} \begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix}_{\mathbb{C}}. \end{aligned}$$

In this problem, the constraint matrix is

$$[\mathbb{C}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -I_1/I_3 & -I_2/I_3 \end{bmatrix}.$$

Using Equations 6.130 and 6.131, the transformed mass matrix is given by

$$[\Lambda] = [\mathbb{C}]^T [I] [\mathbb{C}] = \frac{1}{I_3} \begin{bmatrix} I_1(I_1 + I_3) & I_1 I_2 \\ I_1 I_2 & I_2(I_2 + I_3) \end{bmatrix}$$

and the transformed stiffness matrix is given by

$$\begin{aligned} [\Psi] &= [\mathbb{C}]^T [K] [\mathbb{C}] \\ &= \frac{1}{I_3^2} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Psi_{11} &= k_1 I_3^2 + k_2 I_1^2 \\ \Psi_{12} &= -k_1 I_3^2 + k_2 I_1 (I_2 + I_3) \\ \Psi_{21} &= -k_1 I_3^2 + k_2 I_1 (I_2 + I_3) \\ \Psi_{22} &= (k_1 + k_2) I_3^2 + k_2 I_2 (2I_3 + I_2), \end{aligned}$$

and where $k_i = GJ_i/L_i$, $i = 1, 2, 3$. We use these matrices in the associated constrained eigenvalue problem to find

the two natural frequencies and modes, which are then added to the rigid-body mode for the complete solution.

If the torsional stiffnesses are $k_1 = k$ N-m/rad, $k_2 = 2k$ N-m/rad and the inertias are $I_1 = I$ kg-m²/rad, $I_2 = 2I$ kg-m²/rad, $I_3 = 3I$ kg-m²/rad, then

$$[\Lambda] = \frac{2I}{3} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix},$$

and

$$[\Psi] = \frac{k}{9} \begin{bmatrix} 11 & 1 \\ 1 & 59 \end{bmatrix}.$$

Solving the eigenvalue problem for this two degree-of-freedom system, the characteristic equation is

$$\omega^4 - 3.17k\omega^2/I + 2.00(k/I)^2 = 0, \quad (6.134)$$

and

$$\omega_1 = 0.93\sqrt{\frac{k}{I}}, \quad \{\Theta\}_{1\mathcal{C}} = \begin{Bmatrix} 1.0 \\ 0.13 \end{Bmatrix} \quad (6.135)$$

$$\omega_2 = 1.52\sqrt{\frac{k}{I}}, \quad \{\Theta\}_{2\mathcal{C}} = \begin{Bmatrix} 0.77 \\ -1.0 \end{Bmatrix}. \quad (6.136)$$

Using the constraint matrix,

$$[\mathcal{C}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/3 & -2/3 \end{bmatrix},$$

we can recover the elastic modes by multiplying the modes $\{\Theta\}_{1\mathcal{C}}$ and $\{\Theta\}_{2\mathcal{C}}$ by the constraint matrix $[\mathcal{C}]$,

$$\begin{aligned} \{\Theta\}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/3 & -2/3 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 0.13 \end{Bmatrix} \\ &= \begin{Bmatrix} 1.0 \\ 0.13 \\ -0.42 \end{Bmatrix} \\ \{\Theta\}_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/3 & -2/3 \end{bmatrix} \begin{Bmatrix} 0.77 \\ -1.0 \end{Bmatrix} \\ &= \begin{Bmatrix} 0.77 \\ -1.0 \\ 0.41 \end{Bmatrix}. \end{aligned}$$

Figure 6.46 shows the rigid-body mode as well as the elastic modes. Note that the elastic modes and frequencies are the same as those for the respective two degree-of-freedom system. By this we mean that the elastic

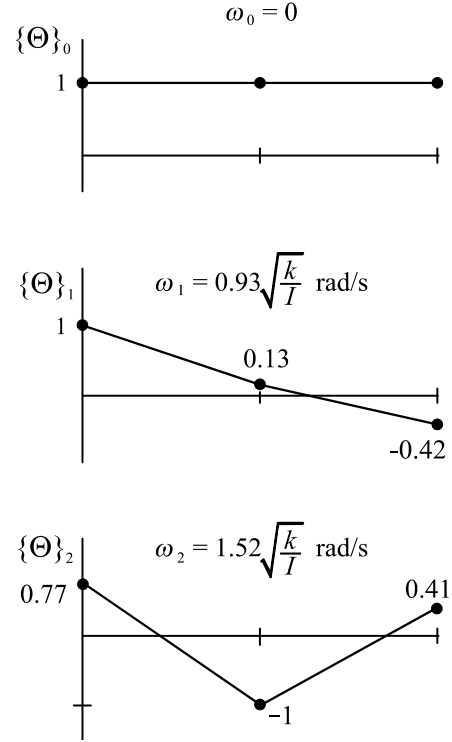


Figure 6.46: Three modes of an unrestrained system. The top figure is the rigid-body mode; the middle figure is the first mode; the bottom figure is the second mode.

modes and frequencies are numerically the same as those we would find if the rotor shafts were fixed at the end bearings and prevented from rotating.

The complete solution is given by the summation

$$\{\theta(t)\} = (C_1 t + C_2)\{\Theta\}_0 + \sum_{i=1}^2 A_i \{\Theta\}_i \cos(\omega_i t - \phi_i).$$

For this three degree-of-freedom system, three initial displacements and three initial velocities are required to solve for the six constants: C_1 , C_2 , A_1 , A_2 , ϕ_1 , and ϕ_2 . ■

Example 6.21 Unrestrained Translation

Solve the three degree-of-freedom unrestrained translational system shown in Figure 6.45 in a direct manner without using a constraint equation.

Solution: The equation of motion is

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

The characteristic equation for this system is

$$\lambda \left(\lambda - 3\frac{k}{m} \right) \left(\lambda - \frac{k}{m} \right) = 0, \quad (6.137)$$

with the roots $\lambda_{0,1,2} = 0$, k/m and $3k/m$, respectively. The case $\lambda_0 = \omega_0^2 = 0$ represents the rigid-body motion. Next, we find the eigenvector corresponding to $\omega_0 = 0$ by substituting ω_0 into the characteristic matrix equation,

$$\begin{bmatrix} k/m & -k/m & 0 \\ -k/m & 2k/m & -k/m \\ 0 & -k/m & k/m \end{bmatrix} \begin{Bmatrix} u_{10} \\ u_{20} \\ u_{30} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

from which we obtain

$$\begin{aligned} u_{10} &= u_{20} \\ u_{20} &= u_{30}. \end{aligned}$$

Thus, the rigid-body mode must be $\{u\}_0 = \{1 \ 1 \ 1\}^T$. By the same procedure, the other two modes are determined to be $\{u\}_1 = \{1 \ 0 \ -1\}^T$ and $\{u\}_2 = \{1 \ -2 \ 1\}^T$. The rigid-body mode is found by integrating $\ddot{Y}_0(t) = 0$ twice giving $Y_0(t) = (C_1 t + C_2)$. The complete solution is

$$\{x(t)\} = (C_1 t + C_2)\{u\}_0 + \sum_{i=1}^2 A_i \{u\}_i \cos(\omega_i t - \phi_i).$$

For this three degree-of-freedom system, three initial displacements and three initial velocities are required to solve for the six constants: C_1 , C_2 , A_1 , A_2 , ϕ_1 , and ϕ_2 . ■

The two example problems demonstrate that we can consider an unrestrained system as (a) a reduced order constrained system or (b) an ordinary vibrating system. As a constrained system, we can more easily see that the vibrations of the system are superimposed onto the dynamic behavior. The first mode behaves as though the system is a rigid body.

6.9.3 Repeated Frequencies

Next, we consider the possibility that two of the system natural frequencies are equal. Recall the modal Equations 6.30 and 6.31,

$$\begin{aligned} \frac{u_{21}}{u_{11}} &= -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} \\ \frac{u_{22}}{u_{12}} &= -\frac{k_{11} - \omega_2^2 m_1}{k_{12}}. \end{aligned}$$

What happens to the mode shapes if $\omega_1 = \omega_2$? If this occurs, the eigenvectors are *not defined uniquely* and the problem becomes *indeterminate*. Since both eigenvectors have the same natural frequency, any initial displacement

of the system will oscillate at ω_1 . If an n degree-of-freedom system has $n - 1$ eigenvectors because two of the frequencies are identical, then the n^{th} independent eigenvector must be constructed. The details will not be discussed here, but it is noted that numerically determined eigenvalues that are more closely spaced than the round-off errors of the computer are viewed as repeated values. The physical meaning of repeated eigenvalues is discussed further from a geometrical perspective in the next section.

6.10 Eigenvalue Geometry

It is possible to add insight to the matrix manipulations that led to the eigenvalue problem and the normal modes. In particular, this section explores some of the geometric underpinnings⁴⁶ of the eigenvalue problem.

Operations with matrices of the kind that were used in the previous sections to derive the eigenvalues and eigenvectors of a vibratory system are closely linked to the analytical geometry of *second-order surfaces* such as ellipsoids and hyperboloids. Such second-order surfaces can be depicted visually in three dimensions, but the analogies that are drawn are valid for higher order dimensional spaces, and the interpretations will hold for any n degree-of-freedom system and its eigenproblem. Thus, “*the entire theory of linear operators – whether they appear as systems of linear algebraic equations, or as linear ordinary or partial differential equations, or linear integral equations – can be formulated as a geometrical problem, associated with a certain second-order surface.*”⁴⁷

Consider the equation of an ellipse in two dimensions,

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 = 1, \quad (6.138)$$

the equation of an ellipsoid in three dimensions,

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 1, \quad (6.139)$$

and the equation of an n -dimensional generalized ellipsoid,

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2 = 1. \quad (6.140)$$

These forms of the surface equations are such that *the major and minor axes coincide with the axes of our frame of reference*. However, the frame of reference of a vibration problem is generally determined by the physical problem, and the ellipsoid will be in a slanted position with respect to all the axes of the n -dimensional space, as in Figure 6.47. We can anticipate that the orientation is directly related to the physical parameters of the particular problem. Each problem has a different orientation.

⁴⁶This discussion is based on the book **Applied Analysis**, C. Lanczos, Dover Publications, 1988, pp.81-95.

⁴⁷Lanczos, p.82.

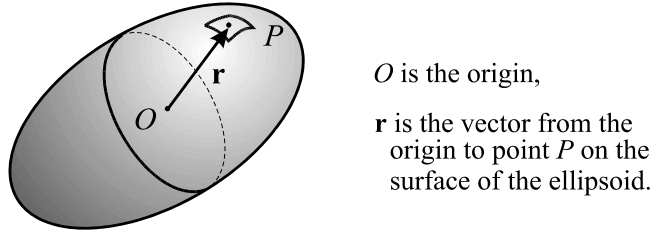


Figure 6.47: Arbitrarily oriented ellipsoid.

For this arbitrary orientation, the equation for the surface will include products of the coordinate axes. The equations above become, respectively, for the ellipse,

$$(a_{11}x_1 + a_{12}x_2)x_1 + (a_{21}x_1 + a_{22}x_2)x_2 = 1,$$

for the ellipsoid,

$$\begin{aligned} &(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)x_1 \\ &+ (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)x_2 \\ &+ (a_{31}x_1 + a_{32}x_2 + a_{33}x_3)x_3 = 1, \end{aligned}$$

and for the n -dimensional generalized ellipsoid,

$$\begin{aligned} &(a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n)x_1 \\ &+ (a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n)x_2 \\ &+ \dots \\ &+ (a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n)x_n = 1. \end{aligned} \quad (6.141)$$

The terms $x_i x_j$ can be combined with $x_j x_i$ by adding coefficients $a_{ij} + a_{ji}$. Since this sum is symmetric with respect to an exchange of indices i, j , then $a_{ij} = a_{ji}$. Such a *symmetric* matrix is invariant under a transposition of rows and columns: $[A]^T = [A]$.

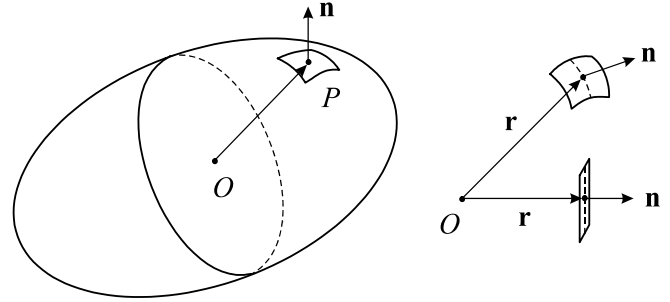
Using matrix notation, Equation 6.141 becomes

$$\{x\}^T [A] \{x\} = 1. \quad (6.142)$$

The vector $\{x\}$ has the significance of a *radius vector*, connecting an arbitrary point on the surface P with the origin O . Our goal is to find the coordinate system in which the surface equation has no *cross terms* and has the form of Equations 6.138 to 6.140. We expect these axes to have special properties.

We need to explore what characterizes the principal axes of a quadratic surface. Figure 6.48 depicts the vector $\mathbf{n}$ normal to the tangent plane at point P . (Such a normal can be constructed at every point on the surface.) The radius vector $\mathbf{r}$ is generally not parallel to the normal vector. Only in exceptional cases is $\mathbf{r} \parallel \mathbf{n}$. For this particular set of directions the name *principal axes* is given.

In Equation 6.142, the matrix-vector product $[A]\{x\}$ can be interpreted as a transformation of the vector $\{x\}$ into some new vector $\{y\}$, that is, $[A]\{x\} = \{y\}$. Thus,

Figure 6.48: Normal vector $\mathbf{n}$ at the surface: general case and principal case. We search for the principal axes of a quadratic surface where $\mathbf{r}$ is parallel to $\mathbf{n}$.

the original direction of vector $\{x\}$ is transformed to a new magnitude and direction $\{y\}$. In the case where the radius vector is parallel to the normal at the surface, the *direction is not changed*. Only the *magnitude is changed*. Therefore, following our interpretation of the above matrix-vector product, we must have

$$[A]\{x\} = \lambda\{x\}. \quad (6.143)$$

This equation is quite familiar as it is the eigenvalue problem considered earlier in this chapter. Here, the equation arises as a result of the search for the principal axes of a quadratic surface.

Consider now Equations 6.142 and 6.143. Matrix $[A]$ can be eliminated, leaving $\lambda\{x\}^T\{x\} = 1$, or $\{x\}^T\{x\} = 1/\lambda$. Thus λ_i is the reciprocal of the square of the distance from a point on the surface to the center.

Now that we have given the geometric significance of the principal axes, we also see that the frame of reference of the associated matrix can be changed to one where the matrix is in its principal form and diagonal. The counterpart in vibration is the simultaneous diagonalization of the mass and stiffness matrices in modal analysis, where the modal coordinates are the principal coordinates.

Repeated Frequencies

We can also consider the interpretation of repeated roots or frequencies.⁴⁸ We have not studied this problem analytically because it is a topic that is too specialized for an introductory text. But some physical insight can be imparted with the discussion that follows.

As two or more distinct roots approach each other, leading to repeated or multiple frequencies, the ellipse of the last section gradually becomes a circle. The principal axes do not disappear, but become two mutually perpendicular *diameters* of the circle. That same circle can be the limiting position of any ellipse and therefore any two perpendicular diameters of the circle can serve as principal axes. This argument can be extended to the sphere,

⁴⁸Lanczos, p.94.

where any three perpendicular diameters can serve as the principal axes, and to any n -dimensional case.

The existence of multiple roots does not invalidate the existence of mutually perpendicular axes. It only means that some of these axes are no longer determined uniquely and therefore can be replaced by other equally valid axes. The collapse of certain frequency values into one is not connected with a corresponding collapse of the associated axes since the mutual perpendicularity of the principal axes prevents them from ever collapsing into one. However, multiplicity of frequencies (or closely spaced frequencies) can cause numerical difficulties. Research has been conducted to develop computational approaches for such problems.⁴⁹

In the remaining sections of this chapter, we study topics that apply what we have already learned to a few problems of interest. The first of these is the periodic structure.

6.11 Periodic Structures

Periodic structures are those with a repetitive pattern. Each periodic subsystem known as a *bay* is designed to be identical and joined to the next bay in the same manner. Examples of such structures are sections of an aircraft fuselage that have repetitive stiffeners on a shell, turbine blades that have circular periodicity, and antenna dishes. A schematic of a periodic structure is shown in Figure 6.49.

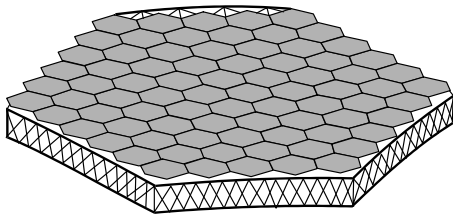


Figure 6.49: A network of mirrors that are supported by a framework is an example of a structure with periodic geometric properties.

It is generally assumed that each bay of a repetitive structure is identical. Then, only one bay plus the boundary conditions need to be analyzed in free vibration, or in forced vibration, if the loading is also symmetric. However, in actual structures, the periodic nature can not be exact and there can be very small differences in material and geometric properties when moving from one bay to the next. Recent research has shown that even small imperfections can result in significant changes

in structural response for such *near-periodic* structures. We will first examine the behavior of an exactly periodic structure, and then study how an imperfection in the periodicity affects the structural response.

6.11.1 Perfect Lattice Models

Perfectly periodic discrete structures are sometimes called *lattice* models because, historically, such spring-mass systems looked like lattices to the physicists who used them to model the interactions of atoms in a solid. Here, a 10 degree-of-freedom structure undergoing longitudinal motion (along the axis of the structure) is formulated and some numerical results are presented and discussed.

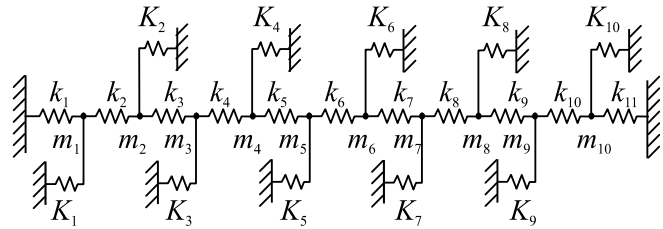


Figure 6.50: Ten bay structure in longitudinal motion. There are ten masses m_i , eleven coupling springs k_i , and ten fixed springs K_i .

Consider the ten mass structure of Figure 6.50. Each mass represents the inertial properties of a *substructure* or *bay*. A mass m_i is attached to neighboring masses by coupling springs k_i and k_{i+1} that represent the coupling stiffnesses between substructures. To represent the stiffness of a substructure, each mass m_i is also attached to spring K_i that is fixed to an immovable point. This is meant to be a conceptual model of certain classes of structures that are weakly coupled internally, but are attached to a much stiffer base or external structure.

Additional examples include space-frame structures such as the space station, solar arrays attached to a satellite by highly stiff supports, and rotating machinery or other circular symmetric systems, where flexible blades are attached to a very stiff shaft. In Figure 6.50, if k_1 and k_{11} are the same spring, then this model can be used for circular symmetric structures as well.

The equation of motion for the periodic structure is

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\},$$

where

$$[M] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & & m_{10} \end{bmatrix}, \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{10} \end{Bmatrix}$$

⁴⁹ A discussion of repeated frequencies is given beginning on p.245 of **Mechanical and Structural Vibrations, Theory and Applications**, J.H. Ginsberg, Wiley, 2001. Also discussed in this text, and of relevance here, are the topics of periodic structures and mode localization, addressed in the next section.

and

$$[K] = \begin{bmatrix} d_1 & -k_2 & \cdots & 0 \\ -k_2 & d_2 & & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & & & -k_{10} \\ 0 & \cdots & -k_{10} & d_{10} \end{bmatrix},$$

where the diagonal terms are

$$\begin{aligned} d_1 &= k_1 + K_1 + k_2 \\ d_2 &= k_2 + K_2 + k_3 \\ &\vdots \\ d_{10} &= k_{10} + K_{10} + k_{11}. \end{aligned}$$

The stiffness matrix is *tridiagonal*, meaning that nonzero elements appear only on the main and the two adjacent diagonals. The main diagonal is of the form $k_i + K_i + k_{i+1}$.

The procedures we have learned for free vibration analysis apply here for the evaluation of natural frequencies, modes, and response. Our purpose is to examine the displacement response for each mass where we take for this example $m_i = 10$ kg and $K_i = 100$ N/m for all i . A parameter found to be important in the behavior of such systems is the *coupling stiffness ratio*, defined as

$$\text{CSR} = \frac{k_i}{K_i}.$$

Once the CSR is prescribed, k_i can be determined since K_i is already known. For example, a *weakly coupled* structure may have a coupling stiffness ratio of $\text{CSR} = 0.01$ or 1%, and, therefore, $k_i = 1$ N/m. The degree of coupling between bays, k_i , affects how fast energy can propagate from one bay to the next. Energy is exchanged due to the compression and elongation of the springs during oscillation. For higher k_i values, energy from one mass is transferred faster to the next mass. This coincides with our understanding of the behavior of coupled pendula.

Figure 6.51 shows the displacement response of each of the ten masses to a unit initial displacement applied at mass m_1 . The time history is 600 s long, and we can see how the wave travels from position one to ten and then reflects back from the right end. Since the periodic system is perfectly periodic with no discontinuities or imperfections, there are no locations where a *mismatch* between the properties of adjacent cells or bays would result in some reflected energy.

The next section discusses the effects of an imperfection on the character of the response.

6.11.2 Effects of Imperfection

The effects of imperfection can be studied by introducing a parameter that is a measure of the physical differences

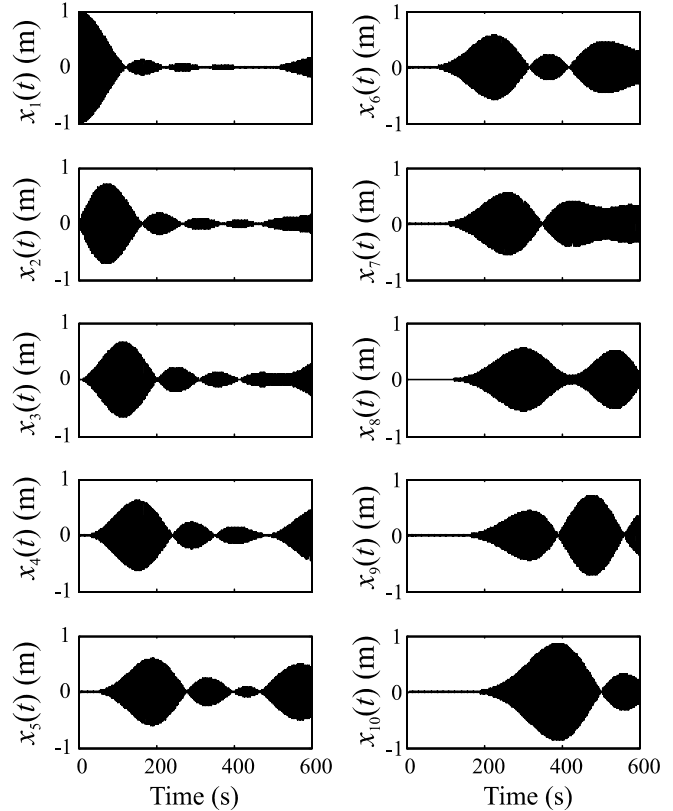


Figure 6.51: Displacement response of ideal structure to $x_1(0) = 1$ and all other initial conditions equal zero. We see the initiation of a displacement in the top left graph. This displacement profile moves to the right, as can be seen in the subsequent graphs. Due to the timescale the response is compressed and appears solid.

between adjacent bays.⁵⁰ Assume that imperfection is introduced due to differences between bay stiffnesses K_i . This *stiffness imperfection ratio* is defined as

$$\text{SIR} = \frac{K_d - K}{K},$$

where K_d is the disordered bay stiffness and K is the ideal bay stiffness. For example, if $\text{SIR} = +10\%$ for an imperfectly periodic structure with ideal bay stiffness $K = 100$ N/m, then this implies that

$$\begin{aligned} K_d &= K(\text{SIR} + 1) \\ &= 100(0.10 + 1) = 110 \text{ N/m}. \end{aligned}$$

If $\text{SIR} = -10\%$, then $K_d = 90$ N/m. Performing a free vibration analysis for a ten bay structure with 10% stiffness imperfection, for example, we find that in the fifth bay vibrational energy is *localized* about the fifth mass for a $\text{CSR} = 1\%$. A unit initial displacement at the first

⁵⁰ See S. Mester and H. Benaroya, "A Parameter Study of Localization," *Shock and Vibration*, Vol.3, No.1, 1996, pp.1-10.

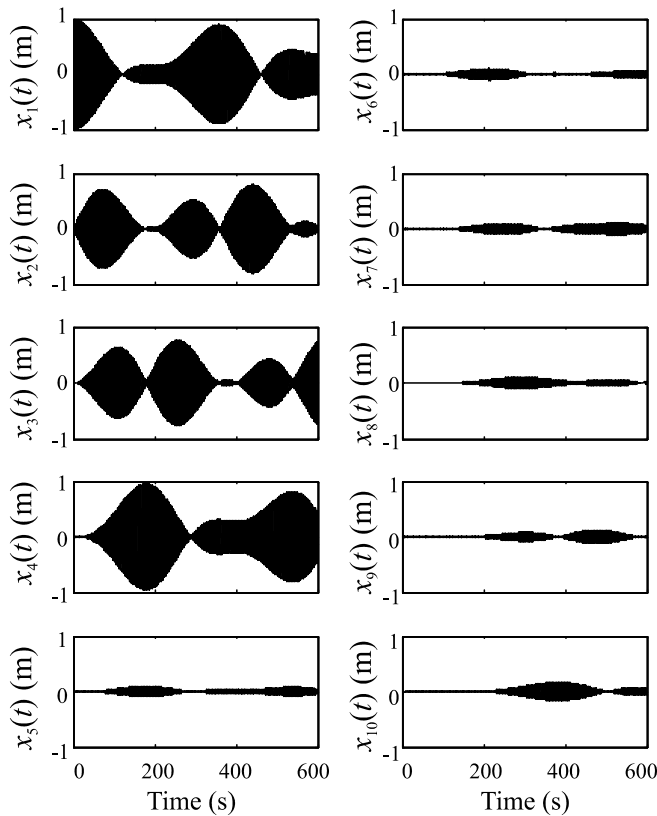


Figure 6.52: Displacement response of a structure with $SIR=10\%$ located at the fifth bay, $CSR=1\%$, with first bay loaded by $x_1(0) = 1$ with all other initial conditions equal to zero. We see that the wave cannot pass through location 5 due to the imperfection at that location. Due to the timescale the response is compressed and appears solid.

mass is used to initiate a free vibration of the system. The resulting responses are shown in Figure 6.52.

The modes become distorted due to imperfections, as seen in Figure 6.53. There is a change from a smooth mode shape to an irregular or distorted mode due to the addition of imperfections.

6.12 Inverse Vibration Problem

Inverse vibration problems come in many forms. Such problems are called inverse because what is known and what is evaluated are reversed from the problems we have been studying. For example, in previous problems the system mass, damping, and stiffness were known quantities, and the known input force was used to solve for the response. In an inverse problem, the force and response are known and are used to evaluate the system mass, damping, and stiffness properties. Such problems are more difficult to solve than the usual *forward* problems because there may be more than one solution. That

is, there may be more than one combination of system properties that satisfies the force-response relation, and so the problems may not have a *unique* solution.

A possible application of this type of analysis includes a method for nondestructive testing and evaluating structural integrity. Measurements at regular time intervals would be able to detect shifts in the spectral (frequency) properties of a given structure over time. In particular, such techniques could be utilized to estimate and locate changes in stiffness due to structural aging.

In this section we consider the use of eigenvalue data to calculate the properties of a linear dynamic system. Suppose a set of experiments is run to estimate the natural frequencies of a structure. For example, consider a multi degree-of-freedom system driven by a variable-frequency load. At each resonance, there is a peak response and the phase angle is $\pi/2$ rad. (See Chapter 3.) Can this frequency data be used to evaluate the mass and stiffness properties of the structure? If not, then what additional information is necessary? This is an elementary description of an inverse vibration problem.

We will examine this problem using two approaches. The first, based on the work of Gladwell,⁵¹ is a deterministic approach that assumes all frequency data are exact. Such a study provides us with a new way of thinking about the relationships between structural properties and their free vibration characteristics.

In the second approach, it is more realistically assumed that the data contain small errors regardless of the sophistication of the experimental setup. We are interested in finding the mass and stiffness *sensitivities* to uncertainties in the frequency data. That is, we wish to know whether the mass and stiffness values will vary substantially or not as a result of frequency parameter uncertainties. This is shown in Section 9.13.

In a typical forward vibration problem, the physical parameters of the system are known at least approximately. These parameters are the masses and spring constants for a discrete system, or the density, modulus of elasticity, and physical dimensions for a continuous system. From an analysis of these parameters, the natural frequencies and the response to a particular excitation can be determined.

In an *inverse* vibration problem, the physical parameters of a system are determined from the spectral data, that is, frequencies and mode shapes. Alternatively, eigenvalues and eigenvectors are used to estimate the mass and stiffness values. Consider the two-mass, two-spring system shown in Figure 6.54. From vibration theory, it is known that this system has two distinct positive eigenvalues, λ_1 and λ_2 , that are the roots of the

⁵¹**Inverse Problems in Vibration**, G.M.L. Gladwell, Martinus Nijhoff Publishers, 1986, and G.M.L. Gladwell, "Inverse Problems in Vibration," *Applied Mechanics Reviews*, Vol.39, No.7, July 1986, pp.1013-1018.

characteristic equation,

$$\lambda^2 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right] \lambda + \frac{k_1 k_2}{m_1 m_2} = 0.$$

The respective natural frequencies are equal to $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$. The equation yields the following relations between the eigenvalues:

$$\lambda_1 + \lambda_2 = \frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}, \quad \lambda_1 \lambda_2 = \frac{k_1 k_2}{m_1 m_2}. \quad (6.144)$$

In inverse vibration problems, the goal is to use the eigenvalue data to reconstruct the physical system properties. For this example, there are two equations for the four unknown values k_1 , k_2 , m_1 , and m_2 . This implies that there are an infinite number of two degree-of-freedom systems that have the eigenvalues λ_1 and λ_2 . It is therefore necessary to introduce two more equations so that the system can be determined uniquely.

In order to obtain a third equation, consider the system shown in Figure 6.55. This system is identical to the previous one, except that the right end has been fixed, which reduces the system to a single degree of freedom. This constrained system has a single known eigenvalue, λ_3 , given by

$$\lambda_3 = \frac{k_1 + k_2}{m_1}. \quad (6.145)$$

Constraining the system is one way of obtaining an additional equation.

By algebraically manipulating Equations 6.144 and 6.145, the ratios between the system properties can be found to be

$$R_1 \equiv \frac{k_2}{m_2} = \lambda_1 + \lambda_2 - \lambda_3 \quad (6.146)$$

$$R_2 \equiv \frac{k_1}{m_1} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} \quad (6.147)$$

$$R_3 \equiv \frac{k_2}{m_1} = \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}{\lambda_1 + \lambda_2 - \lambda_3}, \quad (6.148)$$

where the ratios are the squares of frequencies. These equations are valid only if the eigenvalues are distinct and $\lambda_1 \neq \lambda_3 - \lambda_2$. Such degenerate cases require other techniques. Ratios R_1 , R_2 , and R_3 must all be positive if there is to be a corresponding physical system, since all masses and stiffnesses are positive quantities. This requires that the eigenvalues satisfy

$$0 < \lambda_1 < \lambda_3 < \lambda_2, \quad (6.149)$$

which is predicted for this type of system by the *inclusion principle*.⁵²

⁵²The inclusion principle, sometimes also called the *Sturmian separation theorem*, is a statement of how the natural frequencies of a system decrease as the number of degrees of freedom increases.

The ratios obviously reveal a great deal about the dynamic properties of the system, but they do not uniquely identify it. In order to do this, additional information is needed. For example, if the total mass of the system, $m = m_1 + m_2$, is known, then the parameters can be solved uniquely using Equations 6.146 to 6.148, resulting in

$$k_1 = \frac{R_1 R_2}{R_1 + R_3} m \quad (6.150)$$

$$k_2 = \frac{R_1 R_3}{R_1 + R_3} m \quad (6.151)$$

$$m_1 = \frac{R_1}{R_1 + R_3} m \quad (6.152)$$

$$m_2 = \frac{R_3}{R_1 + R_3} m. \quad (6.153)$$

The system's total mass serves only to scale the results, so in many cases it may be sufficient to assume a value if it is not known explicitly.

Example 6.22 A Two Degree-of-Freedom Inverse Vibration Problem

For the system of Figure 6.54, assume that the following data were obtained from two experiments,

$$\begin{aligned} \omega_1 &= \sqrt{\lambda_1} = 2 \text{ Hz} \\ \omega_2 &= \sqrt{\lambda_2} = 22 \text{ Hz} \\ \omega_3 &= \sqrt{\lambda_3} = 5 \text{ Hz}, \end{aligned}$$

and that $m_1 + m_2 = 11$ kg. (The first experiment provided ω_1 and ω_2 , and the second experiment, corresponding to the case of Figure 6.55, provided ω_3 .) Solve for k_1 , k_2 , m_1 , and m_2 .

Solution: Note that ω_3 is the natural frequency of the structure when the second mass is fixed, as per Equation 6.145. Using Equations 6.146 to 6.148, we find

$$R_1 = 463, \quad R_2 = 4.18, \quad R_3 = 20.8,$$

all of which have units of $(\text{rad/s})^2$, and using Equations 6.150 to 6.153, the physical parameters are found to be

$$\begin{aligned} k_1 &= 44.0 \text{ N/m}, \quad k_2 = 219 \text{ N/m} \\ m_1 &= 10.5 \text{ kg}, \quad m_2 = 0.473 \text{ kg}. \end{aligned}$$

These results make sense physically since we expect a large disparity between mass and/or stiffness properties

For example, assume that there are two mathematical models of the same structure. One model is with three degrees of freedom, and the other is with two degrees of freedom. The first will have eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ while the other will have eigenvalues $\Lambda_1 \leq \Lambda_2$. The inclusion principle can be used to show that $\lambda_1 \leq \Lambda_1 \leq \lambda_2 \leq \Lambda_2 \leq \lambda_3$. This makes physical sense since the structure becomes less stiff, or more flexible, with more degrees of freedom, resulting in lower frequencies of oscillation.

if there is a significant difference between the natural frequencies. ■

While it has been shown that it is possible to derive closed-form solutions to the inverse vibration problem associated with a two degree-of-freedom system, for larger systems more intricate numerical approaches are necessary. Here, we restrict ourselves to problems that we can demonstrate analytically.

This problem is considered in Section 9.13 with uncertainties.

6.13 Fluid Sloshing in Container

Understanding the vibration of structures *containing* fluids is important for a broad range of applications. Studies of liquid behavior in containers such as the one shown schematically in Figure 6.56 have been used to evaluate the response of reservoirs and tanks to various kinds of excitation (such as earthquakes). Liquid motion is a concern in moving vehicles such as aircraft, spacecraft, ships, railroad cars, and trucks transporting liquid cargo.

The general goal in understanding fluid-structure interactions is to limit sloshing motions and thus prevent structural damage or interference with normal operations. One vital concern is the possibility that some of the frequencies associated with the fluid oscillations will overlap with structural natural frequencies, causing resonances and vibration magnification. In critical situations, sloshing can result in structural failure or dynamic instability.

Two specific critical examples are the sloshing of fuel in aircraft and in spacecraft. In spacecraft, the sloshing of liquid fuel due to space operations, shown schematically in Figure 6.57, occurs (a) during the launch phase at full gravity under high acceleration, and (b) during maneuvers in space under low gravity.

In low gravity, fluid *capillary forces* dominate and the fluid equilibrium is a curved surface rather than the flat surface we are familiar with at full gravity. Analysis of sloshing can provide information about fluid motion during vehicle maneuvers, about spacecraft stability, and for the design of propellant management devices for reducing sloshing. These are difficult but important areas of ongoing research.⁵³

There are analytical and computational approaches for

⁵³A comprehensive early review of this area of engineering research can be found in **The Dynamic Behavior of Liquids in Moving Containers**, H.N. Abramson, Editor, NASA Special Publication SP-106, U.S. Government Printing Office, 1966. Even though the information is not state-of-the-art, the volume discusses the key problem areas and presents some preliminary analyses.

A recent and excellent monograph is **Liquid Sloshing Dynamics: Theory and Applications**, R.A. Ibrahim, Cambridge University Press, 2005.

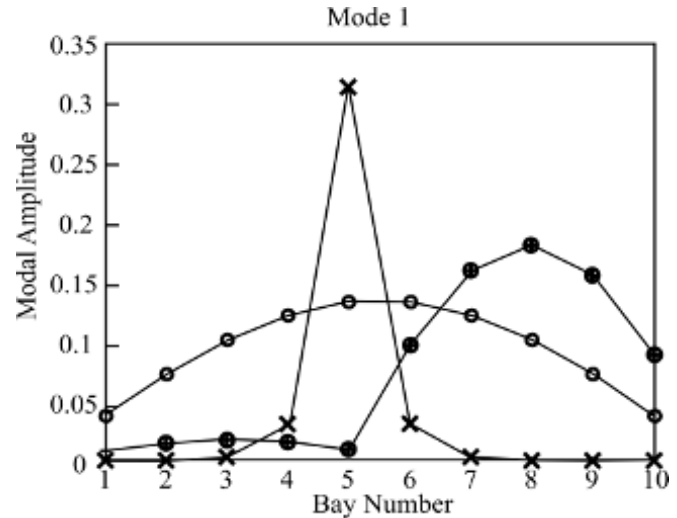


Figure 6.53: Distorted mode due to the imperfection at bay 5; $[- \times -]$ for negative disorder, $[- \circ -]$ for no imperfection, $[- + -]$ for positive disorder $SIR = 0.1$. Asymmetry in the disordered curve is due to the fact that disorder is off-center.

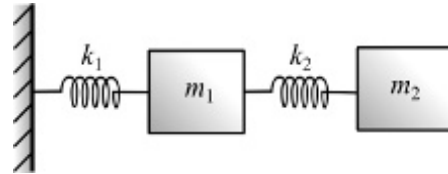


Figure 6.54: A two degree-of-freedom system.

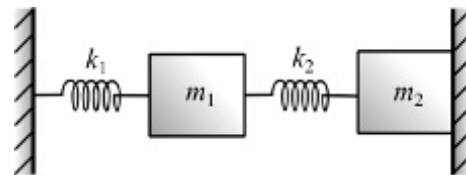


Figure 6.55: A two degree-of-freedom system constrained (fixed) at the right.

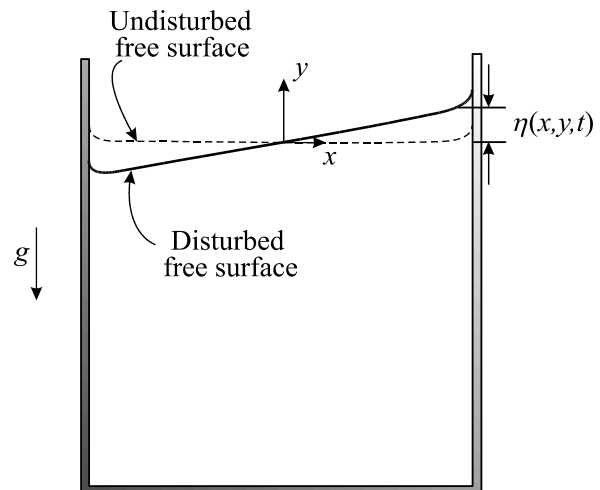


Figure 6.56: Container with moving liquid.

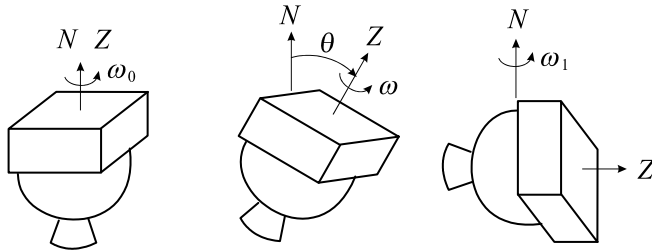


Figure 6.57: A schematic drawing representing a spinning satellite.

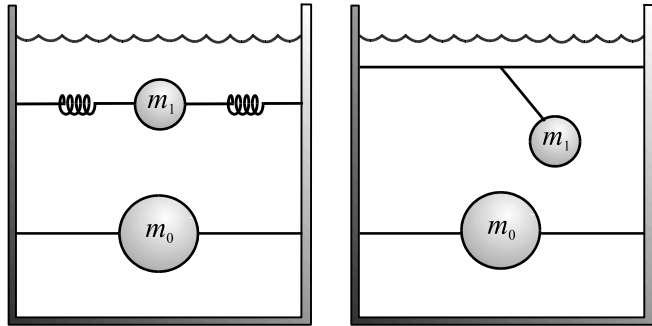


Figure 6.58: Simple mechanical models of linear, lateral sloshing, comprised of a spring-mass system on the left and a pendulum-mass system on the right.

studying the interaction between sloshing fluid and a vibrating structure. Earlier work attempted to use *equivalent linear mechanical models* to represent the fluid motion and the forces the fluid exerts on the structure. Such efforts, which continue to this day, replace the interior fluid by an equivalent single or multi degree-of-freedom system of masses, dampers, and springs, where the values of m , c , and k are determined experimentally from scale models. Generally, these equations of motion are nonlinear. As computational power improved, analytical mechanical models were partially replaced by computational models such as those based on the finite element method. However, even to this day there is much that is not understood about sloshing behavior, especially in severe environments such as impact or earthquake loading and in a low gravity environment.

Two linear equivalent mechanical models are shown in Figures 6.58 and 6.59, although current state-of-the-art models are nonlinear. Such mechanical models can provide significant insights into the fluid-structure interaction behavior.

Figures 6.58 and 6.59 show schematic representations of equivalent mechanical representations of a fluid. These can be very simple models with one sloshing mass m_1 and one mass m_0 representing that part of the fluid that does not slosh, as in Figure 6.58. Or the model may be more complex with many sloshing masses, as in Figure 6.59. Here, the analyst has more flexibility in modeling fluid behavior to fit the data. Pendula are also popular models

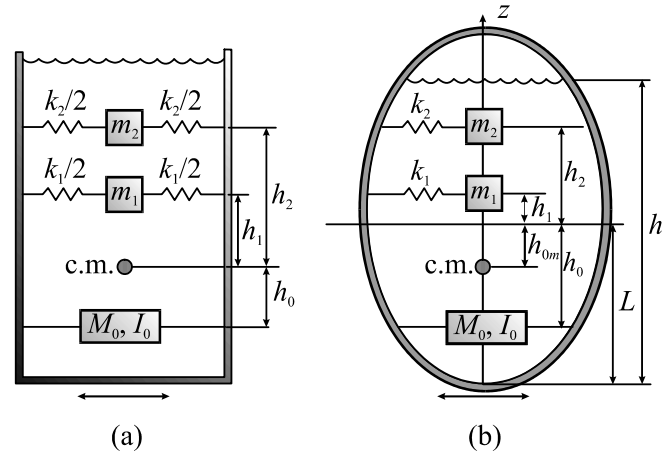


Figure 6.59: Multi degree-of-freedom mechanical models of linear, lateral sloshing in (a) a rectangular tank and (b) an ellipsoidal tank. Only two sloshing masses are shown, although any number of masses can be used. The parameters shown in the figures are determined by fitting the physical system and data.

for a sloshing liquid.

The equivalent mechanical model of Figure 6.58 is a two degree-of-freedom system with one equation of motion for each mass, m_0 and m_1 . The container along with the fluid that does not appreciably move (represented by m_0) would have an equation of motion that includes any external forces acting on the container. This is coupled to the mechanical model of the sloshing fluid (represented by m_1). The degree of sophistication of the mechanical models is based on the available data. In order to capture the more complex behavioral characteristics of the fluid, more degrees of freedom (and nonlinear characteristics for the fluid) are necessary, as shown in Figure 6.59.

6.14 Stability of Motion

The concepts of stability and equilibrium are central to vibration. In our studies of control in Chapter 10, we discuss that a primary goal of a control system is to stabilize an unstable dynamic system. Stability of motion can be understood as the property of a system that keeps it from undergoing undesirably large displacements. Here we wish to explore some aspects of the stability of motion by way of an example. In Chapter 11, an introduction to nonlinear oscillations and stability is provided.

When we study the vibration of a structure, we inherently assume that its oscillation is stable, that is, the oscillation is about a stable equilibrium. Otherwise, instability dominates the structural behavior and is, therefore, of greater concern than any mode of vibration. In the following example, we show how a system that looks mathematically similar to those studied earlier can have

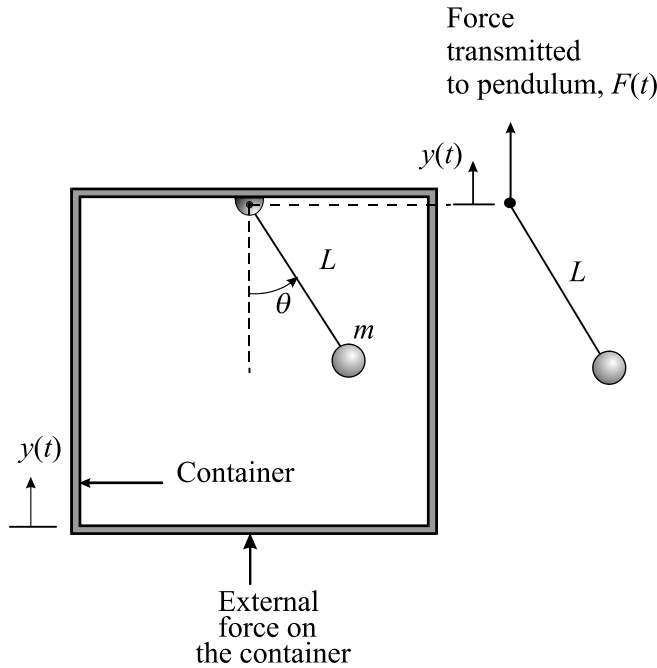


Figure 6.60: Pendulum forced at the base. Such a model can be used to represent a variety of applications. The isolated pendulum is shown at the right.

a very different governing equation and may become unstable for certain combinations of parameter values.

Example 6.23 Pendulum Forced at the Base

Figure 6.60 is a sketch of a pendulum rotating about a vertically translating point. This model can be an equivalent mechanical representation of a fluid in a container undergoing vertical motion. Examples include liquid fuel in a rocket, or stores of liquid in a tank on the ground excited by an earthquake. Derive the system equations of motion and discuss system behavior.

Solution: This problem seems no different than many earlier problems. However, its equation of motion has a unique feature that demonstrates a behavior that is new.

The governing equations are derived using Lagrange's equation for each generalized coordinate, y and θ . We assume that the container is rigid and only vertical motion is significant.

The kinetic energy can be expressed as

$$\begin{aligned} T &= \frac{1}{2}m[(L\dot{\theta}\cos\theta)^2 + (\dot{y} + L\dot{\theta}\sin\theta)^2] \\ &= \frac{1}{2}m[L^2\dot{\theta}^2 + 2L\dot{y}\dot{\theta}\sin\theta + \dot{y}^2], \end{aligned}$$

where Figure 6.61 shows the velocity components. The total potential energy equals the sum due to vertical displacement y and rotation θ ,

$$V = mg[L(1 - \cos\theta) + y],$$

where V is measured positive from position $\theta = 0$.

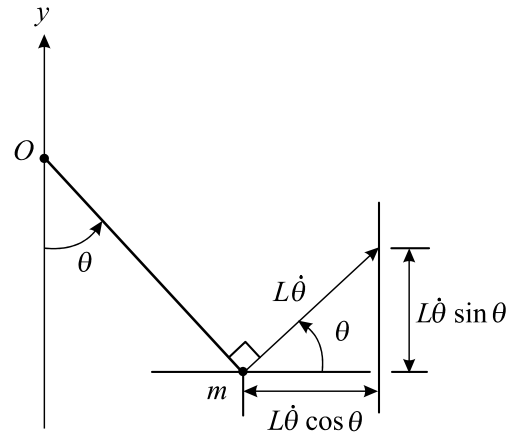


Figure 6.61: Velocity components of a pendulum with base motion.

The base motion results in a transmitted force, $F(t)$, on the pendulum pivot in the vertical direction y . Since there is no force in the θ direction, the virtual work simplifies, as follows,

$$\begin{aligned} \delta W &= F_\theta \delta\theta + F_y \delta y \\ &= 0 \cdot \delta\theta + F \delta y. \end{aligned}$$

We can now evaluate Lagrange's equation for each generalized coordinate. For the θ coordinate,

$$\begin{aligned} \frac{\partial T}{\partial \theta} &= \frac{1}{2}m[2L^2\dot{\theta} + 2L\dot{y}\sin\theta] \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) &= \frac{1}{2}m[2L^2\ddot{\theta} + 2L\ddot{y}\sin\theta + 2L\dot{y}\cos\theta] \\ \frac{\partial T}{\partial \theta} &= \frac{1}{2}m[2L\dot{\theta}\cos\theta] \\ \frac{\partial V}{\partial \theta} &= mgL\sin\theta. \end{aligned}$$

Substituting these into Lagrange's equation leads to

$$mL^2\ddot{\theta} + mL\ddot{y}\sin\theta + mgL\sin\theta = 0,$$

which can be simplified to

$$L\ddot{\theta} + (\ddot{y} + g)\sin\theta = 0. \quad (6.154)$$

For the y coordinate,

$$\begin{aligned} \frac{\partial T}{\partial \dot{y}} &= m[\dot{y} + L\dot{\theta}\sin\theta] \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= m[\ddot{y} + L\ddot{\theta}\sin\theta + L\dot{\theta}\cos\theta] \\ \frac{\partial T}{\partial y} &= 0 \\ \frac{\partial V}{\partial y} &= mg. \end{aligned}$$

The resulting equation of motion is

$$\ddot{y} + L\ddot{\theta} \sin \theta + L\dot{\theta}^2 \cos \theta + g = \frac{F}{m}. \quad (6.155)$$

Next, we look for the stability of motion about the equilibrium position $\theta = 0$. In the region about small θ , Equation 6.154 can be rewritten as

$$\ddot{\theta} + \frac{1}{L}(\ddot{y} + g)\theta = 0, \quad (6.156)$$

which shows the coupling between the generalized coordinates. For small θ , Equation 6.155 becomes

$$\ddot{y} + L\ddot{\theta} + L\dot{\theta}^2 + g = \frac{F}{m}, \quad (6.157)$$

which is simplified but still nonlinear. If θ and $\dot{\theta}$ are small, we can neglect higher order terms such as $\dot{\theta}^2$ and $\ddot{\theta}\theta$. In this case, Equation 6.157 is linearized to

$$\ddot{y} + g = \frac{F}{m}, \quad (6.158)$$

and is decoupled from the θ motion.

If we assume that the support motion is harmonic, $y(t) = A \cos \omega t$, we have an example of the base motion of a fluid filled container. Substituting the second derivative of $y(t)$ into Equation 6.158, we find that the force required to create the base motion is

$$F(t) = -mA\omega^2 \cos \omega t + mg.$$

Equation 6.156 then becomes

$$\ddot{\theta} + \frac{1}{L}(g - A\omega^2 \cos \omega t)\theta = 0, \quad (6.159)$$

which is an equation with a time-varying coefficient of θ . The term $(g - A\omega^2 \cos \omega t)$ is a harmonic function that can have negative or positive values depending on the values of A and ω .

If $g > A\omega^2$ then $g > A\omega^2 \cos \omega t$ and the coefficient of θ is always positive with a response $\theta(t)$ that is always stable. If $g < A\omega^2$, we obtain solutions with coefficients of the form $\exp(+\omega T)$ that grow without bound. Equations of this type, known as the Mathieu equation, are discussed further in Section 11.5. ■

Such problems of dynamic stability are important because they potentially affect all vibration studies. Equations such as 6.159, where the loading appears within a coefficient on the left-hand side of the governing equation rather than the right-hand side, are known as *parametrically excited* systems. We will discuss these systems in Chapter 11 when covering nonlinear oscillations.

6.15 Rayleigh's Quotient

In general, the vibration response of engineering structures cannot be solved exactly. There are several reasons. The number of degrees of freedom may be prohibitively large. The structural material may be nonhomogeneous and the variability may not be known. The structure may be geometrically too complex for an explicit mathematical description. The forcing may be too intricate for a simple formulation. Finally, there may be uncertainties in the system or forcing that preclude an exact deterministic or probabilistic formulation or solution.

In such cases, alternate approaches are necessary. We emphasize, however, that our previous work can be brought to bear in the development of such approaches. We build on the analytical models and generalize them for computational tractability. We account for material variability and geometrical complexity by techniques such as discretization that, in the limit, approach the actual system. The most prominent and well developed group of such modeling techniques goes by the name *finite element methods*. The reader is urged to learn about such methods from specialized texts on the subject.⁵⁴ We will not discuss these techniques in this book.

An intermediate step between analytical models and computational models is the development of *approximate models* that permit us to go beyond the simplest models without needing to resort to large scale computation.

In this section, *Rayleigh's quotient* for estimating the lower frequencies of a vibrating body is introduced. In Chapter 8 we will continue the study of approximation tools by applying these and other techniques to continuous systems.

We begin with the eigenvalue problem in the form

$$[K]\{u\}_r = \omega_r^2 [M]\{u\}_r, \quad r = 1, \dots, n,$$

for an n degree-of-freedom system where $\{u\}_i$ is a modal vector. Pre-multiplying each side by $\{u\}_r^T$ and solving for ω_r^2 we find

$$\omega_r^2 = \frac{\{u\}_r^T [K] \{u\}_r}{\{u\}_r^T [M] \{u\}_r}, \quad r = 1, \dots, n. \quad (6.160)$$

This equation is known as *Rayleigh's quotient*, which can be interpreted physically as the *ratio of the maximum of the potential energy to the maximum of the kinetic energy*. A "reasonable" eigenvector is substituted for $\{u\}_r$, resulting in an estimate of the eigenvalue ω_r . A reasonable eigenvector is one that satisfies boundary conditions such as on the slope and deflection. While r may represent any eigenvalue, it is almost impossible to provide a

⁵⁴ An introductory book on computational modeling that is theoretically comprehensive is **Mechanics of Structures: Variational and Computational Methods**, W.D. Pilkey and W. Wunderlich, CRC Press, 1994.

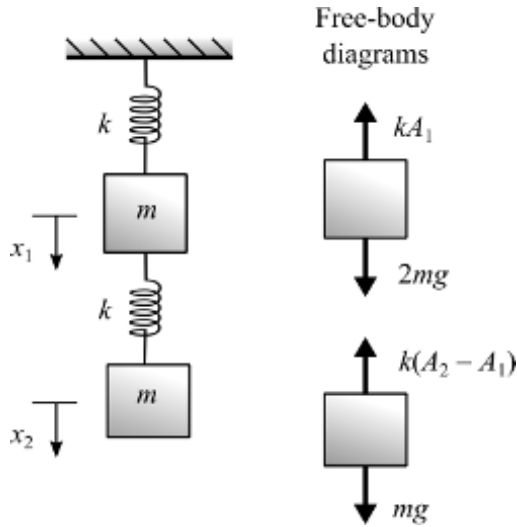


Figure 6.62: Two degree-of-freedom oscillator along with respective free-body diagrams in static equilibrium.

reasonable guess of the eigenvector for modes beyond the first several. Therefore, this approach is useful if we are interested in estimates of the lowest few modes of a system. Note that in Equation 6.160 normal modes are not used because they are not known. Generally, we guess the mode shape.

It can be shown that Rayleigh's quotient has a *minimum* in the neighborhood of a *fundamental* mode. If the assumed vector differs from the eigenvector $\{u\}_r$ by a small quantity of *first order* – say ε – then the ratio differs from the eigenvalue ω^2 by a quantity of the *second order*, ε^2 . As a practical matter, this means that even if our guess of the mode shape is not very accurate, the calculated frequency can still be accurate. The example below demonstrates this property.

Two additional important characteristics are that:

- Rayleigh's quotient for the lowest frequency is never lower than the fundamental frequency, and
- Rayleigh's quotient for the highest frequency is never higher than the highest frequency.

These characteristics are manifestations of the inclusion principle cited earlier. For intermediate frequency values, the respective Rayleigh's quotient can be above or below the actual value. These properties have practical implications.

Example 6.24 Rayleigh's Quotient for a Two Degree-of-Freedom System

Apply Rayleigh's quotient to the two degree-of-freedom oscillator shown in Figure 6.62 and estimate the frequencies of oscillation for the two modes.

Solution: The usefulness of Rayleigh's quotient depends on the ability of the analyst to guess the mode shape for

the frequency of interest, especially for the higher mode shapes. Here we equate the maximum kinetic energy to the maximum potential energy (as we once did for a single degree-of-freedom oscillator) to obtain Rayleigh's quotient. The maximum kinetic energy for a harmonic oscillator without damping occurs when the harmonic function is at its maximum. Suppose

$$x_1(t) = A_1 \cos \omega t, \quad x_2(t) = A_2 \cos \omega t$$

giving

$$\dot{x}_1(t) = -\omega A_1 \sin \omega t, \quad \dot{x}_2(t) = -\omega A_2 \sin \omega t.$$

In these expressions, we are considering *a single mode* for both masses and, therefore, do not distinguish their frequencies since in each mode the motion is synchronous. Then, the kinetic energy is

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2,$$

with its maximum value given by

$$T_{max} = \frac{1}{2} m (A_1 \omega)^2 + \frac{1}{2} m (A_2 \omega)^2.$$

The corresponding potential energy expressions are

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2$$

$$V_{max} = \frac{1}{2} k A_1^2 + \frac{1}{2} k (A_2 - A_1)^2.$$

Equating T_{max} to V_{max} we find

$$\omega^2 = \frac{k}{m} \frac{A_1^2 + (A_2 - A_1)^2}{(A_1^2 + A_2^2)}.$$

This equation is valid for any frequency and mode shape. Since the mode shape is given by the ratio A_2/A_1 , we have “only” to guess this ratio to find an estimate of ω .

For the first mode, we use the *static deflection* of the two masses as the guess. From the free-body diagrams, the static deflections are

$$A_1 = \frac{2mg}{k}, \quad A_2 = \frac{3mg}{k},$$

which can then be used to estimate the square of the fundamental frequency,

$$\omega_1^2 = \frac{k}{m} \frac{(2mg/k)^2 + (3mg/k)^2}{(2mg/k)^2 + (3mg/k)^2}$$

$$= \frac{5k}{13m} = 0.385 \frac{k}{m}.$$

The exact solution is $0.36 k/m$, meaning that the estimate is quite accurate.

For the second mode, we know that the masses will be displaced in opposite directions. Therefore, we let

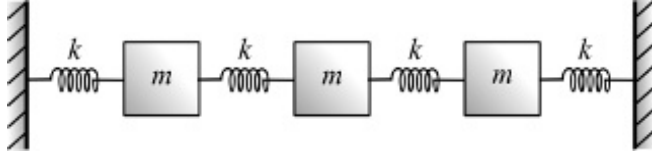


Figure 6.63: Three degree-of-freedom oscillator.

$A_1 = 2mg/k$ and $A_2 = -A_1$. Proceeding as before, we obtain the estimate

$$\omega_2^2 = 2.50 \frac{k}{m},$$

where the exact result is $2.62 k/m$. Again an accurate estimate is obtained.

Higher modes are more difficult to evaluate by this method since it is challenging to guess the higher modal ratios. We can ask what would the results look like if we made a poorer guess for the assumed mode? Let us try a more naïve first mode: $A_1 = A_2 = mg/k$. A simple computation gives $\omega_1^2 = 0.5 k/m$. This result is not as accurate as the one based on the better physical understanding of the relative displacement of the masses. Both eigenvalue estimates of ω_1^2 are above the exact one, as we expect. One practical conclusion is that a variety of mode shapes can be guessed, with the smallest resulting eigenvalue being the best estimate.

The highest eigenvalue (here the second frequency ω_2) is below the exact value, as expected. ■

Example 6.25 Rayleigh's Quotient for a Three Degree-of-Freedom System

For the three degree-of-freedom system of Figure 6.63, use Rayleigh's quotient to estimate the three natural frequencies of vibration.

Solution: We approach this problem in the same manner as the last one. The first mode is estimated using the ratio of static displacements. The mass and stiffness matrices are

$$[M] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } [K] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

To obtain the ratio of static displacements, assume the force vector $\{F\} = \{1 \ 1 \ 1\}^T$. Then, for the static sys-

tem,

$$\begin{aligned} \{u\}_1 &= [K]^{-1}\{F\} \\ &= \frac{1}{k} \begin{bmatrix} 0.75 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \\ 0.25 & 0.50 & 0.75 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \\ &= \frac{1}{k} \begin{Bmatrix} 1.5 \\ 2.0 \\ 1.5 \end{Bmatrix}. \end{aligned}$$

We can ignore the factor $1/k$ since we only need the ratios of the displacements to use Rayleigh's quotient. From Equation 6.160,

$$\begin{aligned} \omega_1^2 &= \frac{\{u\}_1^T [K] \{u\}_1}{\{u\}_1^T [M] \{u\}_1} = \frac{10k}{17m} \\ &= 0.5882 \frac{k}{m}, \end{aligned}$$

giving $\omega_1 = 0.7670\sqrt{k/m}$, which essentially matches the exact value of $\omega_1 = 0.7669\sqrt{k/m}$. (This is less than a 0.01% error!)

To estimate the second natural frequency, we use the knowledge that the second mode possesses one node at the center and assume $\{u\}_2 = \{1 \ 0 \ -1\}^T$. Proceeding as above, we find $\omega_2 = \sqrt{2k/m}$, which coincides with the exact value.

To estimate the third natural frequency, whose mode possesses two nodes, we assume $\{u\}_3 = \{1 \ -1 \ 1\}^T$, and find $\omega_3 = 1.8257\sqrt{k/m}$. This result compares well (with only a 1.2% error) with the exact value of $1.8477\sqrt{k/m}$. (The approximate value is slightly lower than the exact value.) This example demonstrates the high accuracy with which we can estimate the natural frequencies of a system with a few degrees of freedom. ■

6.16 Concepts Summary

In this chapter we studied the vibration of systems idealized as discrete multi degree-of-freedom models. The concept of modes was introduced, the direct method for solving a set of coupled ordinary differential equations of motion was developed, and techniques for decoupling these equations using the modal transformation were demonstrated and explained. Such a transformation is possible for undamped systems as well as systems that have the property of proportional damping, wherein the property matrices are diagonalized simultaneously. A new phase transformation was introduced as a decoupling method for general systems.

The major advantage of modal analysis as compared to the direct method is that in modal analysis only the

significant modes of vibration need to be included, thus reducing the dimension of the problem considerably.

Special problems in multi degree-of-freedom vibration have been investigated. Two problems are structures with periodicity and the inverse problem. Briefly mentioned are the important but more advanced problems of dynamic stability and the vibration of structures with internal fluid.

Finally, Rayleigh's quotient is derived and used to estimate the lowest natural frequencies of a structure.

6.17 Quotes

- "If facts conflict with a theory, either the theory must be changed or the facts." (Benedict Spinoza)
- "Theory is the distilled essence of practice." (Rankine)
- "Teague's Paradox: Nothing is so simple that it cannot be misunderstood."
- "How wonderful it is that nobody need wait a single moment before starting to repair the world!" (Anne Frank)
- "Science is a way of knowing. Technology is a way of doing." (Paul G. Hewitt)
- "I tell people that my mass is the same; it's just that gravity is increasing, and hence my weight keeps growing." (Mark Nagurka)
- "Engineering is the art of modelling materials we do not wholly understand, into shapes we cannot precisely analyze so as to withstand forces we cannot properly assess, in such a way that the public has no reason to suspect the extent of our ignorance." (AR Dykes, British Institution of Structural Engineers)
- "Engineers like to solve problems. If there are no problems handily available, they will create their own problems." (Scott Adams)
- "We can't change the wind but we can adjust our sails."
- "Innovate, Integrate, Collaborate, Communicate."
- "Inertia is a pain in the mass." (Mark Nagurka)
"Inertia happens." (Anita Nagurka)
- "The fact that no one understands you doesn't mean you're an artist."
- "It's not what you look at that matters, it's what you see..." (Henry David Thoreau)

- "Not everything that can be counted, counts and not everything that counts can be counted." (Albert Einstein)

6.18 Problems

Problems for Section 6.2 – The Concepts of Stiffness and Flexibility

1. The two degree-of-freedom system in Figure 6.64 undergoes rectilinear motion. (a) Derive the flexibility influence coefficients. (b) Derive the stiffness influence coefficients. (c) Find the inverse of the flexibility matrix and show that this equals the stiffness matrix. (d) Write the matrix equation of motion.

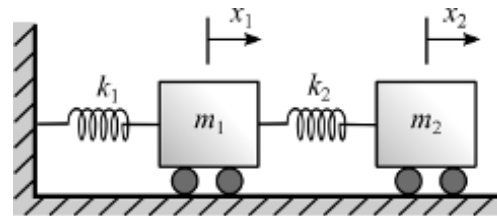


Figure 6.64: Two degree-of-freedom system.

2. For the triple pendulum of Figure 6.65, determine the flexibility influence coefficients that relate the horizontal forces and the horizontal displacements. Find the inverse of the matrix of flexibility coefficients and then write the matrix equation of motion for this system.

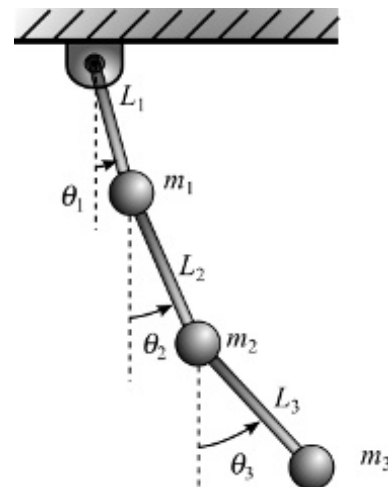


Figure 6.65: Triple pendulum.

Problems for Section 6.3 – Derivation of Equations of Motion

3. For the double pendulum of Figure 6.66, derive the equations of motion using (a) Newton's second law, and (b) Lagrange's equation.

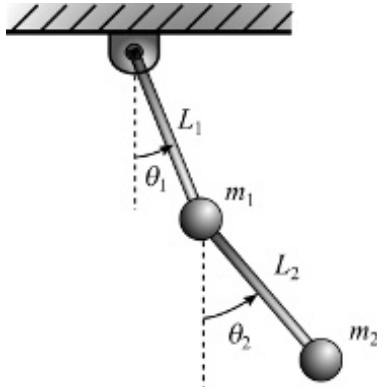


Figure 6.66: Double pendulum.

4. Derive the characteristic equation and the modal ratios for the system shown in Figure 6.66.
5. The system shown in Figure 6.67 consists of two linear springs with stiffnesses k_1 and k_2 and a uniform bar of mass m and length of $2L$. If the force F is acting at a distance a from the centerline as shown, derive Lagrange's equation of motion. Assume small motions.

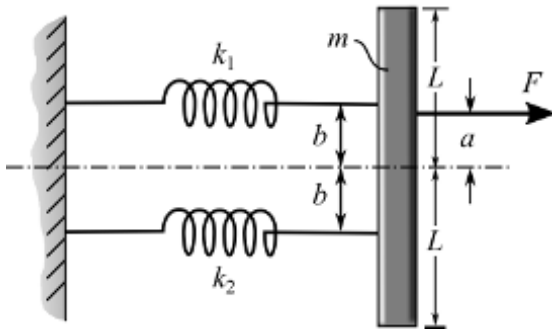


Figure 6.67: A bar restrained by two springs with asymmetric load.

6. A simple lumped-parameter model of a building, shown as the inverted link in Figure 6.68, can be used for preliminary study of earthquake dynamics. Derive the equations of motion for this two degree-of-freedom system using (a) Newton's second law, and (b) Lagrange's equation. Let $y(t)$ be the prescribed ground motion, and $x(t)$ be the absolute position of the mass m .
7. Derive the equation of motion for the elastically restrained rigid beam shown in Figure 6.69 using (a) Newton's second law, and (b) Lagrange's equation.

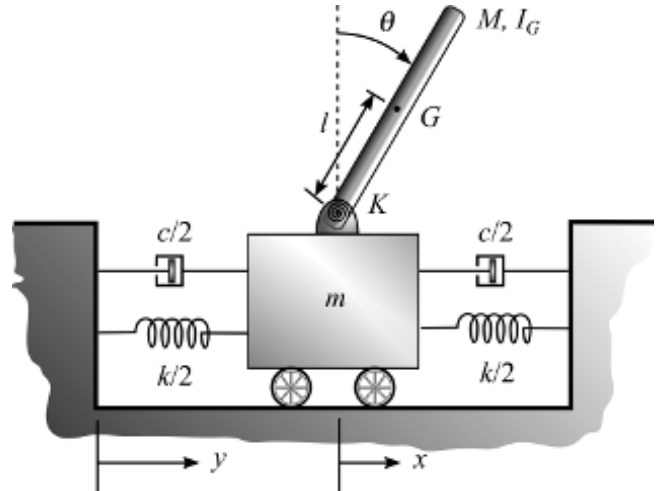


Figure 6.68: An earthquake loading model showing a simple model of building (the inverted beam) under base excitation.

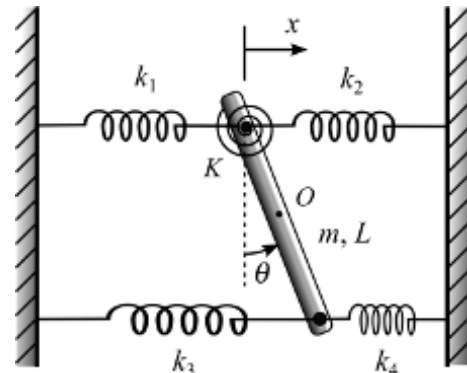


Figure 6.69: Elastically restrained rigid beam suspended with a torsional spring.

8. Use Hamilton's principle to derive the equations of motion for the structures shown in Figures (a) 6.66, (b) 6.68, and (c) 6.69.
9. Derive the equations of motion for the problem of Example 6.5 using (a) Newton's second law, (b) Lagrange's equation, and (c) Hamilton's principle, and then solve for the response.

Problems for Section 6.4 – Undamped Vibration

10. In Example 6.7 evaluate $\bar{C}_1$, $\bar{C}_2$, ϕ_1 , and ϕ_2 using the given initial conditions: $x_1(0) = 1$ cm, $x_2(0) = -1$ cm, $\dot{x}_1(0) = 3$ cm/s, $\dot{x}_2(0) = -2$ cm/s. Use $\sqrt{k/m} = 10$ rad/s.
11. Consider the problem of the "sliding pendulum" drawn in Figure 6.70, where a mass slides along a frictionless rod which is assumed to be uniform and thin. Use (a) Newton's second law, and (b) Lagrange's equation to derive the equations of motion.

Can the governing equations be linearized? Under what conditions? What happens when $r = l$?

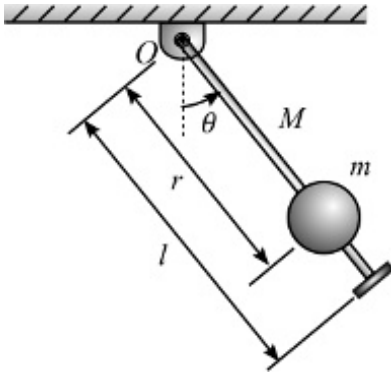


Figure 6.70: Sliding pendulum.

12. Linearize and solve the equations of motion derived in Problem 11. State all assumptions necessary for the linearization and explain them physically.
13. For the system shown in Figure 6.13, solve for the responses where $k_1 = k_2 = k_3 = k$, $m_1 = m_2 = m$, and where the initial conditions are $x_1(0) = 1$, $x_2(0) = 1$, and zero initial velocities, all in appropriate units. Plot the modes, identifying the nodes. Plot $\omega_{1,2}$ vs. m and k .
14. Solve for the general response of the system given in Problem 1:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0.$$

15. Solve for the general response of the system given in Problem 2.
16. Linearize the governing equations of Problem 3 shown below and find the natural frequencies and mode shapes for $m_1 = m_2 = m$ and $l_1 = l_2 = l$. State all assumptions in the linearization process:

$$\begin{aligned} (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 [\ddot{\theta}_2 \cos(\theta_2 - \theta_1) \\ - \dot{\theta}_2^2 \sin(\theta_2 - \theta_1)] + (m_1 + m_2) g \sin \theta_1 = 0 \\ m_2 l_2 \ddot{\theta}_2 + m_2 l_1 [\ddot{\theta}_1 \cos(\theta_2 - \theta_1) \\ + \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)] + m_2 g \sin \theta_2 = 0. \end{aligned}$$

17. For the two degree-of-freedom system shown in Figure 6.71, determine the natural frequencies of oscillation about equilibrium and the respective modal ratios. Assume the following parameter values:

$m_1 = 10$ kg, $m_2 = 14$ kg, $k_1 = 4$ N/m, $k_2 = 5$ N/m, and $k_3 = 6$ N/m.

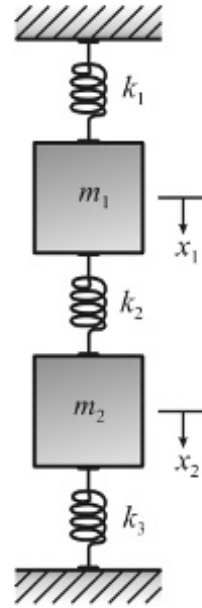


Figure 6.71: Two degree-of-freedom system.

18. In Problem 6, using the governing equations of motion given below, assume a harmonic base motion and find the undamped response:

$$\begin{aligned} (m + M) \ddot{x} + Ml (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) \\ - k(y - x) - c(\dot{y} - \dot{x}) = 0 \\ (I_G + Ml^2) \ddot{\theta} + Ml \ddot{x} \cos \theta \\ + K\theta - Mgl \sin \theta = 0. \end{aligned}$$

19. Find the response of the freely vibrating system of Problem 7,

$$\begin{aligned} m \left(\ddot{x} + \frac{L}{2} \ddot{\theta} \right) + (k_1 + k_2)x \\ + (k_3 + k_4)(x + L\theta) = 0 \\ \frac{mL}{2} \ddot{x} + (I_O + m \frac{L}{2}) \ddot{\theta} + (k_3 + k_4)(x + L\theta)L \\ + mg \frac{L}{2} \theta + K\theta = 0. \end{aligned}$$

20. Suppose the system of Equation 6.38,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix},$$

is forced by the vector $\{F_1(t) \ F_2(t)\}^T = \{f_1 \cos \omega_1 t \ f_2 \cos \omega_2 t\}^T$. Solve for the general response using the direct method.

21. For the undamped vibration absorber problem governed by Equations 6.45 and 6.46,

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 &= f_1 \sin \omega t \\ m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 &= 0, \end{aligned}$$

assume that the absorber mass is also forced by $f_2 \sin \omega t$. Solve for the responses $x_1(t)$ and $x_2(t)$. Obtain $A(\omega)$ and $B(\omega)$ as a function of ω/ω_a .

22. A cargo ship containing many boxes is at sea and experiences rough motion. Consider two masses in the cargo hold that are in close proximity when mass m_1 is subjected to a force $F = A \cos(\omega t + \phi)$ in line with the springs, as shown in Figure 6.72. The two masses are connected to each other via spring k_2 , and mass m_1 is connected to the ship's hull via a spring and a damper. Derive the equation for the relative motion of the two masses.

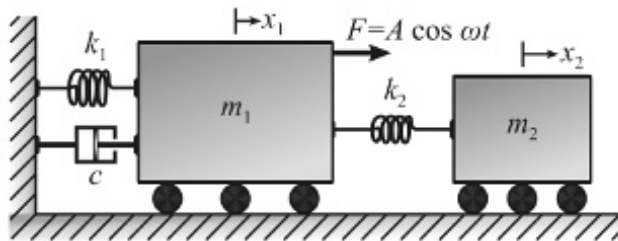


Figure 6.72: A two degree-of-freedom model of two masses on a cargo ship.

23. The vibration characteristics of an airplane are very complex due to its intricate system of structures and substructures. Suppose we consider the much simplified model of Figure 6.73, which is a three degree-of-freedom representation of the fuselage and wings. Derive the equations of motion using both Newton's second law and Lagrange's equation. Initially assume arbitrary θ_1 and θ_2 . Then consider two simplifications: (a) symmetrical vibration of the wing-body combination, that is, $\theta_1 = \theta_2 = \theta$, $m_1 = m_2 = m$, and $K_1 = K_2 = K$, and (b) small θ . Solve for the free vibration response after simplifying and linearizing the equations of motion. Discuss your assumptions.

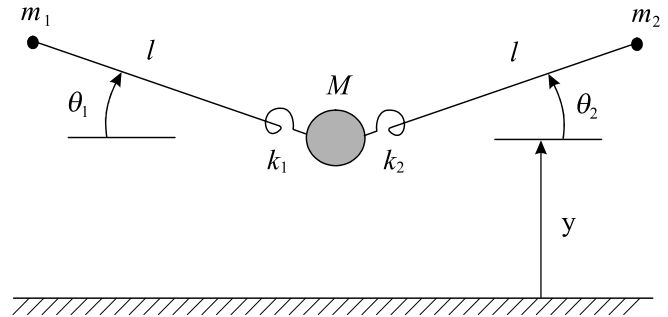


Figure 6.73: A highly simplified airplane model.

24. What will change in our formulation and solution of Problem 23 if the symmetry assumption in the previous problem is removed? Discuss fully.
25. Two uniform rods AB and CD of linear density ρ are suspended and connected by a spring as shown in Figure 6.74. The rods are in equilibrium when hanging in the vertical direction. (a) Derive the equations of motion of the system. (b) Find the natural frequencies of oscillation. (c) If bar AB is rotated α due to a static force applied to rod AB , find the corresponding rotation of CD and the force in the spring. (d) Given $l_1 = 0.5$ m, $l_2 = 0.8$ m, $k = 3$ kN/m, $\rho = 4$ kg/m, and $\alpha = 5$ degrees, evaluate the previous questions numerically.

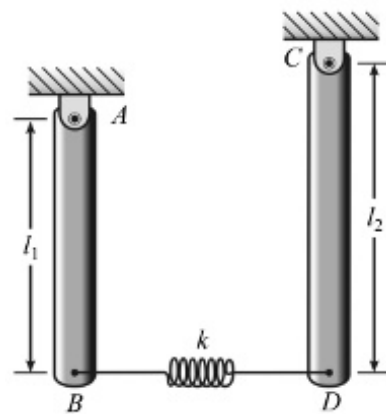


Figure 6.74: Two physical pendula coupled by a spring.

26. For the two degree-of-freedom system shown in Figure 6.75 derive the equations of motion. Assume that there is no slip between the cords that connect the masses to the pulley. The moment of inertia of the pulley about its point of rotation is I . How would the formulation be different if slippage does occur?

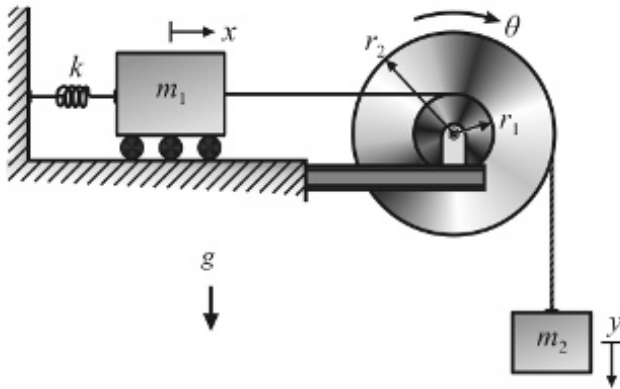


Figure 6.75: A system of two masses coupled through a pulley.

27. Complete the solution of Equation 6.55,

$$\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t - \phi_1) + C_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t - \phi_2),$$

if the initial conditions are given by

$$\begin{aligned} \theta_1(0) &= \theta_a, & \theta_2(0) &= \theta_b, \\ \dot{\theta}_1(0) &= \Omega_a, & \dot{\theta}_2(0) &= \Omega_b. \end{aligned}$$

Problems for Section 6.5 – Direct Method: Free Vibration with Damping

28. Derive Equations 6.69 and 6.70,

$$\begin{aligned} x_1(t) &= e^{-a_1 t} (r_1 C_1 \cos \omega_{d1} t + r_1' C_2 \sin \omega_{d1} t) \\ &\quad + e^{-a_2 t} (r_2 C_3 \cos \omega_{d2} t + r_2' C_4 \sin \omega_{d2} t) \\ x_2(t) &= e^{-a_1 t} (C_1 \cos \omega_{d1} t + C_2 \sin \omega_{d1} t) \\ &\quad + e^{-a_2 t} (C_3 \cos \omega_{d2} t + C_4 \sin \omega_{d2} t). \end{aligned}$$

29. Derive Equations 6.71 and 6.72,

$$\begin{aligned} x_1(t) &= B_1' e^{-a_1 t} \cos(\omega_{d1} t - \phi_{d1}') \\ &\quad + B_2' e^{-a_2 t} \cos(\omega_{d2} t - \phi_{d2}') \\ x_2(t) &= B_1 e^{-a_1 t} \cos(\omega_{d1} t - \phi_{d1}) \\ &\quad + B_2 e^{-a_2 t} \cos(\omega_{d2} t - \phi_{d2}). \end{aligned}$$

30. Use the direct method to solve the equations of motion for a system with the following property matrices,

ces,

$$\begin{aligned} [M] &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & [C] &= \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}, \\ [K] &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Sketch this system.

31. Use the direct method to solve the equations of motion for a system with the following property matrices,

$$\begin{aligned} [M] &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & [C] &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \\ [K] &= \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}. \end{aligned}$$

Sketch this system.

32. Use the direct method to solve the equations of motion for a system with the following property matrices,

$$\begin{aligned} [M] &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & [C] &= \begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix}, \\ [K] &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Sketch this system.

33. Derive Equation 6.78 showing all steps.

Problems for Section 6.6 – Modal Analysis

34. Derive Equation 6.79.

35. Show that the orthogonality relations hold when there is coupling in both mass as well as in stiffness matrices as long as they are symmetric. This requires a derivation of the characteristic matrix and equation, the eigenvalues, and the eigenvectors.

36. For the two degree-of-freedom system in Figure 6.76 undergoing longitudinal motion, derive the equations of motion utilizing each of the following approaches: (a) flexibility coefficients, (b) Newton's second law of motion, (c) Lagrange's equation, and (d) Hamilton's principle. Then, with the equations of motion in matrix form, derive the natural frequencies and mode shapes, and solve for the responses.

Assume the values $m_1 = m_2 = 1$ kg and all stiffnesses are 1 N/m. Solve via (i) the direct method and (ii) modal analysis.

Find the response of each mass if the initial velocities equal zero and the initial displacements are in the ratio of the first mode. For a design application, it is necessary that $|x_2 - x_1| \geq x_{cr}$, where x_{cr} is some critical separation distance between the masses. Describe how to check that this condition is met, and what options exist for redesign of the system or feedback control if the condition is not met.

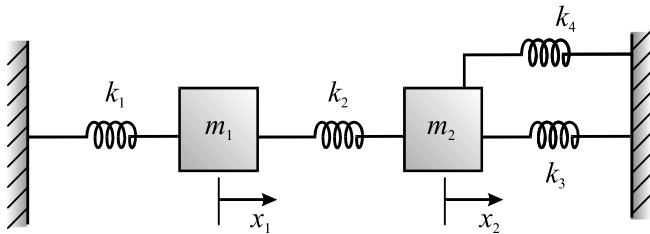


Figure 6.76: Two degree-of-freedom system.

37. In Problem 36, the system is forced with $F_1(t) = A_1 \cos \Omega_1 t$ and $F_2(t) = A_2 \sin \Omega_2 t$ acting on mass m_1 and m_2 , respectively. Using modal analysis, solve for the forced response assuming initial conditions $x_1(0) = 0$, $\dot{x}_1(0) = v_1$, $x_2(0) = d_2$, and $\dot{x}_2(0) = 0$.

38. If Problem 36 is modified to include damping between the two masses as shown in Figure 6.77, solve for the case with no forcing using (a) modal analysis, and (b) the direct method.

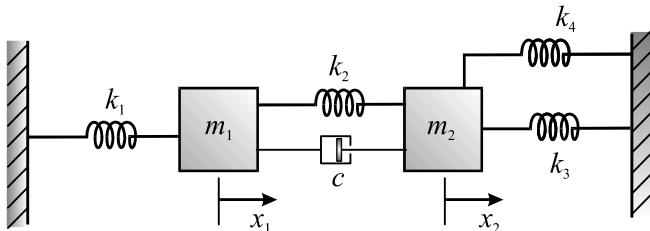


Figure 6.77: Two degree-of-freedom system with damping.

39. The two degree-of-freedom damped and forced system of Figure 6.78 oscillates about its equilibrium position. Use Lagrange's equation to formulate the problem and then solve the equations of motion for $F(t) = A \cos \omega t$. Check whether proportional damping is a valid model here. If not, how can the system be modified so that such a model is valid? Make this modification, and then solve via modal analysis.

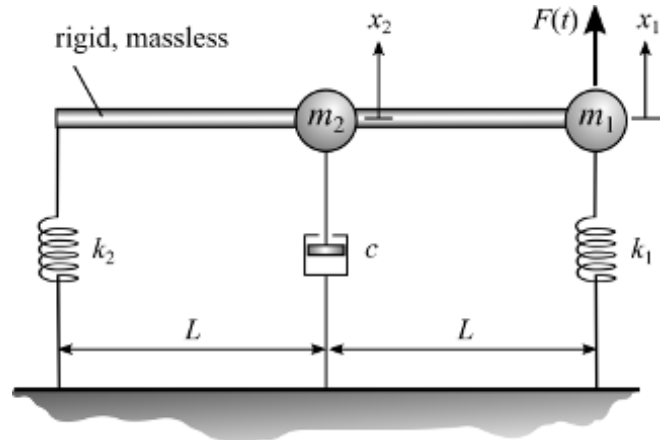


Figure 6.78: Two degree-of-freedom system with damping.

40. Solve [Example 6.14](#) with the equation of motion

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix},$$

for $F_1(t) = \cos(0.66\sqrt{k/mt})$ and $F_2(t) = 0$. Plot the displacement time histories.

41. In Problem 40, solve the same equation of motion, except with $F_1(t) = \cos(2\sqrt{k/mt})$ and $F_2(t) = 0$. Plot the displacement time histories.

42. In Problem 40, solve for the response where the system damping is (a) $[C] = 2[M]$, (b) $[C] = 3[K]$, (c) $[C] = 2[M] + 3[K]$. In each discuss the results, especially regarding how the damping for each mode depends on the proportional damping model. Let $\sqrt{k/m}$ be 1 rad/s if necessary.

43. A vibrating mass is found to be oscillating with an amplitude that is too large. To reduce this amplitude, an auxiliary system is added as shown in Figure 6.79. This problem generally occurs when the forcing frequency is too close to the natural frequency of the primary system.

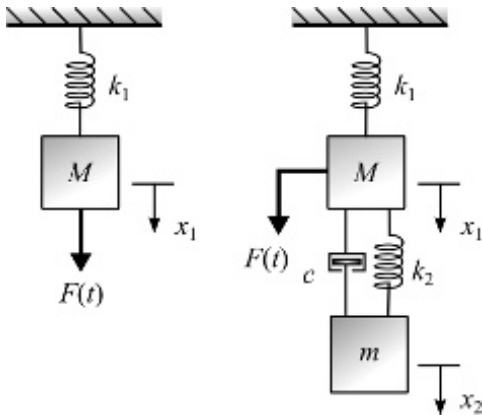


Figure 6.79: Original system and system with vibration absorber.

The auxiliary system acts as a damped vibration absorber. (a) Derive the equation of motion of the system plus absorber and then solve to determine what values of absorber mass m , stiffness k_2 , and damping c must be selected in order to minimize the vibration amplitude of primary mass M . Assume that $m = 0.25M$ and $F(t) = F_0 \cos \omega t$. (b) Suppose that we can accept a design where $k_2 M / k_1 m$ is selected so that neither natural frequency of the combined system, ω_1 and ω_2 , is closer than 5% to the driving frequency ω . Devise the system that achieves this criterion.

Problems for Section 6.9.1 – Unrestrained Systems

44. Figure 6.80 may be used as a simple model of a mobile robotic arm undergoing forced motion. The forcing is at the base of the arm. Derive the matrix equation of motion, and solve it in general for the response if (a) $F(t) = u(t)$, and (b) $F(t) = \cos 2t$. Solve this problem in two ways: (i) using a constraint matrix, and (ii) directly without a constraint matrix.

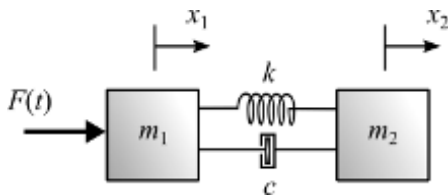


Figure 6.80: Simple model of a mobile robotic arm.

45. The system shown in Figure 6.81 undergoes rotational motion. Derive the equations of motion and solve for the response in terms of the initial conditions, given by $\theta_1(0)$, $\dot{\theta}_1(0)$, $\theta_2(0)$, and $\dot{\theta}_2(0)$. Solve this problem in two ways: (a) using a constraint matrix, and (b) directly without a constraint matrix.

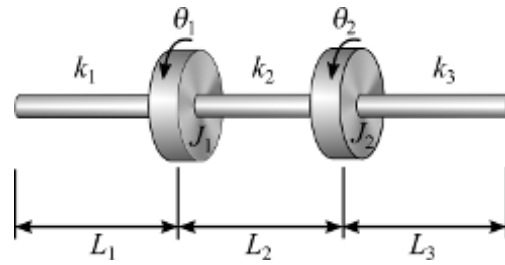


Figure 6.81: Two disk shaft system.

46. Two identical disks are connected by an elastic shaft of stiffness K . The moment of inertia of each disk with respect to the axis of the shaft is J . The system is at rest when a constant moment M is applied instantaneously to one of the disks. Determine the motion of the system assuming that the mass of the disks is much greater than that of the shaft. Solve this problem in two ways: (a) using a constraint matrix, and (b) directly without a constraint matrix.
47. In Example 6.20, derive $[K]$.
48. In Example 6.20, derive the eigenvalue problem for the constrained system and show that the frequencies and modes are given by Equations 6.135 and 6.136.
49. Solve Example 6.20, where $k_1 = 3 \text{ N-m/rad}$, $k_2 = 10 \text{ N-m/rad}$, $I_1 = 5 \text{ kg-m}^2/\text{rad}$, $I_2 = 2I_1$, and $I_3 = I_2/2$.
50. In Example 6.21, show how Equation 6.137 arises.
51. For Example 6.21, show the steps in the derivation of the three modes of vibration.

Problems for Section 6.11 – Periodic Structures

52. (a) For the three degree-of-freedom system shown in Figure 6.82, derive the equations of motion and solve for the frequencies and modes of vibration for the arbitrary stiffnesses k_1, k_2, k_3, k_4 and masses m_1, m_2, m_3 . Then simplify these results for the case where all the stiffnesses have the value $k = 1 \text{ N/m}$ and all the masses have the value $m = 1 \text{ kg}$. Plot the modes and responses for $x_1(0) = 0.5 \text{ cm}$ and assume that all other initial conditions are equal to zero.
- (b) For the previous system, suppose that $k_1 = k_3 = k_4 = k = 1 \text{ N/m}$, $m_1 = m_2 = m_3 = m = 1 \text{ kg}$, and $k_2 = (1 + \epsilon)k = (1 + 0.05)k$, where ϵ signifies a stiffness imperfection. Solve for the frequencies and modes of vibration and plot the modes and responses for $x_1(0) = 0.5 \text{ cm}$ and all other initial conditions equal zero.

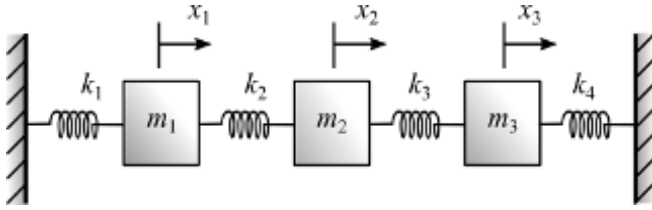


Figure 6.82: Three degree-of-freedom periodic system.

Compare the results with and without imperfections. How do these results compare with those discussed in Section 6.11.2? Discuss.

53. In Problem 52, part (b), solve assuming $k_1 = k_2 = k_3 = k_4 = k$ and $m_1 = m_3 = m$ and $m_2 = m(1 + \epsilon)$. Use the same parameter values. Discuss.
54. Derive Equation 6.144.

Problems for Section 6.12 – Inverse Vibration: Estimation of Mass and Stiffness

55. Derive Equations 6.150 to 6.153.
56. Suppose that for the two degree-of-freedom system of Figure 6.54 we obtain the data: $f_1 = 4$ Hz, $f_2 = 10$ Hz, $f_3 = 8$ Hz, and $m_1 + m_2 = 20$ kg. Solve for the system parameters k_1 , k_2 , m_1 , m_2 .

Problems for Section 6.13 – Sloshing of Fluids in Containers

57. Since liquids have damping characteristics, suggest possible equivalent mechanical models for a sloshing liquid that includes damping.
58. Suppose a container with interior fluid is in an aircraft. The aircraft leaves on its flight with full tanks and arrives at its destination with almost empty tanks. How could this fact be incorporated in the equivalent mechanical model?
59. Suppose a container with interior fluid is in a spacecraft in orbit around the Earth. What additional considerations need to be incorporated in the equivalent mechanical model?

Problems for Section 6.14 – Stability of Motion

60. In Example 6.23, solve for $\theta(t)$ analytically for the case $g < A\omega^2$.
61. Derive Equations 6.154 and 6.155 using Newton's second law of motion.

Problems for Section 6.15 – Rayleigh's Quotient

62. For a two disk torsional system (see Figure 6.81) with its two ends fixed to walls, follow the procedure of Example 6.24 to evaluate the two approximate frequencies of oscillation. Explain the procedure used to guess the respective mode shapes. Check that the approximate fundamental frequency is above the exact frequency and that the second frequency is below the exact frequency.
63. How would Rayleigh's quotient need to be altered for an unrestrained system? Apply the method to a general two degree-of-freedom undamped system in unrestrained rectilinear motion.



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Chapter 7

Continuous Models for Vibration

“Where reality is approached.”

Continuous models of vibrating systems are more realistic since the properties of actual systems are distributed rather than concentrated at discrete points. The price we pay for increased realism is increased mathematical complexity. The governing equations of motion change from ordinary differential equations for the discrete models to partial differential equations for the continuous ones, since displacement is a function of both time and position. For elementary applications, deriving and solving the partial differential equations do not pose an undue challenge. But, when structures with varying cross sections and nonuniform material properties need to be modeled, approximate techniques of the type we began to study in the last chapter are needed.

In this chapter, we focus on models of strings and beams, which are spatially one dimensional. Membranes and plates, which are spatially two dimensional, are the object of the next chapter. In modeling these systems, direct and modal solutions are explored.



Figure 7.1: A fishing rod is an example of a continuous system. Its deflection can be studied using a flexible rod model. To be accurate, nonlinear large-deformation theory would be needed. (Photograph by M. Nagurka)

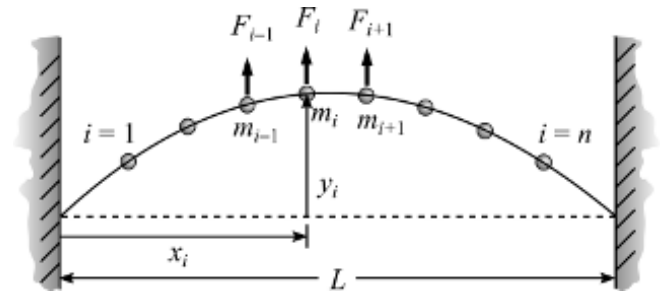


Figure 7.2: Massless string with point masses shown in exaggerated configuration. The external forces on three arbitrary sequential masses are shown.

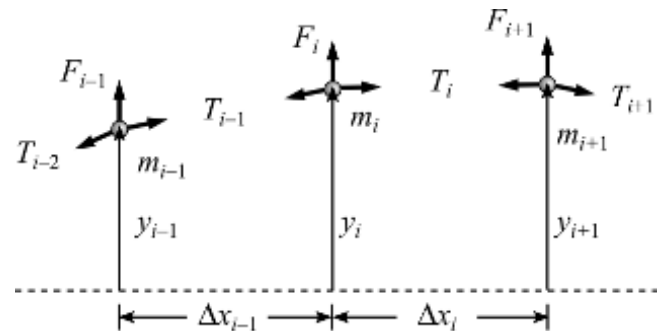


Figure 7.3: Free-body diagrams of three sequential masses.

7.1 Discrete to Continuous

We derive the governing equations of motion of a continuous system by starting with a discrete version of the system and then taking appropriate limits. In this way, the continuous model is understood to be a limiting case of an increasingly higher order discrete model.

As an example of this limiting procedure, we examine the derivation of the equation of motion of a *string*. We start with a massless string to which point masses are fixed at specified locations, as shown in Figures 7.2 and 7.3. Point mass m_i is acted on by external force F_i and internal tensions T_{i-1} and T_i ; all forces are functions of time. Applying Newton’s second law of motion in the vertical direction, y , and assuming small separation distances, Δx_i , we obtain the equation of motion for the i^{th}

mass,

$$T_i \frac{y_{i+1} - y_i}{\Delta x_i} - T_{i-1} \frac{y_i - y_{i-1}}{\Delta x_{i-1}} + F_i = m_i \frac{d^2 y_i}{dt^2}. \quad (7.1)$$

Here, the assumption of small slopes θ allows the approximation $\sin \theta \simeq \tan \theta \simeq \theta$, where θ is the angle of a string segment with respect to the horizontal.

Defining $\Delta y_i \equiv y_{i+1} - y_i$ and $\Delta y_{i-1} \equiv y_i - y_{i-1}$, Equation 7.1 becomes

$$T_i \frac{\Delta y_i}{\Delta x_i} - T_{i-1} \frac{\Delta y_{i-1}}{\Delta x_{i-1}} + F_i = m_i \frac{d^2 y_i}{dt^2}, \quad i = 1, 2, \dots, n, \quad (7.2)$$

for each point mass. The first two terms on the left-hand side represent the change in the vertical force on point mass m_i , and so Equation 7.2 can be rewritten,

$$\Delta \left(T_i \frac{\Delta y_i}{\Delta x_i} \right) + F_i = m_i \frac{d^2 y_i}{dt^2}, \quad i = 1, 2, \dots, n,$$

where Δ denotes the change in the expression within the parentheses. Dividing both sides by Δx_i , we obtain

$$\frac{\Delta}{\Delta x_i} \left(T_i \frac{\Delta y_i}{\Delta x_i} \right) + \frac{F_i}{\Delta x_i} = \frac{m_i}{\Delta x_i} \frac{d^2 y_i}{dt^2}, \quad i = 1, 2, \dots, n.$$

Next, we let the number n of point masses m_i increase to ∞ and the spaces between the masses decrease to zero, such that the average mass per unit length remains constant. As a result, the position x_i becomes x , and as $\Delta x_i \rightarrow 0$ the difference equation becomes a partial differential equation,

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial y(x,t)}{\partial x} \right] + f(x,t) = m(x) \frac{\partial^2 y(x,t)}{\partial t^2}. \quad (7.3)$$

Equation 7.3 governs the displacement $y(x,t)$ of the string, $0 \leq x \leq L$, where $f(x,t)$ is the limiting value of the force per unit length and $m(x)$ is the limiting value of the mass per unit length.

This formulation serves to demonstrate that the discrete and continuous models are different idealizations of the same system. Therefore, techniques developed for discrete systems can be applied to continuous ones. Taking the limit $n \rightarrow \infty$ above means that the continuous system is *infinite-dimensional*, that is, it has an infinite number of degrees of freedom.

Equation 7.3 is the equation of motion of the string in the vertical direction. The motion in the horizontal direction governs the tension in the string. Summing the forces in the horizontal direction and equating to zero, since there is no movement or acceleration in this direction, results in

$$T_{i+1} \cos \theta_{i+1} - T_i \cos \theta_i = 0.$$

This is a statement that the horizontal component of force, $T \cos \theta$, is constant.

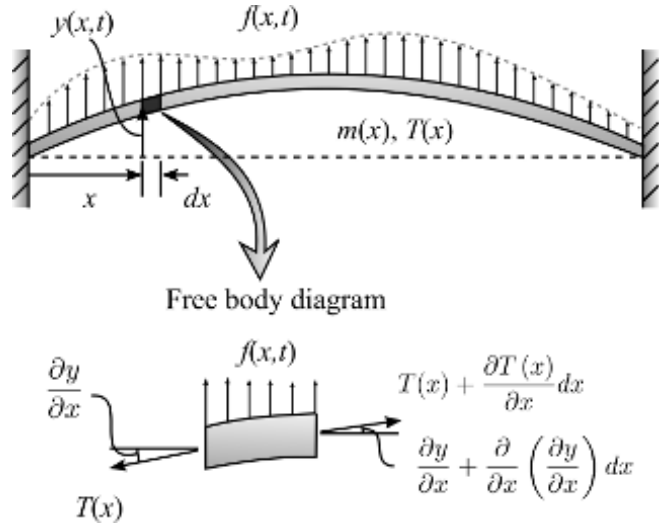


Figure 7.4: A string fixed at both ends and acted on by an external distributed load per unit length $f(x, t)$ (top) and the free-body diagram of a differential element (bottom). Oscillation is about the equilibrium position. The free-body diagram for an element of length dx depicts the tensions as well as the slopes at each end of the element.

Equation 7.3 can also be derived using the free-body diagram of a *continuous* element. This is shown in the next section.

7.2 Vibration of Strings

A string¹ is a valuable model for understanding the dynamic behavior of continuous systems. Strings have been used for simplified models of telephone wires, conveyor belts, cables, and even models of human DNA. When a string is stretched from its equilibrium position, the tension acts as the restoring force that brings the string back to its original undeformed position.²

¹ Physically speaking, strings are similar to cables and hence the dynamics of strings are similar to those of cables of small sag-to-span ratios. Strings are tensioned cables. Cables are loose strings. Early linear cable theories adopting the assumption of inextensibility failed to reconcile with the taut string theory.

Cables are widely used in aerospace, mechanical, civil, and ocean engineering systems because of their high flexibility, light weight, and high packaging efficiency. Well-known applications include suspension and cable-stayed bridges, power transmission lines, mooring cables, cable nets, hoisting cables of cranes, etc. Moreover, there are many innovative applications of cables in recent decades. For example, NASA succeeded in using a 1000 ft (305 m) long cable and a C-141A airplane to tow a reusable launch vehicle (a QF-106A airplane) to a high altitude to decrease the launch cost and increase the payload. However, because of high flexibility and zero buckling strength, large in-operation deformations and nonlinear vibrations of cables seriously impair the performance of mechanical systems with cables.

² A transversely vibrating *beam* derives its restoring force from bending stiffness as shown in Section 7.5.1. The string cannot transmit a bending moment but can resist axial tension.

It is straightforward to derive the governing equation of motion using the free-body diagram for a section of the displaced string, as shown in Figure 7.4. The variables are $T(x)$, the tension in the string, $f(x, t)$, the applied transverse force per unit length, and $m(x)$, the mass per unit length. Using Newton's second law of motion, neglecting weight, and assuming small displacements allowing the approximation $\sin \theta \simeq \theta$, where θ is the slope at a point on the string, the sum of the forces in the transverse direction y is equal to the mass times the acceleration of the string element in the y direction,

$$\left(T(x) + \frac{\partial T(x)}{\partial x} dx \right) \left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} dx \right) + f(x, t) dx - T(x) \frac{\partial y}{\partial x} = m(x) dx \frac{\partial^2 y}{\partial t^2}.$$

Expanding the products on the left-hand side, ignoring second-order terms $(dx)^2$ (assuming that for linear vibration these terms are not significant³), and dividing by dx gives the governing equation of linear motion,

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial y}{\partial x} \right] + f(x, t) = m(x) \frac{\partial^2 y}{\partial t^2}. \quad (7.4)$$

The free vibration problem can be used to solve for natural frequencies and modes by setting $f(x, t) = 0$,

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial y}{\partial x} \right] = m(x) \frac{\partial^2 y}{\partial t^2}. \quad (7.5)$$

With the constant tension approximation, $T(x) = T$, and assuming a uniform string where $m(x) = m$, the result is the *wave equation*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}, \quad (7.6)$$

where $c = \sqrt{T/m}$ is the velocity of wave propagation with units of length per unit time. The wave equation is derived using Hamilton's principle in Section 7.2.2.

Since the wave equation is a second-order equation in terms of x and t , two boundary conditions and two initial conditions are needed to determine the solution. For the fixed-fixed string, we take for the boundary conditions the displacements at both ends as zero.

Two general approaches to solving continuous system equations are (a) the modal or *standing wave* approach and (b) the *wave propagation* approach. The standard solution technique for linear systems of finite length is based on the modal approach. It is instructive, however, to consider first the wave propagation solution, since the two methods are complementary and offer us two perspectives. The modal solution is developed in Section 7.2.4.

³The assumptions must be verified numerically, experimentally, or by comparison to more exact analytical solutions.

7.2.1 Wave Propagation Solution

Wave propagation solutions are particularly useful for systems with one or more dimensions that are *infinite* or *semi-infinite*. Examples include systems large enough that the effects of the boundaries can be ignored, such as very long strings and the ground around a structure that is subjected to seismic loading. We will also show that finite length systems can be studied using waves.

Assume that a string undergoing small transverse vibration has constant tension T , constant mass per unit length m , and is governed by the wave equation, Equation 7.6, rewritten here using a more compact notation,⁴

$$y_{,xx} = \frac{1}{c^2} y_{,tt}, \quad (7.7)$$

where $c = \sqrt{T/m}$.

The general solution is found by assuming that the displacement response $y(x, t)$ is equal to the sum of two different waves traveling in opposite directions,⁵

$$y(x, t) = F_1(x - ct) + F_2(x + ct), \quad (7.8)$$

where $F_1(x, t)$ and $F_2(x, t)$ are two arbitrary functions that can be differentiated twice with respect to x and t . Taking the derivatives,

$$\begin{aligned} y_{,tt} &= c^2 F_{1,tt} + c^2 F_{2,tt} \\ y_{,xx} &= F_{1,xx} + F_{2,xx}, \end{aligned}$$

and substituting them into Equation 7.7 yields

$$F_{1,xx} + F_{2,xx} = \frac{1}{c^2} (c^2 F_{1,tt} + c^2 F_{2,tt}). \quad (7.9)$$

Equation 7.9 demonstrates that the governing equation is satisfied by *any* two different functions of $x \mp ct$. To demonstrate this for specific functions, assume the harmonic solution

$$y(x, t) = A \sin \frac{2\pi}{\lambda} (x - ct) + B \cos \frac{2\pi}{\lambda} (x + ct). \quad (7.10)$$

Differentiating this solution with respect to x and t as necessary, we see that the wave equation and thus Equation 7.9 are satisfied.

To better understand each term in Equation 7.10, we consider the position of the harmonic function $y(x, t) = \sin(x - t)$ at two times, $t = 0$ and $t = \pi/2$ s, as shown in Figure 7.5. Based on the value of y , the wave has traveled *to the right* a distance of $\pi/2$ in this time. Similarly, one can show that $\sin(x + t)$ will travel to the left.

⁴The shorthand notation is, for example,

$$y_{,tt} \equiv \frac{\partial^2 y}{\partial t^2}, \quad y_{,xx} \equiv \frac{\partial^2 y}{\partial x^2}, \quad y_{,xt} \equiv \frac{\partial^2 y}{\partial x \partial t}.$$

⁵We do not need the sum of two functions. Either function alone is also a solution.

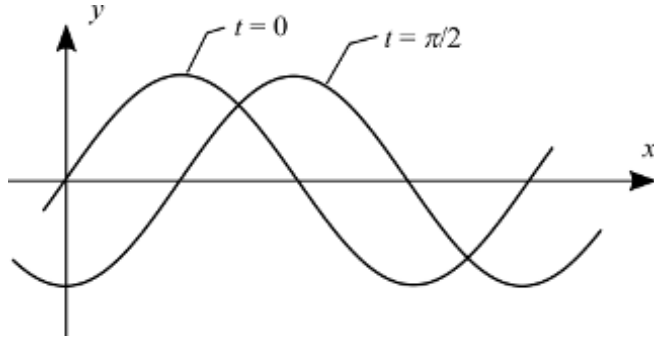


Figure 7.5: Example of a wave propagating to the right. $y(x, t) = \sin(x - t)$ at two times, $t = 0$ and $t = \pi/2$ s.

Consider a wave propagating to the right given by the function $y(x, t) = A \sin \frac{2\pi}{\lambda}(x - ct)$. The *period* P can be defined for any location x as the time for the value of y to repeat.⁶ It can be related to the frequency ω by the relation $P = 2\pi/\omega$. For the right-going wave,

$$\omega = \frac{2\pi c}{\lambda}$$

and therefore $P = \lambda/c$, or the *wavelength* $\lambda = Pc$, which has length units. The number of waves per unit length, known as the *wave number* κ , equals the inverse of the wavelength, $\kappa = 1/\lambda$, with units of length^{-1} . With these parameters, the right-going wave can be written as

$$y(x, t) = A \sin(2\pi\kappa x - \omega t), \quad (7.11)$$

where A is the amplitude, $2\pi\kappa$ is the spatial frequency, and ωt is the shift to the right.

Standing Waves

Suppose that two waves of equal amplitude A and frequency ω travel in opposite directions. Then, the wave propagation solution is the sum of the two waves,

$$\begin{aligned} y(x, t) &= A \sin(2\pi\kappa x - \omega t) + A \sin(2\pi\kappa x + \omega t) \\ &= 2A \sin(2\pi\kappa x) \cos \omega t. \end{aligned} \quad (7.12)$$

We see that the two oppositely moving propagating waves become a single non-propagating wave. It is no longer a traveling wave, but rather a *standing wave*, that is, one with a profile that is *oscillating* rather than propagating. Again, the two waves traveling in opposite directions add to become a standing wave.

In a semi-infinite or infinite medium, traveling wave solutions are used because the waves generally travel in only one direction. When a bounded system is considered, such as a string or a beam supported at both ends, we have the possibility of waves traveling in both directions due to reflections at the boundaries, resulting

⁶We use the symbol P for period instead of T (used in previous chapters) since the latter has been used for string tension.

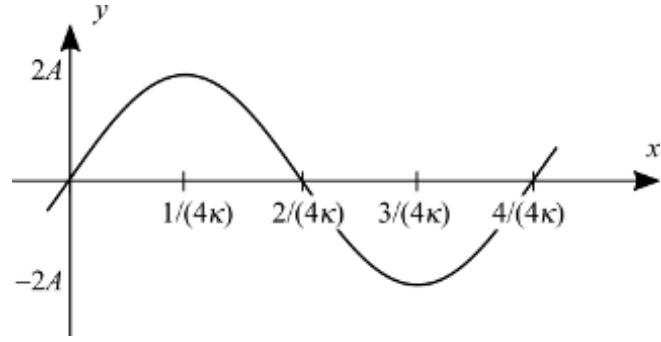


Figure 7.6: Superposition of traveling waves. The nodes, where $y(x, t) = 0$, are at $x = 0$, $x = (1/2\kappa)$, $x = (1/\kappa), \dots$, for $t = 0$.

in standing waves. As shown by Equation 7.12, traveling waves and standing waves can be interchanged. Figure 7.6 depicts a sketch of the standing wave for $t = 0$.

For this standing wave, there are *nodes* where $y = 0$ at $\sin 2\pi\kappa x = 0$, and *anti-nodes* or maximum amplitudes where $\sin 2\pi\kappa x = \pm 1$. Although this holds for *any* ω , once the boundary conditions are specified, for example for a string fixed at both ends, only certain frequencies satisfy the above conditions for nodes and maxima. We will learn much more about satisfying the boundary conditions for standing waves later in this chapter. Next, we examine how traveling waves in bounded media can satisfy these conditions as well.

For the standing wave given by Equation 7.12, we assume that the displacements are fixed for all time at the locations $x = 0$ and $x = L$,

$$\begin{aligned} y(0, t) &= 0 \\ y(L, t) &= 0 = 2A \sin(2\pi\kappa L) \cos \omega t. \end{aligned}$$

The first equation is satisfied identically and provides no new information. The second equation can be valid for all t only if $\sin 2\pi\kappa L = 0$. Then, $2\pi\kappa L = r\pi$ for integer r , providing the necessary condition

$$\kappa = \frac{r}{2L}. \quad (7.13)$$

Since

$$\kappa = \frac{1}{\lambda} = \frac{\omega}{2\pi c}, \quad (7.14)$$

we can equate Equations 7.13 and 7.14 to find

$$\omega_r = \frac{r\pi c}{L} = \frac{r\pi}{L} \sqrt{\frac{T}{m}}, \quad r = 1, 2, \dots, \quad (7.15)$$

where ω_r are the *allowable vibration frequencies* for the string for the given boundary conditions. We will revisit these results using the standing wave approach in the next section. The purpose of showing both approaches is to demonstrate that such problems can be cast in different forms providing useful alternate perspectives.

If the two amplitudes leading to Equation 7.12 are unequal, in addition to the equivalent standing wave there would also exist a wave traveling in the direction of the wave with the larger amplitude. Its amplitude would be the difference of the amplitudes of the components, as the following example shows.

Example 7.1 Addition of Two Waves with Different Amplitudes

Find the sum of the two waves of different amplitudes traveling in opposite directions, defined by

$$\begin{aligned} y_1(x, t) &= A_1 e^{i(2\pi\kappa x - \omega t)} \\ y_2(x, t) &= A_2 e^{i(2\pi\kappa x + \omega t)}, \end{aligned}$$

where we have used the complex exponential form for the traveling wave.

Solution: If we write the relation between the two amplitudes as $A_2 = A_1 + \alpha$, then the sum of the two waves is

$$\begin{aligned} y_1(x, t) + y_2(x, t) &= A_1 e^{i(2\pi\kappa x - \omega t)} + A_1 e^{i(2\pi\kappa x + \omega t)} + \alpha e^{i(2\pi\kappa x + \omega t)} \\ &= 2A_1 \cos(2\pi\kappa x) \cos \omega t + \alpha \cos(2\pi\kappa x + \omega t), \end{aligned}$$

where $\alpha = A_2 - A_1$ can be of any magnitude, and only the real parts of the complex exponentials are retained. The α term represents a propagating wave in the y_2 direction with amplitude α . This extra term may be written as

$$\begin{aligned} \alpha \cos(2\pi\kappa x + \omega t) &= \alpha \frac{y_2}{A_2} \\ &= y_2 \left(1 - \frac{A_1}{A_2} \right), \end{aligned}$$

where the real part of y_2 is implied. The extra term can be seen to be a fraction of the propagating wave y_2 . ■

7.2.2 Wave Equation via Hamilton's Principle

Continuing our practice of examining different formulations of governing equations, we derive the wave equation for a string using Hamilton's principle. Figure 7.7 depicts the system. To avoid confusion, since $T(x)$ represents the string tension, we will use $\tau(t)$ to represent the kinetic energy required for the application of Hamilton's principle for conservative forces,

$$\delta \int_{t_1}^{t_2} (\tau - V) dt = 0.$$

The advantage of Hamilton's principle over other methods to formulate the equations of motion is that it yields the boundary conditions for the system.

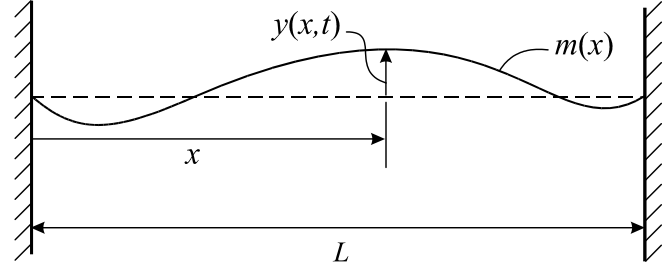


Figure 7.7: Schematic of string.

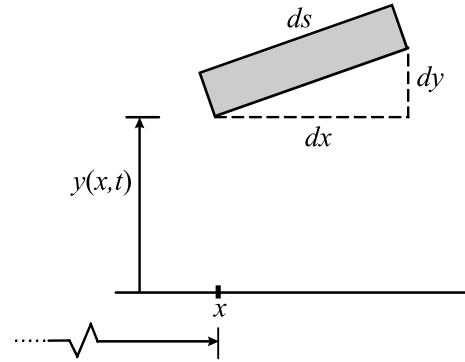


Figure 7.8: Schematic of a segment of a string showing the geometry of an elongation.

As before, the string mass per unit length is denoted by $m(x)$ and the transverse displacement by $y(x, t)$. The kinetic energy for an infinitesimal element of string of length dx is

$$\frac{1}{2} m(x) dx \left[\frac{\partial y(x, t)}{\partial t} \right]^2,$$

and the total kinetic energy is found by integrating over the length L of the string,

$$\tau(t) = \frac{1}{2} \int_0^L m(x) \left[\frac{\partial y(x, t)}{\partial t} \right]^2 dx.$$

The strain energy $V(t)$ is proportional to the increase in string length ds when compared to the string at rest. The change in strain energy equals the work done by the tension $T(x)$ in elongating the string by an amount Δ , that is, $V(t) = \int_0^L T(x) d\Delta$, where Δ will be shown to be a function of x . (It is assumed that tension is only a function of position. The tension is assumed constant when an element of string of length dx is stretched to a length ds . The approximation holds when the static tension is much higher than the dynamic tension.)

For the element of original length dx shown in Figure

7.8, the increase in length⁷ is given by $d\Delta = ds - dx$,

$$\begin{aligned} ds - dx &= \left\{ 1 + \left[\frac{\partial y(x, t)}{\partial x} \right]^2 \right\}^{1/2} dx - dx \\ &\simeq \frac{1}{2} \left[\frac{\partial y(x, t)}{\partial x} \right]^2 dx, \end{aligned}$$

where the last approximation is valid for *small slopes*, using the *binomial series* for the square root. Therefore, the strain energy for the entire string is

$$V(t) = \frac{1}{2} \int_0^L T(x) \left[\frac{\partial y(x, t)}{\partial x} \right]^2 dx.$$

We substitute these expressions into Hamilton's principle, which states that $\delta I = 0$, where

$$\begin{aligned} I &= \int_{t_1}^{t_2} [\tau(t) - V(t)] dt \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \left[m(x) \left(\frac{\partial y}{\partial t} \right)^2 - T(x) \left(\frac{\partial y}{\partial x} \right)^2 \right] dx dt. \end{aligned} \quad (7.16)$$

Since $y = 0$ at the end points, the variation of y is subject to the conditions that at $t = t_1$ and $t = t_2$,

$$\delta y(x, t) = 0. \quad (7.17)$$

To proceed, we interchange the order of integration with respect to t and x , interchange δ and ∂ , and integrate by parts. When integrating the time integrals by parts, we recall that the variations vanish at the end-times. The operators δ , $\partial/\partial t$, and $\partial/\partial x$ are commutative, and $\int dt$ and $\int dx$ interchange due to the uniform convergence of the limiting processes that define the integrals.

We consider separately the variation of each term in Equation 7.16, noting that the first term is a time derivative and the second term is a space derivative. Integrating the variation of the first term by parts with respect to t , we have

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \frac{1}{2} m(x) \left(\frac{\partial y}{\partial t} \right)^2 dt \\ &= \int_{t_1}^{t_2} m(x) \frac{\partial y}{\partial t} \delta \left(\frac{\partial y}{\partial t} \right) dt \\ &= \int_{t_1}^{t_2} m(x) \frac{\partial y}{\partial t} \frac{\partial(\delta y)}{\partial t} dt \\ &= \left[\left(m(x) \frac{\partial y}{\partial t} \right) \delta y \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(m(x) \frac{\partial y}{\partial t} \right) \delta y dt \\ &= - \int_{t_1}^{t_2} m(x) \frac{\partial^2 y}{\partial t^2} \delta y dt, \end{aligned} \quad (7.18)$$

⁷We use the right triangle with hypotenuse ds , along with the two sides that represent the difference in coordinates between the two ends of the segment, as shown in Figure 7.8, where $ds^2 = dx^2 + dy^2$, or $ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + (dy/dx)^2}$.

where use has been made of Equation 7.17.

Taking the variation of the second term in the integrand of Equation 7.16 with respect to x yields

$$\begin{aligned} &\delta \int_0^L \frac{1}{2} T(x) \left(\frac{\partial y}{\partial x} \right)^2 dx \\ &= \int_0^L T(x) \frac{\partial y}{\partial x} \delta \left(\frac{\partial y}{\partial x} \right) dx \\ &= \int_0^L T(x) \frac{\partial y}{\partial x} \frac{\partial(\delta y)}{\partial x} dx \\ &= \left[\left(T(x) \frac{\partial y}{\partial x} \right) \delta y \right]_0^L - \int_0^L \frac{\partial}{\partial x} \left(T(x) \frac{\partial y}{\partial x} \right) \delta y dx. \end{aligned} \quad (7.19)$$

Substituting Equations 7.18 and 7.19 into Hamilton's principle, we find

$$\begin{aligned} &\int_{t_1}^{t_2} \int_0^L \left\{ \left(\frac{\partial}{\partial x} \left[T(x) \frac{\partial y}{\partial x} \right] - m(x) \frac{\partial^2 y}{\partial t^2} \right) \delta y dx \right. \\ &\quad \left. - \left[\left(T(x) \frac{\partial y}{\partial x} \right) \delta y \right]_0^L \right\} dt = 0. \end{aligned} \quad (7.20)$$

Except for the requirement that it vanishes at the ends, the variation δy is arbitrary over the domain $0 < x < L$, and therefore the expression in the first parentheses must equal zero,

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial y}{\partial x} \right] - m(x) \frac{\partial^2 y}{\partial t^2} = 0. \quad (7.21)$$

This is the wave equation derived previously using Newton's second law of motion. The second term in Equation 7.20 must vanish at the ends due to the condition that δy equals zero at the boundaries,

$$\left[\left(T(x) \frac{\partial y}{\partial x} \right) \delta y \right]_0^L = 0.$$

If an end is fixed, then $\delta y = 0$ at that end. If an end of the string is free, then $T(x)\partial y/\partial x = 0$ at that end.

Next, we develop the solution of the wave equation.

7.2.3 Boundary Value Problem

To find the natural modes of vibration of a string, we assume that the solution is *separable*,

$$y(x, t) = Y(x)F(t), \quad (7.22)$$

where $Y(x)$ is called the *eigenfunction* or *modal function* and $F(t)$ is the time-dependent displacement. This solution implies that behaviors in time and space are independent; the same approach was used in Chapter 6 when the eigenvalue problem was first presented.

Since Equation 7.22 must satisfy the governing equation, we take the appropriate derivatives,⁸ $y' = Y'F$, $\ddot{y} = Y\ddot{F}$, and substitute them into governing Equation 7.21 to find

$$\underbrace{\frac{d}{dx} [T(x)Y'(x)F(t)]}_{\text{I}} = \underbrace{m(x)Y(x)\ddot{F}}_{\text{II}}$$

$$\frac{1}{m(x)Y(x)} \frac{d}{dx} [T(x)Y'(x)] = \frac{\ddot{F}}{F} = -\omega^2.$$

In the last equation, the time-dependent variables and the space-dependent variables have been placed on opposite sides of the equal sign. Since part I is a function only of x , and part II is a function only of t , it must be that each is equal to the same constant, say $-\omega^2$, so that when $\ddot{F} + \omega^2 F = 0$ is solved, the result is simple harmonic motion with frequency ω .

The equation governing $Y(x)$ is then

$$-\frac{d}{dx} [T(x)Y'(x)] = \omega^2 m(x)Y(x), \quad (7.23)$$

which must be satisfied in the domain $0 \leq x \leq L$. In addition, the *boundary conditions* must be specified⁹ to completely define the problem. Equation 7.23 is known as a *Sturm-Liouville* eigenvalue problem.¹⁰

7.2.4 Modal Solution for Fixed-Fixed Boundary Conditions

In this section, we derive the natural frequencies and mode shapes for the uniform string that is fixed at both ends via an eigenvalue/eigenfunction analysis.

Eigenvalues - Natural Frequencies

Consider a uniform vibrating string that is fixed at both ends and under constant tension. We wish to find the modes of vibration. A uniform string is one where the geometry and material properties do not vary along the length. The constant tension approximation is reasonable for a string oscillating with small amplitudes.

Equation 7.23 simplifies considerably for constant T and m . The eigenvalue problem becomes

$$Y'' + \beta^2 Y = 0, \quad \beta^2 = \frac{\omega^2 m}{T}, \quad (7.24)$$

⁸Prime denotes differentiation with respect to x and overdot represents differentiation with respect to t .

⁹The behavior of the system is based strongly on the boundary conditions. They can be grouped into two basic types: *geometric boundary conditions* and *natural boundary conditions*. Geometric (or imposed) boundary conditions are deflections or rotations. Natural (or dynamic) boundary conditions are a result of moment or shear force balances.

¹⁰In this book, we study vibratory systems that are *self-adjoint*, a property that leads to the symmetry of the property matrices. Therefore, the self-adjoint property is tied to the symmetry of the mass and stiffness matrices of the previous chapter.

with the boundary conditions $Y(0) = 0$ and $Y(L) = 0$. The wave speed c is given by $c = \sqrt{T/m}$ and therefore $\beta = \omega/c$, which is the wave number.

Equation 7.24 is a second-order constant coefficient differential equation with a periodic solution in x ,

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x, \quad (7.25)$$

and satisfying the boundary conditions requires that the following two equations be solved,¹¹

$$Y(0) = 0 = C_2$$

$$Y(L) = 0 = C_1 \sin \beta L.$$

Therefore, $C_2 = 0$ and we ignore the possibility that amplitude $C_1 = 0$ since then the solution is the trivial solution $Y(x) = 0$. If $C_1 \neq 0$, then $\sin \beta L = 0$, resulting in the possible values for β ,

$$\beta_r L = r\pi, \quad r = 1, 2, \dots,$$

where there is a countably infinite¹² number of values for β . The parameter r is used as an identifying index.

Substituting $\beta_r = \omega_r/c$ and $c = \sqrt{T/m}$ yields the allowable frequencies of oscillation,

$$\omega_r = \frac{r\pi}{L} \sqrt{\frac{T}{m}}, \quad r = 1, 2, \dots, \quad (7.26)$$

which is the same expression derived for the traveling wave solution, Equation 7.15. The ratio $\sqrt{T/m}/L$ has units of time^{-1} . It is interesting to note that the string has an infinite number of discrete natural frequencies, which confirms that the continuous system has an infinite number of degrees of freedom.

The frequencies given by Equation 7.26 depend on the boundary conditions and the geometric and material properties. This equation also signifies that the higher the string tension, the higher the frequencies. Further, if the string has a higher mass per unit length or if it is longer, the frequencies will be lower, in proportion to $1/\sqrt{m}$ and $1/L$.

This behavior is confirmed by our experience, whether playing a guitar or just experimenting with a rubber band; a tighter string plays a musical tone of higher frequency and a heavier string plays a lower tone. Also, since all the frequencies are integer multiples of the fundamental frequency ($r = 1$), the vibration of the string will yield a musical tone.

¹¹At the fixed end $x = 0$, the string does not displace and therefore $y(0, t) = Y(0)F(t) = 0$, or $Y(0) = 0$. Similarly, at the fixed end $x = L$.

¹²Any set that can be put in a one-to-one correspondence with the natural numbers (or integers) so that a prescription can be given for identifying its members one at a time is called a countably infinite (or denumerably infinite) set. Once one countable set S is given, any other set that can be put into a one-to-one correspondence with S is also countable. Examples of countable sets include the integers, algebraic numbers, and rational numbers.

Eigenfunctions - Mode Shapes

Once the natural frequencies of the system are known, the mode shapes (or eigenfunctions, since the mode is a continuous function) given by Equation 7.25 with $C_2 = 0$ can be determined,

$$Y_r(x) = C_r \sin \omega_r \sqrt{\frac{m}{T}} x = C_r \sin \frac{r\pi x}{L}.$$

The mode shapes $Y_r(x)$ are accurate only to a constant value since, if $Y_r(x)$ is a solution of Equation 7.24, then so is $C_r Y_r(x)$, where C_r is any constant. To specify C_r , a normalization is usually carried out according to the convention¹³

$$\int_0^L m Y_r^2(x) dx = 1, \quad r = 1, 2, \dots$$

Therefore,

$$C_r^2 \int_0^L m \sin^2 \frac{r\pi x}{L} dx = 1 \quad \text{giving} \quad C_r^2 = \frac{2}{mL},$$

and the normalized modes then have the specific form

$$\hat{Y}_r(x) = \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L}, \quad r = 1, 2, \dots$$

Figure 7.9 shows the first three modes for an aluminum string of 1 in² cross section with $m = 0.1 \text{ lb-s}^2/\text{in}^2$ and $L = 120 \text{ in}$. Using the normalization, which is customary, different analysts can obtain the same numerical results. Note that each mode, in this case, has the same peak amplitude, $\sqrt{2/mL}$, since C_r is independent of r .

Complete Solution

The spatial-time history of each mode r can be written using Equation 7.22,

$$\begin{aligned} y_r(x, t) &= \hat{Y}_r(x) F_r(t) \\ &= \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L} (A_r \sin \omega_r t + B_r \cos \omega_r t), \end{aligned}$$

with $\omega_r = (r\pi/L)\sqrt{T/m} = r\pi c/L$. The complete solution is found by summing all the modal components,

$$\begin{aligned} y(x, t) &= \sum_{r=1}^{\infty} \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L} \left(A_r \sin \frac{r\pi c}{L} t, \right. \\ &\quad \left. + B_r \cos \frac{r\pi c}{L} t \right), \end{aligned} \quad (7.27)$$

where the units are consistent.¹⁴

¹³This convention is valid for systems without discrete masses.

¹⁴The units of the derived equation make sense. From Equation 7.27, the response $y(x, t)$ is a displacement with length units.

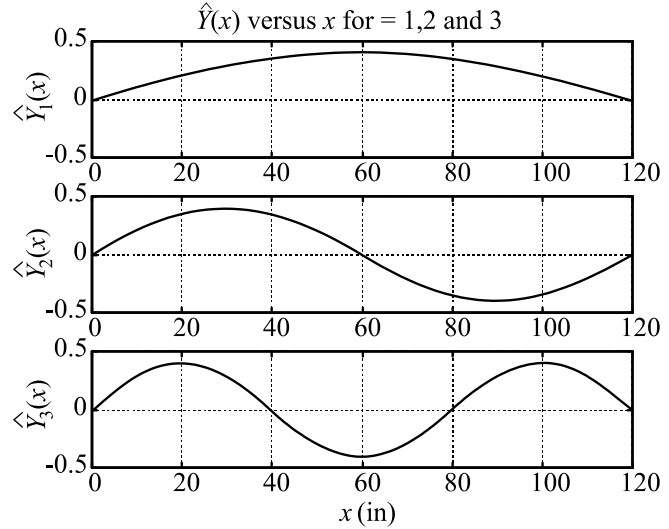


Figure 7.9: Normal modes of an aluminum string of 1 in² cross section with $m = 0.1 \text{ lb-s}^2/\text{in}^2$ and $L = 120 \text{ in}$. The first three modes are shown (top to bottom).

For example, assume that the initial displacement of the string is given by $y(x, 0) = d(x)$ and the initial velocity by $\dot{y}(x, 0) = 0$. The arbitrary constants A_r and B_r are functions of the initial conditions. Using Equation 7.27, we satisfy the initial velocity first to find

$$\begin{aligned} \dot{y}(x, 0) &= \sum_{r=1}^{\infty} \sqrt{\frac{2}{mL}} \omega_r \sin \frac{r\pi x}{L} (A_r \cos \omega_r t - B_r \sin \omega_r t)|_{t=0} \\ &= \sum_{r=1}^{\infty} \sqrt{\frac{2}{mL}} A_r \omega_r \sin \frac{r\pi x}{L} \\ &= 0. \end{aligned}$$

This equality can hold for all values of x only if $A_r = 0$ for all values of r .

Satisfying the initial displacement condition yields the relation

$$y(x, 0) = d(x) = \sum_{r=1}^{\infty} \sqrt{\frac{2}{mL}} B_r \sin \frac{r\pi x}{L}, \quad (7.28)$$

where $d(x)$ is the initial displacement function. Equation 7.28 is a Fourier sine expansion¹⁵ with the coefficients B_r .

Therefore, the right-hand side must also have such units. In the factor $\sqrt{2/mL}$, m has units of mass/length and L units of length. Then, $\sqrt{2/mL}$ has units of $1/\sqrt{\text{mass}}$. This means that the coefficients A_r and B_r have units of $\sqrt{\text{mass}} \times \text{length}$ as verified below in Equation 7.29. Thus, Equation 7.27 is correct dimensionally. Checking dimensional consistency provides an additional validation of the mathematical derivation.

¹⁵ B_r can be obtained by multiplying both sides of Equation 7.28 by $\sin(r\pi x/L)$ and integrating over the domain x from 0 to L . The procedure uses the orthogonality properties of the sine function.

given by

$$\sqrt{\frac{2}{mL}} B_r = \frac{2}{L} \int_0^L d(x) \sin \frac{r\pi x}{L} dx,$$

or

$$B_r = \sqrt{\frac{2m}{L}} \int_0^L d(x) \sin \frac{r\pi x}{L} dx. \quad (7.29)$$

Therefore, the complete solution satisfying the given initial conditions is

$$\begin{aligned} y(x, t) &= \sum_{r=1}^{\infty} \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L} \left(\sqrt{\frac{2m}{L}} \int_0^L d(x) \sin \frac{r\pi x}{L} dx \right) \cos \omega_r t \\ &= \frac{2}{L} \sum_{r=1}^{\infty} \sin \frac{r\pi x}{L} \left(\int_0^L d(x) \sin \frac{r\pi x}{L} dx \right) \cos \omega_r t. \end{aligned}$$

This equation can be solved once the initial displacement $d(x)$ is known.¹⁶ From Figure 7.10, the complete solution is approached by adding partial sums for the assumed parameter values $m = 0.1 \text{ lb-s}^2/\text{in}^2$, $L = 120 \text{ in}$, initial displacement

$$d(x) = \frac{x}{L} \left(\frac{x}{L} - \frac{1}{3} \right) \left(\frac{x}{L} - \frac{3}{4} \right) \left(\frac{x}{L} - 1 \right) \text{ in},$$

and $\omega_r = (r\pi/L)\sqrt{T/m}$, where $T = 100 \text{ lb}$.

Beams are Next Our next topic of study is the beam. We have two options regarding the order of topics. From a behavior perspective, given that we have just completed studying the transversely vibrating string, it makes sense to proceed to the transversely vibrating beam. From a mathematical perspective, and from a pedagogical perspective, it makes sense to study the beam in axial vibration and in torsional vibration since the governing equations for these cases are identical to that of the string. Also, the governing equations and boundary conditions are less difficult to formulate and solve than those for the bending beam. Therefore, we proceed with the axially vibrating beam.

¹⁶ A dimension check of Equation 7.29 verifies that B_r has units of $\sqrt{\text{mass} \times \text{length}}$.

$y(x, t)$ versus t for $x/L = 1/4$ $y(x, t)$ versus t for $x/L = 1/2$

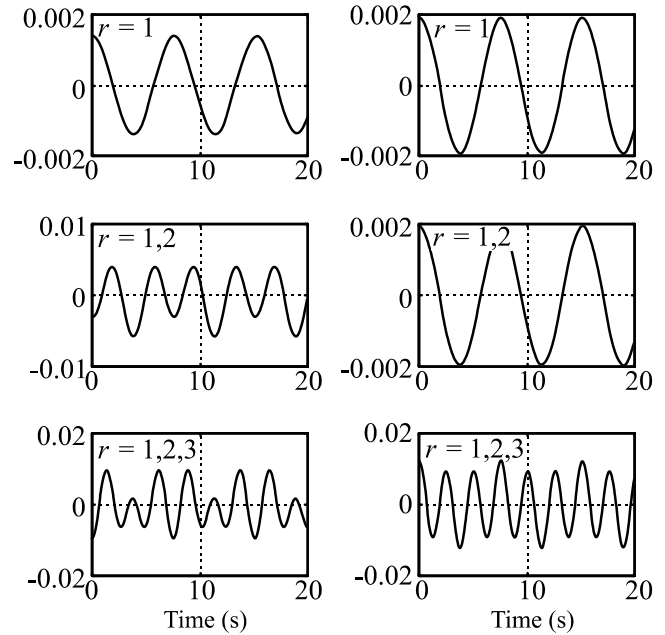


Figure 7.10: Partial sum solutions for the string for $r = 1$, $r = 1, 2$, $r = 1, 2, 3$ for $x/L = 1/4$ and $1/2$. The more terms that are retained in the summation, the more accurate the result. In general, we do not need many terms to attain reasonable accuracy, although this figure does not show this.

DOROTHY JOHNSON VAUGHAN

(September 20, 1910 – November 10, 2008)



Contribution: Dorothy Johnson Vaughan was an American mathematician who worked at the National Advisory Committee for Aeronautics (NACA), the predecessor agency to NASA.

Life Sketch: Johnson was born in Kansas City, Missouri to Leonard H. and Annie A. Johnson. In 1917, the family moved to Morgantown, West Virginia. She graduated from Beechurst H.S. in 1925.

She earned a full-tuition scholarship to Wilberforce University, a private Black college in Xenia, Ohio. (Wilberforce was

founded in 1865 as “one of the destination points of [the] railroad” when the Ohio Underground Railroad was established.) At Wilberforce, Johnson earned “splendid grades” and chose math as her major. When she was an upperclassman, one of Johnson’s professors at Wilberforce recommended her for graduate study in mathematics at Howard University.

Johnson welcomed the chance to prove herself in a competitive graduate school. But the economic reality that confronted her when she came out of college made graduate study seem like an irresponsible extravagance. With the onset of the Great Depression, her parents, like a third of all Americans, found steady work hard to come by. An extra income would help keep the household solvent and improve the odds that Dorothy’s sister might be able to follow her path to college.

At the age of 19 Johnson felt it was her responsibility to ensure that the family could make its way through the hard times, even though it meant forfeiting her own ambitions. She opted to earn a degree in education and pursue teaching, the most stable career for a Black woman with a college degree. She graduated from Wilberforce University in 1929 with a B.S. degree. After two years she went to work as a mathematics teacher at R.R. Moton High School in Farmville, Virginia.

In the Spring 1943, Vaughan filled out two job applications: one to work at Camp Pickett, where the need for physical labor was so great that there was virtually no possibility that they would not hire her, and the other to work at a federal agency in Hampton, Virginia that was seeking women to fill mathematical jobs having to do with airplanes. That second application to the National Advisory Committee for Aeronautics’ personnel department was most certainly meant for white, well-to-do students at the all-female State Teachers College in Farmville. Vaughan was hired by NACA’s Langley Memorial Aeronautical Laboratory in 1943.

Her career at Langley spanned twenty-eight years. She was assigned to the West Area Computers, a work group composed entirely of African American female mathematicians. They were placed in a segregated section and were responsible for doing the mathematical computations for the engineers conducting aeronautical experiments. Using slide rulers, calculators, and film readings, they provided the engineers with the data needed to conduct various performance testing, such as the variables affecting drag and lift of the aircraft. These experiments were conducted in the wind tunnels at Langley.

In 1949, Vaughan became the head of the West Area Computers. Mathematician Katherine Johnson was assigned to Vaughan’s group before being transferred to Langley’s Flight Research Division. Vaughan continued at Langley after NACA became NASA, specializing for the rest of her career in electronic computing and FORTRAN programming. She worked in the Langley Research Center’s Analysis and Computation Division, and also participated in Scout Project (Solid Controlled Orbital Utility Test system) tests at Wallops Flight Facility.

She married Howard S. Vaughan Jr. and raised a family

while working at Langley, and one of her children went on to also be employed at NASA.

She retired from NASA in 1971, and died in 2008 at the age of 98. She was preceded in death by her husband and two sons.

Noteworthy: In a 1994 interview, she recalled that working at Langley during the Space Age felt like being on “the cutting edge of something very exciting.”

Vaughan is featured in the 2016 film *Hidden Figures* about three African-American women at NASA who calculated flight trajectories for Project Mercury and Apollo 11 in the 1960s. It is based on the book by the same name by author Margot Lee Shetterly that documented the NASA careers and contributions of Katherine Johnson, Mary Jackson, and Vaughan.

7.3 Axial Vibration of Beams

Beams¹⁷ are fundamental components in structures and machines. They are also useful as idealized models for complex system behavior. We will derive and solve the governing equations for beams undergoing axial (that is, longitudinal), torsional, and transverse motion. We will assume small amplitude motion for linear behavior. The equation of motion will be formulated using Newton’s second law of motion and also using Hamilton’s principle.

7.3.1 Axial Vibration: Newton’s Approach

Consider the schematics of the beam in [Figure 7.11](#). Displacements, strains, and stresses are assumed uniform at a given cross section. From the free-body diagram in the figure, internal force P acts to the left and P plus an undetermined increment dP acts to the right. For the static problem, $dP = 0$ and the internal load becomes identical to the external load. For the dynamic problem, the sum of the forces equals the product of mass and acceleration.

Let the element in the free-body diagram have a mass per unit length $m(x) = \rho(x)A(x)$, where $\rho(x)$ is the density and $A(x)$ is the area of the cross section at x . Then, by Newton’s second law of motion for an element of length dx ,

$$[P(x, t) + dP(x, t)] - P(x, t) = m(x)dx \frac{\partial^2 u(x, t)}{\partial t^2}. \quad (7.30)$$

¹⁷We have found the term *beam* used interchangeably with the alternatives *bar* and *rod*. Numerous texts have been examined, and there appears to be no accepted standard, even within the same text! In general *beam* is used for transverse motion, *rod* for axial motion, and *shaft* for rotational motion. We use *beam* in our discussions except for the case of a beam under torsional vibration. Such instances are likely to be in application to rotating machinery, and we adopt the conventional term *shaft* for such cases.

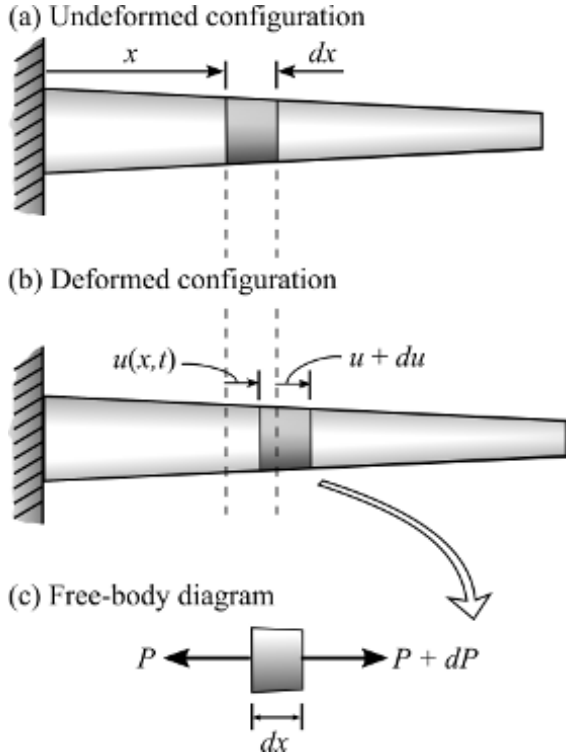


Figure 7.11: Axial deformation due to internal force P . The top figure (a) shows the beam with element of length dx before the axial force is applied. The second figure (b) depicts the equilibrium and deformed positions of the beam after the axial force is applied. The last figure (c) is the free-body diagram of the beam element dx .

From strength of materials,

$$P = EA\varepsilon = EA \frac{\partial u}{\partial x},$$

where E is the modulus of elasticity (Young's modulus) and ε is the strain and, therefore, for a given t ,

$$\begin{aligned} dP(x, t) &= \frac{\partial P(x, t)}{\partial x} dx \\ &= \frac{\partial}{\partial x} \left(EA(x) \frac{\partial u(x, t)}{\partial x} \right) dx. \end{aligned}$$

Equation 7.30 then becomes

$$\frac{\partial}{\partial x} \left(EA(x) \frac{\partial u(x, t)}{\partial x} \right) = m(x) \frac{\partial^2 u(x, t)}{\partial t^2}. \quad (7.31)$$

If the beam is uniform, then $m(x) = m = \rho A$ and the area A can be canceled from both sides of the equation, leading to the wave equation,

$$c^2 \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial^2 u(x, t)}{\partial t^2},$$

where $c^2 = E/\rho$ has units of (speed)². Before solving this equation, we derive it using Hamilton's principle, in part to show that the method chosen may involve more or less work.

7.3.2 Axial Vibration: Hamilton's Approach

Hamilton's principle is an alternative method to derive the governing equation of motion for the axial vibration of the beam. Considering the deformed beam of Figure 7.11, the change in the length dx is given by

$$\Delta(dx) = du = \frac{\partial u}{\partial x} dx = \varepsilon dx,$$

where x is the direction of motion, $u(x, t)$ is the axial displacement, and ε is the strain. We require expressions for the strain and kinetic energies.

The strain energy associated with the deformation is

$$dV = \frac{1}{2} P du = \frac{1}{2} P \varepsilon dx.$$

The axial stress at a particular location x is $\sigma = P/A$, and since $\sigma = E\varepsilon$, we have

$$dV = \frac{1}{2} \frac{P^2}{EA(x)} dx = \frac{1}{2} EA(x) \varepsilon^2 dx.$$

For the whole beam, the strain energy is given by the relation

$$V = \frac{1}{2} \int_0^L EA(x) \varepsilon^2 dx = \frac{1}{2} \int_0^L EA(x) \left(\frac{\partial u}{\partial x} \right)^2 dx,$$

where $EA(x)$ is known as the *axial stiffness*.

The kinetic energy for the complete beam is given by

$$T = \frac{1}{2} \int_0^L m(x) \left(\frac{\partial u}{\partial t} \right)^2 dx,$$

where $m(x)$ is the mass per unit length.¹⁸

Substituting V and T into Hamilton's principle,

$$\begin{aligned} \delta \int_{t_1}^{t_2} (T - V) dt &= \delta \int_{t_1}^{t_2} \left\{ \frac{1}{2} \int_0^L m(x) \left(\frac{\partial u}{\partial t} \right)^2 dx \right. \\ &\quad \left. - \frac{1}{2} \int_0^L EA(x) \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} dt \\ &= \int_{t_1}^{t_2} \left\{ \int_0^L m(x) \frac{\partial u}{\partial t} \delta \left(\frac{\partial u}{\partial t} \right) dx \right. \\ &\quad \left. - \int_0^L EA(x) \frac{\partial u}{\partial x} \delta \left(\frac{\partial u}{\partial x} \right) dx \right\} dt \\ &= 0. \end{aligned}$$

Performing the usual interchanges and integration by parts and recalling that δu vanishes at $t = t_1$ and $t = t_2$, we find the resulting expression to be

$$\begin{aligned} \int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{\partial}{\partial x} \left(EA(x) \frac{\partial u}{\partial x} \right) - m(x) \frac{\partial^2 u}{\partial t^2} \right] \delta u dx \right. \\ \left. + \left[\left(EA(x) \frac{\partial u}{\partial x} \right) \delta u \right]_0^L \right\} dt = 0. \quad (7.32) \end{aligned}$$

¹⁸Here, we resume using the notation T for kinetic energy.

Since δu is arbitrary in $0 < x < L$, the first square bracket must equal zero,

$$\frac{\partial}{\partial x} \left(EA(x) \frac{\partial u}{\partial x} \right) - m(x) \frac{\partial^2 u}{\partial t^2} = 0. \quad (7.33)$$

This is the equation of motion, which has the same form as the governing equation for the vibrating string.¹⁹ From Equation 7.32 the boundary conditions are given by

$$\left[\left(EA(x) \frac{\partial u}{\partial x} \right) \delta u \right]_0^L = 0. \quad (7.34)$$

For example, a clamped end at $x = 0$ results in zero displacement δu over all time t , that is, $u(0, t) = 0$. A free end at $x = L$ implies that a moment cannot be resisted and $EA(L) \partial u(L) / \partial x = 0$.

7.3.3 Simplified Eigenvalue Problem

The assumed solution for the fixed-free axially vibrating beam is $u(x, t) = U(x)F(t)$, where $U(x)$ is the mode of vibration and $F(t)$ defines the harmonic behavior. Differentiating this assumed solution and substituting the respective terms into Equation 7.33 gives

$$\frac{d}{dx} \left[EA(x) \frac{dU(x)}{dx} \right] = m(x) U(x) \frac{\ddot{F}}{F} = -\omega^2 m(x) U(x), \quad (7.35)$$

where, in the second equality, the assumption of harmonic behavior results in the replacement of $\ddot{F}/F$ by $-\omega^2$, as was done in the string problem. The fixed-free boundary conditions to be satisfied are

$$\begin{aligned} U(0) &= 0 \\ EA(L) \frac{dU(x)}{dx} \Big|_{x=L} &= 0. \end{aligned}$$

The second boundary condition is derived²⁰ from Equation 7.34. It is further assumed that the beam is prismatic and uniform, that is, $A(x) = A$ and $m(x) = m$.

Equation 7.35 can be transformed into the following simple ordinary differential equation for the modes,

$$\frac{d^2 U}{dx^2} + \beta^2 U = 0,$$

where $\beta^2 = \omega^2 / c^2$ and $c = \sqrt{EA/m}$ is the wave speed. The general modal solution is

$$U(x) = C_1 \sin \beta x + C_2 \cos \beta x. \quad (7.36)$$

¹⁹In Equation 7.4, replace y by u , and T by EA .

²⁰The boundary condition is due to

$$EA(L) \frac{\partial u(x)}{\partial x} \Big|_{x=L} = 0 \quad \text{or} \quad EA(L) \frac{dU(x)}{dx} \Big|_{x=L} = 0,$$

for all t after dividing by $F(t)$.

The coefficients C_1 and C_2 are determined from the given boundary conditions,

$$\begin{aligned} U(0) &= 0 = C_2 \\ EA(L) \frac{dU(x)}{dx} \Big|_{x=L} &= 0 = C_1 \beta \cos \beta L. \end{aligned}$$

From the first boundary condition, $C_2 = 0$. Two possible solutions exist for the second boundary condition: $C_1 = 0$ or $\beta \cos \beta L = 0$. The first results in the trivial solution $U(x) = 0$, which is of no use. The second equation defines β , that is, $C_1 \neq 0$ and $\beta \cos \beta L = 0$, signifying an infinite number of roots for this equation. Therefore, $\beta_r L = (2r - 1)\pi/2$, or

$$\beta_r = (2r - 1) \frac{\pi}{2L}, \quad r = 1, 2, \dots$$

From the definition of β , we can solve for ω for the r^{th} mode,

$$\begin{aligned} \omega_r^2 &= \beta_r^2 \frac{EA}{m} \\ \omega_r &= (2r - 1) \frac{\pi}{2} \sqrt{\frac{EA}{mL^2}}. \end{aligned}$$

Equation 7.36 for the r^{th} mode becomes

$$U_r(x) = C_r \sin \frac{(2r - 1)\pi}{2L} x, \quad r = 1, 2, \dots$$

Since the modes can only be specified to a constant, it is customary to normalize them according to

$$\int_0^L m U_r^2 dx = 1,$$

which yields $\hat{U}_r(x)$, the normalized mode, and $C_r = \sqrt{2/mL}$.

The complete solution is then

$$\begin{aligned} u(x, t) &= \sum_{r=1}^{\infty} u_r(x, t) \\ &= \sum_{r=1}^{\infty} \hat{U}_r(x) F_r(t), \end{aligned}$$

and with the respective substitutions,

$$\begin{aligned} u(x, t) &= \sum_{r=1}^{\infty} C_r \sin \beta_r x (A_r \sin \omega_r t + B_r \cos \omega_r t) \\ &= \sum_{r=1}^{\infty} \sqrt{\frac{2}{mL}} \sin \frac{(2r - 1)\pi}{2L} x \\ &\quad \times \left(A_r \sin \frac{(2r - 1)\pi}{2L} \sqrt{\frac{EA}{m}} t \right. \\ &\quad \left. + B_r \cos \frac{(2r - 1)\pi}{2L} \sqrt{\frac{EA}{m}} t \right). \quad (7.37) \end{aligned}$$

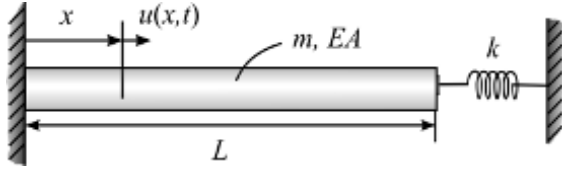


Figure 7.12: Uniform beam with elastic restraint in longitudinal motion.

Coefficients A_r and B_r are found by satisfying the initial displacement $u(x, 0)$ and initial velocity $\dot{u}(x, 0)$ using Equation 7.37 and its time derivative.

Example 7.2 Axial Vibration of Uniform Beam with Elastic Boundary

Consider the axially vibrating beam shown in Figure 7.12, where the right end of the beam is restrained elastically. Derive the eigenvalue problem and solve for the eigenvalues.

Solution: The differential equation of motion for the uniform beam in longitudinal vibration is

$$EA \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2}.$$

The boundary conditions are

$$\begin{aligned} u(0, t) &= 0 \\ EA \frac{\partial u(L, t)}{\partial x} &= -ku(L, t). \end{aligned}$$

The respective eigenvalue problem is

$$EA \frac{d^2 U}{dx^2} = -\omega^2 m U. \quad (7.38)$$

The boundary conditions are

$$\begin{aligned} U(0) &= 0 \\ EA(L) \frac{dU(x)}{dx} \Big|_{x=L} &= -kU(L), \end{aligned} \quad (7.39)$$

where the last condition is a force balance at the interface between the right end of the beam and the left end of the spring. A positive displacement results in spring force opposition.

Equation 7.38 can be written as

$$\frac{d^2 U}{dx^2} + \beta^2 U = 0,$$

with $\beta^2 = \omega^2 m / EA$. We have solved this problem previously, except here there is an elastic boundary condition at the end $x = L$. The modal equation is $U(x) = C_1 \sin \beta x + C_2 \cos \beta x$, with $U(0) = 0 = C_2$ and at $x = L$ Equation 7.39 becomes

$$C_1 \beta \cos \beta L = -\frac{k}{EA} C_1 \sin \beta L.$$

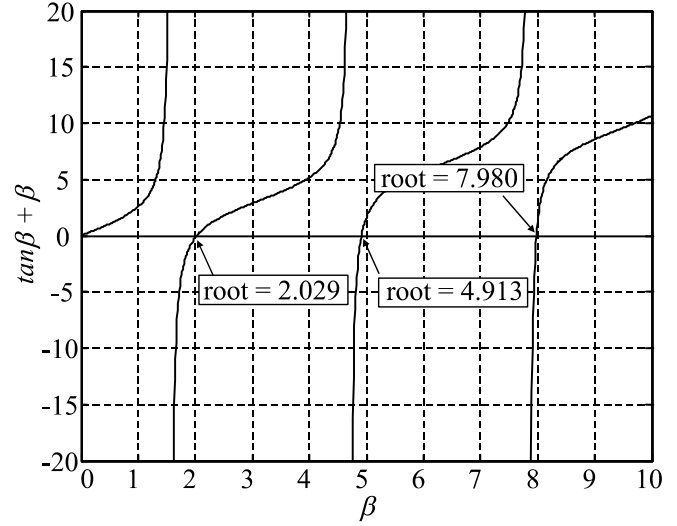


Figure 7.13: Plot of $\tan \beta + \beta$ vs. β .

This expression can be solved for βL by rewriting it as the transcendental equation

$$\tan \beta L = -\frac{\beta L E A}{k L},$$

which has an infinite number of roots $\beta_r L$.

For the arbitrary set of parameter values, $k = EA$ and $L = 1$, we have $\tan \beta = -\beta$, that can be solved numerically. The function $f(\beta) = \tan \beta + \beta$ is plotted in Figure 7.13. The first three positive roots of this equation are 2.029, 4.913, and 7.980.²¹ With β evaluated, the frequencies can be found using

$$\omega_r^2 = \beta_r^2 \frac{EA}{m}.$$

The modes can then be expressed explicitly, leading to the complete solution in time and space. ■

7.3.4 Eigenfunction Expansion Method

Consider a uniform beam that is loaded axially by a distributed force $F(x, t)$, as shown in Figure 7.14. We demonstrate how this problem can be solved using modal techniques in a very similar way to how we solved multi degree-of-freedom vibration problems.

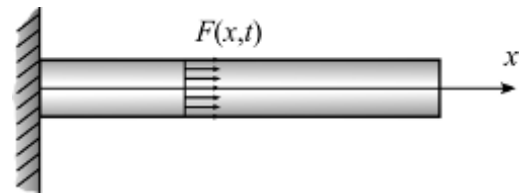


Figure 7.14: Beam under general forcing $F(x, t)$.

²¹ Another way to find the roots is to superpose the curves $\tan \beta$ and $-\beta$. The intersections are the roots.

The equation of motion is

$$EA \frac{\partial^2 u}{\partial x^2} - m \frac{\partial^2 u}{\partial t^2} + F(x, t) = 0. \quad (7.40)$$

We assume a product solution for $u(x, t)$, with the resulting eigenvalue problem,

$$\frac{d^2 U}{dx^2} + \beta^2 U = 0, \quad (7.41)$$

where $\beta^2 = \omega^2 m / EA$, with the solution

$$U(x) = C_1 \sin \beta x + C_2 \cos \beta x. \quad (7.42)$$

The frequencies ω_j and $U_j(x)$ can be obtained by satisfying the boundary conditions. The normalized modes $\hat{U}_j$ can be found as before.

As part of this problem and solution, we demonstrate the orthogonality of the modes. We motivate the more formal discussion of [Section 7.5.4](#) on the orthogonality of the modes by tackling this solution with an expansion of the solution and the forcing function in terms of the normalized modes, via the expansion theorem,²²

$$u(x, t) = \sum_{j=1}^{\infty} u_j(t) \hat{U}_j(x) \quad (7.43)$$

$$F(x, t) = \sum_{j=1}^{\infty} F_j(t) \hat{U}_j(x). \quad (7.44)$$

The modes are harmonic functions, and they have orthogonality properties that can be used to determine the expansion functions $u_j(t)$ and $F_j(t)$. Multiplying both sides of [Equations 7.43](#) and [7.44](#) by $m \hat{U}_k(x)$ and integrating over the x domain from 0 to L , we have

$$\int_0^L m u(x, t) \hat{U}_k(x) dx = \sum_{j=1}^{\infty} m u_j(t) \int_0^L \hat{U}_j(x) \hat{U}_k(x) dx.$$

Since $\hat{U}_j(x)$ is harmonic, the integrals on the right-hand side equal zero except for $k = j$. Thus,

$$\begin{aligned} u_j(t) &= \int_0^L m u(x, t) \hat{U}_j(x) dx \\ F_j(t) &= \int_0^L m F(x, t) \hat{U}_j(x) dx. \end{aligned} \quad (7.45)$$

Substituting [Equations 7.43](#) and [7.44](#) into [Equation 7.40](#) leads to the summation

$$\sum_{j=1}^{\infty} \left[EA u_j(t) \frac{d^2 \hat{U}_j}{dx^2} - m \frac{d^2 u_j}{dt^2} \hat{U}_j + F_j(t) \hat{U}_j \right] = 0. \quad (7.46)$$

²²This procedure is the *method of eigenfunction expansion*. It is also known as a *modal expansion*.

From the eigenvalue problem, we have

$$-EA \frac{d^2 \hat{U}_j}{dx^2} = \omega_j^2 m \hat{U}_j,$$

which can be substituted into [Equation 7.46](#), resulting in the equation

$$\sum_{j=1}^{\infty} \left[-\omega_j^2 m u_j(t) - m \frac{d^2 u_j}{dt^2} + F_j(t) \right] \hat{U}_j(x) = 0.$$

Since $\hat{U}_j(x) \neq 0$, the terms in the square brackets must add to zero,

$$\ddot{u}_j + \omega_j^2 u_j = \frac{F_j(t)}{m}, \quad j = 1, 2, \dots$$

The solution for the j^{th} term is

$$\begin{aligned} u_j(t) &= A_j \cos \omega_j t + B_j \sin \omega_j t \\ &+ \frac{1}{m \omega_j} \int_0^t F_j(\tau) \sin \omega_j(t - \tau) d\tau, \end{aligned}$$

where any force $F(x, t)$ can be substituted into [Equation 7.45](#) for the evaluation of $F_j(t)$. The complete solution is the infinite sum,

$$u(x, t) = \sum_{j=1}^{\infty} u_j(t) \hat{U}_j(x).$$

The number of terms needed for an accurate representation depends on the loading characteristics. The initial conditions are satisfied by solving for A_j and B_j .

An interesting application of the axial vibration of beams and cables is examined next.



THE SPACE ELEVATOR.

There has been much interest in the concept of the space elevator, also known as the orbital tower. It is essentially a long beam extending from the surface of the Earth to a point in space such that its center of gravity is at the geosynchronous (GEO) location.²³ The tower is climbed by “elevators” into space, carrying people and equipment. The beam is also

²³A geosynchronous orbit may be defined as one with an orbital period (the time needed to orbit once around the Earth) that matches the rotation rate of the Earth. This is called a *sidereal* day, which is 23 hours 56 minutes and 4 seconds, and represents the time taken for the Earth to rotate once about its polar axis relative to a distant fixed point. (The sidereal day is about four minutes shorter than the common day length of 24 hours, which is relative to the sun.) A geostationary orbit is a special case of a geosynchronous orbit. A satellite is in a geostationary orbit when it appears stationary from the point of view of an observer on the Earth's surface. This can occur only when the orbit is geosynchronous, the orbit is a circle, and the orbit lies in the plane of the Earth's equator. A geostationary satellite orbits at a height of 35,785 km (22,236 miles) above the equator.

called a ribbon or a string, since due to its length its mechanical characteristics are more like a string.

There are a number of key elements in a space elevator. The most important is the ribbon, upon which the autonomous climber makes its way into space. The ribbon was the fundamental stumbling block to a reliable and practical elevator design until carbon nanotubes (CNTs) were discovered in the early 1990s. Prior to that there were no suitable materials that could be utilized to make a ribbon of a length stretching from the surface of the Earth to a point more than a third of the way to the Moon, a length of 144,000 km (89,500 mi).²⁴

CNTs were a new kind of material and much stronger than the strongest material known in the 1990s. CNTs are less dense than other very strong materials such as steel and Kevlar, and have a tensile strength over 25 times stronger.

The essential idea is to deploy a cable/ribbon from GEO in two directions, “up” and “down.” It would be deployed down to the Earth’s surface, and up well beyond GEO. Both parts would be in tension, with the equilibrium position at GEO. If the upper part would need to be shorter, then a counterweight could be placed at the end so that the whole structure would be in force balance. The highest tension would be in the middle and the least tension would be at both ends. As such, the ribbon would be tapered from the middle to the ends.

A climber moving to the outer end can be given enough energy to escape from Earth’s gravity well and proceed to the Moon or another destination. In addition to the ribbon, there is a climber, and an anchoring structure on the Earth – although anchor is not the proper term since the space elevator is an orbital structure, orbiting over a fixed spot on the surface of the Earth. Key to the success of the space elevator is a powerful and reliable energy source. Two discussions of the design of a space elevator are given by Pearson²⁵ (1975 - pre-CNT) and by Edwards²⁶ (2000 - post-CNT).

Many design considerations need to be addressed before the early space elevator pioneers build their first prototype. These are only summarized here. For a structure of this length, there are numerous environmental conditions that act upon it. Some of the most serious concerns are within the Earth’s atmosphere and low Earth orbit. In that region there are weather considerations, and at the orbital altitudes a danger is the possibility of collisions with debris. Lightning strikes hitting the tether could destroy it due to extreme heating. Atmospheric conditions would limit where over the Earth the tether is orbited. In addition to the possibility of micrometeorite impacts, the ribbon is susceptible to radiation damage in the Earth’s radiation belts as well as atomic oxygen erosion. Due to the electromagnetic properties of the

²⁴The average distance from the center of the Earth to the center of the Moon is 384,400 km (238,900 miles).

²⁵J. Pearson, “The Orbital Tower: A Spacecraft Launcher Using the Earth’s Rotational Energy,” *Acta Astronautica*, Vol.2, 1975, pp.785-799.

²⁶B.C. Edwards, “Design and Deployment of a Space Elevator,” *Acta Astronautica*, Vol.47, No.10, 2000, pp.735-744.

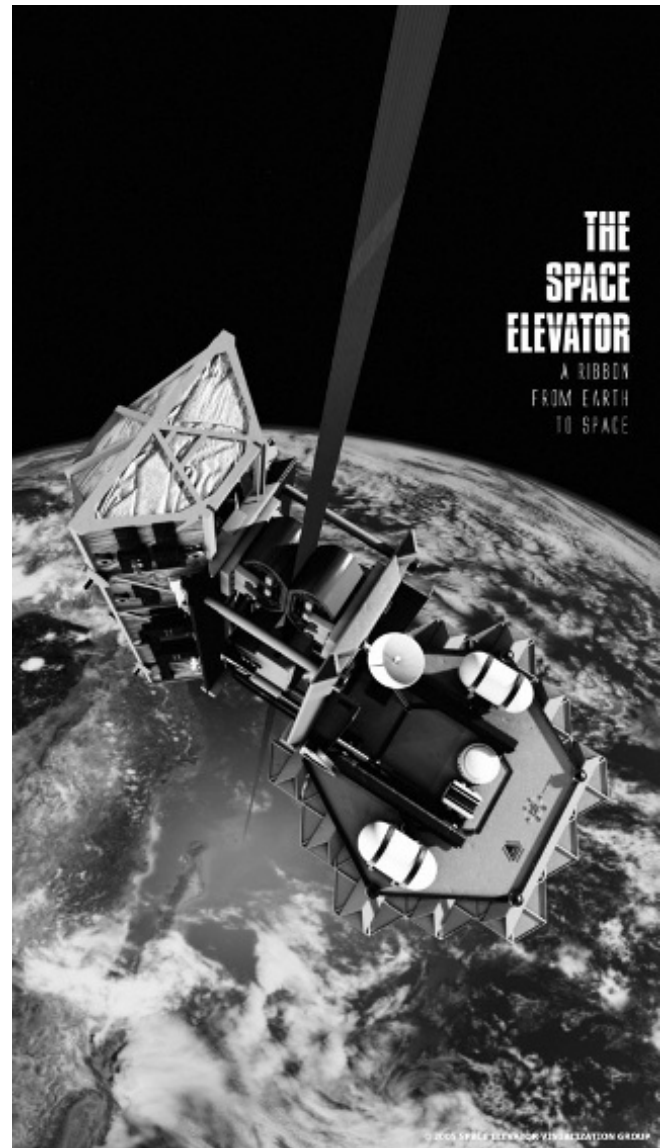


Figure 7.15: An artist rendering of a space elevator concept (Courtesy Alan Chan).

CNT ribbon, it is heated due to the magnetic field induced electrical currents.

Structural oscillations of the ribbon are a design concern. The ribbon may oscillate along the axis as well as in a direction perpendicular to the axis. The frequencies and periods of such oscillations depend on the tether’s geometric and material properties. These values affect the allowable speeds of the climber because of resonance concerns, with large amounts of energy transmitted to the ribbon from the climber inducing very large and dangerous oscillations.

As stated by Pearson, “the ‘orbital tower’ could be built only by overcoming the three problems of buckling, strength, and dynamic stability. The buckling problem could be solved by building the tower outward from the geostationary point so that it remains balanced in tension and stabilized by the

gravity gradient until the lower end touches the Earth and the upper end reaches 144,000 km altitude. The strength problem could be solved by tapering the cross-sectional area of the tower as an exponential function of the gravitational and inertial forces, from a maximum at the geostationary point to a minimum at the ends. The strength requirements are extremely demanding, but the required strength-to-weight ratio is theoretically available in perfect-crystal whiskers of graphite.

“The dynamic stability is investigated and the tower is found to be stable under the vertical forces of lunar tidal excitations and under the lateral forces due to payloads moving along the tower. By recovering the excess energy of returning spacecraft, the tower would be able to launch other spacecraft into geostationary orbit with low power to overcome frictional and conversion losses. By extracting energy from the Earth’s rotation, the orbital tower would be able to launch spacecraft without rockets from the geostationary orbit to reach all the planets or to escape the solar system.” Pearson called his structure an orbital tower, and referred to perfect-crystal whiskers of graphite because CNTs had not yet been discovered. The taper ratio needs to be a reasonable number, on the order of 2, so that the ribbon is practical.

We will outline some of the equations used by Pearson to analyze the dynamics of his “tower.” As this can only be a brief summary, we refer the reader to his paper.

One concern for the design of the space elevator ribbon is the effect on it by the tidal forces of the Moon, which have a period of 12.5 h. To assess this potential problem, the periods of several modes of vibration were calculated. The physical model used was that of a tapered rod in tension, with the following two assumptions: (a) displacements, strains, and stresses are uniform at a given cross section, that is, sections remain plane during deformation, and (b) The vibration is linear and elastic. The first assumption is justified because the cross section of the ribbon is many orders of magnitude smaller than its length.

Using a beam or rod model, the governing equation of axial motion is

$$\frac{\partial}{\partial r} \left[EA(r) \frac{\partial u(r, t)}{\partial r} \right] = \rho A(r) \frac{\partial^2 u(r, t)}{\partial t^2}, \quad (7.47)$$

where we use Pearson’s notation: $u(r, t)$ is the axial deflection from equilibrium, r equals the distance from the Earth’s center, $A(r)$ is the cross-sectional area of the rod, and ρ is the material density.

By balancing the forces at a section of the beam, it is found that the cross-sectional area can be written as

$$A(r) = A_s \exp(3r_0^2/2hr_s) \cdot \exp[(-r_0/h)(r_0/r + r_0 r^2/2r_s^3)],$$

where A_s is the cross-sectional area of the tower at the synchronous point, r_s is the synchronous orbit radius, r_0 equals the Earth’s radius, and $h = \sigma/\rho g_0$, where σ is the stress at location r , and g_0 is Earth’s gravity at the surface r_0 . Substituting $A(r)$ into Equation 7.47, and assuming E is constant,

the equation of longitudinal motion becomes (HERE)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{h} \left(\frac{r_0^2}{r^2} - \frac{r_0^2 r}{r_s^3} \right) \frac{\partial u}{\partial r} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2}.$$

The assumed solution via separation of variables is $u(r, t) = U(r)F(t)$, and substituting it into the equation of motion results in the following two ordinary differential equations,

$$\begin{aligned} \frac{d^2 U}{dr^2} + \frac{1}{h} \left(\frac{r_0^2}{r^2} - \frac{r_0^2 r}{r_s^3} \right) \frac{dU}{dr} + \frac{\omega^2 \rho}{E} U &= 0 \\ \frac{d^2 F}{dt^2} + \omega^2 F &= 0. \end{aligned}$$

Using a numerical solution, Pearson found that for a taper ratio of ten, the period of the first longitudinal mode is approximately 21.5 h, and the amplification factor for the first mode is about 0.5. For the ribbon design, these values seem reasonable as there appears to be no amplification.

Other related problems were also studied. One is whether the waves generated in the cable by the traveling/climbing elevator limit the speed allowable for such transit. A set of calculations determines that these critical speeds are above 1000 km/s, and thus not an issue.



Figure 7.16: Rolling contact bearings, such as the tapered roller bearing shown, are used in rotating machinery. Many bearings fail prematurely because of contamination, poor lubrication, temperature extremes, poor fitting, unbalance, and misalignment. All of these factors lead to an increase in bearing vibration. To detect degrading bearings before they fail catastrophically (with the associated costs of downtime or damage to other parts of the machine), engineers often rely on “condition monitoring” software that indicates changes in acoustic emission or bearing temperature. (Photograph by M. Nagurka)

7.4 Torsional Vibration of Shafts

The torsional vibration of shafts has numerous applications. Shafts are the backbone of power transmission systems and rotor-bearing systems. Two examples are a pump connected to a motor through a coupling shaft, and a turbopump driven by a shaft in a helicopter main engine. Rotor-bearing systems are often modeled as flexible rotors subjected to torques. Thrust forces may act on rotating shafts in the axial (longitudinal) direction as well.

We assume that there is no *warping* of the cross-sectional planes of the shaft. Warping may occur for non-uniform or non-circular shafts where the plane cross sections distort out of plane. Sketches of a shaft and the free-body diagram of an element of length dx are shown in Figure 7.17.

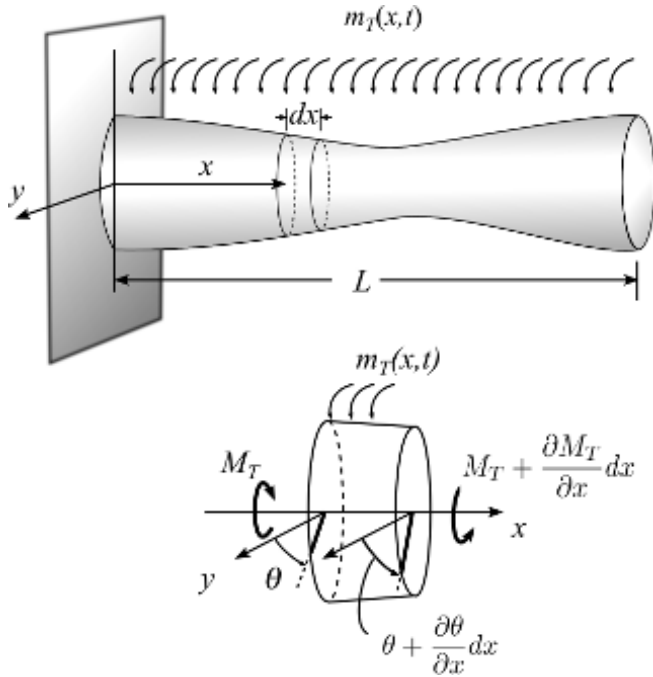


Figure 7.17: Shaft with a free-body diagram of a representative section.

We define the following properties at section x : shear modulus G , polar area moment of inertia $J(x)$ for a circular cross section, angle of twist $\theta(x, t)$, internal twisting moment $M_T(x, t)$, polar mass moment of inertia per unit length $I(x)$, and external twisting moment per unit length $m_T(x, t)$. For a solid cylinder with variable radius $r(x)$,

$$J(x) = \frac{\pi r(x)^4}{2} \text{ and } I(x) = \frac{\rho \pi r(x)^4}{2}.$$

From strength of materials, at a specific time and location on the shaft, the internal twisting moment is given

by

$$M_T(x, t) = GJ(x) \frac{\partial \theta(x, t)}{\partial x}. \quad (7.48)$$

Applying the Newton-Euler equation $\sum M = I\ddot{\theta}$ to the free-body diagram, the torsional equation of motion is found,

$$\left[M_T + \frac{\partial M_T}{\partial x} dx \right] + m_T(x, t) dx - M_T = I(x) dx \frac{\partial^2 \theta}{\partial t^2},$$

and, after simplification,

$$\frac{\partial M_T}{\partial x} + m_T(x, t) = I(x) \frac{\partial^2 \theta}{\partial t^2}. \quad (7.49)$$

Substituting Equation 7.48 into Equation 7.49 to eliminate the internal twisting moment from the equation, we find

$$\frac{\partial}{\partial x} \left[GJ(x) \frac{\partial \theta}{\partial x} \right] + m_T(x, t) = I(x) \frac{\partial^2 \theta}{\partial t^2}. \quad (7.50)$$

For free torsional vibration, we set $m_T(x, t) = 0$. As examples of possible boundary conditions, a clamped end at $x = 0$ permits no rotation, $\theta(0, t) = 0$, and a free end at $x = L$ implies no moment resistance,

$$M_T(L, t) = GJ(L) \frac{\partial \theta(L, t)}{\partial x} = 0.$$

These equations have the same mathematical form as those for a *string* and for the longitudinal vibration of a *beam*.²⁷ The torsional vibration problem is solved in the same way as the string and beam vibration problems.

Example 7.3 Harmonic Torque Applied to Shaft

A torque $T = T_0 \sin \omega t$ is applied to the free end of a fixed-free shaft. Determine the steady-state amplitude of the free end of the shaft.

Solution: Equation 7.50 for a uniform shaft is

$$GJ \frac{\partial^2 \theta}{\partial x^2} = I \frac{\partial^2 \theta}{\partial t^2},$$

with the boundary conditions

$$GJ \frac{\partial \theta(L, t)}{\partial x} = T_0 \sin \omega t \quad \theta(0, t) = 0,$$

where the applied torque is seen in the free end boundary condition. The steady-state response is harmonic,

$$\theta(x, t) = \Theta(x) \sin \omega t,$$

where the phase is zero since this is an undamped system. We substitute the response into the governing equation

²⁷If $I(x)$ is replaced by $mJ(x)$, the governing equation is the wave equation with torsional wave speed $c = \sqrt{G/m}$.

as well as into the boundary conditions to find

$$\begin{aligned} GJ \frac{d^2 \Theta}{dx^2} + \omega^2 I \Theta &= 0 \\ GJ \frac{d\Theta(L)}{dx} &= T_0 \\ \Theta(0) &= 0, \end{aligned}$$

where $\sin \omega t$ has been canceled from all terms.

The solution of the differential equation is

$$\Theta(x) = C_1 \cos \left(\omega \sqrt{\frac{I}{GJ}} x \right) + C_2 \sin \left(\omega \sqrt{\frac{I}{GJ}} x \right).$$

The application of the boundary conditions results in the following expressions for the integration constants,

$$\begin{aligned} C_1 &= 0 \\ GJ C_2 \omega \sqrt{\frac{I}{GJ}} \cos \left(\omega \sqrt{\frac{I}{GJ}} L \right) &= T_0, \end{aligned}$$

or

$$C_2 = \frac{T_0}{\omega \sqrt{GJI}} \sec \left(\omega \sqrt{\frac{I}{GJ}} L \right).$$

Thus, the steady-state amplitude at the free end of the shaft is

$$\Theta(L) = \frac{T_0}{\omega \sqrt{GJI}} \tan \left(\omega \sqrt{\frac{I}{GJ}} L \right),$$

and the particular response is given by

$$\begin{aligned} \theta(x, t) &= \frac{T_0}{\omega \sqrt{GJI}} \sec \left(\omega \sqrt{\frac{I}{GJ}} L \right) \sin \left(\omega \sqrt{\frac{I}{GJ}} x \right) \sin \omega t. \end{aligned} \quad (7.51)$$

The solution will grow without bound when

$$\cos \left(\omega \sqrt{\frac{I}{GJ}} L \right) = 0,$$

that is, when the forcing frequency is such that

$$\omega \sqrt{\frac{I}{GJ}} L = n \frac{\pi}{2}, \quad n = 1, 3, \dots$$

The natural frequencies of the fixed-free torsional beam are

$$\omega_n = \frac{n\pi}{2} \sqrt{\frac{GJ}{IL^2}}, \quad n = 1, 3, \dots$$

Equation 7.51 is valid only when the loading frequency $\omega \neq \omega_n$. When $\omega = \omega_n$, the response will grow linearly in

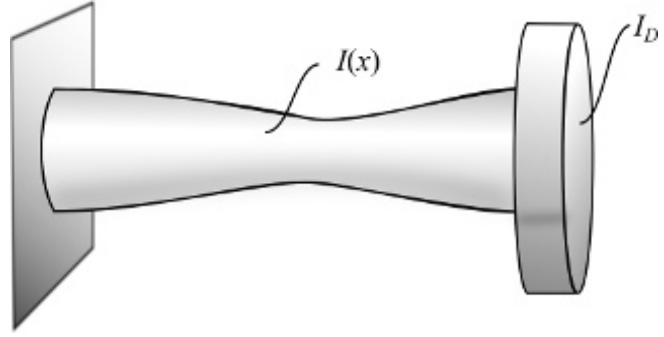


Figure 7.18: Shaft with rigid disk attached at one end.

time, as we have shown in the case of resonance of single degree-of-freedom systems.

Equation 7.51 needs to be added to the homogeneous solution to obtain the complete solution.

In Section 7.6.5 we consider the general problem of harmonic boundary conditions. ■

7.4.1 Torsion of Shaft with Rigid Disk at One End

Consider the torsional oscillations of a shaft with a rigid disk attached at the free end, as shown in Figure 7.18. This model can be used to represent the torsional oscillations of power transmission shafts in machinery.

The previous analysis is still valid except for the boundary condition at the rigid disk, which acts as an applied moment. From Equation 7.48, for $x = L$,

$$M_T(L, t) = GJ(L) \frac{\partial \theta(L, t)}{\partial x}.$$

Using the Newton-Euler equation, the moment due to the end mass is also given by

$$M_T(L, t) = -I_D(L) \frac{\partial^2 \theta(L, t)}{\partial t^2},$$

where I_D is the mass moment of inertia of the disk. Equating these two expressions results in the boundary condition

$$GJ(L) \frac{\partial \theta(L, t)}{\partial x} = -I_D(L) \frac{\partial^2 \theta(L, t)}{\partial t^2}, \quad (7.52)$$

where the negative sign is due to the fact that the shaft undergoing positive $\theta'(L, t)$ at its right end exerts a negative moment on the connecting face of the disk.

The solution of the eigenvalue problem is via separation of variables,

$$\theta(x, t) = \Theta(x) F(t),$$

where $F(t)$ is known to be harmonic. Following our previous procedures, the eigenvalue problem becomes

$$\frac{d}{dx} [GJ(x) \Theta'(x)] = -\omega^2 I(x) \Theta(x), \quad (7.53)$$

with boundary conditions $\Theta(0) = 0$ and, from Equation 7.52,

$$GJ(L)\Theta'(L) = \omega^2 I_D(L)\Theta(L). \quad (7.54)$$

The second boundary condition is a function of frequency ω due to the mass at the right boundary and is known as a *frequency-dependent boundary condition*.

To solve Equation 7.53, we assume the shaft to be uniform, $J(x) = J$, $I(x) = I$, and define $\beta^2 = \omega^2 I/GJ$. Making these substitutions results in the simplified eigenvalue problem

$$\frac{d^2\Theta}{dx^2} + \beta^2\Theta = 0,$$

with the solution

$$\Theta(x) = C_1 \sin \beta x + C_2 \cos \beta x.$$

Applying the two boundary conditions results in the two equations needed to evaluate C_1 and C_2 . The fixed left boundary condition leads to $\Theta(0) = 0 = C_2$. Then substituting $\Theta'(L)$ and $\Theta(L)$ into Equation 7.54 yields

$$GJC_1\beta \cos \beta L = \omega^2 I_D C_1 \sin \beta L. \quad (7.55)$$

Substituting $\omega^2 = \beta^2 GJ/I$ in Equation 7.55 and simplifying give the relation

$$\tan \beta L = \frac{I}{I_D \beta}.$$

This equation can be rewritten in a transcendental form,

$$\tan \beta L = \frac{IL}{I_D} \frac{1}{\beta L}, \quad (7.56)$$

that can be solved numerically for βL . There are an infinite number of solutions for β , so the subscript r is added to represent these. The natural frequencies are then

$$\omega_r^2 = \frac{\beta_r^2 GJ}{I}, \quad r = 1, 2, \dots$$

We note that ω_r are not integer multiples of the fundamental frequency.

If in Equation 7.56 we assume $IL/I_D = 1$, then $\beta L \tan \beta L = 1$. From a plot of $\beta L \tan \beta L - 1$ vs. βL , shown in Figure 7.19, we find the following first three roots: $\beta_r L = 0.869, 3.426$, and 6.437 .

Next, we study the transversely vibrating beam. This is a new class of problems that is governed by a fourth-order partial differential equation. However, the solution technique is similar to that used for the earlier problems.

7.5 Transverse Vibration of Beams

The modeling of a transversely vibrating beam needs to include the effects of shear distortion, rotary inertia, and

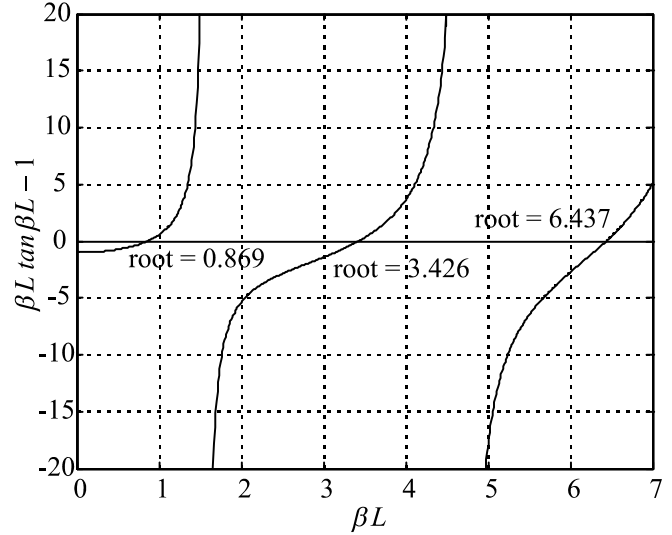


Figure 7.19: Graph of $\beta L \tan \beta L - 1$ vs. βL for the first three roots.

bending moment. Timoshenko²⁸ was the first to include *shear distortion effects* in the modeling of beams in bending, and therefore his name is attached to such models. The formulation and basic solutions along with simple boundary conditions are studied in this section. In Section 7.6, special loading and boundary conditions are explored and possible applications discussed.²⁹

When rotary inertia and shear effects are ignored, the beam is called a *Bernoulli-Euler* beam. It is a special case of the more general Timoshenko beam.



Figure 7.20: An airplane wing can be viewed as a tapered cantilever beam that is subject to aerodynamic and propulsive loading causing transverse vibration. (Photograph by M. Nagurka)

²⁸A biography of Timoshenko appears later in this chapter.

²⁹An extensive discussion of various beam theories can be found in S.M. Han, H. Benaroya, and T. Wei, "Dynamics of Transversely Vibrating Beams Using Four Engineering Theories," *J. Sound and Vibration*, Vol.225, No.5, 1999, pp.935-988.



Figure 7.21: An extended Hummer would experience multiple modes of vibration. It can be modeled as a simply supported beam. (Milwaukee, WI; photograph by M. Nagurka)

7.5.1 Timoshenko Beam

The equation governing the transverse vibration of a beam of length L is derived here. The following properties are assumed at section x : $m(x)$ is the mass per unit length, $A(x)$ is the cross-sectional area, and $I(x)$ is the area moment of inertia. We assume small deflection $y(x, t)$ and slope $\partial y / \partial x$, and include the effects of bending $M(x, t)$ and shear $Q(x, t)$.

We will derive two coupled second-order differential equations, one for the transverse deflection $y(x, t)$ and the other for the rotation $\psi(x, t)$. Under simplifying assumptions, the two equations can be combined into one fourth-order differential equation for either $y(x, t)$ or $\psi(x, t)$.

We consider a free-body of a section of length dx , as shown in Figure 7.22. Its slope $\partial y / \partial x$ has a component $\psi(x, t)$ due to bending and a component $\alpha(x, t)$ due to shear distortion,

$$\frac{\partial y}{\partial x} = \psi(x, t) + \alpha(x, t).$$

From elementary beam theory, the bending moment³⁰ is related to the derivative of the resulting slope by

$$M(x, t) = EI(x) \frac{\partial \psi(x, t)}{\partial x},$$

and the shear is related to the resulting slope by

$$Q(x, t) = s \alpha A(x) G,$$

where s is a number³¹ that depends on the shape of the

³⁰The exact relation is

$$\frac{M}{EI} = \frac{y''}{[1 + (y')^2]^{3/2}}.$$

For small displacements and slopes, y' is very small when compared to 1 and can be ignored.

³¹See **Vibration Problems in Engineering**, W. Weaver, S. Timoshenko, and D. Young, 5th ed., McGraw-Hill, 1990, pp.434-436 for additional values of s . This simplified relation between Q and α is introduced to make the equations analytically tractable, while accounting for the fact that shear is parabolically distributed over a cross section. Therefore, the equations of this section are valid for thin beams, where the ratio of length to depth is at least 10:1.

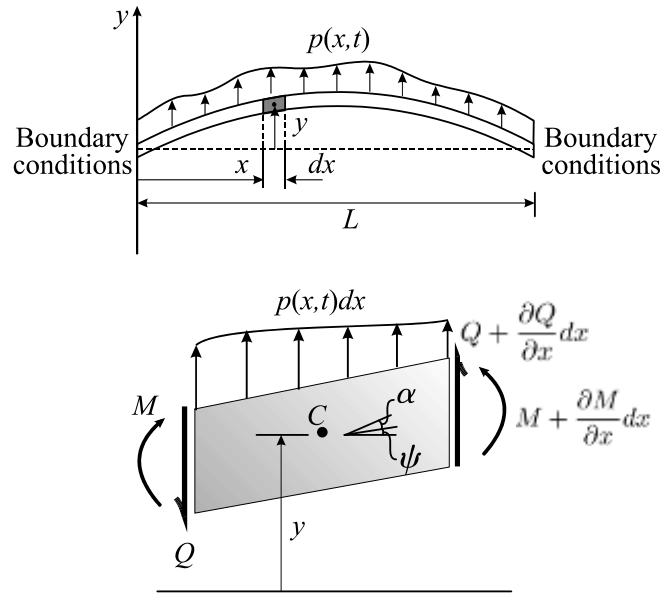


Figure 7.22: Transverse deflection of a beam (top) and the free-body diagram of a differential element of length dx (bottom). The beam is loaded by the continuous force $p(x, t)$ and has arbitrary boundary conditions. The free-body diagram of the element includes a segment of the external force as well as the internal shear Q and bending moment M .

cross section, and G is the constant shear modulus of the material. $Q(x, t)$ may then be rewritten as

$$Q(x, t) = s \left(\frac{\partial y(x, t)}{\partial x} - \psi(x, t) \right) A(x) G. \quad (7.57)$$

The product $s A(x)$ is called the *reduced section* and is computed from classical beam theory. For example, $s = 5/6$ for a plane rectangular cross section and $s = 1/1.175$ for a plane circular cross section.

Hamilton's variational principle, $\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0$, is used to derive the equations of motion and boundary conditions. For this we need to derive the kinetic and strain energies, and their variations. The kinetic energy due to translation and rotation of the whole beam is given by

$$T(t) = \frac{1}{2} \int_0^L m(x) \left(\frac{\partial y}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^L J(x) \left(\frac{\partial \psi}{\partial t} \right)^2 dx,$$

where

$$J(x) = \rho(x) I(x) = \frac{m(x)}{A(x)} I(x) = k^2(x) m(x), \quad (7.58)$$

where $J(x)$ is the mass moment of inertia per unit length, and $k(x)$ is the radius of gyration, both with respect to

the neutral bending axis. In the second integral above for the rotary inertia, *only the effect of bending is included since the shear results only in a distortion, not a rotation*. Using the chain rule, the variation of the kinetic energy δT is given by

$$\begin{aligned}\delta T(t) &= \int_0^L m(x) \frac{\partial y}{\partial t} \delta \left(\frac{\partial y}{\partial t} \right) dx \\ &+ \int_0^L k^2(x) m(x) \frac{\partial \psi}{\partial t} \delta \left(\frac{\partial \psi}{\partial t} \right) dx.\end{aligned}\quad (7.59)$$

To calculate the virtual work δW done on the beam we partition the work into a *conservative* part that is equal to the change in strain energy and a *nonconservative* part that includes the work done by external forces $p(x, t)$,

$$\begin{aligned}\delta W(t) &= \delta W_c(t) + \delta W_{nc}(t) \\ &= -\delta V(t) + \int_0^L p(x, t) \delta y(x, t) dx.\end{aligned}\quad (7.60)$$

The change in strain energy equals the work done by the conservative actions due to the moment and the shear deformation,

$$\begin{aligned}V(t) &= \frac{1}{2} \int_0^L M(x, t) \frac{\partial \psi}{\partial x} dx + \frac{1}{2} \int_0^L Q(x, t) \alpha dx \\ &= \frac{1}{2} \int_0^L EI(x) \left(\frac{\partial \psi}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^L s \alpha^2 A(x) G dx.\end{aligned}$$

The variation $\delta V(t)$ is given by

$$\begin{aligned}\delta V(t) &= \int_0^L EI(x) \frac{\partial \psi}{\partial x} \delta \left(\frac{\partial \psi}{\partial x} \right) dx \\ &+ \int_0^L s A(x) G \alpha \delta \alpha dx,\end{aligned}\quad (7.61)$$

where the number of variables is reduced by introducing $\delta \alpha$, given by the relation

$$\delta \alpha = \delta \left(\frac{\partial y}{\partial x} - \psi \right).$$

Substituting [Equations 7.59, 7.60, and 7.61](#) into Hamilton's principle, we find

$$\begin{aligned}\int_{t_1}^{t_2} \left\{ \left[\int_0^L m(x) \frac{\partial y}{\partial t} \delta \left(\frac{\partial y}{\partial t} \right) dx \right. \right. \\ \left. \left. + \int_0^L k^2(x) m(x) \frac{\partial \psi}{\partial t} \delta \left(\frac{\partial \psi}{\partial t} \right) dx \right] \right. \\ \left. - \left[\int_0^L EI(x) \frac{\partial \psi}{\partial x} \delta \left(\frac{\partial \psi}{\partial x} \right) dx \right. \right. \\ \left. \left. + \int_0^L s A(x) G \left(\frac{\partial y}{\partial x} - \psi \right) \delta \left(\frac{\partial y}{\partial x} - \psi \right) dx \right] \right. \\ \left. + \int_0^L p(x, t) \delta y(x, t) dx \right\} dt = 0,\end{aligned}$$

where the variation δ operates on the function that immediately follows it.

Performing the usual interchanges and integration by parts and combining terms we find

$$\begin{aligned}\int_{t_1}^{t_2} \left[\int_0^L \left\{ \frac{\partial}{\partial x} \left[sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \right] \right. \right. \\ \left. \left. - m(x) \frac{\partial^2 y}{\partial t^2} + p(x, t) \right\} \delta y dx \right. \\ \left. + \int_0^L \left\{ \frac{\partial}{\partial x} \left(EI(x) \frac{\partial \psi}{\partial x} \right) + sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \right. \right. \\ \left. \left. - k^2(x) m(x) \frac{\partial^2 \psi}{\partial t^2} \right\} \delta \psi dx \right. \\ \left. - \left[\left\{ EI(x) \frac{\partial \psi}{\partial x} \right\} \delta \psi \right]_0^L \right. \\ \left. - \left[\left\{ sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \right\} \delta y \right]_0^L \right] dt = 0.\end{aligned}\quad (7.62)$$

To proceed we note that $\delta \psi$ and δy are arbitrary and independent for $0 < x < L$. It then follows that the governing equations of motion for the vibration of a Timoshenko beam are

$$\begin{aligned}\frac{\partial}{\partial x} \left[sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \right] \\ - m(x) \frac{\partial^2 y}{\partial t^2} + p(x, t) = 0\end{aligned}\quad (7.63)$$

$$\begin{aligned}\frac{\partial}{\partial x} \left(EI(x) \frac{\partial \psi}{\partial x} \right) + sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \\ - k^2(x) m(x) \frac{\partial^2 \psi}{\partial t^2} = 0,\end{aligned}\quad (7.64)$$

with the possible boundary conditions defined by

$$\left[\left\{ EI(x) \frac{\partial \psi}{\partial x} \right\} \delta \psi \right]_0^L = 0 \quad (7.65)$$

$$\left[\left\{ sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \right\} \delta y \right]_0^L = 0. \quad (7.66)$$

These boundary conditions are interpreted to mean that *either* the term in the braces *or* the variation parameter equals zero at $x = 0$ and $x = L$.

The term in the braces in [Equation 7.65](#) is a moment and the term $\delta \psi$ is a rotation. Therefore, the relevant boundary condition is that *either* the moment *or* the rotation equals zero at an end. Similarly, in [Equation 7.66](#), the term in the braces is the shear force and δy is a deflection. Then, *either* the shear force *or* the deflection equals zero at a boundary. The boundary conditions are applied in the following subsections.

Free Vibration of Timoshenko Beam The free vibration problem can be solved³² beginning with the assumed harmonic responses,

$$y(x, t) = Y(x) \exp(i\omega t) \quad \text{and} \quad \psi(x, t) = \Psi(x) \exp(i\omega t).$$

These equations are substituted into [Equations 7.63](#) and [7.64](#) which can then be solved for $y(x, t)$ and $\psi(x, t)$. We address next the simpler problem of the uniform beam.

Uniform Timoshenko Beam Consider the uniform beam where $A(x) = A$, $m(x) = m$, $k(x) = k$, and $I(x) = I$. [Equations 7.63](#) and [7.64](#) for the uniform beam can be simplified,

$$sGA \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} - \psi \right) - m \frac{\partial^2 y}{\partial t^2} + p(x, t) = 0 \quad (7.67)$$

$$EI \frac{\partial^2 \psi}{\partial x^2} + sGA \left(\frac{\partial y}{\partial x} - \psi \right) - k^2 m \frac{\partial^2 \psi}{\partial t^2} = 0. \quad (7.68)$$

We choose to eliminate ψ by combining these two equations leading to an equation governing $y(x, t)$. We first differentiate [Equation 7.68](#) with respect to x ,

$$EI \frac{\partial^3 \psi}{\partial x^3} + sGA \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) - k^2 m \frac{\partial^3 \psi}{\partial x \partial t^2} = 0. \quad (7.69)$$

From [Equation 7.67](#), we solve for $\partial \psi / \partial x$ as well as the other derivatives required in [Equation 7.69](#),

$$\frac{\partial \psi}{\partial x} = \frac{\partial^2 y}{\partial x^2} - \frac{m}{sGA} \frac{\partial^2 y}{\partial t^2} + \frac{p(x, t)}{sGA} \quad (7.70)$$

$$\frac{\partial^3 \psi}{\partial x^3} = \frac{\partial^4 y}{\partial x^4} - \frac{m}{sGA} \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{1}{sGA} \frac{\partial^2 p}{\partial x^2} \quad (7.71)$$

$$\frac{\partial^3 \psi}{\partial x \partial t^2} = \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{m}{sGA} \frac{\partial^4 y}{\partial t^4} + \frac{1}{sGA} \frac{\partial^2 p}{\partial t^2}. \quad (7.72)$$

Substituting [Equations 7.70](#), [7.71](#), and [7.72](#) into [7.69](#), and after a bit of algebra, the governing equation is found,

$$\begin{aligned} & EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} \\ & - \left(k^2 m + \frac{EI m}{sGA} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{(km)^2}{sGA} \frac{\partial^4 y}{\partial t^4} \\ & = p(x, t) + \frac{k^2 m}{sGA} \frac{\partial^2 p}{\partial t^2} - \frac{EI}{sGA} \frac{\partial^2 p}{\partial x^2}, \end{aligned} \quad (7.73)$$

where the external loading $p(x, t)$ has been retained, even though for the eigenvalue problem it is set to zero.

Each term in [Equation 7.73](#) represents a different physical aspect of beam behavior. If rotary effects can be neglected, $k = 0$ in [Equation 7.73](#). However, if the shear effect can be neglected, $s = 0$ in [Equations 7.67](#) and [7.68](#), and $Q = 0$ in [Equation 7.57](#).

The *Bernoulli-Euler* beam equation that is used for many simplified studies assumes $k = 0$ and $s = 0$,

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = p(x, t). \quad (7.74)$$

7.5.2 Boundary Conditions

We summarize the possible boundary conditions that are derivable from [Equations 7.65](#) and [7.66](#). These are used in subsequent problems.

Fixed End A fixed end does not allow displacement or rotation for all time. At $x = 0$,

$$y(0, t) = 0$$

$$\psi(0, t) = 0.$$

Hinged End A hinged end does not allow displacement, but has no moment resistance and therefore permits rotation. Thus, at $x = 0$,

$$y(0, t) = 0$$

$$M(0, t) = \left[EI(x) \frac{\partial \psi}{\partial x} \right]_{x=0} = 0.$$

Free End A free end has no moment or shear resistance. Thus, at $x = 0$,

$$M(0, t) = \left[EI(x) \frac{\partial \psi}{\partial x} \right]_{x=0} = 0$$

$$Q(0, t) = \left[sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \right]_{x=0} = 0.$$

These are the basic boundary conditions. In practical applications, we might choose to mix these properties. We consider hybrid boundary conditions in [Section 7.6](#). Finally, the boundary conditions above can be written in the same form for a boundary at $x = L$.

7.5.3 Bernoulli-Euler Beam

In order to be able to analytically tackle the eigenvalue problem, some reasonable simplifying assumptions are needed. When the cross-sectional dimensions are much smaller than the length,³³ the shear distortion effect and the rotary inertia effect can be neglected.³⁴ Also, for the eigenvalue problem, external forces $p(x, t)$ are set equal to zero.

³³The rule of thumb is 1:10. The approximation becomes better for smaller ratios and less accurate for larger ratios.

³⁴The radius of gyration, $k(x)$, which is related to $J(x)$, is approximately equal to zero in [Equation 7.58](#) and in subsequent equations.

³²The solution is provided in **Mechanical Vibrations: Theory and Applications to Structural Dynamics**, M. G rardin and D. Rixen, Wiley, 1994, p.197.

Under the simplifying assumptions, Equations 7.63 and 7.64 become

$$\frac{\partial}{\partial x} \left[sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) \right] - m(x) \frac{\partial^2 y}{\partial t^2} = 0 \quad (7.75)$$

$$\frac{\partial}{\partial x} \left(EI(x) \frac{\partial \psi}{\partial x} \right) + sGA(x) \left(\frac{\partial y}{\partial x} - \psi \right) = 0. \quad (7.76)$$

We solve Equation 7.76 for $sGA(x)(\partial y/\partial x - \psi)$, and substitute the result into Equation 7.75. Due to the assumption of no shear distortion, $\partial y/\partial x = \psi(x, t) + \alpha(x, t)$ can be written as $\partial \psi/\partial x = \partial^2 y/\partial x^2$ since $\alpha = 0$.

The resulting governing equation for $y(x, t)$ is the Bernoulli-Euler beam with variable properties,

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y}{\partial x^2} \right] = -m(x) \frac{\partial^2 y}{\partial t^2}. \quad (7.77)$$

We assume the product solution,

$$y(x, t) = Y(x)F(t), \quad (7.78)$$

which can be differentiated and substituted into the governing equation to obtain

$$\frac{d^2}{dx^2} [EI(x)Y''(x)F(t)] = -m(x)Y(x)\ddot{F}.$$

Separation of variables results in

$$\frac{1}{m(x)Y(x)} \frac{d^2}{dx^2} [EI(x)Y''(x)] = -\frac{\ddot{F}}{F}. \quad (7.79)$$

Using the same argument as for the vibrating string, we equate each side of Equation 7.79 to the constant ω^2 so that the solution is harmonic in time,

$$\ddot{F} + \omega^2 F = 0, \quad (7.80)$$

and the eigenfunction $Y(x)$ is governed by

$$\frac{d^2}{dx^2} [EI(x)Y''(x)] = \omega^2 m(x)Y(x). \quad (7.81)$$

This equation, known as the eigenvalue/eigenfunction problem, is solved for a particular set of four boundary conditions to determine the eigenvalues ω_r and the eigenfunctions $Y_r(x)$, as shown below. Equation 7.80 requires two initial conditions and Equation 7.81 requires four boundary conditions for a complete solution.

The solution to Equation 7.80 is harmonic,

$$F(t) = A \cos(\omega t - \phi), \quad (7.82)$$

where A and ϕ are found by satisfying the initial conditions once the complete solution is formulated.

Possible idealized boundary conditions are described below.

Fixed End At a fixed or clamped end, deflection and rotation cannot occur and the boundary conditions are

$$\begin{aligned} y(0, t) &= Y(0)F(t) = 0 \\ y(L, t) &= Y(L)F(t) = 0 \\ y'(0, t) &= Y'(0)F(t) = 0 \\ y'(L, t) &= Y'(L)F(t) = 0. \end{aligned}$$

Since $F(t) \neq 0$, it must be that $Y(0) = Y(L) = 0$ and $Y'(0) = Y'(L) = 0$ for all t .

Hinged End A hinged end cannot deflect and cannot resist a rotation. The zero deflection equations have been provided already. For zero moment resistance, the moment equation, $M = EI(x)\partial^2 y/\partial x^2 = EI(x)Y''(x)F(t)$, yields the following conditions,

$$\begin{aligned} EI(0)Y''(0)F(t) &= 0 \\ EI(L)Y''(L)F(t) &= 0. \end{aligned}$$

Since $F(t) \neq 0$, and EI can be divided out of the expressions, it must be that $Y''(0) = Y''(L) = 0$ for all t .

Free End A free end has no resistance to a moment or shear force. The moment conditions have been provided above. For zero shear resistance, the shear equation,

$$Q = -\frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 y}{\partial x^2} \right] = -\frac{d}{dx} [EI(x)Y''(x)]F(t), \quad (7.83)$$

yields the following conditions,

$$\begin{aligned} \frac{d}{dx} [EI(0)Y''(0)] F(t) &= 0 \\ \frac{d}{dx} [EI(L)Y''(L)] F(t) &= 0. \end{aligned}$$

Since $F(t) \neq 0$, and EI can be divided out of the expressions, it must be that

$$\begin{aligned} Y'''(0) &= 0 \\ Y'''(L) &= 0 \end{aligned}$$

for all t .

Since *no boundary is exactly hinged or completely fixed* and is somewhere in between these extremes, these are idealizations.

Example 7.4 A Beam That is Hinged (Simply Supported) at Both Ends

For a beam with hinged-hinged boundary conditions, solve for the response. Assume that the beam is uniform.

Solution: For the hinged beam, the boundary conditions are $Y(0) = Y(L) = 0$ and $Y''(0) = Y''(L) = 0$. Since the beam is uniform

$$I(x) = I, \quad m(x) = m,$$

and therefore Equation 7.81 becomes

$$EI \frac{d^4}{dx^4}[Y(x)] = \omega^2 m Y(x). \quad (7.84)$$

Defining $\beta^4 = \omega^2 m / EI$, the eigenfunction $Y(x)$ is governed by

$$Y''''(x) - \beta^4 Y(x) = 0, \quad (7.85)$$

with the solution

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x. \quad (7.86)$$

Applying the two $x = 0$ boundary conditions to Equation 7.86, we find

$$\begin{aligned} Y(0) = 0 &= C_2 + C_4 \\ Y''(0) = 0 &= -C_2 + C_4, \end{aligned}$$

requiring that $C_2 = C_4 = 0$. Physically, this result makes sense since the cosine and the hyperbolic cosine cannot equal zero at $x = 0$, as required by the hinged condition.

Using Equation 7.86 again for the two $x = L$ boundary conditions leads to

$$Y(L) = 0 = C_1 \sin \beta L + C_3 \sinh \beta L \quad (7.87)$$

$$Y''(L) = 0 = -C_1 \beta^2 \sin \beta L + C_3 \beta^2 \sinh \beta L, \quad (7.88)$$

where the factor β^2 cancels out in the second equation. Adding and subtracting Equations 7.87 and 7.88 yields the relations

$$\begin{aligned} 2C_3 \sinh \beta L &= 0 \\ 2C_1 \sin \beta L &= 0. \end{aligned}$$

There are several possible solutions to consider here. The first is that $C_1 = C_3 = 0$, but this is the trivial solution. The second possibility is that $C_1 = 0$, and $C_3 \neq 0$ with $\sinh \beta L = 0$, which is true only for $\beta = 0$, and again trivial. We are then left with the case $C_3 = 0$, and $C_1 \neq 0$ with $\sin \beta L = 0$. For this last equality to hold, it must be that

$$\beta_r L = r\pi, \quad r = 1, 2, \dots$$

Substituting the expression for β into this equation, we find

$$\begin{aligned} \omega_r^2 &= \beta_r^4 \frac{EI}{m} \\ &= \left(\frac{r\pi}{L}\right)^4 \frac{EI}{m} \end{aligned}$$

giving the eigenvalues ω_r ,

$$\omega_r = \left(\frac{r\pi}{L}\right)^2 \sqrt{\frac{EI}{m}}, \quad r = 1, 2, \dots \quad (7.89)$$

Equation 7.89 specifies the frequencies that can be accommodated in free vibration for the given boundary conditions.³⁵ The eigenfunctions become

$$Y_r(x) = C_1 \sin \left(\frac{r\pi x}{L}\right). \quad (7.90)$$

To fix the value of C_1 , Equation 7.90 is generally normalized according to the rule

$$\begin{aligned} \int_0^L m Y_r^2(x) dx &= 1 \\ \int_0^L C_1^2 m \sin^2 \left(\frac{r\pi x}{L}\right) dx &= 1 \\ \Rightarrow C_1 &= \sqrt{\frac{2}{mL}}. \end{aligned}$$

With C_1 independent of index r , each mode has the same maximum amplitude. The normalized mode shapes are then

$$\hat{Y}_r(x) = \sqrt{\frac{2}{mL}} \sin \left(\frac{r\pi x}{L}\right), \quad r = 1, 2, \dots \quad (7.91)$$

These are plotted in Figure 7.23 for $m = 0.1$ slug/in and $L = 120$ in. The notation $\hat{Y}_r(x)$ signifies that the modal function has been normalized.

The response for each mode, with index r , is obtained by forming the product $F_r(t)\hat{Y}_r(x)$, as per Equation 7.78,

$$y_r(x, t) = A_r \sqrt{\frac{2}{mL}} \cos(\omega_r t - \phi_r) \sin \left(\frac{r\pi x}{L}\right),$$

for $r = 1, 2, \dots$. Summing over r , we find the complete response to be

$$y(x, t) = \sum_{r=1}^{\infty} \left[A_r \sqrt{\frac{2}{mL}} \cos(\omega_r t - \phi_r) \sin \left(\frac{r\pi x}{L}\right) \right], \quad (7.92)$$

where A_r and ϕ_r are determined by satisfying the initial conditions. In practice, one retains as many terms as necessary in Equation 7.92 to attain the desired accuracy.

We complete this problem by considering the assumed initial displacement $y(x, 0) = Cx(1 - x/L)$ and zero initial velocity $\dot{y}(x, 0) = 0$. These initial conditions can be satisfied by solving for A_r and ϕ_r , for which we need the orthogonality properties of the modes, a topic to be discussed in Section 7.5.4. This problem will be revisited in Example 7.7, where vibration-induced stresses are also discussed. In the following example

³⁵From Equation 7.89 we note that $\omega_r = r^2 \omega_1$.

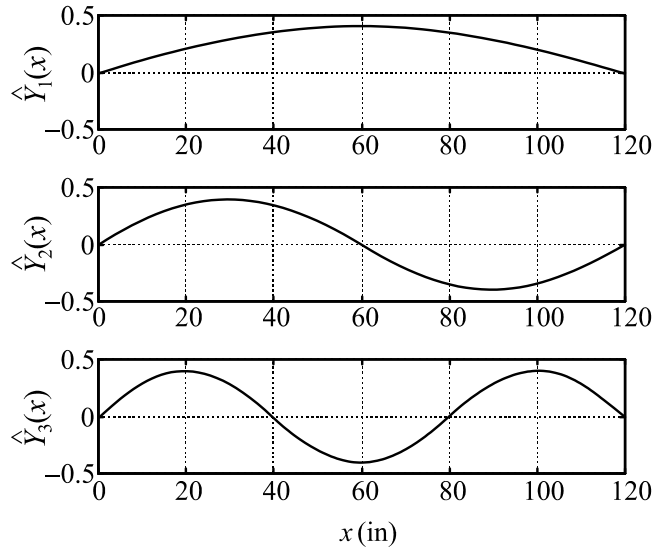


Figure 7.23: First three normalized mode shapes for a hinged-hinged beam for $m = 0.1$ slug/in and $L = 120$ in. The peak amplitude for each mode is $\sqrt{2/mL} = \sqrt{2/12} = 0.41$ in.

alternate boundary conditions are considered. ■

Example 7.5 Solution for the Fixed-Free Case

For a beam with fixed-free boundary conditions, solve for the response starting with the eigenfunction.

Solution: We begin with Equation 7.86,

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x.$$

For a beam fixed at $x = 0$ and free at $x = L$, the boundary conditions for the eigenfunctions are

$$Y(0) = 0, \quad Y'(0) = 0, \\ Y''(L) = 0, \quad Y'''(L) = 0.$$

At the fixed end, the displacement and slope equal zero, and at the free end, the moment and shear force equal zero. Applying the boundary conditions for $x = 0$, we find

$$C_2 + C_4 = 0 \\ C_1 + C_3 = 0,$$

so that the eigenfunction is reduced to

$$Y(x) = C_1(\sin \beta x - \sinh \beta x) + C_2(\cos \beta x - \cosh \beta x). \quad (7.93)$$

Applying the two conditions at $x = L$ results in

$$C_1(\sin \beta L + \sinh \beta L) + C_2(\cos \beta L + \cosh \beta L) = 0 \quad (7.94)$$

$$C_1(\cos \beta L + \cosh \beta L) - C_2(\sin \beta L - \sinh \beta L) = 0. \quad (7.95)$$

Solving Equation 7.95 for C_2 in terms of C_1 and substituting the result into Equation 7.93 yields

$$Y(x) = \frac{C_1}{\sin \beta L - \sinh \beta L} \times [(\sin \beta L - \sinh \beta L)(\sin \beta x - \sinh \beta x) + (\cos \beta L + \cosh \beta L)(\cos \beta x - \cosh \beta x)], \quad (7.96)$$

where C_1 is eventually incorporated into the complete solution.³⁶

To establish the frequencies that the beam can sustain in free vibration, we rewrite Equations 7.94 and 7.95 in matrix-vector form,

$$\begin{bmatrix} (\sin \beta L + \sinh \beta L) & (\cos \beta L + \cosh \beta L) \\ (\cos \beta L + \cosh \beta L) & -(\sin \beta L - \sinh \beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The determinant of the matrix must be equal to zero for a nontrivial solution to exist, that is,

$$(\sin \beta L + \sinh \beta L)(\sin \beta L - \sinh \beta L) + (\cos \beta L + \cosh \beta L)^2 = 0.$$

This is the characteristic equation and it simplifies to the transcendental equation

$$\cos \beta L \cosh \beta L = -1,$$

which has no analytic solution and must be solved numerically for βL . We find the first three roots to be $\beta L = 1.875$, 4.694 , and 7.855 and, using Equation 7.89, the natural frequencies are

$$\omega_1 = (1.875)^2 \sqrt{\frac{EI}{mL^4}} \\ \omega_2 = (4.694)^2 \sqrt{\frac{EI}{mL^4}} \\ \omega_3 = (7.855)^2 \sqrt{\frac{EI}{mL^4}}.$$

The first three modes of Equation 7.96 are plotted in Figure 7.24 for $L = 120$ in, assuming $C_1 = 1$. For example, using Equation 7.96, for $\beta L = 1.875$, the first mode

³⁶We could have also used Equation 7.94 to solve for C_2 in terms of C_1 .

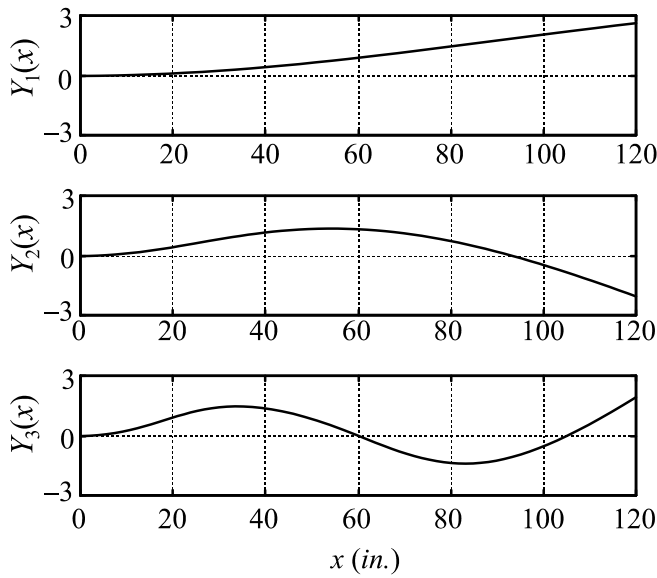


Figure 7.24: First three eigenfunctions for a clamped-free beam with $C_1 = 1$ and $L = 120$ in.

is given by

$$Y_1(x) = \frac{1}{-2.230}[-2.230(\sin \beta x - \sinh \beta x) + 3.037(\cos \beta x - \cosh \beta x)],$$

where $\beta = 1.875/120 = 0.0156 \text{ in}^{-1}$. Based on Equation 7.78, each modal response is given by

$$y_r(x, t) = B_r Y_r(x) F_r(t) = \bar{C}_r Y_r(x) \cos(\omega_r t - \phi_r), \quad (7.97)$$

where $\bar{C}_r = AB_r$. The complete solution is then the sum

$$y(x, t) = \sum_{r=1}^{\infty} y_r(x, t),$$

where $\bar{C}_r$ and ϕ_r are determined by satisfying the initial conditions. Note that having assumed $C_1 = 1$ there was no need to normalize the mode. ■

In the next section, the orthogonality properties of the modes are explored further.

7.5.4 Orthogonality of the Modes

The orthogonality of the modes is used to solve the governing equations of motion using the normal mode or modal analysis method.

Each class of problems has its own set of orthogonality relations that reflect the governing equation of motion. The orthogonality of the eigenfunctions is demonstrated for the bending beam. The eigenvalue problem is defined by Equation 7.81, and each eigenfunction will satisfy this

relation. We denote two distinct solutions by $Y_r(x)$ and $Y_s(x)$, so that

$$\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 Y_r}{dx^2} \right] = \omega_r^2 m(x) Y_r(x) \quad (7.98)$$

$$\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 Y_s}{dx^2} \right] = \omega_s^2 m(x) Y_s(x), \quad (7.99)$$

where $\omega_r \neq \omega_s$. We multiply Equation 7.98 by $Y_s(x)$ and integrate both sides over the domain $0 \leq x \leq L$. The left-hand side is integrated twice by parts to find

$$\begin{aligned} \int_0^L Y_s(x) \frac{d^2}{dx^2} \left[EI(x) \frac{d^2 Y_r}{dx^2} \right] dx &= \left[Y_s(x) \frac{d}{dx} \left[EI(x) \frac{d^2 Y_r}{dx^2} \right] \right]_0^L \\ &\quad - \left[Y_s'(x) EI(x) \frac{d^2 Y_r}{dx^2} \right]_0^L \\ &\quad + \int_0^L EI(x) \frac{d^2 Y_r}{dx^2} \frac{d^2 Y_s}{dx^2}(x) dx \\ &= \omega_r^2 \int_0^L m(x) Y_r(x) Y_s(x) dx. \end{aligned} \quad (7.100)$$

Similarly, we multiply Equation 7.99 by $Y_r(x)$, again integrate the left-hand side twice by parts to obtain an equation of the same form as Equation 7.100, and then subtract this second equation from Equation 7.100, to find

$$\begin{aligned} (\omega_r^2 - \omega_s^2) \int_0^L m(x) Y_r(x) Y_s(x) dx &= \left[Y_s(x) \frac{d}{dx} \left[EI(x) \frac{d^2 Y_r}{dx^2} \right] \right]_0^L \\ &\quad - \left[Y_s'(x) EI(x) \frac{d^2 Y_r}{dx^2} \right]_0^L \\ &\quad - \left[Y_r(x) \frac{d}{dx} \left[EI(x) \frac{d^2 Y_s}{dx^2} \right] \right]_0^L \\ &\quad + [Y_r'(x) EI(x) Y_s''(x)]_0^L. \end{aligned} \quad (7.101)$$

For any combination of fixed, hinged, or free ends, all the terms on the right-hand side vanish, resulting in

$$(\omega_r^2 - \omega_s^2) \int_0^L m(x) Y_r(x) Y_s(x) dx = 0. \quad (7.102)$$

Since this derivation assumes that $\omega_r \neq \omega_s$, the factor $(\omega_r^2 - \omega_s^2)$ can be divided out to find the first orthogonality relation,

$$\int_0^L m(x) Y_r(x) Y_s(x) dx = 0, \quad r \neq s. \quad (7.103)$$

If the modal functions have been normalized, then this relation can be generalized to include the case $r = s$ by writing it as

$$\int_0^L m(x) \hat{Y}_r(x) \hat{Y}_s(x) dx = \delta_{rs}, \quad r, s = 1, 2, \dots, \quad (7.104)$$

where $\hat{Y}_r(x)$ is the normalized mode and δ_{rs} is the *Kronecker delta* defined as

$$\delta_{rs} = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{if } r \neq s. \end{cases}$$

For $r = s$, Equation 7.104 becomes

$$\int_0^L m(x) \hat{Y}_r^2(x) dx = 1,$$

yielding the identity $(\omega_r^2 - \omega_r^2) \cdot 1 = 0$ for $r = s$ in Equation 7.102.

Substituting Equation 7.104 into 7.100 results in the equation

$$\int_0^L \hat{Y}_s(x) \frac{d^2}{dx^2} \left[EI(x) \frac{d^2 \hat{Y}_r}{dx^2} \right] dx = \omega_r^2 \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.105)$$

After integration by parts and using the same procedure and assumptions as before, we find another orthogonality property,

$$\int_0^L EI(x) \frac{d^2 \hat{Y}_r}{dx^2} \frac{d^2 \hat{Y}_s}{dx^2} dx = \omega_r^2 \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.106)$$

Such orthogonality properties are used in modal analysis solutions of continuous vibration problems.

The next example introduces a viscous damping term and an external forcing term in the equation of motion.

Example 7.6 Transverse Vibration Response of a Forced and Damped Uniform Beam

Solve for $y(x, t)$ for a beam with uniform properties with the equation of motion

$$EI \frac{\partial^4 y}{\partial x^4} + c \frac{\partial y}{\partial t} + m \frac{\partial^2 y}{\partial t^2} = p(x, t), \quad (7.107)$$

where the natural frequencies are given by Equation 7.89 and the normal modes by Equation 7.91.

Solution: To solve this problem, we use the normal mode method, which utilizes the modal orthogonality properties. First, we expand the applied force $p(x, t)$ in terms of the normal modes,

$$p(x, t) = \sum_{k=1}^{\infty} p_k(t) \hat{Y}_k(x).$$

Multiplying both sides of this equation by $m \hat{Y}_j(x)$ and then integrating over the beam span, we have

$$\int_0^L m p(x, t) \hat{Y}_j(x) dx = \sum_{k=1}^{\infty} p_k(t) \int_0^L m \hat{Y}_j(x) \hat{Y}_k(x) dx.$$

For the right-hand side of this equation, the orthogonality properties of the modes imply that the integral equals zero for $k \neq j$, and otherwise equals 1 for $k = j$. Therefore,

$$\int_0^L m p(x, t) \hat{Y}_j(x) dx = p_j(t). \quad (7.108)$$

Applying the same procedure to the displacement response,

$$y(x, t) = \sum_{k=1}^{\infty} y_k(t) \hat{Y}_k(x), \quad (7.109)$$

and multiplying and integrating as above, we find

$$\int_0^L m y(x, t) \hat{Y}_j(x) dx = y_j(t).$$

The displacement $y(x, t)$ can be differentiated with respect to x and t so that all the terms in the equation of motion, Equation 7.107, can be put into the modal expansion form,

$$\begin{aligned} \sum_{j=1}^{\infty} \left[EI y_j(t) \frac{d^4 \hat{Y}_j}{dx^4} + c \frac{dy_j}{dt} \hat{Y}_j + m \frac{d^2 y_j}{dt^2} \hat{Y}_j \right] \\ = \sum_{j=1}^{\infty} p_j(t) \hat{Y}_j(x). \end{aligned} \quad (7.110)$$

The modes are defined by, and satisfy, Equation 7.84: $EI d^4 \hat{Y}_j / dx^4 = \omega_j^2 m \hat{Y}_j$. Substituting the first term into Equation 7.110 leads to

$$\sum_{j=1}^{\infty} [\omega_j^2 m y_j + c \dot{y}_j + m \ddot{y}_j - p_j(t)] \hat{Y}_j(x) = 0.$$

Since the eigenfunctions $\hat{Y}_j(x)$ do not equal zero except for unrestrained motion, the expression in the square brackets must vanish for every j ,

$$\ddot{y}_j + \frac{c}{m} \dot{y}_j + \omega_j^2 y_j = \frac{1}{m} p_j(t), \quad j = 1, 2, \dots \quad (7.111)$$

As for single degree-of-freedom systems, we write $c/m = 2\zeta_j \omega_j$, thus defining the damping ratio,

$$\zeta_j = \frac{c}{2m\omega_j}.$$

For every j , Equation 7.111 is the equation of a damped, forced, harmonic oscillator, with the convolution solution

$$y_j(t) = \int_0^t p_j(\tau) g_j(t - \tau) d\tau, \quad (7.112)$$

where $g_j(t)$ is the impulse response function for a damped oscillator. For an underdamped oscillator, the impulse response function is given by

$$g_j(t) = \frac{1}{m \omega_j \sqrt{1 - \zeta_j^2}} e^{-\zeta_j \omega_j t} \sin \omega_j \sqrt{1 - \zeta_j^2} t.$$

Once solved, $y_j(t)$ is substituted into the expansion equation

$$\begin{aligned} y(x, t) &= \sum_{j=1}^{\infty} y_j(t) \hat{Y}_j(x) \\ &= \sum_{j=1}^{\infty} \hat{Y}_j(x) \int_0^t p_j(\tau) g_j(t - \tau) d\tau. \end{aligned}$$

Substituting Equation 7.108 for $p_j(t)$ results in

$$\begin{aligned} y(x, t) &= \sum_{j=1}^{\infty} \hat{Y}_j(x) \\ &\times \int_0^t \left[\int_0^L m p(\xi, \tau) \hat{Y}_j(\xi) d\xi \right] g_j(t - \tau) d\tau. \end{aligned} \quad (7.113)$$

This is the complete steady-state time-domain solution. If the transient response to nonzero initial conditions is included, then it can be added to Equation 7.113. ■

DANIEL BERNOULLI
(January 29, 1700 – July 27, 1782)



Contribution: Daniel Bernoulli was one of the many prominent mathematicians in the Bernoulli family. He is particularly remembered for his applications of mathematics to mechanics, especially fluid mechanics, and for his pioneering work in probability and statistics. Bernoulli's work is still studied throughout the world.

Bernoulli conducted important research in hydrodynamics. The term itself is based on the title of his book *Hydrodynamica*. This work contained the first correct analysis of water flowing from a hole in a container, based on the principle of conservation of energy. Bernoulli also discussed pumps and other machines to raise water and presented the basic laws for the kinetic theory of gases, which he applied to explain Boyle's law. Although not fully detailed, he arrived at the equation of state, credited to Van der Waals a century later.

He worked with Euler on elasticity and the development of the Euler-Bernoulli beam equation, which is widely used in mechanical, civil, and aeronautical engineering. He pointed out the advantage of resolving compound motion into translational and rotational motion. He defined the nodes and frequencies of oscillation of dynamic systems. He showed that the movements of strings of musical instruments are composed of an infinite number of harmonic vibrations all superimposed on the string.

He determined the shape that a string assumes when acted upon by two forces, one of which is vertical to the curve and the other parallel to a given direction. He derived the equations for the entire series of such curves known as the *velaria*, *lintearia*, and the *catenaria*.

An important aspect of Bernoulli's work that proved important in the development of mathematical physics was his acceptance of many of Newton's theories and his use of these together with the more powerful calculus of Leibniz. In his study of mechanics he used the principle of conservation of energy, based on an integral of Newton's second law. He also studied the movement of bodies in a resisting medium using Newton's methods.

Bernoulli also contributed to the fields of probability and political economics. He presumed that the moral value of the increase in a person's wealth is inversely proportional to the amount of that wealth. He then assigned probabilities to the various means that a person has to make money and deduced an expectation of increase in moral expectation. Bernoulli applied some of his deductions to insurance.

Life Sketch: Daniel Bernoulli was born in Groningen, Netherlands into a family of leading mathematicians. He was the son of Johann Bernoulli, the nephew of Jacob Bernoulli, the younger brother of Nicolaus II Bernoulli, and the older brother of Johann II. All three sons would go on to study mathematics.

When Bernoulli was five years old, the family returned to their native city of Basel and his father filled the chair of mathematics left vacant by the death of his uncle Jacob.

Bernoulli wanted to pursue an academic career like his father. He applied for two chairs at Basel. His application for the chair of anatomy and botany was decided by drawing of lots and he was unlucky in this game of chance. The next chair to become vacant was the chair of logic. Bernoulli applied for it but again the game of chance went against him. Having failed to obtain an academic post, Bernoulli went to Venice to study practical medicine.

While in Venice, Bernoulli designed an hourglass specifically to be used at sea. Its trickle of sand was constant even when a ship was rolling in heavy seas. He submitted his idea to the Paris Academy. In 1725 he returned from Italy to Basel to learn that he had won a prize from the Paris Academy for his idea. Through this and his work *Mathematical Exercises* Bernoulli attained fame and was invited to take up a chair of mathematics at St. Petersburg. His brother Nicolaus was also offered a chair of mathematics at St. Petersburg and in late 1725 the two brothers travelled together.

Within eight months of taking up their appointments, Nicolaus died of fever. Greatly saddened by the loss of his brother and unhappy with the harsh climate, Bernoulli wrote to his father telling him of his desire to return to Basel. Instead, Johann Bernoulli arranged for one of his best pupils, Leonard Euler, to work with Daniel in St. Petersburg. Euler arrived in 1727 and this period in St. Petersburg, which ended in 1733, was Bernoulli's most productive time.

Bernoulli wrote many papers on various mechanical questions, especially on problems connected with vibrating strings, and the solutions given by Taylor and by d'Alembert. Bernoulli and Euler became famous for their pioneering work on the mechanics of flexible and elastic bodies.

In 1750, Bernoulli returned to the University of Basel, where he successively held the chairs of medicine, metaphysics, and natural philosophy until his death. He was known for remarkable physics lectures with experiments performed during the lectures.

He died in Basel, Switzerland.

Noteworthy: Bernoulli had a troubled relationship with his father. After both of them entered and tied for first place in a scientific contest at the University of Paris, Johann banned Daniel from his house (presumably unable to bear the "shame" of being compared to his offspring). Johann also tried to take Daniel's book *Hydrodynamica* as his own and rename it *Hydraulica*. Despite Daniel's attempts at reconciliation, his father carried the grudge until his death.

Bernoulli received many honors. He was elected to most of the leading scientific societies of his day, including those in Bologna, St. Petersburg, Berlin, Paris, London, Bern, Turin, Zurich, and Mannheim. Bernoulli was awarded the Grand Prize of the Paris Academy 10 times for topics in astronomy and nautical science. He won in 1740 (jointly with Euler) for work on Newton's theory of the tides; in 1743 and 1746 for essays on magnetism; in 1747 for a method to determine time at sea; in 1751 for an essay on ocean currents; in 1753 for the effects of forces on ships; and in 1757 for proposals to reduce the pitching and tossing of ships in high seas.

Bernoulli conjectured laws that were not verified until many years later. Among these was Coulomb's law in electrostatics.

Bernoulli's 1766 analysis of smallpox morbidity and mortality data demonstrated the value of vaccination. It was one of the earliest attempts to analyze data via statistical methods.

His strengths and weaknesses were summarized as follows: "Bernoulli's active and imaginative mind dealt with the most varied scientific areas. Such wide interests, however, often prevented him from carrying some of his projects to completion. It is especially unfortunate that he could not follow the rapid growth of mathematics that began with the introduction of partial differential equations into mathematical physics. Nevertheless he assured himself a permanent place in the history of science through his work and discoveries in hydrodynamics, his anticipation of the kinetic theory of gases, a novel method for calculating the value of an increase in assets, and the demonstration that the most common movement of a string in a musical instrument is composed of the superposition of an infinite number of harmonic vibrations..." (*Dictionary of Scientific Biography*, H. Straub, Charles Scribner's, 1970, pp.36-46).

Example 7.7 Evaluation of Arbitrary Constants and Vibration-Induced Stresses

Continue [Example 7.4](#) by evaluating the arbitrary constants A_r and ϕ_r of [Equation 7.92](#). Also, connect vibration and stress to the fatigue life of structures.

Solution: We use the following form for the solution,

$$y(x, t) = \sum_{p=1}^{\infty} \sqrt{\frac{2}{mL}} [B_p \cos \omega_p t + D_p \sin \omega_p t] \sin \left(\frac{p\pi x}{L} \right),$$

and evaluate the constants B_p and D_p . The initial conditions are assumed to be

$$y(x, 0) = Cx \left(1 - \frac{x}{L} \right) \\ \dot{y}(x, 0) = 0,$$

where C is dimensionless.

To satisfy the initial displacement,

$$y(x, 0) = Cx \left(1 - \frac{x}{L} \right) = \sum_{p=1}^{\infty} \sqrt{\frac{2}{mL}} B_p \sin \frac{p\pi x}{L}, \quad (7.114)$$

and to satisfy the initial velocity,

$$\dot{y}(x, 0) = 0 = \sum_{p=1}^{\infty} \sqrt{\frac{2}{mL}} D_p \omega_p \sin \frac{p\pi x}{L}.$$

From the velocity condition, $D_p = 0$ for all p .

We multiply [Equation 7.114](#) by $\sin r\pi x/L$, where r is another index, and integrate over the length of the beam,

$$\int_0^L Cx \left(1 - \frac{x}{L} \right) \sin \frac{r\pi x}{L} dx \\ = \sum_{p=1}^{\infty} \sqrt{\frac{2}{mL}} \int_0^L B_p \sin \frac{p\pi x}{L} \sin \frac{r\pi x}{L} dx,$$

in order to take advantage of the orthogonality properties. The integral on the right-hand side equals zero except for $p = r$. Performing the integrations leads to

$$\frac{2CL^2}{(r\pi)^3}(1 - \cos r\pi) = \sqrt{\frac{2}{mL}} B_r \frac{L}{2}.$$

The term $(1 - \cos r\pi)$ equals zero for even r and equals two for odd r . Therefore,

$$B_r = \frac{8C}{(r\pi)^3} \sqrt{\frac{mL^3}{2}}, \quad r = 1, 3, \dots,$$

and the response is given by

$$y(x, t) = \sum_{r=1,3,\dots}^{\infty} \frac{8LC}{(r\pi)^3} \cos \omega_r t \sin \frac{r\pi x}{L}. \quad (7.115)$$

The displacement of the beam as a function of space and time can be related to the stresses that develop in a vibrating beam. Stress oscillations are related directly to the fatigue life and reliability of a structure and its components.

Mechanics of materials relates the bending moment M to the second derivative of the slope by the equation

$$M = EI \frac{\partial^2 y}{\partial x^2}.$$

The stress σ_M at an element of beam a distance e from the neutral axis of the beam is

$$\sigma_M = \frac{Me}{I} = Ee \frac{\partial^2 y}{\partial x^2}.$$

Then, using Equation 7.115,

$$\sigma_M(x, t) = Ee \sum_{r=1,3,\dots}^{\infty} \frac{-8C}{r\pi L} \sin \frac{r\pi x}{L} \cos \omega_r t.$$

The stress can be evaluated for particular values of x and t . For a simply supported beam, the maximum stress occurs at the midspan, $x = L/2$. Then,

$$\sin \frac{r\pi x}{L} = \sin \frac{r\pi}{2},$$

which is nonzero for odd r ,

$$\sigma_M \left(\frac{L}{2}, t \right) = -Ee \sum_{r=1,3,\dots}^{\infty} \frac{8C}{r\pi L} \sin \frac{r\pi}{2} \cos \omega_r t.$$

The argument $\omega_r t$ can be written in terms of the fundamental frequency ω_1 . Since

$$\omega_r = r^2 \left(\frac{\pi}{L} \right)^2 \sqrt{\frac{EI}{m}} = r^2 \omega_1,$$

then

$$\omega_r t = r^2 \omega_1 t.$$

The fundamental frequency is related to the fundamental period T_1 by $\omega_1 = 2\pi/T_1$. Therefore, at $t = T_1/2$, we have $\omega_r t = r^2\pi$, and $\cos \omega_r t = \cos r^2\pi = -1$ for odd r . The stress is then given³⁷ by

$$\begin{aligned} \sigma_M \left(\frac{L}{2}, \frac{T_1}{2} \right) &= Ee \sum_{r=1,3,\dots}^{\infty} \frac{8C}{r\pi L} \sin \frac{r\pi}{2} \\ \sigma_M \left(\frac{L}{2}, \frac{T_1}{2} \right) &= \frac{2EeC}{L}, \end{aligned} \quad (7.116)$$

with the same units as the modulus of elasticity E .

We conclude with a specific case. Assuming that the beam is a pipe of length $L = 200$ in, the modulus of elasticity $E = 30 \times 10^6$ psi, the initial displacement constant $C = 1$, and the distance from the neutral axis $e = 2$ in, the stress calculation using Equation 7.116 yields

$$\begin{aligned} \sigma_M \left(\frac{L}{2}, \frac{T_1}{2} \right) &= \frac{2 \times (30 \times 10^6) \times 2 \times 1}{200} \text{ psi} \\ &= 0.6 \times 10^6 \text{ psi}. \end{aligned}$$

We note that the stress is linearly related to the value of C , which was given as part of the initial displacement. ■

STEPHEN P. TIMOSHENKO

(December 22, 1878 – May 29, 1972)



Contribution: Timoshenko was a brilliant mechanical engineer, and by many considered the father of modern engineering mechanics. He investigated new problems in the areas

³⁷ We make use of the identity

$$\sum_{r=1,3,\dots}^{\infty} \frac{1}{r} \sin \frac{r\pi}{2} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right) = \frac{\pi}{4}.$$

of mechanics, elasticity, and strength of materials. Many of his solutions are widely used today.

The Timoshenko beam theory, one of his many contributions, takes into account shear deformation and rotational inertia effects, making it suitable for describing the behavior of short beams, sandwich composite beams, and beams subject to high-frequency excitation when the wavelength approaches the thickness of the beam. The added effects lower the stiffness of the beam. If the shear modulus of the beam material approaches infinity - and thus the beam becomes rigid in shear - and if rotational inertia effects are neglected, Timoshenko beam theory converges to ordinary beam theory, that is, Bernoulli-Euler theory.

Timoshenko's influence on the study of applied mechanics and engineering was immense. He changed the way engineering students, especially in Europe and America, were taught. His commitment to engineering education earned him many awards. He was recognized as a superb teacher, and the author of engineering textbooks that remain in use and are still important references. His books in the areas of strength of materials and mechanics include: *Strength of Materials*, *Theory of Elasticity*, *Elastic Stability*, *Engineering Mechanics*, *Theory of Plates and Shells*, *Theory of Structures*, and *Advanced Dynamics*. He also published many articles on elasticity, stability, and vibration.

Life Sketch: Timoshenko was born into the family of a land surveyor in the village of Shpotivka, Russia (currently Sumy Oblast, Ukraine). He was a student in Romny (in Sumy Oblast) from 1889 to 1896, and then continued his education towards a university degree at the St. Petersburg Ways of Communication Institute. After graduating in 1901, he taught there until 1903.

In 1903 he joined the St. Petersburg Polytechnic as a senior laboratory assistant in applied mechanics and taught students the statics of structures. In 1904 he travelled to Europe to become better acquainted with German technical school achievements and teaching methods. In 1905 he went to the University of Göttingen where he worked for one year under Ludwig Prandtl.

In the fall of 1906 he was appointed to the Chair of Strength of Materials at the Kiev Polytechnic Institute. The return to his native Ukraine turned out to be an important part of his career and also influenced his future personal life. From 1907 to 1911 as a professor at the Polytechnic Institute he did research on an earlier variant of the finite element method of elastic calculations called Rayleigh's method.

Timoshenko was influenced greatly by Rayleigh's book *The Theory of Sound*. He became particularly captivated by the approximate methods of calculating vibration frequencies of complex structures and developed his own solutions to many of these kinds of problems. In 1907 he developed and delivered a course on strength of materials, the lecture notes of which were later published. He was elected dean of the Division of Structural Engineering in 1909.

In 1911 he signed a protest against the minister of educa-

tion and was fired from the Kiev Polytechnic Institute. In 1911 he was awarded the D.I. Zhuravski prize of the St. Petersburg Ways of Communication Institute that helped him survive after losing his job.

He went to St. Petersburg where he worked as a lecturer and then a professor in the Electrotechnical Institute and the St. Petersburg Institute of the Railways (1911-1917). During this period he developed the theory of elasticity and the theory of beam deflection, and conducted research on buckling.

Prompted by a bridge disaster in Canada, Timoshenko started working on the theory of stability of composite beams. He discovered simpler methods of solving stability problems. In 1912, he traveled to England, where he found the facilities at Cambridge University less desirable than those in German laboratories.

In 1918 he returned to Kiev and assisted in establishing the Ukrainian Academy of Sciences - the oldest academy among the Soviet republics outside of Russia.

After General Denikin of the Imperial Russian Army took over Kiev in 1919, the Ukrainian Academy of Sciences was closed and Timoshenko lost his job. In 1920, after the Bolshevik takeover of Kiev, Timoshenko emigrated to Yugoslavia. He became a professor at the Zagreb Polytechnic Institute where he worked until 1922. He is remembered for delivering lectures in Russian while using as many words in Croatian as he could so the students could understand him.

In 1922 Timoshenko moved to the U.S. He worked as a technical consultant for the Westinghouse Electric Corporation from 1923 to 1927, after which he became a professor at the University of Michigan where he created the first bachelor's and doctoral programs in engineering mechanics. While at Michigan, he published books on strength of materials, elasticity, and stability.

From 1936 onward he was a professor at Stanford University. In 1960 he moved to Wuppertal (W. Germany) to be with his daughter. He died in 1972 and his ashes were buried in Palo Alto, California.

Noteworthy: Timoshenko was known as a remarkable teacher. His lectures on applied mechanics attracted a large number of students as well as other professors. When he was at the University of Michigan, European scientists, including Prandtl, Southwell, Westergaard, and von Kármán, came to meet him.

Timoshenko was elected to the National Academy of Sciences and the Royal Society. He received honorary doctoral degrees from various universities such as Lehigh, Zurich Tech Institute, and Glasgow University. In 1935 the American Society of Mechanical Engineers (ASME) awarded Timoshenko the Worcester Reed Warner Medal for his outstanding achievements in the field of mechanics. Stanford University named their laboratory of engineering mechanics after him.

In 1957 ASME established a medal named after Timoshenko; he became its first recipient. "The Timoshenko Medal honors Stephen P. Timoshenko as the world-renowned

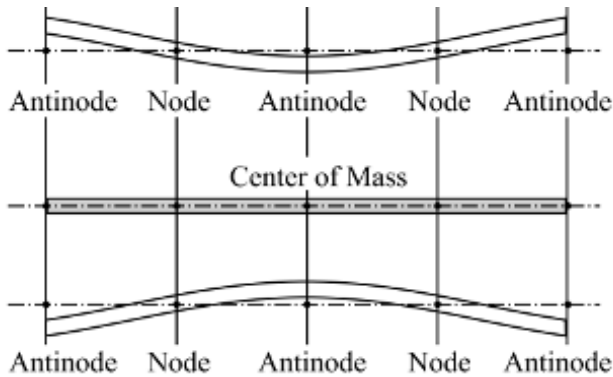


Figure 7.25: A beam vibrates with its middle and ends moving back and forth in opposite directions when it is struck by a mallet at the center. The beam is supported at the nodal points.

authority in the field of mechanical engineering and it commemorates his contributions as author and teacher.” The Timoshenko Medal is awarded annually for distinguished contributions in applied mechanics. The actual award is a bronze medal and honorarium.

In his book *History of Strength of Materials* he traced ideas from the time of Leonardo da Vinci and Galileo to modern times related to strength of materials, elasticity, and structures in general.

In addition to engineering textbooks, Timoshenko wrote two other books, *Engineering Education in Russia* and *As I Remember*, the latter an autobiography published in Russian in 1963 with its English translation appearing in 1968.

His textbooks have been published in 36 languages. His first textbooks and papers were written in Russian; later in his life, he published mostly in English.

7.5.5 Nodes and Antinodes

Consider a thin beam, like a xylophone bar, supported at two interior locations, struck by a mallet. Figure 7.25 shows the result. The beam bends back and forth rapidly with its ends and center moving in opposite directions. There are two points, however, that do not move. These are the *nodes*. The points that move farthest are the *antinodes*.

The study of transversely vibrating beams continues in the next section for beams with special boundary conditions and forces.

7.6 Other Transverse Beam Vibration Cases

This section introduces special beam vibration problems. More realistic forces and boundary conditions are considered in order to demonstrate the power of these models

to predict the behavior of a large class of practical engineering problems.

7.6.1 Beam with Axial Forces

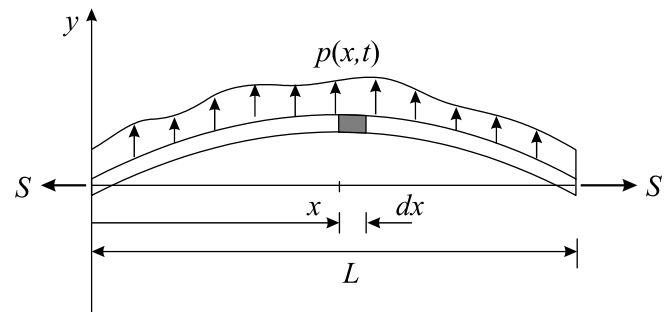


Figure 7.26: Beam with transverse continuous force $p(x, t)$ and axial forces S .

Consider the lateral vibration of a beam loaded both laterally and axially, as shown in Figure 7.26. Our interest lies in the added effect of the axial forces on the response. Whenever a beam or column is compressed, there is concern about its buckling. When it is placed under tension there is the possibility of yielding and failure.

To formulate this problem, we use the free-body diagram of an arbitrary section of the beam and show all external forces, noting that there is an additional moment term Sy due to the constant axial force S , where y is the deflection at the center of the section under consideration. This is shown in Figure 7.27.

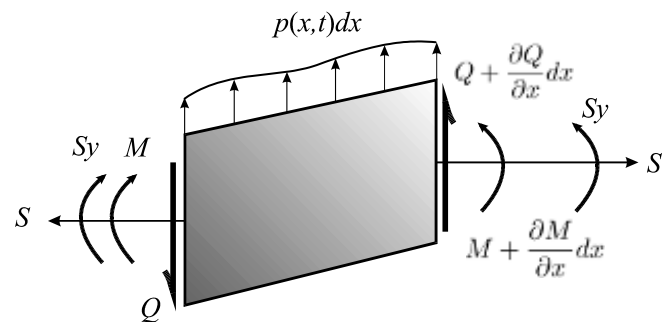


Figure 7.27: Free-body diagram of a beam element of length dx for the transverse vibration of a beam subjected to an axial force.

While the equation of motion in the vertical y direction remains the same (since S acts approximately perpendicular to y), the moment equation about the center of the cross section now must include the moment due to S , that is Sy ,

$$M + Sy = EI(x) \frac{\partial^2 y}{\partial x^2}.$$

Governing Equation 7.77 becomes

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y}{\partial x^2} - Sy \right] + p(x, t) = -m(x) \frac{\partial^2 y}{\partial t^2}.$$

Adopting the uniformity assumption as before, and setting $p(x, t) = 0$ for the analysis of the eigenvalue problem, the equivalent of Equation 7.84 becomes

$$EI \frac{d^4}{dx^4} Y(x) - S \frac{d^2}{dx^2} Y(x) = \omega^2 m Y(x), \quad (7.117)$$

where the effect of axial force S is found to be significant. For this problem, we assume

$$Y_r(x) = C_r \sin \frac{r\pi x}{L}, \quad r = 1, 2, \dots,$$

take the necessary derivatives, and substitute these into Equation 7.117, to find

$$\begin{aligned} EIC_r \left(\frac{r\pi}{L} \right)^4 \sin \frac{r\pi x}{L} + SC_r \left(\frac{r\pi}{L} \right)^2 \sin \frac{r\pi x}{L} \\ = \omega_r^2 m C_r \sin \frac{r\pi x}{L}, \end{aligned}$$

where the coefficients C_r can be canceled. Thus,

$$\omega_r = \left(\frac{r\pi}{L} \right)^2 \sqrt{\frac{EI}{m}} \sqrt{1 + \frac{S}{EI} \left(\frac{L}{r\pi} \right)^2}, \quad r = 1, 2, \dots$$

For a tensile axial force $+S$, the effect is an increase in the frequencies of free vibration.³⁸

Had a compressive force $-S$ been applied, the frequencies would be given by

$$\omega_r = \left(\frac{r\pi}{L} \right)^2 \sqrt{\frac{EI}{m}} \sqrt{1 - \frac{S}{EI} \left(\frac{L}{r\pi} \right)^2}, \quad (7.118)$$

resulting in lower natural frequencies. In this case, for what compressive load will the frequencies shift down so that the fundamental frequency becomes zero? For $r = 1$, the term

$$\frac{S}{EI} \left(\frac{L}{r\pi} \right)^2$$

is the ratio of S to the Euler buckling load. If $SL^2/EI\pi^2 \rightarrow 1$, the lowest mode of vibration approaches zero frequency and transverse buckling occurs for $S = EI\pi^2/L^2$. A plot of ω_1 as a function of S using Equation 7.118 is shown in Figure 7.28.

The zero frequency in this system signifies the onset of structural instability, unlike the zero frequency in an unrestrained system that represents rigid-body motion.

³⁸This confirms our intuition, and it is comparable to the tension in the string, where higher tension results in a higher set of frequencies.

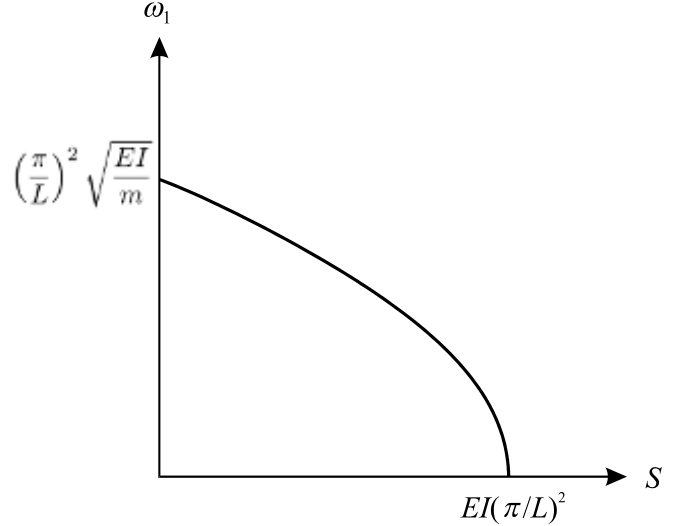


Figure 7.28: Natural frequency ω_1 as a function of axial force S .

7.6.2 Beam with Elastic Restraints

To demonstrate the importance of the boundary conditions, we consider the effect of the degree to which the supports are fixed. In previous problems, the boundaries were clearly defined as fixed, hinged, or free. Suppose, however, that a boundary is in reality somewhere between fixed and hinged. What would the effect be and how could this be modeled in the equation for the boundary condition?

Physically, such a boundary resists deflection and rotation, with characteristics that are a mix of the fixed and the hinged ends. One possibility for modeling such intermediate conditions is to restrain the end of the beam using a combination of translational and torsional springs as shown in Figure 7.29. The springs at the boundary resist the moment and shear, generally given by

$$M = EI \frac{\partial^2 y}{\partial x^2}, \quad Q = EI \frac{\partial^3 y}{\partial x^3},$$

for a uniform beam.

The eigenvalue problem and the general solution are identical to previous results, except for the following boundary conditions for the shear force Q and the moment M ,

$$Q(0) = EIY'''(0) = -k_1 Y(0)$$

$$Q(L) = EIY'''(L) = k_3 Y(L)$$

$$M(0) = EIY''(0) = k_2 Y'(0)$$

$$M(L) = EIY''(L) = -k_4 Y'(L).$$

The sign convention used above takes positive shear to be up on the left end and down on the right end of the beam, and a positive pair of bending moments results in a concave up beam.

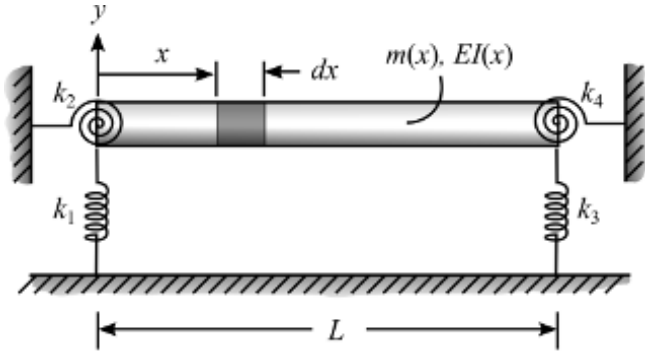


Figure 7.29: Beam with elastic restraints, where k_2 and k_4 are torsional stiffness constants and k_1 and k_3 are translational stiffness constants.

We begin with the general eigenfunction Equation 7.86,

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x,$$

along with the appropriate derivatives,

$$\begin{aligned} Y'(x) &= \beta(C_1 \cos \beta x - C_2 \sin \beta x \\ &\quad + C_3 \cosh \beta x + C_4 \sinh \beta x) \\ Y''(x) &= \beta^2(-C_1 \sin \beta x - C_2 \cos \beta x \\ &\quad + C_3 \sinh \beta x + C_4 \cosh \beta x) \\ Y'''(x) &= \beta^3(-C_1 \cos \beta x + C_2 \sin \beta x \\ &\quad + C_3 \cosh \beta x + C_4 \sinh \beta x). \end{aligned}$$

Substituting these into the equations for the boundary conditions leads to the following four simultaneous algebraic equations,

$$\begin{aligned} EI\beta^3 C_1 - k_1 C_2 - EI\beta^3 C_3 - k_1 C_4 &= 0 \\ (-EI\beta^3 \cos \beta L - k_3 \sin \beta L)C_1 \\ + (EI\beta^3 \sin \beta L - k_3 \cos \beta L)C_2 \\ + (EI\beta^3 \cosh \beta L - k_3 \sinh \beta L)C_3 \\ + (EI\beta^3 \sinh \beta L - k_3 \cosh \beta L)C_4 &= 0 \\ -k_2 C_1 - EI\beta C_2 - k_2 C_3 + EI\beta C_4 &= 0 \\ (-EI\beta \sin \beta L + k_4 \cos \beta L)C_1 \\ + (-EI\beta \cos \beta L - k_4 \sin \beta L)C_2 \\ + (EI\beta \sinh \beta L + k_4 \cosh \beta L)C_3 \\ + (EI\beta \cosh \beta L + k_4 \sinh \beta L)C_4 &= 0. \end{aligned}$$

As before, a nontrivial solution exists if and only if the 4×4 determinant of the matrix of coefficients $C_1, \dots, C_4$ is equal to zero,

$$\begin{vmatrix} EI\beta^3 & -k_1 & -EI\beta^3 & -k_1 \\ k_{21} & k_{22} & k_{23} & k_{24} \\ -k_2 & -EI\beta & -k_2 & EI\beta \\ k_{41} & k_{42} & k_{43} & k_{44} \end{vmatrix} = 0, \quad (7.119)$$

where

$$\begin{aligned} k_{21} &= -EI\beta^3 \cos \beta L - k_3 \sin \beta L \\ k_{22} &= EI\beta^3 \sin \beta L - k_3 \cos \beta L \\ k_{23} &= EI\beta^3 \cosh \beta L - k_3 \sinh \beta L \\ k_{24} &= EI\beta^3 \sinh \beta L - k_3 \cosh \beta L \end{aligned}$$

and

$$\begin{aligned} k_{41} &= -EI\beta \sin \beta L + k_4 \cos \beta L \\ k_{42} &= -EI\beta \cos \beta L - k_4 \sin \beta L \\ k_{43} &= EI\beta \sinh \beta L + k_4 \cosh \beta L \\ k_{44} &= EI\beta \cosh \beta L + k_4 \sinh \beta L. \end{aligned}$$

This results in the characteristic equation for the frequencies, which are then used to find the modes of vibration.

Note that the fixed and free boundary conditions can be recovered here by setting the appropriate spring constants to zero or infinity. For example, letting $k_1 = k_2 \rightarrow \infty$ and $k_3 = k_4 \rightarrow 0$ results in the fixed-free beam.

FLOOR VIBRATION. Floors in buildings can vibrate and the vibration can be objectionable depending on the amplitudes and frequencies as well as the purpose of the room. Floor vibrations in an operating room or in a manufacturing facility may be problematic, whereas these same vibrations in floors of residential and commercial buildings may be acceptable. Floors can be excited by people walking, in particular, by the heel strike phase of normal human gait. (Walking frequencies above 2.5 Hz are uncommon; a 2 Hz walking frequency is a common design specification.) The input force during walking depends on the weight and walking style of the individual and is a fraction of the individual's weight (approximately half at 2 Hz). Floors can be excited by rhythmic vibration, such as in dance and sport events. Vibrations of floors can be caused by the operation of machinery in the building (on the same or different floor) and by vehicular traffic outside. And, vibrations can result from transient impulse loads due to earthquakes, aerodynamic (wind) loads on buildings, or other loads.

In most cases, the perception of floor vibration is related to the response acceleration at different natural frequencies of the floor. Humans are most sensitive to vibrations at frequencies between 4 to 8 Hz. Larger accelerations can be tolerated at higher and lower frequencies.

7.6.3 Beam on an Elastic Foundation

The boundary conditions considered up to now were at discrete points, namely, at the ends of the beam. It is

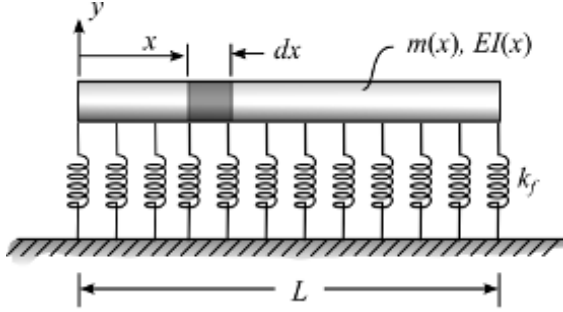


Figure 7.30: Beam on an elastic foundation modeled as a continuous elastic medium with a stiffness constant per unit length k_f .

possible that a continuous boundary effect occurs along the length of the beam. In such cases, the boundary condition becomes part of the governing equation.

An important example of such a problem is a beam on an elastic foundation, a schematic of which appears in Figure 7.30. Applications include machine vibration, the vibration of a structure on a foundation, and structural response to ground shock such as earthquakes and explosives.³⁹

Here, the elastic restraint against transverse motion is distributed continuously along the length of the beam and damping effects are ignored. Following Equation 7.77 for an element of length dx , the Bernoulli-Euler equation becomes

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y}{\partial x^2} \right] dx = -k_f y dx - \rho A(x) dx \frac{\partial^2 y}{\partial t^2},$$

where the restraining force for an element of length dx is given by the term $k_f y dx$. The parameter k_f is the constant stiffness per unit length of foundation, and $\rho A(x) = m(x)$ is the mass of the beam per unit length. If the elastic foundation has nonuniform stiffness properties then the stiffness would be a function of x , that is, $k_f(x)$. For a *prismatic beam* (uniform cross section),

$$EI \frac{\partial^4 y}{\partial x^4} + k_f y = -\rho A \frac{\partial^2 y}{\partial t^2}.$$

The foundation stiffness force $k_f y$ is included in the governing equation.

Assuming a product solution and performing the usual derivatives and substitutions, the equation governing the modal function $Y(x)$ is found to be

$$EI \frac{d^4 Y}{dx^4} - (\rho A \omega^2 - k_f) Y = 0,$$

or

$$\frac{d^4 Y}{dx^4} + \left(\frac{-\rho A}{EI} \omega^2 + \frac{k_f}{EI} \right) Y = 0. \quad (7.120)$$

³⁹The example of the rotating shaft that is immersed in a lubricated sleeve is one where there is continuous damping along the length, instead of continuous stiffness studied in this example.

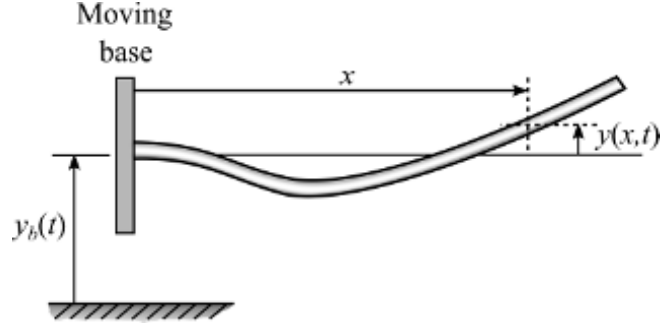


Figure 7.31: Side view of a beam on a vertically moving support. The beam position is given by the sum of the base displacement y_b and the elastic displacement y .

Letting $m = \rho A$, we redefine β to include the foundation stiffness,

$$\beta^4 = (\omega^2 m - k_f) / EI,$$

and rewrite Equation 7.120 in standard form,

$$Y''''(x) - \beta^4 Y(x) = 0. \quad (7.121)$$

This is the same equation as Equation 7.85, except for the definition of β , and, therefore, the equation for ω becomes

$$\omega_r = \left(\frac{r\pi}{L} \right)^2 \sqrt{\frac{EI}{m} + \frac{k_f}{m} \left(\frac{L}{r\pi} \right)^4} \quad (7.122)$$

for the hinged-hinged case. A conclusion we can draw is that the *elastic end restraints* of the previous section result in changes in both frequencies and mode shapes, whereas an *elastic foundation* changes the frequencies but not the eigenfunction shape. This is not a general conclusion and depends on the boundary conditions.

The importance of the term $k_f(L/r\pi)^4$ is problem-specific and depends on the value of EI . For most values of k_f , it can be ignored when compared with EI . The importance of the elastic foundation is associated primarily with the lowest frequencies.

7.6.4 Beam with a Moving Support

An interesting problem is that of a vibrating horizontal beam on a moving vertical support, as shown schematically in Figure 7.31. This could be representative of a helicopter rotor blade.

The governing equation for such a beam is Equation 7.77, except that $y(x, t)$ is replaced by $y(x, t) + y_b(t)$,

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 (y + y_b)}{\partial x^2} \right] = -m(x) \frac{\partial^2 (y + y_b)}{\partial t^2},$$

where $y_b(t)$ represents the vertical motion of the base translating as a rigid-body in addition to the displacement $y(x, t)$ due to bending motion. Since y_b is only a

function of time, the governing equation can be simplified to resemble the base-excited structure of Chapter 3,

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y}{\partial x^2} \right] + m(x) \frac{\partial^2 y}{\partial t^2} = -m(x) \frac{\partial^2 y_b}{\partial t^2}.$$

The steady-state solution to this equation for a uniform beam is given by Equation 7.113, repeated here,

$$y(x, t) = \sum_{j=1}^{\infty} \hat{Y}_j(x) \int_0^t \left[\int_0^L p(\xi, \tau) \hat{Y}_j(\xi) d\xi \right] g_j(t - \tau) d\tau,$$

with $p(x, t) = -m(\partial^2 y_b / \partial t^2)$, where $g_j(t)$ is the impulse response function for the undamped oscillator. If the base excitation is harmonic, then

$$y_b(t) = B \sin \Omega t,$$

and

$$p(x, t) = p(t) = mB\Omega^2 \sin \Omega t.$$

Once solved, the total motion is given by $y(x, t) + y_b(t)$.



Figure 7.32: Ziplining is an example of a moving load on a flexible cable. It can induce vibration in the line. (Milford, PA; the subject in the photograph is M. Nagurka)

7.6.5 Different Boundary Conditions

When a continuous system is subjected to forces, the equation of motion becomes nonhomogeneous. In this case it is possible to use the method of eigenfunction expansion and convolution integral, as was already done. Sometimes, the applied forces appear as time-dependent boundary conditions. We saw an example of this in Example 7.3 for a harmonic torque at the free end of a



Figure 7.33: The Milwaukee Art Museum's signature wings form a movable sunshade with a 66-m wingspan. The wings consist of 72 steel fins, pinned at a central spine and ranging in length from 8 to 32 m. The mass of the structure is 81,600 kg. It takes 3.5 minutes for the wings to open or close. Sensors on the fins monitor wind speed and direction; if the speed exceeds 37 km/h for more than 3 s, the wings close. (Photograph by M. Nagurka)

shaft. When these forces are harmonic, as we see next, the method of separation of variables proceeds as usual. Otherwise, different techniques⁴⁰ such as those of Section 8.4.1 must be used.

Harmonic Boundary Conditions

Consider a uniform cantilever beam of length L fixed at the left end and subjected at the right free end to a harmonic load⁴¹ acting down in the positive y direction equal to $P \cos \Omega t$. The transverse beam vibration equation is still valid, but with the following boundary conditions,

$$\begin{aligned} y(0, t) &= 0, & \frac{\partial^2 y(L, t)}{\partial x^2} &= 0 \\ \frac{\partial y(0, t)}{\partial x} &= 0, & EI \frac{\partial^3 y(L, t)}{\partial x^3} &= -P \cos \Omega t. \end{aligned}$$

The last boundary condition is a statement that the shear force on the right end of the beam is equal and opposite to that of the applied force. Otherwise, the shear force would equal zero at a free end, as we saw earlier.

We look for a solution with response frequency equal to the forcing frequency Ω , such that

$$y_p(x, t) = Y(x) \cos \Omega t.$$

⁴⁰Dynamics of Continuous Elements, S.K. Clark, Prentice Hall, 1972.

Analytical Methods in Vibrations, L. Meirovitch, MacMillan Publishing, 1967.

⁴¹This case is similar to Example 7.3 where the loading is applied at a boundary.

Upon substituting $y_p(x, t)$ into the equation of motion, we find the eigenfunction equation,

$$EIY''''(x) - m\Omega^2 Y(x) = 0,$$

with the solution

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x \\ + C_3 \sinh \beta x + C_4 \cosh \beta x,$$

where

$$\beta^4 = \frac{\Omega^2 m}{EI}.$$

The constants are determined by satisfying the four boundary conditions above. $Y(0) = 0$ leads to $C_2 + C_4 = 0$. $Y'(0) = 0$ leads to $C_1 + C_3 = 0$, $Y''(0) = 0$, and

$$EIY'''(L) \cos \Omega t = -P \cos \Omega t,$$

where $\cos \Omega t$ can be eliminated from both sides of the equation,

$$EIY'''(L) = -P. \quad (7.123)$$

We now have the four equations needed to evaluate the constants C_i .

Another way to solve this problem is to move the forcing function from the boundary condition to the right-hand side of the differential equation of motion. In this approach, the governing equation becomes

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = -\delta(x - L)P \cos \Omega t, \quad (7.124)$$

where $\delta(x - L)$ is the Dirac delta function, and the boundary conditions are as shown above except that

$$EI \frac{\partial^3 y(L, t)}{\partial x^3} = 0.$$

Equation 7.124 is solved using the eigenfunction expansion method discussed earlier in this chapter.

Displacement Boundary Conditions

Suppose that the displacement at a boundary is specified,

$$y(L, t) = h(t),$$

where $h(t)$ is an arbitrary function. To move the boundary condition to the governing equation is more difficult. In this case, the governing equation and boundary conditions are given by

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = 0 \\ y(0, t) = 0, \quad \frac{\partial y(0, t)}{\partial x} = 0 \\ \frac{\partial^2 y(L, t)}{\partial x^2} = 0, \quad y(L, t) = h(t).$$

The goal is to move the nonhomogeneous term from the boundary condition to the governing equation. We do this by defining the modified displacement profile $v(x, t)$,

$$v(x, t) = y(x, t) - h(t)f(x), \quad (7.125)$$

where $f(x)$ is a polynomial. This polynomial is determined so that $v(x, t)$ satisfies the equivalent homogeneous boundary conditions,

$$v(0, t) = 0, \quad \frac{\partial v(0, t)}{\partial x} = 0 \\ \frac{\partial^2 v(L, t)}{\partial x^2} = 0, \quad v(L, t) = 0.$$

That is,

$$v(0, t) = y(0, t) - h(t)f(0) = 0 \\ \frac{\partial v(0, t)}{\partial x} = \frac{\partial y(0, t)}{\partial x} - h(t) \frac{df(0)}{dx} = 0 \\ \frac{\partial^2 v(L, t)}{\partial x^2} = \frac{\partial^2 y(L, t)}{\partial x^2} - h(t) \frac{d^2 f(L)}{dx^2} = 0 \\ v(L, t) = y(L, t) - h(t)f(L) = 0.$$

After substituting the original boundary conditions for $y(x, t)$, the polynomial must satisfy

$$f(0) = 0 \\ \frac{df(0)}{dx} = 0 \\ \frac{d^2 f(L)}{dx^2} = 0 \\ f(L) = 1.$$

We try a polynomial with four unknown coefficients, or a third order polynomial, $f(x) = ax^3 + bx^2 + cx + d$. After applying the conditions for $f(x)$, we find

$$f(0) = d = 0 \\ \frac{df(0)}{dx} = c = 0 \\ \frac{d^2 f(L)}{dx^2} = 6aL + 2b = 0 \\ f(L) = aL^3 + bL^2 = 1.$$

The coefficients a and b are then given by

$$a = -\frac{1}{2} \frac{1}{L^3} \quad \text{and} \quad b = \frac{3}{2} \frac{1}{L^2},$$

and the polynomial is

$$f(x) = -\frac{1}{2} \left(\frac{x}{L}\right)^3 + \frac{3}{2} \left(\frac{x}{L}\right)^2.$$

The corresponding governing equation and boundary conditions are given by

$$EI \frac{\partial^4 v}{\partial x^4} + m \frac{\partial^2 v}{\partial t^2} = -m \ddot{h} f(x)$$

$$v(0, t) = 0, \quad \frac{\partial^2 v(L, t)}{\partial x^2} = 0$$

$$\frac{\partial v(0, t)}{\partial x} = 0, \quad v(L, t) = 0,$$

which can be solved for $v(x, t)$ in the usual way, and then $y(x, t)$ can be found using Equation 7.125.

The same procedure can be used when the slope and moment are specified. The key is to figure out how to choose $v(x, t)$ so that the nonhomogeneous terms are removed from the boundary conditions.

7.6.6 Beam with Traveling Force

A brief introduction is provided here to traveling force problems. A number of important applications can be cast in this form. Examples include vehicle-structure interaction problems⁴² such as automobile-bridge, jet-aircraft carrier, structure-ocean wave, and train-track are familiar ones. High-speed transportation systems based on magnetically levitated (maglev) vehicles result in electromagnetic coupling between the vehicle and guideway span.⁴³ Figures 7.34 and 7.35 hint at how preliminary models may be formulated.

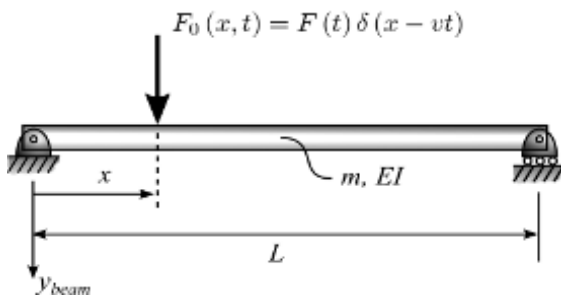


Figure 7.34: Traveling point force $F_0(x, t)$ with speed v on a beam.

⁴²Interaction problems are generally those where two dissimilar systems are coupled in some way, and, therefore, their vibration characteristics must be solved simultaneously.

⁴³M.L. Nagurka, S.K. Wang, "A Superconducting Maglev Vehicle/Guideway System with Preview Control. Part I: Vehicle, Guideway, and Magnet Modeling," *ASME Journal of Dynamic Systems, Measurement and Control*, Vol.119, No.4, December 1997, pp.638-643.

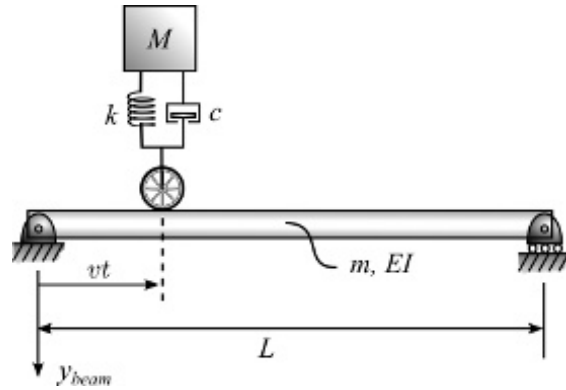


Figure 7.35: Single degree-of-freedom vehicle traveling with speed v on a beam.

An n degree-of-freedom model can be used instead of the single degree-of-freedom model shown in Figure 7.35 for the vehicle, if a more comprehensive model is warranted. The beam model may be as simple as the Bernoulli-Euler beam, or may include the additional effects of the Timoshenko beam. For a rail system, the beam may be on a foundation and damping can be added. Many options exist, depending on the application, and models of different complexities can be developed by adding some of the individual effects of the more basic models we have studied.

Wave methods may also be used if the application requires the modeling of an infinite length beam, as in a train-track system.

7.7 Concepts Summary

This has been another chapter of many new ideas and problems. Continuous systems are introduced as a limiting case of an n degree-of-freedom discrete system, where $n \rightarrow \infty$.

The simplest continuous system is the string, which was modeled in different ways, demonstrating that vibrating continuous systems can be viewed as either modal or wave systems. It was shown that either approach yields identical results. The wave approach led to the use of Newton's second law of motion or Hamilton's principle. In all these models of one-dimensional continuous systems, a modal approach was used to solve for free-vibration responses. Some beam problems with special loading or boundary conditions were also derived and discussed.

The ideas developed in this chapter are extended in the next chapter to two-dimensional systems, in particular, membranes and plates.

7.8 Quotes

- “Color vibrates at a very specific frequency, like music. If the notes are off even a little, it ruins the whole orchestra.”
- “Success is not final, failure is not fatal. It is the courage to continue that counts.”
- “Mary, I know what I’m gonna do tomorrow, and the next day, and next year, and the year after that. I’m shakin’ the dust of this crummy little town off my shoes and I’m gonna see the world! ... Then I’m gonna come back home and go to college and see what they know, and then I’m gonna build things. I’m gonna build airfields, I’m gonna build skyscrapers a hundred stories high, I’m gonna build bridges a mile long!” (James Stewart, in *It’s a Wonderful Life*)
- “Some day it might be possible to tax them.” (Michael Faraday, replying to Gladstone on being asked what use his discoveries were)
- “A theory may be so rich in descriptive possibilities that it can be made to fit any data.” (Phillip Johnson-Laird)
- “Judge a man by his questions rather than his answers.” (Voltaire)
- “The wisdom of life consists in the elimination of non-essentials.” (Lin Yutang)
- “However beautiful the strategy, you should occasionally look at the results.”

7.9 Problems

Problems for Section 7.2 – Vibration of Strings

1. Derive Equation 7.4,

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial y}{\partial x} \right] + f(x, t) = m(x) \frac{\partial^2 y}{\partial t^2},$$

using Newton’s second law of motion.

2. Solve for the string response, Equation 7.27, given the following initial conditions:
 - (a) $y(x, 0) = 0$, $\dot{y}(x, 0) = g(x)$
 - (b) $y(x, 0) = \sin \pi x/L$, $\dot{y}(x, 0) = 0$
 - (c) $y(x, 0) = 0$, $\dot{y}(x, 0) = \sin \pi x/L$
 - (d) $y(x, 0) = \sin \pi x/L$, $\dot{y}(x, 0) = \sin \pi x/L$.

In all cases, sketch the initial conditions and plot the response $y(x, t)$.

3. Derive the expression for the tension required in a simply supported transmission line modeled as a string of length l and linear density ρ , such that its fundamental frequency for transverse vibration is f_1 . What is the value of the tension where $l = 20$ m, $\rho = 5$ kg/m, and $f_1 = 15$ Hz?
4. A string is stretched between $x = 0$ and $x = L$ and has a variable density $\rho = \rho_0 + \varepsilon x$, where ρ_0 and ε are constants. The initial displacement is $f(x)$, and the string is released from rest.

(a) If the tension T is constant, then show that the governing equation of motion is

$$T \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2},$$

for $0 < x < L$, and $t > 0$, with the boundary conditions $y(0, t) = 0$, $y(L, t) = 0$, and the initial conditions $y(x, 0) = f(x)$, $dy(x, 0)/dt = 0$.

(b) Show that the frequencies of normal mode vibration are given by $f_n = \omega_n/2\pi$, where ω_n are the positive roots of

$$J_{1/3}(\alpha\omega)J_{-1/3}(\beta\omega) = J_{1/3}(\beta\omega)J_{-1/3}(\alpha\omega),$$

where

$$\alpha = \frac{2\rho_0}{3\varepsilon} \sqrt{\frac{\rho_0}{T}}$$

$$\beta = \frac{2(\rho_0 + \varepsilon L)}{3\varepsilon} \sqrt{\frac{\rho_0 + \varepsilon L}{T}}.$$

5. Consider the uniform string with constant mass per unit length m , constant tension T , and length L , shown in Figure 7.36. One end of the string is fixed, and the other end is constrained to move in the vertical direction. Find the frequency equation, the first three natural frequencies, and plot the first three mode shapes.

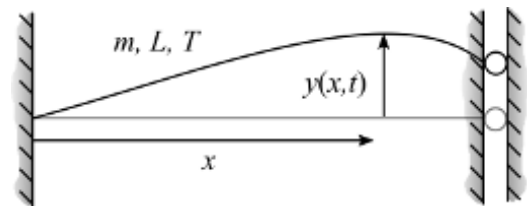


Figure 7.36: String fixed at one end and constrained at the other end.

6. Consider the uniform string with constant mass per unit length m , constant tension T , and length L shown in Figure 7.37. One end of the string is fixed, and the other end is attached to a spring with stiffness k that is constrained to move in the vertical direction. Find the frequency equation, the first three

natural frequencies, and plot the first three mode shapes. Use the ratio $kL/T = 3$.

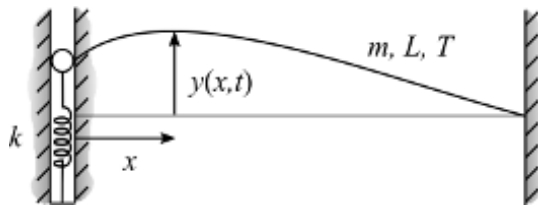


Figure 7.37: String spring-constrained at one end and fixed at the other end.

7. Consider the uniform string with constant mass per unit length m , constant tension T , and length L shown in Figure 7.38. One end of the string is fixed, and the other end is attached to a mass M that is constrained to move in the vertical direction. Find the frequency equation, the first three natural frequencies, and plot the first three mode shapes. Use the ratio $mL/M = 2$.

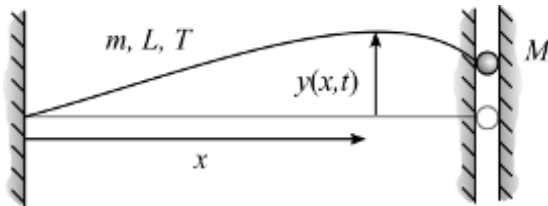


Figure 7.38: String fixed at one end and mass-constrained at the other end.

8. Consider the uniform string with constant mass per unit length m , constant tension T , and length L shown in Figure 7.39. One end of the string is attached to a mass M and the other end of the string is constrained by a spring with stiffness k . Both ends are constrained to move in the vertical direction. Find the frequency equation, the first three natural frequencies, and plot the first three mode shapes. Use the ratios $mL/M = 2$ and $kL/T = 100$.

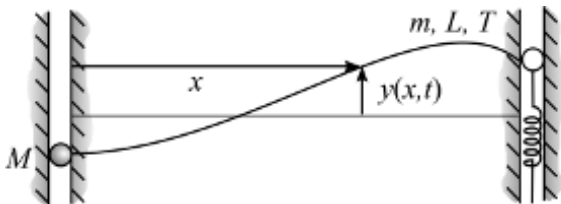


Figure 7.39: String mass-constrained at one end and spring-constrained at the other end.

9. Consider the fixed-fixed uniform string made of two sections shown in Figure 7.40. The left section of the string has a constant mass per unit length m , and the right section of the string has the constant mass

per unit length $2m$. (a) From a free-body diagram and Newton's second law of motion, under what assumptions can the tension be considered constant? (b) Assuming constant tension in both strings, find the frequency equation, the first three natural frequencies, and plot the first three mode shapes.

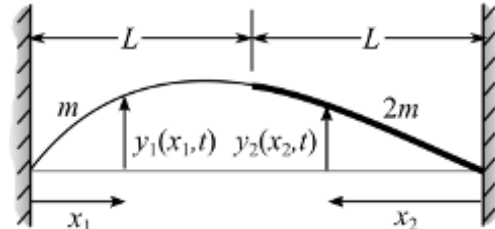


Figure 7.40: Discontinuous string.

10. Plot the displacement response of the string in Figure 7.36 at $x = L$ as a function of time if the initial displacement is $y(x,0) = 0.05x/L$ and the initial velocity is zero.
11. Plot the displacement response of the string in Figure 7.36 at $x = L$ if the string is initially horizontal and is subject to the sinusoidal input force at $x = L$, $F(t) = F_0 \sin \omega_f t$. Use $L = 1$ m, $T = 100$ N, $m = 0.2$ kg/m, $\omega_f = 10$ rad/s, and $F_0 = 5$ N.
12. Plot the displacement response of the string in Figure 7.38 at $x = L$ as a function of time if the initial displacement is $y(x,0) = 0.05x/L$ and the initial velocity is zero. The orthogonality condition for the eigenfunctions must be obtained first.

Problems for Section 7.3 – Axial Vibration of Beams

13. Consider a system consisting of discrete masses connected by springs in series as shown in Figure 7.41. Derive the equation of motion for mass m_i using Newton's second law, arrange the equation in incremental form, take the limit, and derive the differential equation of motion for the longitudinal vibration of a thin beam.

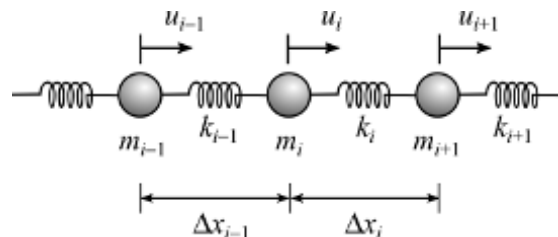


Figure 7.41: Discrete mass-spring system constrained to move horizontally.

14. Derive Equation 7.32.

15. For the uniform fixed-fixed beam in longitudinal vibration shown in Figure 7.42 derive the expressions for the natural frequencies and the mode shapes. Sketch the first three modes.

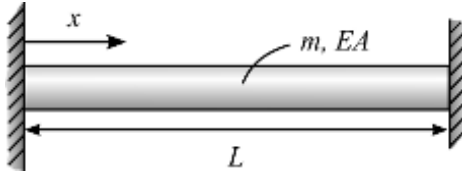


Figure 7.42: Fixed-fixed beam in longitudinal vibration.

16. For Equation 7.37, solve for the constants A_r and B_r by satisfying the initial conditions.
17. Complete the solution of Example 7.2 for the modes and the complete solution.
18. Consider the free axial motion of a beam with the elastic boundary conditions of Example 7.2, and solve for the first four natural frequencies, modes, and the complete solution, for the following cases:
- $k = EA \text{ N/m}$, $L = 10 \text{ m}$
 - $k = 10EA \text{ N/m}$, $L = 1 \text{ m}$
 - $k = EA/10 \text{ N/m}$, $L = 1 \text{ m}$
 - $k = EA/10 \text{ N/m}$, $L = 10 \text{ m}$.
19. An unrestrained beam lies on a horizontal and smooth surface and at $t = 0$ is forced at one end by $F(t)$, as shown in Figure 7.43. Derive the general response for any forcing function using the known eigenvalues and eigenfunctions. Then evaluate the specific steady-state response for the following specific forcing functions:
- $F(t) = F_0 u(t)$, where $u(t)$ is the unit step function
 - $F(t) = \sin \Omega t$, for the case where $\Omega = \omega_n/2$
 - $F(t) = 1 - t/t_0$, $0 \leq t \leq t_0$
 - $F(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function.

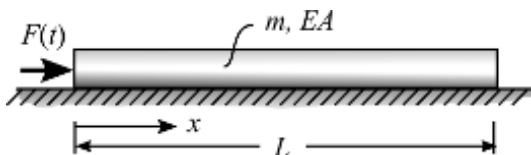


Figure 7.43: Beam on smooth (frictionless) surface.

20. Find the response of a uniform beam clamped at $x = 0$ and free at $x = L$ subjected to the longitudinal force,

$$f(x, t) = F_0 \delta[x - (L - \varepsilon)] u(t).$$

21. Derive the natural frequencies and mode shapes of the longitudinally vibrating uniform beam of Figure 7.44. Sketch the first three mode shapes. What will change in the analysis if the beam is vertical?
22. A cantilever beam has its free end stretched uniformly so that the original length L becomes L_0 , and then it is released at $t = 0$, as shown in Figure 7.44. Begin with the general solution for the axial response of a beam,

$$u(x, t) = \sum_{r=1}^{\infty} \sin \frac{(2r-1)\pi x}{2L} \times \left(A_r \sin \frac{(2r-1)\pi x}{2L} \sqrt{\frac{EA}{m}} t + B_r \cos \frac{(2r-1)\pi x}{2L} \sqrt{\frac{EA}{m}} t \right),$$

then satisfy the boundary conditions to fix the arbitrary constants, and finally derive the particular response.

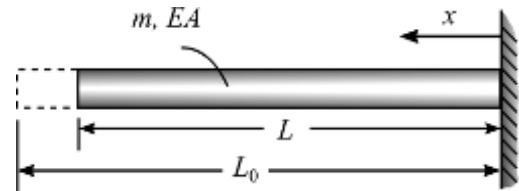


Figure 7.44: Stretched cantilever beam with original length L .

23. Consider the stepped shaft made of the same material of density ρ and modulus E , shown in Figure 7.45. Determine the characteristic equation for the natural frequencies of the system as it undergoes axial motion. Let $u_1(x, t)$ denote the displacement in the shaft with the fixed end, and $u_2(x, t)$ be the displacement in the shaft with the free end.

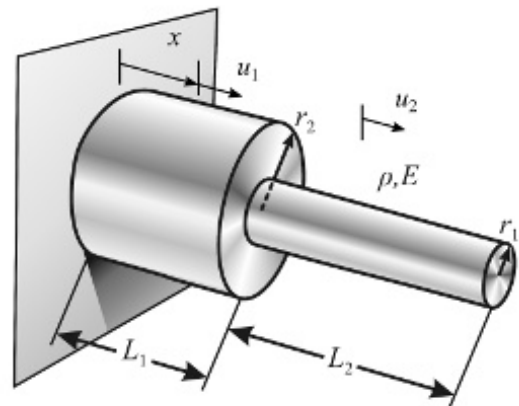


Figure 7.45: Stepped shaft undergoing axial motion.

24. Consider the uniform beam shown in Figure 7.46. Solve for the lowest natural frequency of axial motion of the steel beam given the following parameter values: $E = 210 \text{ GN/m}^2$, $\rho = 7850 \text{ kg/m}^3$, $L = 1.2 \text{ m}$, $k = 100 \text{ kN/m}$, and $A = 3 \text{ cm}^2$.

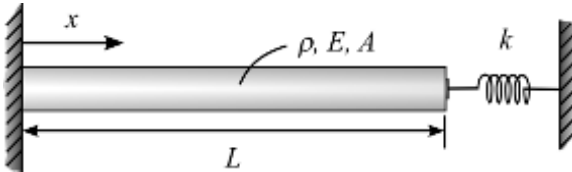


Figure 7.46: Axially vibrating beam with spring boundary.

25. Solve for the first three modes and frequencies of the nonuniform longitudinally vibrating beam of Figure 7.47 beginning with the known equation of motion. The following properties hold:

$$EA(x) = 2EA(1 - x/L),$$

$$\text{and } m(x) = 2m(1 - x/L).$$

- (a) Solve this problem for arbitrary k , and (b) then reduce the solution for the case where $k = 0$. How far can this problem be solved analytically?

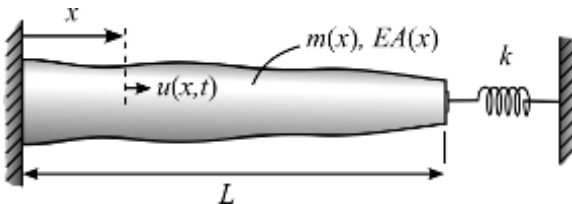


Figure 7.47: Elastic constraint of a longitudinally vibrating beam. At position x , the beam elongates an amount $u(x,t)$. At the right end there is a spring of stiffness k .

26. Consider the longitudinally vibrating uniform beam shown in Figure 7.48. One end of the beam is fixed and the other end is free with an attached concentrated mass M . Derive the frequency equation, the first three natural frequencies, and plot the first three mode shapes for axial vibration. Use $mL/M = 1$.

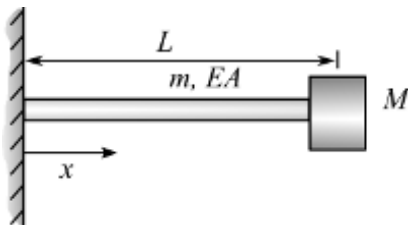


Figure 7.48: Axially vibrating cantilever beam fixed at one end with a point-mass at the other end.

27. Derive the orthogonality condition for the eigenfunctions of the beam shown in Figure 7.48.
28. Consider the longitudinally vibrating uniform beam shown in Figure 7.49. The ends of the beam are supported by linear springs with stiffnesses k_1 and k_2 . Derive the frequency equation, the first three natural frequencies, and plot the first three mode shapes for axial vibration. Use $k_1L/EA = 0.5$ and $k_2L/EA = 2$.

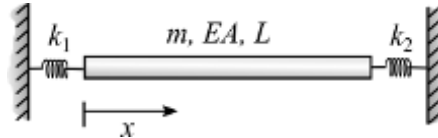


Figure 7.49: Axially vibrating beam supported by springs at both ends.

29. Consider the two identical longitudinally vibrating uniform beams shown in Figure 7.50 connected by a linear spring. Derive the frequency equation, the first three natural frequencies, and plot the first three mode shapes for axial vibration. Use $kL/EA = 0.1$.

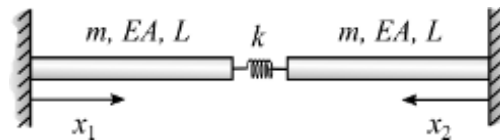


Figure 7.50: Two identical cantilever beams connected by a spring.

30. Plot the response of the concentrated mass in Figure 7.48 due to the initial displacement $u(x,0) = 0.02x/L$ and zero initial velocity.

Problems for Section 7.4 – Torsional Vibration of Shafts

31. Consider a fixed-fixed uniform torsional beam with length L , torsional rigidity GJ , and mass per unit length m . Derive the frequency equation, the first three natural frequencies, and plot the first three mode shapes.
32. The uniform shaft of Figure 7.51 is subjected to the torque $T_0 \sin \omega t$ at its free end. (a) Find the steady-state vibration response by solving the free vibration problem, including the forcing function in a boundary condition. (b) Solve this problem as a forced vibration problem where the free end is left unforced and the forcing function is an external loading in the equation of motion.

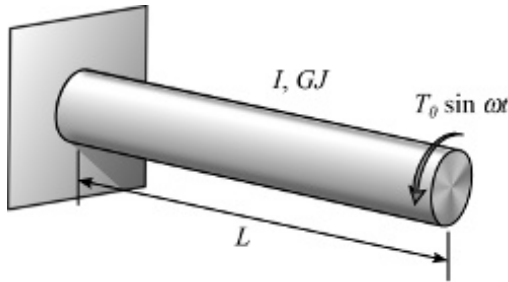


Figure 7.51: Torsional vibration of a shaft.

33. For the torsion problem of Section 7.4.1, complete the general solution for $\theta(x, t)$, assuming the arbitrary initial conditions $\theta(x, 0)$ and $\dot{\theta}(x, 0)$.
34. For the torsion problem of Section 7.4.1, solve for the frequencies, the modes and the total response $\theta(x, t)$ of the system for the cases: (a) $IL = I_D$, (b) $IL = 10I_D$, (c) $IL = I_D/10$.
35. Consider the stepped shaft consisting of two uniform segments of lengths L_1 and L_2 , torsional rigidities GJ_1 and GJ_2 , and mass moments of inertia per unit length I_1 and I_2 , shown in Figure 7.52. Set up the eigenvalue problem for the torsional vibration of the system and derive the frequency equation. Show that as $L_2 \rightarrow 0$,

$$\lim_{L_2 \rightarrow 0} I_2 L_2 = I_D,$$

where I_D is the mass moment of inertia of a disk. Find the frequency equation and the orthogonality condition.

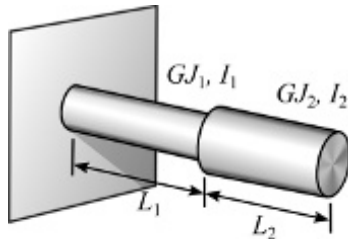


Figure 7.52: Stepped shaft undergoing torsion.

36. In the oil drilling industry, a drill-tube is used for well-boring. It is a tube of outer diameter D_o and inner diameter D_i and is made of steel of density ρ and modulus of rigidity G . At the end of the drill-tube there is a drill bit that does the actual drilling. (The bit rotates, enabling it to cut through the toughest of material. The bits are generally made of manufactured diamonds.) Assuming a tube length l and a bit moment of inertia I_b , derive an expression for the first two natural frequencies of torsional vibration, assuming that the tube is fixed at the upper end.

37. Derive Equations 7.53 and 7.54.

38. A circular elastic shaft of length L has one end clamped and a rigid disk attached at the other end, as shown in Figure 7.53. If G denotes the modulus of rigidity, J the area polar moment of inertia of the shaft, I the mass polar moment of inertia of the shaft per unit length, and I_D the mass moment of inertia of the disk about its center axis, calculate the potential and kinetic energy expressions associated with the rotational motion, and derive the equation of motion using Lagrange's equation.

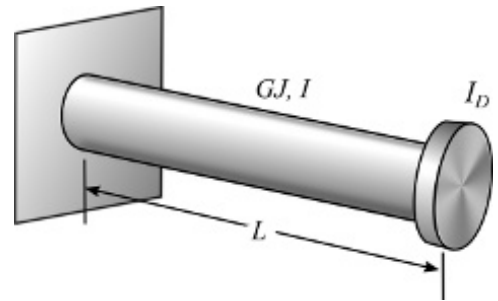


Figure 7.53: Torsional vibration of a shaft with a rigid disk.

Problems for Section 7.5 – Transverse Vibration of Beams

39. Derive the frequency equation, the first three natural frequencies, and plot the first three mode shapes of a clamped-clamped uniform beam undergoing transverse vibration.
40. Derive the frequency equation, the first three natural frequencies, and plot the first three mode shapes of a clamped-pinned uniform beam undergoing transverse vibration.
41. Derive the frequency equation, the first three natural frequencies, and plot the first three mode shapes of a clamped-uniform beam with a concentrated mass M at the free end undergoing transverse oscillation. Use $M/mL = 1$.
42. Perform the necessary variations to derive Equation 7.62.
43. Equations 7.63 and 7.64 are solved using the assumed solutions $y(x, t) = Y(x) \exp(i\omega t)$ and $\psi(x, t) = \Psi(x) \exp(i\omega t)$. Find the characteristic equation.
44. In Equations 7.67 and 7.68, eliminate $y(x, t)$ and derive the governing equation for $\psi(x, t)$. What physical motion does this equation govern?

45. Derive the Bernoulli-Euler beam equation, Equation 7.74. State all assumptions.
46. For the simply supported uniform beam of Figure 7.54 undergoing transverse motion, derive the natural frequencies and the natural modes of vibration. Sketch the first three mode shapes. What will change in the analysis if the beam is on an inclined surface?

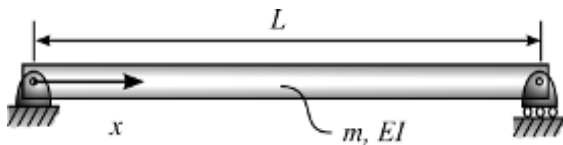


Figure 7.54: Simply supported uniform beam.

47. For the transverse vibration of a simply supported uniform beam, solve for the transient response if the initial conditions are given by

$$y(x, 0) = B \left(\frac{x}{L} - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} \right) \quad \dot{y}(x, 0) = 0.$$

48. A two-span uniform beam may be used as a simple model of a bridge. For this model, sketched in Figure 7.55, derive the frequency equation.

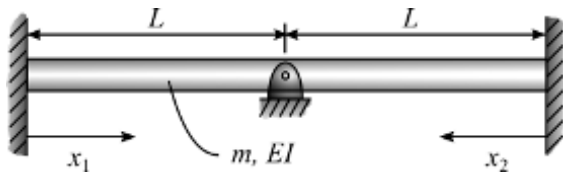


Figure 7.55: Two-span continuous uniform beam.

49. The beam of Figure 7.56 is released and rotates as a rigid body and impacts at an angular speed of Ω rad/s. Assume that upon impact the beam latches onto the support and there is no rebound and no loss in energy. At impact the beam transfers all of its rotational kinetic and potential energies into transverse vibrational motion. (a) Derive the response after impact. (b) How would the analysis proceed if there is a rebound?

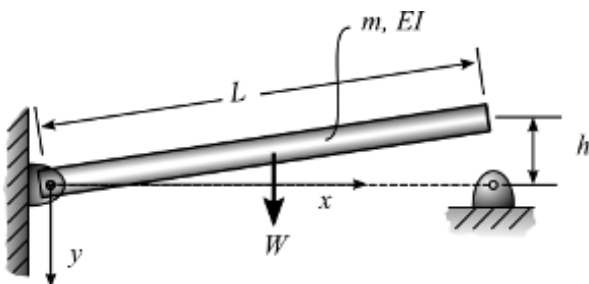


Figure 7.56: A freely falling rotating beam.

50. Determine the characteristic equation for the transverse vibration of a beam that is pinned at one end and free at the other end.
51. List the boundary conditions for the transversely vibrating beams of Figures 7.57 and 7.58.

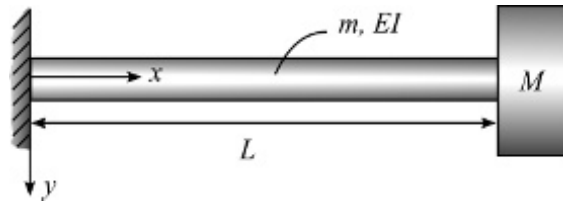


Figure 7.57: Transversely vibrating beam with end mass.

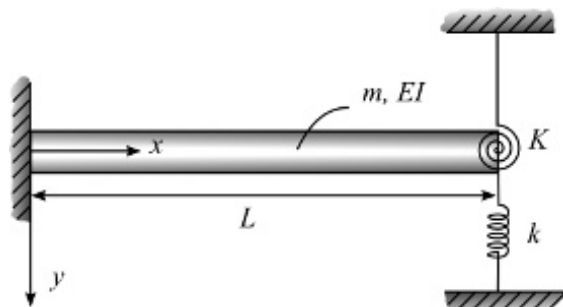


Figure 7.58: Transversely vibrating beam with elastic support.

52. For the transversely vibrating beam of Figure 7.59, find the modes and the first three natural frequencies of vibration.

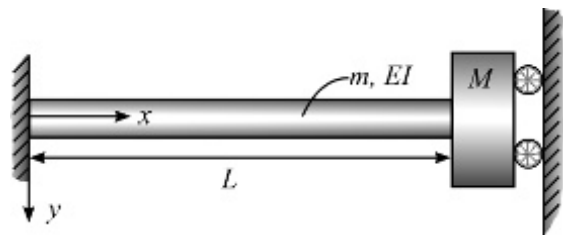


Figure 7.59: Transversely vibrating beam with end roller.

53. Complete the solution of Equation 7.97 for the initial conditions $y(x, 0) = d(x)$ and $\dot{y}(x, 0) = v(x)$.
54. Derive Equation 7.102 beginning with Equation 7.101.
55. Derive Equation 7.106 starting with Equation 7.105.
56. Fill in the steps in the derivation of Equation 7.110.
57. Solve Equation 7.113 for zero initial conditions and for each of the following forcing functions:

- (a) $p(x, t) = u(t)$, where $u(t)$ is the unit step function
- (b) $p(x, t) = \sin \Omega t$
- (c) $p(x, t) = e^{-\varphi t}$, for $t \geq 0$
- (d) $p(x, t) = \sin \Omega t$, for $0 \leq \Omega \leq \pi/2$.

58. Derive the equation of motion and boundary conditions for a transversely vibrating beam using Hamilton's principle. Assume small deflections. Let I_p equal the mass moment of inertia per unit length of the beam about its neutral axis. Figure 7.60 provides a sketch of the beam along with positive sign conventions.

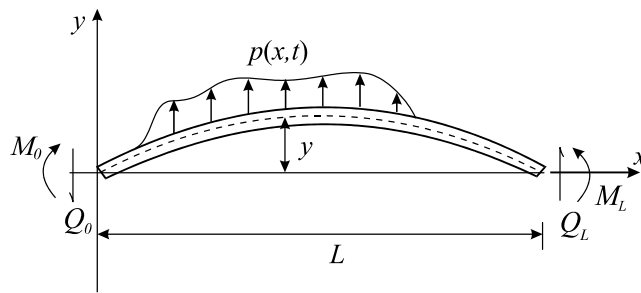


Figure 7.60: Transversely vibrating beam subjected to an arbitrary continuous load $p(x, t)$ along with boundary moments and shears.

59. Solve for the response of the cantilever beam sketched in Figure 7.61.

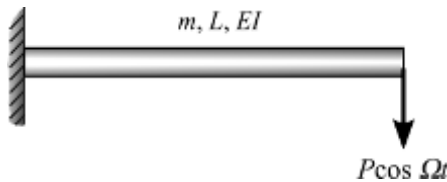


Figure 7.61: Beam with harmonic end load.

- 60. In Example 7.7, solve for the case where $t = \tau_1/4$.
- 61. Find the response of a simply supported beam subjected to the external moment at $x = L$,

$$p(x, t) = M(t) u''(x - L).$$

Problems for Section 7.6 – Other Transverse Beam Vibration Cases

62. Derive the equation of motion for the transversely vibrating beam with axial force by applying Hamilton's principle.

- 63. Solve Equation 7.119 by setting the determinant of the coefficients equal to zero, resulting in a polynomial that is solved for the frequencies. Use the following parameter values: $E = 30 \times 10^6$ lb/in², $I = 10 \times 10^4$ in⁴, $k_1 = 100$ lb/in, $k_2 = 1000$ lb/in, $k_3 = 500$ lb/in, $k_4 = 2000$ lb/in, and $L = 1000$ in.
- 64. Derive Equation 7.120.
- 65. Derive Equation 7.122.
- 66. Discuss the modal participation factor for a vibrating beam that is subjected to loads as shown in Figure 7.62. (Refer to Section 6.6.2 for a discussion of the modal participation factor.)

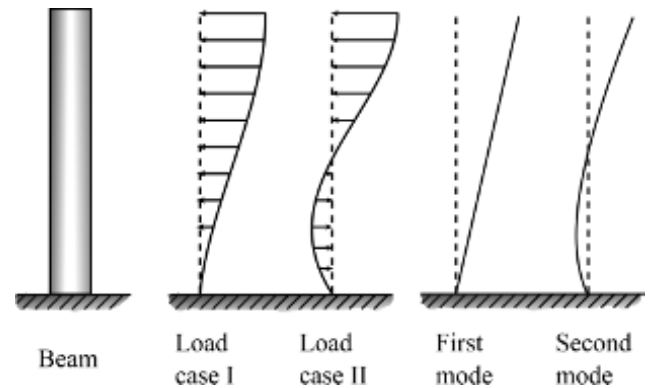


Figure 7.62: Modal participation factor.



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Chapter 8

Continuous Models for Vibration: Advanced Models

“Reality becomes more complex.”

8.1 Vibration of Membranes

In this chapter, the modeling of membranes and plates is introduced. Membranes and plates are two-dimensional extensions of strings and beams, respectively. Similarities to earlier models appear as we derive and solve the equations of motion. We do not consider the damping of membranes and plates.

Membranes have numerous applications, for example, inflatable structures, parachutes, and many biomedical components. Plates are used for foundations, structural panels, and composite structures, among many other mechanical applications.

8.1.1 Rectangular Membranes

Consider a *thin uniform* rectangular membrane under constant tension along the boundary, which coincides with the x and y axes. The transverse displacement w is of interest and it is assumed to be *small*. The governing equation of motion in the w direction is derived with arbitrary initial conditions on displacements and velocities. The procedure is the same as for the string, with a second dimension needed to define a membrane. As was true for strings, there is no moment resistance in membranes.

The free-body diagram of a membrane element of area $dx dy$ is shown in Figure 8.1. Summing the forces in the transverse direction w and substituting these into Newton’s second law of motion gives

$$\begin{aligned} & \left[T dy + \frac{\partial T}{\partial x} dx dy \right] \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right] + p(x, y, t) dx dy \\ & - T \frac{\partial w}{\partial x} dy + \left[T dx + \frac{\partial T}{\partial y} dx dy \right] \left[\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} dy \right] \\ & - T \frac{\partial w}{\partial y} dx = m dx dy \frac{\partial^2 w}{\partial t^2}, \quad (8.1) \end{aligned}$$

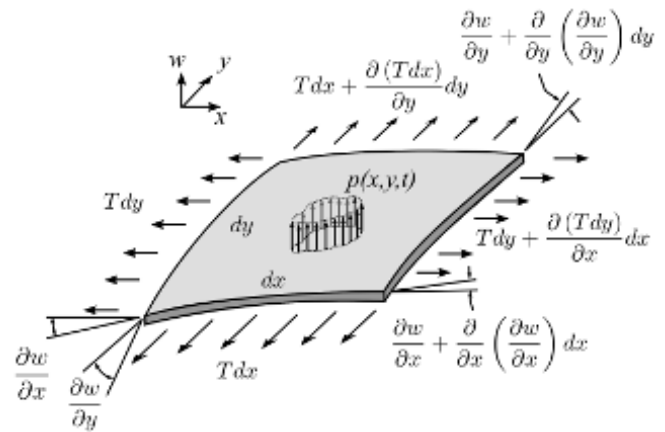


Figure 8.1: Free-body diagram of a rectangular membrane element of area $dx dy$. The sketch shows an external force $p(x, y, t)$, the resisting internal forces, and the slopes of the deflection $w(x, y, t)$, which are out of the plane.

where the left-hand side terms are the forces on the free-body and the right-hand side term equals the membrane element mass times its acceleration. In this derivation m is the mass per unit area, assumed constant, and the assumption of small displacements permits the approximation that sines of angles can be replaced by angles.

Expanding the products, simplifying, and assuming that T is constant, Equation 8.1 becomes

$$\begin{aligned} & T dy \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right] - T dy \frac{\partial w}{\partial x} \\ & + T dx \left[\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} dy \right] - T dx \frac{\partial w}{\partial y} + p(x, y, t) dx dy \\ & = m dx dy \frac{\partial^2 w}{\partial t^2}, \end{aligned}$$

neglecting higher order terms (third-order in dx and dy). Simplifying further leads to the governing equation of

motion,

$$\frac{T}{m} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] + \frac{1}{m} p(x, y, t) = \frac{\partial^2 w}{\partial t^2},$$

which is the *wave equation* in two dimensions. For the free vibration analysis, we define $T/m = c^2$ and set $p(x, y, t) = 0$, to find

$$c^2 \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] = \frac{\partial^2 w}{\partial t^2}.$$

This is the two-dimensional generalization of the equation of motion of the string.

To solve this partial differential equation we assume a product solution,

$$w(x, y, t) = W(x, y)F(t), \quad (8.2)$$

and substitute the appropriate derivatives into the equation of motion, to find

$$c^2 \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right] F = W \ddot{F}.$$

We separate $F(t)$ and its second derivative from $W(x, y)$ and its derivatives,

$$\frac{c^2}{W} \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right] = \frac{\ddot{F}}{F},$$

and since W is a function only of x and y , and F is a function only of t , the only way both sides of the equation can equal each other is if each equals a constant, say $-\omega^2$,

$$\frac{c^2}{W} \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right] = -\omega^2$$

$$\frac{\ddot{F}}{F} = -\omega^2.$$

This results in the partial differential equation governing $W(x, y)$ and the ordinary differential equation governing $F(t)$. The solution for $F(t)$ is a harmonic function in time,

$$F(t) = B_1 \cos \omega t + B_2 \sin \omega t. \quad (8.3)$$

Rewriting the equation governing $W(x, y)$, we have an eigenvalue problem,

$$\nabla^2 W(x, y) + \beta^2 W(x, y) = 0, \quad \text{plus boundary conditions,} \quad (8.4)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\beta^2 = \left(\frac{\omega}{c} \right)^2 = \omega^2 \left(\frac{m}{T} \right).$$

∇ is the *Laplacian* operator called *nabla*¹ or *del*.

To solve the eigenvalue problem, another separation of variables is required. We substitute $W(x, y) = X(x)Y(y)$ and its derivatives into Equation 8.4 to find

$$\frac{X''}{X} + \frac{Y''}{Y} + \beta^2 = 0.$$

As before, we separate the variables that are functions of x from those that are functions of y ,

$$\frac{X''}{X} = -\frac{Y''}{Y} - \beta^2 = -\alpha^2.$$

The negative sign is chosen for the constant α^2 to ensure that the solution will be harmonic in x and y . Two equations must be solved,

$$X'' + \alpha^2 X = 0$$

$$Y'' + \gamma^2 Y = 0,$$

where $\gamma^2 = \beta^2 - \alpha^2$. The solutions to these equations are

$$X(x) = A_1 \sin \alpha x + A_2 \cos \alpha x$$

$$Y(y) = A_3 \sin \gamma y + A_4 \cos \gamma y,$$

and, therefore,

$$W(x, y) = C_1 \sin \alpha x \sin \gamma y + C_2 \sin \alpha x \cos \gamma y$$

$$+ C_3 \cos \alpha x \sin \gamma y + C_4 \cos \alpha x \cos \gamma y. \quad (8.5)$$

The coefficients C_i are determined by satisfying the boundary conditions. Then Equations 8.5 and 8.3 are substituted into Equation 8.2, and B_1 and B_2 are determined by satisfying the initial conditions.

Simply Supported Boundary Conditions

We assume here that the membrane is simply supported along all four sides, and is of dimensions a by b . Of the four components in Equation 8.5, only the first term is nonzero for the following simply supported boundary conditions:

$$W(0, y) = 0, \quad W(a, y) = 0,$$

$$W(x, 0) = 0, \quad W(x, b) = 0.$$

We can demonstrate this finding by examining the conditions at each boundary:

Along $x = 0$,

$$W(0, y) = C_3 \sin \gamma y + C_4 \cos \gamma y = 0,$$

which is true for arbitrary y only if $C_3 = C_4 = 0$.

¹The symbol nabla was adopted by Hamilton. It looks like the frame of a harp (*naval* in Hebrew, *nabla* in Aramaic; from the letters: *nun*, *bet*, *lamed*).

Along $x = a$,

$$W(a, y) = C_1 \sin \alpha a \sin \gamma y + C_2 \sin \alpha a \cos \gamma y = 0,$$

which yields two possible solutions. The first is that $C_1 = C_2 = 0$, which is the trivial solution and not of interest. The other possible solution is $\sin \alpha a = 0$.

Along $y = 0$,

$$W(x, 0) = C_2 \sin \alpha x + C_4 \cos \alpha x = 0,$$

which is true for arbitrary x only if $C_2 = C_4 = 0$.

Along $y = b$,

$$W(x, b) = C_1 \sin \alpha x \sin \gamma b + C_3 \cos \alpha x \sin \gamma b = 0,$$

which also has two possible solutions. The trivial solution is $C_1 = C_3 = 0$. The other possibility is that $\sin \gamma b = 0$.

To summarize, the only solution that can satisfy all the boundary conditions is that $C_1 \neq 0$ along with the following *characteristic equations* and roots,

$$\begin{aligned} \sin \alpha a = 0 &\implies \alpha_j a = j\pi \\ \sin \gamma b = 0 &\implies \gamma_k b = k\pi, \end{aligned}$$

where $j, k = 1, 2, \dots$. Since $\beta^2 = \alpha^2 + \gamma^2$, we can write

$$\beta_{jk} = (\alpha_j^2 + \gamma_k^2)^{1/2} = \pi \left[\left(\frac{j}{a} \right)^2 + \left(\frac{k}{b} \right)^2 \right]^{1/2}$$

and

$$\omega_{jk} = c\beta_{jk} = \sqrt{\frac{T}{m}}\beta_{jk}, \quad j, k = 1, 2, \dots,$$

with the eigenfunctions or modes given by

$$W_{jk}(x, y) = C_{jk} \sin \frac{j\pi x}{a} \sin \frac{k\pi y}{b}, \quad j, k = 1, 2, \dots$$

Two modes are drawn in Figures 8.2 and 8.3 for a square membrane with $a = 3$ and $b = 3$ (both in length units). Figure 8.2 is the mode for $j = k = 1$ and Figure 8.3 is the mode for $j = k = 2$. Note that *two* modes can exist at the *same* frequency; this is a possibility for two-dimensional distributed parameter systems. For the case here $a = b$, $\beta_{jk} = \beta_{kj}$, $\omega_{jk} = \omega_{kj}$, and $W_{jk} = W_{kj}$.

To specify the values of coefficients C_{jk} , we normalize according to the rule,

$$\int_0^a \int_0^b m C_{jk}^2 \sin^2 \left(\frac{j\pi x}{a} \right) \sin^2 \left(\frac{k\pi y}{b} \right) dy dx = 1,$$

with the result that for all j and k ,

$$C_{jk} = \frac{2}{\sqrt{mab}}.$$

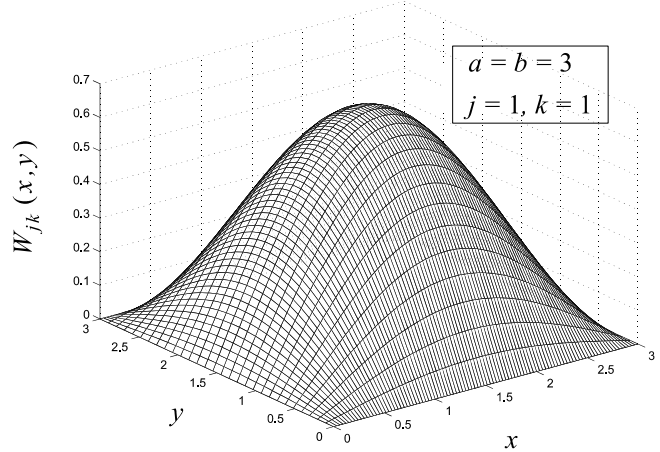


Figure 8.2: Mode $W_{11}(x, y)$ for a square membrane.

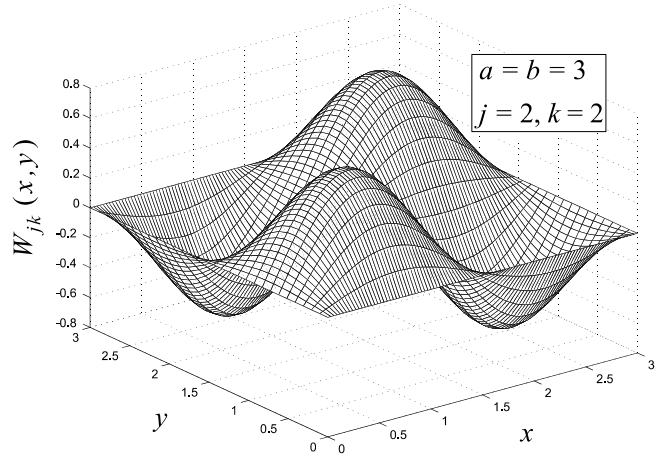


Figure 8.3: Mode $W_{22}(x, y)$ for a square membrane.

The complete solution in space and time can now be recovered for each mode. The function in time needs to be generalized as follows,

$$F_{jk}(t) = B_{jk1} \cos \omega_{jk} t + B_{jk2} \sin \omega_{jk} t. \quad (8.6)$$

For the jk^{th} mode, using $w_{jk}(x, y, t) = W_{jk}(x, y)F_{jk}(t)$,

$$\begin{aligned} w_{jk}(x, y, t) &= \frac{2}{\sqrt{mab}} \sin \frac{j\pi x}{a} \sin \frac{k\pi y}{b} \\ &\quad \times (B_{jk1} \cos \omega_{jk} t + B_{jk2} \sin \omega_{jk} t), \\ &\quad j, k = 1, 2, \dots, \end{aligned} \quad (8.7)$$

where B_{jk1} and B_{jk2} are determined next using the initial conditions. The complete solution is the double sum

$$w(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} w_{jk}(x, y, t).$$

We can satisfy the initial conditions by appropriately choosing the coefficients B_{jk1} and B_{jk2} .

We denote the initial conditions by

$$\begin{aligned} w(x, y, 0) &= d(x, y) \\ \frac{\partial w(x, y, 0)}{\partial t} &= v(x, y), \end{aligned}$$

where $d(x, y)$ is the initial displacement field and $v(x, y)$ is the initial velocity field. Applying the initial displacement field to Equation 8.7, we find the double Fourier series

$$d(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{2}{\sqrt{mab}} B_{jk1} \sin \frac{j\pi x}{a} \sin \frac{k\pi y}{b}, \quad (8.8)$$

where $d(x, y)$ must be continuous and have finite derivatives $\partial d/\partial x$, $\partial d/\partial y$, and $\partial^2 d/\partial x \partial y$.

We examine each part of the double Fourier series. Defining

$$K_j(y) = \sum_{k=1}^{\infty} \frac{2}{\sqrt{mab}} B_{jk1} \sin \frac{k\pi y}{b}, \quad (8.9)$$

we rewrite Equation 8.8 as

$$d(x, y) = \sum_{j=1}^{\infty} K_j(y) \sin \frac{j\pi x}{a}. \quad (8.10)$$

For fixed y , Equation 8.10 is a Fourier sine series with

$$K_j(y) = \frac{2}{a} \int_0^a d(x, y) \sin \frac{j\pi x}{a} dx.$$

Equation 8.9 is also a Fourier sine series with

$$\frac{2}{\sqrt{mab}} B_{jk1} = \frac{2}{b} \int_0^b K_j(y) \sin \frac{k\pi y}{b} dy.$$

Combining expressions and solving for B_{jk1} , we find

$$B_{jk1} = 2\sqrt{\frac{m}{ab}} \int_0^b \int_0^a d(x, y) \sin \frac{j\pi x}{a} \sin \frac{k\pi y}{b} dx dy, \quad j, k = 1, 2, \dots \quad (8.11)$$

Equation 8.11 is sometimes called the *generalized Euler formula*.

Following the same procedure, and applying the initial velocity field, leads to

$$B_{jk2} = \frac{2}{\omega_{jk}} \sqrt{\frac{m}{ab}} \int_0^b \int_0^a v(x, y) \sin \frac{j\pi x}{a} \sin \frac{k\pi y}{b} dx dy, \quad j, k = 1, 2, \dots, \quad (8.12)$$

where this is a frequency-dependent coefficient. Once initial displacement $d(x, y)$ and initial velocity $v(x, y)$ are specified, Equations 8.11 and 8.12 can be evaluated and substituted into Equation 8.7.

8.1.2 Circular Membranes

Circular membranes are found in specialized inflatable structures, solar reflectors, and circular elements of bio-mechanical devices such as heart valves. The free-body diagram of a section of a circular membrane sector is shown in Figure 8.4, where *polar coordinates* are used to define the geometry.

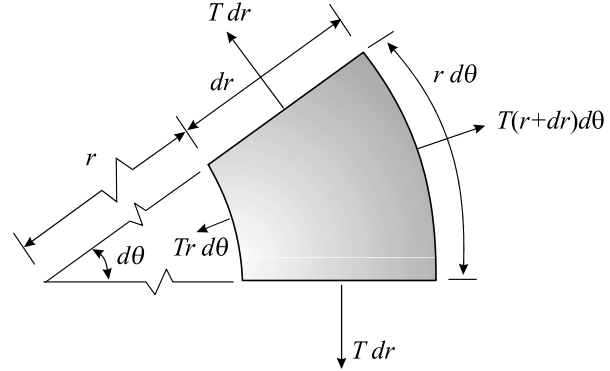


Figure 8.4: Free-body diagram of a section from a sector of a circular membrane. The position is defined by the polar coordinate variables r and θ . The free-body diagram shows the tension forces T acting on the section.

In similarity to the rectangular case, the general solution in polar coordinates can be written as $W(r, \theta)F(t)$. In polar coordinates, the eigenvalue problem is

$$\nabla^2 W(r, \theta) + \beta^2 W(r, \theta) = 0, \quad \text{plus boundary conditions,} \quad (8.13)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2},$$

and $\beta = \omega/c$ is the wave number. Assuming a solution of the form

$$W(r, \theta) = R(r)\Theta(\theta), \quad (8.14)$$

and substituting this along with its respective derivatives into Equation 8.13 yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + \beta^2 R\Theta = 0.$$

Collecting like variables, we find

$$\frac{r^2}{R}(R'' + \frac{1}{r}R' + \beta^2 R) = -\frac{\Theta''}{\Theta} = j^2. \quad (8.15)$$

A positive sign is chosen for the constant j^2 so that Θ has a harmonic solution governed by the equation $\Theta'' + j^2\Theta = 0$, as required by the circular geometry. Also, j must be an integer to satisfy the requirement that $\Theta(\theta) = \Theta(\theta + 2\pi n)$ and $\Theta'(\theta) = \Theta'(\theta + 2\pi n)$, where n is an integer. In this way, continuity and uniqueness are maintained in the circumferential direction around the circular membrane. Thus,

$$\Theta_j(\theta) = A_{1j} \sin j\theta + A_{2j} \cos j\theta, \quad j = 0, 1, 2, \dots, \quad (8.16)$$

where the coefficients A_{1j} and A_{2j} are to be determined.

The equation in $R(r)$,

$$R'' + \frac{1}{r}R' + \left(\beta^2 - \frac{j^2}{r^2}\right)R = 0,$$

can now be solved. This equation has an expansion solution² known as the *Bessel function*,

$$R_j(r) = B_{1j}J_j(\beta r) + B_{2j}Y_j(\beta r), \quad (8.17)$$

where J_j is the Bessel function of order j of the *first kind*, and Y_j is the Bessel function of order j of the *second kind*. From the assumed product solution, Equation 8.14, for each order j , we have

$$\begin{aligned} W_j(r, \theta) &= R_j(r)\Theta_j(\theta) \\ &= C_{1j}J_j(\beta r)\sin j\theta + C_{2j}J_j(\beta r)\cos j\theta \\ &\quad + C_{3j}Y_j(\beta r)\sin j\theta + C_{4j}Y_j(\beta r)\cos j\theta, \\ j &= 0, 1, 2, \dots, \end{aligned} \quad (8.18)$$

where $C_{ij} = A_{ij}B_{ij}$ and there are an infinite number of wavenumber values β_{jk} , as we determine next. The boundary conditions are used to find C_{lj} and β_{jk} .

Fixed Boundary Conditions

Consider the membrane fixed on the rim at $r = a$, that is, $W_j(a, \theta) = 0$. Since there is a finite deflection at the center of the membrane, and since $Y_j(\beta r)$ has the property that $Y_j(0) = \infty$, then for physical reasons it must be that $C_{3j} = C_{4j} = 0$ in Equation 8.18.

The modes of vibration for a circular membrane fixed at the rim are then given by

$$\begin{aligned} W_j(r, \theta) &= C_{1j}J_j(\beta r)\sin j\theta + C_{2j}J_j(\beta r)\cos j\theta, \\ j &= 0, 1, 2, \dots \end{aligned} \quad (8.19)$$

To satisfy the boundary condition at the rim, set $r = a$,

$$W_j(a, \theta) = C_{1j}J_j(\beta a)\sin j\theta + C_{2j}J_j(\beta a)\cos j\theta = 0,$$

for all θ , and $j = 0, 1, 2, \dots$. Since the solution $C_{1j} = C_{2j} = 0$ is trivial, it must be that

$$J_j(\beta a) = 0, \quad j = 0, 1, 2, \dots \quad (8.20)$$

Equation 8.20 specifies the allowable frequencies of free vibration for each order j and represents the characteristic equation for each j for this problem. There are an *infinite number of characteristic equations*, one for each j , and, for each of these characteristic equations, the Bessel function has an *infinite number of roots*, and therefore also frequencies.

² Assume an expansion solution of the general polynomial form: $y = a_0 + a_1x + a_2x^2 + \dots$. Substitute this into the governing equation, and find the power series solution defined as the *Bessel function*.

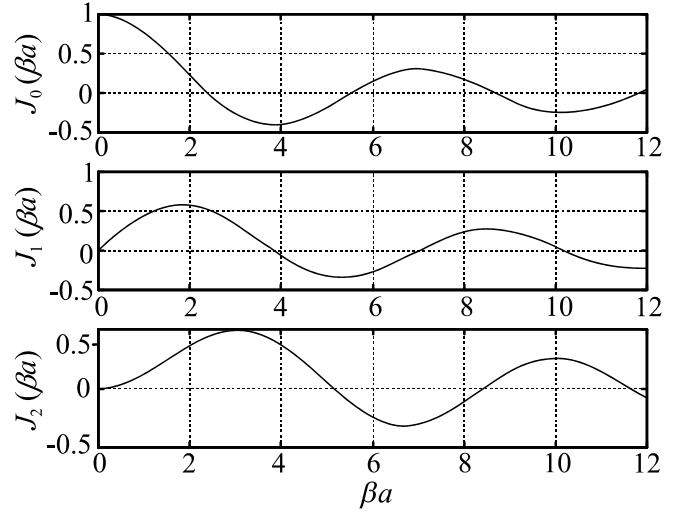


Figure 8.5: Roots of the Bessel function $J_j(\beta a) = 0$, for $j = 0, 1$, and 2 , are the intersections of the curves with the zero horizontal axis.

The first three roots of $J_0(\beta a)$ are 2.405, 5.520, and 8.654; of $J_1(\beta a)$ are 3.832, 7.016, and 10.173; and of $J_2(\beta a)$ are 5.134, 8.417, and 11.620. These roots are shown in the graphs of Figure 8.5. In order to make clear this multiplicity of roots for each j , we use the following notation for β ,

$$\beta_{jk} = \frac{\omega_{jk}}{c}.$$

For each frequency there are two distinct modes, with the exception of the case $j = 0$. For other values of j , two mode shapes correspond to a single frequency. These are known as *degenerate modes*, and for these modes Equation 8.19 becomes

$$W_{1jk}(r, \theta) = C_{1jk}J_j(\beta_{jk}r)\sin j\theta \quad (8.21)$$

$$W_{2jk}(r, \theta) = C_{2jk}J_j(\beta_{jk}r)\cos j\theta, \quad (8.22)$$

$$j, k = 0, 1, 2, \dots$$

For $j = 0$, $\sin j\theta = 0$ and $\cos j\theta = 1$, and, therefore, W_{20k} is simply written as W_{0k} . This reduced case is considered separately when evaluating the coefficients C_{ljk} . The natural modes³ can be normalized according to the rule

$$\int_D mW_{0k}^2 dD = 1,$$

where D is a two-dimensional domain of integration. For $j = 0$,

$$\begin{aligned} \int_0^{2\pi} \int_0^a mC_{0k}^2 J_0^2\left(\frac{\omega_{0k}}{c}r\right) r dr d\theta &= 1 \\ \Rightarrow C_{0k}^2 &= \frac{1}{\pi m a^2 J_1^2(\omega_{0k}a/c)}. \end{aligned} \quad (8.23)$$

³ **Analytical Methods in Vibrations**, L. Meirovitch, MacMillan, 1967, pp.176-177.

For all other j , the natural modes are degenerate,

$$\int_0^{2\pi} \int_0^a m C_{1jk}^2 J_j^2 \left(\frac{\omega_{jk}}{c} r \right) \sin^2(j\theta) r dr d\theta = 1$$

$$\Rightarrow C_{1jk}^2 = \frac{2}{\pi m a^2 J_{j+1}^2(\omega_{jk} a/c)}, \quad (8.24)$$

$$\int_0^{2\pi} \int_0^a m C_{2jk}^2 J_j^2 \left(\frac{\omega_{jk}}{c} r \right) \cos^2(j\theta) r dr d\theta = 1$$

$$\Rightarrow C_{2jk}^2 = \frac{2}{\pi m a^2 J_{j+1}^2(\omega_{jk} a/c)}. \quad (8.25)$$

The normal modes are, therefore,

$$\hat{W}_{0k}(r, \theta) = \frac{1}{\sqrt{\pi m a} J_1(\omega_{0k} a/c)} J_0 \left(\frac{\omega_{0k}}{c} r \right)$$

$$\hat{W}_{1jk}(r, \theta) = \frac{\sqrt{2}}{\sqrt{\pi m a} J_{j+1}(\omega_{jk} a/c)} J_j \left(\frac{\omega_{jk}}{c} r \right) \sin j\theta$$

$$\hat{W}_{2jk}(r, \theta) = \frac{\sqrt{2}}{\sqrt{\pi m a} J_{j+1}(\omega_{jk} a/c)} J_j \left(\frac{\omega_{jk}}{c} r \right) \cos j\theta,$$

$$j, k = 1, 2, \dots,$$

where the notation $\hat{W}$ signifies that the mode has been normalized. Several modal amplitudes are depicted in [Figures 8.6 to 8.8](#).

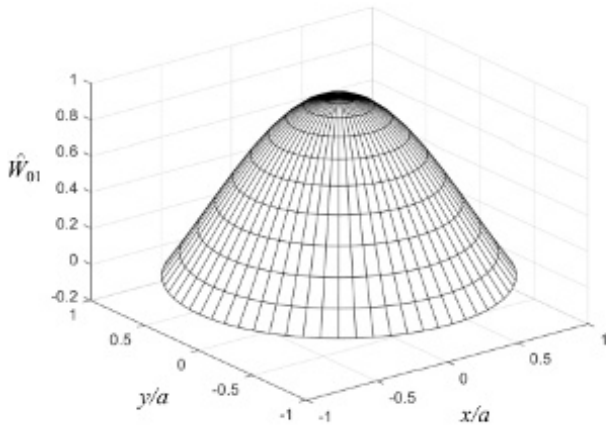


Figure 8.6: Mode shape $\hat{W}_{01}(r, \theta)$ of a circular membrane for $j = 0, k = 1$.

Nodal lines of zero displacement are given by circles of constant radius r and by diametrical lines of constant angle θ .

[Equations 8.21](#) and [8.22](#) can now be written as the sum

$$W_{jk}(r, \theta) = \begin{cases} W_{0k} & \text{for } j = 0 \\ W_{1jk} + W_{2jk} & \text{for } j, k = 1, 2, 3, \dots, \end{cases}$$

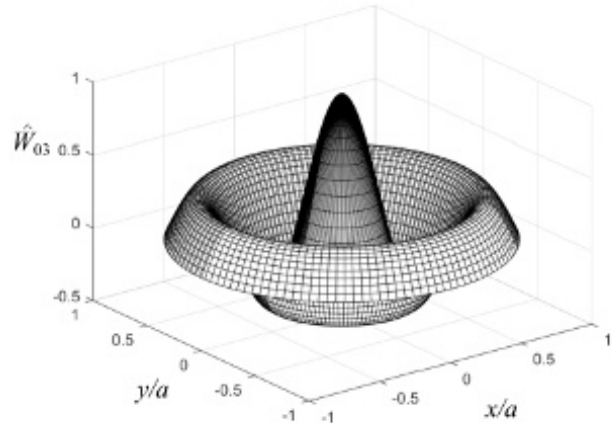


Figure 8.7: Mode shape $\hat{W}_{03}(r, \theta)$ of a circular membrane for $j = 0, k = 3$.

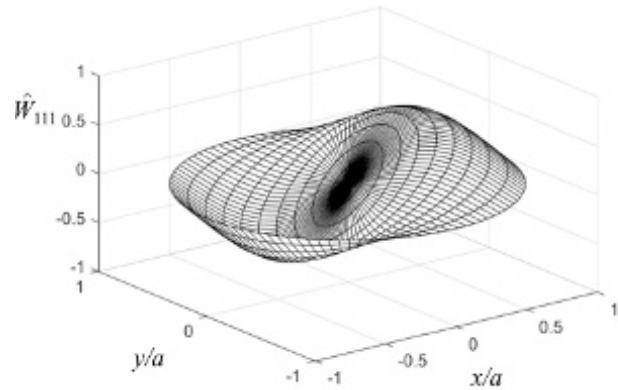


Figure 8.8: Mode shape $\hat{W}_{111}(r, \theta)$ of a circular membrane for $j = 1, k = 1$.

with the complete response given by

$$w(r, \theta, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} w_{jk}(r, \theta, t),$$

$$\text{where } w_{jk}(r, \theta, t) = W_{jk}(r, \theta) F_{jk}(t),$$

and $F_{jk}(t)$ is a harmonic function such as [Equation 8.6](#) with arbitrary coefficients that can be established by satisfying the initial conditions.

FRIEDRICH WILHELM BESSEL

(July 22, 1784 – March 17, 1846)



Contribution: Bessel was a German mathematician and astronomer who made significant contributions to mathematics. He invented cylindrical functions, now known as Bessel functions, which he used in his studies of planetary perturbations. Bessel functions appear as coefficients in the series expansion of the indirect perturbation of a planet. (Jacob Bernoulli, Daniel Bernoulli, Euler, and Lagrange had studied elliptical orbits before Bessel.)

Bessel was a pioneer in precision astronomy. He improved knowledge of the positions of stars and planets, providing a reference system for their positions. He derived an invaluable catalog of 3222 stars, which he published in 1818.

Bessel studied nutation, precision, aberration, and refraction. Between 1821 and 1833, he undertook the monumental task of determining the positions and proper motions of over 50,000 stars. This led to his discovery of a parallax figure for the “Flying Star” 61 Cygni in 1838. Bessel determined a parallax value of 0.314 arcseconds, “corresponding to a distance of about 10 light years. The modern parallax value of 61 Cygni is 0.292 arcseconds.”

In 1841, Bessel announced that Sirius and Procyon, two of Maskelyne’s 36 fundamental stars, each had an invisible companion because of variations in their proper motion. Thus, he was the first to predict the existence of “dark stars.”

Bessel impacted the study of physics by correcting the seconds pendulum, a pendulum with a period of exactly two seconds. He directed geodetical measurements of meridian arcs in East Prussia and derived the shape of Earth as an oblate spheroid with ellipticity $1/299$. This eventually led to the introduction of a new Prussian measurement system. Bessel also founded the potential theory.

Life Sketch: Bessel was born in Minden, Westphalia (now Germany). His father was a civil servant and his mother was the daughter of a pastor. Bessel attended the Gymnasium in Minden for four years, but he lacked talent and struggled to master Latin. (He later taught himself Latin suggesting that the Gymnasium failed to inspire him.)

At the age of 14, Bessel left school and entered into a 7-

year apprenticeship contract with an import-export business. At first, Bessel received no wages, but as his accounting skills developed he received a small salary. Interest in overseas business activities led Bessel to spend his evenings studying geography, navigation, astronomy, mathematics, Spanish, and English. In particular, the company’s reliance on cargo ships led him to problems in navigation. This in turn led to an interest in astronomy as a way of determining longitude.

In 1804, Bessel wrote a paper on Halley’s Comet and calculated the orbit using data from observations made in 1607. Bessel continued to make further observations and concentrated his efforts on astronomy, celestial mechanics, and mathematics.

In 1806, Bessel accepted a post at the Lilienthal Observatory, a private observatory near Bremen. After considerable thought Bessel left the affluence that was guaranteed in his commercial job and chose instead the near poverty of the Observatory post. However, the Lilienthal Observatory gave him valuable experience observing planets, in particular Saturn, its rings and satellites. He also observed comets and continued his study of celestial mechanics. In 1807, he began to work on reducing James Bradley’s Greenwich observations of the positions of 3222 stars made around 1750.

Bessel’s impressive work was quickly recognized, and he was made offers at different posts. In 1809 Bessel accepted an invitation by Frederick William III of Prussia to serve as director of the new Königsberg Observatory and be a professor of astronomy. Thus, at the young age of 26, Bessel became the director. However, it was not possible for Bessel to receive a professorship without first being granted the title of doctor. A doctorate was awarded by the University of Göttingen based on the recommendation of Gauss, who had met Bessel in Bremen in 1807 and recognized his talents.

Although the Observatory at Königsberg was still under construction, Bessel took up his new post in May 1810. He continued to work on Bradley’s observations until construction ended in 1813.

By this time, Bessel’s work had gained international fame. Bessel had pinned down the position of over 50,000 stars during his time at Königsberg. He was awarded the honorary Lalande Prize from the Institut de France for his tables of refraction, which were based on Bradley’s observations. And in 1812, he was elected to the Berlin Academy.

On completion of the Observatory in 1813 Bessel continued to pursue research and teaching. He declined the directorship of the Berlin Observatory out of fear of greater administrative and social responsibilities.

Bessel worked out a method of mathematical analysis involving what is now known as the Bessel function. He introduced this in 1817 in his study of the problem of determining the motion of three bodies moving under mutual gravitation.

In 1830 Bessel published the mean and apparent positions of 38 stars over the 100-year period 1750-1850. These 38 stars were the 36 “fundamental stars” of Maskelyne together with two further polar stars. From periodic variations in the

proper motions of Sirius and Procyon, two of Maskelyne's 36 fundamental stars, Bessel deduced that they had companion stars in orbit that had not been observed. He announced that Sirius had a companion in 1841, thus being the first to predict the existence of "dark stars." Ten years later the orbit of the companion was computed and it was observed in 1862.

Bessel was the first to use parallax in calculating the distance to a star. Astronomers had believed for some time that parallax would provide the first accurate measurement of interstellar distances – in fact, in the 1830s there was a fierce competition between astronomers to be the first to measure a stellar parallax accurately. In 1838 Bessel won the race, announcing that 61 Cygni had a parallax of 0.314 arcseconds; which, given the diameter of the Earth's orbit, indicated that the star was about 3 parsecs (9.8 light years) away.

Bessel's precise measurements allowed him to notice deviations in the motions of Sirius and Procyon, which he deduced must be caused by the gravitational attraction of unseen companions. His announcement of Sirius's "dark companion" in 1844 was the first correct claim of a previously unobserved companion by positional measurement, and eventually led to the discovery of Sirius B.

Despite his many successes, his personal life had its tragedies and challenges. His happy marriage in 1812 was darkened by the early death of both of his sons and an ongoing illness.

In 1842 he traveled to England and participated in the Congress of the British Association in Manchester. There, he met with English scientists who inspired him to finish and publish his important work.

He died of cancer in Königsberg, Prussia (now Kaliningrad, Russia) at age 61.

8.2 Vibration of Plates

We next develop the equations governing the dynamic behavior of plates.⁴ Plates are useful models not only for structural components, but also for internal combustion engine components such as pistons and cylinder heads, diaphragms used in artificial hearts, pumping devices, and lenses used in optical devices.

We begin by looking at the most basic plate model, one of rectangular cross section and uniform thickness.

⁴An advanced book on the vibration of plates is **The Bending and Stretching of Plates**, E.H. Mansfield, Cambridge University Press, 2nd ed., 1989. It includes studies of plates of various shapes, variable rigidity problems, approximate methods, and large-deflection theory. Another book, **Vibrations of Elastic Plates**, Y.-Y. Yu, Springer-Verlag, 1996, provides an introduction to linear and nonlinear dynamical modeling of sandwich plates, laminated composites, and piezoelectric layered plates. Two recommended monographs on plates and shells are the **Theory of Plates and Shells**, S. Timoshenko and S. Woinowsky-Krieger, McGraw-Hill, 1959, and **Vibrations of Plates and Shells**, W. Soedel, Marcel Dekker, 1981.

Plates are more complex than membranes because plates can resist bending moments. We encountered an analogous situation when considering the restoring forces of the string in contrast to the moment resistance of a beam.

The following assumptions are made so that a simplified model can be developed:

1. plate thickness h is small when compared with other dimensions (a rule of thumb is a ratio of less than 1:10),
2. there is no strain along the middle surface during bending,
3. plane sections do not warp and therefore remain plane,
4. normal stresses in the direction transverse to the plate can be ignored due to the small relative thickness,
5. the load is normal to the surface,
6. deflections are small,
7. the influences of shear and rotary inertia are neglected,⁵ and
8. damping is neglected.

8.2.1 Rectangular Plates

For a rectangular plate, we consider the free-body diagrams of Figure 8.9 for load and shear forces and Figure 8.10 for moments. We take the positive coordinate direction as those of the axes in the figures, and use the right-hand rule for positive moments. Our development makes use of results from solid mechanics.⁶

For the plate, the net shear forces per unit length are given by

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz, \quad Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz,$$

where Q_x is the net shear force per unit length on the face with a perpendicular in the x direction, and similarly for Q_y . Next, from Newton's second law of motion, we set the sum of the forces in the z direction (perpendicular to the xy plane of the plate) equal to the mass of the element times its acceleration in the z direction,

$$\left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx - Q_y dx + \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy - Q_x dy + p(x, y, t) dx dy = \rho h dx dy \frac{\partial^2 w}{\partial t^2},$$

⁵More general problems with fewer assumptions require the use of approximate and computational models.

⁶For a reference, see, for example, **Foundations of Solid Mechanics**, Y.C. Fung, Prentice Hall, 1965, and **Applied Elasticity**, C.-T. Wang, McGraw-Hill, 1953.

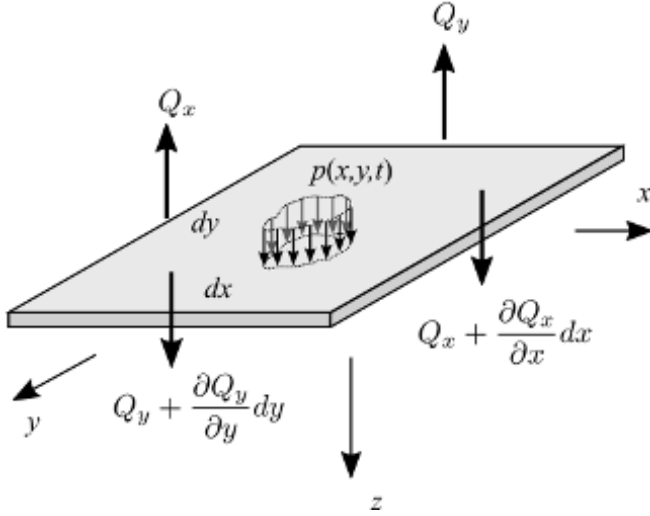


Figure 8.9: Free-body diagram for a rectangular plate element showing only external force $p(x, y, t)$ and shear forces. The figure shows the middle plane of the plate.

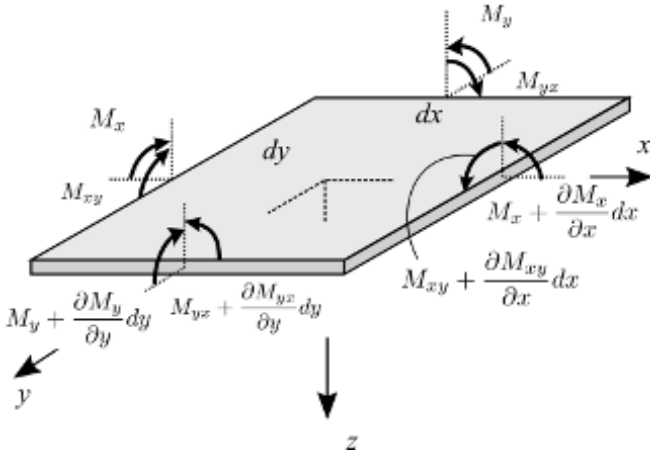


Figure 8.10: Free-body diagram for a rectangular plate element showing only resisting moments. The figure shows the middle plane of the plate.

or, simplifying,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p(x, y, t) = \rho h \frac{\partial^2 w}{\partial t^2}, \quad (8.26)$$

where $w(x, y, t)$ is the deflection of the middle plane⁷ in the z direction, $p(x, y, t)$ is the load intensity per unit area, ρ is the density, and ρh is the mass per unit area.

The Newton-Euler equation is applied next to the sum of the moments about the x and y axes. The notation M_y , for example, indicates the bending moment due to normal stress σ_y , and M_{xy} indicates a twisting moment due to shear stress τ_{xy} . The intermediate equation shown next represents the sum of the moments about the x axis,

neglecting rotary inertia,⁸

$$\begin{aligned} M_y dx - \left(M_y + \frac{\partial M_y}{\partial y} dy \right) dx - M_{xy} dy \\ + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \right) dy + Q_y dx \frac{dy}{2} \\ + \left(Q_y dx + \frac{\partial Q_y}{\partial y} dx dy \right) \frac{dy}{2} = 0, \end{aligned}$$

which becomes

$$-\frac{\partial M_y}{\partial y} dy dx + \frac{\partial M_{xy}}{\partial x} dx dy + Q_y dx dy = 0.$$

Higher order products such as $dx(dy)^2$ have been neglected based on our earlier assumptions. For example, the term

$$\frac{\partial Q_y}{\partial y} dx \frac{(dy)^2}{2}$$

has been neglected because it is much smaller than the other terms in the equation.

The same procedure is used for the moments about the y axis. The moment equations about the x and y axes are then, respectively,

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \quad (8.27)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0. \quad (8.28)$$

We perform the appropriate differentiations of [Equations 8.27](#) and [8.28](#) in order to obtain expressions for $\partial Q_x / \partial x$ and $\partial Q_y / \partial y$, which are then substituted into [Equation 8.26](#), resulting in

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = -p(x, y, t) + \rho h \frac{\partial^2 w}{\partial t^2}, \quad (8.29)$$

where $M_{xy} = -M_{yx}$.

The moments are related to the plate deflection w . To see this, let u and v be components of displacement at any point in the plate parallel to the x, y axis,

$$u = -z \frac{\partial w}{\partial x}, \quad v = -z \frac{\partial w}{\partial y}.$$

Figure 8.11 shows a schematic of the displacement u .

The strains are related to the displacements in the

⁷Note that w is not a function of z , but only of x, y , and time.

⁸Details of this derivation can be found in the book **Mechanics of Elastic Structures**, J.T. Oden and E.A. Ripperger, McGraw-Hill, 1981.

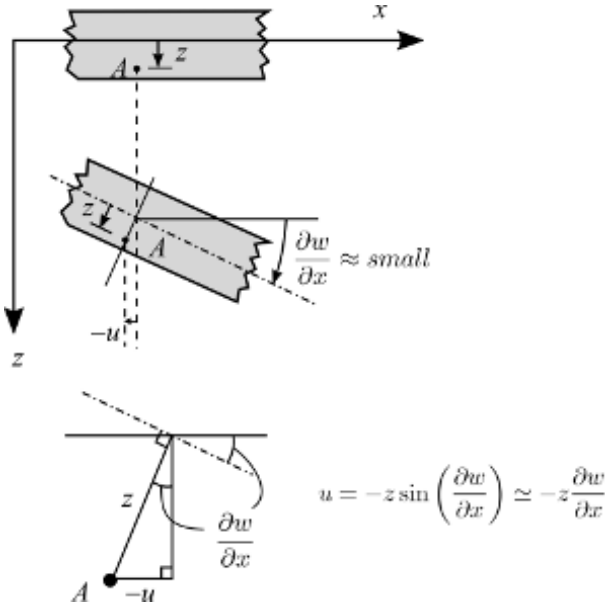


Figure 8.11: Schematic of strain-displacement for a plate. The top figure shows the displacement of an arbitrary point A , when the plate is in an undeformed position on the x axis, and then in its deformed state. The bottom figure shows the geometry.

following way,

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2} \\ \varepsilon_y &= \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ &= -2z \frac{\partial^2 w}{\partial x \partial y}.\end{aligned}$$

Hooke's law for plane stress relates these strains to the stress resultants,

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = -\frac{Ez}{1-\nu^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ \sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = -\frac{Ez}{1-\nu^2} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ \tau_{xy} &= G\gamma_{xy} = -2Gz \frac{\partial^2 w}{\partial x \partial y} \\ &= -\frac{Ez}{1+\nu} \frac{\partial^2 w}{\partial x \partial y},\end{aligned}$$

where ν is Poisson's ratio, E is Young's modulus of elasticity, and G is the shear modulus.

Each stress resultant is multiplied by its respective mo-

ment arm, yielding the following moments,

$$M_x = \int_{-h/2}^{h/2} z \sigma_x dz = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (8.30)$$

$$M_y = \int_{-h/2}^{h/2} z \sigma_y dz = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (8.31)$$

$$\begin{aligned}M_{xy} &= - \int_{-h/2}^{h/2} z \tau_{xy} dz = -M_{yx} \\ &= D(1-\nu) \frac{\partial^2 w}{\partial x \partial y},\end{aligned} \quad (8.32)$$

where $D = Eh^3/12(1-\nu^2)$ is called the *flexural rigidity* of the plate. Equations 8.30 to 8.32 relate the moments to the deflection w . Substituting the appropriate derivatives of these moments into Equation 8.29 we find⁹

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = p(x, y, t), \quad (8.33)$$

or

$$D \nabla^4 w + \rho h \ddot{w} = p(x, y, t), \quad (8.34)$$

where ∇^4 is called the *biharmonic* operator,

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

Equation 8.33 or 8.34 is the equation of motion for the transverse vibration of a plate given the stated assumptions.

8.2.2 Eigenvalue Problem

The eigenvalue problem is obtained by setting $p(x, y, t) = 0$, and then assuming the solution to be separable in space and time,

$$w(x, y, t) = W(x, y)F(t).$$

Substituting the assumed solution and its derivatives into Equation 8.34, we find

$$DF(t) \nabla^4 W(x, y) + \rho h W(x, y) \ddot{F}(t) = 0,$$

or

$$\frac{D}{\rho h} \frac{\nabla^4 W}{W} = -\frac{\ddot{F}}{F} = \omega^2 = \text{a constant}.$$

⁹The derivation uses the following:

$$\begin{aligned}\frac{\partial^2 M_x}{\partial x^2} &= -D \left(\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) \\ \frac{\partial^2 M_y}{\partial y^2} &= -D \left(\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) \\ -2 \frac{\partial^2 M_{xy}}{\partial x \partial y} &= -2D(1-\nu) \frac{\partial^4 w}{\partial x^2 \partial y^2}.\end{aligned}$$

The eigenvalue problem is an equation in W ,

$$\nabla^4 W - \beta^4 W = 0, \quad (8.35)$$

where $\beta^4 = \omega^2 \rho h / D$. Equation 8.35 can be solved by recognizing its equivalent factored form,

$$(\nabla^2 + \beta^2)(\nabla^2 - \beta^2)W = 0.$$

The two possible solutions to this equation, W_1 and W_2 , that satisfy the equality to zero, are, respectively,

$$\begin{aligned} (\nabla^2 + \beta^2)W_1 &= 0 \\ (\nabla^2 - \beta^2)W_2 &= 0. \end{aligned} \quad (8.36)$$

The complete solution by linear superposition is

$$W(x, y) = W_1(x, y) + W_2(x, y).$$

The solution W_1 is the same as that for the membrane, as is the solution W_2 if Equation 8.36 is written as

$$(\nabla^2 + (i\beta)^2)W_2 = 0.$$

Equation 8.5 is the solution for W_1 , reproduced here,

$$\begin{aligned} W_1(x, y) &= C_1 \sin \alpha x \sin \gamma y + C_2 \sin \alpha x \cos \gamma y \\ &+ C_3 \cos \alpha x \sin \gamma y + C_4 \cos \alpha x \cos \gamma y. \end{aligned} \quad (8.37)$$

Similarly,¹⁰

$$\begin{aligned} W_2(x, y) &= C_5 \sinh \alpha_1 x \sinh \gamma_1 y + C_6 \sinh \alpha_1 x \cosh \gamma_1 y \\ &+ C_7 \cosh \alpha_1 x \sinh \gamma_1 y + C_8 \cosh \alpha_1 x \cosh \gamma_1 y. \end{aligned} \quad (8.38)$$

From the membrane solutions, recall that $\beta^2 = \alpha^2 + \gamma^2$ and $\beta_1^2 = \alpha_1^2 + \gamma_1^2$.

Simply Supported Square Plate In order to proceed analytically,¹¹ we consider a simply supported plate of dimension a by b . In this case, all terms in Equations 8.37 and 8.38 must equal zero except for the term $C_1 \sin \alpha x \sin \gamma y$. None of the other terms can satisfy the conditions of no moment resistance and no deflection at all boundaries. Therefore,

$$W(x, y) = C_1 \sin \alpha x \sin \gamma y. \quad (8.39)$$

The set of displacement boundary conditions requires that

$$W(0, y) = W(a, y) = W(x, 0) = W(x, b) = 0.$$

¹⁰We use the identities $\sin(ix) = i \sinh x$ and $\cos(ix) = \cosh x$ in Equation 8.37 and incorporate any imaginary numbers into the arbitrary constants.

¹¹More complex problems can, of course, be solved using approximate and numerical procedures.

By Equation 8.39, this leads to the conditions on αa and γb ,

$$\begin{aligned} \sin \alpha a = 0 &\implies \alpha_j a = j\pi \\ \sin \gamma b = 0 &\implies \gamma_k b = k\pi, \quad j, k = 1, 2, \dots \end{aligned}$$

Therefore, the natural frequencies are

$$\begin{aligned} \omega_{jk} &= \beta_{jk}^2 \sqrt{\frac{D}{\rho h}} = (\alpha_j^2 + \gamma_k^2) \sqrt{\frac{D}{\rho h}} \\ &= \pi^2 \left[\left(\frac{j}{a}\right)^2 + \left(\frac{k}{b}\right)^2 \right] \sqrt{\frac{D}{\rho h}}, \end{aligned}$$

and the modes are given by

$$W_{jk}(x, y) = C_{jk} \sin \alpha_j x \sin \gamma_k y,$$

or

$$W_{jk}(x, y) = C_{jk} \sin \frac{j\pi x}{a} \sin \frac{k\pi y}{b}, \quad (8.40)$$

where, by using the usual normalization procedure, $C_{jk} = 2/\sqrt{\rho h a b}$. The complete solution is the double sum

$$w(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \hat{W}_{jk}(x, y) F_{jk}(t), \quad (8.41)$$

where $\hat{W}_{jk}(x, y)$ are the normalized modes, and the remainder of the solution follows identically to that for the membrane.

Example 8.1 Forced, Simply Supported Rectangular Plate

Find the response of a uniform, simply supported plate¹² that is loaded at one point (x_1, y_1) with the harmonic loading function

$$P(x, y, t) = \sin \Omega t \delta(x - x_1, y - y_1),$$

where the δ is the Dirac delta function, defined as

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x_1, y - y_1) dx dy \\ &= \begin{cases} 1 & \text{if } x = x_1 \text{ and } y = y_1 \\ 0 & \text{if } x \neq x_1 \text{ and/or } y \neq y_1. \end{cases} \end{aligned}$$

Solution: From the previous formulation, the normal modes $\hat{W}_{jk}(x, y)$ and frequencies ω_{jk} are known. Assuming zero initial conditions, the response is given by Equation 8.41, where

$$\begin{aligned} &F_{jk}(t) \\ &= \int_0^t \left[\int_0^a \int_0^b P(\zeta, \eta, \tau) \hat{W}_{jk}(\zeta, \eta) d\zeta d\eta \right] g_{jk}(t - \tau) d\tau. \end{aligned}$$

¹²See Example 7.6 for the analogous beam problem.

When substituting the two-dimensional Kronecker delta function for the force $P(x, y, t)$, the double integral is reduced to a single value at $x = x_1$ and $y = y_1$, with all other values equal to zero,

$$F_{jk}(t) = \int_0^t \hat{W}_{jk}(x_1, y_1) g_{jk}(t - \tau) d\tau \\ = \int_0^t \frac{2}{\sqrt{\rho h a b}} \sin \Omega \tau \sin \frac{j \pi x_1}{a} \sin \frac{k \pi y_1}{b} g_{jk}(t - \tau) d\tau.$$

The function $g_{jk}(t)$ is the impulse response function, which for an undamped oscillator is

$$g_{jk}(t) = \frac{1}{\omega_{jk}} \sin \omega_{jk} t, \quad t \geq 0.$$

The complete solution is then

$$w(x, y, t) = \frac{4}{\rho h a b} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\omega_{jk}} \sin \frac{j \pi x_1}{a} \sin \frac{k \pi y_1}{b} \\ \times \sin \frac{j \pi x}{a} \sin \frac{k \pi y}{b} \int_0^t \sin \Omega \tau \sin \omega_{jk}(t - \tau) d\tau.$$

Note that for certain values of x_1 and y_1 there are modes that cannot be excited by the given force. These are the modes where $j x_1/a$ and $k y_1/b$ are integers and, therefore, sine equals zero. If the loading function is more complicated, a more intricate algebraic procedure is needed to evaluate $F_{jk}(t)$. ■

Problems with circular plates¹³ can be approached using the above techniques. The algebra can become more complicated due to the use of polar coordinates.

MARIE-SOPHIE GERMAIN

(April 1, 1776 – June 26, 1831)



Contribution: Sophie Germain was a French mathematician who made important contributions to the fields of differential geometry and number theory.

One of Germain's major discoveries in number theory was the following theorem: if x, y , and z are integers, and $x^5 + y^5 = z^5$ then either x, y , or z has to be divisible by five. This proof, which she first described in a letter to Gauss, became quite significant as it restricted the possible solutions to Fermat's Last Theorem.

Another discovery was what is called today the *Sophie Germain Prime*, which is a prime number p where $2p + 1$ is also prime. One of her most famous identities, commonly known as *Sophie Germain's Identity*, states that for any two numbers x and y :

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy) \cdot (x^2 + 2y^2 - 2xy).$$

Life Sketch: Sophie Germain was born in Paris, France. She was the second of three daughters of a wealthy French silk merchant. Sophie lived an isolated existence at her family home and was supported financially by her father and his estate until her death. Sophie's home was a meeting place for those interested in liberal reforms and she was exposed to political and philosophical discussions during her early years.

At age 13, she read about Archimedes in a book in her father's library. The book included the legend that during the Roman invasion of Syracuse, Archimedes was so engrossed in his mathematics that he ignored a Roman soldier who questioned him. The soldier speared him to death without comprehending his fame. The story touched her deeply. If Archimedes could be so consumed by mathematics as to not realize somebody was about to take his life, it must be incredibly interesting.

Sophie wanted to become a mathematician, even though this was a profession and role reserved for men. As a woman, Germain was barred from formal education in mathematics and struggled to learn the subject from books and via correspondences with leading mathematicians.

Germain was particularly drawn to Lagrange's teachings. She submitted letters and papers to him under the pseudonym "Antoine-August Le Blanc" or "Monsieur Le Blanc," one of his former students. Lagrange was so impressed that he asked to meet with Le Blanc. Germain admitted her identity to him.

She corresponded with Gauss, again using a pseudonym. Between 1804 and 1809 she wrote him a dozen letters, initially adopting the pseudonym "M. LeBlanc." In his correspondence, Gauss gave her number theory proofs high praise, an evaluation he repeated in letters to his colleagues.

Germain's true identity was revealed to Gauss only after the 1806 French occupation of his hometown of Braunschweig (Brunswick). She worried that Gauss would meet the same fate as that of Archimedes. Fearing for his safety, she contacted a French commander who was a friend of her family to personally ensure Gauss's safety. When Gauss was told that

¹³ See the comprehensive book **Theory and Analysis of Elastic Plates**, J.N. Reddy, Taylor & Francis, 1999.

Germain had requested his protection, he was puzzled as had never heard of her. For a second time – this time to Gauss – Germain revealed her identity. When Gauss learned that the intervention was due to Germain, who he knew only as “M. LeBlanc,” he gave her even more praise.

He sent her a letter of admiration and they corresponded for years. In 1808 Gauss was appointed professor of astronomy at the University of Göttingen. His interest shifted to applied mathematics, and he stopped replying to her letters.

In 1809, the French Academy of Sciences announced a competition to explain the intricate patterns observed on the surfaces of vibrating elastic plates. Germain anonymously entered the contest in 1811. She kept working on her analysis to explain the underlying mathematical laws of a vibrating plate. She submitted and refined her solutions multiple times before being awarded the prize in 1816 by the French Academy of Sciences. To avoid revealing her identity, she did not attend the ceremony. Her analysis of surface vibrations, though imperfect (her lack of formal education was evident in the anonymous paper she submitted), was a visionary effort made all the more extraordinary by her circumstances.

When Sophie was in her early 50s, she was diagnosed with breast cancer. She died at the age of 55 right before she was to be awarded an honorary doctorate and meeting with Gauss.

Noteworthy: Sophie Germain was allowed to attend sessions at the French Academy of Sciences, the first woman with this privilege.

Gauss had lobbied to have an honorary doctorate awarded to Germain by Göttingen University, but she died before it was awarded.

8.3 Approximate Methods

In [Section 6.15](#) the approximate method known as Rayleigh’s quotient was examined. This method is also applicable to continuous systems. Two methods are introduced in this section: the more general *Rayleigh-Ritz* method, which is an extension of the Rayleigh quotient, and the *Galerkin* method, one of the most general approximate methods. Many computational approaches are based on these methods.

We use a beam problem to demonstrate the methods.

8.3.1 Rayleigh’s Quotient

For the discrete systems of [Section 6.15](#), Rayleigh’s quotient was derived beginning with the eigenvalue problem. The same approach is used here for the continuous parameter problem. The procedure is demonstrated for the transverse vibration of a beam, beginning with the eigenvalue equation, Equation 7.100, where the orthogonality of the normal modes was already demonstrated.¹⁴

For Rayleigh’s quotient, we work with one mode, letting $r = s$,

$$\begin{aligned} & \int_0^L Y_r(x) \frac{d^2}{dx^2} \left[EI(x) \frac{d^2 Y_r}{dx^2} \right] dx \\ &= \omega_r^2 \int_0^L m(x) [Y_r(x)]^2 dx, \end{aligned}$$

and solving for ω_r^2 , we find

$$\omega_r^2 = \frac{\int_0^L Y_r(x) \frac{d^2}{dx^2} \left[EI(x) \frac{d^2 Y_r}{dx^2} \right] dx}{\int_0^L m(x) [Y_r(x)]^2 dx}. \quad (8.42)$$

This expression yields the exact frequency if the actual eigenfunction $Y_r(x)$ is used. When an approximate function is used in place of the eigenfunction, [Equation 8.42](#) is called Rayleigh’s quotient and yields an approximate value for ω_r . The numerator is proportional to the strain energy and the denominator to the kinetic energy of the mode if harmonic motion is assumed.¹⁵

Usually, we can only guess the eigenfunction $Y_r(x)$. If $\mathcal{Y}_r(x)$ represents that guess, Rayleigh’s quotient is then

$$\Omega_r^2 = \frac{\int_0^L \mathcal{Y}_r(x) \frac{d^2}{dx^2} \left[EI(x) \frac{d^2 \mathcal{Y}_r}{dx^2} \right] dx}{\int_0^L m(x) [\mathcal{Y}_r(x)]^2 dx}. \quad (8.43)$$

The better the guess, the closer (from above) the approximate frequency will be to the actual value. For the first mode, we can obtain a reasonable approximation of ω_1 . For the cantilever beam, for the first eigenfunction $Y_1(x)$, we guess the function $\mathcal{Y}_1(x)$ that equals zero at $x = 0$, $\mathcal{Y}_1(0) = 0$. At the free end, the deflection and slope must not equal zero, and the guess $\mathcal{Y}_1(x)$ must satisfy $\mathcal{Y}_1(L) \neq 0$ and $\mathcal{Y}_1'(L) \neq 0$. Several possibilities for the eigenfunction are possible, such as the simple one

$$\mathcal{Y}_1(x) = a \left(1 - \cos \frac{\pi x}{2L} \right),$$

where a is a constant. The 2 in the denominator of the argument of the cosine guarantees appropriate values for $\mathcal{Y}_1(x)$ at the boundaries.¹⁶ Assuming that the beam is uniform,¹⁷ and substituting $\mathcal{Y}_1(x)$ into [Equation 8.43](#), we

¹⁴The same procedure would be followed for axial vibration and torsional vibration.

¹⁵This becomes clearer when the Rayleigh-Ritz procedure is discussed in the next section.

¹⁶Try evaluating $\mathcal{Y}_1(0)$, $\mathcal{Y}_1(L)$, $\mathcal{Y}_1'(0)$, and $\mathcal{Y}_1'(L)$, and compare with the actual boundary conditions.

¹⁷The benefits of such approximate techniques are for problems where an exact solution is not possible, for example, components with non-uniform sections. Here, the purpose is to demonstrate the technique.

find the approximation for ω_1 , denoted as Ω_1 ,

$$\Omega_1^2 = \frac{\int_0^L EI \left(1 - \cos \frac{\pi x}{2L}\right) \left(\left[\frac{\pi}{2L}\right]^2 \cos \frac{\pi x}{2L}\right) dx}{\int_0^L m \left(1 - \cos \frac{\pi x}{2L}\right)^2 dx}$$

$$\Omega_1 = \frac{3.667}{L^2} \sqrt{\frac{EI}{m}}.$$

The exact result is the same except with a coefficient of 3.516. (Thus, the approximation has an error of 4%.) As expected, the approximate value is larger than the actual value since the assumed mode, if it is not the actual mode, is always stiffer than the actual mode. If the beam is not uniform, then $EI(x)$ is included in the integration with respect to x in Equation 8.43.

To estimate ω_2 , a guess for $Y_2(x)$ is needed. This becomes a difficult process. A more effective approach is available using the Rayleigh-Ritz method, introduced next.

LORD JOHN WILLIAM STRUTT RALEIGH

(November 12, 1842 – June 30, 1919)



Contribution: Acclaimed as the last of the great British classical physicists, Rayleigh made significant contributions in the field of vibrations. He also advanced the fields of acoustics, optics, and electromagnetism. Raleigh introduced dissipation effects in Lagrange's equations in the same manner that inertia and conservative forces are described. He discovered and isolated argon, for which he won the Nobel Prize in 1904.

Life Sketch: John William Strutt, third Baron Rayleigh, was born in Langford Grove, Essex, as the son of John James Strutt, second Baron. He was one of the very few members of higher nobility who won fame as an outstanding scientist.

Rayleigh was frail and his prospects of attaining maturity appeared precarious. His education was repeatedly interrupted by ill health. He showed little sign of being anything other than a child of average ability.

In 1861 he entered Trinity College, Cambridge, where he studied mathematics. His coach at Cambridge was Edward Routh, who, in addition to being the most famous of the Cambridge coaches at that time (perhaps of all time), was an applied mathematician making important contributions to dynamics. There is no doubt that the grounding in mathematical techniques which Rayleigh gained from Routh was an important factor in his outstanding scientific career.

Another key influence on Rayleigh during his undergraduate years at Cambridge was Stokes, the Lucasian professor of mathematics. Stokes inspired Rayleigh with lectures that combined theory and practice – many physical experiments were carried out during the lectures. At the time students did not undertake physics experiments themselves, so seeing Stokes perform experiments in his course on light was Rayleigh's only exposure to the experimental side of science.

At Cambridge he proved to be far from an average student. He was awarded an astronomy scholarship in 1864, then in the Tripos examinations of 1865 he was Senior Wrangler (the top First Class student) and in the same year he was the first Smith's prizeman.

His first paper was inspired by reading Maxwell's 1865 paper on electromagnetic theory. He was a voracious reader of scientific literature, and from that Rayleigh tried to determine the important research problems to tackle. He was strongly influenced by the work of Helmholtz (in particular reading Helmholtz's 1860 results on the acoustic resonator).

In 1866 Rayleigh was elected a Fellow of Trinity College, Cambridge, which he held until 1871. A severe attack of rheumatic fever in 1872 made him spend the winter in Egypt and Greece. Shortly after his return his father died (1873) and he succeeded to the barony, taking up residence in the family seat, Terling Place, at Witham, Essex. He now found himself compelled to devote part of his time to the management of his estates (7000 acres).

The combination of general scientific knowledge with acquired knowledge of agriculture made his practice in estate management ahead of his time. Nevertheless, in 1876 he left the entire management of the land to his younger brother.

From then on, he devoted his full time to science. In 1879 he was appointed to follow James Clerk Maxwell as Professor of Experimental Physics and Head of the Cavendish Laboratory at Cambridge. In 1884 he left Cambridge to continue his experimental work at his country seat at Terling, Essex, and from 1887 to 1905 he was Professor of Natural Philosophy in the Royal Institution of Great Britain, following Tyndall. He became chancellor of Cambridge University in 1908.

He served for six years as President of a Government Committee on Explosives, and from 1896 to 1919 he was Scientific Advisor to Trinity House. He was Lord Lieutenant of Essex from 1892 to 1901.

Rayleigh's early research was mainly mathematical, concerning optics and vibrating systems, but his later work ranged over almost the whole field of physics, covering sound, wave theory, color vision, electrodynamics, electromagnetism,

light scattering, flow of liquids, hydrodynamics, density of gases, viscosity, capillary action, elasticity, and photography. His patient and delicate experiments led to the establishment of the standards of resistance, current, and electromotive force. Much of his later work concentrated on electric and magnetic problems.

In 1871 Rayleigh married Evelyn Balfour, the sister of Arthur James Balfour (a leading member of the Conservative Party for 50 years and Prime Minister of Britain 30 years later). They had three sons. He died in Witham, Essex at the age of 76.

Noteworthy: Rayleigh was a Justice of the Peace and the recipient of honorary science and law degrees. He was a Fellow of the Royal Society (1873) and served as Secretary from 1885 to 1896 and as President from 1905 to 1908. He was an original recipient of the Order of Merit (1902) and made a Privy Councillor (1905). He was awarded the Copley, Royal, and Rumford Medals of the Royal Society, and the Nobel Prize in 1904.

Rayleigh was a modest and generous man. He donated the proceeds of his Nobel Prize to the University of Cambridge to build an extension to the Cavendish laboratories.

The usual course of action for young British men of social standing at this time was to take a European tour - the grand tour as it was called. Rayleigh made a very different, and for that time unusual, tour for he set out on a trip to the United States. One advantage of Rayleigh's privileged social position was that he did not need an academic post to earn his living. Rather, when he returned from the U.S., he purchased equipment for undertaking scientific experiments and set it up on the family estate at Terling.

In the laboratory he had set up he made impressive discoveries, but one should not think that this was because the rich Rayleigh was able to have better equipment than anyone else. On the contrary, he obtained impressive experimental results with cheap equipment. Rayleigh was always one to economize and make do with unsophisticated equipment. Also, he was not as well off as might have been expected, for the 1870s were a time of economic problems for farming in England, and, as a consequence, his income was far less than might otherwise have been the case.

Many phenomena are named after him, including Rayleigh waves, Rayleigh scattering, and the Rayleigh criterion. Rayleigh's theory of scattering, published in 1871, was the first correct explanation of why the sky is blue.

Rayleigh was an excellent instructor and, under his active supervision, a system of practical instruction in experimental physics was devised at Cambridge, developing from a class of five or six students to an advanced school of some seventy experimental physicists.

He had a fine sense of literary style. Every paper he wrote, even on the most abstruse subject, is a model of clearness and simplicity of diction. His *Theory of Sound* was published in two volumes during 1877-1878, and his other extensive studies are reported in his *Scientific Papers* - six volumes issued

during 1889-1920. The 446 papers reprinted in his collected works show his capacity for understanding everything just a little more deeply than anyone else.

In addition to the more usual topics of applied mathematics and physics (papers on Bessel functions, the relationship between Laplace functions and Bessel functions, and Legendre functions), he wrote on more unusual topics such as "Insects and the color of flowers" (1874), "On the irregular flight of a tennis ball" (1877), "The soaring of birds" (1883), "The sailing flight of the albatross" (1889), and "The problem of the Whispering Gallery" (1910). He also contributed articles to the *Encyclopaedia Britannica*.

In his paper 1885 paper "On waves propagated along the plane surface of an elastic solid" Rayleigh wrote: "It is proposed to investigate the behavior of waves upon the plane surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wavelength. It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance."

Although a member of the House of Lords, Rayleigh intervened in debate only on rare occasions and never allowed politics to interfere with science. His recreations were travel, tennis, photography, and music.

Craters on Mars and the Moon are named in his honor. The asteroid 22740 Rayleigh was named in his honor on June 1, 2007.

8.3.2 Rayleigh-Ritz Method

The *Rayleigh-Ritz* method may be viewed as an extension of Rayleigh's quotient that provides a method to estimate higher frequencies. As with Rayleigh's quotient, an assumption is made regarding modal deformation while satisfying geometric boundary conditions. The modal deformation is approximated by the finite sum,

$$Y_r(x) = \sum_{i=1}^n c_i \mathcal{Y}_i(x), \quad (8.44)$$

where $\mathcal{Y}_i(x)$ are a set of assumed *trial* functions and c_i are to be determined in the manner described below.

We begin again with Equation 7.100 for any two modes of a transversely vibrating beam with vanishing boundary conditions,

$$\int_0^L EI(x) Y_r''(x) Y_s''(x) dx = \omega_r^2 \int_0^L m(x) Y_r(x) Y_s(x) dx,$$

where ω_r^2 is given by

$$\omega_r^2 = \frac{\int_0^L EI(x) Y_r''(x) Y_s''(x) dx}{\int_0^L m(x) Y_r(x) Y_s(x) dx}. \quad (8.45)$$

Equation 8.45 is an exact expression for the frequency if the eigenfunctions are substituted on the right-hand side. But, if a finite sum guess is used, then this equation for the frequency becomes an approximate one,

$$\begin{aligned}\Omega_r^2 &= \frac{\int_0^L EI(x) \left[\sum_{i=1}^n c_i \mathcal{Y}_i'' \right] \left[\sum_{j=1}^n c_j \mathcal{Y}_j'' \right] dx}{\int_0^L m(x) \left[\sum_{i=1}^n c_i \mathcal{Y}_i \right] \left[\sum_{j=1}^n c_j \mathcal{Y}_j \right] dx} \\ &= \frac{\sum_{i=1}^n \sum_{j=1}^n c_i c_j \int_0^L EI(x) \mathcal{Y}_i'' \mathcal{Y}_j'' dx}{\sum_{i=1}^n \sum_{j=1}^n c_i c_j \int_0^L m(x) \mathcal{Y}_i \mathcal{Y}_j dx} \equiv \frac{N}{D},\end{aligned}$$

where N and D represent the expressions in the numerator and denominator, respectively, for $i, j = 1, 2, \dots, n$, and where Ω_r represent the approximate values of the frequencies. An n -term finite sum leads to estimates of the first n frequencies, as we show below. To simplify the notation, as well as to better understand the physical nature of the equations, we define the following,

$$m_{ij} = \int_0^L m(x) \mathcal{Y}_i \mathcal{Y}_j dx \quad (8.46)$$

$$k_{ij} = \int_0^L EI(x) \mathcal{Y}_i'' \mathcal{Y}_j'' dx. \quad (8.47)$$

Then,

$$\Omega_r^2 = \frac{N}{D} = \frac{\sum_{i=1}^n \sum_{j=1}^n c_i c_j k_{ij}}{\sum_{i=1}^n \sum_{j=1}^n c_i c_j m_{ij}}. \quad (8.48)$$

Our task is to determine the optimal values of the c_i for the best estimate of the frequency. Knowing that the exact frequency is always smaller than the approximate value, we seek to *minimize* Ω_r^2 , as defined by Equation 8.48, by differentiating it with respect to c_i and setting the resulting expression equal to zero,

$$\frac{\partial}{\partial c_i} \left(\frac{N}{D} \right) = \frac{D \partial N / \partial c_i - N \partial D / \partial c_i}{D^2} = 0, \quad i = 1, 2, \dots, n.$$

The only way that this equation can equal zero is if the numerator equals zero (since D is never equal to zero). The numerator can be written as

$$\frac{\partial N}{\partial c_i} - \frac{N}{D} \frac{\partial D}{\partial c_i} = 0, \quad i = 1, \dots, n. \quad (8.49)$$

In this way, the infinite degree-of-freedom system has been replaced by an n degree-of-freedom system.

To proceed further, we need the trial function $\mathcal{Y}_i(x)$.¹⁸ From the equations above we see that this function must

¹⁸When selecting a trial function for use in an approximate method, it is recognized that only the eigenfunction will yield the exact results. Any other function leads to an approximate solution. As we know, the eigenfunctions satisfy the governing differential equation and all the boundary conditions. If the trial function satisfies all geometric (displacement and slope) and natural (force and moment) boundary conditions, it is called a *comparison function*.

be twice differentiable and still be a function of x after the second derivative. Equation 8.44 for a two-term approximation for the fixed-free beam can be the following,

$$\begin{aligned}Y_r(x) &= c_1 \mathcal{Y}_1(x) + c_2 \mathcal{Y}_2(x) \\ &= c_1 \left(\frac{x}{L} \right)^2 + c_2 \left(\frac{x}{L} \right)^3.\end{aligned}$$

Next, we evaluate N and D as follows,

$$\begin{aligned}N &= \sum_{i=1}^2 \sum_{j=1}^2 c_i c_j k_{ij} \\ &= \sum_{i=1}^2 [c_i c_1 k_{i1} + c_i c_2 k_{i2}] \\ &= c_1^2 k_{11} + c_1 c_2 k_{12} + c_2 c_1 k_{21} + c_2^2 k_{22} \\ &= c_1^2 k_{11} + 2c_1 c_2 k_{12} + c_2^2 k_{22}.\end{aligned}$$

Similarly,

$$\begin{aligned}D &= \sum_{i=1}^2 \sum_{j=1}^2 c_i c_j m_{ij} \\ &= c_1^2 m_{11} + 2c_1 c_2 m_{12} + c_2^2 m_{22},\end{aligned}$$

where we have made use of the symmetries of m_{ij} and k_{ij} . We can now evaluate the derivatives needed in Equation 8.49,

$$\frac{\partial N}{\partial c_1} = 2c_1 k_{11} + 2c_2 k_{12} \quad (8.50)$$

$$\frac{\partial N}{\partial c_2} = 2c_1 k_{12} + 2c_2 k_{22} \quad (8.51)$$

$$\frac{\partial D}{\partial c_1} = 2c_1 m_{11} + 2c_2 m_{12} \quad (8.52)$$

$$\frac{\partial D}{\partial c_2} = 2c_1 m_{12} + 2c_2 m_{22}, \quad (8.53)$$

where

$$\left. \begin{aligned}m_{11} &= \int_0^L m \mathcal{Y}_1^2 dx = \frac{mL}{5} \\ m_{12} &= \int_0^L m \mathcal{Y}_1 \mathcal{Y}_2 dx = \frac{mL}{6} = m_{21} \\ m_{22} &= \int_0^L m \mathcal{Y}_2^2 dx = \frac{mL}{7} \\ k_{11} &= \int_0^L EI [\mathcal{Y}_1'']^2 dx = \frac{4EI}{L^3} \\ k_{12} &= \int_0^L EI \mathcal{Y}_1'' \mathcal{Y}_2'' dx = \frac{6EI}{L^3} = k_{21} \\ k_{22} &= \int_0^L EI [\mathcal{Y}_2'']^2 dx = \frac{12EI}{L^3}.\end{aligned} \right\} \quad (8.54)$$

These have to be differentiable as many times as the order of the governing equation. If the trial function satisfies only geometric boundary conditions, then it is called an *admissible function*. It is easier to find functions that satisfy only the geometric conditions and not the natural conditions.

Therefore, Equation 8.49 for $i = 1, 2$, in conjunction with Equations 8.50 to 8.53, can be written in matrix-vector form as

$$\begin{bmatrix} k_{11} - \Omega_r^2 m_{11} & k_{12} - \Omega_r^2 m_{12} \\ k_{12} - \Omega_r^2 m_{12} & k_{22} - \Omega_r^2 m_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (8.55)$$

or, in compact matrix notation, as

$$[[K] - \Omega_r^2 [M]] \{c\} = \{0\}. \quad (8.56)$$

This is the same matrix eigenvalue problem we have solved previously when considering multi degree-of-freedom systems. The values of c_1 and c_2 can be found by solving the 2×2 determinant $[[K] - \Omega_r^2 [M]] = 0$. The evaluation of this determinant provides us with estimates of the two natural frequencies Ω_1^2 and Ω_2^2 , since a two-term approximate solution is taken, resulting in a two degree-of-freedom approximate system. Then, using the two natural frequencies, we proceed to find the values c_{1r} and c_{2r} , $r = 1, 2$, in Equation 8.55. The complete solution is then given by

$$\begin{aligned} y(x, t) &= \sum_{r=1}^2 Y_r(x) F_r(t) \\ &= [c_{11} \mathcal{Y}_1(x) + c_{21} \mathcal{Y}_2(x)] F_1(t) \\ &\quad + [c_{12} \mathcal{Y}_1(x) + c_{22} \mathcal{Y}_2(x)] F_2(t), \end{aligned}$$

where $F_1(t)$ is harmonic in Ω_1 and $F_2(t)$ is harmonic in Ω_2 . The first and second modal functions have been approximated by

$$\begin{aligned} Y_1(x) &= c_{11} \mathcal{Y}_1(x) + c_{21} \mathcal{Y}_2(x) \\ Y_2(x) &= c_{12} \mathcal{Y}_1(x) + c_{22} \mathcal{Y}_2(x). \end{aligned}$$

This approach replaces an infinite-dimensional continuous system by an n -dimensional discrete system. The discrete system can then be solved using the techniques of Chapter 6.

Physical Meaning of Equation 8.48

We explore Equation 8.48 further for a better physical understanding of its meaning in application to a bending beam. The strain energy of a bending beam is

$$V(t) = \frac{1}{2} \int_0^L EI(x) \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx,$$

and its kinetic energy is

$$T(t) = \frac{1}{2} \int_0^L m(x) \left(\frac{\partial y}{\partial t} \right)^2 dx.$$

Using the product solution, $y(x, t) = Y(x)F(t)$, $V(t)$ and $T(t)$ become

$$\begin{aligned} V(t) &= \frac{1}{2} F^2(t) \int_0^L EI(x) [Y''(x)]^2 dx \\ T(t) &= \frac{1}{2} \dot{F}^2 \int_0^L m(x) Y^2(x) dx. \end{aligned}$$

If $F(t)$ is harmonic, say $A \cos \omega t$, then the maximum $V(t)$ and the maximum $T(t)$ are given by

$$\begin{aligned} V_{max} &= \frac{1}{2} A^2 \int_0^L EI(x) [Y''(x)]^2 dx \\ T_{max} &= \frac{1}{2} A^2 \omega^2 \int_0^L m(x) Y^2(x) dx. \end{aligned}$$

The kinetic energy for $\omega = 1$ rad/s is customarily defined as T_{max}^* . Then,

$$T_{max} = \omega^2 T_{max}^*.$$

For a system with no dissipation, such as those without friction or damping, the maximum strain energy equals the maximum kinetic energy,

$$V_{max} = \omega^2 T_{max}^*,$$

or

$$\omega^2 = \frac{N}{D} = \frac{V_{max}}{T_{max}^*},$$

from Equation 8.48.

Rayleigh's quotient and its extension, the Rayleigh-Ritz procedure, are essentially measures of the ratio between strain energy and kinetic energy. Physically, it makes sense that this ratio is related to the frequency of oscillation, since the rate at which energy is being exchanged between kinetic and strain can be obtained by knowing the maxima of each and the time it takes to go between maxima.

If $Y_i(x)$ is not known exactly, we then approximate this function by $\mathcal{Y}_i(x)$. To obtain the minimum estimate of the frequency, we differentiate the approximate expressions for V_{max} and T_{max}^* with respect to c_i ,

$$\begin{aligned} \frac{\partial N}{\partial c_i} &= \frac{\partial V_{max}}{\partial c_i} \\ \frac{\partial D}{\partial c_i} &= \frac{\partial T_{max}^*}{\partial c_i}. \end{aligned}$$

We use the approach outlined above to estimate the fundamental and second harmonic frequencies for a uniform cantilever beam. We assume the approximate solution to be

$$\mathcal{Y}(x) = c_1 x^2 + c_2 x^3.$$

The maximum strain energy is given by

$$\begin{aligned} V_{max} &= \frac{1}{2}EI \int_0^L (2c_1 + 6c_2x)^2 dx \\ &= \frac{1}{2}EI(4c_1^2L + 12c_1c_2L^2 + 12c_2^2L^3). \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial V_{max}}{\partial c_1} &= EI(4c_1L + 6c_2L^2) = \sum_{j=1}^2 2k_{1j}c_j \\ \frac{\partial V_{max}}{\partial c_2} &= EI(6c_1L^2 + 12c_2L^3) = \sum_{j=1}^2 2k_{2j}c_j, \end{aligned}$$

where the second set of equalities is from Equations 8.50 and 8.51. Expanding the summations yields $k_{11} = 2EIL$, $k_{12} = 3EIL^2$, $k_{21} = 3EIL^2$, and $k_{22} = 6EIL^3$. Similarly for T_{max}^* ,

$$\begin{aligned} T_{max}^* &= \frac{m}{2} \int_0^L (c_1x^2 + c_2x^3)^2 dx \\ &= \frac{m}{2} \left(c_1^2 \frac{L^5}{5} + 2c_1c_2 \frac{L^6}{6} + c_2^2 \frac{L^7}{7} \right). \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial T_{max}^*}{\partial c_1} &= \frac{mL^5c_1}{5} + \frac{mL^6c_2}{6} = \sum_{j=1}^2 2m_{1j}c_j \\ \frac{\partial T_{max}^*}{\partial c_2} &= \frac{mL^6c_1}{6} + \frac{mL^7c_2}{7} = \sum_{j=1}^2 2m_{2j}c_j, \end{aligned}$$

where, from Equation 8.52 and 8.53, $m_{11} = mL^5/10$, $m_{12} = m_{21} = mL^6/12$, and $m_{22} = mL^7/14$.

Combining these terms yields the eigenvalue formulation,

$$\begin{bmatrix} 2EIL - \frac{\Omega^2 mL^5}{10} & 3EIL^2 - \frac{\Omega^2 mL^6}{12} \\ 3EIL^2 - \frac{\Omega^2 mL^6}{12} & 6EIL^3 - \frac{\Omega^2 mL^7}{14} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Taking the matrix determinant and setting it equal to zero yields

$$\Omega^4 - 1224 \frac{EI}{mL^4} \Omega^2 + 15120 \left(\frac{EI}{mL^4} \right)^2 = 0.$$

The two estimated frequencies are given by

$$\begin{aligned} \Omega_1 &= \frac{3.533}{L^2} \sqrt{\frac{EI}{m}} \quad \text{and} \\ \Omega_2 &= \frac{34.81}{L^2} \sqrt{\frac{EI}{m}}, \end{aligned}$$

with the exact numerical coefficients being 3.516 and 22.03, respectively. The estimate for ω_1 is very close (0.5% error), whereas the estimate for ω_2 is poor (58% error). It is generally found that accurate estimates are obtained for the first $n/2$ frequencies, where n equals the number of terms in the approximate solution. If a better approximation is necessary for ω_2 , then a four-term approximation would be appropriate.

BORIS GRIGORYEVICH GALERKIN

(March 4, 1871 – July 12, 1945)



Contribution: Galerkin was a world-famous mathematician, scientist, and engineer. In 1915, he developed a method of approximate integration of differential equations, now known as Galerkin's Method. The method is used in mechanics, vibrations, thermodynamics, electromagnetism, and hydrodynamics.

Galerkin is famous for his work on thin elastic plates. His major monograph *Thin Elastic Plates* was published in 1937.

Galerkin contributed to the field of industrial construction through his scientific research. He was often recruited as a consultant in the planning and building of many of the Soviet Union's largest power plants. In 1929, Galerkin used trapezoidal profiles to investigate stresses in dams and retaining walls. His results were used to plan the construction of the Dnepr dam and hydroelectric station.

From 1940 until his death, Galerkin served as head of the Institute of Mechanics at the Soviet Academy of Sciences.

Life Sketch: Galerkin was born in Polotsk, Russia, now part of Belarus. His parents owned a house in the town, but they did not bring in enough money and Galerkin had a hard time making ends meet during his years of education. At the age of 12 Galerkin started working as calligrapher in the court. He attended secondary school in Minsk and entered the Petersburg Technological Institute in 1893. There he was enrolled in the mechanics department and studied mathematics and engineering. In order to earn money, he took on private tutoring and later worked as a draftsman and designer. After graduating from the Technological Institute in 1899, he worked at the Kharkov Locomotive Plant.

Like many other students of his day, he became a member of the Russian Social-Democratic Party (future Communist Party). Perhaps this provides an explanation for his frequent job changes. The first three years after graduation Galerkin was an engineer at the Russian Mechanical and Steam-locomotive Union factory in Kharkov, while simultaneously teaching workers special courses. From the end of 1903 he was an engineer on the construction of the China Far East Railway. Half a year later he became the technical head at the "Northern Mechanical and Boiler Plant." He participated in organizing the Union of Engineers in St. Petersburg and, in 1905, he was arrested for organizing a strike among the engineers. In 1906, he became a member of the Social-Democratic Party's St. Petersburg Committee and did not work anywhere else.

In August 1906 the police surrounded his house and arrested him and other Committee members. In March 1907, the St. Petersburg Court Chamber passed a sentence, which was surprisingly light, given that at the time of the arrest some of the Committee members fired at policemen. Galerkin ("Zakhar" according to his underground nickname) was sentenced to 1.5 years imprisonment.

In prison, he lost interest in revolutionary activities and devoted himself to science and engineering. Later, in Soviet questionnaires he would not give clear answers on questions about membership in different parties. Of course, he was familiar with the fate of old Party members, but the main reason for it was that he had been elected to the Committee from the Mensheviks (a Party group with non-radical views, whose members later were accused of contra-revolutionary activities and repressed).

Galerkin was released from prison in the end of 1908. In March 1909, he became a teacher at the St. Petersburg Polytechnical Institute, at the same year his first scientific work was published by the Institute's *Transactions*. The article, written in prison, was titled "A theory of longitudinal curving and an experience of longitudinal curving theory application to many-storied frames, frames with rigid junctions and frame systems." The length of the title was indicative of the length of the work itself, 130 pages.

In the summer of 1909, Galerkin traveled abroad to see constructions and buildings which interested him. During the next four years, before World War I, he visited Germany, Austria, Switzerland, Belgium, and Sweden.

In 1913, Galerkin worked on the design of a boiler power plant in St. Petersburg. It was the first building in Russia with a metallic frame under large loads. It was considered one of his unique engineering projects.

Galerkin regularly published his works in the Institute's *Transactions* and in *Engineering News*. Before 1915, pivot systems were at the center of his scientific interest; later he started researching plates. In 1915, Galerkin published an article in which he put forward an idea of an approximate method for solving differential equations with boundary values. He applied his method to a large number of pivot and

plate analysis problems.

In January 1919, Galerkin became a professor in the 2nd (formerly Women's) Polytechnical Institute, remaining a teacher of structural mechanics in the 1st Polytechnical Institute mechanical department. In March 1920, a chaired professor in structural mechanics was established at the department, and Galerkin was selected. In the summer of 1921, Galerkin took part in a competition for a chair in civil engineering. In the beginning of 1922, he left the mechanical faculty for the faculty of civil engineering, which was nearer to his engineering activities.

In December 1923, Galerkin was elected dean of the Polytechnical Institute's civil engineering faculty. It happened during a very important period of the Institute's history, when a group of deans resigned from their posts, protesting the unceremonious intervention of so-called "student representatives," controlled by the trade-unions and the Communist party committees in the educational process. Galerkin showed himself to be a talented leader of the faculty. He managed to neutralize "assistants," who were appointed against his will, and he did not hurry to fulfill the orders of incompetent leaders. In 1924-1929, Galerkin was also a professor in the Railway Engineers Institute and at the St. Petersburg University. In 1924, he made his last trip abroad. He participated in a Congress on applied mechanics in the Netherlands.

In January 1928, Galerkin was appointed as a corresponding member elected at the USSR Academy of Sciences. In October 1929 he left the dean's post. After this, the civil engineering faculty divided into two parts: the hydrotechnical and irrigation sections became the water industry faculty, and the rest became the Civil and Industrial Engineering Institute. The water industry faculty soon became the Hydrotechnical Institute. Galerkin was a professor at both institutes.

By the 1920s, Galerkin was already a world-famous scientist. He had become an authority among engineers and designers. He was often recruited as a consultant for the design and construction of industrial buildings in northwest Russia (heat power plants, Volkhov hydro power plant, Kondopoga pulp and paper mill and others).

In 1934, Galerkin received two doctoral degrees as well as the title "Honored Worker in Science and Engineering." In the beginning of 1936, he was elected a member of the USSR Academy of Sciences. He also became a member of the highest Certifying Commission in the State Committee on higher technical education, a chairman of the technical mechanics group in the USSR Academy of Sciences technical section, the headmaster of the USSR Academy of Sciences Institute of Mechanics, and the chairman of the Civil Engineers scientific society and its Leningrad section. In April 1936, according to a governmental order, Galerkin was appointed chairman of the Governmental Commission for the examination of the Moscow Palace of the Soviets' steel frame walls.

Though having many titles, Galerkin remained a professor of the structural mechanics and theory of elasticity depart-

ment at the hydrotechnical faculty (the Hydrotechnical Institute was returned to the Polytechnical Institute as a faculty in 1934). Mostly he taught theory of elasticity, which was difficult for students who had weak training in mathematics.

In the summer of 1941, after the beginning of the war, the Commission on Defensive Installations Construction was created by the city government. Galerkin was involved with construction engineering and effectively became the supervisor of the work for the Commission.

Later, he was evacuated to Moscow, where he joined the military engineering commission of the USSR Academy of Sciences. Hard non-stop work was undermining his health. Not long after the World War II, Galerkin died in Moscow.

Noteworthy: Galerkin's name is forever attached to the method of approximate integration of differential equations, known as the Galerkin method. He published his finite element method in 1915.

Galerkin was short and had a weak voice. Students claimed that his image did not correspond to the status of a serious scientist with authority from the government.

8.3.3 Galerkin Method

While the Rayleigh-Ritz method is a powerful technique, an alternative approach, called the *Galerkin* method, is more general and powerful. It is able to model nonself-adjoint problems, such as nonlinear partial differential equations.¹⁹ In this approach, an approximate solution is written as

$$u_a(x, t) = \sum_{i=1}^n \mathcal{Y}_i(x) q_i(t),$$

where the functions $\mathcal{Y}_i(x)$ are taken to satisfy the boundary conditions. The approximate solution does not exactly satisfy the governing equation of motion, and there is an error. This error is minimized in a procedure known as the Galerkin method.

Using the axial vibration of a uniform beam to demonstrate the method, we substitute the approximate solution $u_a(x, t)$ into the governing equation,

$$EA \frac{\partial^2 u_a}{\partial x^2} - m \frac{\partial^2 u_a}{\partial t^2} = R_{error}, \quad (8.57)$$

where R_{error} , known as the *residual error*, exists if $u_a(x, t)$ is not the exact solution. Otherwise, $R_{error} = 0$. In the Rayleigh-Ritz procedure, the expression for the natural frequency is minimized. Here, an expression that

includes the residual error is minimized. However, since the error depends on the quality of the chosen function $\mathcal{Y}_i(x)$, the minimization equations are taken with respect to these functions.

We multiply Equation 8.57 by the variation of the assumed solution,

$$\delta u_a(x, t) = \sum_{i=1}^n \mathcal{Y}_i(x) \delta q_i(t),$$

and integrate the resulting equation over the domain,

$$\int_0^L \left[EA \frac{\partial^2 u_a}{\partial x^2} - m \frac{\partial^2 u_a}{\partial t^2} \right] \delta u_a dx = \int_0^L R_{error} \delta u_a dx.$$

Substituting the expression for δu_a , we find

$$\begin{aligned} & \sum_{i=1}^n \int_0^L \left[EA \frac{\partial^2 u_a}{\partial x^2} - m \frac{\partial^2 u_a}{\partial t^2} \right] \mathcal{Y}_i(x) \delta q_i(t) dx \\ &= \sum_{i=1}^n \int_0^L R_{error} \mathcal{Y}_i(x) \delta q_i(t) dx, \end{aligned}$$

or

$$\sum_{i=1}^n \left\{ \int_0^L \left[EA \frac{\partial^2 u_a}{\partial x^2} - m \frac{\partial^2 u_a}{\partial t^2} - R_{error} \right] \times \mathcal{Y}_i(x) dx \right\} \delta q_i(t) = 0.$$

Since the variation $\delta q_i(t)$ is arbitrary, the term in the braces must equal zero,

$$\begin{aligned} & \int_0^L \left[EA \frac{\partial^2 u_a}{\partial x^2} - m \frac{\partial^2 u_a}{\partial t^2} \right] \mathcal{Y}_i(x) dx \\ &= \int_0^L R_{error} \mathcal{Y}_i(x) dx, \quad i = 1, 2, \dots, n. \end{aligned} \quad (8.58)$$

To minimize the error in u_a , the integral of the weighted residual error must be minimized. The criterion in selecting $\mathcal{Y}_i(x)$ is that the integral on the right-hand side be as close to zero as possible. The $\mathcal{Y}_i(x)$ act as *weighting functions* selected so that the residual error is minimum over the domain, as specified in Equation 8.58.

Such techniques are part of a group of approximate methods called *methods of weighted residuals*. (The finite element method is an example.) Equation 8.58 represents the work done per unit length by the force R_{error} undergoing a deflection $\mathcal{Y}_i(x)$. For the exact solution, this work must equal zero since then $R_{error} = 0$.

Therefore, we set up the equations

$$\int_0^L \left[EA \frac{\partial^2 u_a}{\partial x^2} - m \frac{\partial^2 u_a}{\partial t^2} \right] \mathcal{Y}_i(x) dx = 0, \quad i = 1, 2, \dots, n,$$

where the approximate mode is used.²⁰ Note again that the term in the square brackets equals the residual error, R_{error} , as per Equation 8.57.

²⁰ Had we included an external force, it would appear on the right-hand side of this equation.

¹⁹ The Rayleigh-Ritz procedure and the Galerkin method are essentially the same if the system is *self-adjoint*. For nonself-adjoint systems, only the Galerkin method can be used. These and other related issues are addressed by H.H.E. Leipholz, "On Some Developments in Direct Methods of the Calculus of Variations," *Applied Mechanics Reviews*, Vol.40, No.10, Oct.1987, pp.1379-1392.

Example 8.2 Second Order Galerkin's Equations for an Axially Vibrating Beam

Assume a general two-term approximate solution for the axial displacement of a beam and derive the discretized (matrix-vector form) equation of motion.

Solution: We assume the two-term solution approximation

$$u_a(x, t) = \mathcal{Y}_1(x)q_1(t) + \mathcal{Y}_2(x)q_2(t). \quad (8.59)$$

The residual error is

$$R_{error} = EA \frac{\partial^2 u_a}{\partial x^2} - m \frac{\partial^2 u_a}{\partial t^2}. \quad (8.60)$$

We multiply both sides of this equation by $\mathcal{Y}_i(x)$ and integrate over the beam length, that is,

$$\int_0^L R_{error} \mathcal{Y}_i(x) dx = 0, \quad i = 1, 2. \quad (8.61)$$

Therefore, the two equations to solve are

$$\begin{aligned} \int_0^L R_{error} \mathcal{Y}_1(x) dx &= 0 \\ \int_0^L R_{error} \mathcal{Y}_2(x) dx &= 0. \end{aligned}$$

Specifically, by substituting Equation 8.59 into 8.60, and then into 8.61, these two equations become

$$\begin{aligned} EA \int_0^L [\mathcal{Y}_1'' q_1 + \mathcal{Y}_2'' q_2] \mathcal{Y}_1 dx \\ - m \int_0^L [\mathcal{Y}_1 \ddot{q}_1 + \mathcal{Y}_2 \ddot{q}_2] \mathcal{Y}_1 dx &= 0 \\ EA \int_0^L [\mathcal{Y}_1'' q_1 + \mathcal{Y}_2'' q_2] \mathcal{Y}_2 dx \\ - m \int_0^L [\mathcal{Y}_1 \ddot{q}_1 + \mathcal{Y}_2 \ddot{q}_2] \mathcal{Y}_2 dx &= 0. \end{aligned}$$

In matrix-vector form, we have

$$\begin{aligned} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \\ = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (8.62) \end{aligned}$$

where

$$\begin{aligned} m_{11} &= - \int_0^L m \mathcal{Y}_1^2 dx \\ m_{12} &= m_{21} = - \int_0^L m \mathcal{Y}_1 \mathcal{Y}_2 dx \\ m_{22} &= - \int_0^L m \mathcal{Y}_2^2 dx \\ k_{11} &= \int_0^L EA \mathcal{Y}_1'' \mathcal{Y}_1 dx \\ &= EA \mathcal{Y}_1' \mathcal{Y}_1 \Big|_0^L - \int_0^L EA \mathcal{Y}_1' \mathcal{Y}_1' dx \\ k_{12} &= \int_0^L EA \mathcal{Y}_2'' \mathcal{Y}_1 dx \\ &= EA \mathcal{Y}_2' \mathcal{Y}_1 \Big|_0^L - \int_0^L EA \mathcal{Y}_2' \mathcal{Y}_1' dx \\ k_{21} &= \int_0^L EA \mathcal{Y}_1'' \mathcal{Y}_2 dx \\ &= EA \mathcal{Y}_1' \mathcal{Y}_2 \Big|_0^L - \int_0^L EA \mathcal{Y}_1' \mathcal{Y}_2' dx \\ k_{22} &= \int_0^L EA \mathcal{Y}_2'' \mathcal{Y}_2 dx \\ &= EA \mathcal{Y}_2' \mathcal{Y}_2 \Big|_0^L - \int_0^L EA \mathcal{Y}_2' \mathcal{Y}_2' dx, \end{aligned}$$

where the boundary values are equal to zero, following Equation 7.34. At this point we can select $\mathcal{Y}_1$ and $\mathcal{Y}_2$ that satisfy the boundary conditions and then solve the eigenvalue problem based on matrix Equation 8.62 to find $\omega_{1,2}$ and $q_{1,2}(t)$, as we have done before. ■

The real power of such approximate methods is their utility in solving problems, including most problems of practical interest, that have no exact analytical solution. For example, the analysis of the tapered cantilever beam of Figure 8.12 can be carried out using the approximate methods of this section. Depending on the degree of the taper, one may have different expressions for $\alpha(x)$ and $\beta(x)$ in the relations

$$\begin{aligned} EA(x) &= EA \left(1 - \alpha(x) \frac{x}{L} \right) \\ m(x) &= m \left(1 - \beta(x) \frac{x}{L} \right), \end{aligned}$$

where $\alpha(x)$ and $\beta(x)$ represent the geometrical and mass distributions, respectively, and $\alpha(L) = 1$ and $\beta(L) = 1$. For a uniform fixed-free beam, the first mode is given by $\mathcal{Y}_1(x) = \sin(\pi x/2L)$, which satisfies the above boundary conditions, and can be used in an approximate model.

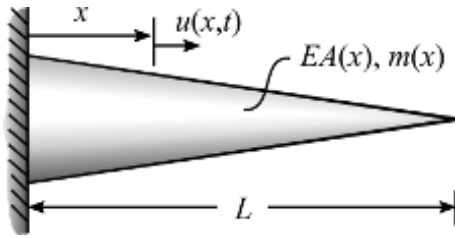


Figure 8.12: Tapered beam in axial vibration.

8.4 Variables Not Separating

The solution of partial differential equations in this chapter has been based on the method of separation of variables. Many problems cannot be solved with this approach. Separation of variables works as a solution method for cases in which the time function is harmonic. The product of the time function and the space function for the complete solution then results in synchronous motion, also called modal behavior. The lack of synchronous motion implies a wave propagation solution.

This section discusses two classes of problems that do not have separable solutions. The first class is vibration with *time-dependent boundary conditions*, where the time dependence is *not* harmonic; separation of variables then fails in the boundary conditions. The second class of problems includes those for which separation of variables fails in the equation of motion due to a *mixed derivative* term. Examples are the vibration of a pipe with internal flow and the motion of a moving taut string.

8.4.1 Nonharmonic, Time-Dependent Boundary Conditions

In the problems studied earlier in this chapter, a variety of boundary conditions, including harmonic loading, were considered. For nonharmonic boundary conditions, the separation of variables approach fails not in the equation of motion but, rather, in the boundary conditions. One could use the Laplace transform method,²¹ but since we did not emphasize this approach for continuous systems, an approach is chosen here that transforms the equation to another coordinate system where the separation of variables succeeds.

To demonstrate the difficulty, we consider the axially vibrating cantilever beam with a force $P(t)$ acting on its free end in the axial direction. The boundary condition

on the free end is

$$EA \partial u(L, t) / \partial x = P(t).$$

Assuming separation of variables $u(x, t) = X(x)F(t)$, this boundary condition becomes

$$EAX'(L)F(t) = P(t).$$

If the loading is harmonic then the analysis can proceed as usual by separation of variables. If $P(t)$ is some arbitrary time function, another technique is required.

We follow the development given in Clark (based on the method of Mindlin and Goodman),²² who develops the technique for a bending beam. The *Mindlin-Goodman* technique transforms the original homogeneous differential equation with nonhomogeneous boundary conditions into a nonhomogeneous differential equation with homogeneous boundary conditions that can then be solved using modal analysis. The approach works as well for a forced system, where the transformation leads to a more complicated forcing function.

As an example, we consider the transverse vibration of a uniform Bernoulli-Euler beam subjected to arbitrary forcing given by $q(x)p(t)$. The equation of motion is

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = q(x)p(t),$$

where $m = \rho A$. This equation has four boundary conditions,

$$\begin{aligned} y(X, t) &= f_0(t) \\ \frac{\partial y(X, t)}{\partial x} &= f_1(t) \\ \frac{\partial^2 y(X, t)}{\partial x^2} &= f_2(t) \\ \frac{\partial^3 y(X, t)}{\partial x^3} &= f_3(t), \end{aligned}$$

where X in the boundary conditions may be either $X = 0$ or L . There are sixteen possible combinations of boundary conditions. Some of the $f_i(t)$ may be zero, but at least one must be an arbitrary function of time. If the functions are harmonic, a standard separation of variables will succeed. In addition to the boundary conditions, there are two initial conditions,

$$\begin{aligned} y(x, 0) &= y_0(x) \\ \dot{y}(x, 0) &= v_0(x). \end{aligned}$$

Due to the arbitrary time functions $f_i(t)$, a direct separation of variables will not work, and, therefore, we proceed to examine the possibility that the response variable

²¹For an introduction to the application of Laplace transform techniques to partial differential equations, see Chapter 12 of **Elementary Applied Partial Differential Equation**, R. Haberman, Prentice Hall, 2nd ed., 1987. The text **Operational Mathematics**, R.V. Churchill, 2nd ed., McGraw-Hill, 1958, is devoted to this subject. Other transform techniques are discussed in **Fourier Transforms**, I.N. Sneddon, McGraw-Hill, 1951. All three books discuss applications to mechanical vibration problems.

²²See Chapter 11 of **Dynamics of Continuous Elements**, S.K. Clark, who references R.D. Mindlin, L.E. Goodman, "Beam Vibrations with Time Dependent Boundary Conditions," *J. Applied Mechanics*, Vol.17, 1950, pp.377-380 for originally developing the method.

$y(x, t)$ can be transformed into a new domain where homogeneity of the boundary conditions can be enforced. This benefit will come at the cost of a more complicated forcing function.

We assume the following relation,

$$y(x, t) = z(x, t) + \sum_{i=0}^3 f_i(t) g_i(x), \quad (8.63)$$

where $z(x, t)$ is the transformed displacement, $f_i(t)$ are the general boundary conditions, and $g_i(x)$ are chosen so that the boundary conditions of $z(x, t)$ are homogeneous and, therefore, separable. An example is

$$g_i(x) = \alpha_i + \beta_i x + \gamma_i x^2 + \delta_i x^3 + \epsilon_i x^4.$$

We show below how to relate the number of terms necessary in such a polynomial to the boundary conditions of a specific problem.

We substitute Equation 8.63 for $y(x, t)$ into the governing equation of motion, using the additional relations,

$$\begin{aligned} \ddot{y} &= \ddot{z} + \sum_{i=0}^3 \ddot{f}_i g_i(x) \\ y^{iv} &= z^{iv} + \sum_{i=0}^3 f_i(t) g_i^{iv}(x), \end{aligned}$$

to find

$$\begin{aligned} EI z^{iv} + m \ddot{z} &= q(x) p(t) \\ &- \sum_{i=0}^3 [EI f_i(t) g_i^{iv}(x) + m \ddot{f}_i g_i(x)], \end{aligned} \quad (8.64)$$

where the superscripts denote partial differentiations of that order with respect to x , that is,

$$z^{iv} = \frac{\partial^4 z(x, t)}{\partial x^4}.$$

The transformation equation must also satisfy all boundary and initial conditions, where, again, $X = 0$ or $X = L$,

$$\begin{aligned} y(X, t) &= f_0(t) \\ &= z(X, t) + \sum_{i=0}^3 f_i(t) g_i(X) \end{aligned} \quad (8.65)$$

$$\begin{aligned} \frac{\partial y(X, t)}{\partial x} &= f_1(t) \\ &= \frac{\partial z(X, t)}{\partial x} + \sum_{i=0}^3 f_i(t) \frac{dg_i(X)}{dx} \end{aligned} \quad (8.66)$$

$$\begin{aligned} \frac{\partial^2 y(X, t)}{\partial x^2} &= f_2(t) \\ &= \frac{\partial^2 z(X, t)}{\partial x^2} + \sum_{i=0}^3 f_i(t) \frac{d^2 g_i(X)}{dx^2} \end{aligned} \quad (8.67)$$

$$\begin{aligned} \frac{\partial^3 y(X, t)}{\partial x^3} &= f_3(t) \\ &= \frac{\partial^3 z(X, t)}{\partial x^3} + \sum_{i=0}^3 f_i(t) \frac{d^3 g_i(X)}{dx^3}, \end{aligned} \quad (8.68)$$

and

$$\begin{aligned} y(x, 0) &= y_0(x) = z(x, 0) + \sum_{i=0}^3 f_i(0) g_i(x) \\ \dot{y}(x, 0) &= v_0(x) = \dot{z}(x, 0) + \sum_{i=0}^3 \dot{f}_i(0) g_i(x). \end{aligned}$$

The procedure is to select the $g_i(x)$ so that all boundary conditions on $z(x, t)$ vanish. That is, with appropriate choices of $g_i(x)$, we can obtain the homogeneous boundary conditions $z(X, t) = 0$, $z'(X, t) = 0$, $z''(X, t) = 0$, and $z'''(X, t) = 0$. Based on Equations 8.65 to 8.68, $g_i(x)$ must be a polynomial in x of an order equal to the highest derivative. There is no unique choice of functions; all lead to the same solution.

Having selected the $g_i(x)$ that satisfy the boundary conditions on z , then for $X = 0$ or $X = L$, all z and derivative terms at the boundaries drop out and Equations 8.65 to 8.68 become

$$\begin{aligned} f_0(t) - \sum_{i=0}^3 f_i(t) g_i(X) &= 0 \\ f_1(t) - \sum_{i=0}^3 f_i(t) \frac{dg_i(X)}{dx} &= 0 \\ f_2(t) - \sum_{i=0}^3 f_i(t) \frac{d^2 g_i(X)}{dx^2} &= 0 \\ f_3(t) - \sum_{i=0}^3 f_i(t) \frac{d^3 g_i(X)}{dx^3} &= 0. \end{aligned}$$

These equations ensure that the boundary conditions for $z(x, t)$ are homogeneous.

With $g_i(x)$ defined, we now solve for $z(x, t)$, which can be accomplished via separation of variables. Based on the above discussion,

$$z(x, t) = \sum_{n=1}^{\infty} X_n(x) T_n(t), \quad (8.69)$$

where $X_n(x)$ are the modes over $[0, L]$ of the transformed problem. As in earlier problems, we can expand the force

factor²³ $q(x)$ in terms of the modes,

$$q(x) = \sum_{n=1}^{\infty} Q_n X_n(x), \quad (8.70)$$

where Q_n are coefficients evaluated by orthogonality considerations using²⁴

$$Q_n = \frac{\int_0^L q(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}. \quad (8.71)$$

Similarly, we can expand $g_i(x)$ and $g_i^{iv}(x)$ in terms of the modes,

$$g_i(x) = \sum_{n=1}^{\infty} G_{in} X_n(x),$$

$$G_{in} = \frac{\int_0^L g_i(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}; \quad (8.72)$$

$$g_i^{iv}(x) = \sum_{n=1}^{\infty} H_{in} X_n(x),$$

$$H_{in} = \frac{\int_0^L g_i^{iv}(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}, \quad (8.73)$$

where G_{in} and H_{in} are constants. Substituting all these expressions into governing Equation 8.64, we find

$$EI \sum_{n=1}^{\infty} X_n^{iv} T_n + m \sum_{n=1}^{\infty} X_n \ddot{T}_n = \sum_{n=1}^{\infty} X_n p(t) Q_n - \sum_{i=0}^3 [EI f_i(t) \sum_{n=1}^{\infty} H_{in} X_n + m \ddot{f}_i \sum_{n=1}^{\infty} G_{in} X_n],$$

where the equality must hold for each n independently,

$$EI X_n^{iv} T_n + m X_n \ddot{T}_n = X_n p(t) Q_n - \sum_{i=0}^3 [EI f_i(t) H_{in} X_n + m \ddot{f}_i G_{in} X_n],$$

$n = 1, 2, \dots$. Dividing both sides by $X_n T_n$ gives

$$EI \frac{X_n^{iv}}{X_n} + m \frac{\ddot{T}_n}{T_n} = \frac{p(t)}{T_n} Q_n - \sum_{i=0}^3 \frac{EI f_i(t) H_{in} + m \ddot{f}_i G_{in}}{T_n}.$$

²³We could have defined the loading as $p(x, t)$ instead of $q(x)p(t)$ and proceeded as we did in Example 7.6.

²⁴Note that the integral relation in the denominators can be normalized to equal one, as we have done in the previous problems. Here, they have not been normalized to demonstrate that such normalization is arbitrary in such problems, whereas in modal analysis the normalization is done by convention.

The time-dependent functions can be separated from the spatially dependent functions in the governing equation and then set equal to ω_n^2 ,

$$\frac{EI}{m} \frac{X_n^{iv}}{X_n} = -\frac{\ddot{T}_n}{T_n} + \frac{p(t)}{m T_n} Q_n - \sum_{i=0}^3 \frac{(EI/m) f_i(t) H_{in} + \ddot{f}_i G_{in}}{T_n} = \omega_n^2,$$

resulting in the equations

$$X_n^{iv} - \frac{m \omega_n^2}{EI} X_n = 0 \quad (8.74)$$

$$\ddot{T}_n + \omega_n^2 T_n = \frac{p(t)}{m} Q_n - \sum_{i=0}^3 \left[\frac{EI f_i(t) H_{in}}{m} + \ddot{f}_i G_{in} \right], \quad (8.75)$$

$n = 1, 2, \dots$. Defining the right-hand side of Equation 8.75 as $P(t)$ to ease the subsequent algebra, the solution to Equation 8.74 is

$$X_n(x) = C_{1n} \cos \frac{m_n x}{L} + C_{2n} \sin \frac{m_n x}{L} + C_{3n} \cosh \frac{m_n x}{L} + C_{4n} \sinh \frac{m_n x}{L}, \quad (8.76)$$

where

$$\frac{m_n}{L} \equiv \left(\frac{\omega_n^2 m}{EI} \right)^{1/4}.$$

Equation 8.75 is a harmonic equation satisfied by

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{1}{\omega_n} \int_0^t P(\tau) \sin \omega_n (t - \tau) d\tau.$$

The complete solution of the transformed problem is given by Equation 8.69.

To satisfy the initial conditions $T_n(0)$ and $\dot{T}_n(0)$, we use the following,

$$z(x, 0) = \sum_{n=1}^{\infty} A_n X_n(x)$$

$$\dot{z}(x, 0) = \sum_{n=1}^{\infty} B_n \omega_n X_n(x).$$

Using the orthogonality properties of the modes, the con-

stants are found to be

$$A_n = \frac{\int_0^L z(x, 0) X_n(x) dx}{\int_0^L X_n^2(x) dx} = \frac{\int_0^L \left(y_0(x) - \sum_{i=0}^3 f_i(0) g_i(x) \right) X_n(x) dx}{\int_0^L X_n^2(x) dx} \quad (8.77)$$

$$B_n = \frac{\int_0^L \dot{z}(x, 0) X_n(x) dx}{\omega_n \int_0^L X_n^2(x) dx} = \frac{\int_0^L \left(\dot{v}_0(x) - \sum_{i=0}^3 \dot{f}_i(0) g_i(x) \right) X_n(x) dx}{\omega_n \int_0^L X_n^2(x) dx}. \quad (8.78)$$

Now we have all the terms that we need to set up the solution,

$$y(x, t) = z(x, t) + \sum_{i=0}^3 f_i(t) g_i(x).$$

We apply this procedure in the following example.

Example 8.3 Cantilever Beam with Transverse Displacement at Free End

Consider a cantilever beam subjected to a transverse time-dependent displacement at its free end, shown in Figure 8.13. Solve for the displacement $y(x, t)$.

Solution: For this case, we have the following boundary conditions,

$$\begin{aligned} y(0, t) &= f_0(t) && \text{prescribed displacement at free end} \\ \frac{\partial^2 y(0, t)}{\partial x^2} &= 0 && \text{zero moment resistance at free end} \\ \frac{\partial y(L, t)}{\partial x} &= 0 && \text{zero slope at fixed end} \\ y(L, t) &= 0 && \text{zero displacement at fixed end.} \end{aligned}$$

To simplify the algebra, we assume zero initial conditions,

$$\begin{aligned} y(x, 0) &= 0 \\ \frac{\partial y(x, 0)}{\partial t} &= 0. \end{aligned}$$

We proceed with the general transformation, as developed above,

$$y(x, t) = z(x, t) + \sum_{i=0}^3 f_i(t) g_i(x)$$

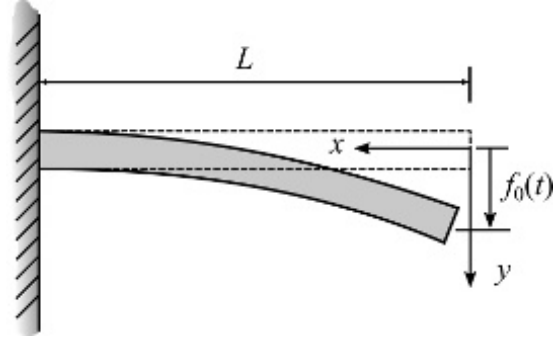


Figure 8.13: Exaggerated drawing of a cantilever beam subject to nonharmonic boundary condition.

(except that in this case, only $f_0(t) \neq 0$),

$$z(0, t) = f_0(t) - \sum_{i=0}^3 f_i(t) g_i(0) \quad (8.79)$$

$$z(L, t) = - \sum_{i=0}^3 f_i(t) g_i(L) \quad (8.80)$$

$$\frac{\partial z(L, t)}{\partial x} = - \sum_{i=0}^3 f_i(t) \frac{dg_i(L)}{dx} \quad (8.81)$$

$$\frac{\partial^2 z(0, t)}{\partial x^2} = - \sum_{i=0}^3 f_i(t) \frac{d^2 g_i(0)}{dx^2}. \quad (8.82)$$

Given that the boundary conditions are known to the second order, we assume

$$g_i(x) = \alpha_i + \beta_i x + \gamma_i x^2 + \delta_i x^3.$$

This equation can be simplified significantly by recognizing that the only remaining $g_i(x)$ will be for $i = 0$. In Equations 8.79 to 8.82, we set all boundary conditions on $z(x, t)$ to zero. Therefore,

$$f_0(t) - \sum_{i=0}^3 f_i(t) g_i(0) = 0 \quad (8.83)$$

$$- \sum_{i=0}^3 f_i(t) g_i(L) = 0 \quad (8.84)$$

$$- \sum_{i=0}^3 f_i(t) \frac{dg_i(L)}{dx} = 0 \quad (8.85)$$

$$- \sum_{i=0}^3 f_i(t) \frac{d^2 g_i(0)}{dx^2} = 0. \quad (8.86)$$

From Equation 8.83, we have

$$f_0(t) = f_0(t) g_0(0),$$

where $g_0(0) = \alpha_0$ and, therefore, $\alpha_0 = 1$. From Equation 8.86, for $x = 0$,

$$0 = -f_0(t)(2\gamma_0),$$

or $\gamma_0 = 0$. From Equation 8.85, for $x = L$,

$$0 = -f_0(t)(\beta_0 + 3\delta_0 L^2),$$

and from Equation 8.84, for $x = L$,

$$0 = -f_0(t)(1 + \beta_0 L + \delta_0 L^3),$$

from which we find $\beta_0 = -3/2L$ and $\delta_0 = 1/2L^3$, giving

$$g_0 = 1 - \frac{3}{2L}x + \frac{1}{2L^3}x^3.$$

Now we use the general solution, Equation 8.76, for $X_n(x)$ to evaluate constants C_{in} by satisfying the boundary conditions. We note that all boundary conditions for $z(x, t)$ equal zero,

$$\begin{aligned} z(0, t) &= 0 \\ z'(L, t) &= 0 \\ z''(0, t) &= 0 \\ z(L, t) &= 0. \end{aligned}$$

Therefore, $X_n(0) = 0$, $X_n(L) = 0$, $dX_n(L)/dx = 0$, and $d^2X_n(0)/dx^2 = 0$. Applying this to Equation 8.76 leads to the following equations,

$$\begin{aligned} C_{1n} &= 0, \quad C_{3n} = 0 \\ 0 &= C_{2n} \sin m_n + C_{4n} \sinh m_n \end{aligned} \quad (8.87)$$

$$0 = C_{2n} \cos m_n + C_{4n} \cosh m_n. \quad (8.88)$$

The latter two equations allow us to find the characteristic equation, $\tan m_n = -\tanh m_n$, leading to the values of m_n . They can be evaluated recursively or via plotting. With the values of m_n determined, the modes are given by

$$X_n(x) = C_{2n} \sin \frac{m_n x}{L} + C_{4n} \sinh \frac{m_n x}{L}.$$

Using either Equation 8.87 or 8.88 we can relate the two remaining coefficients. From Equation 8.87, we have

$$C_{4n} = -C_{2n} \frac{\sin m_n}{\sinh m_n},$$

and then

$$\begin{aligned} X_n(x) &= \frac{C_{2n}}{\sinh m_n} \left(\sinh m_n \sin \frac{m_n x}{L} \right. \\ &\quad \left. - \sin m_n \sinh \frac{m_n x}{L} \right). \end{aligned} \quad (8.89)$$

For compactness, we can define $C'_{2n} \equiv C_{2n}/\sinh m_n$.

Next, we evaluate all the coefficients and parameters that are part of the solution. Since the external load acts at the end of the beam, there is no spatial variation, and therefore $Q_n = 0$ and $q(x) = 0$ in Equations 8.71 and 8.70, respectively.

From Equation 8.72,

$$G_{0n} = \frac{\int_0^L g_0(x) X_n(x) dx}{\int_0^L X_n^2(x) dx},$$

where $g_0(x)$ and $X_n(x)$ have both been evaluated. Substituting, integrating, and simplifying leads to

$$G_{0n} = \frac{2}{m_n C'_{2n} (\sinh m_n - \sin m_n)}.$$

Since all the other $g_i(x) = 0$, we have $G_{2n} = G_{3n} = G_{4n} = 0$. Similarly, from Equation 8.73, $H_{in} = 0$.

Equations 8.77 and 8.78, with $y_0(x) = 0$ and $v_0(x) = 0$, are used to evaluate A_n and B_n ,

$$\begin{aligned} A_n &= \frac{\int_0^L \left(-\sum_{i=0}^3 f_i(0) g_i(x) X_n(x) \right) dx}{\int_0^L X_n^2(x) dx} = -f_0(0) G_{0n} \\ B_n &= \frac{\int_0^L \left(-\sum_{i=0}^3 \dot{f}_i(0) g_i(x) X_n(x) \right) dx}{\omega_n \int_0^L X_n^2(x) dx} = -\frac{\dot{f}_0(0) G_{0n}}{\omega_n}. \end{aligned}$$

The complete solution is then

$$\begin{aligned} y(x, t) &= z(x, t) + \sum_{i=0}^3 f_i(t) g_i(x) \\ &= \sum_{n=1}^{\infty} X_n(x) \left[-f_0(0) G_{0n} \cos \omega_n t \right. \\ &\quad \left. - \frac{\dot{f}_0(0) G_{0n}}{\omega_n} \sin \omega_n t \right. \\ &\quad \left. + \frac{1}{\omega_n} \int_0^t P(\tau) \sin \omega_n(t - \tau) d\tau \right] \\ &\quad + f_0(t) \left(1 - \frac{3x}{2L} + \frac{x^3}{2L^3} \right), \end{aligned}$$

where

$$\begin{aligned} P(t) &= \frac{p(t) Q_n}{m} - \sum_{i=0}^3 \left[\frac{EI}{m} f_i(t) H_{in} + \ddot{f}_i G_{in} \right] \\ &= -\ddot{f}_0(t) G_{0n}. \end{aligned}$$

For zero initial conditions, we have $f_0(0) = 0$ and $\dot{f}_0(0) = 0$, leading to

$$\begin{aligned} y(x, t) &= f_0(t) \left(1 - \frac{3x}{2L} + \frac{x^3}{2L^3} \right) \\ &\quad - \sum_{n=1}^{\infty} \left[\frac{2X_n(x)}{m_n C'_{2n} (\sinh m_n - \sin m_n)} \right. \\ &\quad \left. \times \frac{1}{\omega_n} \int_0^t \ddot{f}_0(\tau) \sin \omega_n(t - \tau) d\tau \right], \end{aligned}$$

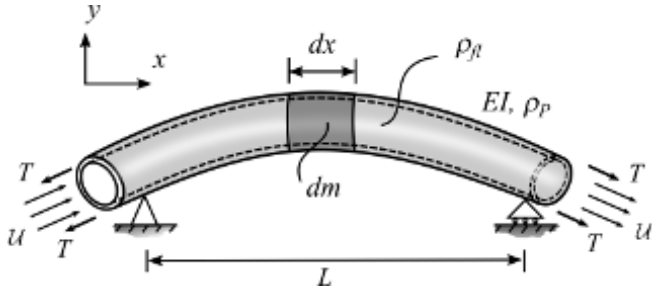


Figure 8.14: Pipe section with internal flow, with tension T at either end, and flow velocity $\mathcal{U}$.

where the factor C'_{2n} in the denominator cancels with that factor within $X_n(x)$ in the numerator, as per Equation 8.89. The convolution integral can be integrated by parts twice so that

$$\begin{aligned} \frac{1}{\omega_n} \int_0^t \ddot{f}_0(\tau) \sin \omega_n(t - \tau) d\tau \\ = f_0(t) - \omega_n \int_0^t f_0(\tau) \sin \omega_n(t - \tau) d\tau. \end{aligned}$$

The problem is solved except for the substitution of a particular function for $f_0(t)$. ■

Transformations such as those in the example problem are useful analytical tools. We often see problems transformed in which a complicating feature is moved to another part of the problem, where it is handled more easily.²⁵

8.4.2 Pipe Flow with Constant Tension

We consider internal flow in a transversely vibrating pipe, as shown schematically in Figure 8.14, where the flow velocity $\mathcal{U}$ is assumed constant, the fluid has a constant mass per unit length of pipe ρ_{fl} , the tension T in the pipe wall due to the internal flow is constant, and the pipe has constant bending rigidity EI and constant mass per unit length ρ_P . In solving, we will assume that linear vibration theory holds.

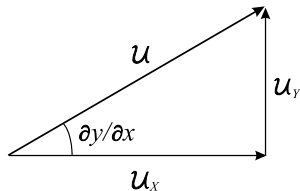


Figure 8.15: Velocity vector diagram.

For an arbitrary pipe section, the vector diagram of velocities is shown in Figure 8.15, where $\mathcal{U}_x$ and $\mathcal{U}_y$ are

²⁵ One example from earlier studies is the Laplace transform. The differential equation is transformed to an algebraic equation that is much easier to solve. But, the inverse transformation from the s -domain into the t -domain can be difficult.

the x and y components of $\mathcal{U}$, respectively. From this diagram, for small slope $\partial y/\partial x$, the flow velocity components are approximated to be

$$\begin{aligned} \mathcal{U}_x &= \mathcal{U} \cos \left(\frac{\partial y}{\partial x} \right) \simeq \mathcal{U} \\ \mathcal{U}_y &= \mathcal{U} \sin \left(\frac{\partial y}{\partial x} \right) \simeq \mathcal{U} \frac{\partial y}{\partial x}. \end{aligned}$$

These velocities are for a static *slightly* bent beam. They must be added to the beam vibratory motion to find the complete flow velocity.

The coordinate x in these equations refers to the element of fluid dm_{fl} moving through the position which is a distance x from the fixed left support at time t . Thus, any derivatives that are taken with respect to time must be *total* or *material* derivatives.²⁶ For an arbitrary function $f(x, t)$, the total derivative is given by

$$\begin{aligned} \frac{D}{Dt} f(x, t) &= \frac{\partial f(x, t)}{\partial t} + \frac{\partial f(x, t)}{\partial x} \frac{\partial x}{\partial t} \\ &= \frac{\partial f(x, t)}{\partial t} + \dot{x} \frac{\partial f(x, t)}{\partial x}, \end{aligned}$$

or, using comma-subscript notation for partial derivatives,

$$\frac{D}{Dt} f(x, t) = f_{,t} + \dot{x} f_{,x}.$$

Therefore, the velocity of the fluid in the transverse y direction is given by

$$\begin{aligned} \frac{D}{Dt} y(x, t) &= v \\ &= \frac{\partial y(x, t)}{\partial t} + \frac{\partial y(x, t)}{\partial x} \frac{\partial x}{\partial t} \\ &= y_{,t} + y_{,x} \mathcal{U}, \end{aligned} \quad (8.90)$$

where y is the position of the fluid and $\mathcal{U}_{y,x} = \mathcal{U}_y$.

Physically, the velocity of the fluid in the transverse direction equals the sum of the flow velocity and the pipe velocity in the transverse direction. It is assumed that the flow is uniform and there is no *cavitation*, that is, no low-pressure regions within the pipe where the fluid separates from the enclosing structure. It is also assumed that the pipe has no motion along the axial direction.

From Newton's second law of motion for element dm , that includes the pipe and internal fluid, the sum of the forces in the transverse direction is given by

$$F_y = dm_{fl} \frac{dv}{dt} + dm_P \frac{d}{dt} (y_{,t}), \quad (8.91)$$

²⁶ The total derivative Df/Dt arises from the chain rule of higher-dimensional calculus. The total differential Df of a function of two or more variables, by the chain rule, accounts for all the contributions from each of the variables. Such derivatives arise where a property changes with time and space, as in a flowing medium.

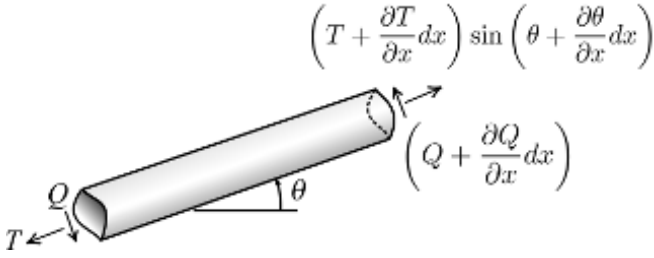


Figure 8.16: Free-body diagram of a pipe element of length dx showing shear force Q and axial tension T .

where $dm_{fl} = \rho_{fl} A_{fl} dx$ and $dm_P = \rho_P A_P dx$ are the respective element masses, and ρ are the densities (mass/volume) and A are the cross-sectional areas. Since $v = v(x, t)$,

$$\begin{aligned} \frac{dv}{dt} &= \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} \\ &= \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \mathcal{U} \\ &= \left(\frac{\partial}{\partial t} + \mathcal{U} \frac{\partial}{\partial x} \right) v. \end{aligned}$$

Substituting for v from Equation 8.90,

$$\begin{aligned} \frac{dv}{dt} &= \left(\frac{\partial}{\partial t} + \mathcal{U} \frac{\partial}{\partial x} \right) (y_{,t} + \mathcal{U} y_{,x}) \\ &= y_{,tt} + 2\mathcal{U} y_{,tx} + \mathcal{U}^2 y_{,xx}. \end{aligned} \quad (8.92)$$

Next, we find the explicit expression for F_y in Equation 8.91. The free-body diagram of the pipe is shown in Figure 8.16. The initial assumptions of constant tension and small angle bending lead to the following approximations,

$$\begin{aligned} \frac{dT}{dx} &\simeq 0 \\ \sin \theta &\simeq \theta \\ \sin(\theta + \theta_{,x} dx) &\simeq \theta + \theta_{,x} dx. \end{aligned}$$

The net tension in the pipe is approximated by

$$F_T = T(\theta + \theta_{,x} dx) - T\theta = T\theta_{,x} dx,$$

and, using $\theta = \partial y / \partial x$, we have

$$F_T = T y_{,xx} dx.$$

The net shear in the pipe is approximated by

$$F_S = - \left(Q + \frac{\partial Q}{\partial x} \right) dx + Q dx.$$

Substituting $M = EI \partial^2 y / \partial x^2$ and $Q = \partial M / \partial x$, the net shear becomes

$$F_S = -EI \frac{\partial^4 y}{\partial x^4} dx.$$

Substituting the sum of the tension and shear forces on the left-hand side of Equation 8.91 and using Equation 8.92, we have

$$\begin{aligned} -EI y_{,xxxx} + T y_{,xx} &= \rho_{fl} A_{fl} (y_{,tt} + 2\mathcal{U} y_{,tx} + \mathcal{U}^2 y_{,xx}) \\ &\quad + \rho_P A_P y_{,tt}. \end{aligned}$$

Combining terms, the governing equation of motion is

$$\begin{aligned} EI y_{,xxxx} - (T - \rho_{fl} A_{fl} \mathcal{U}^2) y_{,xx} \\ + 2\rho_{fl} A_{fl} \mathcal{U} y_{,tx} + (\rho_{fl} A_{fl} + \rho_P A_P) y_{,tt} = 0. \end{aligned} \quad (8.93)$$

If $\mathcal{U} = 0$, then $T \rightarrow 0$ since T is the tension in the pipe due to fluid flow. As the fluid speed approaches zero, the tension does so as well, and we recover the Bernoulli-Euler beam equation. If $\rho_{fl} A_{fl} \mathcal{U}^2 = T$, then $y_{,xx}$ drops out, where $\rho_{fl} A_{fl} \mathcal{U}^2$ is a *centrifugal force* arising from the curvature of the pipe. The cross-derivative term $y_{,tx}$ is a *Coriolis force* and the last term $y_{,tt}$ is the inertia term. The term $EI y_{,xxxx}$ is due to pipe bending rigidity.

The centrifugal and Coriolis forces are sometimes called *fictitious forces* as they are not external applied forces. The centrifugal force is due to rotational motion and the Coriolis force is present when a body undergoes motion relative to a rotating coordinate system.

Attempting a separation of variables solution will not work due to the cross-derivative term, as we will see next.

Nonseparable Solution

We start this solution by proceeding as though we did not know that the term $y_{,tx}$ prevents a separation of variables. Assuming $y(x, t) = Y(x)F(t)$, Equation 8.93 becomes

$$\begin{aligned} EI Y_{,xxxx} F - (T - \rho_{fl} A_{fl} \mathcal{U}^2) Y_{,xx} F \\ + 2\rho_{fl} A_{fl} \mathcal{U} Y_{,x} F_{,t} + (\rho_{fl} A_{fl} + \rho_P A_P) Y F_{,tt} = 0. \end{aligned}$$

Dividing by $Y(x)F(t)$ results in the equation

$$\begin{aligned} EI \frac{Y_{,xxxx}}{Y} - (T - \rho_{fl} A_{fl} \mathcal{U}^2) \frac{Y_{,xx}}{Y} \\ + 2\rho_{fl} A_{fl} \mathcal{U} \frac{Y_{,x} F_{,t}}{Y F} + (\rho_{fl} A_{fl} + \rho_P A_P) \frac{F_{,tt}}{F} = 0, \end{aligned}$$

which is not separable due to the third expression on the left-hand side. The lack of synchronous motion, or modes, means that any two points on the beam will vibrate *out of phase*, and the beam behaves as a *wave*. This is similar to the nonproportional damping problem.

Based on this reasoning, we use the wave solution, Equation 7.11,

$$y(x, t) = A \sin(2\pi \kappa x - \omega t).$$

Taking the appropriate derivatives, substituting these into Equation 8.93, canceling common factors A and

$\sin(2\pi\kappa x - \omega t)$, we obtain the following fourth-order expression in κ ,

$$EI(2\pi\kappa)^4 + (T - \rho_{fl}A_{fl}\mathcal{U}^2)(2\pi\kappa)^2 + 2\rho_{fl}A_{fl}\mathcal{U}(2\pi\kappa)\omega - (\rho_{fl}A_{fl} + \rho_P A_P)\omega^2 = 0. \quad (8.94)$$

Recalling $2\pi\kappa = \omega/c$, there are several possible forms for Equation 8.94. With the rigidity term EI retained, this equation can be solved only numerically with a solution that would see wave-like beam deflections propagating back and forth between the pipe supports rather than the synchronous modal behavior to which we are accustomed.

To proceed analytically, we assume that the rigidity of the pipe is very small, as it would be in a water hose used in fire fighting. In this case, Equation 8.94 becomes

$$A\kappa^2 + B\kappa + C = 0,$$

where

$$\begin{aligned} A &= T - \rho_{fl}A_{fl}\mathcal{U}^2 \\ B &= \frac{\rho_{fl}A_{fl}\mathcal{U}\omega}{\pi} \\ C &= -\frac{(\rho_{fl}A_{fl} + \rho_P A_P)\omega^2}{(2\pi)^2}. \end{aligned}$$

This quadratic equation for wave number κ can be solved for its two roots,

$$\kappa_{1,2} = \frac{1}{2A} \left(-B \pm \sqrt{B^2 + 4AC} \right), \quad (8.95)$$

and $(\omega/\pi)^2$ can be factored out in the numerator. Since the number of waves per unit length, κ , is a real number, the argument of the square root must be positive semi-definite, that is,

$$T(\rho_{fl}A_{fl} + \rho_P A_P) \geq \rho_{fl}A_{fl}\rho_P A_P \mathcal{U}^2.$$

No boundary conditions have yet been imposed. The pipe (or string) is fixed to prevent transverse displacement at two points a distance L apart: $y(0, t) = 0$ and $y(L, t) = 0$. The complete general solution for the two values κ_1 and κ_2 is

$$y(x, t) = A_1 \sin(2\pi\kappa_1 x - \omega t) + A_2 \sin(2\pi\kappa_2 x - \omega t).$$

The first boundary condition leads to $A_1 = -A_2$. The second boundary condition leads to

$$\begin{aligned} \sin(2\pi\kappa_1 L - \omega t) &= \sin(2\pi\kappa_2 L - \omega t) \\ &= \sin(2\pi\kappa_2 L - \omega t + 2n\pi). \end{aligned}$$

The second equality is just a statement of the 2π -periodicity of the sine function. Then, $2\pi\kappa_1 L = 2\pi\kappa_2 L + 2n\pi$, or

$$\kappa_1 = \kappa_2 + \frac{n}{L}, \quad n = 1, 2, \dots$$

Using Equation 8.95,

$$\kappa_1 - \kappa_2 = \frac{n}{L} = \frac{\omega \sqrt{T(\rho_{fl}A_{fl} + \rho_P A_P) - \rho_{fl}A_{fl}\rho_P A_P \mathcal{U}^2}}{\pi(T - \rho_{fl}A_{fl}\mathcal{U}^2)},$$

from which the allowable propagation frequencies are found to be

$$\omega_n = \frac{n\pi(T - \rho_{fl}A_{fl}\mathcal{U}^2)}{L \sqrt{T(\rho_{fl}A_{fl} + \rho_P A_P) - \rho_{fl}A_{fl}\rho_P A_P \mathcal{U}^2}}.$$

As $\mathcal{U} \rightarrow \sqrt{(T/\rho_{fl}A_{fl})}$, the frequency $\omega_n \rightarrow 0$. Setting the denominator equal to zero and solving for $\mathcal{U}$ provides a stability limit.

The respective equation for a string, with $\rho_P = 0$ and $\rho_{fl}A_{fl} = \rho_{string}A_{string}$, is

$$\begin{aligned} \omega_n &= \frac{n\pi}{L} \frac{(T - \rho_{string}A_{string}\mathcal{U}^2)}{\sqrt{T\rho_{string}A_{string}}} \\ &= \frac{n\pi}{L} \frac{(T/(\rho_{string}A_{string}) - \mathcal{U}^2)}{\sqrt{T/(\rho_{string}A_{string})}} \\ &= \frac{n\pi}{L} \frac{(c^2 - \mathcal{U}^2)}{c}, \end{aligned}$$

where $c^2 = T/(\rho_{string}A_{string})$, and as $\mathcal{U} \rightarrow c$, frequency $\omega_n \rightarrow 0$.

While these have been simplified results, they provide some indication of how such systems behave.

Example 8.4 Equation of Motion for the Fluid-Conveying Pipe Using Lagrange's Equation

Formulate the equation of motion for the fluid-conveying pipe using Lagrange's equation. Assume constant tension in the pipe, constant fluid velocity $\mathcal{U}$, linear material properties, and linear vibration. Then simplify the equation for the moving string equation.

Solution: The strain energy is due to work done by bending and work done by tension. It is given, respectively, by

$$\begin{aligned} V_b &= \int_0^L \frac{1}{2} EI y_{,xx}^2 dx \\ V_t &= \int_0^L T \Delta s = \int_0^L T(ds - dx). \end{aligned}$$

Referring to Figure 7.8 to relate ds to dx and dy , we can approximate the square root to find

$$ds - dx \simeq \frac{1}{2} y_{,x}^2 dx.$$

The total strain is then

$$V = V_b + V_t = \frac{1}{2} \int_0^L (EI y_{,xx}^2 + T y_{,x}^2) dx.$$

The kinetic energy is due to the vibrating pipe and to the motion of the fluid. The total kinetic energy is

$$\begin{aligned} T &= T_{fl} + T_P \\ &= \frac{1}{2} \int_0^L [\rho_{fl} A_{fl} [\mathcal{U}^2 + (y_{,t} + \mathcal{U} y_{,x})^2] + \rho_P A_P y_{,t}^2] dx. \end{aligned}$$

These expressions are substituted into Lagrange's equation to derive the equation of motion,

$$\begin{aligned} EI y_{,xxxx} - (T - \rho_{fl} A_{fl} \mathcal{U}^2) y_{,xx} \\ + 2\rho_{fl} A_{fl} \mathcal{U} y_{,tx} + (\rho_{fl} A_{fl} + \rho_P A_P) y_{,tt} = 0, \end{aligned} \quad (8.96)$$

matching Equation 8.93.

Simplifying for the moving string, we set

$$E = 0, \quad \rho_P = \rho_{fl} = \rho, \quad A_P = A_{fl} = A,$$

and the equation of motion becomes

$$y_{,tt} + 2\mathcal{U} y_{,tx} + (\mathcal{U}^2 - c^2) y_{,xx} = 0, \quad (8.97)$$

with $c^2 = T/\rho A$. ■

Traveling String: A Related Problem

If the bending rigidity is taken to be very small, as for a string or a fire hose, $EI y_{,xxxx} \simeq 0$, and if we then let the pipe *disappear* and replace the whole cross section including the fluid inside by a string, then the equation of motion for a moving string with zero rigidity is obtained,

$$-(T - \rho_{string} \mathcal{U}^2) y_{,xx} + 2\rho_{string} \mathcal{U} y_{,tx} + \rho_{string} y_{,tt} = 0.$$

Rearranging terms results in

$$y_{,tt} + 2\mathcal{U} y_{,tx} - \left(\frac{T}{\rho_{string}} - \mathcal{U}^2 \right) y_{,xx} = 0,$$

where T/ρ_{string} equals the wave speed squared, c^2 . The stability of motion of the moving string depends on the sign of the difference in the parenthesis.



VIBRATIONS IN AXIALLY MOVING WEB. A web is defined as a thin continuous film of material, such as paper, plastic, polymer, textile, and even metal. The web is transported through a web handling machine under tension. The tensile force exerted on the web is needed to overcome web and roller inertias, bearing friction, bending hysteresis, and air drag (which can be a significant factor at high web speeds).

In a web handling machine, web vibrations cause significant problems to product quality. These web vibrations are attributed to tension changes (both axial and transverse), mechanical drive speed variations, air impingement, and roller eccentricity. The interaction of these influences can result in

web imperfections such as surface scratches, tears, creases, and coating disturbances. Reduction, if not elimination, of these imperfections is essential to the consistent manufacture of web that conforms to product specifications. The mechanical vibrations, particularly in the transverse direction, of the web limit quality and productivity. Most modern web-handling systems rely on vibration control and speed tracking control.



8.5 Concepts Summary

This chapter has investigated two-dimensional membrane and plate models, which were derived and solved modally. We saw that membranes can be viewed as two-dimensional strings, and plates as two-dimensional beams. Rayleigh's approximate method was introduced, and the more powerful Rayleigh-Ritz and Galerkin methods were presented and applied to example problems. The problems of nonharmonic boundary conditions and of pipe vibration with internal flow were also introduced as special cases when separation of variables does not work as a solution method. The pipe problem reduced to a wave equation.

8.6 Quotes

- "The purpose of computing is insight, not numbers." (Richard W. Hamming)
- "Troubles are the tools by which God fashions us for better things." (Henry Ward Beecher)
- "Science is about cosmic order. Religion is about cosmic purpose." (Paul G. Hewitt)
- "I am an old man now, and when I die and go to Heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics and the other is the turbulent motion of fluids. And about the former I am rather more optimistic." (Sir Horace Lamb)
- "Mach 2 travel feels no different," a passenger commented on an early Concorde flight. "Yes," Sir George replied. "That was the difficult bit." (Sir George Edwards, co-director of Concorde development)
- "A knowledge of the existence of something we cannot penetrate, of the manifestations of the profoundest reason and the most radiant beauty, which are only accessible to our reason in their most elementary forms - it is this knowledge and this emotion

that constitute the truly religious attitude; in this sense, and in this alone, I am a deeply religious man.” (Albert Einstein)

8.7 Problems

Problems for Section 8.1 – Vibration of Membranes

1. Identify applications where a membrane model would apply.
2. Derive Equation 8.8.
3. Derive Equation 8.11.
4. Derive Equation 8.12.
5. Derive Equation 8.15. Explain the need for $\theta = \theta_0 + 2\pi j$.
6. Derive Bessel’s equation and solve it in general. (Information about Bessel’s equation can be found in differential equations textbooks.)
7. Derive Equations 8.23, 8.24, and 8.25. (Symbolic manipulation code such as Mathematica may be useful for the derivations.)
8. Derive by means of Hamilton’s principle the differential equation of motion and the associated boundary conditions for a thin rectangular membrane. Give the physical interpretation of the boundary conditions.
9. Derive the frequency equation for a uniform annular membrane defined over the domain $b \leq r \leq a$, with fixed boundaries at $r = b$ and $r = a$.
10. For the simply supported rectangular membrane of Section 8.1.1, suggest possible initial displacements $d(x, y)$. Select one and solve for the complete solution. Assume zero initial velocity.
11. Find the response of a uniform rectangular membrane to the initial displacement

$$f(x, y) = Axy(a - x)(b - y),$$

with zero initial velocity.

12. Repeat Problem 10 for an assumed initial velocity $v(x, y)$, with zero initial displacement.
13. Convert the wave equation from Cartesian coordinates,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

to polar coordinates,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right),$$

using the transformation $x = r \cos \theta$ and $y = r \sin \theta$.

Problems for Section 8.2 – Vibration of Plates

14. Identify applications where a plate model would apply.
15. Derive by means of Hamilton’s principle the equation of motion of a square plate.
16. Derive Equations 8.27 and 8.28.
17. Derive Equation 8.29.
18. Derive Equation 8.33.
19. Derive Equation 8.38.
20. Show that Equation 8.39 must be true for the given boundary conditions.
21. Derive coefficients

$$C_{jk} = \frac{2}{\sqrt{\rho ab}}$$

using the normalization procedure given for W_{jk} by Equation 8.40.

22. Derive the differential equation of motion for a nonuniform plate with varying thickness. Assume that the material properties such as Young’s modulus and Poisson’s ratio remain constant.

Problems for Section 8.3 – Approximate Methods

23. Derive Equations 8.54.
24. In Section 8.3.2, show how the parameters k_{ij} and m_{ij} are derived.
25. Derive Rayleigh’s quotient for a longitudinally vibrating beam.
26. Estimate the fundamental frequency for the tapered beam of Figure 8.17 where

$$m(x) = m(1 - x/L)$$

and

$$EA(x) = EA(1 - x/L).$$

Compare your result to the exact value of $\omega_1 = 2.40\sqrt{EA/mL^2}$.

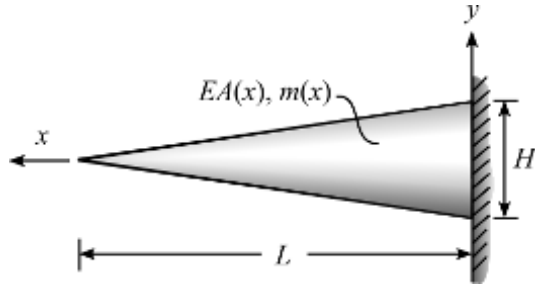


Figure 8.17: Tapered beam in longitudinal vibration.

27. For Problem 26, estimate the first two natural frequencies using the Rayleigh-Ritz procedure. Assume a trial function of the form $\mathcal{Y}(x) = c_1x^2 + c_2x^3$. Compare the fundamental frequencies estimated by the Rayleigh-Ritz and the Rayleigh quotient methods.
28. For the tapered beam sketched in Figure 8.17 undergoing transverse vibration, estimate the first two natural frequencies using the Rayleigh-Ritz procedure. Assume a trial function of the form $\mathcal{Y}(x) = c_1x^2 + c_2x^3$ and beam properties,

$$m(x) = \rho A(x) = \rho H(1 - x/L)$$

and

$$EI(x) = (E/12)[H(1 - x/L)]^3.$$

Problems for Section 8.4 – Variables Not Separating

29. In Example 8.3, derive the general solution for $f_0(t) = x \exp(-t/L)$.
30. Solve the problem of Example 8.3 where the only nonzero boundary condition is $f_1(t) = \partial^2 y(0, t)/\partial x^2$.
31. Derive Equation 8.96.
32. Derive Equation 8.97.

Chapter 9

Random Vibration: Probabilistic Forces

“The uncertainties must be confronted.”

9.1 Introduction

The main goal for studying dynamic systems under random loading is to predict the response (output) statistics given the loading (input) statistics. *Statistics* is the discipline that organizes data in a form that is meaningful and useful.

We return to the governing equation for a linear second-order system with loading, that is, a forced oscillator, given by [Equation 2.5](#),

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{1}{m}F(t), \quad (9.1)$$

where the force $F(t)$ is now a random function of time. If $F(t)$ is a deterministic function of time, we solve the convolution integral for the vibration response. We now face the problem where $F(t)$ is not deterministic.

Suppose the force oscillates in a complex manner as shown in [Figure 9.1](#). One possibility is to carry out many experiments and gather data on $F(t)$ in the form of time histories. Then, the time history with the largest amplitude can be used for the deterministic analysis and design. But, if the largest amplitude force occurs only infrequently, the system would be over-designed. In other words, the system would be designed to be stronger than it needs to be, making it uneconomical.

We distinguish between inherently random molecular forces, such as the Brownian motion experienced by atoms on a molecular scale, and the environmental forces of concern here. Environmental forces may not be inherently random, although they undergo very complex cycles. Natural randomness is called *aleatory variability*, and scientific uncertainty in the model of a process due to limited data or knowledge is called *epistemic uncertainty*. We are unable to model such complex phenomena using deterministic techniques. Therefore, we adopt the tools of *probability* and statistics to provide us with a way to quantify our complexities.

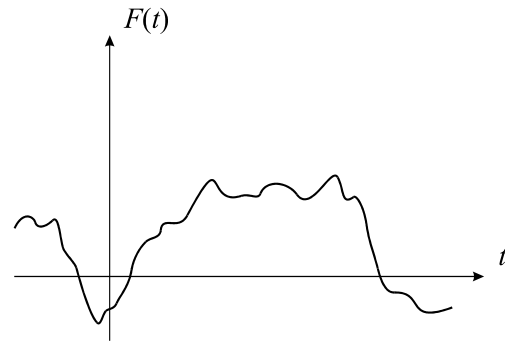


Figure 9.1: An example of a random time history.

What if all the time histories were averaged and this *average* or *mean value time history* is used as the deterministic load in the convolution integral, introduced in [Chapter 4](#)? This would be a good start, but the response calculated in this manner would underestimate the actual response too often. How often depends on the scatter, or standard deviation, of possible time histories.

The next question of interest is how much scatter there is above and below the mean value response. If we knew the mean value response as well as a measure of the scatter, this information could be used to achieve a safe and economical design. This approach makes sense.

Finally, we can ask how the engineer knows the frequency of occurrence of very large amplitude forces. If a very large force, such as an earthquake, occurs only once in 100 years, how is that fact used in a design? What the designer needs is a way to give more *weight* to more likely events, without completely ignoring more severe but less likely events. We will learn that the *probability density function* acts as such a weighting function.

All of the ideas described above are actually probabilistic concepts. We will first develop some introductory ideas with motivating examples. This will set the stage for our effort at random vibration modeling.

Before proceeding, it should be noted that other tools exist for evaluating how parameter variations affect changes in system response. A powerful tool for doing this is known as *sensitivity analysis*, introduced

in Section 10.6 of Chapter 10. Sensitivity analysis can be used to understand whether small variations in certain parameters translate into small or large variations in other parameters of interest or in the response. Sensitivity analysis can be viewed as complementing – not replacing – the probabilistic approach introduced in this chapter.

A Definition for Probability

The definition of probability is itself a debate. Some view it as subjective, and others believe that only through experimentation can a rigorously derived probability be possible. A difficulty is that it is often not possible to perform enough experiments to arrive at that rigorous probability and then judgment must be used. Fortunately, for our purposes, we do not need to resolve this debate. We will assume that in some manner it is possible to obtain the probabilities necessary for our computations, and usually these will be based on data analysis.¹

Think of a vibrating oscillator exhibiting random behavior, meaning the response amplitudes and frequencies are unpredictable. They appear to vary from one instant of time to the next. How can we answer a question such as: What is the probability that the displacement amplitude A is greater than a specific value A_0 ? Using probability notation for this question, we have

$$\Pr\{A > A_0\} = ?$$

This probability is related to the fraction of time the oscillator displacement has an amplitude greater than A_0 , implying a *fraction* or *frequency* interpretation for probability. We can look at a long time history of the oscillation and determine the amount of time the amplitude A is greater than A_0 . That *excursion frequency* is the probability estimate,

$$\Pr\{A > A_0\} = \frac{\text{amount of time } A > A_0}{\text{total time history}}.$$

For example, if the oscillation time history is 350 hours and for 37 of those hours $A > A_0$, then our estimate is

$$\Pr\{A > A_0\} = \frac{37}{350} = 0.106.$$

The probability that $A > A_0$ is estimated to be 10.6%. This is an estimate only since we expect that if the test lasted 3500 h instead of 350 h there would be some change in the estimated probability. A sign of an accurate estimate is that the probability estimate has approximately converged after a certain amount of time.

¹**Random Data: Analysis and Measurement Procedures**, J.S. Bendat and A.G. Piersol, 2nd ed., Wiley, 1986. This book provides an excellent development of the theory and techniques of data analysis.

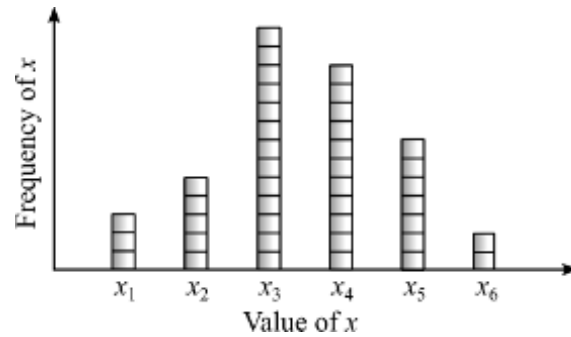


Figure 9.2: A histogram showing the relative frequency of a particular value x_i .

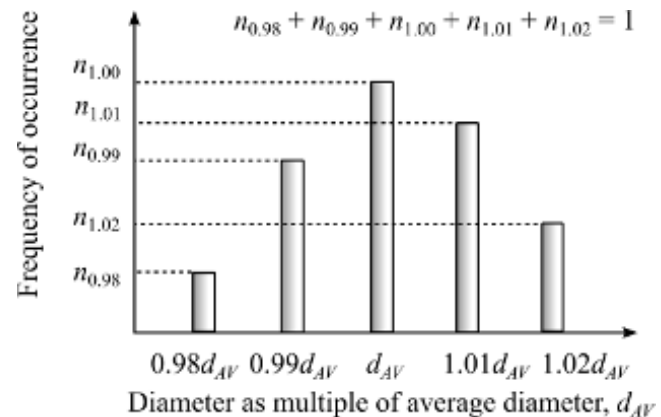


Figure 9.3: Histogram of machine shaft diameters. Each histogram is a fraction of one, and the sum of all the fractions equals one. The notation n_d denotes the number of shafts n of diameter d .

Figure 9.2 depicts a *histogram* that can be used to estimate probabilities using frequencies of occurrence, which is the most common approach to estimating probabilities, and the one we use here.

While the discussion above of the meaning of probability involved time, this is not always the case. Probabilistic models of mechanical systems are a natural result of the observation that most physical variables may take on a range of possible values. For example, if 100 machine shafts are manufactured, there will be 100 different diameters if measurements include enough significant figures. Figure 9.3 depicts a histogram of diameter data. As expected, the diameters are very close in value, but not exactly the same.

How do we account for such a spread of values if we are interested in estimating the strength of the shaft in torsion? What numbers should be substituted into the stress-strain relation? Similarly, running ultimate tensile-strength tests a number of times on “identical specimens” will show no two identical results. Small differences in dimensions, material properties, and boundary conditions make it impossible to exactly duplicate experimental results. There will always be some scatter.

How should this information be utilized and organized?

Randomness is possible for constants as well as functions (of time or space). A constant with a scatter of possible values is called a *random variable*. A function with scatter is generally called a *random process* or a *stochastic process*.² Random variables are those that can be prescribed only to a specific level of certainty. An example is the material yield stress that defines the transition from elastic to plastic behavior. *Random processes* are time-dependent (or space-dependent, or both) phenomena that, with repeated observation under essentially identical conditions, do not show the same time histories.

It is increasingly important to understand and model uncertainties and qualitative information in engineering analysis and design. An example of qualitative information is a verbal description of size or strength. Developing the ability to analyze uncertainties allows the engineer to decide for which applications uncertainties are significant and cannot be ignored, and for which they are insignificant and may be ignored.

Chapter Outline

A steep learning curve exists before it is possible to consider even the simplest random vibration problem. A new way of thinking with uncertainty as our *paradigm*³ must be learned. In some ways the probabilistic paradigm is very uncomfortable for engineers since we are trained to believe that, given enough experiments and theoretical development, any problem is solvable exactly – or at least to within measurement tolerances. We are about to learn that certainty exists only in idealized models, not in the physical systems that must be understood and designed. Although nothing is exact, sometimes uncertainty can be ignored for particular applications.

When uncertainty cannot be ignored, some basic concepts in probability must be learned. Be patient and do not be discouraged at what may at first reading appear to be a diversion from what we are primarily interested in: *vibration*. This is no more a diversion than was our study of ordinary differential equations. We include an introduction to probability since it is not commonly covered by engineers studying vibration.

We will provide an introduction to probability in [Sections 9.2 to 9.7](#) and begin to apply these analytical tools to vibration in [Section 9.8](#). We consider some motivating examples first.

²From the Greek we also have *stochastic* ($\sigma\tau\omicron\kappa\omicron\varsigma$) process.

³A paradigm is a way of thinking. It may be viewed as the beliefs, values, and techniques shared by a particular group of people. Therefore, a new paradigm in a technical area implies a new way of thinking about that area. An example of a *paradigm shift* was the development of the field of *chaos* in nonlinear dynamics.

9.2 Motivation

In order to demonstrate the importance of uncertainty modeling in mechanical systems, we present examples touching on a range of topics, including random vibration, fatigue life, ocean wave forces, wind forces, and material properties.

9.2.1 Random Vibration

The discipline of random vibration⁴ of structures was born of the need to understand how structures respond to dynamic loads that are too complex to model deterministically. Essentially, the question that must be answered is: Given the statistics (uncertainties) of the loading, what are the statistics (most likely values with bounds) of the response? Generally, for engineering applications the statistics of greatest usefulness are the mean or average value and the variance or scatter.

Example 9.1 Aerodynamic Loads

Suppose that we are aircraft designers designing a wing for a new airplane. As engineers, we are very familiar with the strength of materials and can size the wing for static loads. Also, we have vibration experience and can evaluate the response of the wing to harmonic or impulsive forcing. But, this wing will be attached to an airplane flying through a turbulent atmosphere. How might we approach the problem of designing under uncertainty?

Solution: Turbulence is a very complicated physical process. In fact, the fluid (air) motion is so complicated that probabilistic models are required in order to make any progress. Here, a plausibly deterministic but very complicated dynamic process is taken to be random for purposes of modeling.

For wing design, estimates are needed of the loads resulting from the interaction of the air and the wing. An *averaging* of forces and moments is needed. The question is how to make sense of such intricate fluid motion.

The next step is to run some scale model tests. A wing section is set up in the wind tunnel and *representative* aerodynamic forces are generated. Data on wind forces and structural response are gathered and analyzed. With additional data analysis, we begin to have some measures

⁴There are a number of worthwhile texts on random vibration. A useful textbook that includes a broad spectrum of theory and application is **Probabilistic Methods in the Theory of Structures**, I. Elishakoff, Wiley-Interscience, 1983. Two early books on random vibration are **Random Vibration in Mechanical Systems**, S.H. Crandall and W.D. Mark, Academic Press, 1963 and **An Introduction to Random Vibration**, J.D. Robson, Elsevier, 1964. See also **An Introduction to Random Vibrations and Spectral Analysis**, D.E. Newland, Longman, 1975, with its third edition now available through Dover Publications, and **Probabilistic Models in Engineering and Science**, H. Benaroya, S. Han, and M. Nagurka, CRC Press, 2nd edition, 2013.

of the expected force magnitudes. Estimates of likely or mean values of these forces are possible as well as of the range of possible forces.

Now we can begin to study the behavior of the wing under a variety of realistic loading scenarios using the tools of probability to model this complex physical problem. ■



Figure 9.4: Excessive vibration and noise are common powerboat problems. (Milwaukee, WI; photograph by M. Nagurka)



Figure 9.5: The waves at the shore exhibit random behavior. (Corpus Christi, TX; photograph by M. Nagurka)

9.2.2 Fatigue Life

The fatigue life of mechanical components and structures⁵ depends on many factors such as material properties, temperature, corrosion environment, and vibration history. A first step in estimating fatigue life involves the characterization of the cycles the structure has experienced. How many cycles did the structure undergo; what were the amplitude ranges; was the loading har-

monic or of broad frequency band? The fatigue life of a structure is a function of these characteristics.

Fatigue life estimates are intimately linked to the reliability of machines and structures. They determine how often components need to be replaced, the economics of the operation of the machine, and the insurance rates.

Anyone studying fatigue life data will be struck by the significant scatter. Components normally considered to be identical can have a wide range of lives. As engineers, we seek a rigorous basis for estimating the fatigue lives of ostensibly identical manufactured components. Eventually, it is necessary to relate the life estimate of the structure to those of its components. This is a difficult task, one that requires the ability to evaluate the structural and machine response to random forces.

Miner's Rule for Fatigue Damage

An early paper on the estimation of fatigue life was written by Miner,⁶ who was a strength test engineer with the Douglas Aircraft Company. *Miner's rule* is a deterministic way to deal with the uncertainties of structural damage and fatigue.

The phenomenon of cumulative damage under repeated loads is assumed to be related to the total work absorbed by a component. If a component can undergo N cycles before failure, and it has undergone n cycles, then the fraction n/N is taken to be the fraction of its useful structural life expended. Then $1 - n/N$ gives the remaining life cycles.

When the total damage reaches the value one, the component is taken to fail. Miner presented experimental verification using aluminum sheets.

At a certain stress level for a specific material and geometry,⁷ Miner's rule estimates the number of cycles to failure. Mathematically, this can be written as

$$\frac{n}{N} \leq 1, \quad (9.2)$$

where n equals the number of cycles undergone by the structure at a specific stress level and N equals the experimentally known number of cycles to failure at that stress level. When $n = N$ the component or structure is assumed to fail.

Since most structures undergo a mixture of loading cycles at different stress levels, Equation 9.2 is written for each stress level i as follows,

$$\sum_i \frac{n_i}{N_i} \leq 1, \quad (9.3)$$

⁶M.A. Miner, "Cumulative Damage in Fatigue," *Journal of Applied Mechanics*, September 1945, pp.A159-A164.

⁷Corners and discontinuities cause high stress concentrations resulting in lower fatigue life.

⁵A useful book in the study of fatigue is **Prediction of Service Life for Machines and Structures**, V.V. Bolotin, ASME Press, 1989.

where each fraction represents the percentage of life used up at each stress level. Suppose we have two stress levels, $i = 1, 2$, with corresponding lives, $N_1 = 100$ and $N_2 = 50$. According to Equation 9.3, the following relation holds

$$\frac{n_1}{100} + \frac{n_2}{50} \leq 1$$

between the number of possible cycles n_1 and n_2 for each stress level. There are numerous combinations that lead to failure. For example: $(n_1, n_2) = (50, 25)$, $(n_1, n_2) = (100, 0)$, $(n_1, n_2) = (0, 50)$, with others easy to find.

Miner realized that these summations were only approximations. His experiments showed that sometimes a component failed before the sum totaled one. Other times it did not fail until the sum was greater than one. Furthermore, failure by this rule is independent of the ordering of the stress cycles. This means that fatigue life, as estimated by Miner's rule, is the same whether high stress cycles precede or follow lower stress cycles. We know, however, that stress history affects fatigue life.

In the 70+ years since Miner's paper, a vast amount of work has been expended to better understand fatigue, but Miner's rule and its variants are still a widely utilized practical method.

9.2.3 Ocean Wave Forces

Similar approaches can be used to model ocean wave forces on structures and wind forces on structures. The differences are primarily due to the added mass of the water and to differences in structural types found in the ocean. This topic is a subset of ocean engineering,⁸ a specialty that draws from many engineering disciplines.

The estimation of wave forces on offshore oil drilling platforms, ships, and other ocean and hydraulic structures – such as water channel spillways and dams – is very important. Without these estimates there is no way to analyze or design the structure. The estimation of loads is always first on the list of tasks for an engineer, known as ocean engineering.

Example 9.2 Wave Forces on an Oil Platform

The need to drill for oil in the oceans has driven interest in designing ocean structures for sites of ever increasing depths. Many fixed-bottom ocean structures, when taken with their foundations, are taller than our tallest skyscrapers. Some are on the order of 500 m tall. As might be expected, the dynamic responses of these structures to ocean waves and currents are significant

⁸ Some useful books are the following: **Dynamics of Offshore Structures**, J.F. Wilson, 2nd ed., Wiley, 2003;

Sea Loads on Ships and Offshore Structures, O.M. Faltinsen, Cambridge University Press, 1990; **A Course in Ocean Engineering**, S. Gran, Elsevier Science, 1992;

Stochastic Dynamics of Marine Structures, A. Naess, T. Moan, Cambridge University Press, 2013.

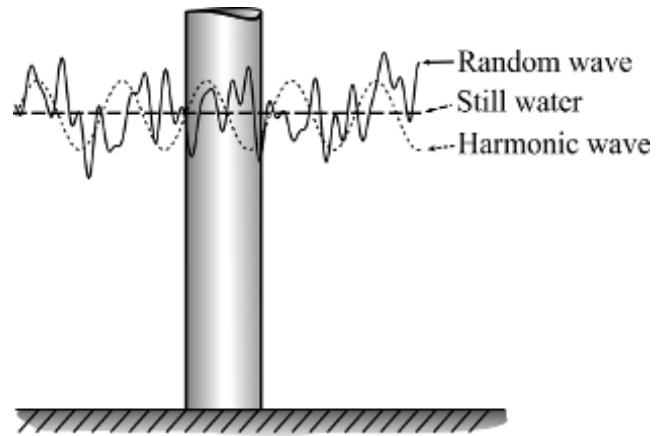


Figure 9.6: Schematic of waves moving past a cylindrical structure. The still water level is shown as is a harmonic wave and a random wave.

and need to be understood and analyzed. We consider the effect of ocean waves on a simple structure, shown in Figure 9.6. How might we approach the modeling of the forces due to these waves?

Solution: The most important paper,⁹ written over half a century ago, on the force exerted by ocean waves on fixed structures derived what is known as the *Morison equation*. After much experimental work, Morison and colleagues determined that the force exerted by *unbroken* surface waves on a circular cylindrical column that extends from the bottom of the ocean upward above the wave crest is made up of two components:

(a) a drag force proportional to the square of the wave particle velocity, with proportionality represented by a drag coefficient having substantially the same value as for steady flow, and

(b) an inertia force proportional to the horizontal component of the inertia force exerted on the mass of water displaced by the column, with proportionality represented by an *inertia coefficient*.

The Morison equation is an empirical model summing these two force contributions. The drag force on an element of a structural cylinder of length dx is given by

$$dF_D(t) = C_D \rho D \frac{u|u|}{2} dx,$$

where C_D is the experimentally determined drag coefficient, ρ is the density of water, D is the diameter of the cylinder, and u is the instantaneous horizontal water particle velocity. The term $u|u|$ ensures that the direction of the force on the structure is in the direction of the flow.

The inertia force on an element of a structural cylinder

⁹ J.R. Morison, M.P. O'Brien, J.W. Johnson, and S.A. Schaff, "The Force Exerted by Surface Waves on Piles," *Petroleum Transactions*, Vol.189, 1950, pp.149-154.

of length dx is given by

$$dF_I(t) = C_I \rho \frac{\pi D^2}{4} \dot{u} dx,$$

where $\dot{u}$ is the instantaneous horizontal water particle acceleration and C_I is an inertia coefficient. The dimensionless drag and inertia coefficients, C_D and C_I , respectively, are functions of flow characteristics, cylinder diameter, and fluid density. Depending on the application, they may be effectively constant or vary randomly.

The Morison force appears as the forcing function in fluid-structure interaction problems,

$$F(t) = \int dF_D(t) + \int dF_I(t).$$

Principles of classical deterministic fluid mechanics are used to derive wave particle velocities and accelerations.

Many tall ocean structures oscillate appreciably in response to the forces acting on them. To take this into account, the relative velocity and acceleration between the fluid and the structure is used in the Morison equation, where u is replaced by $(u - \dot{x})$ and $\dot{u}$ is replaced by $(\dot{u} - \ddot{x})$, where $\dot{x}$ and $\ddot{x}$ are the structural velocity and acceleration, respectively.

Also, in order to better characterize the complexity of the wave motion, fluid velocity, acceleration, and the resulting force are modeled as random functions of time. We will explore the concept of random forces in detail later in this chapter. More details on the vibration of structures in fluids can be found in numerous books.¹⁰ ■

9.2.4 Wind Forces

Engineering structures such as cooling towers, aircraft, skyscrapers, rockets, and bridges are all exposed to wind and aerodynamic loads. Wind is the natural movement of the atmosphere due to temperature and pressure gradients. Aerodynamic loads are the atmospheric forces resulting from the interaction of the wind and the structure. While we know how to write an equation for a harmonic force, what does an equation for wind force look like? Due to the complexity of the fluid mechanics of wind, it is generally necessary to approximate the force due to the wind. There are various levels of approximate relations, depending on the application. In all instances, the force relation includes at least one experimentally determined parameter or coefficient. Such

¹⁰These two books are recommended: **Flow-Induced Vibration**, R.D. Blevins, van Nostrand Reinhold, 1977 and its second edition; **Estuary and Coastline Hydrodynamics**, A.T. Ippen, Editor, McGraw-Hill, 1966.

Also consider **Nonlinear and Stochastic Dynamics of Compliant Offshore Structures**, S. Han and H. Benaroya, Kluwer Academic Publishers, 2002.

semi-empirical force equations are valuable in engineering practice. They look very much like the Morison equation.



Figure 9.7: The aerial bucket on a boom truck can experience significant vibration due to motion for positioning as well as aerodynamic loading. (Charlotte, NC; photograph by M. Nagurka)



Figure 9.8: Hurricanes cause loss of life as well as collapse of buildings. This house escaped with only slight damage due to Hurricane Katrina in 2005. (Tamarac, FL; photograph by M. Nagurka)

9.2.5 Material Properties

While the modeling of randomness in material properties is beyond our scope, we mention this type of modeling because of the importance of many new materials that have *effective* properties, that is, properties that are an average over a cross section. These include various *compos-*

ites and *tailored materials*, such as carbon fiber. These modern materials are designed for particular structural applications, especially where high strength, durability, and low weight are needed. The design requires that a complicated mix of fibers and substrates be organized to yield particular properties. The difficulty then is to model these materials so that their properties are included in a vibration analysis. Defining stress-strain relations and Young's modulus, for example, is not straightforward. It is sometimes necessary that properties be averaged or effective properties be defined.

Soil is a naturally occurring material that is very complex and cannot be modeled in a traditional manner. It is common that two nearby volumes of soil have very different mechanical properties. Therefore, in structural dynamics applications, such as earthquake engineering, the loading is effectively random, in part, because by the time it reaches the structure, the force has traversed through a complex topology of earth.¹¹

Data on the variability of material properties are tabulated in numerous references. From Haugen,¹² for example, hot rolled 1035 steel round bars of diameters in the range 1 to 9 in have yield strengths of between 40,000 to 60,000 psi, with an average yield of just under 50,000 psi. In addition, the variability can change appreciably depending on temperature. For example, a titanium-aluminum-lead alloy has an ultimate shear strength of between 88,000 to 114,000 psi at 90°F, but at 1000°F the strength drops to between 42,000 to 60,000 psi. This variability, which can be significant, is a function of different causes and it is therefore necessary to know the environment where the structure will operate during its lifetime. Sometimes, a structure or machine will operate in many very different environments and temperatures, making the design more challenging. While temperature and thermal effects are not discussed here, these can be critical factors in many advanced aerospace and machine designs.

9.2.6 Statistics and Probability

The previous examples of natural forces have one factor in common: they depend on experimentally determined parameters. Just as linearity assumes small oscillations, these semi-empirical equations are valid only for a particular domain of data. While deterministic models also depend greatly on experimental data for their formulation (and ultimately their validity), random models are an attempt to deal explicitly with observed scatter in the

data and with the resulting intricate dynamic behavior. Random models also show how data scatter affects response scatter.

Data are always our link to valid probabilistic models, their derivation and validation. The step of acquiring data is the basis for any valid probabilistic model used in applications.

From Data to Model and Back to Data

As we have implied, modeling can be as much an art as a science. Engineers are generally handed a problem that needs to be solved, not an equation, and often not even a well thought out description of the problem. For example: *We need to go to the Moon in ten years!*

Engineering is predicated on understanding how structures and materials behave under various operating conditions. This understanding is based on theory and *data*. Many experiments have been performed to get us to our current level of understanding and intuition about vibration. They may suggest *cause* and *effect* relationships between variables. They provide us with parameter values. Finally, they are the basis for the equations we derive.

Data has scatter and the significance of scatter in a particular problem determines whether it can or cannot be ignored. If it cannot be ignored, then data is used to estimate the statistics of the randomness. The resulting probabilistic model is used to study the particular problem at hand. The model's validity is established by comparing its predictions with available data. Such comparisons help define the limits of model validity.

In this way, a full circle has been achieved. Data give birth to understanding and parameter values, which lead to mathematical models and their predictions. Validity of these models is established by comparing their predictions with new data that are not part of the original set.

¹¹This area of research is known as *earthquake engineering* and the specific study of how energy propagates through complex materials such as soils is part of the study of *waves in random media*.

¹²**Probabilistic Mechanical Design**, E.B. Haugen, Wiley-Interscience, 1980, includes applications of probability to mechanical engineering, primarily based on the Gaussian distribution.

MARY WINSTON JACKSON

(April 9, 1921 – February 11, 2005)



Contribution: Mary Winston Jackson was an American mathematician and aerospace engineer at the National Advisory Committee for Aeronautics (NACA), which later became the National Aeronautics and Space Administration (NASA).

Life Sketch: Mary Jackson was born to Frank and Ella Scott Winston in Hampton, Virginia. She grew up and attended high school there, where she excelled. She earned her bachelor's degrees in mathematics and physical science from Hampton Institute in 1942. After graduating, Jackson taught school in Maryland.

She joined the National Advisory Committee for Aeronautics (NACA) in 1951. She began her career there as a research mathematician, or “computer,” at the Langley Research Center in her hometown of Hampton, Virginia. In 1953 she moved to the Compressibility Research Division. After five years and after taking several additional courses, she joined a special training program and was promoted to aerospace engineer. Her work was analyzing data from wind tunnel experiments and real-world aircraft flight experiments at the Theoretical Aerodynamics Branch of the Subsonic-Transonic Aerodynamics Division at Langley. Her goal was to understand air flow, including thrust and drag forces. Many years later, she was assigned to work with the flight engineers at NASA.

Jackson worked to help women and other minorities to advance their careers, including advising them how to study so that they could change their titles from “mathematician” to “engineer” to increase their chances of promotion.

After 34 years at NASA, Jackson reached the highest level of engineer that was possible without becoming a supervisor. She decided to take a pay cut and change positions to become an administrator in the Equal Opportunity Specialist field. After undergoing training at NASA Headquarters, she returned to Langley where she worked to make changes and highlight women and other minorities who were accomplished in their fields. She served as both the Federal Women's Program Manager in the Office of Equal Opportunity Programs, and as the Affirmative Action Program Manager. She worked at NASA until her retirement in 1985.

Jackson married and had two children.

Noteworthy: Jackson is featured in the 2016 film *Hidden Figures* about three African American women at NASA who calculated flight trajectories for Project Mercury and Apollo 11 in the 1960s. It is based on the book by the same name by author Margot Lee Shetterly that documented the NASA careers and contributions of Katherine Johnson, Dorothy Vaughan, and Jackson.

engineering analysis and design. Mathematical models of physical phenomena are essentially relationships between variables. Where some of these variables have associated uncertainties, there are a multiplicity of possible values for each random variable. An example is the set of possible values of Young's moduli determined from a series of experiments on “identical” test specimens. This multiplicity is represented by the *probability distribution function*, introduced in the following section. A random variable may be *discrete*, *continuous*, or *mixed*. If a parameter is a random variable, its probability distribution function provides a complete description of its variability.

In the following discussion, we adopt the notation that random variables are represented as capital letters. Exceptions are sometimes made for variables with Greek letters. We use lower case letters to denote *realizations*, or specific values, of random variables.



Figure 9.9: Aerodynamic effects on a flag can be viewed as an example of random vibration. (La Crosse, WI; photograph by M. Nagurka)

9.3.1 Probability Distribution

The likelihood that a random variable takes on a particular range of values is defined by its *cumulative distribution function*,¹³ $F_X(x)$,

$$F_X(x) = \Pr\{X \leq x\},$$

where $\Pr\{X \leq x\}$ is the probability that the random variable X is less than or equal to a particular value x , which is called the *realization*¹⁴ of X . This probability is a function of the particular value x , shown in Figure

¹³ $F_X(x)$ is *cumulative* since probability is accumulated as x becomes larger.

¹⁴ A *realization* is one of many possible values of a random variable.

9.3 Random Variables

We begin to explore the properties of *random variables*. Probability affords us a framework for defining and utilizing random variables in the models developed for en-

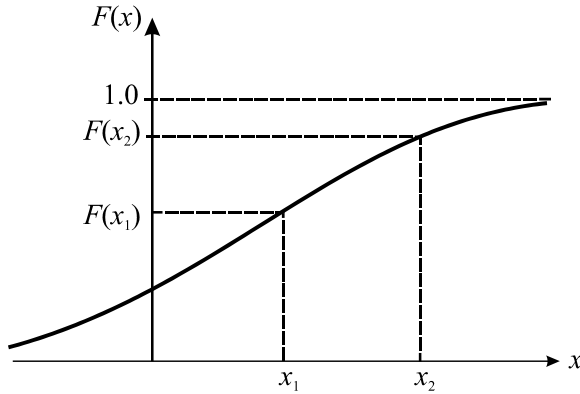


Figure 9.10: Cumulative distribution function, $F_X(x) = \Pr\{X \leq x\}$.

9.10. We will omit the subscript in $F_X(x)$ where clarity is not an issue.

Based on the axioms of probability,¹⁵ it can be shown that $F(x)$ is an increasing function of x , and is bound by 0 and 1. An *impossible* event has a zero probability, and a *certain* event has a probability of one. In particular,

$$F(-\infty) = 0,$$

since $\Pr\{X < -\infty\} = 0$. All realizations of the random variable must be greater than negative infinity. Similarly,

$$F(+\infty) = 1,$$

since $\Pr\{X < +\infty\} = 1$. All realizations of the random variable must be less than positive infinity. Thus, bounds on $F(x)$ are $0 \leq F(x) \leq 1$, and for $x_1 < x_2$,

$$F(x_1) \leq F(x_2),$$

since $\Pr\{X \leq x_1\} \leq \Pr\{X \leq x_2\}$. The probability distribution function is *non-decreasing*.

The cumulative distribution function is one way to probabilistically describe a random variable. We introduce the probability density function to answer the question posed at the beginning of this chapter: How is the more likely force given added *weight* in the computation of structural response?

9.3.2 Probability Density Function

The *probability density function* contains the same information as the probability distribution function. As-

¹⁵ An axiom is a rule that is assumed to be true and upon which further rules and facts are deduced. For engineering, the deduced facts must conform to reality. An excellent book on the basics of probabilistic modeling is **Probability, Random Variables, and Stochastic Processes**, A. Papoulis, McGraw-Hill, 2002. There are several editions. We find the first edition (1965) most readable. We also recommend other fine texts by Papoulis on probability and stochastic processes. A different approach to explaining probability through problem solving is offered in **The Probability Tutoring Book: An Intuitive Course for Engineers and Scientists (and everyone else!)**, C. Ash, IEEE Press, 1993.

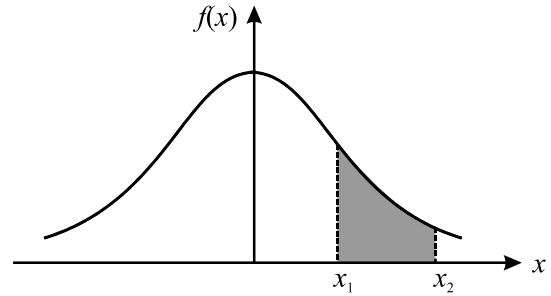


Figure 9.11: Probability density function $f(x)$. The probability that X is between x_1 and x_2 equals the area of the shaded region.

suming continuity of the distribution,¹⁶ the probability density function $f_X(x)$ is defined as

$$f_X(x) = \frac{dF_X(x)}{dx},$$

and is sketched in Figure 9.11. Alternately, by integrating both sides and rearranging,

$$F(x) = \Pr\{X \leq x\} = \int_{-\infty}^x f(\xi) d\xi, \quad (9.4)$$

where we have omitted the subscripts.

Equation 9.4 provides a useful interpretation of the density function: the probability that a continuous random variable X has a value less than or equal to the number x is equal to the area under the density function for values less than or equal to x . Similarly, for arbitrary x_1 and x_2 , the probability that $x_1 < X \leq x_2$ is

$$\Pr\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f(x) dx. \quad (9.5)$$

It is customary to write $\Pr\{x_1 < X \leq x_2\}$ with $<$ rather than $\leq$ on the left side so that there is no overlap between probability ranges, that is, between $\Pr\{x_0 < X \leq x_1\}$ and $\Pr\{x_1 < X \leq x_2\}$.

Note the important *normalization* property,

$$\int_{-\infty}^{+\infty} f(x) dx = 1, \quad (9.6)$$

signifying that the density function represents all possible outcomes or realizations of the random variable. Therefore, the area under the density function needs to be *normalized* to one. Since probability is numerically in the range 0 to 1, the density function must be a positive semi-definite¹⁷ function: $f(x) \geq 0$. For a continuous

¹⁶ The distribution function does not have to be a continuous function. In many instances it may have discrete jumps where a finite probability exists for a certain realization. It is just easier to work with a continuous function.

¹⁷ A function that is *positive definite* has all values greater than zero. A function that is *positive semi-definite* may also have values that are equal to zero.

random variable, the probability $\Pr\{X = x\} = 0$ since there is no area under the density function at a point x .

The random variable is a static property – the shape of the density function does not change with time. Where the density function is time-dependent, the variable is called a *random*, or *stochastic*, *process*. This more advanced concept is discussed in [Section 9.7](#).

Example 9.3 Use of the Density Function

The probability density function of random variable X is experimentally found to be $f(x) = ce^{-|x|}$. Evaluate the constant c and then find $\Pr\{-2 < X \leq 2\}$.

Solution: The constant c must be determined first by the normalization property before the density function can be used to derive probabilities of events. From [Equation 9.6](#), the area under the density function must equal one,

$$\begin{aligned} c \int_{-\infty}^{\infty} e^{-|x|} dx &= 1 \\ 2c \int_0^{\infty} e^{-x} dx &= 1 \quad \Rightarrow c = \frac{1}{2}, \end{aligned}$$

making $f(x) = \frac{1}{2}e^{-|x|}$. Then,

$$\Pr\{-2 < X \leq 2\} = \int_{-2}^2 \frac{1}{2}e^{-|x|} dx = 1 - e^{-2} = 0.865.$$

Such density functions have numerous engineering applications. The exponential density is used for reliability analysis and is introduced in [Section 9.5.2](#). ■

Before examining some commonly used densities, we define an averaging procedure known as the mathematical expectation for probabilistic variables.

9.4 Mathematical Expectation

The single most important descriptor of a random variable is its *mean* or *expected value*. This defines the most likely value of a variable. However, random variables may have the same mean, but their *spread of possible values*, or their *variance*, can be considerably different. This explains the need for the variance in addition to the mean value.

The mean and variance of a random variable are statistical averages and are evaluated using the concept of the *mathematical expectation of a function of random variable* X , defined as

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x)f(x)dx. \quad (9.7)$$

The *expected* or *mean value* is defined, using [Equation 9.7](#), as

$$\mu = E\{X\} = \int_{-\infty}^{\infty} xf(x)dx, \quad (9.8)$$

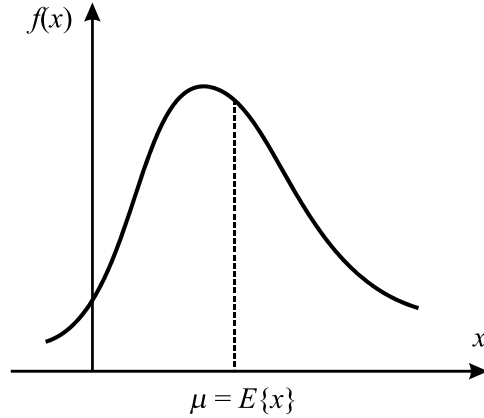


Figure 9.12: The expected value shown here for a generic density function is at the “center of mass” of the area under the curve.

where x is a realization of X . The integral is over all possible realizations with $f(x)$ acting as a weighting function. The expected value is a constant, *first-order* statistic and is also known as the *first moment* because the variable X appears to the first power. The term moment is used by analogy to the center of mass in mechanics. $E\{X\}$ is the center of “probability mass,” as shown in [Figure 9.12](#).

The density function, acting as a probabilistic “weighting” function, is a larger factor in the integral for more probable values of the random variable. From the definition of expectation, the expected value of a constant is that constant.

Example 9.4 Expected Value

[Equation 9.8](#) is written for a continuous random variable. There are instances when the variable of interest is discrete. How is a discrete variable analyzed?

Solution: For a discrete random variable, the integral in the mathematical expectation becomes a summation. Suppose an experiment for the yield strength of a material is performed ten times giving the data:

10.0, 9.8, 11.1, 9.1, 9.9, 9.7, 10.3, 10.1, 9.9, 10.0.

To find the expected value we use the discrete counterpart to [Equation 9.8](#),

$$\mu = E\{X\} = \sum_{i=1}^{10} x_i f(x_i),$$

where x_i are the test results and $f(x_i)$ are the probability weights – in this case the fraction of times that a particular value occurred. For the data listed, test results that occur once have a probability of 1/10. Results such as 9.9 and 10.0 that occur twice have a probability of 2/10. Note that $\sum_{i=1}^{10} f(x_i) = 1$ signifying that all

possible outcomes have been included. Then,

$$\begin{aligned}\mu &= \left(10.0 \times \frac{2}{10}\right) + \left(9.8 \times \frac{1}{10}\right) + \left(11.1 \times \frac{1}{10}\right) \\ &+ \left(9.1 \times \frac{1}{10}\right) + \left(9.9 \times \frac{2}{10}\right) + \left(9.7 \times \frac{1}{10}\right) \\ &+ \left(10.3 \times \frac{1}{10}\right) + \left(10.1 \times \frac{1}{10}\right) \\ &= \frac{99.9}{10} = 9.99.\end{aligned}$$

On the other hand, if the yield strength is assumed to be continuous between the values 9.1 and 11.1 with uniform density $f(x) = 1/(11.1 - 9.1) = 1/2$, then

$$\mu = \int_{9.1}^{11.1} x \cdot \frac{1}{2} dx = 10.1.$$

This result is slightly different than that for the discrete case due to the uneven discrete distribution. The uniform density is introduced in the next section. ■

Example 9.5 Expected Value of a Function

Consider the case where the random variable X is governed by the density function $f(x) = cx^2$, $2 < x \leq 3$. Find the expected value of the function $g(X) = X^3$.

Solution: The first step in this solution is to calculate the normalizing constant c , as follows,

$$\begin{aligned}\int_2^3 cx^2 dx &= 1 \\ \frac{19}{3}c &= 1 \implies c = \frac{3}{19}.\end{aligned}$$

Then,

$$\begin{aligned}E\{g(X)\} &= \int_2^3 x^3 \cdot \frac{3}{19} x^2 dx \\ &= 17.5.\end{aligned}$$

The point of this problem is to emphasize that the mathematical expectation of a function of a random variable is calculated in the same way as that of the random variable itself. The density function acts as a weighting function. ■

In the same way we calculated the mean value, we can derive an equation that provides a measure of the scatter about the mean value.

9.4.1 Variance

The *variance* is a *second-order moment*. It is defined as

$$\begin{aligned}\text{Var}\{X\} &= E\{(X - E\{X\})^2\} \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx.\end{aligned}$$

Expanding the squared term and integrating term by term, we find the variance equal to

$$\text{Var}\{X\} = E\{X^2\} - (E\{X\})^2, \quad (9.9)$$

which is the difference between the *mean square value* and the *mean value squared*. Here, the second moment analogy is with the *mass moment of inertia*.

In order that the measure of dispersion has the same dimensions as the random variable, the *standard deviation* is defined as the positive square root of the variance,

$$\sigma = +\sqrt{\text{Var}\{X\}}. \quad (9.10)$$

A related and important dimensionless parameter is the *coefficient of variation*,

$$\delta = \frac{\sigma}{\mu}.$$

It is used as a non-dimensional measure of the degree of uncertainty in a parameter, that is, the scatter of its data. In engineering practice, one expects a δ value of between 0.05 to 0.15, or 5 to 15%. Values larger than this imply a serious lack of knowledge about the system itself and its underlying physics. If this is the case, then experiments are warranted before one can consider the analysis and design of such a system.

Example 9.6 Variance of a Random Variable

The random variable X is governed by the density function $f(x) = ce^{-x}$, where $5 < x \leq 13$. Evaluate the variance of this variable.

Solution: The first step is to calculate the normalization constant c ,

$$\begin{aligned}\int_5^{13} ce^{-x} dx &= 1 \\ c(e^{-5} - e^{-13}) &= 1 \\ \implies c &= \frac{1}{(e^{-5} - e^{-13})} = 148.5.\end{aligned}$$

Then, the variance can be evaluated,

$$\text{Var}\{X\} = E\{X^2\} - (E\{X\})^2,$$

where

$$\begin{aligned}E\{X\} &= \int_5^{13} (x) 148.46 \exp(-x) dx = 5.997 \\ E\{X^2\} &= \int_5^{13} (x^2) 148.46 \exp(-x) dx = 36.95,\end{aligned}$$

and, therefore, $\text{Var}\{X\} = 36.95 - 5.997^2 = 0.9860$. ■

To engineer a product such as a structure or a machine, we need to understand the behavior of materials,

the dynamic characteristics of the system, and the external forces. Usually, the largest uncertainties are with the loading. Even so, in practice we expect probability densities to have most of their area about the mean value – that is, with a small variance.

Sometimes in engineering the uncertainty is such that we only know the *high* and *low* values of a variable. In this instance it is assumed that all intermediate values are equally probable. This leads us to the uniform probability density, which we will study next. Other times our experience tells us that parameter values significantly different from the mean can happen, even if these are unlikely. A possible density is the Gaussian density (also studied below). What we see is that data from testing and design are crucial to the decision regarding the choice of the most physically realistic probability density.

9.5 Useful Probability Densities

It turns out that a handful of density functions are sufficient for probabilistic modeling in many engineering applications. Here, five are discussed: the uniform, exponential, normal (or Gaussian), lognormal, and the Rayleigh densities. As mentioned, our notation is a capital letter for the random variable and a lower case letter for its realization.

9.5.1 Uniform Density

The *uniform density* is an appropriate model for a variable with known upper and lower bounds and *equally likely* values within the range. From Figure 9.13 it can be seen that for any range of possible realizations Δx , the area under the uniform density curve (a *horizontal line*) is the same. According to Equation 9.5, there is equal probability for the variable to be in any range.

Suppose that X is a continuous random variable that can have any value in the interval $[a, b]$, where both a and b are finite. The probability density function is given by

$$f(x) = \begin{cases} 1/(b-a) & \text{if } a \leq x \leq b \\ 0 & \text{otherwise,} \end{cases} \quad (9.11)$$

and X is called *uniformly distributed*. The probability distribution function for a uniformly distributed random variable is found to be

$$\begin{aligned} F(x) = \Pr\{X \leq x\} &= \int_{-\infty}^x f(s) ds \\ &= \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \leq x < b \\ 1 & x \geq b. \end{cases} \end{aligned}$$

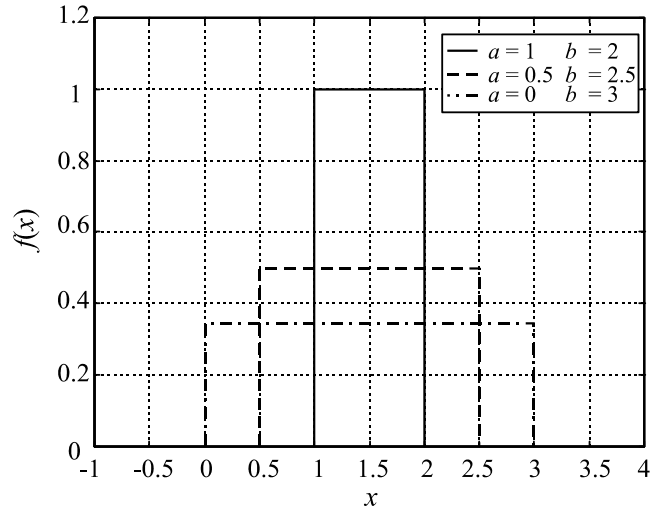


Figure 9.13: Uniform density function for three sets of upper and lower bounds.

The mean and mean square values of a uniformly distributed random variable X are given by

$$\begin{aligned} E\{X\} &= \int_a^b x \frac{1}{b-a} dx = \frac{b+a}{2} \\ E\{X^2\} &= \int_a^b x^2 \frac{1}{b-a} dx = \frac{b^2 + ab + a^2}{3}. \end{aligned}$$

The standard deviation is then found to be

$$\sigma = \frac{b-a}{\sqrt{12}} = 0.289(b-a).$$

As expected, the statistics are functions only of the upper and the lower bounds.

Example 9.7 Uniform Density

A constant force Z is known to have a value of between 10 lb and 25 lb, with no additional information. All values in the range are of equal probability. Find the most likely value and calculate the probability that the force $Z > 20$ lb. Also, find the variance and coefficient of variation.

Solution: For such cases, where any value in a range is equally likely, a uniform density is chosen, $f_Z = c$, where c is a constant. The normalization property of the density can be used to find the value of c ,

$$\int_{10}^{25} f(z) dz = \int_{10}^{25} c dz = 1,$$

resulting in the value $c = 1/15$. The mean value for a variable with uniform density is the midpoint between the upper and lower bounds, $\mu_Z = (25 + 10)/2 = 17.5$ lb. Or, we can reach the same result using the definition of mean value, $E\{Z\} = \int_{10}^{25} z f(z) dz$.

The probability that force $Z > 20$ lb is

$$\Pr\{Z > 20\} = \int_{20}^{25} \frac{1}{15} df = \frac{1}{3}.$$

The variance can be evaluated from the relation

$$\sigma_Z^2 = E\{Z^2\} - \mu_Z^2 = 18.75 \text{ lb}^2,$$

and the coefficient of variation is then

$$\delta = \frac{\sigma_Z}{\mu_Z} = \frac{4.3}{17.5} = 0.25$$

or 25%. This is a relatively large scatter about the mean value. In engineering applications, coefficients of variation greater than 15% imply a need for further data gathering. ■

Example 9.8 Quadratic Density

Suppose that instead of being uniformly distributed, Z is distributed according to a quadratic law, $f(z) = \alpha z^2$, also with $10 \text{ lb} < Z \leq 25 \text{ lb}$. Calculate $\Pr\{Z > 20\}$ and $\text{Var}\{Z\}$.

Solution: Following the above procedure, it is straightforward to find $\alpha = 0.00021$, $\mu_Z = 19.98 \text{ lb}$, and

$$\Pr\{Z > 20\} = \int_{20}^{25} 0.00021 z^2 dz = 0.53,$$

which makes sense, when compared to the uniform density results, since much more of the area under the quadratic density function is located near the upper end of the range. We have found that 53% of the area is in the range $20 < Z \leq 25$.

Here, the variance is $\sigma_Z^2 = 6.76 \text{ lb}^2$, a much smaller value than that for the uniform density, and the coefficient of variation is $\delta = 0.13$ or 13%, again signifying that the spread of values is much smaller for the quadratic density. ■

9.5.2 Exponential Density

For mechanical reliability,¹⁸ the exponential density is most commonly used to estimate failure times. The (failure) density is

$$f(t) = \lambda e^{-\lambda t}, \quad \lambda > 0, \quad t \geq 0,$$

where λ is a constant (failure) rate per unit time, and $1/\lambda$ is the mean (time to failure).

¹⁸A starting point for studying reliability is **Mechanical Reliability: Theory, Models and Applications**, B.S. Dhillon, AIAA, 1988.

Example 9.9 Time to Failure

A pump is known to fail according to an exponential density with a mean of 1000 h. Then $\lambda = 1/1000$. Suppose that a critical mission requires the pump to operate for 200 h. Calculate the failure probability.

Solution: For an exponential density, the probability distribution function is

$$F(t) = 1 - e^{-\lambda t}.$$

The probability that failure will occur before some specific time $t \leq t_0$ is given by $F(t_0)$. Here, $t_0 = 200 \text{ h}$, and

$$F(200) = 1 - e^{-200/1000} = 0.181.$$

The probability that the pump will fail during the first 200 h is 0.181 or 18.1% according to the distribution. Knowing this value will help in making the decision whether a backup pump needs to be on hand. ■

9.5.3 Normal (Gaussian) Density

Many physical variables are assumed to be governed by the normal or Gaussian density. There are two reasons for this: the Gaussian density is mathematically tractable and tabulated, and the *central limit theorem* is broadly applicable. The central limit theorem states that under very general conditions, as the number of variables in a sum becomes large, the density of the sum of random variables approaches the Gaussian density in the limit regardless of the individual densities.

Examples of variables that arise as the sum of a number of random effects, where no one effect dominates, are noise generated by falling rain, the effects of a turbulent boundary layer, and the response of linear structures to a turbulent environment. Many naturally occurring physical processes approach a Gaussian density.

The Gaussian density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad (9.12)$$

where $-\infty < x < \infty$, and the meanings of μ and σ are found by taking the expected value¹⁹ of X and X^2 , respectively,

$$E\{X\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-y^2/2} dy = \mu \quad (9.13)$$

$$E\{X^2\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu)^2 e^{-y^2/2} dy = \mu^2 + \sigma^2. \quad (9.14)$$

¹⁹We make use of the transformation of variables: $y = (x - \mu)/\sigma$ and note that $dx = \sigma dy$.

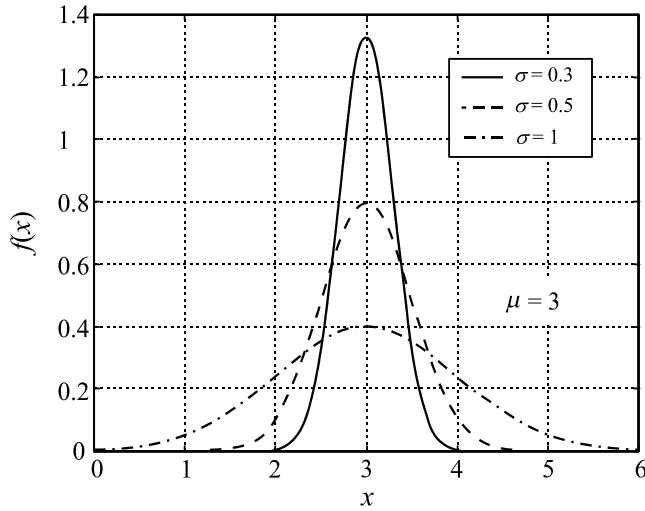


Figure 9.14: The normal or Gaussian density function for $\mu = 3.0$ with three possible values of σ : 0.3, 0.5, 1.0.

We see then that the mean value is μ and, using Equation 9.10, the standard deviation is σ .

Note that the Gaussian density extends from $-\infty$ to ∞ , and therefore cannot represent any physical variable. It is an *approximate* representation since there are no physical parameters that can take on all possible values on the real number line. The Gaussian approximation is valid in many instances.

For example, consider a positive-definite random variable X that is modeled as a Gaussian random variable with coefficient of variation $\delta = 0.20$, or $\mu = 5\sigma$. How significant is the area under the density function in the negative X region? Integrating for $x < 0$, one finds an area of approximately 24×10^{-8} , a negligible probability for almost all purposes. Thus, the suitability of a Gaussian model depends on the application and how far the *tails* extend into physically forbidden regions.

When it is not possible to accept any negative values,²⁰ we can resort to a truncated Gaussian density with the following density function,

$$f(x) = \frac{A}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right], \quad 0 \leq x_1 \leq x \leq x_2,$$

and zero elsewhere, as shown in Figure 9.15. (If $x_1 \rightarrow -\infty$ and $x_2 \rightarrow +\infty$, then $A \rightarrow 1$, and X becomes a Gaussian random variable with $E\{X\} = x_0$ and $\text{Var}(X) = \sigma^2$.)

For ease in applications, the Gaussian variable X is sometimes transformed to a new variable S with *zero mean* and *unit variance*,

$$S = \frac{X - \mu_X}{\sigma_X}.$$

²⁰ This is especially true in reliability calculations where the probabilities of failure may be very small, even on the order of 10^{-8} , and extra care must be taken to ensure that the density function is suitable.

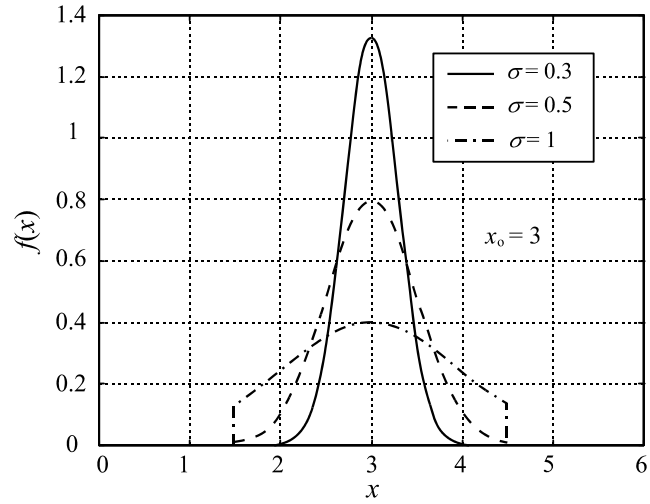


Figure 9.15: The truncated Gaussian density function for mean value $x_0 = 3.0$ with three possible values of σ : 0.3, 0.5, 1.0.

The resulting *standard normal density* is given by

$$f_S(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}. \quad (9.15)$$

The probability distribution is then

$$F_S(s) = \Pr\{S \leq s\} = \int_{-\infty}^s \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds, \quad (9.16)$$

where $F_S(s)$ can be found in Gaussian tables.

JOHANN CARL FRIEDRICH GAUSS

(April 30, 1777 – February 23, 1855)



Contribution: Gauss was a German mathematician and scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, electrostatics, astronomy, and optics. Gauss had a remarkable influence and is ranked as one of history's most influential mathematicians. He has been called the "prince of mathematics."

Life Sketch: Gauss was born in Brunswick, Duchy of Brunswick (now Germany). Gauss was a child prodigy. There are many anecdotes pertaining to his precocity while a toddler. At the age of seven, Gauss started elementary school, and his potential was noticed almost immediately.

In 1788 Gauss began his education at the Gymnasium, where he learned High German and Latin. He made groundbreaking mathematical discoveries while still a teenager. After receiving a stipend from the Duke of Brunswick, Gauss entered Brunswick Collegium Carolinum in 1792. There, Gauss independently discovered Bode's law, the binomial theorem, the arithmetic-geometric mean, as well as the law of quadratic reciprocity, and the prime number theorem.

In 1795 Gauss left Brunswick to study at Göttingen University. He completed *Disquisitiones Arithmeticae* (translation from Latin is *Arithmetical Investigations*), his magnum opus, in 1798 at the age of 21, though it would not be published until 1801. This work was fundamental in consolidating number theory as a discipline and has shaped the field to the present day. There were seven sections, all but the last section being devoted to number theory.

Although Gauss left Göttingen in 1798 without a diploma, by this time he had made one of his most important discoveries – the construction of a regular 17-gon by ruler and compasses. This was the most significant advance in this field since the time of Greek mathematics and was published as *Section VII* of *Disquisitiones Arithmeticae*.

Gauss returned to Brunswick where he received a degree in 1799. After the Duke of Brunswick had agreed to continue Gauss's stipend, he requested that Gauss submit a doctoral dissertation to the University of Helmstedt. He already knew Pfaff, who was chosen to be his advisor. Gauss's dissertation was a discussion of the fundamental theorem of algebra.

With his stipend to support him, Gauss did not need to find a job and so devoted himself to research.

In June 1802 Gauss visited Olbers (a German astronomer and physician), who had discovered Pallas in March of that year, and Gauss investigated its orbit. Olbers requested that Gauss be made director of the proposed new observatory in Göttingen, but no action was taken. Gauss began corresponding with Bessel, whom he did not meet until 1825, and with Sophie Germain.

Gauss married in October 1805, and had a happy personal life. His benefactor, the Duke of Brunswick, was killed fighting for the Prussian army, and in 1807 Gauss left Brunswick to take up the position of director of the Göttingen observatory. In 1807 he was appointed Professor of Astronomy and Director of the astronomical observatory in Göttingen, a post he held for the remainder of his life.

In 1808 his father died, and a year later his wife died after giving birth to their second son, who died soon after her. (Gauss had six children with his first wife.) Gauss was shattered and wrote to Olbers asking him to give him a home for a few weeks, "to gather new strength in the arms of your friendship." Gauss was married for a second time the next

year and, although they had three children, this marriage seemed to be one of convenience.

Gauss's work never seemed to suffer from his personal tragedy. He published his second book in 1809, a major two-volume treatise on the motion of celestial bodies. In the first volume he discussed differential equations, conic sections, and elliptic orbits, while in the second volume, the main part of the work, he showed how to estimate and then to refine the estimation of a planet's orbit. Gauss's contributions to theoretical astronomy stopped after 1817, although he went on making observations until the age of 70.

Much of Gauss's time was spent on a new observatory, completed in 1816, but he still found the time to work on other subjects. His publications during this time included a rigorous treatment of series and the introduction of the hypergeometric function, a practical essay on approximate integration, a discussion of statistical estimators, and an investigation of geodesic problems principally concerned with potential theory. In the 1820s, Gauss found himself more and more interested in geodesy (a branch of earth sciences that deals with the measurement and representation of the Earth, including its gravitational field).

Gauss had been asked in 1818 to carry out a geodesic survey of the state of Hanover to link up with the existing Danish grid. Gauss was pleased to accept and took personal charge of the survey, making measurements during the day and reducing them at night, using his extraordinary mental capacity for calculations.

Because of the survey, Gauss invented the heliotrope, which worked by reflecting the Sun's rays using a design of mirrors and a small telescope. However, inaccurate base lines were used for the survey and an unsatisfactory network of triangles. Gauss often wondered if he would have been better advised to have pursued some other occupation, but he published over 70 papers between 1820 and 1830.

In 1822 Gauss won the Copenhagen University Prize with the idea of mapping one surface onto another so that the two were similar in their smallest parts. A paper in 1823 (with a supplement in 1828) was devoted to mathematical statistics, in particular to the least squares method.

From the early 1800s Gauss had an interest in the question of the possible existence of a non-Euclidean geometry. In a book review in 1816 he discussed proofs which deduced the axiom of parallels from the other Euclidean axioms, suggesting that he believed in the existence of non-Euclidean geometry, although he was rather vague. Gauss believed his reputation would suffer if he admitted in public that he believed in the existence of non-Euclidean geometry.

Gauss had a major interest in differential geometry, and published many papers on the subject. His 1828 paper arose from his geodesic interests, but it contained such geometrical ideas as Gaussian curvature.

The period 1817-1832 was a particularly distressing time for Gauss. He took in his sick mother in 1817, who stayed until her death in 1839, while he was arguing with his wife

and her family about whether they should go to Berlin. He had been offered a position at Berlin University and his wife wanted to move there. Gauss, however, never liked change and decided to stay in Göttingen. In 1831 Gauss's second wife died after a long illness.

Gauss worked on problems of physics, publishing papers which contained the principle of least constraint and discussed forces of attraction. These papers were based on Gauss's potential theory, which proved of great importance in his work on physics. He later came to believe his potential theory and his method of least squares provided vital links between science and nature.

In 1831 Gauss developed a fruitful collaboration with the physics professor Wilhelm Weber, leading to new knowledge in magnetism (including finding a representation for the unit of magnetism in terms of mass, length, and time) and the discovery of Kirchhoff's circuit laws in electricity. They constructed the first electromagnetic telegraph in 1833, which connected the observatory with the institute for physics in Göttingen. Gauss ordered a magnetic observatory to be built in the garden of the observatory, and with Weber founded the *Magnetischer Verein* (magnetic club in German), which supported measurements of Earth's magnetic field in many regions of the world. He developed a method of measuring the horizontal intensity of the magnetic field which has been in use well into the second half of the 20th century and worked out the mathematical theory for separating the inner (core and crust) and outer (magnetospheric) sources of Earth's magnetic field.

Gauss showed that there can only be two poles in the globe and went on to prove an important theorem, which concerned the determination of the intensity of the horizontal component of the magnetic force along with the angle of inclination. Gauss used the Laplace equation to aid him with his calculations, and ended up specifying a location for the magnetic South Pole.

Gauss spent the years from 1845 to 1851 updating the Göttingen University widow's fund. This work gave him practical experience in financial matters, and he went on to make his fortune through shrewd investments in bonds issued by private companies.

Gauss presented his golden jubilee lecture in 1849, fifty years after his diploma had been granted by Helmstedt University. It was a variation on his dissertation of 1799. From the mathematical community only Jacobi and Dirichlet were present, but Gauss received many messages and honors.

From 1850 Gauss's work was again nearly all of a practical nature although he did approve Riemann's doctoral thesis and heard his probationary lecture. He was also able to attend the opening of the new railway link between Hanover and Göttingen, but this proved to be his last outing. His health deteriorated slowly, and Gauss died in his sleep at the age of 77 in Göttingen, Hanover.

Noteworthy: Since his father wanted him to follow in his footsteps and become a mason, he was not supportive of

Gauss's schooling in mathematics and science. Gauss was primarily supported by his mother in this effort and by the Duke of Braunschweig.

In his 1799 doctorate in absentia, *A new proof of the theorem that every integral rational algebraic function of one variable can be resolved into real factors of the first or second degree*, Gauss proved the fundamental theorem of algebra which states that every non-constant single-variable polynomial over the complex numbers has at least one root. Mathematicians including d'Alembert had produced false proofs before him, and Gauss's dissertation contains a critique of d'Alembert's work. Ironically, by today's standard, Gauss's own attempt is not acceptable, owing to implicit use of the Jordan curve theorem. However, he subsequently produced three other proofs, the last one in 1849 that was more rigorous. His attempts clarified the concept of complex numbers.

Gauss's personal life was overshadowed by the early death of his first wife in 1809, soon followed by the death of one child. Gauss plunged into a depression from which he never fully recovered.

Gauss was an ardent perfectionist and a hard worker. He was never a prolific writer, refusing to publish works which he did not consider complete and above criticism. This was in keeping with his personal motto *pauca sed matura* ("few, but ripe"). His personal diaries indicate that he had made several important mathematical discoveries years or decades before his contemporaries published them.

Though he did take in a few students, Gauss was known to dislike teaching. It is said that he attended only a single scientific conference, which was in Berlin in 1828.

The CGS unit for magnetic induction was named gauss in his honor. A crater on the Moon is named *Gauss*. Asteroid 1001 is called *Gaussia*.

9.5.4 Lognormal Density

When it is important to strictly limit possible values of a parameter to the positive number range, the lognormal density is commonly used. Applications where variables are strictly positive include material strength, fatigue life, loading intensity, time to the occurrence of an event, and volume, length, and area. A random variable X has a *lognormal* probability density function if $\ln X$ is normally distributed, that is,

$$f(x) = \frac{1}{x\zeta\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\frac{\ln x - \lambda}{\zeta} \right)^2 \right\}, \quad 0 < x < \infty, \quad (9.17)$$

where $\lambda = E\{\ln X\}$ is the mean value and $\zeta = \sqrt{\text{Var}(\ln X)}$ is the standard deviation of $\ln X$. The resulting function is plotted in Figure 9.16.

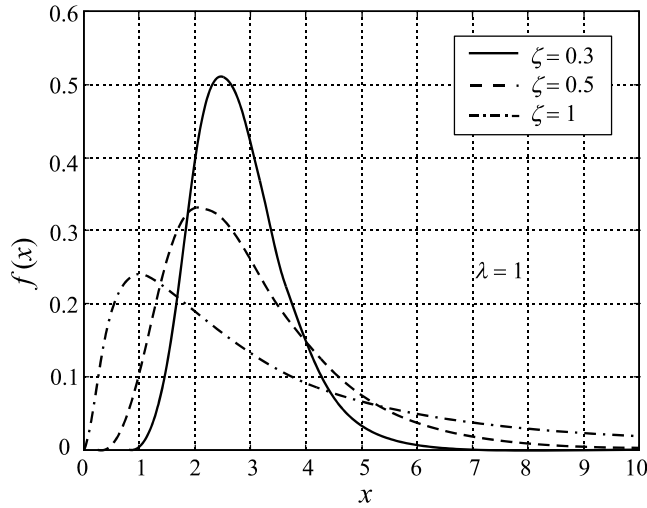


Figure 9.16: The lognormal density function for $\lambda = 1$ and three values of ζ : 0.3, 0.5, 1.0.

9.5.5 Rayleigh Density

The *Rayleigh* density, like the lognormal, is also limited to strictly positive realizations and is given by

$$f(x) = \frac{x}{\sigma^2} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \quad x > 0.$$

Figure 9.17 shows example density curves. The first-order and second-order statistics can be derived,

$$\begin{aligned} E\{X\} &= \sqrt{\frac{\pi}{2}}\sigma \\ E\{X^2\} &= 2\sigma^2 \\ \sigma_X &= \sqrt{\frac{4-\pi}{2}}\sigma = 0.655\sigma. \end{aligned}$$

Note that the Rayleigh density function requires only the parameter σ for its specification, whereas the Gaussian and lognormal require two parameters each.

As an example, the peak amplitudes of the response of a random oscillator driven by a Gaussian force are Rayleigh distributed.

9.6 Two Random Variables

When a problem includes more than one random variable, it becomes necessary to generalize the basic probability definitions. Consider the two random variables X and Y . The *joint cumulative distribution function* is defined as

$$F_{XY}(x, y) = \Pr\{X \leq x, Y \leq y\}.$$

This function defines the probability that the random variable X is less than or equal to the realization x and

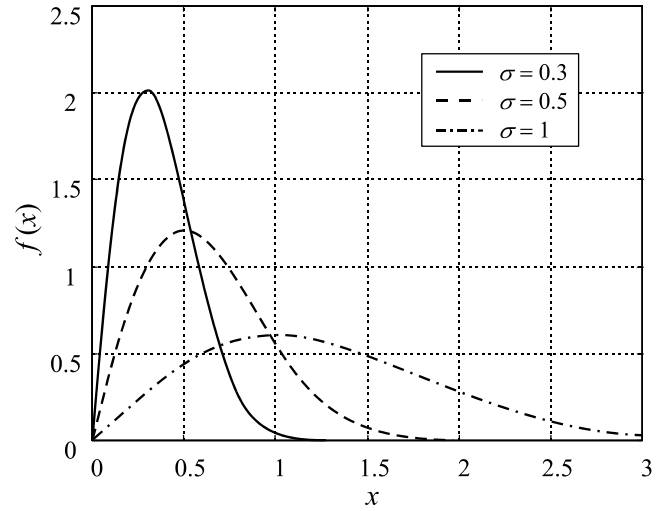


Figure 9.17: The Rayleigh density function for $\sigma = 0.3, 0.5, 1.0$.

the random variable Y is less than or equal to the realization y . Generalizing from our study of one random variable, the *joint density function* can be defined as

$$f_{XY}(x, y)dx dy = \Pr\{x < X \leq x + dx, y < Y \leq y + dy\},$$

or

$$F_{XY}(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{XY}(u, v) du dv.$$

Conversely,

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

and

$$\Pr\{a < X \leq b, c < Y \leq d\} = \int_c^d \int_a^b f_{XY}(u, v) du dv. \quad (9.18)$$

Equation 9.18 defines the probability that two random variables are *simultaneously* within a certain range, specifically, that $a < X \leq b$ and $c < Y \leq d$. This double integral equals the volume under the joint density function $f_{XY}(x, y)$, as shown in Figure 9.18. In applications it is common that two or more parameters define the system, and, thus, joint probability information is needed.

9.6.1 Covariance and Correlation

Proceeding from models with one random variable to those with two also requires the introduction of the concept of covariance and the related parameter, the correlation coefficient. This coefficient characterizes how *linear* the relationship is between the two random variables. We

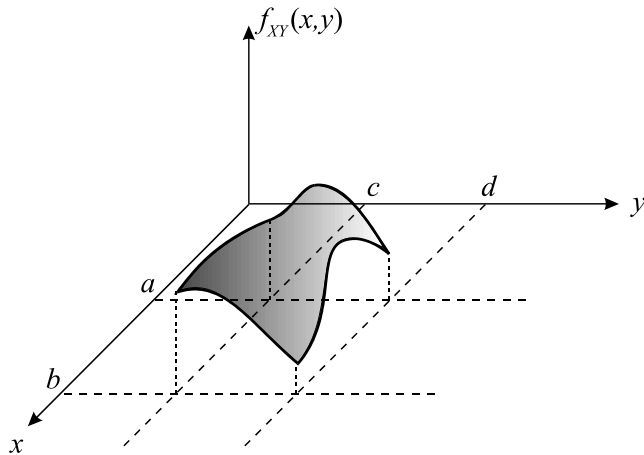


Figure 9.18: Sketch of the domain of two random variables X and Y . The probability $\Pr\{a < X \leq b, c < Y \leq d\}$ is equivalent to the volume under the curve.

begin by considering the two random variables X and Y with their respective *joint second moment*,

$$E\{XY\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy. \quad (9.19)$$

If X and Y are *statistically independent* then the joint density function can be separated into the product of the respective *marginal densities*, $f_{XY}(x, y) = f_X(x)f_Y(y)$, and using Equation 9.19, $E\{XY\} = E\{X\}E\{Y\}$. The *covariance* is defined as the joint second moment about the mean values μ_X and μ_Y ,

$$\begin{aligned} \text{Cov}(XY) &= E\{(X - \mu_X)(Y - \mu_Y)\} \\ &= E\{XY\} - \mu_X \mu_Y. \end{aligned} \quad (9.20)$$

Note that if the variables are independent, $\text{Cov}(XY) = 0$. The *correlation coefficient* ρ is defined as the normalized and dimensionless $\text{Cov}(XY)$, that is,

$$\rho_{XY} = \frac{\text{Cov}(XY)}{\sigma_X \sigma_Y}. \quad (9.21)$$

To better understand the correlation coefficient, we assume that X and Y are linearly related by the equation $X = aY$, where a is a positive constant. Then,

$$E\{XY\} = aE\{Y^2\},$$

and

$$\text{Cov}(XY) = aE\{Y^2\} - aE^2\{Y\}.$$

Using the definition of variance, the covariance becomes $\text{Cov}(XY) = a\sigma_Y^2$, and Equation 9.21 becomes

$$\rho_{XY} = \frac{a\sigma_Y^2}{\sigma_X \sigma_Y} = \frac{a\sigma_Y}{\sigma_X} = +1, \quad (9.22)$$

since $\sigma_X = a\sigma_Y$. The random variables X and Y are completely correlated. The last equality in Equation 9.22

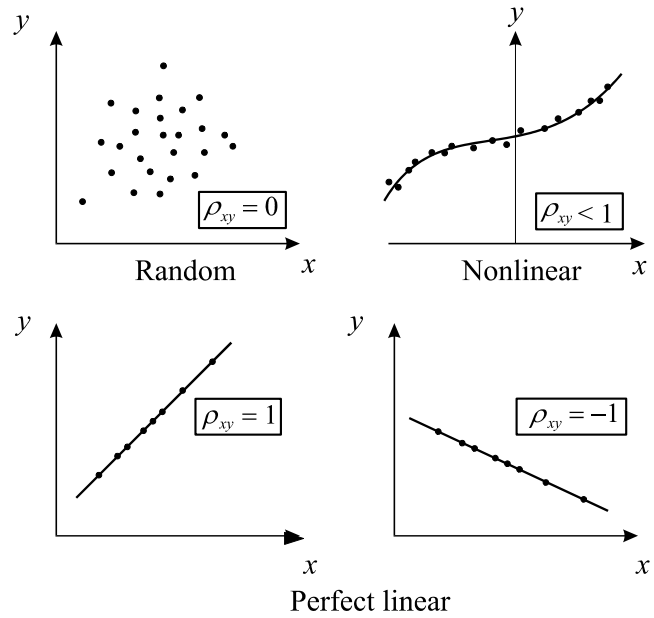


Figure 9.19: Correlation coefficients for random, nonlinear, and linear relationships between two random variables X and Y .

is found using the property of the variance of a random variable multiplied by a constant: If Y has variance σ_Y , then aY has variance $a\sigma_Y$. Had we defined $X = -aY$, then we would have found that $\rho_{XY} = -1$, or X and Y are completely negatively correlated.²¹ We conclude that $-1 \leq \rho_{XY} \leq +1$.

Figure 9.19 depicts representative correlations between data points for random, nonlinear, and perfect linear relationships.

Example 9.10 Jointly Distributed Variables

Two random variables X and Y are jointly distributed according to the joint density $f_{XY}(x, y) = \frac{1}{2}e^{-y}$, $y > |x|$, $-\infty < x < \infty$, as plotted in Figure 9.20. Compute the marginal densities and the covariance.

Solution: The marginal densities are defined as

$$\begin{aligned} f_X(x) &= \int f_{XY}(x, y) dy \\ &= \int_{|x|}^{\infty} \frac{1}{2} e^{-y} dy = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty, \\ f_Y(y) &= \int f_{XY}(x, y) dx \\ &= \int_{-y}^y \frac{1}{2} e^{-y} dx = ye^{-y}, \quad y > 0. \end{aligned}$$

²¹ A high value for ρ may indicate strong correlation, but not direct *cause* and *effect* since X and Y may be correlated by virtue of being related to a third variable. Also, if X and Y are independent, $\rho = 0$, as we see in Example 9.10. The converse is not necessarily true. $\rho = 0$ indicates the absence of a *linear* relationship; a random or a nonlinear functional relationship between X and Y is still a possibility.

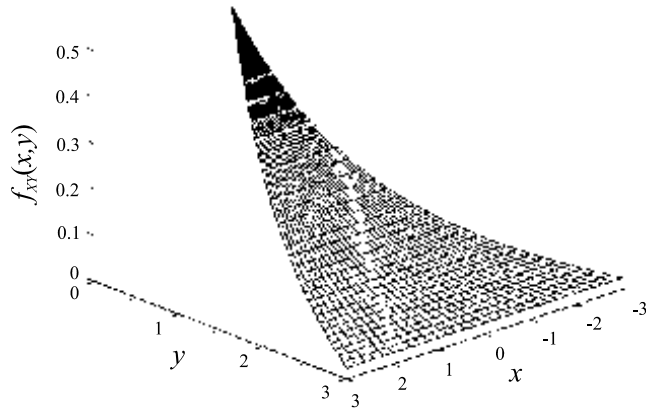


Figure 9.20: The joint density function is a three-dimensional surface, with the volume underneath representing the probability of occurrence in a range, $f_{XY}(x, y) = \frac{1}{2}e^{-y}$, $y > |x|$, $-\infty < x < \infty$.

Since the joint density is not equal to the product of the two marginal densities, the variables are not statistically independent. Covariance Equation 9.20 requires an evaluation of the second joint moment,

$$\begin{aligned} E\{XY\} &= \int_0^\infty \int_{-y}^y \frac{1}{2} x y e^{-y} dx dy \\ &= \frac{1}{2} \int_0^\infty y e^{-y} \int_{-y}^y x dx dy = 0, \end{aligned}$$

and the respective mean values

$$\begin{aligned} E\{X\} &= \int_{-\infty}^\infty x f_X(x) dx \\ &= \int_{-\infty}^0 \frac{1}{2} x e^x dx + \int_0^\infty \frac{1}{2} x e^{-x} dx = 0. \end{aligned}$$

By a similar procedure, $E\{Y\} = 2$. Therefore, $\text{Cov}(XY) = 0$, not because the two variables are independent, but because the right-hand side equals zero. ■

We have drawn a number of our examples from the discipline of reliability, which relates to all aspects of engineering. Reliability is also a function of vibration characteristics, which depend on uncertainties in loading and material parameter values.

Example 9.11 Correlation Coefficient and Reliability

Assume that we have been tasked with the job of estimating the reliability of a component. We have run tests on both the component and the loading to gather data. How do we estimate reliability?

Solution: Define the *strength* of the component as X

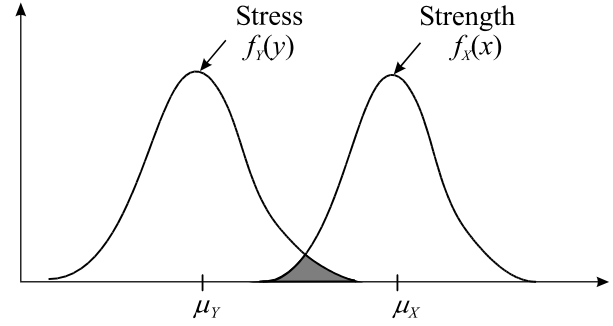


Figure 9.21: Overlap of strength and stress densities for reliability. The shaded region represents realizations for which $Y > X$.

and the *stress* it experiences due to loading as Y . The strength may be a yield stress or an ultimate stress. We test a sufficient number of “identical” components in order to establish the component strength probability density function and determine it to be $f_X(x)$. Similarly, loading data leads us to a particular loading or stress density function $f_Y(y)$.

The strength is designed to exceed the stress for all but the most rare of cases, as one would expect. This is shown schematically in Figure 9.21. The shaded region in the figure represents the realizations where the loading stress Y is greater than the component strength X . *This is defined as a failure of the component.* The probability of failure is equal to $\Pr\{(X - Y) \leq 0\}$. If we define $Z = X - Y$, then we need to find $\Pr\{Z \leq 0\}$. The reliability of the component is then defined as $R = 1 - \Pr\{Z \leq 0\}$, which is designed to be a very large number.

From Equations 9.20 and 9.21 we have

$$E\{XY\} = \rho \sigma_X \sigma_Y + \mu_X \mu_Y,$$

and using Equation 9.9, the variance of Z is

$$\text{Var}\{Z\} = \text{Var}\{X\} + \text{Var}\{Y\} - 2\rho \sigma_X \sigma_Y. \quad (9.23)$$

Since strength and loading stress are uncorrelated, $\rho = 0$ and the variance of Z equals the variance of X plus the variance of Y , or

$$\sigma_Z = +\sqrt{\sigma_X^2 + \sigma_Y^2}.$$

The mean value of Z is

$$\mu_Z = \mu_X - \mu_Y.$$

We have derived the mean and variance of the probability of failure. If we can establish the density function $f_Z(z)$, then the reliability of the component is given by

$$R = 1 - \int_{-\infty}^0 f_Z(z) dz. \quad (9.24)$$

Suppose that both X and Y are Gaussian. Then, Z is also Gaussian (it is not proven here). In this

instance, $f_Z(z)$ is fully defined given μ_Z and σ_Z , and Equation 9.24 can then be evaluated. ■

9.7 Random Processes

The study of the response of a single degree-of-freedom system to random forcing requires the extension of random variable concepts to include time-dependent random variables, or random processes. This will further involve an understanding of the notion of correlation and power spectrum. A random process can be thought of informally as a random variable that varies with time in a probabilistic way. With this in mind, we can extend the earlier definitions of random variables to those for random processes. With these tools, the random force can be probabilistically modeled.

The basic introduction in the following subsections provides us with what we need to start understanding the study of random vibration, initiated in Section 9.8.

9.7.1 Random Process Descriptors

A *random process* may be understood to be a time-dependent random variable. For a specific time t , $X(t)$ is a random variable with distribution function

$$F_{X(t)}(x; t) = \Pr\{X(t) \leq x\},$$

where it is customary to separate spatial- and time-dependent variables using a semi-colon. This is the *first-order distribution* of the process $X(t)$. It defines how the probability characteristics of the random process change with time as well as how the mean value and variance change with time.

The corresponding *first-order density* is given by

$$f_{X(t)}(x; t) = \frac{\partial F_{X(t)}(x; t)}{\partial x}.$$

The *second-order distribution* for $X(t_1)$ and $X(t_2)$ is the joint distribution

$$F(x_1, x_2; t_1, t_2) = \Pr\{X(t_1) \leq x_1, X(t_2) \leq x_2\} \quad (9.25)$$

with corresponding density

$$f_{X_1 X_2}(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_{X_1 X_2}}{\partial x_1 \partial x_2}. \quad (9.26)$$

We have used the short-hand notations $X_1 \equiv X(t_1)$ and $X_2 \equiv X(t_2)$. Also, subscripts may be dropped if it is clear which variables are meant.

Equation 9.25 is the probability that at time t_1 , the random process $X(t)$ will have a value less than or equal to the realization x_1 and that at time t_2 , it will have

a value less than or equal to the realization x_2 . Equation 9.26 has the same information in an alternate form: the volume under the second-order density equals the probability. Equations 9.25 and 9.26 are the starting points for the mathematical modeling of the probabilistic evolution of $X(t)$ in time. For random vibration, we are interested in the random force.

The question considered next is: If the random function is represented by many possible time histories, how can averages be determined?

9.7.2 Ensemble Averaging

The random function of time $X(t)$ with probability density $f(x; t)$ is representative of many possible time histories, also known as a *sample population*. Theoretically, there are an infinite number of samples $X_i(t)$ with statistical properties governed by the density function $f(x; t)$. As Figure 9.22 shows, for any time t_1 there exists an infinite number of possible values: $x_1(t_1)$, $x_2(t_1)$, $x_3(t_1)$, ... To account for all these possible values, we average them to obtain the most likely value of the function at time t_1 . This process is known as *ensemble averaging* since the group of samples is known as an *ensemble*.

The averaging procedure uses the same mathematical expectation of Equation 9.8,

$$\mu_{X(t)}(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x f(x; t) dx, \quad (9.27)$$

except that $\mu_{X(t)}(t)$ and $f(x; t)$ are now functions of time. In the integration x is the dummy variable. A possible time-dependent probability density function is $f(x; t) = ce^{-xt}$, where c is a normalization constant. In general, random processes are governed by time-dependent density functions and have time-dependent mean values.

Similarly, second-order averages that are very important in random vibration can be evaluated. The motivation for such averages is the question: How does the value of the process $X(t)$ at $t = t_1$ affect its value at later time $t = t_2$? Knowing this helps us understand how rapidly a process varies. For a slowly varying function, it is expected that if t_1 and t_2 are not *too far* apart then $X(t_2)$ can be estimated given $X(t_1)$ (as shown in Figure 9.23(a)) and the values are correlated. On the other hand, if $X(t)$ varies rapidly (as in Figure 9.23(b)) any estimate of future values is not accurate and the values are much less correlated.

To address this question, we consider the random process $X(t)$ at two arbitrary times, $X(t_1)$ and $X(t_2)$. The second-order average is given by

$$E\{X(t_1)X(t_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2) dx_1 dx_2, \quad (9.28)$$

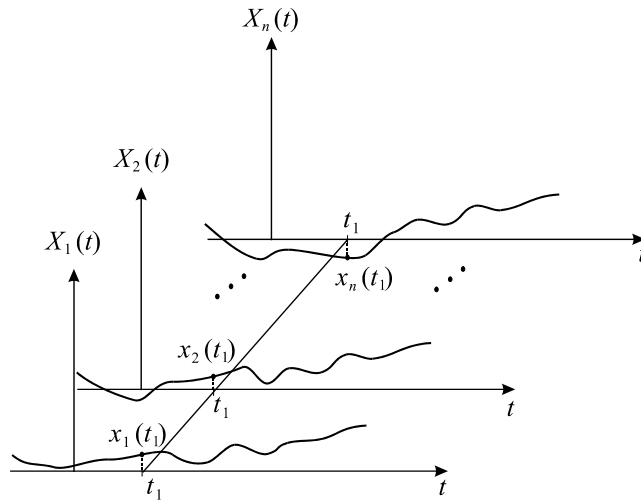


Figure 9.22: Ensemble averaging of n time functions governed by the density function $f(x; t)$.

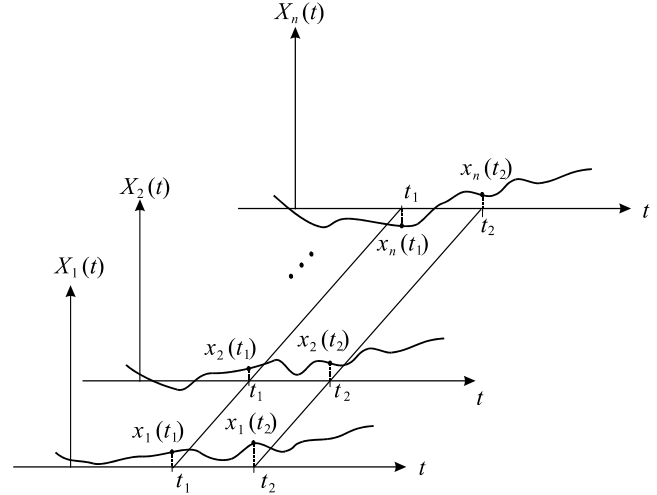


Figure 9.24: Second-order ensemble averaging of n realizations of random process $X(t)$.

where the joint density function $f(x_1, x_2; t_1, t_2)$ is required. Here, x_1 and x_2 are realizations and dummy variables of integration. We have omitted the subscripts on the density function.

The second-order average defined by Equation 9.28 and shown in Figure 9.24 is called the *correlation function* and is given the shorthand notation $R_{XX}(t_1, t_2)$,

$$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}, \quad (9.29)$$

where the end result of the double integral is a function of t_1 and t_2 .

In Equation 9.28, it is necessary to emphasize that at specific times t_1 and t_2 , the random process is a random variable. At each instant of time, the value of a random process is governed by a probability density. The density $f(x; t)$ changes with time, but once a specific time is chosen, the density function is only a function of the realization x , that is, $f(x; t_1) = f(x)$, as shown in Figure 9.25.

The correlation function, similar in purpose to the earlier correlation coefficient, is a measure of the *similarity between the values of one stochastic process at two times or between different stochastic processes at two time instances*. For different stochastic processes, the *cross correlation function*, $R_{XY}(t_1, t_2) = E\{X(t_1)Y(t_2)\}$, is defined.

We can see from Equations 9.28 and 9.29 that the correlation is calculated by multiplying the corresponding values of the functions and then averaging these products using the expectation operator. If the two functions have similar shapes, then it is expected that a larger correlation will be found, since otherwise some of the products will be smaller or negative, leading to a smaller average. The largest correlation is found for $t_1 = t_2$, $R_{XX}(t_1, t_1) = E\{X(t_1)X(t_1)\}$.

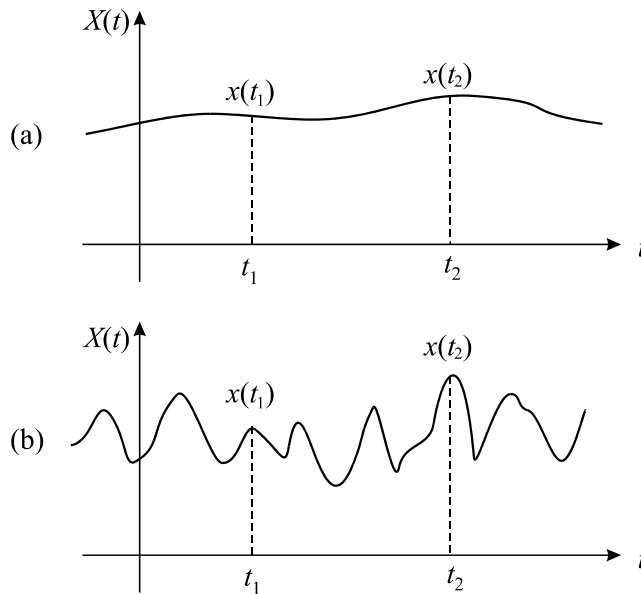


Figure 9.23: (a) Slowly varying and (b) rapidly varying random processes. The slowly varying process is more highly correlated.

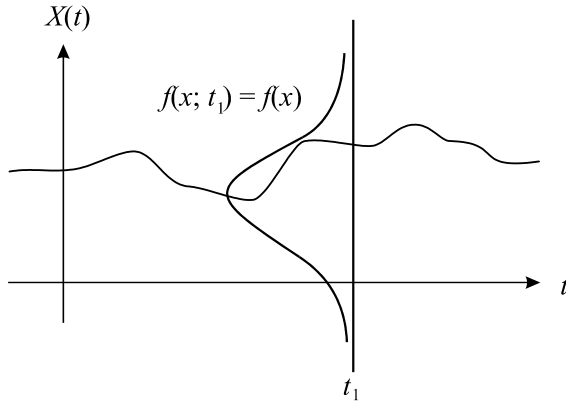


Figure 9.25: For a random process the density function $f(x; t)$ varies with time. The density of a random process at specific time $t = t_1$ is the same as the density of a random variable $f(x)$ since the random process is a random variable at each instant of time.

The *autocovariance* of $X(t)$ is defined as

$$C_{XX}(t_1, t_2) = E\{[X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)]\},$$

which equals

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2).$$

The *variance* is defined for $t_1 = t_2 = t$ as

$$\sigma_{X(t)}^2 = C_{XX}(t, t) = R_{XX}(t, t) - \mu_X^2(t).$$

For a normal random process, for example, the time-dependent probability density function is

$$f_X(x; t) = \frac{1}{\sqrt{2\pi C_{XX}(t, t)}} \exp \left[-\frac{(x - \mu_X(t))^2}{2C_{XX}(t, t)} \right].$$

Generally, specific correlation functions $R_{XX}(t_1, t_2)$ are derived experimentally. In practice, preliminary analyses assume a reasonable function for $R_{XX}(t_1, t_2)$ that is representative of the physical process under study. Predictions using this model are then verified with experimental data.

9.7.3 Stationarity

Equations 9.27 and 9.28 can be difficult to evaluate mathematically and due to the difficulty in obtaining the necessary data to define the joint density function. To begin to understand random vibration problems, we make the assumption of *stationarity*. If the statistical properties of a random process are invariant under translation in time, the process is called *stationary*. While this assumption may appear to limit applicability, stationarity is a viable assumption for numerous practical applications. To use vibration terminology, the assumption of stationarity implies steady-state behavior in the statistical sense.

For a stationary random process, the mean value becomes a constant (as for random variables),

$$E\{X(t)\} = \mu_X,$$

and the correlation function becomes a function of time difference,

$$R_{XX}(t_1, t_2) = E\{X(t)X(t + \tau)\} = R_{XX}(\tau),$$

$\tau = t_2 - t_1$, rather than a function of the specific times t_1 and t_2 . For physical processes, the correlation is an even function, $R_{XX}(\tau) = R_{XX}(-\tau)$. Similarly, for two stationary processes, the *cross correlation* function is $R_{XY}(\tau)$.

For $\tau = 0$, the *mean-square* value of $X(t)$ is

$$R_{XX}(\tau = 0) = E\{X^2(t)\} = \sigma_X^2 + \mu_X^2,$$

using Equation 9.9. The mean-square value has the following physical interpretation. If $X(t)$ is a displacement, then $E\{X^2(t)\}$ is a measure of strain energy, $V(t) \sim kX^2(t)$, where k is a stiffness parameter. If $X(t)$ is a velocity, then $E\{X^2(t)\}$ is a measure of kinetic energy, $T(t) \sim mX^2(t)$, where m is a mass parameter. The average energy of a stationary process is independent of time and equals the autocorrelation at $\tau = 0$. This knowledge is useful for interpreting the meaning of the *spectral density* in Section 9.7.4. In physical processes, as $\tau \rightarrow \infty$, the correlation function approaches the mean-square value,

$$R_{XX}(\tau \rightarrow \infty) \rightarrow \mu_X^2.$$

As the time difference τ becomes larger, less correlation exists between the two respective values of the process, and the limit of the correlation becomes the square of the mean value.

Stationarity also simplifies the autocovariance,

$$C_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2,$$

where, if $\tau = 0$, then $C_{XX}(0) = R_{XX}(0) - \mu_X^2 = \sigma_X^2$. The *correlation coefficient* is defined as

$$r_{XX}(\tau) = C_{XX}(\tau)/C_{XX}(0).$$

The *correlation time* of a stochastic process can be defined as

$$\tau_c = \frac{1}{C_{XX}(0)} \int_0^\infty C_{XX}(\tau) d\tau.$$

The concept of correlation time is useful in physical problems where there are several timescales.

Ergodicity

As a practical matter, one rarely has the benefit of numerous experiments, but usually must make the best use

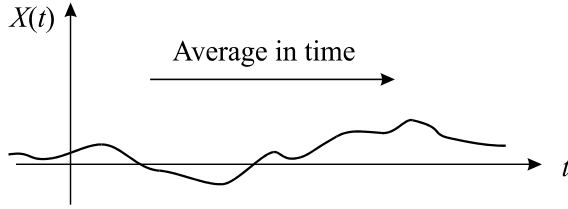


Figure 9.26: Ergodicity is the property of a stationary random process where averages in time rather than across an ensemble result in the same averages.

of *one* trial. This is especially true for expensive testing environments such as for space or for the ocean. Utilizing one trial requires the introduction of the concept of *ergodicity*. A stationary random process is said to be an *ergodic* process if the *time average* of a *single* record is approximately equal to the ensemble average. It is thus possible to average over a long *single* time history (as shown in Figure 9.26) rather than trying to obtain numerous records over which to perform an ensemble average.

The mean value is then given by

$$\mu_X \simeq \bar{X} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) dt, \quad (9.30)$$

where it is assumed that $X(t)$ is one particular realization $x(t)$ of the random process. Such an average makes sense only if μ_X is constant. Otherwise μ_X will be a function of T and the initial assumption is no longer valid. An ergodic process is always stationary; the opposite is not always true.

The corresponding ergodic definition for the autocorrelation function is

$$R_{XX}(\tau) \simeq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t)X(t+\tau) dt. \quad (9.31)$$

Now that we have an understanding of the autocorrelation, we proceed to study its Fourier transform, the spectral density.

9.7.4 Power Spectrum

A measure of the “energy” of the stochastic process $X(t)$ is given by its *power spectrum*, or *spectral density*, $S_{XX}(\omega)$, which is the *Fourier transform*²² of its autocor-

²²The Fourier transform of function $q(x)$ is defined as

$$\mathcal{F}\{q(x)\} = Q(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(x) e^{-i\omega x} dx.$$

The inverse Fourier transform of $Q(\omega)$ is

$$\mathcal{F}^{-1}Q(\omega) = q(x) = \int_{-\infty}^{\infty} Q(\omega) e^{i\omega x} d\omega.$$

$Q(\omega)$ and $q(x)$ are known as a Fourier transform pair. Sometimes the factor $(1/2\pi)$ appears on the other transform, and sometimes the factor $1/\sqrt{2\pi}$ appears in both transforms.

relation function,

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-i\omega\tau) d\tau,$$

and, thus,

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) \exp(i\omega\tau) d\omega. \quad (9.32)$$

These equations are known as the *Wiener-Khintchine* formulas.

Since $R_{XX}(-\tau) = R_{XX}(\tau)$, $S_{XX}(\omega)$ is not a complex function but a real and even function. For $\tau = 0$,

$$\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = R_{XX}(0) = E\{X(t)^2\} \geq 0,$$

where $S_{XX}(\omega) \geq 0$ since, as a measure of energy, it must be positive semi-definite. The integral of the power spectrum equals the “average or mean square power” of the process $X(t)$, confirming our opening statement that it is an energy measure. Where there is no chance of confusion, the subscripts used above can be omitted.

cross spectral densities are similarly defined,

$$S_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XY}(\tau) \exp(-i\omega\tau) d\tau = S_{YX}(\omega),$$

as are cross correlation functions,

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(\omega) \exp(i\omega\tau) d\omega.$$

Some experimentally derived examples of $R_{XX}(\tau)$ and $S_{XX}(\omega)$ that have broad physical application are presented next, and in the following sections.

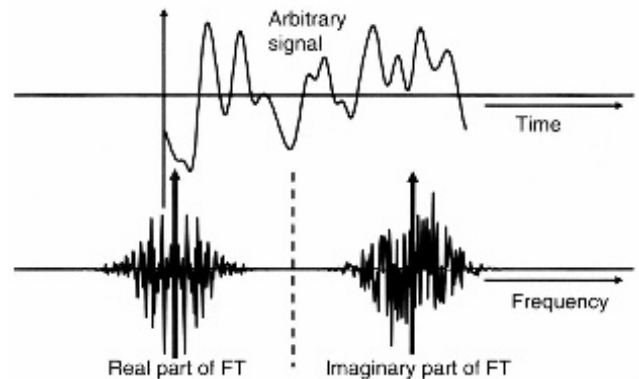


Figure 9.27: Fourier transform of an arbitrary waveform. The real part is symmetric, and the imaginary part is antisymmetric. The magnitudes of the peaks indicate the amplitudes of the sinusoids in the original signal, and their positions on the axis determine their frequencies. Pairs of spikes on either side of the axis on the real part give the cosine components, and imaginary pairs give the sine contributions.

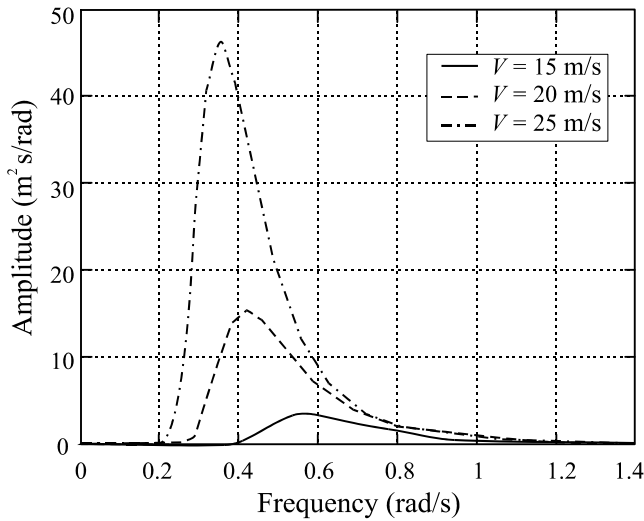


Figure 9.28: The Pierson-Moskowitz ocean wave height spectrum for three possible wind velocities.

Ocean Wave Height Spectra

The height of ocean waves generated by wind can be modeled as a random process. A widely accepted spectral density of ocean wave height $\eta(t)$ is the *Pierson-Moskowitz* spectrum,

$$S_{\eta\eta}(\omega) = \frac{8.1 \times 10^{-3} g^2}{\omega^5} \exp -0.74 \left(\frac{g}{V\omega} \right)^4 \text{ m}^2\text{s}, \quad (9.33)$$

where $\omega > 0$, g is the acceleration due to gravity, and V is the wind speed at a height of 19.5 m above the still water level. Any consistent set of units for g and V can be used, with ω in rad/s. The Pierson-Moskowitz is an experimentally determined spectrum, as are most if not all spectra used in applications. Representative spectra are shown in Figure 9.28.

The fluid particle velocities and accelerations are derivable from the wave height elevation. Other ocean wave height spectral densities are used in engineering practice. Each is specific to a particular part of the ocean for a particular time of year.

Wind Velocity Spectra

All structures must be designed to withstand wind forces. One can classify structures subjected to wind as either *streamlined* or *bluff bodies*. Streamlined structures, such as wings, have a high *aspect ratio* (ratio of one dimension to another) and are optimally shaped so a streamline can naturally flow around them. They are designed so that the interaction of their shape with the wind results in a desirable configuration of forces. Bluff bodies, such as tall buildings or offshore drilling platforms, are generally of low aspect ratio and may have corners or sharp edges. They are designed more for strength than for fitting within the streamlines.

An example of a wind spectrum for horizontal velocity is²³

$$S_v(f) = 4\kappa \bar{V}_{10}^2 \frac{L/\bar{V}_{10}}{(2 + \bar{f}^2)^{5/6}} \left(\frac{\text{ft}^2}{\text{s}} \right), \quad (9.34)$$

where f is the frequency in Hz, $\bar{f} = fL/\bar{V}_{10}$ is a dimensionless frequency, L is a length scale²⁴ of approximately 4000 ft, $\bar{V}_{10}$ is the mean wind speed at 10 ft above the ground, and κ is a dimensionless number in the range $0.005 \leq \kappa \leq 0.05$ that depends on wind profile expected in the region. As with wave height spectra, wind characteristics have wide variability depending on location.

Earthquake Ground Motion Spectra

For earthquakes, the difficulty in specifying ground motion spectra for use as input to the structure is due to the significant variability in soil and geologic properties, even for two sites very near each other. Photographs of earthquake-related devastation often show two adjacent similar structures, one with little damage and the other with complete destruction. Differences can be due to even minor differences in structures and soil/foundation dynamic characteristics.

The vibration ground motion of the El Centro, California earthquake²⁵ of May 18, 1940 has been used as input to numerous structural designs to test their survivability to anticipated ground motion.

TORRE MAYOR OFFICE TOWER²⁶

The 57-story Torre Mayor building is a dominant structure in Mexico City and is presently the third tallest building in Mexico. The developer for the project had obtained a site in an area of the city where few sites were available, and desired a building of at least 55 stories in height. Unfortunately, the site was in an area noted for extremely soft soil, and since much of Mexico City is within the bowl of an ancient volcano, bedrock was hundreds of meters below the ground level. In addition, Mexico City is in a region of high seismicity, requiring that this new building be capable of withstanding multiple major earthquakes during its life.

All of this resulted in the design team determining that with conventional construction, it was not possible to build Torre Mayor higher than approximately 38 stories due to soil

²³See A.G. Davenport and M. Novak in **Shock & Vibration Handbook**, C.M. Harris, Editor, McGraw-Hill, 1988, Chapter 29, p.23.

²⁴A length scale is a dimension that is representative of the system being modeled. For example, if the wavelength of a harmonic force is λ , then this is a length scale of the problem. Similarly, structural dimensions can be used as length scales.

²⁵See W.J. Hall, in **Shock & Vibration Handbook**, C.M. Harris, McGraw-Hill, 1987, Chapter 24, p.5.

²⁶This case study is courtesy of Douglas P. Taylor of Taylor Devices, Inc.

bearing stress limitations – far below the developer’s requirements. Because of seismic considerations, the proposed building had a relatively heavy frame, and the project engineers wondered what would result if fluid dampers were added to reduce stress – thereby offering the possibility of reducing the story weight.

Most tall buildings have relatively low structural damping, in the range of 1% critical, and this amount of damping was assumed for the seismic analysis. When damping is to be added to a building, the type of damping device must be considered. In general, fluid dampers are preferred, because they vary their output force as a function of stroking velocity, thus making their output out of phase with the normal flexural bending and shear stresses in a building frame subjected to horizontal motion from earthquakes or high wind loads. This out of phase response gives the fluid damper the ability to effectively reduce both stress and deflection in the building simultaneously, a feature not found with other damping devices using friction elements or elastomers.

The approach selected by the design team was to add fluid dampers to reduce frame stresses, then lighten the damped building frame until the stresses increased up to the code allowable, and use the weight saved to add additional floors. Starting with the undamped frame at 1% critical, damping was added in 2% increments and the weight savings used to add additional floors. The analysis revealed that when the added damping in the structure reached 10%, the resulting building was 55 stories, satisfying the owner, with essentially the same weight and maximum seismic stresses as the conventional design of only 38 stories.

Some compromises were needed in the frame configuration due to increased flexibility of the damped design. Figures 9.29 and 9.30 show the special brace design that was selected for the long walls of the building, using “mega-braces,” where heavy diagonal braces spanning up to 6 floors were arranged diagonally in a diamond-like pattern. The finished building design used 24 large dampers of 570 MT output on the long walls, and 74 smaller dampers of 280 MT force in conventional diagonal braces for the short walls. (MT is metric ton, and is a non-SI metric unit of mass equal to 1000 kg.) Since completion in 2002, the Torre Mayor with fluid dampers has experienced numerous earthquakes with no damage.

9.7.5 Units

The power spectrum represents a real dynamic physical process, but Equation 9.32 is an integral over negative frequencies! Of course, these negative frequencies do not exist except mathematically, just as the complex exponential is used for mathematical ease. The angular frequency ω has units of rad/s. In applications, frequencies f with units of Hz (cycles/s) are more common.

The relation $f = \omega/2\pi$ permits us to define an equivalent one-sided spectral density function $W_{XX}(f)$. By



Figure 9.29: Torre Mayor under construction showing the cross-braced dampers. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)



Figure 9.30: Mega-brace dampers for Torre Mayor building. (Courtesy of Douglas P. Taylor of Taylor Devices, Inc.)

equating shaded areas in Figure 9.31, we find

$$2S_{XX}(\omega) d\omega = W_{XX}(f) \frac{1}{2\pi} d\omega, \quad (9.35)$$

or $W_{XX}(f) = 4\pi S_{XX}(\omega)$. In most applications, since densities depend on experimental data, the one-sided density function is more common. The wave height and wind spectra are one-sided.

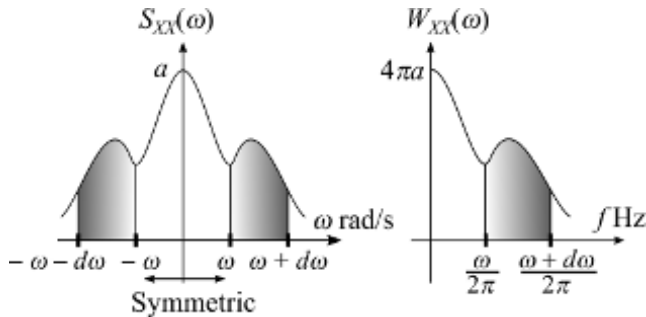


Figure 9.31: Equivalent one-sided spectrum $W_{XX}(f)$ where the shaded areas under the left density function equal the shaded area under the right density function.

TUSTIN'S DESCRIPTION OF RANDOM VIBRATION. "I've heard people describe a continuous spectrum, say 10-20000 Hz, as 1990 sine waves 1 Hz apart. No, that is close but not quite correct. Since waves have constant amplitude and phase, cycle after cycle. Suppose that there were 1990 of them. Would the totality be random? No. For the totality to be random, the amplitude and starting phase of each sine would have to vary randomly, unpredictably. Unpredictable variations are what we mean by random. Broad-spectrum random vibration contains no sinusoids but rather a continuum of vibrations."²⁷

9.7.6 Narrow-Band and Broad-Band Processes

The power spectrum can be interpreted to be representative of the distribution of vibratory energy as a function of frequency. It is then possible to define general categories of vibration according to how its energy is distributed. A structure vibrating at a single constant frequency ω_0 can be represented in the time-domain and frequency-domain as shown in Figure 9.32.

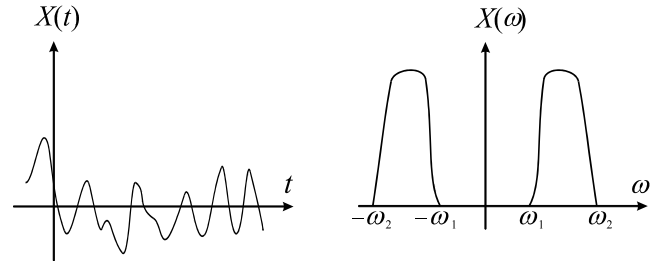


Figure 9.33: A narrow-band process in time-domain and frequency-domain.

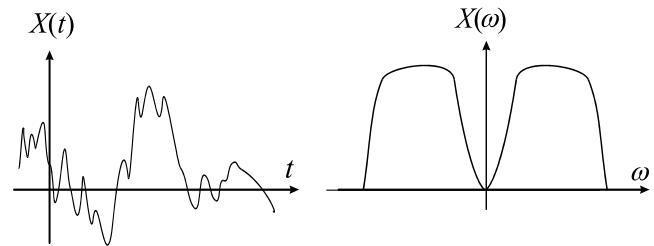


Figure 9.34: A broad-band process in time domain and frequency domain.

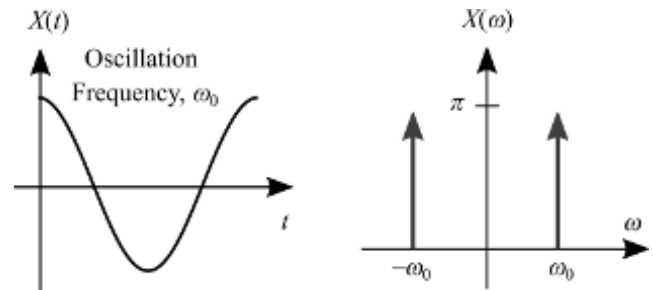


Figure 9.32: A constant frequency process in time-domain and frequency-domain.

Two processes of practical importance are *narrow-band* (Figure 9.33) and *broad-band* (Figure 9.34) processes. Narrow and broad indicate the breadth of the respective frequency bands.

A *narrow-band* process is an *almost*-harmonic oscillator. Instead of vibrating at one distinct frequency as a harmonic oscillator, it vibrates with frequencies in a narrow range: $\omega_1 \leq \omega \leq \omega_2$. The spectral density of a narrow-band process can be idealized by the spectrum in Figure 9.35. It has a flat spectrum in the frequency band, with constant magnitude S_0 .

The autocorrelation function for such an idealized

²⁷W. Tustin, **Random Vibration & Shock Testing**, Equipment Reliability Institute, Santa Barbara, CA, 2005.

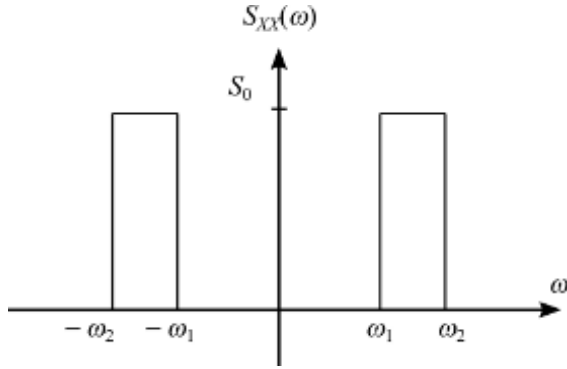


Figure 9.35: Spectral density of an idealized narrow-band process.

process is evaluated as follows,

$$\begin{aligned} R_{XX}(\tau) &= \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega \\ &= 2 \int_{\omega_1}^{\omega_2} S_0 \cos \omega\tau d\omega \\ &= \frac{2S_0}{\tau} (\sin \omega_2\tau - \sin \omega_1\tau) \end{aligned} \quad (9.36)$$

$$= \frac{4S_0}{\tau} \cos \left[\frac{\omega_1 + \omega_2}{2} \tau \right] \sin \left[\frac{\omega_2 - \omega_1}{2} \tau \right], \quad (9.37)$$

where the real part of the complex exponential is retained using the symmetry of the power spectrum.²⁸ The mean-square value can be obtained by taking the limit of the right-hand side of Equation 9.37,

$$\begin{aligned} E\{X^2\} &= R_{XX}(0) = \lim_{\tau \rightarrow 0} \text{rhs} \\ &= 2S_0(\omega_2 - \omega_1). \end{aligned}$$

Alternatively, in this case, it is easy to evaluate the mean square value by calculating the area under the spectral density curve,

$$E\{X^2\} = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = 2S_0(\omega_2 - \omega_1).$$

A *broad-band* vibration is one with a significant range of frequencies, having a broader frequency range than the narrow-band process. The equations above are valid, except $\omega_2 - \omega_1$ is a larger range.

9.7.7 White-Noise Process

A *white-noise* process is an idealization that is used to simplify the algebra of an analysis, as we will see below. The term *white* is adopted from optics to signify that all

²⁸If the random process $X(t)$ is real, then $R_{XX}(\tau)$ is even and real. Therefore, $S_{XX}(\omega)$ is also even, $S_{XX}(-\omega) = S_{XX}(\omega)$, and in this case the exponential can be replaced by the cosine.

frequencies are part of such a process, much like white light is composed of the whole color spectrum.

The power spectrum ranges from $-\infty$ to ∞ , with the autocorrelation evaluated by setting $\omega_1 = 0$ and $\omega_2 \rightarrow \infty$ in Equation 9.36,

$$\lim_{\omega_1 \rightarrow 0} \frac{4S_0}{\tau} (\sin \omega_2\tau - \sin \omega_1\tau) = 2S_0 \frac{\sin \omega_2\tau}{\tau},$$

with the result

$$R_{XX}(\tau) = \lim_{\omega_2 \rightarrow \infty} 2S_0 \frac{\sin \omega_2\tau}{\tau} = 2\pi S_0 \delta(\tau),$$

and with the Fourier transform

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi S_0 \delta(\tau) e^{-i\omega\tau} d\tau = S_0.$$

This is the flat spectrum anticipated and shown in Figure 9.36. The figure includes the equivalent one-sided spectrum.

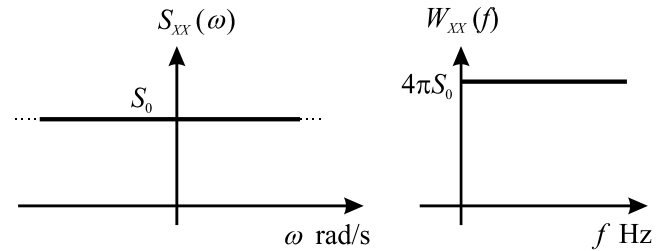


Figure 9.36: White-noise spectrum, on the left in angular frequency (rad/s) and on the right in circular frequency (Hz).

A random process that has a band-limited spectrum is called *colored noise*, following the optics analogy.

VIBRATION TESTING. The vibration testing industry uses sinusoidal and random inputs to help understand how a structure vibrates and to help determine its resonant frequencies. These frequencies are important as they are the frequencies at which the structure vibrates with the greatest amplitude and thus may be the most harmful to the structure.

Sinusoidal vibration testing is used primarily to determine damage to structures. It can identify structural resonances and then focus on one or more of them to determine modal properties and to determine fatigue life associated with each mode. Because “real-world” vibrations are not simple sinusoids, sinusoidal vibration testing is limited. Due to its simplicity, it is a good point of entry into the study of vibrations.²⁹

Random vibration testing excites all the frequencies in a defined spectrum at any given time. One of the main goals or

²⁹W. Tustin, **Random Vibration & Shock Testing**, Equipment Reliability Institute, Santa Barbara, CA, 2005.

uses of random vibration testing in industry is to bring a device to failure. For example, a company might desire to find out how a particular product may fail due to environmental vibrations it may encounter. The company will simulate these vibrations on a shaker and operate their product under these conditions. Testing the product to failure identifies product weaknesses and prompts ways to avoid them. Random vibration testing is the key testing method for this kind of application.

Random vibration testing is more realistic than sinusoidal vibration testing as it simultaneously includes all the forcing frequencies and excites resonances. Under a sinusoidal test, a particular resonance frequency may be found for one part of the device and at a different frequency another part of the device may resonate. Arriving at separate resonance frequencies at different times may not cause any kind of failure, but when both resonance frequencies are excited at the same time, a failure may occur. Random vibration testing will cause both resonances to be excited at the same time, because all frequency components in the testing range will be present at the same time.³⁰

HIGHLY ACCELERATED LIFE TEST. In a highly accelerated life test (HALT) a product is tested until failure. HALT uses “stress factors” such as exposing the product to significant random vibration and extreme temperatures. During HALT the product is in operation and continuously monitored for failures.

A specialized environmental chamber is required to apply pseudo-random vibration to a test product over a wide range of frequencies. Some HALT chambers apply random vibration to 10,000 Hz in 6 degrees of freedom. HALT chambers are sometimes called repetitive shock chambers because pneumatic air hammers are used to produce the random vibration. In addition to exposing a product to extreme vibration environments, the chambers are capable of rapid changes in temperature, such as 50°C/min or higher rates of change. High power resistive heating elements and liquid nitrogen are used for rapid heating and cooling, respectively.

The goal of HALT is to proactively and quickly find design weaknesses, affording engineers the opportunity to fix them prior to product launch. Because of its accelerated nature, HALT is typically faster and less expensive than traditional testing techniques. During product development, it can expose reliability issues at a time when changes are much less costly to make. By finding weaknesses and making changes early, HALT can lower product development costs and compress time to market. HALT can also be used when a product is being introduced into the market to expose potential problems caused by new manufacturing processes as well as after a product has been introduced into the market to test product reliability caused by changes in components, manufacturing processes, suppliers, etc. HALT is used by most manufac-

turers in a wide range of industries to improve reliability of products.

9.8 Random Vibration

9.8.1 Formulation and Preliminaries

We have arrived at our main goal in the study of systems under random loading: to predict the response statistics given the loading statistics. Single degree-of-freedom systems, when modeled as linear, are governed by the second-order differential equation,

$$\ddot{X}(t) + 2\zeta\omega_n\dot{X}(t) + \omega_n^2 X(t) = \frac{1}{m}F(t). \quad (9.38)$$

Here, we take the input force to be the stationary random process $F(t)$, with mean μ_F and power spectrum $S_{FF}(\omega)$, and the output displacement to be the random process $X(t)$. The stationarity assumption for the forcing means that transient dynamic behavior cannot be considered here.³¹ The initial loading transients of an earthquake, a wind gust, or an extreme ocean wave cannot be considered as stationary. If the character of the loading does not change, steady-state behavior can be assumed to be statistically stationary.

Before proceeding with the problem development, we need to show that $g(t)$ and $G(i\omega)$ are a Fourier transform pair. First, we take the Fourier transform of the equation of motion, Equation 9.38, to obtain

$$\left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right] X(i\omega) = \frac{1}{m}F(i\omega),$$

where the Fourier transforms of time derivatives are used,³² and $F(i\omega)$ is the Fourier transform of $F(t)$. Solving for $X(i\omega)$, we obtain

$$X(i\omega) = \frac{F(\omega)}{m \left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right]}. \quad (9.39)$$

The inverse Fourier transform is the response $x(t)$.

We recognize the frequency-response function, Equation 3.42,

$$G(i\omega) = \frac{1}{m \left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right]}. \quad (9.40)$$

Equation 9.39 then becomes

$$X(i\omega) = G(i\omega) F(i\omega), \quad (9.41)$$

³¹ There are clever ways by which stationary solutions can be utilized in nonstationary cases. One possibility is to multiply the stationary process by a deterministic time function such that the product is an *evolutionary* or nonstationary process. For example, use $A(t)F(t)$ as the forcing function, where $A(t)$ is a deterministic transient function and $F(t)$ is stationary.

³² $\mathcal{F}(dX/dt) = i\omega\mathcal{F}(X)$ and $\mathcal{F}(d^2X/dt^2) = -\omega^2\mathcal{F}(X)$.

³⁰ J.V. Baren, “What is Random Vibration Testing?,” *Sound & Vibration*, February 2012, p.9.

where $G(i\omega)$ is the *transfer function* evaluated at $s = i\omega$ for this system. It relates the input $F(i\omega)$ to the output $X(i\omega)$ in the frequency domain.

The inverse Fourier transform of $X(i\omega)$ is given by

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(i\omega) e^{i\omega t} d\omega \\ &= \int_{-\infty}^{\infty} G(i\omega) F(i\omega) e^{i\omega t} d\omega, \end{aligned} \quad (9.42)$$

where $F(i\omega)$ is the Fourier transform of $F(t)$, or

$$F(i\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt.$$

Substituting $F(i\omega)$ into Equation 9.42, we have

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} G(i\omega) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\tau) e^{-i\omega\tau} d\tau \right] e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega) e^{i\omega t} \left[\int_{-\infty}^{\infty} F(\tau) e^{-i\omega\tau} d\tau \right] d\omega. \end{aligned}$$

By definition, the response to a unit impulse load, $F(t) = \delta(t)$, is called the impulse response function and is denoted by $g(t)$. Replacing $F(\tau)$ with $\delta(\tau)$ and $x(t)$ with $g(t)$, we obtain

$$\begin{aligned} g(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega) e^{i\omega t} \left[\int_{-\infty}^{\infty} \delta(\tau) e^{-i\omega\tau} d\tau \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega) e^{i\omega t} d\omega. \end{aligned}$$

The impulse response function $g(t)$ and $G(i\omega)$ are therefore Fourier transform pairs,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega) e^{i\omega t} d\omega \quad (9.43)$$

$$G(i\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt, \quad (9.44)$$

where $1/2\pi$ appears in the Fourier transform instead of inverse Fourier transform formula.

9.8.2 Mean-Value Response

We return to the equation for the deterministic response of a linear harmonic oscillator, Equation 4.29,

$$X(t) = \int_{-\infty}^{\infty} g(\tau) F(t - \tau) d\tau. \quad (9.45)$$

Here, the stationary random load $F(t)$ is applied as $t \rightarrow -\infty$, that is, long before the present time. The extension of the integration to $-\infty$ does not change the value of $X(t)$ because the impulse response function is zero for negative time, and due to causality is zero for $\tau > t$.

We take the expected value of both sides of Equation 9.45, and use the linear properties of the mathematical expectation and the integral to interchange their order,

$$\begin{aligned} E\{X(t)\} &= \int_{-\infty}^{\infty} g(\tau) E\{F(t - \tau)\} d\tau \\ &= E\{F(t)\} \int_{-\infty}^{\infty} g(\tau) d\tau \\ &= \mu_F \int_{-\infty}^{\infty} g(\tau) d\tau \\ &= \mu_F G(0), \end{aligned}$$

where the stationarity of $F(t)$ is used in the second and third equations,³³ and $G(0) = G(i\omega)|_{\omega=0}$ from Equation 9.44. The mean-value displacement can be written as follows,

$$\mu_X = G(0)\mu_F = \frac{1}{k}\mu_F,$$

where the second equality is found using Equation 9.40. We recognize this equation to be Hooke's law relating force and displacement via the stiffness constant.

We next derive the correlation function, and use that result to derive the output spectral density.

9.8.3 Response Correlations

For a stationary process, the autocorrelation function is given by

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} x(t)x(t + \tau)f_X(x)dx,$$

where $f_X(x)$ is the probability density function of the process $X(t)$. The Fourier transform of this equation cannot be evaluated because the response density function $f_X(x)$ is not known. We can proceed in two other equivalent ways: (a) use the ergodic definition of the autocorrelation,³⁴ or (b) via Equation 9.45. We use the second approach.

First, we derive the cross correlation between $F(t)$ and $X(t)$. Multiplying both sides of Equation 9.45 by $F(t - \alpha_1)$, and taking expected values of both sides, we obtain

$$\begin{aligned} E\{X(t)F(t - \alpha_1)\} \\ = \int_{-\infty}^{\infty} g(\tau_1) E\{F(t - \tau_1)F(t - \alpha_1)\} d\tau_1, \end{aligned}$$

where the autocorrelation of the force is

$$E\{F(t - \tau_1)F(t - \alpha_1)\} = R_{FF}(\tau_1 - \alpha_1)$$

³³The force is stationary and therefore has a constant mean value.

³⁴ $R_{XX}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} X(t)X(t + \tau)dt$

and

$$E\{X(t)F(t - \alpha_1)\} = R_{XF}(\alpha_1),$$

is the cross correlation between loading $F(t)$ and response $X(t)$. Thus,

$$R_{XF}(\alpha_1) = \int_{-\infty}^{\infty} g(\tau_1) R_{FF}(\alpha_1 - \tau_1) d\tau_1, \quad (9.46)$$

and $R_{FF}(\tau)$ is known from experimental data. Next, multiplying both sides of Equation 9.45 by $X(t + \alpha_2)$, and taking expected values of both sides,

$$\begin{aligned} E\{X(t + \alpha_2)X(t)\} \\ = \int_{-\infty}^{\infty} g(\tau_2) E\{X(t + \alpha_2)F(t - \tau_2)\} d\tau_2. \end{aligned} \quad (9.47)$$

By definition then,

$$R_{XX}(\alpha_2) = \int_{-\infty}^{\infty} g(\tau_2) R_{XF}(\tau_2 + \alpha_2) d\tau_2. \quad (9.48)$$

Substituting Equation 9.46 into 9.48 we find³⁵

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha) g(\beta) R_{FF}(\tau + \beta - \alpha) d\alpha d\beta, \quad (9.49)$$

which is a double convolution. To evaluate the variance, we use

$$\begin{aligned} \sigma_X^2 &= E\{X(t)^2\} - E^2\{X(t)\} \\ &= R_{XX}(0) - [G(0)E\{F\}]^2. \end{aligned} \quad (9.50)$$

Response Mean and Variance

Suppose that an analysis resulted in the statistics μ_F and σ_F , where stationarity has been assumed. Let us consider how the response mean and variance (or standard deviation) can be useful in a design procedure.

Stationarity implies that the mean value is not a function of time and that the correlation is only a function of time difference τ . Equation 9.50 was obtained by setting $\tau = 0$ and by substituting the response mean value. The designer needs both the mean value *and* the variance to establish bounds on the possible response. Example bounds are $\mu_F \pm \sigma_F$, $\mu_F \pm 2\sigma_F$, or $\mu_F \pm 3\sigma_F$, where the larger the *sigma bounds* the more likely that all possible responses are covered. Along with a higher probability comes this broader band with its vagueness. There is no way around this *uncertainty-type* principle. These upper and lower bounds are used to define the least and most likely range of responses. If designing for strength, then

³⁵Keep careful track of the *dummy variables* so that appropriate arguments are maintained. Here, let $\alpha_2 \equiv \tau$, $\tau_1 \equiv \alpha$, and $\tau_2 \equiv \beta$ in order to simplify the notation.

the upper sigma bound can be used to size the structural components.

How wide or narrow the sigma bounds are depends on the underlying density function. For parameters governed by the Gaussian probability density, there is a probability of 0.6827 of being within the one-sigma bounds, and a probability of 0.9545 of being within the two-sigma bounds. Different densities have different probabilities for their sigma bounds.

There is no easy or clear cut answer regarding how many sigma bounds to use in a design. The designer must study the data in order to better understand the underlying density. As a practical matter, by retaining larger sigma bounds in the design, it becomes more conservative, which leads to a more costly structure or product.

We proceed to evaluate the response spectral density.

9.8.4 Response Spectral Density

We begin with the Fourier transform relation between power spectrum and correlation function, $S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega\tau) R_{XX}(\tau) d\tau$, substitute Equation 9.49 for $R_{XX}(\tau)$, and let $\lambda = \tau + \beta - \alpha$,

$$\begin{aligned} S_{XX}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega\tau) \\ &\quad \times \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha) g(\beta) R_{FF}(\lambda) d\alpha d\beta \right] d\tau \\ S_{XX}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\alpha) \exp(-i\omega\alpha) d\alpha \\ &\quad \times \int_{-\infty}^{\infty} g(\beta) \exp(+i\omega\beta) d\beta \\ &\quad \times \int_{-\infty}^{\infty} R_{FF}(\lambda) \exp(-i\omega\lambda) d\lambda \\ &= G(i\omega) G^*(i\omega) S_{FF}(\omega), \end{aligned} \quad (9.51)$$

where $*$ denotes complex conjugate, and, therefore,

$$S_{XX}(\omega) = |G(i\omega)|^2 S_{FF}(\omega). \quad (9.52)$$

This is the *fundamental* result for random vibration and linear systems theory that allows us to evaluate the output spectral density, given the input spectral density and the system frequency response function. It is emphasized here that the derivation of Equation 9.52 made use of the convolution equation, which is valid only for linear systems. Any generalization to nonlinear systems requires problem-specific approaches.³⁶

³⁶Two widely used techniques for nonlinear stochastic problems are *stochastic linearization*, which allows the use of linear theory, and *perturbation methods*, which transform a nonlinear equation into an infinite sequence of linear equations, again allowing the use of linear theory.

Example 9.12 Oscillator Response to White Noise

Consider an application of the above ideas to an oscillator. What is the response of a damped oscillator to a force with white-noise probability density?

Solution: The governing equation of motion is

$$\ddot{X}(t) + 2\zeta\omega_n\dot{X}(t) + \omega_n^2 X(t) = \frac{F(t)}{m},$$

where $F(t)$ is the external force, and the squared magnitude of the system function is given by

$$|G(i\omega)|^2 = \frac{1/m^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}.$$

Therefore, given *any* input spectral density $S_{FF}(\omega)$, the response spectral density is

$$\begin{aligned} S_{XX}(\omega) &= |G(i\omega)|^2 S_{FF}(\omega) \\ &= \frac{S_{FF}(\omega)/m^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}. \end{aligned}$$

If, for mathematical simplicity, the forcing is white noise, $S_{FF}(\omega) = S_0$, then

$$S_{XX}(\omega) = \frac{S_0/m^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}, \quad (9.53)$$

and the mean-square response is given by

$$E\{X(t)^2\} = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega. \quad (9.54)$$

To evaluate this integral, we make use of the following formula,

$$\int_{-\infty}^{\infty} \left| \frac{B_0 + i\omega B_1}{A_0 + i\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2},$$

and we write the frequency response function in the form,

$$G(i\omega) = \frac{1/k}{1 + i\omega \cdot 2\zeta/\omega_n - \omega^2 \cdot 1/\omega_n^2}.$$

Therefore,

$$\begin{aligned} &\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \\ &= S_0 \int_{-\infty}^{\infty} \left| \frac{1/k}{1 + i\omega \cdot 2\zeta/\omega_n - \omega^2 \cdot 1/\omega_n^2} \right|^2 d\omega \\ &= S_0 \cdot \frac{\pi [0 + (1/\omega_n^2) \cdot (1/k)^2]}{1 \cdot (2\zeta/\omega_n) \cdot (1/\omega_n^2)} \\ &= \frac{\pi S_0}{ck}. \end{aligned}$$

The mean-square response can also be written in terms of a one-sided spectrum using Equation 9.35,

$$\begin{aligned} E\{X(t)^2\} &= \frac{\pi S_0}{ck} \\ &= \frac{W_0}{4ck}, \end{aligned}$$

using the one-sided density. It needs to be emphasized that if the $1/2\pi$ factor is not “on this side” of the Fourier transform, that is, if Equation 9.54 has the factor $1/2\pi$, the above result becomes

$$\begin{aligned} E\{X(t)^2\} &= \frac{S_0}{2ck} \\ &= \frac{W_0}{8\pi ck}. \end{aligned}$$

Even though infinite mean-square energy is input to the system,³⁷ it responds with finite mean-square energy. Figure 9.37 shows plots of the components of Equation 9.52. White noise is frequently used even though it is non-physical because its use as an approximation of the loading characteristics leads to reasonably accurate results. ■

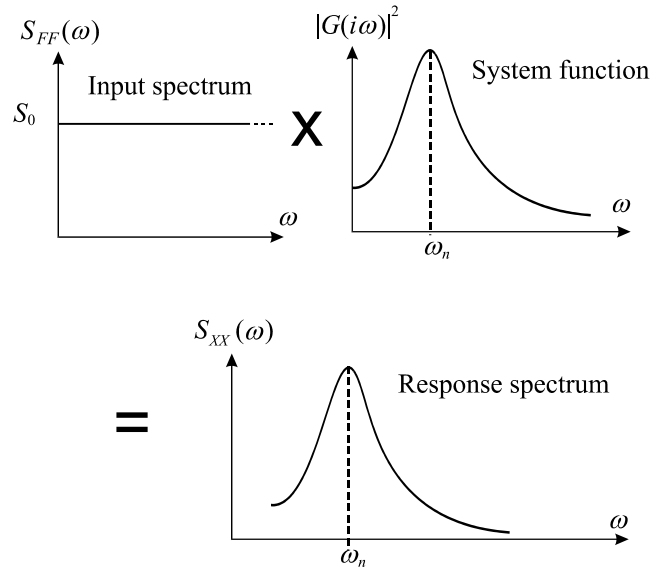


Figure 9.37: With input spectrum $S_{FF}(\omega)$, transfer function $|G(i\omega)|^2$, and output spectrum $S_{XX}(\omega)$, we see how the transfer function acts as a filter, allowing primarily input energy around the natural frequency of the system to pass through, resulting in output spectrum energy primarily around the system natural frequency.

³⁷The energy input equals the area under the spectral density, which for white noise is

$$\int_{-\infty}^{\infty} S_0 d\omega = \infty.$$

Response to Colored Noise

Suppose the system in the last example is subject to more complex loading, where the spectral density of the forcing is not a constant, but a function of ω . How would the analysis above change?

The output spectral density becomes a more complicated function of frequency, for example, if the loading spectral density is more complicated, such as a wind load spectrum. Then, the mean square response must be evaluated numerically. Applications are almost always too complicated to be solved analytically. However, a preliminary simplified analysis can provide insights that lead to a better design.



VIBRATION TESTING OF BICYCLES.³⁸ Ranging all the way from the casual bicycle commuter to the Tour de France racer, today's cyclist highly values a bicycle that maximizes comfort and handling on rough roads and trails. As a result, modern bicycle engineers can put immense efforts into designing and testing new technologies which may effectively mitigate this vibration without significantly penalizing other key characteristics like weight, handling, and responsiveness. Both the mountain bike and the road bike market have a long history of ongoing suspension refinement and innovation. More recently, the road bike market is exploding with a variety of vibration-mitigating technologies to improve comfort and reduce rider fatigue.

To evaluate the vibration performance of new ideas and prototypes, bicycle engineers conduct tests using a variety of sensors, such as accelerometers and linear potentiometers, to quantify the vibration. Sensors, such as cycling powermeters, speedometers, and GPS trackers, are also used to add context to the vibration data.

While laboratory testing or FEA modeling of fundamental structural behaviors can sometimes be useful (such as the modal analysis of a hanging bicycle during free vibration), bicycle engineers must deal with the fact that the moving bicycle-rider-road system is highly complex. As a result, an emphasis is placed on real-world testing and data analysis metrics such as the RMS, Power Spectral Density (PSD), and Frequency Response Function (FRF).

For example, the effectiveness of handlebar prototypes can be compared by installing accelerometers at various points along the front end of the bicycle, as shown in [Figure 9.38](#). This bicycle is then repeatedly ridden across a rough road by various riders at various speeds while collecting acceleration data, as shown in [Figure 9.39](#).

Sample accelerometer data from three runs are overlaid in [Figure 9.40](#). This plot indicates that prototype 2 has the lowest vibration magnitude. It also shows how chaotic and variable real-world data can be, so we must take a statistical

approach that accounts for several riders and several runs. One quick statistical method is to take the RMS of all the data to create a single value that describes the amount of vibration at the hand for each handlebar. While this approach is valid, it provides little insight into the underlying physics of what is actually happening to the bicycle.

More insight can be achieved by performing a spectral analysis such as an FFT or PSD. The PSD shown in [Figure 9.41](#) shows a single strong vibration peak at about 38 Hz, indicating that we are dealing with a single mode and/or a single forced input frequency as the front wheel encounters the cobblestones. We also see that the vibration content in this scenario tapers off by 100 Hz, which has implications on human sensitivity/comfort and the potential effectiveness of various damping technologies.

Taking the analysis one step further, we can effectively normalize for any run-to-run variation by conducting an FRF analysis between two measured points on the structure. This allows us to compare the vibration transmission through the structure in-between these two points, such as through the front fork, handlebar, and seatpost. For example, in [Figure 9.42](#), we see that prototype 2 transmits less vibration through the structure and into the rider's body than either the baseline or prototype 1. The FRF also gives us information about prototype 2's effectiveness at any given frequency. This type of testing and analysis helps guide the engineer's design decisions and, almost as importantly, arms the bicycle marketer with scientific proof that this new design provides a true benefit to the customer.



Figure 9.38: Accelerometers and data acquisition unit mounted to a bicycle. (Courtesy of Paul Harder, Trek Bicycle)

³⁸Courtesy of Paul Harder, Trek Bicycle, Waterloo, WI.



Figure 9.39: Bicycle testing on cobblestone road. (Courtesy of Paul Harder, Trek Bicycle)

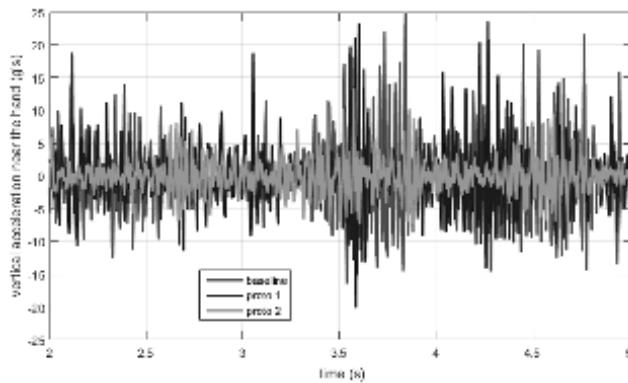


Figure 9.40: Example raw data from three runs with different handlebars. (Courtesy of Paul Harder, Trek Bicycle)

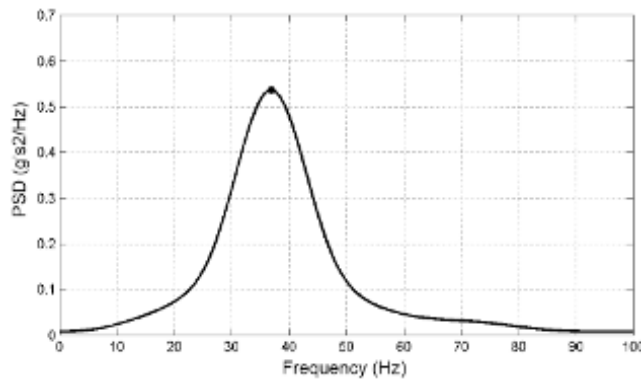


Figure 9.41: PSD for one accelerometer channel of data. (Courtesy of Paul Harder, Trek Bicycle)

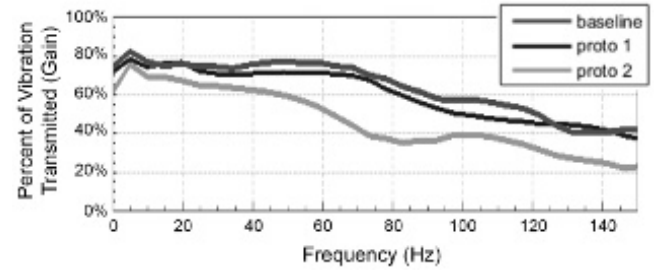


Figure 9.42: FRF plot for the vibration transmission through a bicycle structure (not necessarily a handlebar). (Courtesy of Paul Harder, Trek Bicycle)

9.9 Stochastic Response of a Linear MDOF System

In general, it is difficult to formulate and solve for the displacement response of a multi degree-of-freedom (MDOF) discrete system loaded by random forces. Here, we develop one possible approach based on the modal analysis of such a structure.³⁹

We start the analysis at the point where the modal equations of motion have been formulated, and the assumption of proportional damping has been made. Using capital letters to signify random processes, the modal equations in indicial notation are

$$\ddot{Q}_i + 2\zeta_i\omega_i\dot{Q}_i + \omega_i^2Q_i = F_i(t), \quad i = 1, 2, \dots, n,$$

where the constants have been defined previously. We assume that the modal forces $F_i(t)$ are *ergodic*. The transformation between physical and modal coordinates are given by

$$\{X(t)\} = \sum_{i=1}^n \{\hat{u}\}_i Q_i(t),$$

or

$$X_j(t) = \sum_{i=1}^2 \hat{u}_{ji} Q_i(t) = \hat{u}_{j1} Q_1(t) + \hat{u}_{j2} Q_2(t), \quad (9.55)$$

for each degree-of-freedom j of a two degree-of-freedom system. The goal in this analysis is to find the response autocorrelation $R_{X_j X_j}(\tau)$ and its Fourier transform, the power spectrum $S_{X_j X_j}(\omega)$.

Substituting Equation 9.55 for $X_j(t)$ in the expression

³⁹The approach of this section is standard, but some additional details are available, for example, in Chapter 23 of **Dynamics of Structures**, R.W. Clough and J. Penzien, McGraw-Hill, 2nd ed., 1993.

of the autocorrelation function, we find

$$\begin{aligned} R_{X_j X_j}(\tau) &= E\{X_j(t)X_j(t+\tau)\} \\ &= E\left\{\sum_{l=1}^n \sum_{m=1}^n \hat{u}_{jl}\hat{u}_{jm}Q_l(t)Q_m(t+\tau)\right\}, \end{aligned} \quad (9.56)$$

where

$$Q_i(t) = \int_0^t F_i(\tau)g_i(t-\tau)d\tau \quad (9.57)$$

$$g_i(t) = \frac{1}{\omega_{d_i}} \exp(-\zeta_i \omega_i t) \sin \omega_{d_i} t \quad (9.58)$$

$$\omega_{d_i} = \omega_i(1 - \zeta_i^2)^{1/2}. \quad (9.59)$$

Since the impulse response function $g(t)$ equals zero for $t < 0$, the lower limit of the integral defining $Q(t)$ can become $-\infty$ without changing the value of the integral.

Next, substituting Equations 9.57 to 9.59 into Equation 9.56, and moving the expectation operator to the stochastic terms, we find

$$\begin{aligned} R_{X_j X_j}(\tau) &= \sum_l \sum_m \int_{-\infty}^{t+\tau} \int_{-\infty}^t u_{jl}u_{jm} E\{F_l(t_1)F_m(t_2)\} \\ &\quad \times g_l(t-t_1)g_m(t+\tau-t_2)dt_1dt_2, \end{aligned} \quad (9.60)$$

where t_1 and t_2 are dummy time variables, and the forcing cross correlations, defined as

$$R_{F_l F_m}(t_2 - t_1) = E\{F_l(t_1)F_m(t_2)\},$$

are due to the assumed ergodicity (and thus stationarity) of the forcing.

If the system is lightly damped and has well separated modal frequencies, as is the case for many engineering structures, the response due to $F_l(t)$ is almost statistically independent of the response due to $F_m(t)$. The cross correlation terms that arise in Equation 9.60 are then approximately equal to zero, with the only nonzero terms arising for $m = l$,

$$R_{F_l F_m}(t_2 - t_1) \simeq E\{F_l(t_1)F_l(t_2)\}. \quad (9.61)$$

We proceed to evaluate the response spectral density, from which probabilities of occurrence can be evaluated. To do this, the following transformations of variables⁴⁰ are necessary,

$$\begin{aligned} u_1 &\equiv t - t_1 & u_2 &\equiv t + \tau - t_2 \\ du_1 &= -dt_1 & du_2 &= -dt_2, \end{aligned}$$

resulting in the response autocorrelation,

$$\begin{aligned} R_{X_j X_j}(\tau) &= \sum_l \sum_m \int_0^\infty \int_0^\infty u_{jl}u_{jm} R_{F_l F_m}(u_1 - u_2 + \tau) \\ &\quad \times g_l(u_1)g_m(u_2)du_1du_2. \end{aligned}$$

The power spectral density for the response $X_j(t)$ equals the Fourier transform of its autocorrelation function,

$$S_{X_j X_j}(\omega) = \frac{1}{2\pi} \int_{-\infty}^\infty R_{X_j X_j}(\tau)e^{-i\omega\tau}d\tau.$$

Since the processes are assumed to be ergodic, by averaging in time⁴¹ we obtain the spectral density

$$\begin{aligned} S_{X_j X_j}(\omega) &= \sum_l \sum_m u_{jl}u_{jm} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g_l(u_1)du_1 \right. \\ &\quad \times \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g_m(u_2)du_2 \\ &\quad \times \left. \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{F_l F_m}(u_1 - u_2 + \tau)e^{-i\omega\tau}d\tau \right\}. \end{aligned}$$

The lower limits of the integrals have been set to $-T$ since $g(t)$ is zero for $t < 0$, and thus the change in lower limits does not affect the values of the integrals. Using the following change of variables,

$$\gamma \equiv u_1 - u_2 + \tau, \quad d\gamma = d\tau,$$

we obtain the spectral density relation

$$\begin{aligned} S_{X_j X_j}(\omega) &= \sum_l \sum_m u_{jl}u_{jm} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g_l(u_1) \exp(i\omega u_1)du_1 \right. \\ &\quad \times \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g_m(u_2) \exp(-i\omega u_2)du_2 \\ &\quad \times \left. \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T-u_2+u_1}^{T-u_2+u_1} R_{F_l F_m}(\gamma) \exp(-i\omega\gamma)d\gamma \right\}. \end{aligned} \quad (9.62)$$

For the last integral, we make the physical argument that $R_{F_l F_m}(\gamma) \rightarrow 0$ as $|\gamma|$ increases, and, therefore, the limits

⁴⁰The integration limits must also be transformed as a result of this transform of variables.

⁴¹See Equations 9.30 and 9.31.

can be replaced by $-T$ and T , respectively.⁴² Then,

$$G_l(-i\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g_l(u_1) e^{i\omega u_1} du_1 \quad (9.63)$$

$$G_m(i\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g_m(u_2) e^{-i\omega u_2} du_2 \quad (9.64)$$

$$S_{F_l F_m}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{F_l F_m}(\gamma) e^{-i\omega \gamma} d\gamma, \quad (9.65)$$

with the resulting response spectral density

$$S_{X_j X_j}(\omega) = \sum_l \sum_m u_{jl} u_{jm} G_l(-i\omega) G_m(i\omega) S_{F_l F_m}(\omega),$$

where $G(i\omega)$ is given by Equation 9.40.

For lightly damped systems with well-spaced modal frequencies, the cross terms in the double summation, those where $l \neq m$, contribute very little to the mean-square response, given by $\int_{-\infty}^{\infty} S_{X_j X_j}(\omega) d\omega$. In this case, we can use the approximation

$$S_{X_j X_j}(\omega) \simeq \sum_l u_{jl}^2 |G_l(i\omega)|^2 S_{F_l F_l}(\omega).$$

More details are available in specialized texts.⁴³

9.10 Lunar Seismic Structural Analysis

The analysis and design of structures in seismic environments is challenging.⁴⁴ Uncertainties abound; for example, we do not know when a seismic event will occur, neither its location (epicenter) nor its magnitude. Furthermore, this is a nonstationary event given that the input turns on at a given time, making a random vibration analysis that much more difficult. On Earth, after extensive experience, we have been able to design and retrofit structures, underground pipelines, and power stations so that they can survive certain levels of seismic activity with minimal damage. For the Moon, however, so much less is known about lunar seismicity that any studies performed without a significant presence on site are likely to be at best order of magnitude accurate.

Seismic waves are divided into two classical types, P or primary waves (also called compressional waves) and S or shear waves (also called secondary waves). P waves

are the first arriving waves, having traveled a direct path from the focus of seismic disturbance to the structure (or seismometer) and cause the medium to move back and forth in the direction of propagation. Afterwards, a pulse pP that traveled from the focus gets reflected off the surface, and arrives at the structure. Second to P waves are S waves, which cause the medium to move perpendicular to the direction of propagation. Analogous to P waves, sS waves are the reflected shear waves.

P and S waves, including their subcategories, are known as *body waves* since they travel through the Earth's, or the Moon's, interior. P and S waves are divided into more categories than discussed here.

After the arrival of the body waves, for terrestrial cases, seismograms are dominated by larger longer-period waves. These waves are called surface waves, as these are trapped near the surface of the Earth. In other words, their energy is concentrated near the Earth's surface. These are subcategorized into two types, *Love waves* and *Rayleigh waves*. Love waves result from SH waves (shear waves polarized in the horizontal plane) trapped near the surface, while Rayleigh waves are a combination of P and SV (shear waves polarized in the vertical plane) waves that can exist at the top of a homogeneous half-space.

Earth's continental crust has many small layers and reflectors. These structures cause some of the energy to be scattered and arrive at a receiver later than the initial pulse. The scattered energy arrives from various directions and shows little or no preferred particle motion. Therefore, this scattered energy causes an arrival to have a *coda*, a trail of incoherent energy that decays over seconds or minutes.

Intraplate earthquakes occur within plate interiors where they are not perfectly rigid. In some cases, for example in Hawaii, these earthquakes are associated with intraplate volcanism. These are studied, generally, to provide data about where and how the plate tectonic model does not fully describe tectonic processes.

From the Apollo Passive Seismic Experiments (APSE), a part of the Apollo missions, a network of four seismometers was installed on the surface of the Moon. Data were collected and transmitted back to the Earth from 1969 to 1977. The passive seismic experiment package sensor unit contained three long-period seismometers with resonant frequencies of 1/15 Hz, aligned orthogonally to measure surface motion in the vertical and two horizontal directions, and a single-axis short-period seismometer sensitive to vertical motion at higher frequencies with a resonant frequency of 1 Hz. During the 8 years of network operation, more than 12,000 seismic events were detected.

Four distinct types of natural seismic sources have been identified: deep moonquakes, thermal moonquakes, meteoroid impacts, and shallow moonquakes.

⁴²Physically, this is a statement that as time difference γ increases, there will be an exponentially decaying correlation. This is borne out by experiments on numerous physical systems.

⁴³**Random Vibration: Theory and Practice**, P. Wirsching, T. Paez, and H. Ortiz, Wiley-Interscience, 1995, now a Dover book.

⁴⁴This discussion is based on the paper by S. Mottaghi and H. Benaroya, "Design of a Lunar Structure. II: Seismic Structural Analysis," *Journal of Aerospace Engineering*, Vol.28, Issue 1, Jan. 2015.

Of those seismic events, deep moonquakes were the most abundant, with 7082 confirmed (and 317 unconfirmed) events. Most of their foci occurred within a clearly defined region between depths of 800 km to 1000 km. Their occurrences are strongly correlated with the tidal forces on the Moon due to the Earth and the Sun. These are low frequency, low magnitude events and are unlikely to pose a danger to a surface lunar structure. A body wave magnitude of 1.3 to 3.0 is estimated for such deep moonquakes.

Thermal moonquakes are very unlikely to pose any threat to lunar structures, as these are very small seismic events caused by temperature variations at or near the lunar surface. Although their exact cause is unknown, it is believed they originate in young craters and large rocks.

A total of 1743 meteoroid impacts has been identified from the APSE data – while the actual number is much greater – the long period seismographs were capable of detecting mainly the signals from objects of mass 0.1 kg and higher. The largest meteoroid mass has been estimated at about 2000 kg based on an assumed impact velocity of 22.5 km/s.

Shallow moonquakes, also called high-frequency teleseismic (HFT) events, were the most energetic and also the rarest seismic sources that have been observed on the Moon. A total number of 28 events has been distinguished. The name HFT is used for these events due to their unusually high frequency content and the great distance at which they were observed. Body wave magnitude estimates are in the range of about 5 to 5.5 for the largest shallow moonquake. The epicenters of all the detected HFT events were located outside the Apollo seismic array, and the number of recorded events was small, thus limiting our understanding of these events. The actual depths of the HFT events were undetermined, but are estimated at less than 200 km and more likely below an approximate 55-km thick crust, perhaps in the range of 60 to 100 km. The cause of these events is yet unknown.

While the terrestrial equivalent of the higher magnitude lunar quakes can cause moderate to severe damage to buildings, seismic effects on a lunar structure are not the same since the lunar geologic structure is very different than that of the Earth. The seismic wave train has been observed both on the Moon and the Earth. On the Moon the seismic wave train is less attenuated and more intensely scattered than on Earth. Thus, the lunar seismograms are very long compared to that of Earth. Although the lunar seismic wave trains are very long, the seismic energy is scattered over tens of minutes, likely reducing the potential damage to a lunar structure.

Even though we have much more knowledge about the Earth than the Moon, we still face many challenges in predicting earthquakes and their effects. Due to mini-

mal data and limited knowledge of the Moon's interior, the results of Moonquakes are, at best, very approximate. In the paper referenced, a seismic model was developed based on best available data and then applied to a proposed lunar structure, shown in Figures 9.43 and 9.44.

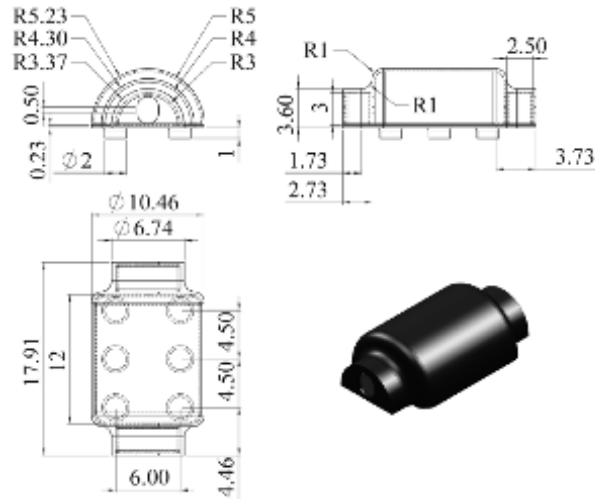


Figure 9.43: Engineering drawings of a lunar structure with dimensions in meters.

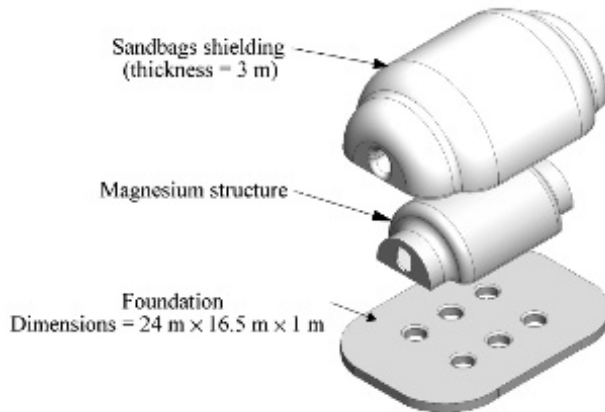


Figure 9.44: Proposed lunar structure. The foundation is composed of lunar concrete. Not shown are six foundation piles.

An elastic pulse propagates without change in shape in an ideal, infinite, homogeneous, non-dispersive, and elastic medium. However, in heterogeneous media, a single seismic pulse is rapidly converted into a long train of pulses. Therefore, the number of possible ray paths becomes quite large and the overall process takes a diffusive form. It is possible to find the power spectrum of the ground velocity, $S_{vv}(\omega)$,

$$S_{vv}(R, \omega, t) = \frac{E_0(\omega)}{\rho (4\pi Dt)^{3/2}} \exp\left(\frac{-R^2}{4Dt} - \frac{\omega t}{Q}\right), \quad t > t_i,$$

where

$$E_0(\omega) = \frac{10^{15.6}}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2} \left(\frac{\omega - 6\pi}{2\pi}\right)^2\right),$$

and where t_i is the time after the start of the seismic event when the first seismic waves reach the structure, R is the distance from the focus, $E_0(\omega)$ is the total seismic energy generated by the seismic event within a unit frequency band around ω at $R = 0$, ρ is the density of the regolith in the vicinity of the structure, Q is the non-dimensional intrinsic quality factor due to anelastic conversion of seismic energy to heat (does not include the loss by scattering), D is the diffusivity, and t is the time the seismic wave traveled. The resulting power spectrum of the ground velocity is depicted in Figure 9.45 for a normally distributed source.

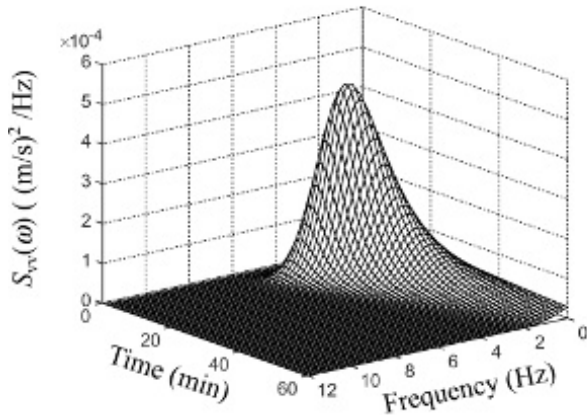


Figure 9.45: Power spectrum of the ground velocity for normally distributed source.

The first 20 modes of the structure were calculated and the first 12 modes were included in a random vibration analysis. Figure 9.46 shows the first six modes along with natural frequencies.

The maximum stresses occurred in the environs of the internal doorways. The von Mises stresses are depicted in Figure 9.47 for the normally distributed source.

The structural response to a lunar seismic event revealed that the risk associated with designs that neglect seismic effects is low since the designs use relatively high factors of safety. This low risk seems reasonable especially since the lunar surface is covered with a layer of regolith, and granular soil is proven to have excellent damping qualities. Moreover, it is very unlikely that a moonquake of such magnitude would occur at a 25 km distance or less, as the estimated depth of these events are between 60 to 100 km.

It is estimated that a lunar structure at a randomly chosen site could experience a shallow moonquake of body wave magnitude greater than 4.5 within a range

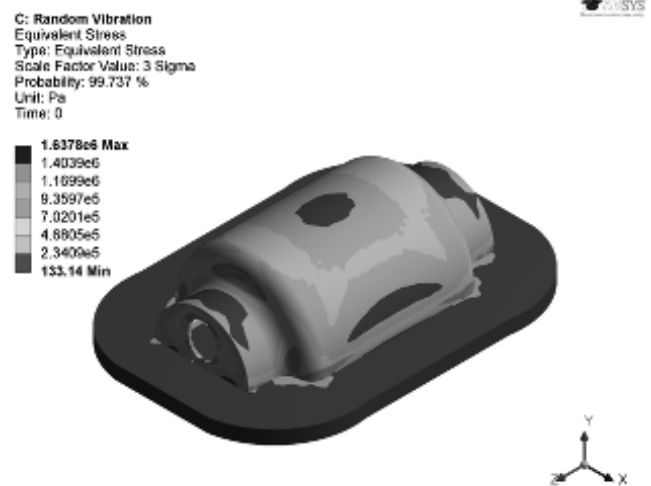


Figure 9.47: Von Mises stresses for a normally distributed source.

of 100 km about once in 400 years. It is envisioned that these structures will be highly damped, reducing the damage due to seismic events. The same statement might not be valid for tall or flexible structures. Also, seismic effects must be considered when designing the airlocks and nodes connecting different structures that require air seals, as these may be more sensitive to such excitation, and to differential motions.

We expect second-generation lunar habitats to be sub-surface structures. These types of structures will require a more detailed seismic analysis since the effects of regolith-structure coupling become more important. Thin-walled structures will unlikely be used in sub-surface habitats.

9.11 Random Vibration of Continuous Structures

While the general problem of random vibration of a continuous structure is quite complex, we can begin by studying the simpler random vibration of an elastic and continuous structure using the modal approach. The algebra can become quite extensive, but it is instructive to carry out the procedures, since computational approaches will be based on similar ideas.

The lateral vibration of a beam is used in this introduction, and we refer to Example 7.6, beginning with Equation 7.107, for several key equations. In order to avoid confusion between the displacement and the modal functions, we will use $Z(x, t)$ for displacement and $\hat{Y}_j(\cdot)$ for the normalized modal function. (Even though $\hat{Y}_j(\cdot)$ is capitalized, it is not a random function.)

We assume that the loading function $P(x, t)$ is random and represents an ensemble of functions in space and

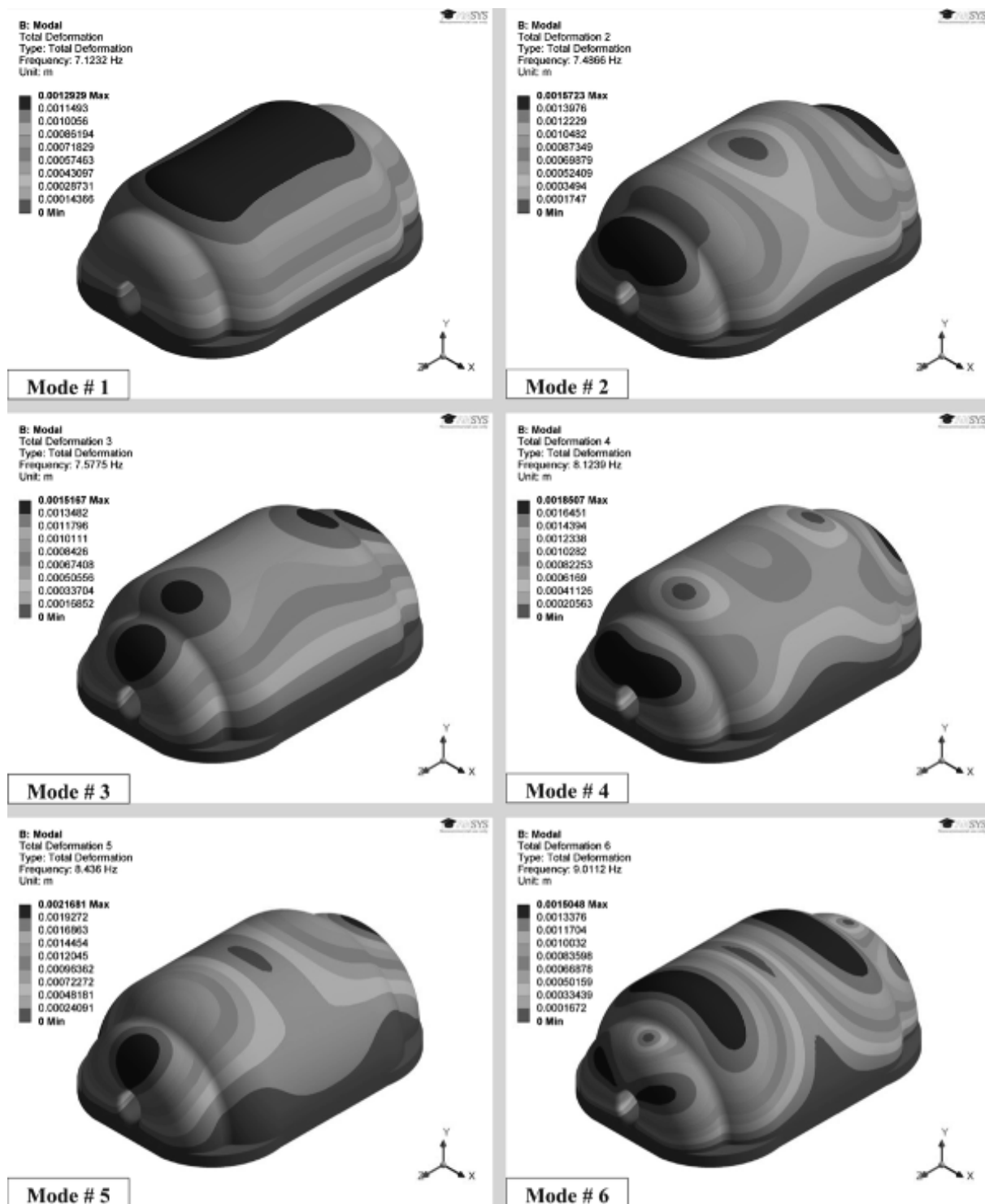


Figure 9.46: The first six modes of a proposed lunar structure.

time. We are given the mean value of the loading,

$$E\{P(x, t)\} = \mu_P(x, t),$$

and its autocorrelation,

$$E\{P(x_1, t_1)P(x_2, t_2)\} = R_{PP}(x_1, x_2; t_1, t_2).$$

The spectral density $S_{PP}(\omega)$ is the Fourier transform of the autocorrelation function. These statistics are available through data-gathering experiments on the structure. The goal in this analysis is to derive the statistics for the response, μ_Z , R_{ZZ} , and S_{ZZ} .

For steady-state vibration, it is reasonable to assume that the statistics of the loading are stationary. We begin with Equation 7.113,

$$Z(x, t) = \sum_{j=1}^{\infty} \hat{Y}_j(x) \int_0^t \left[\int_0^L mP(\xi, t-\tau) \hat{Y}_j(\xi) d\xi \right] g_j(\tau) d\tau,$$

where $g_j(t)$ is the impulse response function and the time shift τ has been placed within $P(x, t)$. We take the expectation of each side of the equation, to find

$$E\{Z(x, t)\} = m \sum_{j=1}^{\infty} \hat{Y}_j(x) \int_0^t \left[\int_0^L E\{P(\xi, t-\tau)\} \hat{Y}_j(\xi) d\xi \right] g_j(\tau) d\tau.$$

Since the force is assumed to be stationary, the expected value is not a function of time,

$$E\{P(x, t-\tau)\} = \mu_P(x).$$

In order to simplify the analysis, we also assume that the loading has zero mean. This does not overly simplify the problem since a non-zero mean can be introduced later by simply shifting the response by the (constant) mean value. For zero mean loading, $\mu_P(x) = 0$, the response is also zero mean,

$$E\{Z(x, t)\} = \mu_Z = 0.$$

The next step in the analysis is to relate the autocorrelation of the response to the autocorrelation of the loading. This requires several steps, as follows. For the loading, we use Equation 7.108,

$$\int_0^L mP(x, t) \hat{Y}_j(x) dx = P_j(t),$$

to find the autocorrelation,

$$\begin{aligned} R_{P_j P_k}(t_1, t_2) &= E\{P_j(t_1)P_k(t_2)\} \\ &= m^2 \int_0^L \int_0^L E\{P(x_1, t_1)P(x_2, t_2)\} \hat{Y}_j(x_1) \hat{Y}_k(x_2) dx_1 dx_2 \\ &= m^2 \int_0^L \int_0^L R_{PP}(x_1, x_2; t_1, t_2) \hat{Y}_j(x_1) \hat{Y}_k(x_2) dx_1 dx_2, \end{aligned} \quad (9.66)$$

where, for a stationary loading,

$$R_{PP}(x_1, x_2; t_1, t_2) = R_{PP}(x_1, x_2; \tau),$$

$\tau = t_2 - t_1$, and, therefore,

$$R_{P_j P_k}(t_1, t_2) = R_{P_j P_k}(\tau).$$

Using Equation 7.109, where $Z(x, t)$ is expanded in terms of the modes $\hat{Y}_k(x)$,

$$Z(x, t) = \sum_{k=1}^{\infty} Z_k(t) \hat{Y}_k(x),$$

and the response autocorrelation is given by

$$\begin{aligned} R_{ZZ}(x_1, x_2; t_1, t_2) &= E\{Z(x_1, t_1)Z(x_2, t_2)\} \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} E\{Z_j(t_1)Z_k(t_2)\} \hat{Y}_j(x_1) \hat{Y}_k(x_2) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} R_{Z_j Z_k}(t_1, t_2) \hat{Y}_j(x_1) \hat{Y}_k(x_2), \end{aligned} \quad (9.67)$$

where,⁴⁵ using Equation 7.112,

$$\begin{aligned} R_{Z_j Z_k}(t_1, t_2) &= E \left\{ \int_{-\infty}^{\infty} P_j(t_1 - \theta) g_j(\theta) d\theta \right. \\ &\quad \left. \times \int_{-\infty}^{\infty} P_k(t_2 - \kappa) g_k(\kappa) d\kappa \right\} \end{aligned}$$

$$\begin{aligned} R_{Z_j Z_k}(t_1, t_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\{P_j(t_1 - \theta)P_k(t_2 - \kappa)\} \\ &\quad \times g_j(\theta)g_k(\kappa) d\theta d\kappa \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{P_j P_k}(t_1 - \theta, t_2 - \kappa) \\ &\quad \times g_j(\theta)g_k(\kappa) d\theta d\kappa. \end{aligned}$$

Since $P(x, t)$ is stationary,

$$R_{P_j P_k}(t_1 - \theta, t_2 - \kappa) = R_{P_j P_k}(t_2 - t_1 + \theta - \kappa),$$

⁴⁵The convolution equation is

$$y_j(t) = \int_0^t p_j(t-\tau)g_j(\tau)d\tau,$$

where $g_j(\tau)$, the impulse response function, is identically zero for $t < \tau$, that is, for the time preceding the excitation of the system. Hence, advancing the upper limit of the integral to infinity does not affect the value of the integral. Similarly, for $t < 0$, $p(t) = 0$ and, therefore, we advance the lower limit to minus infinity. The response can then be written as

$$y_j(t) = \int_{-\infty}^{\infty} p_j(t-\tau)g_j(\tau)d\tau.$$

and $R_{Z_j Z_k}(t_1, t_2) = R_{Z_j Z_k}(\tau)$. Therefore,

$$\begin{aligned} R_{Z_j Z_k}(\tau) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{P_j P_k}(\tau + \theta - \kappa) g_j(\theta) g_k(\kappa) d\theta d\kappa, \end{aligned} \quad (9.68)$$

and

$$R_{ZZ}(x_1, x_2; \tau) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} R_{Z_j Z_k}(\tau) \hat{Y}_j(x_1) \hat{Y}_k(x_2).$$

To complete this analysis, the spectral density of the response, $S_{ZZ}(\omega)$, needs to be evaluated. Formally, this is given by the Fourier transform of $R_{ZZ}(x_1, x_2; \tau)$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ZZ}(x_1, x_2; \tau) e^{-i\omega\tau} d\tau,$$

and using Equation 9.67,

$$\begin{aligned} S_{ZZ}(\omega) &= \frac{1}{2\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[\int_{-\infty}^{\infty} R_{Z_j Z_k}(\tau) e^{-i\omega\tau} d\tau \right] \hat{Y}_j(x_1) \hat{Y}_k(x_2), \end{aligned} \quad (9.69)$$

where the term in the brackets equals $S_{Z_j Z_k}(\omega)$, and is given by

$$S_{Z_j Z_k}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{Z_j Z_k}(\tau) e^{-i\omega\tau} d\tau,$$

where $R_{Z_j Z_k}(\tau)$ is given by Equation 9.68. Taking the Fourier transform of both sides and letting $\lambda = \tau + \theta - \kappa$,

$$\begin{aligned} S_{Z_j Z_k}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(\lambda - \theta + \kappa)} \\ &\times \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{P_j P_k}(\lambda) g_j(\theta) g_k(\kappa) d\theta d\kappa \right] d\lambda, \end{aligned} \quad (9.70)$$

where τ has been replaced by $\lambda - \theta + \kappa$ and $d\tau$ by $d\lambda$. We can rewrite Equation 9.70 in a more useful form by separating the integrals according to dummy variables,⁴⁶

$$\begin{aligned} S_{Z_j Z_k}(\omega) &= \int_{-\infty}^{\infty} g_j(\theta) e^{i\omega\theta} d\theta \int_{-\infty}^{\infty} g_k(\kappa) e^{-i\omega\kappa} d\kappa \\ &\times \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{P_j P_k}(\lambda) e^{-i\omega\lambda} d\lambda. \end{aligned} \quad (9.71)$$

⁴⁶Recall the procedures where similar integrals existed. See, for example, Equation 9.51, where we derived the fundamental relation in random vibration, Equation 9.52. Also, recall the sequence of Equations 9.62 to 9.65 that parallel our development here.

Since the Fourier transform of the impulse response function $g(t)$ is the frequency response function $G(i\omega)$, Equation 9.71 becomes

$$S_{Z_j Z_k}(\omega) = G_j^*(i\omega) G_k(i\omega) S_{P_j P_k}(\omega), \quad (9.72)$$

where

$$G_j(i\omega) = \frac{1}{m \omega_j^2 - \omega^2 + 2i\zeta_j \omega_j \omega},$$

$G_j^*(i\omega)$ is the complex conjugate of $G_j(i\omega)$, and $S_{P_j P_k}(\omega)$ is the spectral density of the modal force components.

$S_{P_j P_k}(\omega)$ is derived assuming that the Fourier transform for $P_j(t)$ exists, as it does for most physical processes,

$$P_j(t) = \int_{-\infty}^{\infty} P_j(\omega) e^{i\omega t} d\omega.$$

Taking the Fourier transform of Equation 9.66 we find

$$S_{P_j P_k}(\omega) = m^2 \int_0^L \int_0^L S_{PP}(\omega) \hat{Y}_j(x_1) \hat{Y}_k(x_2) dx_1 dx_2, \quad (9.73)$$

where $S_{PP}(\omega)$ is the known spectral density of the loading. With additional substitutions, we find

$$\begin{aligned} S_{ZZ}(x_1, x_2; \omega) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} G_j^*(i\omega) G_k(i\omega) S_{P_j P_k}(\omega) \hat{Y}_j(x_1) \hat{Y}_k(x_2). \end{aligned} \quad (9.74)$$

One value of this equation is that the mean-square (MS) displacement can be evaluated,

$$\begin{aligned} Z_{MS}(x) &= R_{ZZ}(x, x; 0) \\ &= \int_{-\infty}^{\infty} S_{ZZ}(x, x; \omega) d\omega, \end{aligned}$$

where we recall that if $\mu_Z(x) = 0$ then $Z_{MS}(x) = \sigma_Z^2(x)$, the variance.

The derivations are now complete, but what do they mean, and how do they help? One of the purposes of a probabilistic analysis is to help the designer bound the uncertainties so that it is possible to design the system even with randomness in the force. Furthermore, this scatter is not haphazard but is defined by a standard deviation and possibly a probability density function. The variance is used to bound the mean value response. It is also possible to convert the spectral density into a representative time history as is done with Borgman's method in Section 9.12.3.



Figure 9.48: Installation of CubeSat OUFTI-1 on an electrodynamic shaker. The CubeSat, built by students at the Université de Liège, Belgium, complies to the CubeSat standard of 10 cm side length and a mass of 1 kg or less. The shaker is used to test the integrity of the CubeSat to simulate high vibration environments. (Courtesy European Space Agency)

9.12 Monte Carlo Simulation

Monte Carlo simulation is the general name for probabilistic computational methods that are based on the repetitive sampling of the distributions of random variables. The term *Monte Carlo* was first introduced by von Neumann and Ulam⁴⁷ during World War II. Monte Carlo methods were initially used in the development of the atomic bomb, where they were applied to problems involving random neutron diffusion in fissile material. The method was named after the resort city on the Riviera where gambling is a major industry.

Monte Carlo simulation generates estimates of the statistics of system variables. The simulation is computationally intensive and consists of generating many values of the variables of interest.

Statistical tests are usually applied to the variables to be modeled to estimate the most likely probability distribution that will produce similar synthesized values. (These values are normally obtained directly from already developed programs.) Such *random numbers* are called *pseudorandom* because they are attained via deterministic equations. Sets of random values for the simulated variable are required in the Monte Carlo process.

Generated random numbers are used in the respective deterministic equations, and this is done for many iterations until a family of results exists for the set of random input variables. Finally, all the results are averaged to

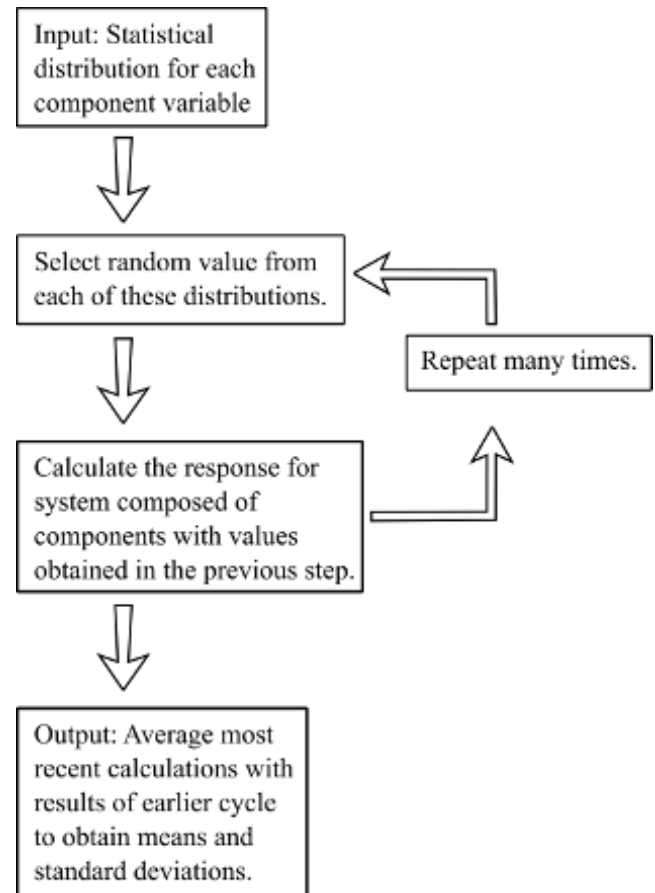


Figure 9.49: Flowchart of the Monte Carlo simulation method. Experimentally derived density functions for each random variable are used to generate realizations. These are substituted into the equations that model the process. The solution is taken to be one data point. This process is repeated until the averages converge.

obtain a mean value and higher moments, from which confidence bounds can be obtained.

Since Monte Carlo results arise from observational data consisting of random numbers, it is necessary that the family of results be sufficiently large to ensure convergence to a statistically significant result. The flow chart in Figure 9.49 illustrates the procedure.

The main drawback to Monte Carlo simulation is that it can be a computationally time-consuming technique. Despite this problem, which is less and less of an issue given the increasing speeds of computers, the Monte Carlo method is now the most powerful and commonly used technique for analyzing complex probabilistic problems. However, the complexity and computational effort required has been increasing, since realism has demanded greater intricacy and more extensive descriptions.⁴⁸

⁴⁷ A readable introduction to Monte Carlo methods is given in the book **Monte Carlo Methods**, J.M. Hammersley and D.C. Handscomb, Wiley, 1964.

⁴⁸ A much more advanced text on this subject is **Simulation and the Monte Carlo Method**, R.Y. Rubinstein, Wiley, 1981.

9.12.1 Random Number Generation

The most frequently used present-day method for generating pseudorandom numbers is the *linear congruential generator*. This generator is based on recursive calculations of the residues of modulus m of a linear transformation. Such a recursive relation may be expressed as

$$X_{i+1} = (aX_i + c)(\text{mod } m),$$

where a , c , and m are nonnegative integers and *mod* is the *modulus* which is defined so that X_{i+1} is the remainder of the division of m into $aX_i + c$. If k_i is defined as the integer part of the ratio $(aX_i + c)/m$,

$$k_i = \text{Int} \left(\frac{aX_i + c}{m} \right),$$

then the corresponding residue of modulus m is given by

$$X_{i+1} = aX_i + c - mk_i. \quad (9.75)$$

Normalizing the values obtained from Equation 9.75 by the modulus m , we obtain

$$u_{i+1} = \frac{X_{i+1}}{m},$$

which constitute a set of random values on the unit interval (0,1) governing the standard uniform probability distribution (mean equals 0.5 and standard deviation equals 0.289).

Such a sequence of pseudorandom numbers is cyclic and will repeat itself in at most m steps. To ensure randomness, the period of the cycle should be as long as possible, and, therefore, in practical applications, a large value of m should be assigned in the generation of u_i . The selection of values for m , a , and c is the most important step for creating a generator of this sort. Table 9.1 lists some choices for these constants that will yield a large period. These values have been tested statistically and are shown to give satisfactory results. The choice of which sets of constants to select from Table 9.1 – as well as the size of the randomizing shuffle – is essentially arbitrary. However, the larger the shuffling array, the less likely it is that sequential correlation will occur. Ultimately, this choice must be balanced with computational time and required storage space for the array.

9.12.2 Generating Random Variates

The next question to be addressed is how to use the uniformly distributed pseudorandom numbers to generate values according to a particular probability density.

The generation of random variates of any density function can be accomplished systematically from the uniform distribution on the interval (0,1). This is carried through one of several methods, such as the *inverse transform* method, *composition* method, and *acceptance-rejection* method.

Table 9.1: Constants for Random Number Generation via the Generating Equation $X_{i+1} = aX_i + c - mk_i$.

m	a	c
7875	421	1663
11979	859	2531
21870	1291	4621
81000	421	17117
86436	1093	18257
117128	1277	24749
121500	4081	25673
134456	8121	28411
243000	4561	51349
259200	7141	54773

Nonstandard Uniform Variable

A nonstandard uniform distribution on any arbitrary interval (a, b) can be obtained using the inverse transform method. By this method, we solve for the random variable by inverting the cumulative distribution function

$$F_x(x) = \begin{cases} 0, & x < a \\ (x - a)/(b - a), & a \leq x \leq b \\ 1, & x > b. \end{cases}$$

The inverse function is

$$x = F_x^{-1}(u) = a + (b - a)u,$$

where $0 \leq u \leq 1$. To apply the inverse transform, the distribution function must exist in a form for which the corresponding inverse transform can be found analytically. Other distribution functions that can be inverted are the Weibull, logistic, and Cauchy distributions.

Normal Variables

Other techniques are employed to generate the random variates in a normal distribution, which has the probability density function

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right], \quad -\infty < x < \infty, \quad (9.76)$$

denoted by $R_N(\mu, \sigma)$, where μ is the mean value and σ is the standard deviation.

It is known that if U_1 and U_2 are two statistically independent standard uniform variates, then the following functions constitute a pair of independent standard nor-

Table 9.2: Recursive Relations to Obtain a Sequence of Random Variables Governed by Exponential, Lognormal, and Normal Probability Density Functions.

PDF	Equation for RV y
$f(y) = \lambda \exp[-\lambda(y - \mu)],$ $\mu \leq y < \infty$	$y = -\frac{1}{\lambda} \ln(1 - R_U) + \mu$
$f(y) = \frac{1}{\sigma y \sqrt{2\pi}} \exp \left[-\left(\frac{\ln y - \mu}{2\sigma} \right)^2 \right],$ $0 \leq y < \infty$	$y = \exp(\mu + \sigma R_N)$
$f(y) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{(y - \mu)^2}{2\sigma^2} \right],$ $-\infty < y < \infty$	$y = \mu + \sigma R_N$

mal variates,⁴⁹

$$S_1 = (-2 \ln U_1)^{\frac{1}{2}} \cos 2\pi U_2,$$

$$S_2 = (-2 \ln U_1)^{\frac{1}{2}} \sin 2\pi U_2.$$

Table 9.2 lists equations that can be used to generate random variates for common probability density functions using standard uniform random values (R_U) generated on the interval (0,1) and standard normal (R_N) distributions. Most engineering parameters fall into the normal, lognormal, or exponential distributions, listed in Table 9.2.

In the next section, a procedure is developed to generate a time history from a power spectrum of a random process.

9.12.3 Generating Time History for Random Process

When computing the vibratory response of a machine or component in a random environment, it is sometimes necessary to work in the time domain rather than in the frequency domain. This is especially true when the differential equation governing system behavior is *nonlinear*. Therefore, we are met with the challenge of utilizing the information stored in a power spectral density for a time-domain numerical analysis. The procedure that is presented next was developed by Borgman⁵⁰ for ocean engineering applications.

A primary characteristic of the ocean model used in offshore structural engineering is the wave elevation or wave height $\eta(x, t)$, which is a function of position x and time t . A realistic model for $\eta(x, t)$ is a random process in time where the possible wave heights are defined by a spectral density $S_\eta(\omega)$ that is determined experimentally.

One widely used spectrum is the *Pierson-Moskowitz*, a generic version of which is

$$S_\eta(\omega) = \frac{A_0}{\omega^5} e^{-B/\omega^4},$$

where the constants A_0 and B depend on the location where the spectrum is measured. See Equation 9.33 for additional details and parameter values.

Instead of using a simple harmonic representation of the wave height, the following function is used,

$$\eta(x, t) = \int_0^\infty \sin(kx - \omega t + \varepsilon) \sqrt{S_\eta(\omega)} d\omega,$$

where ω is the range of frequency components contained in the wave, k is the wave number, and ε is a random phase angle selected from a list of random numbers uniformly distributed over the interval between 0 and 2π . The wave height $\eta(x, t)$ has units of length and the quantity $\sqrt{S_\eta(\omega)} d\omega$ has units of $\sqrt{(\text{length}^2/\text{s}) \cdot (\text{rad/s})} = \text{length}$. Physically, the integral represents the $S_\eta(\omega)$ -weighted average of the sum of harmonics within a particular frequency band $d\omega$.

For computational purposes, the integral is discretized over the range of frequencies where there is significant energy, say $\omega_0 < \omega < \omega_N$. The spectrum is discretized so that each discretized area is *equal* (rather than discretizing so that each $\Delta\omega$ is equal). Such a procedure avoids the presence of periodicities in the resulting time history. The frequency partition is

$$\omega_0 < \omega_1 < \omega_2 < \cdots < \omega_N,$$

and for $\omega < \omega_0$ and $\omega > \omega_N$, it is assumed that $S_\eta(\omega) = 0$. Over each band, averaged quantities are used,

$$\Delta\omega_i = \omega_i - \omega_{i-1}$$

$$\bar{\omega}_i = \frac{\omega_i + \omega_{i-1}}{2}, \quad i = 1, 2, \dots, N,$$

where the overbar denotes average. Using averaged quantities, the average wave height is given by

$$\bar{\eta}(x, t) = \sum_{i=1}^N \sin(\bar{k}_i x - \bar{\omega}_i t + \varepsilon_i) \sqrt{S_\eta(\bar{\omega}_i) \Delta\omega_i}, \quad (9.77)$$

where $\bar{\omega}_i^2 = \bar{k}_i g$ for deep water. The cumulative area under the density function is defined as

$$S(\omega_j) \equiv \sum_{i=1}^{j \leq N} S_\eta(\bar{\omega}_i) \Delta\omega_i. \quad (9.78)$$

Then, the difference between two cumulative areas,

$$S_\eta(\bar{\omega}_i) \Delta\omega_i \simeq S(\omega_i) - S(\omega_{i-1}),$$

⁴⁹A proof is given by Rubinstein, p.86.

⁵⁰L.E. Borgman, "Ocean Wave Simulation for Engineering Design," *J. Waterways and Harbors Division, ASCE*, 95 (WW4), Nov.1969, pp.557-583.

is constant, as discussed in the earlier discretization procedure, and is set equal to a^2 .

Equation 9.77 then becomes

$$\bar{\eta}(x, t) = a \sum_{i=1}^N \sin(\bar{k}_i x - \bar{\omega}_i t + \varepsilon_i). \quad (9.79)$$

The total area of all N discretized regions is given by

$$Na^2 = S(\omega_N) \simeq S(\infty) = \int_0^\infty S_\eta(\omega) d\omega. \quad (9.80)$$

This equation signifies that the total area is equal to the number of discretized bands N times the area of each band a^2 . This total area equals the cumulative area $S(\omega_N)$ according to Equation 9.78 for $j = N$. Since most of the energy is at or below ω_N , this area is approximately equal to $S(\infty)$, the total area under the Pierson-Moskowitz spectrum $S_\eta(\omega)$.

Substituting the expression for the Pierson-Moskowitz density into the last expression in Equation 9.80, and integrating, we find another equation for the total area under the spectrum,

$$\int_0^\infty \frac{A_0}{\omega^5} e^{-B/\omega^4} d\omega = \frac{A_0}{4B} e^{-B/\omega^4} \Big|_0^\infty = \frac{A_0}{4B}.$$

Then, setting both expressions for the total area equal to each other,

$$Na^2 = \frac{A_0}{4B},$$

yields the expression for a^2 ,

$$a^2 = \frac{A_0}{4BN}. \quad (9.81)$$

We do not yet know the partition frequencies, which are determined as follows. For ω_N ,

$$S(\omega_N) = \frac{A_0}{4B} \exp(-B/\omega_N^4). \quad (9.82)$$

Because of energy equipartition,

$$S(\omega_i) = \frac{i}{N} S(\omega_N),$$

and using Equation 9.82,

$$\frac{A_0}{4B} \exp(-B/\omega_i^4) = \frac{i}{N} \frac{A_0}{4B} \exp(-B/\omega_N^4).$$

Solving this last equation for ω_i , we find

$$\omega_i = \left(\frac{B}{\ln(N/i) + (B/\omega_N^4)} \right)^{1/4}. \quad (9.83)$$

Combining Equations 9.79 and 9.81, we have the needed time history,

$$\bar{\eta}(x, t) = \sqrt{\frac{A_0}{4BN}} \sum_{i=1}^N \sin(\bar{k}_i x - \bar{\omega}_i t + \varepsilon_i),$$

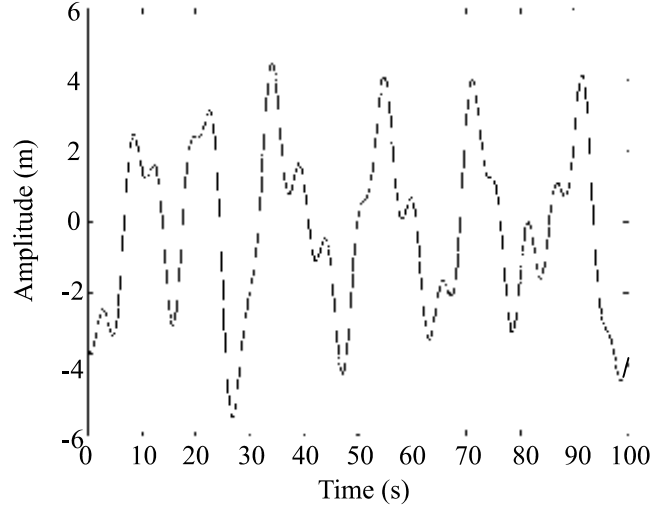


Figure 9.50: Sample wave profile from Borgman's method applied to the Pierson-Moskowitz wave height spectrum for wind velocity $V = 25$ m/s, and $A = 8.1 \times 10^{-3} g^2$, $B = 0.74 g^4 / V$ from Equation 9.33.

where $\bar{\omega}_i = (\omega_i + \omega_{i-1})/2$, with ω_i given by Equation 9.83.

Figure 9.50 shows the time history of a sample wave derived using the above procedure. This wave profile can now be used to generate wave force time histories via the Morison equation, as outlined in Section 9.2.3.

9.13 Inverse Vibration with Uncertain Data

We now introduce some uncertainty into the experimentally determined parameters of the deterministic inverse vibration problem of Section 6.12. Quantities involved in the design and analysis of engineering systems generally exhibit some degree of randomness. This can be attributed to several sources. One source is the uncertainty involved in measurements. A measurement may be made, in some cases, as accurately as is needed for a particular application. In other situations, a measurement can be made only as accurately as the measurement system will allow. In either case, there is some uncertainty in the resulting values.

Even if quantities can be measured exactly, the inherent statistical nature of material properties and production techniques suggest a need for probabilistic methods. Two seemingly identical components will, in general, exhibit slight characteristic differences that may affect the way each performs. Assemblies of such components are even more likely to differ from one another.

Finally, the modeling of engineering systems usually requires some approximations. Reasons for this include a lack of complete understanding of the particular system,

or a need to simplify a particularly complex equation. Such assumptions may introduce some form of uncertainty into the solution. This last form of randomness obviously depends on the particular system. The analysis of this type of uncertainty can be quite difficult.

Here, the system is modeled mathematically using deterministic equations and then randomness is introduced in the variables. It is important to note that a probabilistic analysis of the type shown here not only provides a more realistic model of the system, but also provides the analyst with a tool for quantifying statistical confidence in the analytical results.

We refer to Equations 6.146 to 6.148, repeated here,

$$\begin{aligned} R_1 &\equiv \frac{k_2}{m_2} = \lambda_1 + \lambda_2 - \lambda_3 \\ R_2 &\equiv \frac{k_1}{m_1} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} \\ R_3 &\equiv \frac{k_2}{m_1} = \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}{\lambda_1 + \lambda_2 - \lambda_3}. \end{aligned}$$

The λ_i values are taken to be random variables and, therefore, we need to be able to work with a function of random variables. In order to work analytically, it is necessary to approximate ratios such as R_1, R_2 , and R_3 using a Taylor series representation.

Consider⁵¹ a function R of random variables λ_i , $i = 1, 2, \dots, n$. Each of these variables can be written as

$$\lambda_i = \mu_{\lambda_i} + \epsilon_i,$$

where μ_{λ_i} is the mean value of λ_i and ϵ_i is a (small) random parameter signifying some uncertainty about the actual (mean) value of the frequency. Therefore, $E[\epsilon_i] = 0$ and $E[\epsilon_i^2] = \sigma_{\lambda_i}^2$ since

$$\begin{aligned} \sigma_{\lambda_i}^2 &= E\left\{(\lambda_i - \mu_{\lambda_i})^2\right\} \\ &= E\left\{(\epsilon_i)^2\right\}. \end{aligned}$$

Before proceeding with the general expansion, consider the case of a two degree-of-freedom structure with two distinct roots (frequency squared), each of which has uncertainties,

$$\begin{aligned} \lambda_1 &= \mu_{\lambda_1} + \epsilon_1 \\ \lambda_2 &= \mu_{\lambda_2} + \epsilon_2. \end{aligned}$$

For a general nonlinear function R_1 of both λ_1 and λ_2 , the Taylor series expansion about the mean values λ_1

and λ_2 is

$$\begin{aligned} R_1(\lambda_1, \lambda_2) &= R_1(\mu_{\lambda_1}, \mu_{\lambda_2}) + \frac{\partial R_1(\mu_{\lambda_1}, \mu_{\lambda_2})}{\partial \lambda_1}(\lambda_1 - \mu_{\lambda_1}) \\ &\quad + \frac{\partial R_1(\mu_{\lambda_1}, \mu_{\lambda_2})}{\partial \lambda_2}(\lambda_2 - \mu_{\lambda_2}) \\ &\quad + \frac{1}{2} \left[\frac{\partial^2 R_1(\mu_{\lambda_1}, \mu_{\lambda_2})}{\partial \lambda_1^2}(\lambda_1 - \mu_{\lambda_1})^2 \right. \\ &\quad + \frac{\partial^2 R_1(\mu_{\lambda_1}, \mu_{\lambda_2})}{\partial \lambda_1 \partial \lambda_2}(\lambda_1 - \mu_{\lambda_1})(\lambda_2 - \mu_{\lambda_2}) \\ &\quad \left. + \frac{\partial^2 R_1(\mu_{\lambda_1}, \mu_{\lambda_2})}{\partial \lambda_2^2}(\lambda_2 - \mu_{\lambda_2})^2 \right] + \dots, \end{aligned}$$

where $(\lambda_1 - \mu_{\lambda_1}) = \epsilon_1$ and $(\lambda_2 - \mu_{\lambda_2}) = \epsilon_2$. A similar expression can be written for $R_2(\lambda_1, \lambda_2)$. It is important to observe that all terms in these expressions are evaluated at the respective mean values of λ_1 and λ_2 , which are known quantities.

R_k for an n degree-of-freedom system is a function of all λ s, and can be expanded in a Taylor series as follows,

$$\begin{aligned} R_k(\lambda_1, \lambda_2, \dots, \lambda_n) &= R_k(\mu_{\lambda_1}, \mu_{\lambda_2}, \dots, \mu_{\lambda_n}) \\ &\quad + \sum_{i=1}^n \frac{\partial R_k}{\partial \lambda_i} \epsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 R_k}{\partial \lambda_i \partial \lambda_j} \epsilon_i \epsilon_j + \dots, \end{aligned}$$

for $k = 1, 2, \dots, n$. If the frequencies exhibit only a small degree of randomness, that is, if $\lambda_i - \mu_{\lambda_i} = \epsilon_i \ll 1$, the expansion can be truncated after only a few terms with little error due to small terms such as ϵ_i^2 and $\epsilon_i \epsilon_j$.

We take only the first two terms of the Taylor series to demonstrate this procedure. This is a linear approximation for the actual value of R ,

$$\begin{aligned} R_k(\lambda_1, \lambda_2, \dots, \lambda_n) &\simeq R_k(\mu_{\lambda_1}, \mu_{\lambda_2}, \dots, \mu_{\lambda_n}) + \sum_{i=1}^n \frac{\partial R_k}{\partial \lambda_i} \epsilon_i, \\ k &= 1, 2, \dots, n. \end{aligned}$$

Since $E\{\epsilon_i\} = 0$, taking the expected value of R_k leads to the approximate result

$$\begin{aligned} E\{R_k\} &\simeq R_k(\mu_{\lambda_1}, \mu_{\lambda_2}, \dots, \mu_{\lambda_n}), \\ k &= 1, 2, \dots, n. \end{aligned} \tag{9.84}$$

Thus, the linear or first-order approximation of the mean value of a complicated function can be obtained by substituting the mean values of all random variables into the function.

To obtain an estimate of the standard deviation of R_k , assume that the variables are statistically independent,⁵²

⁵²Such assumptions are generally made as a first approximation. For cases where this is not a valid assumption, it is necessary to somehow estimate, usually by way of experiments, the correlations between the random variables so that the terms $E\{\epsilon_i \epsilon_j\}$ can be evaluated.

⁵¹See D. Moss and H. Benaroya, "A Discrete Inverse Vibration Problem with Parameter Uncertainties," *Applied Mathematics and Computation*, Vol.69, 1995, pp.313-333.

so that $E[\epsilon_i \epsilon_j] = E[\epsilon_i]E[\epsilon_j] = 0$ for $i \neq j$. The standard deviation of R_k is then estimated by

$$\begin{aligned}\sigma_{R_k}^2 &= E\{R_k^2\} - E^2\{R_k\} \\ &= \sum_{i=1}^n \left(\frac{\partial R_k}{\partial \lambda_i} \right)^2 \sigma_{\lambda_i}^2, \quad k = 1, 2, \dots, n, \quad (9.85)\end{aligned}$$

where the partial derivatives on the right-hand side are evaluated at the respective mean values of all the random variables. Equation 9.85 depends only on the mean values and standard deviations of the random variables. It is independent of the particular distributions of these variables, with the only assumption being that they are independent. The method is therefore useful in cases where little is known about the probabilistic nature of the random variables except for estimates of their means and variances.

A more accurate prediction for the statistics of random variable R_k is obtained by retaining the second-order term of the Taylor series,

$$\begin{aligned}R_k(\lambda_1, \lambda_2, \dots, \lambda_n) &\simeq R_k(\mu_{\lambda_1}, \mu_{\lambda_2}, \dots, \mu_{\lambda_n}) \\ &+ \sum_{i=1}^n \frac{\partial R_k}{\partial \lambda_i} \epsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 R_k}{\partial \lambda_i \partial \lambda_j} \epsilon_i \epsilon_j,\end{aligned}$$

for $k = 1, 2, \dots, n$. However, the convergence of such series expansions cannot be taken for granted, meaning that adding more terms to the series does not necessarily lead to an improvement in accuracy.

It is necessary to verify solutions obtained using these approximate expansions since accuracy depends on the smallness of ϵ . To verify the accuracy of the truncated Taylor series, we have two options. The first is to build an experiment that duplicates the vibrating structure and test it under various conditions. The other option is utilizing the Monte Carlo simulation technique, introduced in Section 9.12.

Example 9.13 An Uncertain Two Degree-of-Freedom System

Use the two-term Taylor series approximation and the following data to demonstrate the procedure developed in Section 9.13. The eigenvalues $\lambda_1 = 0.382 \text{ Hz}^2$, $\lambda_2 = 2.618 \text{ Hz}^2$ are for the original system shown in Figure 9.51, and $\lambda_3 = 1.000 \text{ Hz}^2$ is for the system with the fixed end shown in Figure 9.52. Assume $m_1 + m_2 = 20 \text{ kg}$ exactly with zero standard deviation, where the mean values of m_1 and m_2 are each 10 kg , the stiffnesses are $k_1 = k_2 = 10 \text{ kg/cm}$, and all λ values are approximate and assumed random with a 1% variation – that is, with coefficients of variation $\delta = \sigma/\mu = 0.01$.

Solution: The following procedure is used to estimate the mean values and variances of k_i and m_i , given the mean values and variances of the masses and eigenvalues.

We use Equation 9.84 for the mean value calculations and Equation 9.85 for the standard deviation calculations. The procedure is in two parts:

1. Given the mean values and standard deviations of λ_i , and using Equations 6.146 to 6.148, we derive the estimated mean values and variances of each ratio R_i .
2. With these results, we use Equations 6.150 to 6.153 to derive the estimated mean values and variances of each stiffness and mass.

The procedure will be demonstrated only for some of the variables since the algebra becomes tedious. We begin with the ratio $R_1 = \lambda_1 + \lambda_2 - \lambda_3$. The mean value of R_1 is estimated by

$$\begin{aligned}\mu_{R_1} &= E\{R_1\} = R_1(\mu_{\lambda_1}, \mu_{\lambda_2}, \mu_{\lambda_3}) \\ &= \mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3}.\end{aligned}$$

The variance is estimated by

$$\begin{aligned}\sigma_{R_1}^2 &= \left(\frac{\partial R_1}{\partial \lambda_1} \right)^2 \sigma_{\lambda_1}^2 + \left(\frac{\partial R_1}{\partial \lambda_2} \right)^2 \sigma_{\lambda_2}^2 + \left(\frac{\partial R_1}{\partial \lambda_3} \right)^2 \sigma_{\lambda_3}^2 \\ &= (1)^2 \sigma_{\lambda_1}^2 + (1)^2 \sigma_{\lambda_2}^2 + (-1)^2 \sigma_{\lambda_3}^2.\end{aligned}$$

Next, we follow the same procedure for the ratio $R_2 = \lambda_1 \lambda_2 / (\lambda_1 + \lambda_2 - \lambda_3)$. The mean is estimated as

$$\mu_{R_2} = E\{R_2\} = \frac{\mu_{\lambda_1} \mu_{\lambda_2}}{\mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3}},$$

and the variance as

$$\begin{aligned}\sigma_{R_2}^2 &= \left(\frac{\partial R_2}{\partial \lambda_1} \right)^2 \sigma_{\lambda_1}^2 + \left(\frac{\partial R_2}{\partial \lambda_2} \right)^2 \sigma_{\lambda_2}^2 + \left(\frac{\partial R_2}{\partial \lambda_3} \right)^2 \sigma_{\lambda_3}^2 \\ &= \left(\frac{(\mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3})\mu_{\lambda_2} - \mu_{\lambda_1}\mu_{\lambda_2}(1)}{(\mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3})^2} \right)^2 \sigma_{\lambda_1}^2 \\ &+ \left(\frac{(\mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3})\mu_{\lambda_1} - \mu_{\lambda_1}\mu_{\lambda_2}(1)}{(\mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3})^2} \right)^2 \sigma_{\lambda_2}^2 \\ &+ \left(\frac{(\mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3})(0) - \mu_{\lambda_1}\mu_{\lambda_2}(-1)}{(\mu_{\lambda_1} + \mu_{\lambda_2} - \mu_{\lambda_3})^2} \right)^2 \sigma_{\lambda_3}^2,\end{aligned}$$

which can be simplified algebraically. The same procedure can be used to estimate the mean and variance of ratio R_3 .

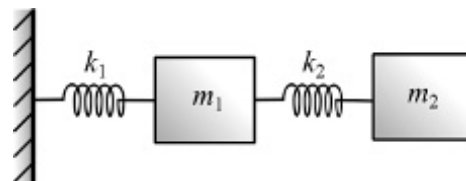


Figure 9.51: A two degree-of-freedom system.

Now that the statistics of each ratio R_i have been estimated, we proceed with step two and use these parameters in the estimation of the statistics of k_i and m_i . Beginning with the relation $k_2 = R_1 R_3 m / (R_1 + R_3)$, we estimate the mean value of k_2 by

$$E\{k_2\} = \frac{\mu_{R_1} \mu_{R_3}}{\mu_{R_1} + \mu_{R_3}} m,$$

where the total mass m is assumed to be an exact value with no variance. For the estimated value of the variance of k_2 , we have

$$\begin{aligned} \sigma_{k_2}^2 &= \left(\frac{\partial k_2}{\partial R_1} \right)^2 \sigma_{R_1}^2 + \left(\frac{\partial k_2}{\partial R_3} \right)^2 \sigma_{R_3}^2 + \left(\frac{\partial k_2}{\partial m} \right)^2 \sigma_m^2 \\ &= \left(\frac{(\mu_{R_1} + \mu_{R_3}) \mu_{R_3} m - \mu_{R_1} \mu_{R_3} m(1)}{(\mu_{R_1} + \mu_{R_3})^2} \right)^2 \sigma_{R_1}^2 \\ &\quad + \left(\frac{(\mu_{R_1} + \mu_{R_3})(0) - \mu_{R_1} \mu_{R_3} m(0)}{(\mu_{R_1} + \mu_{R_3})^2} \right)^2 \sigma_{R_3}^2 \\ &\quad + \left(\frac{(\mu_{R_1} + \mu_{R_3}) \mu_{R_1} m - \mu_{R_1} \mu_{R_3} m(1)}{(\mu_{R_1} + \mu_{R_3})^2} \right)^2 \sigma_m^2, \end{aligned}$$

where the second expression on the right-hand side equals zero since R_2 is not in the equation for k_2 . The same procedure can then be used to estimate the mean values and variances of the remaining parameters, k_1 , m_1 , and m_2 . Substituting the given mean values and variances, we find the numerical results of the first-order expansion as listed in Table 9.3. Comparisons are made in the table between the expansion values and a Monte Carlo (MC) simulation that is considered to be essentially exact. The values under the columns labeled MC show the percent differences between the perturbation results and the Monte Carlo results for mean and standard deviation, respectively. ■

9.14 Concepts Summary

This chapter has introduced basic concepts in applied probability and random processes. In particular, we discovered how a variable that has associated uncertainties in its value can be modeled mathematically through

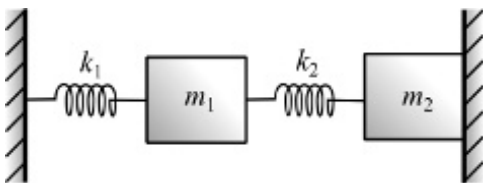


Figure 9.52: A two degree-of-freedom system constrained (fixed) at the right.

Table 9.3: Taylor Expansion Results Compared to Monte Carlo (MC) Simulation.

	μ	σ	$\delta = \sigma/\mu$	MC $\mu\Delta\%$	MC $\sigma\Delta\%$
m_1	10	0.235	0.024	0.07	0.54
m_2	10	0.235	0.024	0.07	0.54
k_1	10	0.212	0.021	0.21	2.45
k_2	10	0.158	0.016	0.22	3.05

the probability distribution function or the probability density function. We also examined several key density functions that are useful in applications.

The concept of the random process was introduced. This led to the correlation function and the power spectrum. The correlation function was found to be physically a measure of how rapidly a function varies, and the power spectrum a measure of the energy distribution as a function of frequency.

Different random processes were discussed and classified according to their spectrum. Broad-band and narrow-band processes are two important cases. The white-noise process was introduced as an idealized model that yields acceptable results in applications, and it was applied in the example of the random vibration of a damped oscillator.

In addition, we considered multi degree-of-freedom systems and continuous systems, as well as the numerical Monte Carlo method.

9.15 Quotes

- “As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.” (Albert Einstein)
- “Probability is common sense reduced to calculation.” (Laplace)
- “Pressure makes diamonds.” (General George Patton Jr.)
- “Strive for perfection in everything you do. Take the best that exists and make it better. When it does not exist, design it.” (Sir Henry Royce, British Engineer, 1863-1933)
- “Learning is a treasure that will follow its owner everywhere.” (Chinese Proverb)
- “You’re braver than you believe, and stronger than you seem, and smarter than you think.” (A.A. Milne, Winnie the Pooh)
- “By swallowing evil words unsaid, no one has ever harmed his stomach.” (Winston Churchill)

- “Beginnings are usually scary, and endings are usually sad, but it’s everything in between that makes it all worth living.” (Bob Marley)
- “Strong people don’t put others down. They lift them up.” (Michael P. Watson)
- “To acquire knowledge, one must study; but to acquire wisdom, one must observe.” (Marilyn Vos Savant)
- “The major difference between a thing that might go wrong and a thing that cannot possibly go wrong is that when a thing that cannot possibly go wrong goes wrong, it usually turns out to be impossible to get at and repair.” (Douglas Adams)
- “Don’t ever mistake silence for ignorance, calmness for acceptance, or kindness for weakness.”
- “You’ve got to think about big things while you’re doing small things, so that all the small things go in the right direction.” (Alvin Toffler)

9.16 Problems

Problems for Section 9.1 – Introduction

- Twenty-five samples of a steel beam were chosen and tested for the Young’s modulus. Eight had a modulus of $E = 30 \times 10^6$ psi. Two had a modulus of $E = 29 \times 10^6$ psi. Fifteen had a modulus of $E = 30.5 \times 10^6$ psi. Estimate the following probabilities:
 - $\Pr\{E > 29.5 \times 10^6 \text{ psi}\}$,
 - $\Pr\{E > 30 \times 10^6 \text{ psi}\}$,
 - $\Pr\{E > 28 \times 10^6 \text{ psi}\}$.

Problems for Section 9.2 – Motivation

- For the following applications, make a list of parameters and forces that are needed to analyze the problem and distinguish between those that can be assumed deterministic and those that must be described probabilistically: (a) airplane design, (b) ship design, (c) turbomachinery design, (d) mechanical watch movements, (e) computer hard drive, (f) automobile internal combustion engine, (g) automobile body.
- Discuss the possible shortcomings of Miner’s rule.
- How would wind forces be different than ocean wave forces?

- How can the differences in material properties at different temperatures be considered for applications where the system operates across a large temperature differential?

Problems for Section 9.3 – Random Variables

- Write the meaning of each of the following mathematical expressions in words only:
 - $F(x)$,
 - $F(-\infty)$,
 - $F(+\infty)$,
 - $\int_{-\infty}^{\infty} f(x)dx = 1$.
- What does the fact that $F(x_1) \leq F(x_2)$ imply about the values of x_1 and x_2 ? Why?
- Can the schematics in Figure 9.53 be cumulative distribution functions? Why?

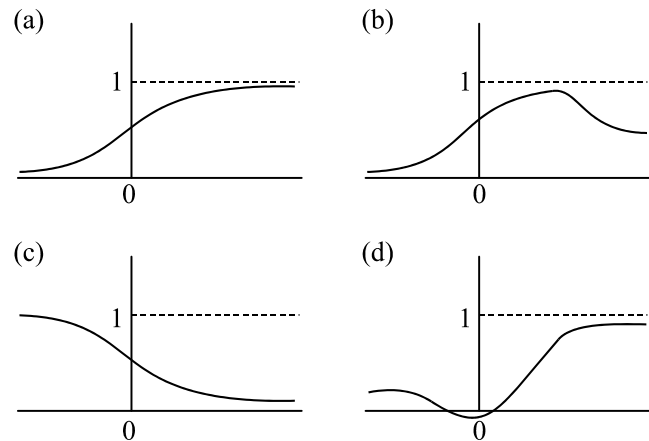


Figure 9.53: Possible cumulative distribution functions.

- Can the schematics in Figure 9.54 be probability density functions? Why?

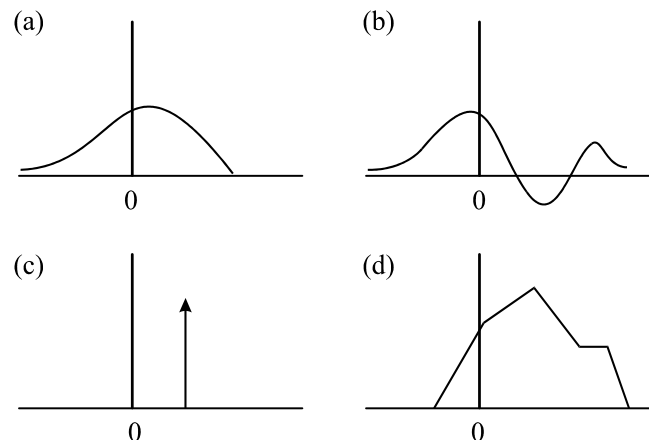


Figure 9.54: Possible probability density functions.

10. Discuss the differences in the ways the cumulative distribution and the density depict random variability. Are there situations where one has the advantage over the other? Provide an example for the advantage of each.
11. If $\Pr\{X \leq x_1\} = 0.1$ and $\Pr\{X \leq x_2\} = 0.2$, sketch the probability density function for random variable X . What is $\Pr\{x_1 < X \leq x_2\}$?
12. For the following density functions, evaluate the normalization constant and the respective probability, where c and a are constants:
 - (a) $ce^{-|x|}$, $-\infty < X < +\infty$; $\Pr\{X \leq 0\}$
 - (b) ce^{-x} , $4 < X < 10$; $\Pr\{X \leq 5\}$
 - (c) $a/(x_2 - x_1)$, $0 < X < 2$; $\Pr\{X \leq 0\}$.
13. Can the following be probability densities?
 - (a) $3/(x_2 - x_1)$, $0 \leq X \leq 1$, (b) $a \exp x$, $0 \leq X \leq \infty$,
 - (c) $c \ln x$, $1 \leq X \leq 2$, (d) b , $2 \leq X \leq 5$, where a , b , and c are constants.
14. Let X have the probability density $f(x) = x^2/9$, $0 \leq x \leq 3$. Is this a legitimate density function? Find the following probabilities:
 - (a) $\Pr\{1 \leq X \leq 2\}$, (b) $\Pr\{X \leq 1\}$, (c) $\Pr\{X \geq 3\}$.
15. What is the probability that a continuous random variable takes on a particular value, that is, $\Pr\{X = x\}$?
16. For the density function $f(x) = cx$, $0 \leq x \leq 1$, evaluate c so that $f(x)$ becomes a probability density function. Then find $\Pr\{X < 0.3\}$.
17. For the density function $f(x) = 1 - |x|$, $-1 \leq x \leq 1$, sketch $f(x)$, show that it is a density, and find $\Pr\{-1/2 < X \leq 1/3\}$.
18. The direction at which the wind strikes a tower is a random variable θ with density function $f(\theta) = c \cos \theta$, $-\pi/4 \leq \theta \leq \pi/4$. The angle θ is measured from due east.
 - (a) Evaluate c . (b) Sketch the density as a function of θ . (c) Calculate the probability that $\theta > |\pi/4|$. (d) Calculate the probability that $-\pi/10 < \theta \leq \pi/10$. (e) Sketch the cumulative distribution function.
20. Find the mathematical expectation of the random variable X with density function $f(x) = c \ln x$, $1 \leq X \leq 3$, where c is a constant to be determined. Also evaluate the mean square value, the variance, and the coefficient of variation.
21. The following values of a variable are known: 2.3, 3.5, 3.5, 4.9, 3.7, 0.7, 4.1. Find the average value, the mean square value, the variance, and the coefficient of variation.
22. Compare the mean values of the following two random variables. For the first random variable we have the data: 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, and 6.0. For the second random variable we have the probability density

$$f(x) = \frac{1}{(x_2 - x_1)}, \quad 3 \leq X \leq 6.$$

Do we expect them to have the same mean values?

Problems for Section 9.5 – Useful Probability Densities

23. (a) The stiffness property of a new material is established by initial testing. The preliminary results show a uniform scatter of data in the range $9 \text{ lb/in} \leq k \leq 11 \text{ lb/in}$. There is no additional information. Calculate the mean value, standard deviation, coefficient of variation, and the probability that the stiffness is greater than 9.5 lb/in .
 - (b) The above spread of data is found to be too large, and so additional testing is carried out. A plot of the data shows the density in Figure 9.55. Calculate the same values as before and draw conclusions.

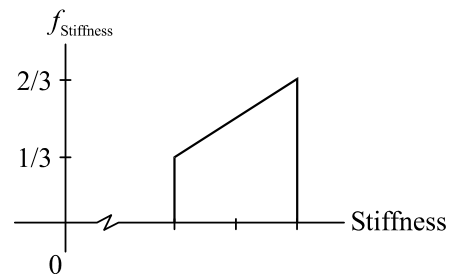


Figure 9.55: Probability density of stiffness data.

Problems for Section 9.4 – Mathematical Expectation

19. Find the mathematical expectation of the random variable X with density function $f(x) = a/(x_2 - x_1)$, $x_1 \leq X \leq x_2$. Also evaluate the mean square value, the variance, and the coefficient of variation.
24. Suppose a random variable has a probability density function with a very broad range of values. Even though the density function is well defined, what options does a designer have when dealing with such a broad parameter variability?
25. Several steel beams delivered to the factory are to be used as columns. The test data for the lot from

which these beams were selected are known to have a Young's modulus in the range $29 \times 10^6 \text{ psi} \leq E \leq 31 \times 10^6 \text{ psi}$, with all values equally likely. If the columns have a cross sectional area of 144 in^2 and are loaded with $100,000 \text{ lb}$, what are the possible stress and strain ranges?

26. Show that the discrete Poisson-distributed random variable X with discrete density

$$\Pr\{X = k\} = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

has an expectation $E\{X\} = \lambda$.

27. If a mechanical component fails according to the exponential distribution with a mean value of 5000 h , what is the probability that the component will fail by 1000 h ? How many additional hours before this probability is doubled? Sketch the density function.
28. Derive Equations 9.13 and 9.14 showing that

$$E\{X\} = \mu$$

and

$$E\{X^2\} = \mu^2 + \sigma^2.$$

29. For the Gaussian random variable X , numerically estimate the probability $\Pr\{X < 0\}$ for (a) mean value $\mu = 10$ and standard deviation $\sigma = 10$, and (b) mean value $\mu = 10$ and standard deviation $\sigma = 1$. Can either of these be used as models for a positive definite variable? Explain.
30. Compare the lognormal and the Rayleigh probability density functions, both of which govern positive-definite variables. Compare the shapes of the plotted functions, and examine how they shift as parameters are varied. Discuss the differences, if any, due to the fact that one is a function of two parameters and the other a function of one parameter.

Problems for Section 9.6 – Two Random Variables

31. Given the joint density function

$$f(x, y) = \frac{1}{(x_2 - x_1)(y_2 - y_1)} \\ 2 \leq x \leq 4, \text{ and } 1 \leq y \leq 3,$$

evaluate $E\{X\}$, $E\{XY\}$, $Cov(XY)$, and ρ . Note that $f(x, y) = f(x)f(y)$.

32. X and Y have a joint probability density given by

$$f(x, y) = e^{-(x+y)}, \quad x \geq 0, \quad y \geq 0.$$

Find $\Pr\{X \geq Y \geq 2\}$ and sketch the region in the x, y plane that defines the region of integration.

33. In Example 9.11 verify Equation 9.23,

$$\text{Var}\{Z\} = \text{Var}\{X\} + \text{Var}\{Y\} - 2\rho\sigma_X\sigma_Y.$$

Problems for Section 9.7 – Random Processes

34. Strain gauges are placed at two locations on a wing that is being tested in a wind tunnel. The first gauge is near the fixed base of the wing and the other gauge is near the free wing tip. Two tests are performed: (a) the wing vibration is recorded for low velocity laminar flow and (b) for high velocity turbulent flow. For each of these tests, discuss the following: (i) Describe and compare the time histories you would expect to see recorded by each strain gauge; (ii) Discuss the type of cross-correlation $R_{X_1 X_2}(\tau)$ you would expect between X_1 and X_2 , the locations of the gauges.

35. In Figure 9.56, discuss whether the processes are stationary.

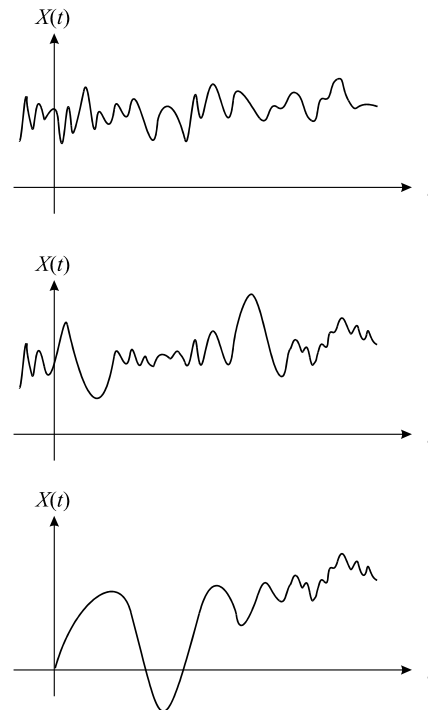


Figure 9.56: Various random processes.

36. From the expressions for wind and ocean wave power spectra, which parameters appear important for a proper characterization of the energy distribution?
37. Referring to the plot of the Pierson-Moskowitz ocean wave height spectrum (Figure 9.28), discuss the physical relationship between wind speed and wave height.

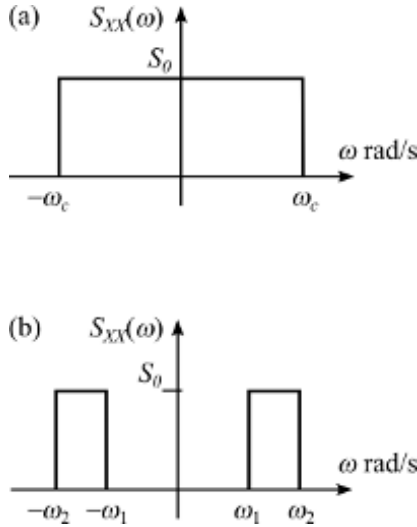


Figure 9.57: Broad-band and narrow-band spectra.

38. For the Pierson-Moskowitz power spectrum, evaluate numerically the areas under the first 20 frequency bands of width $\Delta\omega = 0.1$ rad/s for (a) $V = 10$ m/s, and (b) $V = 20$ m/s. What conclusions can be drawn regarding the effect of wind speed on the frequency distribution of wind energy for this spectrum? Could similar conclusions have been determined by directly studying the equation for the spectrum?
39. Convert the power spectra of Figure 9.57 to one-sided equivalent spectra that are functions of cyclic frequency (Hz).
40. Compare Equations 9.33 and 9.34. In particular, discuss their respective units and how the energy distribution differs.
41. Look up examples of the El Centro earthquake spectrum online. Is there a spectral density? How is the information presented?

Problems for Section 9.8 – Random Vibration

42. Beginning with Equation 9.46,

$$R_{XF}(\alpha_1) = \int_{-\infty}^{\infty} g(\tau_1) R_{FF}(\alpha_1 - \tau_1) d\tau_1,$$

derive Equation 9.49,

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha) g(\beta) R_{FF}(\tau + \beta - \alpha) d\alpha d\beta.$$

43. Following Example 9.12, we would like to examine the sensitivity of the output spectral density

$S_{XX}(\omega)$ and the mean-square response $E\{X(t)^2\}$ to various combinations of parameter values, where S_0 is the input spectrum. For the following cases, plot $|G(i\omega)|^2$ and $S_{XX}(\omega)$, and evaluate $E\{X(t)^2\}$ for unit mass, $0 < \omega < 3.0$ rad/s and S_0 ft²/s:

- (a) $\omega_n = 0.1$ and $\zeta = 0.1$
- (b) $\omega_n = 0.5$ and $\zeta = 0.1$
- (c) $\omega_n = 1.0$ and $\zeta = 0.1$
- (d) $\omega_n = 2.0$ and $\zeta = 0.1$
- (e) $\omega_n = 0.1$ and $\zeta = 0.5$
- (g) $\omega_n = 0.5$ and $\zeta = 0.5$
- (h) $\omega_n = 1.0$ and $\zeta = 0.5$
- (i) $\omega_n = 2.0$ and $\zeta = 0.5$.

Compare the results and draw general conclusions.

44. Using the fundamental relation between input and output spectra, find the response spectrum for the oscillator governed by

$$\ddot{X} + 2\zeta\omega_n\dot{X} + \omega_n^2 X = \frac{1}{m}F(t),$$

with $m = 10$ N, $\zeta = 0.15$, $k = 5$ N/m, where the spectrum for $F(t)$ is (a) the Pierson-Moskowitz (Equation 9.33), with wind speed $V = 25$ km/h, and (b) the wind spectrum (Equation 9.34) where $\bar{V}_{10} = 20$ mi/h, and $\kappa = 0.01$. For each, numerically evaluate the mean-square response and plot each response spectrum.

45. Consider Figure 9.37 and describe in your own words the fundamental result depicted there graphically.

Problems for Section 9.9 – Stochastic Response of a Linear Multi Degree-of-Freedom System

46. Reduce the methodology of this section to the specific case of a two degree-of-freedom system.
47. Following the methodology of this section, study a forced vibrating system with the following property matrices,

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [k] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

where each component force vector $\{F_1(t) \ F_2(t)\}^T$ is white noise. Transform the coupled equations of motion into modal coordinates and proceed with an analysis of this two degree-of-freedom system. Discuss any assumptions needed in progressing through the steps of the analysis.

Problems for Section 9.10 - Lunar Seismic Structural Analysis

48. Refer to Figure 9.45 and the equation that it represents to discuss the physical nature of that description. How do variations in the parameters that define the power spectrum affect the velocity spectrum?
49. Compare the seismic risk for a surface lunar structure as compared to that for a buried lunar structure.

Problems for Section 9.11 – Random Vibration of Continuous Structures

50. Describe three applications where a continuous structure is loaded by a random force.
51. Suppose a simply supported beam is forced by a random load with mean value μ_F and autocorrelation $R_{FF}(\tau) = \exp(-\alpha\tau)$, where α is a constant. Solve for the mean, autocorrelation, and spectral density of the response.
52. For Problem 51, suppose the calculations need to be simplified. Can the autocorrelation of the force be replaced by a triangular spike around $\tau = 0$? Try this replacement, justify the simplification, and solve. Compare with the results of Problem 51.
53. What complications occur in the derivations of this section if the force is not assumed to be stationary?
54. Derive Equation 9.74.

Problems for Section 9.12 – Monte Carlo Simulation

55. Generate 100 random numbers that are governed by an exponential density function.
56. Generate 100 random numbers that are governed by a Gaussian density function.
57. Generate a time history for the Pierson-Moskowitz spectral density where $V = 40$ m/s, using the parameter values of Equation 9.33. Compare these results with those of Figure 9.50.
58. Describe the Borgman method for the generation of a random time history in physical terms. Why does it work?

Problems for Section 9.13 – Inverse Vibration with Uncertain Data

59. For Example 9.13, derive the expressions for μ_{k_1} and $\sigma_{k_1}^2$.

60. In Example 9.13, formulate expressions for the mean value and variance of R_3 .
61. Derive Equation 9.85.
62. In the inverse problem with uncertainty, suppose that instead of a fixed constraint, we take an elastic constraint in order to generate an additional equation (as shown in Figure 9.58). The benefit of using a spring as a constraint, rather than just fixing mass m_2 as in Figure 9.52, is that k_3 can be varied to represent a free end using $k_3 = 0$, or a fixed end using $k_3 = \infty$. Derive the ratios needed to estimate the mean values and variances of the structural parameters. Assume that each mass is exactly 20 kg.

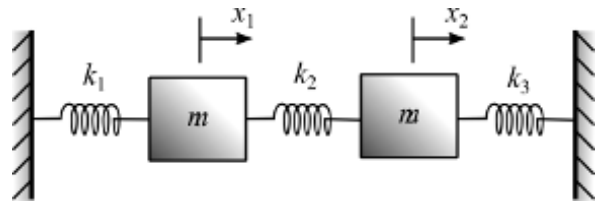


Figure 9.58: Two degree-of-freedom system.

Chapter 10

Vibration Control

“The response is unacceptable, so it is changed.”

Our study of vibration has so far consisted of investigating dynamic system models and the forces or, more generally, the external inputs acting on them. For a given problem, we developed a model of the system and its loading, from which we evaluated the vibration response. We made no judgment whether the response (such as the displacement, velocity, or acceleration) was too large or too small. The possibility of modifying the behavior was addressed only briefly.

This chapter on *vibration control* studies the modification of dynamic systems to achieve the desired response, namely, minimizing undesired vibration. Before altering or controlling the response of a structure or machine, we need to define measures of the performance of the system. Then, if the performance is not met, for example, a response exceeds certain bounds, appropriate forces can be applied to bring the response within acceptable levels. These forces can be viewed as *controlling* the vibration.

In this chapter, we can only hint at the breadth and depth of vibration control, a vast subject that has generated significant interest and has many applications. Our goal is to present a limited background of controls engineering and convey a sense of how *feedback control* can be exploited to achieve desired system performance.

A subset of topics in controls engineering is introduced. For much of this chapter, we concern ourselves with the control of deterministic dynamic systems that (a) can be modeled as linear, time-invariant differential equations and (b) have a single-input and single-output (sometimes abbreviated as SISO systems). In the latter parts, we will consider control of stochastic systems and multi-input and multi-output (MIMO) systems.

Although we present performance measures in the time domain and related issues of stability and sensitivity, we do not develop the frequency domain or graphical methods available for the analysis and design of control systems. These include the *Root-Locus* method, the *Bode* plot, the *Nyquist* diagram, and the *Nichols Chart*. To

provide a description that would be meaningful to the uninitiated reader would require a significant increase in the number of pages and is beyond our scope. Just as Chapter 9 provided a focused view of the probabilistic approach, this chapter provides a selective view of control. The reader is referred to textbooks¹ on system dynamics and control to fill in the many gaps that are unavoidable in this introductory chapter.

10.1 Motivation

We begin by asking why we need to control vibration? In some circumstances, vibration can simply be an annoyance that we wish to avoid. For example, a washing machine operating with an unbalanced load can be noisy and a nuisance, especially if it shakes so violently it moves from its position. In more serious situations, vibration can be catastrophic. The vibration generated by an earthquake can be severe enough to destroy cities in which thousands perish. We, of course, wish to avoid – or, at minimum, mitigate – the negative effects of vibration. Through successful design, we seek to prevent significant damage and perilous destruction.

Humans are acutely sensitive to vibration. Tolerable

¹There are many excellent textbooks on system dynamics and controls. (Although the focus of system dynamics is modeling, it falls within the more general discipline called *systems engineering* and is deeply linked to controls.) We provide a few references (in alphabetical order based on last name of first author).

System dynamics books: **Modeling and Analysis of Dynamic Systems**, C.M. Close, D.K. Frederick, J.C. Newell, 3rd ed., Wiley, 2001. **System Dynamics**, K. Ogata, 4th ed., Prentice Hall, 2003. **System Dynamics**, W.J. Palm III, 3rd ed., McGraw-Hill, 2013.

Control of dynamic systems books: **Modern Control Systems**, R.C. Dorf and R.H. Bishop, 13th ed., Prentice Hall, 2016. **Feedback Control of Dynamic Systems**, G. Franklin, J.D., Powell, and A. Emami-Naeini, 7th ed., Prentice Hall, 2014. **Automatic Control Systems**, F. Golnaraghi and B.C. Kuo, 9th ed., Wiley, 2009. **Control Systems**, M. Gopal, McGraw-Hill, 2006. **Control Systems Engineering**, N.S. Nise, 7th ed., Wiley, 2015. **Modern Control Engineering**, K. Ogata, 5th ed., Prentice Hall, 2009. **Feedback and Control Systems**, A.J. Stubberud, I.J. Williams, and J.J. DiStefano III, 2nd ed., Schaum's Outline Series, McGraw-Hill, 2013.



Figure 10.1: Redesigned hammers offer advantages in terms of attenuated vibration as well as ergonomics. Further advantages in terms of reduced vibration and weight can be achieved using titanium rather than steel. (Titanium has the same strength as steel with 45% less weight and exhibits much less vibration. Some hammer manufacturers claim up to 10 times less vibration.) (Ergonomics Laboratory, Marquette University; courtesy of Richard W. Marklin. Photograph by M. Nagurka)

levels depend on the magnitude and frequency as well as the duration of exposure to the vibration. With sufficient exposure, people may sustain physical injury (to their internal organs and musculoskeletal system, for example) or experience physiological effects (such as numbing due to prolonged holding of a jackhammer or vibrating hand tool). Humans suffer other problems in sustained vibration environments, including loss of concentration, inefficiency, and fatigue. These problems are realities for a person working in a factory near pounding equipment or a driver operating a vehicle. And, they translate into higher likelihood of an accident, leading to the possibility of destruction and even death as a consequence of sustained exposure to vibration.

The efficiency of production in a factory can be affected in other ways. Manufactured goods produced by a vibrating machine may fail to meet tolerance specifications and need to be discarded. Surfaces that were intended to be smooth may exhibit undulations or become rough due to machine vibration. Even minor levels of vibration may interfere with quality in manufacturing, pushing component tolerances out of specification and rendering products useless.

Although today's computer hard disk drives are much more robust than in the past, excessive vibration at the wrong frequency can cause the read-write heads to touch the hard disk, corrupting data and damaging the disk.

There is a limit to the amount of vibration that computers and electronic equipment can sustain before malfunctioning or failing.² Computers and electronic hardware must be *hardened* or protected from undue vibration.

Transportation systems are inherently plagued by vibration problems. Automobiles, motorcycles, airplanes, helicopters,³ trains, ships and powerboats,⁴ and spacecraft generate vibration just by their operation. They contain machines that reciprocate and/or rotate, such as engines, transmissions, compressors, turbines, air conditioners, generators, alternators, and pumps. Vehicles must be designed to account for the vibrational comfort of the operator, passengers, and crew. The internal combustion engine in a car or truck is a built-in vibration generator. More generally, all reciprocating engines are a source of vibration, no matter how well designed, balanced, and mounted.

One might think that shifting from piston-cylinder engines driving propellers to jet engines would solve the vibration problem in modern aircraft. It turns out it causes another problem. The external structural surfaces of aircraft are exposed to intense jet noise inducing vibration and leading to structural fatigue, referred to as *sonic* or *acoustic* fatigue. This problem affects jet planes and satellite booster rockets, where it is a significant concern due to the sizably larger engines and thrust.

Systems suffering the effects of vibration vary widely in size and application. Examples of large systems are (a) an apartment building or office complex shaken by an earthquake and (b) a super-tanker excited by its engines and propellers. Larger still are suspension bridges and skyscrapers that can vibrate excessively due to an earthquake or vibrate dangerously due to wind gusts. Even more extensive in dimension are high-voltage power transmission lines that are also susceptible to wind-excited vibration. At the smaller end of the spectrum, we already mentioned the read-write heads in hard disk drives. Even smaller are miniscule structures, such as cantilevered beams, in nanotechnology sensors and actuators. There are many other examples of vibration

²It is amazing that hard disk drives do not fail more often. Inside is a metal disk rotating at 7200 rpm (some at 10,000 rpm and at 15,000 for the fastest hard drives) with a read-write head floating above it by less than the thickness of a human hair as it passes by at approximately 240 km/h (150 mph). An analogy might be a jet plane flying centimeters off the ground.

³Vibration problems in helicopters have caused catastrophic failures. The forces from the blades on the rotor head fluctuate and are transmitted into the cabin structure and the rotor control mechanism. Cabin vibration leads to passenger and pilot discomfort, and the vibration in the control mechanism is transmitted to the pilot's hands. The instrument panel may vibrate significantly, making the instruments unreadable.

⁴Vibration problems in ships and powerboats are very common. One source is attributed to the ship's propellers. They create periodic hydrodynamic pressures that act on the hull and cause it to vibrate. If the propellers have been damaged, the unbalance itself can be a source of vibration.

problems causing discomfort and damage, with dangerous consequences. Whatever the source, the negative effects of vibration are a reality, making the control of vibration an important necessity.



Taipei 101 is an iconic skyscraper in Taipei, Taiwan. (It is shown in Figure 10.2.) Its height is 508 m, measured from the ground to the tip of the spire on top, and it contains 101 floors (with an additional five basement floors underground). The highest occupied floor is at 438 m. At the time of its construction in 2004, it was the tallest building in the world – a title it held until the Burj Khalifa was finished in 2010.

Sitting just 200 m (660 ft) from a major fault line, Taipei 101 is prone to earthquakes and fierce winds common in its area of the Asia-Pacific. The engineers had to design a structure that could withstand earthquakes as well as gale winds up to 216 km/h.

Typically, skyscrapers must be flexible in strong winds yet remain rigid enough to prevent large sideways movement. Flexibility prevents structural damage while resistance ensures comfort for the occupants and protection of glass, walls, and structural features. Most designs achieve the necessary strength by enlarging critical structural elements such as bracing, but the height of Taipei 101 combined with the demands of its environment called for additional innovations.

To achieve stability and lessen the impact of violent motion, a gigantic *tuned mass damper* is used. The damper (shown in Figure 10.3) consists of a steel sphere 5.5 m in diameter and weighing 660 metric tons, suspended from the 92nd to the 87th floor. Acting like a giant pendulum, the massive steel ball sways to counteract the building's movement.

The ball is supported by 16 steel cables, each 9 cm in diameter and 42 m long, that form a sling, and 8 viscous dampers (seen in Figure 10.4) act like shock absorbers when it shifts. The ball can move 150 cm in any direction and reduce movement of the tower by up to 40 percent. (Two smaller tuned mass dampers are installed at the tip of the spire to provide additional protection against strong wind loads.)

The damper is visible from an indoor observatory located inside the tower, where the photographs were taken. During particularly windy days one can see the damper in action.

Taipei 101's damper was built at a cost of US\$4 million. It passed a real-life test during Taiwan's 6.8-magnitude earthquake in 2002, during construction of the tower. As an aside, there are 61 elevators inside of the tower, each with atmospheric controls to keep passengers' ears from popping. The two fastest elevators move at 1010 m/min (37.7 miles per hour); reaching the 89th floor from ground level takes around 44 seconds.

The damper design of Taipei 101 is also used in the 262-m Trump Tower in New York. Hidden from public view, there is a 600-ton tuned mass damper on the 90th floor of the Tower.



Figure 10.2: The Taipei 101 tower in Taipei, Taiwan, was the world's tallest building from 2004 until 2010. (Photograph by M. Nagurka)

In contrast to passively tuned dampers, the Yokohama Landmark Tower in Japan employs a computer-controlled actively tuned damper.



10.2 Approaches to Controlling Vibration

When an unacceptable vibration problem occurs, it must be decided how it is best solved. Three methods can be identified. In every case, the first effort should be to reduce the vibration input from the source. If, for example, the vibration is attributed to imbalance in a rotating machine, the machine should be balanced as carefully as possible. If the vibration stems from misalignment of bearings, the bearings should be aligned or replaced. The first step is simply to address the source of the problem through proper operation.

The second method is called *passive control*.⁵ In passive control the stiffness, mass, and damping properties of the system are modified to make the system

⁵Passive Vibration Control, D.J. Meade, Wiley, 2000, p.45.

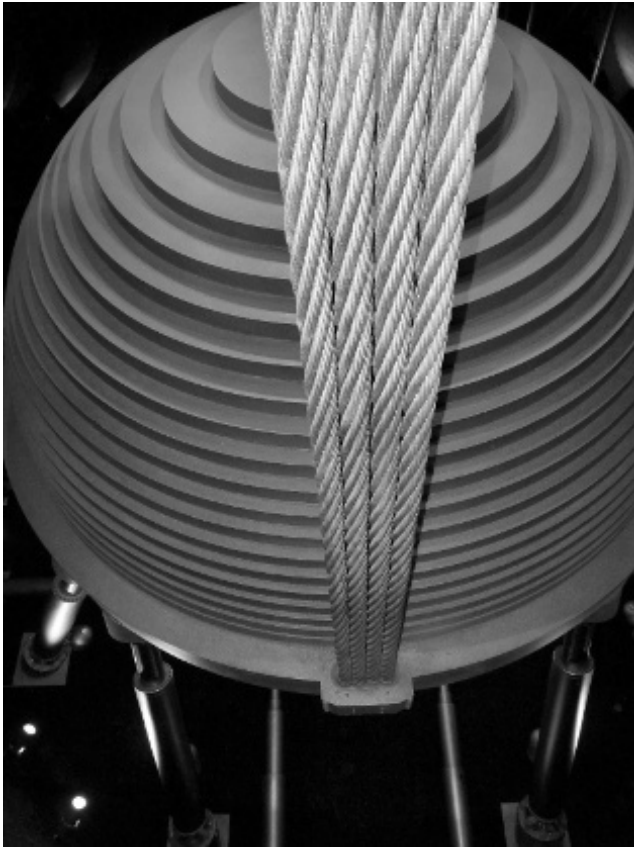


Figure 10.3: The damper consists of a steel sphere suspended from the 92nd to the 87th floor. Acting like a giant pendulum, the massive ball sways to counteract the building's movement due to strong wind gusts or an earthquake. Steel cables form a sling to support the ball. (Photograph by M. Nagurka)

less responsive to its vibratory environment. For example, by implementing basic structural changes the system's dynamic properties can be altered, shifting a natural frequency (or frequencies) away from an operating frequency. Design changes can be made to the structure, such as removing or adding masses, springs, and dampers. These non-powered, non-active elements simply react passively in opposition to the accelerations, deflections, and velocities imposed upon them by the vibration, thereby changing the energy interactions in the system. A classic example is a tuned mass-spring-damper system – a vibration absorber – such as used in Taipei 101 and other tall buildings. We investigated the vibration absorber in [Section 6.4.3](#).

The third, and most, significant approach is called *active control*. Active control systems require (a) sensors to detect the vibration and (b) actuators driven by external power to compensate or control the vibration. The actuators may be electromechanical, electrohydraulic, electropneumatic, piezoelectric, or other types. They impart the necessary forces and motions to generate a *negative*

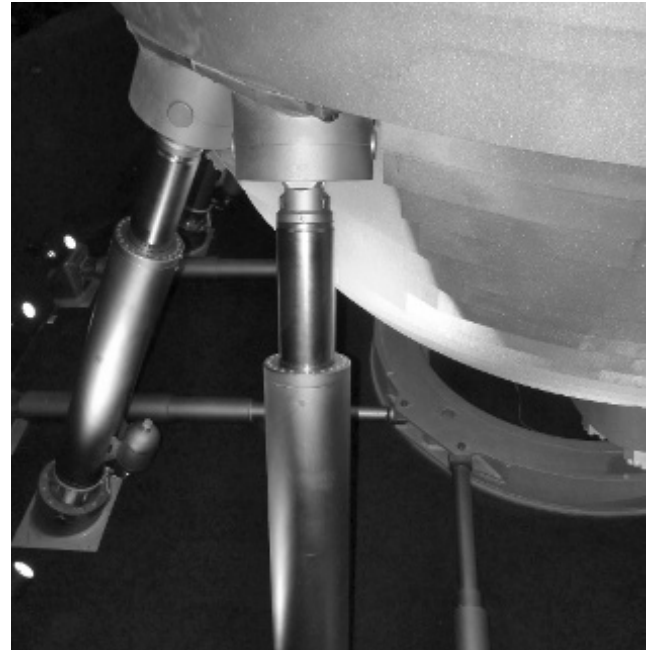


Figure 10.4: Eight viscous dampers act like shock absorbers when the ball shifts. (Photograph by M. Nagurka)

vibration that effectively cancels, to the extent possible, the undesirable vibration. Active control systems rely on sensors, actuators, a source of power, as well as microprocessors or computer controllers, to process the sensor signals and command the actuators to reduce the total vibration.⁶ Active control methods are generally more complex and costly than passive methods, but some vibration problems are not amenable to, or cannot be sufficiently remedied by, passive solutions.

10.2.1 Why Active Control

Machines and structures are designed to perform in ways that meet specifications. For example, a machine may be designed such that its displacement amplitudes remain below a particular value or a structure may be designed such that its operating frequencies remain far from its natural frequency (or frequencies). The system may achieve its specifications over a limited bandwidth,⁷ but if the operating conditions change, the desired performance may not be met. Since the operating conditions of a machine or structure may change with time, the performance cannot always be guaranteed with a *fixed* passive design.

⁶Poor design of an active control system can increase the vibration. Clearly, this would be highly counterproductive!

⁷Bandwidth is used here as a range of frequencies over which the system operates within the design specifications. Formally, the bandwidth is the range of frequencies for which the magnification factor or magnitude ratio is greater than $\sqrt{2}/2$.

It is not just the operating conditions that can change over time. Environmental forces and effects can vary considerably, challenging the success of a fixed design. For example, it is not possible to design an efficient solid wing enabling an airplane to take off, fly, and land effectively. Wings have a variety of control surfaces that alter the flying characteristics needed for stably negotiating turns, accelerating and decelerating, and varying the lift forces.

As another example, we learned in [Chapter 3](#) that as a rotating machine speeds up to its operating speed it has to pass through its resonance frequency. At its resonance frequency, amplitudes can become very large. One solution is to add damping so that amplitudes are limited and cannot grow too fast. For some applications this may be sufficient. In other applications, the addition of damping may not be sufficient, and to achieve the desired performance it is necessary to control the vibration actively. Active control is the focus of this chapter.

We recall from previous chapters that both time and frequency domains provide useful information in the study of vibration. These domains are also useful in solving vibration control problems. The Laplace transform, introduced in [Chapter 4](#), converts a dynamic model from the time-domain representation in terms of time, t , to the (complex) frequency-domain representation in terms of complex variable s .

The *transfer function*⁸ is an input-output representation of the system dynamics, defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, assuming the system has zero initial conditions. Transfer function descriptions form the basis for what is known as *classical controls*.⁹ An alternate approach to characterize the dynamics of systems relies on *state variable models*. They give an internal description of the system dynamics coupling the input and output variables, and form the basis of what is called *modern control*. Both the transfer function and state variable models are used in the study of controls. Analytical results from [Chapters 2](#) and [3](#) for single degree-of-freedom systems provide a foundation for the control studies in this chapter. Specifically, see [Sections 3.7.2](#) and [4.1](#) for related and relevant discussion.

⁸The transfer function represents a linear model of the system, with the input regarded as the *cause* and the output as the *effect*. Although the transfer function serves as a valuable tool for the analysis and design of control systems, it has certain basic limitations. For example, it cannot be applied to models of nonlinear systems or time-varying systems. Furthermore, transfer functions cannot be used efficiently for multivariable systems, that is, systems with multiple inputs and multiple outputs.

⁹A substantial part of what came to be called classical control originated in the work on feedback amplifiers at the Bell Laboratories in the late 1920s and 1930s.



Figure 10.5: In this da Vinci surgical robot the four arms carry surgical tools that are teleoperated by a surgeon at a console. If a tool hits an object or moves too fast, high-frequency vibrations and harmonics may result. The system can remain stable by filtering out these high frequency vibrations, but it is at the cost of making the environment feel “mushy” for the surgeon. Haptic feedback control ensures that the robotic arms move as intended and give meaningful tactile feedback. (Photograph by M. Nagurka)

Example 10.1 Transfer Function of a Base-Excited System

Determine the transfer function of the mass-spring-damper base-excited system shown in [Figure 10.6](#) where the output is the displacement of the mass and the input is the displacement of the base.

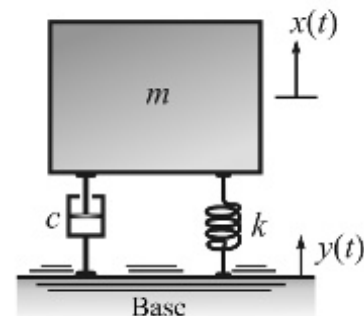


Figure 10.6: A mass-spring-damper model of a base-excited system.

Solution: The equation of motion for the mass in the base-excited system, derived in [Chapter 3](#), is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = c\dot{y}(t) + ky(t).$$

Taking the Laplace transform of this equation with zero initial conditions gives

$$(ms^2 + cs + k)X(s) = (cs + k)Y(s).$$

By rearranging, the transfer function between the response and the input is

$$\frac{X(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k}.$$

The transfer function characterizes the dynamic relationship between the input and output displacements. ■

10.3 Feedback Control

It is customary to call the system that is to be controlled, be it an engine, machine, or building, the *plant*. In an *open-loop* control system the plant operates based on input commands from a controller and does not depend on the actual output of the plant. In other words, in an open-loop control system the control action is independent of the system response.¹⁰ The success of the system depends totally on an accurate model of the system. It also assumes that the system does not change over time and is not influenced by disturbances that potentially could affect the response in an unacceptable manner.

In contrast to an open-loop control system, a *closed-loop* system incorporates *feedback*, that is, the actual system response is measured by a *sensor* and is fed back and compared to the desired response, also known as the *reference input*. The control action, implemented through an *actuator*, is dependent on the sum or difference of the reference input and the actual response.

The purpose of designing a system with feedback is to minimize unwanted behavior that might result from disturbances, inaccurate modeling, and system changes over time. The challenge is to determine the control law that yields acceptable response behavior, both in the short term (transient) and long term (steady-state), to changes in the reference input. Ensuring stability and meeting specifications are the result of a successful design.

Feedback control is responsible for countless engineering achievements, from maintaining the speed in the early steam engine using a mechanical governor¹¹ to the modern automobile with dozens of controlled systems

(engine control, emission control, traction control, anti-lock brakes, air bag deployment, to name a few).

An open-loop system is depicted schematically using a *block diagram* in Figure 10.7. In the figure, the variables are written in terms of *Laplace transforms* of the time-domain variables. The response $Y(s)$ of system $G(s)$ is due to input $U(s)$, all expressed in the s -domain. $G(s)$ is known as the *transfer function* and is the same as the frequency response function of Chapter 3, Equation 3.42. For the single degree-of-freedom oscillator with the equation of motion, $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$, the transfer function between the displacement and force is $G(s) = 1/[ms^2 + cs + k]$.

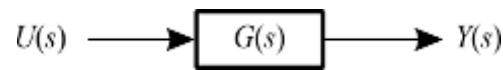


Figure 10.7: Block diagram of an open-loop system.

Block diagrams are a graphically simple means to visualize the interconnections in a system. The blocks represent the system dynamics in terms of transfer functions. For each block, the arrowed lines depict cause-and-effect relationships between an input (for example, a force) and an output (for example, a displacement). As such, a block is a symbol for the mathematical operation on the input signal giving the output signal. The circle with an $\times$ inside, $\otimes$, indicates a summing operation. A plus or minus sign in each sector of the circle indicates whether the signal coming in is added or subtracted, with the result being the line leaving. (The signals to be combined must have identical units.) Block diagram representations provide a very powerful means for portraying the connections and design of systems, from the simplest to the most complex.

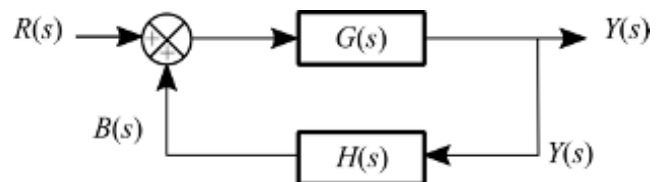


Figure 10.8: Block diagram of a closed-loop positive feedback system.

¹⁰ An open-loop system merely controls the input. An example is a machine tool that simply follows a given guide or pattern.

¹¹ James Watt, the inventor of the steam engine, is credited with the automatic control device called the *flyball governor*. The object of the governor is to keep the speed of the engine constant by regulating the supply of steam to the engine. It incorporates two flyballs that rotate about a vertical axis at a speed proportional to the speed of the engine. Due to centrifugal forces, they tend to move out, and this movement is used to control the supply of steam to the engine through a mechanical linkage to the steam

supply valve. The linkage is designed such that the steam supply is reduced when the engine speed is high and the supply is increased when the speed is low.

It was found that the speed could be kept within narrow limits of a desired speed, although the speed tended to *hunt* or oscillate about that speed. About 100 years later, James Clerk Maxwell (better known for his contributions to electromagnetic field theory) developed a complete mathematical analysis of this problem.

The early history of the governor and other feedback devices is covered in the article by D.S. Bernstein, "Feedback Control: An Invisible Thread in the History of Technology," *IEEE Control Systems Magazine*, Vol.22, Issue 2, April 2002, pp.53-68.

The block diagram¹² of a closed-loop feedback system is depicted in Figure 10.8. The output $Y(s)$ is measured by a sensor with dynamics $H(s)$ and fed back. The signal fed back is $B(s) = H(s)Y(s)$, and, as shown, it is added to the *reference input* $R(s)$, which is typically set as the desired output. This type of feedback system is known as a *positive feedback* system. The system input is a combination of the reference signal $R(s)$ and feedback signal $B(s)$.

Manipulating the signal information from the block diagram gives

$$\begin{aligned} Y(s) &= G(s)[R(s) + B(s)] \\ &= G(s)[R(s) + H(s)Y(s)], \end{aligned} \quad (10.1)$$

where $Y(s)$ appears on both sides of the equation. Solving Equation 10.1 for $Y(s)$ gives

$$Y(s) = \left[\frac{G(s)}{1 - G(s)H(s)} \right] R(s), \quad (10.2)$$

where the bracketed term $[G(s)/(1 - G(s)H(s))]$ is the *closed-loop transfer function*.¹³

A key concern with *positive* feedback is immediately evident. The possibility exists for the response to become infinite, in theory, if the denominator of the transfer function vanishes. In practice, the output grows with each successive cycle around the feedback path, making the system go unstable.¹⁴ An example of this behavior occurs when a microphone is placed too close to a speaker and the result is an ear-splitting screech. Thus, the effect of a positive feedback system is not necessarily *positive* in the sense of being desirable. The name refers to the nature of change rather than the desirability of the outcome.^{15,16}

¹²In the block diagram representations here, we assume that the blocks are *non-interacting* in the sense that there is no *loading* effect of one block on another. This means that for two blocks in series (cascaded blocks), represented by $G_1(s)$ and $G_2(s)$, respectively, the equivalent transfer function is $G_1(s)G_2(s)$.

¹³The open-loop transfer function can be recovered by setting $H(s) = 0$.

¹⁴Stability is discussed later in this chapter and in Chapter 11.

¹⁵The result of positive feedback is amplification, in which a small perturbation results in very large changes. Positive feedback drives the system away from its reference input. Examples of positive feedback include chemical and nuclear fission. Bombarding fissile material with neutrons causes it to emit even more neutrons. The greater the mass of fissile material, the larger the amplification. If the amplification is sufficient, it becomes supercritical; the process accelerates until the fissile material is spent or dispersed by the resulting explosion.

¹⁶Positive feedback was used in early radio receivers to increase their sensitivity, making possible practical radio reception. Edwin Armstrong (December 18, 1890-January 31, 1954), an American electrical engineer, is credited with the idea. In 1912 he constructed electronic amplifiers that worked by positive feedback.

Armstrong is best known as the inventor of frequency modulation (FM) radio. Rather than vary the amplitude of a radio wave to create sound, as done in amplitude modulation (AM) radio, Arm-

There are many intriguing stories about positive and negative feedback and the differences between them. We hope the following stories, offered to help cement ideas about positive feedback, are not in any way offensive, politically incorrect, or inappropriate.

STORY I¹⁷: “The distinction between the stabilizing and destabilizing character of negative and positive feedback loops is neatly captured in the story of the misconnected electric blanket. The newlyweds were given an electric blanket for their queen-size double bed. The blanket had separate temperature settings for the two sides of the bed.... Properly connected, there should have been two separate negative feedback systems, each attempting to control the temperature of the blanket for the comfort of each individual.

“The story goes that the newlyweds misconnected the blanket so that his setting controlled her blanket temperature and hers controlled his. The result ... [was a] nasty positive feedback system. She felt cold, turned up her setting, making his side too warm for him so he turned down his setting, making her even colder, so she raised her setting even further, and so on. How such a scenario would end is left up to the fertile imagination of the reader.”

STORY II: A department in the university decides to take an extended camping trip together over winter break. They are far up north when the professors asked their new chairman if the weather was going to be cold or mild. The chairman felt a natural responsibility to the professors but unfortunately did not have much experience in nature. When she looked at the sky, she couldn't tell what the weather was going to be like. Nevertheless, to be on the safe side, she told the professors that the weather was indeed going to be cold and that they should collect firewood to be prepared.

Being a practical leader, after several days, she got an idea. She called the National Weather Service and asked, “Is the weather going to be cold?” “It looks like it is going to be quite cold,” the meteorologist at the weather service responded. So the chairman told the professors to collect even more firewood in order to be prepared.

strong's method varied the frequency. Compared to AM, FM radio broadcasts sounded clearer, richer, and static-free. Armstrong received a patent on wideband FM on December 26, 1933.

At the time RCA had a very heavy investment in AM radio, and despite Armstrong's patent claimed ownership of FM radio. A patent fight between RCA and Armstrong ensued, lasting more than a decade. RCA's surprising victory in the courts left Armstrong unable to claim royalties on any FM radios sold in the U.S. His costly legal battles brought Armstrong to ruin. Being nearly penniless and emotionally distraught, he committed suicide, leaving behind a note to his wife: “May God help you and have mercy on my soul.” His widow, who had been the secretary of RCA's president David Sarnoff before marrying Armstrong, renewed the patent fight against RCA and finally prevailed.

¹⁷This story is from **Introduction to System Dynamics Modeling with Dynamo**, G. Richardson and A. Pugh, MIT Press, Cambridge, MA, 1981, pp.11-12.

A few days later she called the National Weather Service again. “Does it still look like it is going to be very cold?” “Yes,” the person from the National Weather Service again replied, “it’s going to be very cold.” The chairman again went back to the professors and told them to collect every scrap of firewood they could find.

A few days later, the chairman called the National Weather Service yet again. “Are you absolutely sure that the weather is going to be very cold?” “Absolutely. It’s going to be as cold as we’ve ever seen.” “How can you be so sure?” the chairman asked. The meteorologist replied, “We watch the professors. They’re collecting firewood like crazy.”

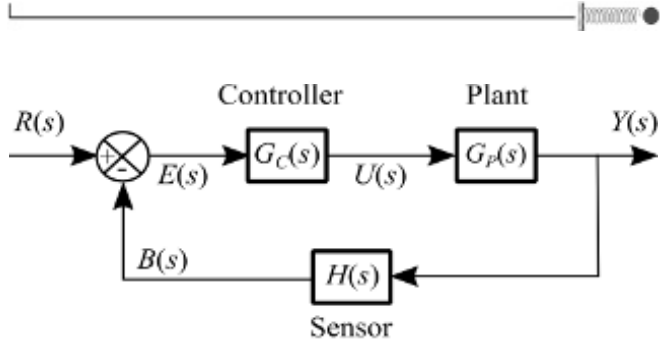


Figure 10.9: Block diagram of a closed-loop negative feedback control system.

Figure 10.9 shows a block diagram of a *negative feedback control system* with the transfer function of the controller, $G_c(s)$, and the transfer function of the plant, $G_p(s)$, in series in what is called the *forward path*. The equivalent transfer function in the forward path is $G(s) = G_c(s)G_p(s)$. The concept of negative feedback was formalized by Harold Black in 1927.

At the summing point, a difference is taken between the reference input and the feedback signal, $B(s)$, meaning that the input and output signals are compared. The difference is an *error*, $E(s)$, that is the input to the controller, where it is used in a control law to generate a control action, $U(s)$. The control action is the output of the controller and the input to the plant. The possibility of the closed-loop system going unstable still exists if a poor control law is chosen. However, with the proper design of the control law, the closed-loop system will be stable and robust. A major advantage of a feedback control system is its ability to ensure that both the transient and steady-state response meet performance specifications.

For the closed-loop negative feedback control system, the Laplace transform of the output is

$$Y(s) = \left[\frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} \right] R(s),$$

and the Laplace transform of the error is

$$E(s) = \left[\frac{1}{1 + G_c(s)G_p(s)H(s)} \right] R(s).$$

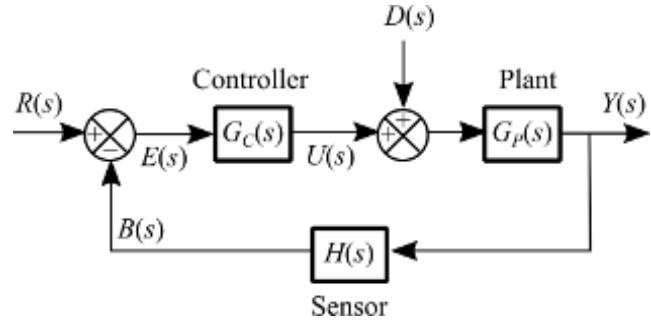


Figure 10.10: Block diagram of a closed-loop negative feedback control system with two inputs (reference and disturbance) and one output.

To reduce the error, we can make the *loop gain*, $G_c(s)G_p(s)H(s)$, large over the frequency range of operation, that is,

$$|G_c(s)G_p(s)H(s)| \gg 1.$$

Most control systems are subject to unwanted disturbance signals. These might include, for example, gusts of wind affecting radar antennas or unaccounted load fluctuations. Figure 10.10 shows a detailed block diagram of a *negative feedback control system* that includes an external disturbance at the input to the plant. This closed-loop system has two inputs, the reference $R(s)$ and the disturbance $D(s)$, and one output $Y(s)$. Two transfer functions can be defined: one between the output and the reference input, $Y(s)/R(s)$, and the other between the output and the disturbance, $Y(s)/D(s)$.

$$Y(s) = \left[\frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} \right] R(s) + \left[\frac{G_p(s)}{1 + G_c(s)G_p(s)H(s)} \right] D(s).$$

A properly designed controller will achieve a near unity transfer function, $Y(s)/R(s)$, and a small or negligible transfer function, $Y(s)/D(s)$, over the frequency range of operation.

HAROLD BLACK

(April 14, 1898 – December 11, 1983)



Contribution: Harold Stephen Black revolutionized telecommunications by inventing systems that eliminated feedback distortion in telephone calls. The major challenge confronting successful telecommunications was elimination of distortion. Instead of reducing the distortion, he initially tried to remove it from the amplifier's output, leaving behind just the original signal. His first solution, which won him the first of his many patents, was the feed-forward amplifier. In this device, the input signal is subtracted from the output, leaving just the distortion. The distortion is then amplified separately and used to cancel out the distortion in the original signal.

The technique worked, but the equipment was touchy and difficult to maintain. Black continued to refine the feed-forward amplifier for three more years, while searching for a more elegant approach. His solution, conceived on August 1, 1927, was the negative feedback amplifier. The concept enabled distortion in a communication signal to be corrected by feeding part of the signal back to the amplifier, comparing it to the original signal, and using the comparison to drive down the distortion.

In his 1934 classical paper "*Stabilized feed-back amplifiers*," he referred to H. Nyquist's theory on stability and its connection to a negative feedback amplifier, which can be unstable and oscillate. Thus, with the help of Nyquist's theory, he was able to demonstrate a stable negative feedback amplifier.

Black's idea of negative feedback is widely recognized as one of the most important and fundamental concepts in engineering. Although the idea was developed for and first applied to long-distance telephone service, it has applications in all fields of engineering and been applied to many fields outside of engineering, for example, psychology. The term feedback is a well-known word in the general lexicon.

The photograph of Black (circa 1941) shows him with amplifying equipment that implemented his idea for reducing distortion by reversing some of the amplifier output and feeding it back into the input. (The photograph is reprinted with permission of Alcatel-Lucent USA Inc. Alcatel-Lucent charged the authors for this permission.)

Life Sketch: Black was born in Leominster, Massachusetts. He earned a BSEE degree from Worcester Polytechnic Institute in 1921. (Years later he received an honorary doctorate in engineering from Worcester Tech.) Following graduation Black joined Western Electric's West Street Labs, which was renamed the Bell Telephone Laboratories (better known as Bell Labs) in 1925, where he was a member of the technical staff until his retirement in 1963. Throughout his lifetime, Black was also a literary critic, teacher, and lecturer. From 1966 to his death he worked on a negative feedback system to aid the blind and deaf. He died in Murray Hill, NJ at the age of 85.

Noteworthy: Black was inducted into the National Inventors Hall of Fame, Akron, OH, in 1981. He was awarded the Worcester Polytechnic Institute's highest honor, the

Robert H. Goddard Award, in 1981 for recognition of outstanding professional achievements. He also received 10 medals, 11 fellowships, nine awards and numerous honors. Black wrote *Modulation Theory*, published in 1953.

The story of how Black discovered negative feedback theory is legendary. In 1927 he was taking the Hudson River Ferry to his office at Bell Laboratories in New York City when he suddenly thought of a solution to the problem of distortion in amplified signals (a fundamental obstacle to economical long-distance telephone service). Having nothing else to write on, he sketched his idea on a copy of the New York Times and then signed and dated it. The patent he won for the negative feedback amplifier was his most famous of 63 U.S. and 278 foreign patents he earned in a long and distinguished career at Bell Labs.

10.3.1 Disadvantages of Feedback

While it might appear that feedback control should be applied to all systems to improve performance, there are costs. These include the following:

- Active control requires additional components, including *sensors*, *actuators*, and *controllers*, which can be costly. To implement these components, a more complicated system must be built and maintained, and there may be reliability issues. (An argument could be made that a controlled system is less expensive in the long run since its behavior is less prone to exceed specifications or to fail.)
- Sensors may introduce a small amount of noise in the system, thereby reducing the overall performance. To circumvent this problem, sensors must be accurate, adding to the overall cost.
- The introduction of feedback may lead to instability of the closed-loop system, even though the open-loop system may be stable. This possibility is the result of inherent time lags in the system. If these lags in the feedback loop are large, what was intended as negative feedback may turn out to be positive feedback at some higher frequency. Thus, the use of feedback increases the possibility of instability. The control system designer must consider stability issues, although for very complicated structures such as aircraft and turbines, it is not straightforward to establish stability under all operating conditions, and extensive testing may be necessary.

Even so, the advantages outweigh the disadvantages, and for some applications there is no choice but to implement feedback control. A major advantage of a feedback control system is that it provides the ability to adjust

Table 10.1: Some Standard Input Signals.

Signal Type	Time Domain	Laplace Transform
Unit Impulse Function	$\delta(t)$	1
Step Function	$A, t > 0$	A/s
Ramp Function	$At, t > 0$	A/s^2
Parabolic Function	$At^2, t > 0$	$2A/s^3$
Sine Wave	$A \sin \omega t$	$\frac{A\omega}{(s^2 + \omega^2)}$

the transient and steady-state performance. Thus, the cost of additional hardware is often worthwhile due to the overall improvements in performance.

10.4 Performance of Feedback Control Systems

In the design of a feedback control system, the desired output response must be evaluated in terms of the system specifications. We will see that *performance* will be related to the *poles* and *zeros* of the transfer function, terms that we will define soon. Some standard input signals, such as those listed in Table 10.1,¹⁸ are used to determine system performance and compare designs.

Many performance measures are defined in terms of the *step response*, studied in Chapter 3. Three important measures are:

- the speed of response, that is, how fast the system responds to changes in the input,
- stability, determined by a bounded output (or other means), and
- steady-state error, which is a measure of accuracy.

The first two measures are related to the transient response; the third is related to the steady-state behavior. These concepts are discussed next with respect to the response to a unit step¹⁹ input; this response is shown generically in Figure 10.11.

The speed of response is generally represented by the *rise time* t_r , the time for the response to first attain its steady-state value. The rise time is the first time the response matches the magnitude of the input, which corresponds to its steady-state value of unity.²⁰ Sometimes

¹⁸The signals in the table equal zero outside the given range.

¹⁹The unit step is a discontinuous function whose value equals zero for negative time and equals one for positive time, for example $A = 1$ in the step function entry in Table 10.1.

²⁰The rise time is sometimes defined as the time for the response to go from 10 to 90% or 5 to 95% of its steady-state value. Books indicate slightly different definitions for the rise time. For underdamped systems, 0 to 100% is normally used. For overdamped systems, 10 to 90% is commonly used.

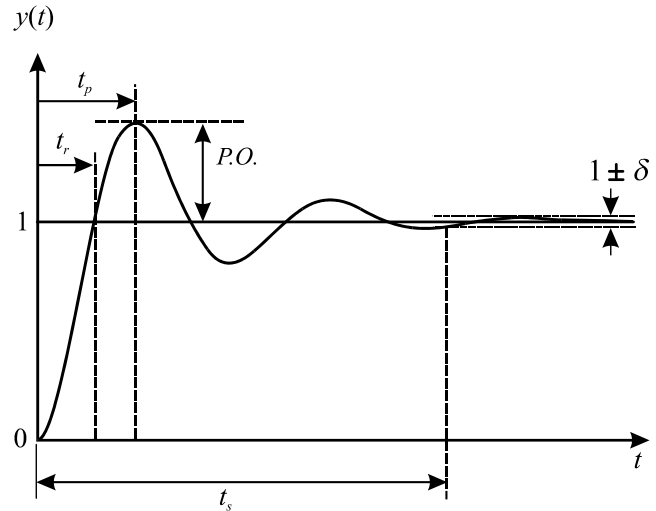


Figure 10.11: Unit step response showing some performance measures.

the rise time as well as the *peak time* t_p , the time to reach the first peak corresponding to the maximum response, are both used as measures of the speed of response. Stability (or relative stability) is often indicated by the *percent peak overshoot* (PO) $(y_{max} - 1) \times 100\%$. The time to reach steady-state behavior is measured by the *settling time* t_s , the time after which the response is within a tolerance of $\pm \delta$ about the steady-state value. Common settling times²¹ are 2% ($\delta = 0.02$), 5% ($\delta = 0.05$), or 10% ($\delta = 0.10$).

For a second-order system described by the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where the input is a unit step, $R(s) = 1/s$ and where $0 < \zeta < 1$, the time-domain solution can be found by taking the inverse Laplace transform, giving

$$\begin{aligned} y(t) &= 1 - \exp(-\zeta\omega_n t) \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right) \\ &= 1 - \frac{\exp(-\zeta\omega_n t)}{\sqrt{1-\zeta^2}} \sin \left(\omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right), \end{aligned} \quad (10.3)$$

for $t \geq 0$, where the solution is valid for $t \geq 0$, $\omega_d = \omega_n \sqrt{1-\zeta^2}$, and $\exp(\lambda) = e^\lambda$.

From Equation 10.3 it is possible to derive expressions for the following performance measures: rise time, peak time, percent peak overshoot, and settling time.

²¹The settling time may jump from one value to another if the response undergoes a very small change. To avoid a discontinuity, the settling time is usually defined for a smooth envelope of the decaying oscillation.

The rise time t_r can be found by letting $y(t_r) = 1$. Then, from Equation 10.3,

$$0 = \exp(-\zeta\omega_n t_r) \left(\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r \right).$$

Since $\exp(-\zeta\omega_n t_r) \neq 0$, we obtain

$$\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r = 0,$$

or

$$\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}.$$

Thus, the rise time is

$$t_r = \frac{1}{\omega_n \sqrt{1-\zeta^2}} \tan^{-1} \left(-\frac{\sqrt{1-\zeta^2}}{\zeta} \right),$$

or, equivalently,

$$t_r = \frac{\alpha}{\omega_n \sqrt{1-\zeta^2}}, \quad \text{where } \cos \alpha = -\zeta, \quad (10.4)$$

with α in radian units.

The peak time t_p can be found by differentiating $y(t)$ with respect to time and setting this derivative equal to zero. Taking the derivative,

$$\begin{aligned} \frac{dy}{dt} &= \zeta\omega_n \exp(-\zeta\omega_n t) \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right) \\ &+ \exp(-\zeta\omega_n t) \left(\omega_d \sin \omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} \cos \omega_d t \right). \end{aligned}$$

The cosine terms cancel and thus dy/dt evaluated at $t = t_p$ can be simplified to

$$\left. \frac{dy}{dt} \right|_{t=t_p} = (\sin \omega_d t_p) \frac{\omega_n}{\sqrt{1-\zeta^2}} \exp(-\zeta\omega_n t_p) = 0,$$

which gives the condition

$$\sin \omega_d t_p = 0,$$

or

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \dots$$

We choose $\omega_d t_p = \pi$ rad since the peak time corresponds to the first peak overshoot, or

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}. \quad (10.5)$$

The peak overshoot occurs at the peak time. Evaluating the response at the peak time, we have

$$\begin{aligned} y(t_p) &= 1 - \exp(-\zeta\omega_n t_p) \left(\cos \omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_p \right) \\ &= 1 - \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \cos \pi \\ &= 1 + \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right). \end{aligned}$$

The percentage peak overshoot, $P.O.$, is defined generally as

$$P.O. = \left[\frac{y(t_p) - y(\infty)}{y(\infty)} \right] \times 100\%.$$

Here, since the steady-state value is unity,

$$P.O. = [y(t_p) - 1] \times 100\%,$$

or substituting $y(t_p)$ from above,

$$P.O. = \exp(-\beta) \times 100\%, \quad \text{where } \beta = \frac{\zeta\pi}{\sqrt{1-\zeta^2}}. \quad (10.6)$$

Rearranging gives an expression for ζ in terms of the $P.O.$,

$$\zeta = \frac{\beta}{\sqrt{\pi^2 + \beta^2}}, \quad \text{where } \beta = -\ln \left(\frac{P.O.}{100} \right).$$

The settling time is the time required for the response to damp out all transients. Theoretically, this may take infinite time. In practice, however, the transient is assumed to be over when the error is reduced below some acceptable value, such as $\pm 2\%$ or $\pm 5\%$ about the final value. From Equation 10.3, the curves $1 \pm \exp(-\zeta\omega_n t) / \sqrt{1-\zeta^2}$ define the upper and lower envelopes bounding the unit-step response. The actual response $y(t)$ always stays within this pair of envelope curves, which has the time constant of $1/\zeta\omega_n$.

In order to find the 2% settling time, we seek the time for which $y(t)$ stays within $\pm 2\%$ of the steady-value (unity), which we express mathematically²² as

$$\frac{\exp(-\zeta\omega_n t_s)}{\sqrt{1-\zeta^2}} = 0.02.$$

Solving for t_s ,

$$t_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}. \quad (10.7)$$

²²This equation is a conservative estimate since we are assuming that the terms in the parentheses in Equation 10.3 equal unity at the settling time.

The numerator of Equation 10.7 varies from 3.91 to 4.74 as ζ varies from 0 to 0.9. For convenience, the 2% settling time is commonly approximated as

$$t_s \simeq \frac{4}{\zeta\omega_n}. \quad (10.8)$$

Similarly, the 5% settling time is commonly approximated as

$$t_s \simeq \frac{3}{\zeta\omega_n}.$$

These approximations are generally reasonable, although may not be sufficient when accurate values are needed.

In determining the success of a control system design we rely on these measures to meet performance specifications and ensure stability. We note that ζ can be determined from the permissible percentage peak overshoot. Then, the settling time can be determined primarily by ω_n , since it is inversely proportional to $\zeta\omega_n$.

In summary, these measures capture essential information for system performance. To assess transient performance, the rise time t_r and the peak time t_p are useful measures of the speed of response. The smaller these values, the faster the system responds to a step input. The percentage peak overshoot *P.O.* is another transient performance measure and is useful for determining relative stability. Values in excess of 40% or 50% indicate that the system is potentially prone to instability. The settling time t_s , related to both stability and speed of response, identifies when steady-state behavior has been reached. Due to inherent trade-offs, although one system (or design) may have a shorter rise time than another system, the latter might have a shorter settling time. Generally, a compromise in these measures is encountered in trying to achieve the desired overall performance.

Example 10.2 Transfer Function of a Second-Order System

Determine the transfer function of a second-order system with peak time of 0.250 s and a peak overshoot of 10.0%.

Solution: From the specification of percent overshoot, we have from Equation 10.6

$$\exp(-\beta) = 0.100, \quad \text{where } \beta = \frac{\zeta\pi}{\sqrt{1-\zeta^2}},$$

or

$$\beta = -\ln(0.100) = 2.30,$$

from which we can find $\zeta = 0.591$. From the specification of peak time, we have from Equation 10.5

$$\omega_n \sqrt{1-\zeta^2} = \frac{\pi}{t_p} \quad \text{or} \quad \omega_n = \frac{\pi}{t_p \sqrt{1-\zeta^2}},$$

from which we can determine $\omega_n = 15.6$ rad/s. Thus, the transfer function meeting the two given specifications is

$$\frac{Y(s)}{R(s)} = \frac{243}{s^2 + 18.4s + 243}.$$

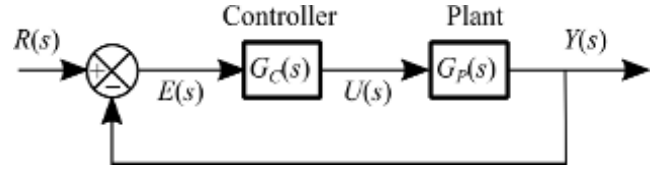


Figure 10.12: Block diagram of a closed-loop unity negative feedback system.

This example demonstrates the process of going from two time-domain specifications to a transfer function model of the system. The rise time and the settling time were not specified, but could have been given. They can be calculated, if desired, once the values of ζ and ω_n are known. ■

Other time-response specifications may be provided. For example, the delay time, t_d , the time for the step response to attain half of its steady-state value, is sometimes given. Exact analytical expressions for specifications, such as the rise time, percentage overshoot, and settling time, are prohibitively complicated for systems of order higher than two.

An important consequence of feedback is that it provides some control of the steady-state error to standard inputs by adjustment of the open-loop gain. As an example, consider the steady-state error to a unit step input. For simplification, we assume a unity negative feedback system, that is, $H(s) = 1$ in the closed-loop system, shown in Figure 10.9 and redrawn in Figure 10.12.

For a unit step input, $R(s) = 1/s$ and the output can be written as

$$Y(s) = \frac{G(s)}{s[1+G(s)]},$$

where $G(s) = G_c(s)G_p(s)$. The error is given by

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= \frac{1}{s} \left[1 - \frac{G(s)}{1+G(s)} \right] = \frac{1}{s[1+G(s)]}. \end{aligned}$$

Applying the final-value theorem, the steady-state error is obtained as

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} [sE(s)] = \frac{1}{1+G(0)}.$$

Thus, the steady-state error may be made small by increasing $G(0) = G_c(0)G_p(0)$, called the DC open-loop gain of the system.²³ However, trade-offs in performance are a reality in control system design. In many cases, a large increase in the loop gain may lead to instability. Hence, the possible decrease in the steady-state error may be limited.

²³This derivation assumes that the poles of the closed-loop transfer function are in the left-half of the s -plane. Otherwise, the final-value theorem cannot be applied.

10.4.1 Poles and Zeros of a Second-Order System

The poles and zeros of a system can be identified from its transfer function. The *poles* are those values of s that make the denominator polynomial of the transfer function equal to zero, and thus render the transfer function infinite. The *zeros* are those values that make the numerator polynomial of the transfer function equal to zero, and thus make the transfer function zero. In other words, the poles are the roots of the denominator polynomial and the zeros are the roots of the numerator polynomial.

Consider the second-order system²⁴ $\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2y(t) = \omega_n^2x(t)$, where ω_n is the undamped natural frequency and ζ is the damping ratio (also called the viscous damping factor). With zero initial conditions, the input-output relation in the s -domain is

$$Y(s) = \left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right] X(s),$$

with the transfer function given by the ratio $Y(s)/X(s)$. The poles of the transfer function are given by the solution of the characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

or $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$. These are the eigenvalues of the system, and correspond to the damped or complex natural frequencies of the system. In Chapter 3, we studied the various cases that are possible for this system,

- If $\zeta > 1$, both poles are *negative* and *real*.
- If $\zeta = 1$, both poles are *negative*, *equal*, and *real*; $s = -\omega_n$.
- If $0 < \zeta < 1$, the poles are *complex conjugates* with *negative real parts*; $s = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2}$ or, equivalently, $s = -\zeta\omega_n \pm i\omega_d$ where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ is the *damped natural frequency*.
- If $\zeta = 0$, the poles are *complex conjugates* and *imaginary*; $s = \pm i\omega_n$.
- If $\zeta < 0$, the poles are in the right-half of the s -plane, and the system is unstable.

We recall from previous studies that the first case, $\zeta > 1$, is called *overdamped*, the second case, $\zeta = 1$, *critically damped*, and the third case, $0 < \zeta < 1$, *underdamped*. This latter case applies to underdamped oscillatory systems, with $s = -\zeta\omega_n \pm i\omega_d$. Equation 3.9 is

²⁴One of the reasons why second-order systems play such an important role is that the response of higher-order systems is often characterized by its dominant poles. We say that a complex conjugate pair of poles is a dominant pair if it has the lowest damping ratio compared with all other poles of the system.

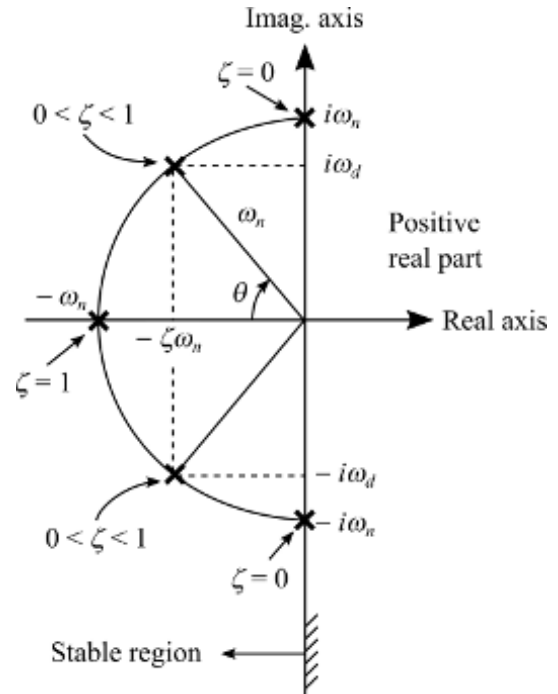


Figure 10.13: Migration of roots (poles) as a function of ζ for a second-order system. The region of stability is in the left-half plane.

the general response for such a system, repeated here,

$$x(t) = Ce^{-\zeta\omega_n t} \cos(\omega_d t - \phi).$$

The term $1/(\zeta\omega_n)$ is known as the *characteristic time constant* and is useful in characterizing the decay of the exponential boundary or envelope of the response given by $Ce^{-\zeta\omega_n t} = Ce^{-t/(\zeta\omega_n)}$.

The roots, or poles, vary in location in the complex plane as a function of ζ and ω_n . The locus of roots as ζ varies with ω_n constant is portrayed in Figure 10.13. The poles are indicated by the symbol $\times$. (Although not shown in this figure, zeros are indicated in the complex plane by the symbol $\circ$.) The distance between the origin and a complex root is always ω_n , and the damping ratio of a pair of complex conjugate poles is related to the angle θ by

$$\zeta = \cos \theta,$$

as shown in the figure.

With no damping, a pair of complex conjugate poles is located on the imaginary axis. This undamped case corresponds to a purely oscillatory time response. With damping, the poles move into the left-half plane. As the damping increases, the pair of complex conjugate poles follows along the circumference of a semicircle in the left-half plane. These values of the poles correspond to underdamped behavior. At a critical value of damping, known as *critical damping*, the poles meet on the real axis. With increased damping, corresponding to the

overdamped case, the poles remain on the real axis, but separate in different directions.²⁵ More generally the connection between the pole location and the impulse response was shown earlier in Figure 3.13.



Figure 10.14: Vibration is a serious problem for rotorcraft designers. Not only debilitating to passengers and crew, vibration is detrimental to avionics and components, reducing reliability and life. The unsteady aerodynamics of rotor blades in forward flight produce vibratory loads that enter the hub, travel down to the gearbox and from there into the airframe. Traditionally, vibration has been reduced using absorbers placed in key locations—initially passive and tuned to a single rotor speed, but more recently active and able to adjust to variable speeds. (Milwaukee, WI; photograph by M. Nagurka)

Example 10.3 Desired Closed-Loop Pole Location

Indicate on the complex plane the location of desired closed-loop poles if it is required that the 2% settling time be less than 0.5 s, the percent overshoot be less than 5%, and the peak time be less than 0.1 s.

Solution: The settling time, peak time, and the percent overshoot are related to the system parameters by

$$t_s = \frac{4}{\zeta\omega_n}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$P.O. = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100\%.$$

The three criteria are satisfied if

$$\frac{4}{\zeta\omega_n} < 0.5$$

$$\frac{\pi}{\omega_d} < 0.1$$

$$\exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) < 0.05,$$

²⁵The *root locus method* is concerned with the movement of the poles of the transfer function of the closed-loop system as an open-loop gain is varied. The method is credited to Walter Evans, and has been used successfully for both analysis and design.

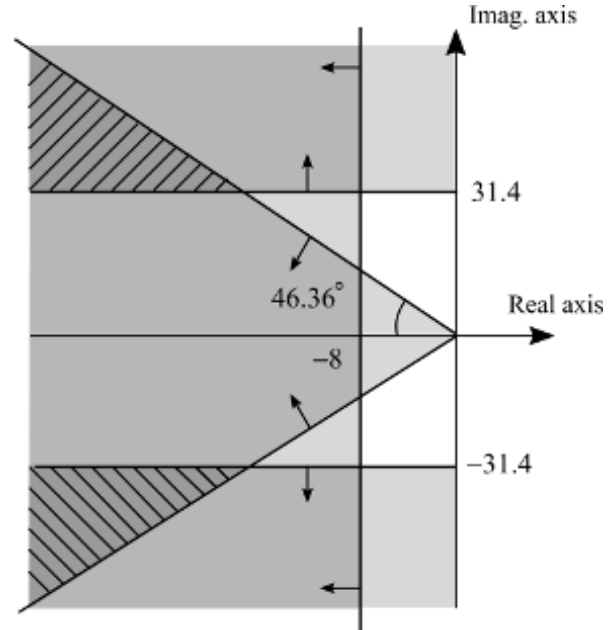


Figure 10.15: The desired characteristic roots are shown in the overlapping hatched regions for Example 10.3.

which can be rewritten as

$$\zeta\omega_n > 8$$

$$\omega_d > \frac{\pi}{0.1} \text{ or } \omega_d > 31.4$$

$$\zeta > 0.6902.$$

Note that $\zeta\omega_n$ is negative of the real part, and ω_d the imaginary part of the complex pole. The condition for the damping ratio can be rewritten in terms of the angle θ in Figure 10.13, so that

$$\theta < 46.36^\circ.$$

The desired complex conjugate pole locations are indicated in the regions with overlapping hatched marks, as shown in Figure 10.15. ■

10.4.2 System Gain

With the system poles and zeros known, the system transfer function can be specified to within a constant. This constant, usually denoted by K , is known as the *system gain*.

Example 10.4 Transfer Function to Differential Equation

Given the transfer function, determine the system differential equation.

$$(a) \ G(s) = \frac{Y(s)}{X(s)} = \frac{2s + 1}{s^2 + 3s + 1},$$

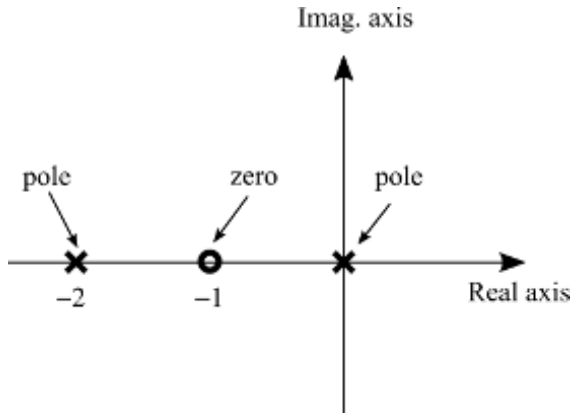


Figure 10.16: Pole-zero locations for Example 10.4(b).

$$(b) \ G(s) = \frac{K(s+a)}{(s+b)(s+c)}.$$

Solution: (a) To solve, cross-multiply giving

$$(s^2 + 3s + 1)Y(s) = (2s + 1)X(s).$$

Taking the inverse Laplace transform yields the system differential equation,²⁶

$$\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2\dot{x}(t) + x(t).$$

The output $y(t)$ and its derivatives appear on the left-hand side of the differential equation; the input $x(t)$ and its derivative appear on the right-hand side.

(b) The transfer function can be specified by giving the zero location $-a$, the pole locations $-b$ and $-c$, and the gain K . If we are given the information that the system gain is 3 and the pole-zero map in the s -plane is as shown in Figure 10.16, then the transfer function is

$$G(s) = \frac{3(s+1)}{s(s+2)},$$

from which the system differential equation can be derived as before.

$$\ddot{y}(t) + 2\dot{y}(t) = 3\dot{x}(t) + 3x(t).$$

This example demonstrates the ease with which one can convert the transfer function into a differential equation relating the input and output time-domain variables. ■

²⁶The system differential equation can be written

$$y(t) = \left(\frac{2D+1}{D^2+3D+1} \right) x(t),$$

where the shorthand notation $D^n \equiv d^n/dt^n$ is sometimes used. Then,

$$(D^2 + 3D + 1)y(t) = (2D + 1)x(t),$$

which corresponds to the differential equation.

10.4.3 Stability of Response

A key reason for controlling the vibration of a structure or a machine is to make it *more stable*. Stability is always a concern in the design of a dynamic system. There are numerous definitions for stability. Two important definitions are:

- A system is *stable* if its impulse response approaches zero as time approaches infinity. The impulse response is a useful input since, by convolution, any load can be represented in terms of the impulse response.
- A system is *stable* if every bounded input produces a bounded output. If an input magnitude is always within a certain finite range, it is called *bounded*.

If the input is random, then there are comparable probabilistic definitions of stability. A necessary condition for the system to be stable is for the roots of the characteristic equation to have *negative real parts*. This ensures that the impulse response will decay exponentially with time. The case $\zeta < 0$ implies that energy is added to the system faster than it can be dissipated, resulting in ever-greater amplitude oscillations and eventual failure.

10.5 Control of Response

10.5.1 Control Actions

The block diagram of Figure 10.17 depicts a generic closed-loop feedback control system. It shows that the error signal is used by the control law to create a command that is typically amplified prior to activating the actuator that controls the plant. A deviation from the desired output, in general, results in the control action.²⁷

Certain control actions are commonly used. These are on-off control, proportional control, integral control, derivative control, and combinations such as proportional-integral (PI) control, proportional-derivative (PD) control, or proportional-integral-derivative (PID) control.²⁸ We briefly describe each of

²⁷The selection of *sensors* and *actuators* is a very important aspect of the control engineer's task. A design is not complete until appropriate sensors and actuators are chosen. Very often the system dynamics are altered due to the presence of sensors and actuators. Such is the case in lightly damped space structures.

²⁸PID controllers are very widely used in practice. As an example of their popularity, we refer to the following quote: "Based on a survey of over eleven thousand controllers in the refining, chemicals and pulp and paper industries, 97% of regulatory controllers utilize PID feedback." (L. Desborough and R. Miller, "Increasing customer value of industrial control performance monitoring - Honeywell's experience," Sixth International Conference on Chemical Process Control, AIChE Symposium Series Number 326, Vol.98, 2002.)

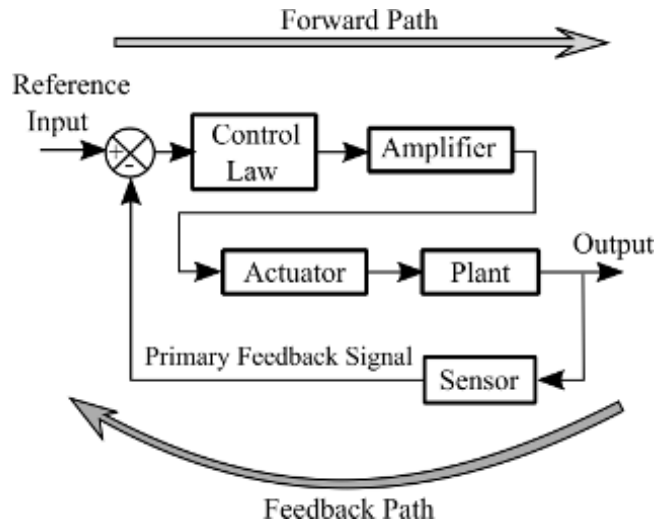


Figure 10.17: Block diagram of a generic negative-feedback control system.

the above control actions, referring to the simplified block diagram of Figure 10.12.

As the name signifies, in *on-off control* there are two possible *actions*, one of which is *on* and one of which is *off*, or no action at all. The sign of the error signal determines which of the two actions is active. For example, if the error signal is positive the control action is *on* and if the error signal is negative the control action is *off*.

Proportional control action is defined by the proportional gain K_p and the transfer function

$$\frac{U(s)}{E(s)} = G(s) = K_p.$$

Integral control action is defined by the integral gain K_i and the transfer function

$$\frac{U(s)}{E(s)} = G(s) = \frac{K_i}{s}.$$

Proportional plus Integral control action (PI) is defined by the transfer function

$$\frac{U(s)}{E(s)} = G(s) = K_p \left(1 + \frac{1}{T_i s} \right),$$

where T_i is the *integral time constant* and $K_p/T_i = K_i$ is the integral gain.

Proportional plus Derivative control action (PD) is defined by the transfer function

$$\frac{U(s)}{E(s)} = G(s) = K_p (1 + T_d s),$$

where T_d is the *derivative time constant* and $K_p T_d = K_d$ is the derivative gain.

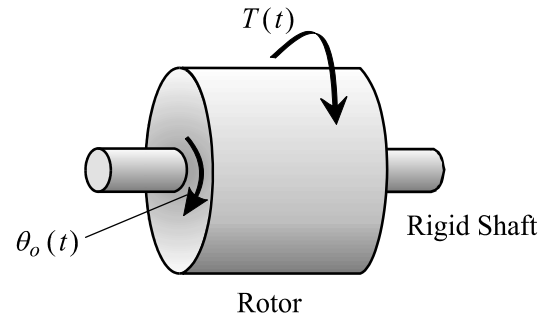


Figure 10.18: Rotational element acted on by external torque. The angular displacement, $\theta_o(t)$, is shown.

Proportional plus Integral plus Derivative control action (PID) is defined by the transfer function

$$\frac{U(s)}{E(s)} = G(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s \right), \quad (10.9)$$

or, in terms of gains K_p , K_i , and K_d , the right-hand side can be rewritten as

$$\frac{U(s)}{E(s)} = G(s) = K_p + \frac{K_i}{s} + K_d s.$$

Transforming Equation 10.9 back into the time domain results in

$$u(t) = K_p \left[e(t) + \frac{1}{T_i} \int_{-\infty}^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right],$$

where $u(t)$ is the output in the time domain and $e(t)$ is the error signal in the time domain. The gains, or time constants, are the control parameters that are *tuned* or adjusted so that the system response meets the design criteria.²⁹

10.5.2 Control of Transient Response

In this section, we consider the design process and the selection of appropriate control actions leading to a desired transient response. We take as an example the control of a rotor modeled as a cylindrical disk with inertia J rotating on a frictionless bearing, as shown in Figure 10.18. Torque $T(t)$ is applied to the rotor. Although we seek the desired displacement $\theta_i(t)$, that is, the input or *reference signal*, the actual displacement is $\theta_o(t)$, and is the response or *output signal*.

With torque $T(t)$ applied to the rotor, its response $\theta_o(t)$ is governed by $J \ddot{\theta}_o(t) = T(t)$, or taking the Laplace

²⁹There are classical methods for tuning PID controllers, such as the Ziegler-Nichols method, as well as many advanced methods that offer increased optimality and/or robustness. A paper that may be of interest to the reader is O. Yaniv and M. Nagurka, "Design of PID Controllers Satisfying Gain Margin and Sensitivity Constraints on a Set of Plants," *Automatica*, Vol.40, No.1, 2004, pp.111-116.

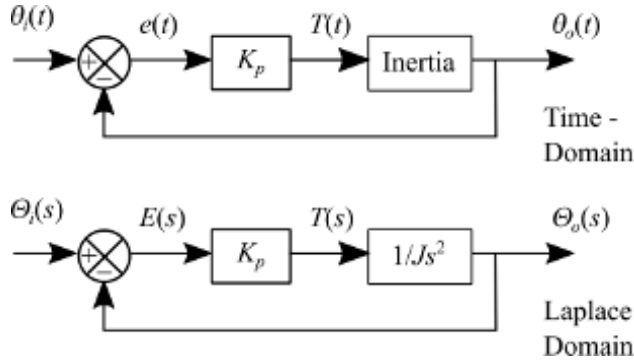


Figure 10.19: Block diagram of rotor with proportional controller. (Top diagram is time-domain representation. Bottom diagram is Laplace-domain representation.)

transform of both sides,

$$Js^2\Theta_o(s) = T(s), \quad (10.10)$$

giving the transfer function³⁰

$$\frac{\Theta_o(s)}{T(s)} = \frac{1}{Js^2}.$$

The problem we face is to modify or control the response keeping the value of J fixed. The only option is to control the applied torque $T(t)$ to achieve the desired angular displacement response. We initially test the effectiveness of using a proportional controller, $T(t) = K_p e(t)$, with the gain K_p , as shown in Figure 10.19.

We can derive the closed-loop transfer function in the following way. In the transformed domain,

$$T(s) = K_p E(s) = K_p (\Theta_i(s) - \Theta_o(s)). \quad (10.11)$$

Equating Equations 10.10 and 10.11 gives

$$Js^2\Theta_o(s) = K_p (\Theta_i(s) - \Theta_o(s))$$

or, collecting like terms,

$$(Js^2 + K_p)\Theta_o(s) = K_p \Theta_i(s). \quad (10.12)$$

The closed-loop transfer function³¹ is then

$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{K_p}{Js^2 + K_p},$$

³⁰We try to use a different symbol for the time-domain variable and its Laplace transform. In general, a small letter is used for the time-domain variable and a capital letter for the Laplace transform of that variable. In this case, we use a capital letter for the time-domain torque, $T(t)$, and the Laplace transform of the torque, $T(s)$, and distinguish the two by the argument (rather than introduce a new symbol, such as $\Gamma(s)$).

³¹In the time-domain, Equation 10.12 is

$$J\ddot{\theta}_o(t) + K_p\theta_o(t) = K_p\theta_i(t).$$

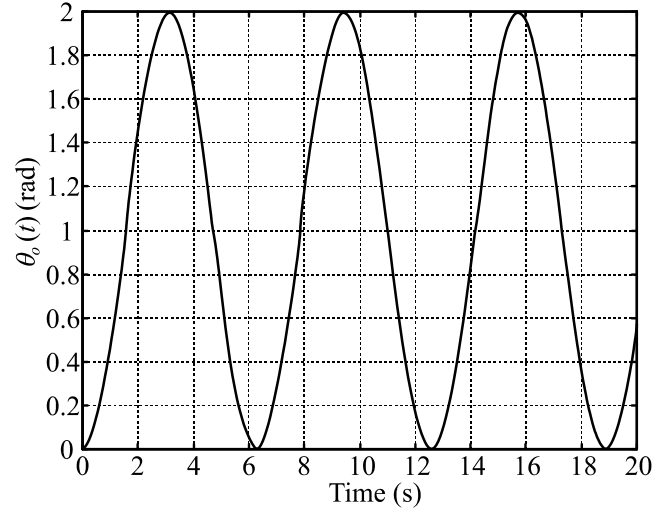


Figure 10.20: Step response with proportional controller.

where the proportional gain K_p acts as a stiffness element in the characteristic equation. The roots of the characteristic equation, $Js^2 + K_p = 0$, are imaginary and the response to a unit step oscillates (theoretically) without decay or growth, as shown in Figure 10.20 for the specific case of $J = 1 \text{ kg-m}^2$ and $K_p = 1 \text{ N-m/rad}$.

This marginally stable response is not acceptable. We next examine the effect of PD control, that is, adding a derivative controller to the proportional controller. The block diagram representation is shown in Figure 10.21. Here,

$$\begin{aligned} T(s) &= K_p(1 + T_d s)E(s) \\ &= K_p(1 + T_d s)(\Theta_i(s) - \Theta_o(s)), \end{aligned}$$

and, in conjunction with the relation $T(s) = Js^2\Theta_o(s)$, we have

$$(Js^2 + K_p T_d s + K_p)\Theta_o(s) = K_p(1 + T_d s)\Theta_i(s), \quad (10.13)$$

where the closed-loop transfer function³² is given by

$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{K_p(1 + T_d s)}{Js^2 + K_p T_d s + K_p}, \quad (10.14)$$

with the characteristic equation $Js^2 + K_p T_d s + K_p = 0$. Since J , K_p , and T_d are positive quantities, the characteristic equation has two roots with negative real parts.

Derivative control introduces an additional term in the characteristic equation, $K_p T_d$, which has a damping effect. The effective viscous torsional damping $C = K_p T_d$ and effective linear torsional stiffness $K = K_p$ can be

³²In the time-domain, Equation 10.13 is

$$J\ddot{\theta}_o(t) + K_p T_d \dot{\theta}_o(t) + K_p \theta_o(t) = K_p \theta_i(t) + K_p T_d \dot{\theta}_i(t).$$

This is the same equation we see for the base-excited structure of Chapter 3.

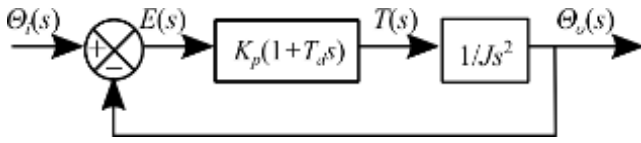


Figure 10.21: Block diagram of rotor with PD controller.

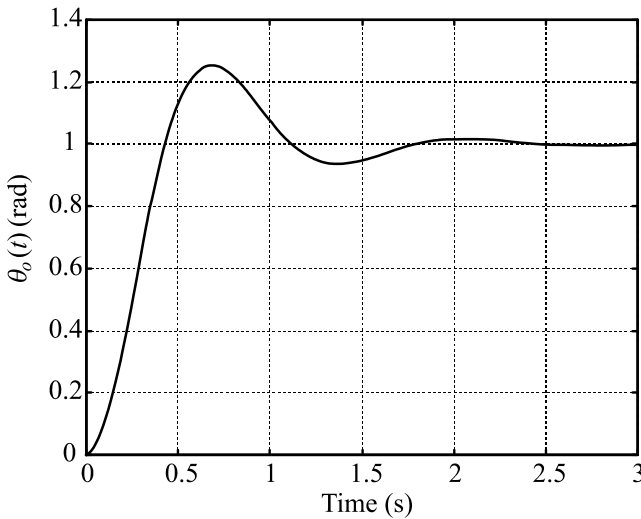


Figure 10.22: Step response with PD controller.

selected to meet the design criteria. The response curve is of the form shown in Figure 10.22. This figure was generated for the specific case of $J = 1 \text{ kg-m}^2$, $K_p = 1 \text{ N-m/rad}$, and $T_d = 4 \text{ rad/s}$, where the roots are $s = -2.00 \pm 3.32i$.

In a similar way, if PD control is applied to the uncontrolled second-order system with external torque $T_{ext}(t)$ governed by the equation

$$J\ddot{\theta}_o(t) + C\dot{\theta}_o(t) + K\theta_o(t) = T_{ext}(t),$$

then the transfer function³³ of Equation 10.14 becomes

$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{K_p(1 + T_d s) + T_{ext}(s)}{Js^2 + (C + K_p T_d)s + (K + K_p)}. \quad (10.15)$$

The block diagram representation is given in Figure 10.23. The designer can adjust K_p and T_d to achieve the desired response.

³³In the time-domain, Equation 10.15 is

$$J\ddot{\theta}_o + (C + K_p T_d)\dot{\theta}_o + (K + K_p)\theta_o = K_p\theta_i + K_p T_d\dot{\theta}_i + T_{ext}(t),$$

where the terms on the right-hand side are due to control actions and external torque.

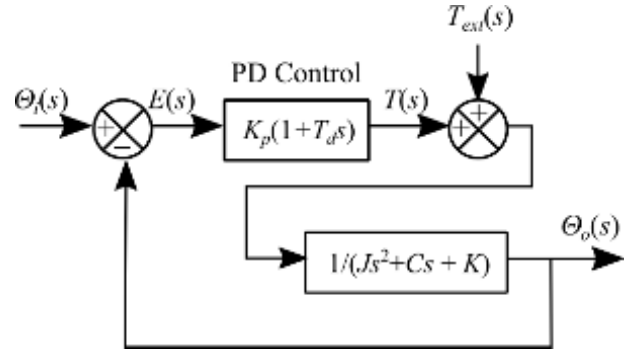


Figure 10.23: Block diagram of second-order system with PD control and external torque.

VIBRATION PERCEIVED AS POWER. Vibration is often perceived as power. If a machine vibrates significantly, it may be viewed as being powerful. But, this commonly held view is perception only. It is certainly possible through good design (such as using passive and active control) to make higher power machines vibrate less than lower power machines. The perception that more vibration equals more power is often exploited by marketing and sales teams. It explains the commercial success of motorcycles, for example, that are loud and shake significantly. A motorcycle engine that is well balanced with appropriate mufflers may indeed be more powerful than a shaky, loud engine, even if the consumer is not convinced.

The design of powered hand tools is also affected by this issue of perception. It is possible to design a powered hand tool, such as a reciprocating saw, with advanced anti-vibration features that isolate the action-reaction forces from the user, making it less fatiguing and easier to control, and quiet, also making it less fatiguing to use. High-end designs of powered hand tools have been made that are more powerful and shake much less than typical designs. Some workers may perceive that these tools that are quieter and shake less are less powerful. Balancing tool performance and low vibration is critical for user acceptance, which is often based on perception.³⁴

10.6 Parameter Sensitivity

From Chapters 4 and 9 we expect uncertainties in the dynamic models that represent physical systems, since physical systems change due to aging, wear, and manufacturing tolerances, for example. One way to account for these uncertainties is to use the probabilistic approach of Chapter 9, as for random loadings. Another way is to examine the *sensitivity of the system response to parameter variations*. The question is: *How does the system*

³⁴Whether or not vibration is annoying or objectionable is highly subjective. It varies from person to person.

response change as a result of a change in a particular parameter value?³⁵ An understanding of the sensitivity of the system response to a parameter variation is useful in the control design process. Parameter sensitivity is also valuable in defining which additional tests are needed to better understand system characteristics.

For linear systems, the transfer function and the frequency response function both describe the system dynamics. The frequency response function can be obtained from the expression for the transfer function by replacing the Laplace variable s with $i\omega$.

The transfer function and the frequency response function are functions of the parameters that define them. For mechanical systems, these parameters are the stiffness, damping, and mass properties, or, in more general terms, the natural frequency ω_n and the damping ratio ζ . The accuracy of the functions depends directly on the accuracy of these parameter values.

The frequency response function can be expressed in terms of its magnitude and phase angle, both of which are functions of the frequency ratio ω/ω_n . The relevant Equations 3.43 to 3.45 are reproduced here,

$$x(t) = A|G(i\omega)|e^{i(\omega t - \phi)},$$

$$|G(i\omega)| = \frac{1/\omega_n^2}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

$$\phi(\omega) = \tan^{-1} \left[\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} \right].$$

Graphs of the magnitude and phase vs. frequency (or frequency ratio) are called *Bode plots*, after H. Bode. It is common for the magnitude plot to be presented using a log-log scale and the phase plot to be presented in a semilog (linear-log) scale. These plots are easy to sketch and interpret, and are rich with information. Since the frequency scale is logarithmic, the plots can cover the behavior of a linear system over a wide frequency range.

The Bode plots are often determined experimentally. In principle, an infinite number of values is required for an exact representation over all frequencies. In practice, a finite number of values over a range of frequencies generally provides sufficient information to obtain a reasonable model. The accuracy of the model then depends on how closely the amplitude and phase graphs approximate the actual functions.

³⁵ For more information about sensitivities, especially regarding how they relate to control, we suggest the detailed book **Introduction to System Sensitivity Theory**, P.M. Frank, Academic Press, 1978.

HENDRIK WADE BODE

(December 24, 1905 – June 21, 1982)



Contribution: Hendrik Wade Bode (pronounced Boh-dee) was an American engineer, researcher, inventor, author, and scientist. He was a pioneer of modern control theory and electronic telecommunications, and revolutionized both the content and methodology of both fields. His research impacted many engineering disciplines and laid the foundation for a diverse array of modern innovations such as computers, robots, and mobile phones, among others.

He is widely known for developing the frequency response magnitude and phase plots that bear his name, the Bode plots. In 1938 he developed his phase and magnitude plots along with rules for asymptotic behavior. His work on automatic feedback control systems introduced innovative methods to the study of system stability that enabled engineers to investigate time domain stability using the frequency domain concepts of gain and phase margin, the study of which was aided by his now famous plots.

Bode's frequency domain-based analysis was faster and simpler than the traditional time-domain-based methods for evaluating stability and performance. His approach provided engineers with a fast and intuitive analysis and system design tool that is as popular today as it was groundbreaking then.

Life Sketch: Bode was born in Madison, Wisconsin and grew up in Illinois. He rapidly advanced through the Urbana school system to graduate from high school at the age of 14.

After graduation from high school he applied for admission to the University of Illinois at Urbana-Champaign, where his father was a professor of education, but was denied because of his age. (In 1977, the University of Illinois granted him an Honorary Sc.D. Degree.)

He eventually applied and was accepted at Ohio State University, where his father also taught, and he received his BA degree in 1924 at age 19 and MA degree in 1926, both in mathematics. After graduation he remained at his alma mater, working as a teaching assistant for an additional year.

Bode was hired by Bell Labs in New York City, where he began his career as a designer of electronic filters and equal-

izers. Subsequently, in 1929, he was assigned to the Mathematical Research Group, where he excelled in research related to electronic networks theory and its application to telecommunications. Sponsored by Bell Laboratories he reentered graduate school, this time at Columbia University, and he completed his PhD in physics in 1935.

With the onset of World War II, Bode turned to military applications of his control systems research, a change of direction that would last in various degrees to the end of his career. He came to the service of his country by working at the National Defense Research Committee (NDRC).

In 1944 Bode was placed in charge of the Mathematical Research Group at Bell Laboratories. He pursued work on electronic communications, especially on filter and equalizer design, and in 1945 he published *Network Analysis and Feedback Amplifier Design*, a book that is considered a classic in the field of electronic telecommunications. It was widely adopted for many graduate programs at various universities as well as for internal training courses at Bell Labs. Bode also authored numerous research papers and technical journals.

Noteworthy: At Bell Labs, Bode developed automatic anti-aircraft control systems. He used radar information to provide data about the location of the enemy aircraft, that was then fed back to the anti-aircraft artillery servomechanisms enabling “automatic radar-augmented enemy-aircraft ballistic tracking” or more simply the automatic shooting down of enemy aircraft with the help of radar. The servomotors used were both electrically and hydraulically powered, the latter being used mainly for positioning the heavy anti-aircraft guns.

In 1945, as World War II was winding down, the NDRC was issuing a summary of technical reports as the prelude to its eventual closing down. Bode coauthored (with Ralph Beebe Blackman and Claude Shannon) a special essay “Data Smoothing and Prediction in Fire-Control Systems” that formally introduced the problem of fire control as a special case of transmission, manipulation, and utilization of intelligence. In other words, it modeled the problem in terms of data and signal processing and thus heralded the coming of the information age. Shannon, considered to be the father of information theory, was greatly influenced by this work.

The *sensitivity* of the system is a measure of the amount by which its transfer function or frequency response function changes from its nominal value when a parameter changes from its nominal value. One measure of sensitivity is given by the ratio of the percent change in the transfer function to the percent change in a parameter value. For example, if the transfer function T changes by amount ΔT due to the small change Δp in parameter p , the *sensitivity* is defined as

$$S = \frac{\Delta T/T}{\Delta p/p} = \frac{\Delta T}{\Delta p} \frac{p}{T}.$$

By definition, the *sensitivity function* is evaluated in the

limit as $\Delta p \rightarrow 0$,

$$S_p^T = \lim_{\Delta p \rightarrow 0} \frac{\Delta T}{\Delta p} \frac{p}{T} = \frac{\partial T}{\partial p} \frac{p}{T}, \quad (10.16)$$

where T is a function of Laplace transform variable s and parameter p .

Consider the frequency response function $G(p)$, where p may be any of the parameters ω , ω_n , or ζ . There are three possible sensitivity functions, or two if the ratio ω/ω_n is taken as a single parameter. The sensitivity function for the frequency response function is then

$$S_p^{G(p)} = \frac{\partial G(p)}{\partial p} \frac{p}{G(p)}, \quad (10.17)$$

which can also be written as

$$S_p^{G(p)} = \frac{d[\ln G(p)]}{d[\ln p]}.$$

Following Equation 10.17, the sensitivity functions for the magnitude and phase angle of the frequency response functions are, respectively,

$$S_p^{|G(p)|} = \frac{\partial |G(p)|}{\partial p} \frac{p}{|G(p)|}$$

$$S_p^{\phi(p)} = \frac{\partial \phi(p)}{\partial p} \frac{p}{\phi(p)}.$$

These two sensitivity functions can be related by the sum

$$S_p^{G(p)} = S_p^{|G(p)|} + i\phi(p)S_p^{\phi(p)}. \quad (10.18)$$

Example 10.5 Sensitivity Relationships

Prove Equation 10.18. That is, show that the sensitivity function for the magnitude of the frequency response function is the real part of the sensitivity function of the frequency response function, and the sensitivity function for the phase angle of the frequency response function is the imaginary part divided by the phase angle evaluated also at $s = i\omega$.

Solution: We can find the sensitivity function of the frequency response function and take the real and the imaginary parts to find the sensitivity functions for the magnitude and the phase angle. The sensitivity of $G(p)$ is defined as

$$S_p^{G(p)} \equiv \frac{\partial G(p)}{\partial p} \frac{p}{G(p)}.$$

The frequency response function $G(i\omega)$ is written in the magnitude and phase form using Euler's formula or

$$G(p) = |G(p)| e^{i\phi(p)}.$$

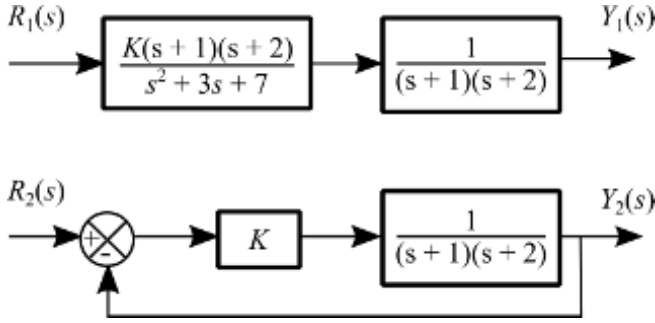


Figure 10.24: Block diagrams showing open-loop (top) and closed-loop (bottom) control of the same plant.

Replacing $G(p)$ with its magnitude and phase form, we have

$$\begin{aligned} S_p^{G(p)} &= \left[\frac{\partial |G(p)|}{\partial p} e^{i\phi(p)} + i |G(p)| \frac{\partial \phi(p)}{\partial p} e^{i\phi(p)} \right] \\ &\quad \times \frac{p}{|G(p)| e^{i\phi(p)}} \\ &= \frac{\partial |G(p)|}{\partial p} \frac{p}{|G(p)|} + i \frac{\partial \phi(p)}{\partial p} p \\ &= \frac{\partial |G(p)|}{\partial p} \frac{p}{|G(p)|} + i \phi(p) \frac{\partial \phi(p)}{\partial p} \frac{p}{\phi(p)}, \end{aligned}$$

where we identify

$$\begin{aligned} S_p^{|G(p)|} &\equiv \frac{\partial |G(p)|}{\partial p} \frac{p}{|G(p)|} \\ S_p^{\phi(p)} &\equiv \frac{\partial \phi(p)}{\partial p} \frac{p}{\phi(p)}. \end{aligned}$$

Therefore,

$$S_p^{G(p)} = S_p^{|G(p)|} + i \phi(p) S_p^{\phi(p)}.$$

The sensitivity of the magnitude is the real part of $S_p^{G(p)}$, and the sensitivity of the phase angle is the imaginary part of $S_p^{G(p)}/\phi(p)$. ■

Example 10.6 Sensitivity in Control Design

Investigate how sensitivity can play a role in control system design by determining the sensitivity of the transfer function for gain K variations in the two system block diagrams in Figure 10.24.

Solution: The transfer functions of the block diagrams are

$$\begin{aligned} T_1(s) &= \frac{Y_1(s)}{R_1(s)} = \frac{K}{s^2 + 3s + 7} \\ T_2(s) &= \frac{Y_2(s)}{R_2(s)} = \frac{K}{(s+1)(s+2) + K}. \end{aligned}$$

For gain value $K = 5$, $T_1(s) = T_2(s)$.

Prior to determining the sensitivity of each transfer function, we first write a general transfer function that includes the above as special cases:

$$T(s) = \frac{C_1(s) + pC_2(s)}{C_3(s) + pC_4(s)},$$

where $C_i(s)$ are polynomials in s , and p is any parameter of interest, as before. Equation 10.16, applied to the general transfer function, results in

$$S_p^T = \frac{p(C_2C_3 - C_1C_4)}{(C_3 + pC_4)(C_1 + pC_2)}.$$

For the example for $T_1(s)$, we have $p = K$, $C_1 = C_4 = 0$, $C_2 = 1$, and $C_3 = s^2 + 3s + 7$. Then, $S_K^{T_1} = 1$, a constant regardless of the value of gain K . For $T_2(s)$, $p = K$, $C_1 = 0$, $C_2 = C_4 = 1$, and $C_3 = s^2 + 3s + 2$. The sensitivity function for $T_2(s)$ is then

$$\begin{aligned} S_K^{T_2} &= \frac{K(s^2 + 3s + 2)}{(s^2 + 3s + 2 + K)K} \\ &= \frac{(s^2 + 3s + 2)}{(s^2 + 3s + 2 + K)}. \end{aligned} \quad (10.19)$$

Here, for the closed-loop system, the sensitivity is a function of gain K and the complex variable s . Thus, to reduce sensitivity to parameter variation, a designer would select an appropriate gain or maintain the frequencies of the input within an appropriate range. Since the sensitivity is a function of complex variable s , the frequency range of the application is important. In the next example, we plot the magnitude and phase of $S_K^{T_2}$ as a function of frequency to determine the frequencies for maximum and minimum sensitivity. ■

Example 10.7 Plot of Sensitivity Function

Plot the sensitivity of the magnitude and the sensitivity of the phase of the transfer function $T_2(s)$ in the previous example.

Solution: The sensitivity is given by

$$S_K^{T_2} = \frac{(s^2 + 3s + 2)}{(s^2 + 3s + 2 + K)}.$$

We substitute $s = i\omega$ so that the sensitivity function for the frequency response function is given by

$$S_K^{G_2} = \frac{(-\omega^2 + i3\omega + 2)}{(-\omega^2 + i3\omega + 2 + K)}.$$

The sensitivity of the magnitude is the real part, or

$$\begin{aligned} S_K^{|G(K)|} &= \text{Re} \left(S_K^{G_2} \right) \\ &= \frac{(2 - \omega^2)(2 + K - \omega^2) - 9\omega^2}{((2 + K - \omega^2)^2 + 9\omega^2)}. \end{aligned}$$

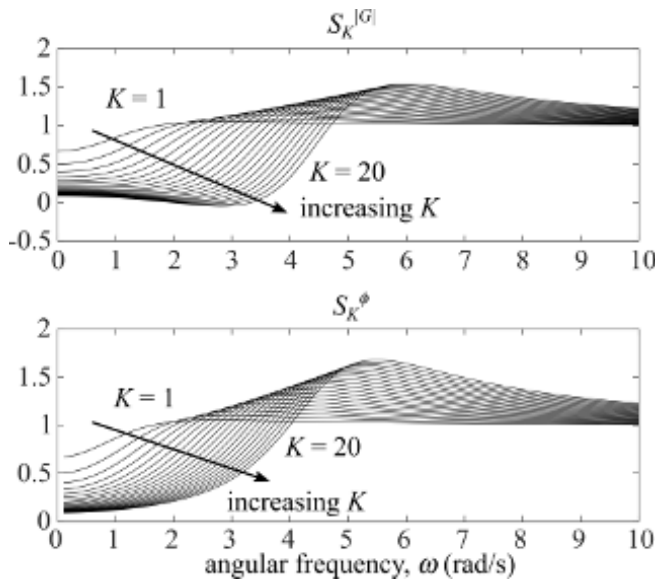


Figure 10.25: The sensitivity function for the magnitude and phase of the second system in Figure 10.24.

The sensitivity of the phase angle is the imaginary part divided by the angle itself, or

$$\begin{aligned} S_K^{|\phi(K)|} &= \text{Im} \left(S_K^{G_2} \right) / \phi(K) \\ &= \frac{1}{\phi(K)} \frac{(2 - \omega^2) 3\omega + (2 + K - \omega^2) 3\omega}{((2 + K - \omega^2)^2 + 9\omega^2)}, \end{aligned}$$

where ϕ is the phase angle of $S_K^{G_2}$,

$$\phi = \angle [-\omega^2 + i3\omega + 2] - \angle [(-\omega^2 + i3\omega + 2 + K)].$$

Figure 10.25 shows the sensitivity of the magnitude and the phase as functions of ω and increasing K . The input frequency at which the maximum sensitivity occurs varies with K . When $K = 20$ and the input frequency is 5.5 rad/s, the sensitivity for the magnitude is about 1.68. This means that a 1% increase in the value of K will result in a 1.68% change in the magnitude of the particular response. ■

An important property of negative feedback is the reduction in the sensitivity to variations in the parameters in the forward path.³⁶

³⁶ As indicated in the biography of H. Black, a significant reason for the development of negative feedback amplifiers at the Bell Telephone Laboratories was to reduce distortion. In particular, the goal was to reduce sensitivity to variations in the parameters of vacuum tubes used in the amplifiers. It was quickly realized that the gains varied considerably, attributable primarily to the variation in the amplification factors of the vacuum tubes. By introducing negative feedback, it was possible to make the gains of the amplifiers relatively insensitive to variations in the parameters of the vacuum tubes.

10.7 State Variable Models

The *state* of a system is defined as the smallest set of variables that must be known at any given time in order that the future response of the system to any specified input may be determined.³⁷ Thus, the state can be regarded as a compact representation of the past history of the system, which can be utilized for predicting its future behavior in response to an external input. Since the complete solution of a differential equation of order n requires exactly n initial conditions, it follows that the state of such a system will be specified by the values of n quantities, called the *state variables*.³⁸

The future behavior of the system can be determined from the governing dynamic equations, which can be written as a set of *first-order differential equations* in terms of the state variables. It is always possible to recast an n^{th} order differential equation governing the system dynamics into a set of n first-order differential equations (for example, a second-order equation of motion can be written equivalently as two coupled first-order differential equations).

There are advantages to writing the model in terms of first-order differential equations. We will see that whether the model consists of two first-order equations or one hundred first-order equations, the same matrix format can be used and it serves as a useful framework for understanding. Such models are amenable to numerical solution, as many numerical integration algorithms are based on solving systems of first-order equations. The same concept is applied to multi degree-of-freedom structural systems in Chapter 6, where matrix methods apply regardless of the number of degrees of freedom.

In this section, the state-space³⁹ model is introduced by way of an example and applied to single degree-of-freedom oscillators with control forces.

³⁷ We include the following quote in the hope that it is helpful: “Intuitively, the state may be regarded as a kind of information storage or memory or accumulation of past causes. We must, of course, demand that the set of internal states be sufficiently rich to carry all information about the past history of [the states] ... to predict the effect of the past upon the future. We do not insist, however, that the state is the *least* such information although this is often a convenient assumption.” (**Topics in Mathematical System Theory**, R.E. Kálmán, P.L. Falb, and M.A. Arbib, McGraw-Hill, 1969.)

³⁸ The number of state variables for a system will be equal to the order of the differential equation required to model it. The number of state variables is also equal to the number of independent energy storage elements in the system. For example, the dynamics of a mass-spring-damper oscillator can be represented by two state variables, since there are two independent energy storage elements: the mass, which can store kinetic energy, and the spring, which can store potential energy.

³⁹ The term *state-space* refers to an n -dimensional space with the state variables as axes. A specific state is a point in the state-space.

Example 10.8 State Variable Equations of a Mechanical Oscillator

Write the equations for the forced motion of a single degree-of-freedom mechanical system in state variable form.

Solution: The equation governing the forced motion of a single degree-of-freedom system is $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F_{ext}(t)$. To write this equation of motion in state variable form, we first need to define the state variables.⁴⁰ Here, we take the *displacement* and *velocity* of the mass m as the state variables, expressed in terms of the variables

$$\begin{aligned}x_1(t) &= y(t) \\x_2(t) &= \frac{dy}{dt} = \dot{y}.\end{aligned}$$

With these definitions, the equation of motion can be written as

$$\dot{x}_2(t) + \frac{c}{m}x_2(t) + \frac{k}{m}x_1(t) = \frac{F_{ext}(t)}{m}.$$

Then, the equivalent two first-order equations are

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t) \quad (10.20)$$

$$\begin{aligned}\dot{x}_2(t) &= \ddot{y}(t) \\&= -\frac{c}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{F_{ext}(t)}{m},\end{aligned} \quad (10.21)$$

or, in matrix-vector form,

$$\begin{aligned}\frac{d}{dt} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \\&+ \begin{Bmatrix} 0 \\ 1/m \end{Bmatrix} F_{ext}(t).\end{aligned} \quad (10.22)$$

The solution of this equation can then be used to find the displacement $y(t)$,

$$y = [1 \ 0] \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix},$$

and the velocity $\dot{y}$,

$$\dot{y} = [0 \ 1] \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}.$$

Equation 10.22 is generalized below. ■

⁴⁰Different sets of state variables can be defined for a given system, meaning that the state-space representation is *not* unique. In practice, it is desirable to select physical variables as the state variables, since they may be measured easily for use in feedback loops. Although the state variables are not unique, the number of state variables is unique.

Example 10.9 State Variable Equations of a Base-Excited System

Write the equations for a base-excited mechanical system in state variable form.

Solution: A structure excited at its base through stiffness and damping elements is governed by the equation

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2y(t) = 2\zeta\omega_n\dot{z}(t) + \omega_n^2z(t),$$

where $z(t)$ is the base displacement and $\dot{z}(t)$ is the base velocity. We are not able to follow the previous procedure to transform the second-order equation into two first-order equations due to the derivative term on the right-hand side. To get around this, we introduce a change of variables,

$$\begin{aligned}x_1(t) &= y(t) \\x_2(t) &= \dot{x}_1(t) - 2\zeta\omega_nz(t),\end{aligned}$$

and make the following transformation. We first substitute for $y(t)$ in the governing equation, giving

$$\ddot{x}_1(t) + 2\zeta\omega_n\dot{x}_1(t) + \omega_n^2x_1(t) = 2\zeta\omega_n\dot{z}(t) + \omega_n^2z(t).$$

Next, we replace $\dot{x}_1(t)$ by $x_2(t) + 2\zeta\omega_nz(t)$ and $\ddot{x}_1(t)$ by $\dot{x}_2(t) + 2\zeta\omega_n\dot{z}(t)$ to find

$$\begin{aligned}\dot{x}_2(t) + 2\zeta\omega_n\dot{z}(t) + 2\zeta\omega_n[x_2(t) + 2\zeta\omega_nz(t)] \\+ \omega_n^2x_1(t) = 2\zeta\omega_n\dot{z}(t) + \omega_n^2z(t),\end{aligned}$$

where the change of variables leads to the cancellation of the $\dot{z}(t)$ term. Simplifying and solving for $\dot{x}_2(t)$ results in

$$\dot{x}_2(t) = -\omega_n^2x_1(t) - 2\zeta\omega_nx_2(t) + (\omega_n^2 - [2\zeta\omega_n]^2)z(t).$$

Then, in matrix-vector form, the two first-order state equations are written as

$$\begin{aligned}\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \\&+ \begin{Bmatrix} 2\zeta\omega_n \\ \omega_n^2 - (2\zeta\omega_n)^2 \end{Bmatrix} z(t)\end{aligned}$$

with

$$y(t) = [1 \ 0] \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}.$$

State variable $x_1(t)$ is the output that can be measured. This procedure can be extended to cases where higher order derivatives appear on the right-hand side. ■

Equation 10.22 can be generalized for n first-order equations,

$$\{\dot{x}(t)\} = [A]\{x(t)\} + \{B\}u(t), \quad (10.23)$$

where $\{x\}$ is an n -dimensional *state vector*, u is the scalar *input* (assuming a single-input system), $[A]$ is an $n \times n$ *system matrix* or *state matrix*, and $\{B\}$ is an n -dimensional column vector ($n \times 1$) called the *control influence vector*. The output of the system is, in general,

$$y(t) = \{C\}\{x(t)\} + Du(t), \quad (10.24)$$

where $\{C\}$ is an n -dimensional row vector ($1 \times n$) called the *output vector* and D , which is most often not present, is a scalar constant called the *direct transmission* that directly links the output to the input. Matrix $[A]$ and vectors $\{B\}$ and $\{C\}$ are constants, that is, they are populated with constant elements, for a time-invariant system. Equation 10.23 is known as the *state equation* and Equation 10.24 is known as the *output equation*.

An important question is how to solve this set of first-order equations. We have a hint from the solution of the single first-order equation,

$$\dot{x}(t) = ax(t) + bu(t). \quad (10.25)$$

Taking the Laplace transform of both sides of Equation 10.25 gives

$$sX(s) - x(0) = aX(s) + bU(s).$$

Solving for $X(s)$, the transform of the solution is

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s),$$

and the inverse transform is obtained using *Borel's theorem*⁴¹ resulting in

$$x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)}b u(\tau)d\tau. \quad (10.26)$$

For n first-order equations in matrix-vector form, we seek the generalization of Equation 10.26.

We can write the state and output equations, 10.23 and 10.24, using a more compact bold notation,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \quad (10.27)$$

$$y(t) = \mathbf{c}\mathbf{x}(t) + Du(t), \quad (10.28)$$

where vectors are indicated by bold small letters and matrices by bold capital letters. Then, the generalization of Equation 10.26 is

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{b}u(\tau)d\tau, \quad (10.29)$$

where the matrix exponential

$$\exp(\mathbf{A}t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^nt^n}{n!} + \cdots$$

⁴¹See Appendix A.

converges for all finite t and any $\mathbf{A}$.

The matrix exponential function describes the unforced response of the system and is called the *state transition matrix*,

$$\Phi(t) = \exp(\mathbf{A}t).$$

Equation 10.29 can be written in terms of $\Phi(t)$,

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{b}u(\tau)d\tau,$$

where the unforced response is found by setting $u(\tau) = 0$,

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0).$$

For the second-order oscillator defined by Equations 10.20 and 10.21, the response is given by

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \exp\left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}t\right) \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} + \int_0^t \exp\left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}(t-\tau)\right) \begin{Bmatrix} 0 \\ \frac{1}{m} \end{Bmatrix} u(\tau)d\tau.$$

Equations in state-space format can be solved numerically by programs such as MATLAB.⁴²

10.7.1 Transfer Function from State Equation

As noted earlier, state equations represent the complete internal description of a system, whereas the transfer function⁴³ is the input-output representation only. The transfer function can be obtained uniquely from the state equations. To show this process, we start with the state-space representation of Equation 10.27 and take the Laplace transform of both sides assuming zero initial conditions to obtain

$$sX(s) = \mathbf{A}X(s) + \mathbf{b}U(s),$$

which can be solved for $X(s)$ to obtain

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s).$$

Substituting this expression in the Laplace transform of Equation 10.28 gives

$$Y(s) = \left[\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D\right]U(s).$$

The transfer function between the output and input can then be written as

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D. \quad (10.30)$$

⁴²MATLAB is a registered trademark of the MathWorks, Inc.

⁴³We recall that the transfer function of a single-input, single-output system is defined as the ratio of the Laplace transforms of the output and the input with zero initial conditions.

Recalling that the inverse of a matrix can be obtained by dividing its adjoint by its determinant, we may write Equation 10.30 as

$$G(s) = \frac{\mathbf{c} \operatorname{adj}(s\mathbf{I} - \mathbf{A}) \mathbf{b}}{\det(s\mathbf{I} - \mathbf{A})} + D = \frac{P(s)}{Q(s)}, \quad (10.31)$$

that is, the transfer function can be written as a ratio of two polynomials. The denominator polynomial $Q(s)$ is the determinant of $(s\mathbf{I} - \mathbf{A})$, and thus a polynomial of degree n if $\mathbf{A}$ is an $n \times n$ matrix. Furthermore, the poles of the transfer function are the roots of $Q(s)$ and the roots of $\det(s\mathbf{I} - \mathbf{A})$ are the eigenvalues of $\mathbf{A}$. This means then that *the poles of the transfer function are identical to the eigenvalues of $\mathbf{A}$.*

The numerator of the transfer function is given by

$$P(s) = \mathbf{c} \operatorname{adj}(s\mathbf{I} - \mathbf{A}) \mathbf{b} + DQ(s), \quad (10.32)$$

where the degree of the first term is less than or equal to $n-1$, and that of the second term is n , since D is a scalar. Hence, the transfer function will always be *proper* if D is nonzero and will be *strictly proper* if $D = 0$. In control theory, a proper transfer function is a transfer function in which the degree of the numerator polynomial does not exceed the degree of the denominator polynomial. A proper transfer function will never grow unbounded as the frequency approaches infinity. A strictly proper transfer function is a transfer function in which the degree of the numerator is less than the degree of the denominator. A strictly proper transfer function will approach zero as the frequency approaches infinity, which is true for all physical systems.

Example 10.10 Finding the Transfer Function

For the state-space model, given below in Equations 10.33 and 10.34, determine the transfer function.

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{Bmatrix} &= \begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} \\ &+ \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} u(t) \end{aligned} \quad (10.33)$$

$$y(t) = \begin{Bmatrix} 2 & 1 & -1 \end{Bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix}. \quad (10.34)$$

Solution: For this third-order system, the denominator polynomial is the characteristic polynomial given by the

determinant of $(s\mathbf{I} - \mathbf{A})$,

$$\begin{aligned} Q(s) &= \det \begin{bmatrix} s+2 & 0 & -1 \\ -1 & s+2 & 0 \\ -1 & -1 & s+1 \end{bmatrix} \\ &= s^3 + 5s^2 + 7s + 1. \end{aligned}$$

The adjoint of $(s\mathbf{I} - \mathbf{A})$ is

$$\begin{aligned} \operatorname{adj} \begin{bmatrix} s+2 & 0 & -1 \\ -1 & s+2 & 0 \\ -1 & -1 & s+1 \end{bmatrix} \\ = \begin{bmatrix} s^2 + 3s + 2 & 1 & s+2 \\ s+1 & s^2 + 3s + 1 & 1 \\ s+3 & s+2 & s^2 + 4s + 4 \end{bmatrix}. \end{aligned}$$

Since $D = 0$, the numerator polynomial is $P(s) = \mathbf{c} \operatorname{adj}(s\mathbf{I} - \mathbf{A}) \mathbf{b}$,

$$\begin{aligned} P(s) &= \begin{Bmatrix} 2 & 1 & -1 \end{Bmatrix} \\ &\times \begin{bmatrix} s^2 + 3s + 2 & 1 & s+2 \\ s+1 & s^2 + 3s + 1 & 1 \\ s+3 & s+2 & s^2 + 4s + 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \\ &= s^2 + 4s + 3. \end{aligned}$$

Thus, the transfer function is

$$G(s) = \frac{P(s)}{Q(s)} = \frac{s^2 + 4s + 3}{s^3 + 5s^2 + 7s + 1},$$

which is strictly proper. ■

It should be noted that the inverse problem of determining the state and output equations of a system from its transfer function has no unique solution. Several different state-space models yield the same transfer function. These state-space models can be written in different forms, known as *realizations*. Two standard forms are *physical realizations* and *canonical realizations*. The interested reader can find a full discussion of this topic in textbooks on system dynamics and modeling.

10.7.2 Controllability and Observability

There are two concepts especially important in the design of control systems and in particular to state-space analysis. These are the dual concepts of *controllability* and *observability*, which were first defined by R.E. Kálmán.⁴⁴ They tell us whether it is possible to control all the state variables of the system completely by

⁴⁴Rudolf E. Kálmán (May 19, 1930 - July 2, 2016) is famous for the Kálmán filter, a widely used mathematical technique to ex-

a suitable choice of an input and whether it is possible to reconstruct the state variables of a system from its input and output. These questions are meaningful to determine connections between the state variables of a system and the input and between the state variables and the output of the system.

Controllability

If an input to a system can be found that takes every state variable from the desired initial state to a desired final state in an arbitrary time interval, the system is said to be *controllable*; otherwise the system is *uncontrollable*. In other words, a system is controllable if all elements of the state vector can be driven from one state to another state by an appropriate application of the input. If a system does not satisfy this test, then one or more of the state variables can change independently of the input. Also, the output could change independently of the input. Hence, the system is uncontrollable.

A general dynamic system can have many modes. The system is said to be controllable if all the modes are controllable. The state vector $\mathbf{x}(t)$ of the system is controllable if the control, that is, input $u(t)$, can drive the initial state $\mathbf{x}(t_0)$ to any final state $\mathbf{x}(t_f)$ in a finite time. To determine whether an n^{th} order system whose state equation is given by Equation 10.27 is completely controllable, we introduce the matrix

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \dots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix}. \quad (10.35)$$

$\mathbf{M}_C$ is called the *controllability matrix*⁴⁵ and n is the number of state variables. The controllability matrix is $n \times n$, that is, it will have as many rows as state variables.

Either of two different mathematical tests can be used to determine if a system is controllable: (a) the rank test or (b) the determinant test. Both tests are based on the controllability matrix defined in Equation 10.35. In the rank test, a system is controllable if the rank of $\mathbf{M}_C$ is equal to n , the number of state variables. In the determinant test, a system is controllable if the determinant of $\mathbf{M}_C$ is nonzero. We note the connection between the two tests, since if the determinant of a matrix is nonzero,

tract a signal from a series of incomplete and noisy measurements. Kálmán's ideas on filtering were initially met with skepticism, so much so that his work was not accepted in electrical engineering journals and he first published his results in a mechanical engineering journal.

⁴⁵For details on the derivation of the controllability matrix, the reader is referred to the book **Control System Design – An Introduction to State-Space Methods**, B. Friedland, McGraw-Hill, 1987.

The controllability matrix is also called the *reachability matrix*. Reachability addresses whether it is possible to reach all points in the state-space in a transient fashion. The interested reader can find more in the book **Feedback Systems: An Introduction for Scientists and Engineers**, K.J. Aström and R.M. Murray, Princeton University Press, 2008.

the matrix is invertible or non-singular and hence attains a full rank.

Example 10.11 Test of Controllability

Determine the controllability of the system described by the state equation

$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u(t).$$

Solution: The controllability matrix for this system is

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix},$$

which is singular. Thus, the system is uncontrollable. It can be explained by writing the two differential equations separately,

$$\begin{aligned} \dot{x}_1(t) &= -3x_1(t) \\ \dot{x}_2(t) &= -2x_2(t) + u(t). \end{aligned}$$

It is clear that although $x_2(t)$ can be changed by $u(t)$, $x_1(t)$ is not influenced by the input, since it is not coupled either directly to the input or to $x_2(t)$. Thus, state variable $x_1(t)$ is uncontrollable. ■

Observability

The concept of *observability* implies an ability to determine all elements of the state vector of the system from knowledge of the input and output over an arbitrary time interval. Observability depends on the relationship between the state variables and the system output, that is, if measurement of the output allows determination of all the state variables. Since the input is assumed known, observability refers to our ability to determine information about all the modes of the system by monitoring the output. A system is observable if and only if the initial state can be determined from knowledge of the input and output over a finite interval of time.⁴⁶

A system is observable if given the input $u(t)$ there exists a finite time such that with knowledge of $u(t)$, $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, and D the initial state $\mathbf{x}(t_0)$ can be determined.

⁴⁶This statement follows since, by definition, if we know the initial state of the system we can determine the future output and states for a specified input.

The observability matrix is defined as

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \vdots \\ \mathbf{cA}^{n-1} \end{bmatrix}$$

or, equivalently,

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{c}^T & (\mathbf{cA})^T & (\mathbf{cA}^2)^T & \dots & (\mathbf{cA}^{n-1})^T \end{bmatrix}^T,$$

where superscript T denotes the transpose and n is the number of state variables.

Just as with controllability, either of two tests can be used to determine if a system is observable: (a) the rank test or (b) the determinant test. In the rank test, a system is observable if the rank of $\mathbf{M}_O$ is equal to n , the number of state variables. In the determinant test, a system is observable if the determinant of $\mathbf{M}_O$ is nonzero. Again, we note the connection between the two tests, since if the determinant of a matrix is nonzero, the matrix is invertible or non-singular and hence attains a full rank.

Example 10.12 Test of Observability

Determine the observability of the system described by the same state equation,

$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u(t),$$

and the output equation,

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}.$$

Solution: From Example 10.11 it was seen that this system is uncontrollable. We now form the observability matrix,

$$\begin{aligned} \mathbf{M}_O &= \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{c}^T & (\mathbf{cA})^T \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \end{aligned}$$

which is singular. The system is unobservable since state variable $x_1(t)$ does not affect the output $y(t)$, nor affect state variable $x_2(t)$, which is coupled to the output. ■

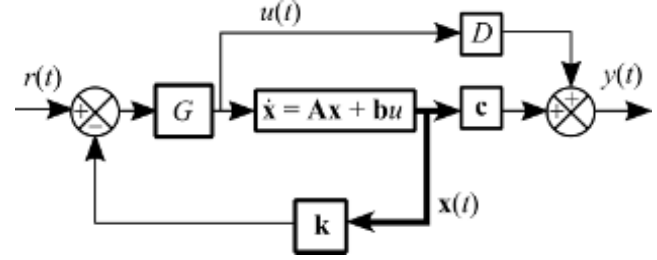


Figure 10.26: Closed-loop system with state-variable feedback. (Thin lines represent scalar signals. Bold lines represent vector signals.)

It can be shown that a linear time-invariant single-input, single-output (SISO) system is completely controllable and observable if and only if the numerator and denominator polynomials of its transfer function are *co-prime*, that is, they do not have a common factor, with the exception of a constant. Thus, the state equations will be both controllable and observable if and only if there are no pole-zero cancellations. In this case, the state equations are said to be a minimal realization of the given transfer function.

10.7.3 State Variable Feedback

Consider a closed-loop system with state-variable feedback, as shown in the block diagram of Figure 10.26. The state vector is fed back⁴⁷ according to the linear control law,

$$u(t) = G[r - \mathbf{k}\mathbf{x}(t)], \quad (10.36)$$

where $\mathbf{k}$ is a vector whose elements $(k_1, k_2, \dots, k_n)$ multiply each corresponding state variable and G is a scalar gain. Equation 10.36 can be written in expanded form as

$$u(t) = Gr - k_1x_1(t) - k_2x_2(t) - \dots - k_nx_n(t).$$

The control law implements negative state variable feedback.

Substituting Equation 10.36 into the state equation 10.27 gives

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}G[r(t) - \mathbf{k}\mathbf{x}(t)],$$

which can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}^*\mathbf{x}(t) + r(t)\mathbf{b}^*,$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{b}^*\mathbf{k} \quad \text{and} \quad \mathbf{b}^* = G\mathbf{b}.$$

Matrix $\mathbf{A}^*$ is the *closed-loop system matrix*.

The elements of $\mathbf{k}$ and scalar G can be viewed as the controller parameters that can be adjusted to achieve the

⁴⁷This assumes that all the elements of the state vector, that is, all the state variables, can be measured.

desired stability and performance. This control problem is called a *pole placement problem* or *eigenvalue assignment problem*.

Consider first the poles or eigenvalues of the open-loop system, which are the roots of the *open-loop characteristic polynomial*,

$$\det(s\mathbf{I} - \mathbf{A}) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n.$$

One or more of the poles of the open-loop system may be in the right-half of the s -plane, meaning the system is unstable, or the poles may be in locations giving undesirable performance. By state variable feedback we are able to shift the poles to desired locations to create a closed-loop system that meets the desired performance. In particular, the poles or eigenvalues of the closed-loop system are the roots of the *closed-loop characteristic polynomial*,

$$\det(s\mathbf{I} - \mathbf{A}^*) = s^n + \beta_1 s^{n-1} + \cdots + \beta_{n-1}s + \beta_n.$$

By adjusting the gains, that is, the elements of $\mathbf{k}$ and G , we can control the behavior of the closed-loop system.

Example 10.13 State Feedback Control

Design a state feedback control law that gives (a) the desired pole locations $s = -6$ and $s = -3 \pm 4i$ and (b) zero steady-state error to a unit step input for the system,

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{Bmatrix} &= \begin{bmatrix} 0 & -6 & -20 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} \\ &+ \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} u(t) \\ y(t) &= \begin{Bmatrix} 0 & 1 & 2 \end{Bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix}. \end{aligned}$$

Solution: The open-loop poles can be found from the roots of the open-loop characteristic polynomial, $\det(s\mathbf{I} - \mathbf{A}) = s^3 + 6s + 20$ and are $s = -2$ and $s = +1 \pm 3i$. Since the complex conjugate pair is in the right-half plane, the open-loop system is unstable. The transfer function⁴⁸ of the open-loop system can be found to be

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^3 + 6s + 20},$$

which, as expected, has the open-pole characteristic polynomial.

⁴⁸The state equation is written in a special form called the *controller canonical form*. In this form, the characteristic polynomial, that is, the denominator of the transfer function, can be obtained by inspection.

We want the poles of the closed-loop system to be $s = -6$ and $s = -3 \pm 4i$. Thus, we can write the transfer function of the closed-loop system as

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{G(s+2)}{(s+6)(s^2 + 6s + 25)} \\ &= \frac{G(s+2)}{s^3 + 12s^2 + 61s + 150}. \end{aligned}$$

For zero steady-state error to step inputs, we must have unity DC gain, from which we find $G = 50$. From the closed-loop transfer function, we can obtain the closed-loop system matrix as

$$\mathbf{A}^* = \begin{bmatrix} -12 & -61 & -150 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\mathbf{A}^* = \mathbf{A} - G\mathbf{b}\mathbf{k}$ we can write

$$\mathbf{b}\mathbf{k} = \frac{1}{G}(\mathbf{A} - \mathbf{A}^*)$$

$$\begin{aligned} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & k_3 \end{Bmatrix} &= \frac{1}{75} \begin{bmatrix} 12 & 61 & 150 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= \frac{1}{75} \begin{bmatrix} 12 & 61 & 150 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

giving

$$\mathbf{k} = \begin{Bmatrix} k_1 & k_2 & k_3 \end{Bmatrix} = \begin{Bmatrix} 0.16 & 0.73 & 1.73 \end{Bmatrix}$$

as the design results. ■

10.8 Multivariable Control

In this section, we consider the control of multivariable systems, sometimes referred to as *multiple-input, multiple-output* (MIMO) systems. State-space models (time-domain models) are preferred over transfer functions (frequency-domain models), as they provide information about the internal behavior of the system as well as the input-output behavior. However, it is possible to represent the input-output relationships in a MIMO system through a transfer function matrix. Each element of this matrix is a transfer function relating a particular output to a specific input, assuming that all other inputs are zero.

The challenge of multivariable control is due to the coupling that exists between different inputs and outputs. This coupling is represented mathematically by

the nonzero off-diagonal elements in the transfer function matrix. One approach – if it is possible – is to decouple the system and then treat each loop independently. Unfortunately, this is not always possible. Historically, controls engineers neglected the coupling terms in the hope that they did not play a significant role. Sometimes this worked; many times it did not.

In addition to the advantage of the internal representation offered by the state-space method, there is an advantage notationally. We can study multivariable systems without the need for changing the notational framework used with *single-input, single-output* (SISO) systems.

10.8.1 State and Output Equations

Generalizing from our discussion in Chapter 10 on vibration control of SISO systems, we can write the state-space representation for an MIMO system in terms of the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (10.37)$$

and the output equation

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \quad (10.38)$$

where, as before, vectors are indicated by bold small letters and matrices by bold capital letters. The vectors are the $n \times 1$ state vector $\mathbf{x}(t)$ (an n -dimensional column vector), the $m \times 1$ input vector $\mathbf{u}(t)$ (an m -dimensional column vector), and the $p \times 1$ output vector $\mathbf{y}(t)$ (a p -dimensional column vector). The matrices are the $n \times n$ system matrix $\mathbf{A}$, the $n \times m$ control influence matrix $\mathbf{B}$, the $p \times n$ output matrix $\mathbf{C}$, and the $p \times m$ direct transmission matrix $\mathbf{D}$.

It is possible to determine the transfer function matrix from the state-space representation, as we did previously. (The procedure from Section 10.7.1 applies for MIMO systems.) Taking the Laplace transform of both sides of Equation 10.37 and assuming zero initial conditions gives

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s),$$

which can be solved for $\mathbf{X}(s)$ to obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s).$$

Substituting this expression in the Laplace transform of Equation 10.38 gives

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}] \mathbf{U}(s).$$

The transfer function matrix between the output and input vectors can then be written as

$$\mathbf{G}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

where $\mathbf{G}(s)$ is a $p \times m$ matrix with a total of pm transfer functions. For every input m there are p transfer functions (one for each output).

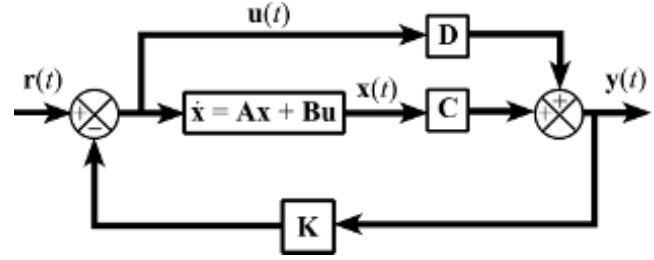


Figure 10.27: Output feedback for MIMO system.

10.8.2 Controllability and Observability

The dual concepts of *controllability* and *observability* generalize for MIMO systems. These properties tell us whether it is possible to control all the state variables of the system completely by a suitable choice of inputs and whether it is possible to reconstruct the state variables of a system from its inputs and outputs.

To determine whether an n^{th} order system whose state equation is Equation 10.37 is completely controllable, we introduce the $n \times n$ *controllability matrix* $\mathbf{M}_C$,

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}.$$

A system is controllable if the rank of $\mathbf{M}_C$ is equal to n , the number of state variables. Equivalently, a system is controllable if the determinant of $\mathbf{M}_C$ is nonzero.

A system is observable if given the input $\mathbf{u}(t)$ there exists a finite time such that with knowledge of $\mathbf{u}(t)$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ the initial state $\mathbf{x}(t_0)$ can be determined. The *observability matrix* $\mathbf{M}_O$ is

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}.$$

A system is observable if the rank of $\mathbf{M}_O$ is equal to n , the number of state variables. Equivalently, a system is observable if the determinant of $\mathbf{M}_O$ is nonzero.

10.8.3 Closed-loop Feedback of MIMO Systems

One method for feedback, called *output feedback*, is to multiply the output by a matrix $\mathbf{K}$, negatively feed the resulting vector back, and add it to the reference input. This scheme is shown in the block diagram of Figure 10.27, where we have used bold lines to indicate vector signals. Mathematically,

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{y}(t) + \mathbf{r}(t).$$

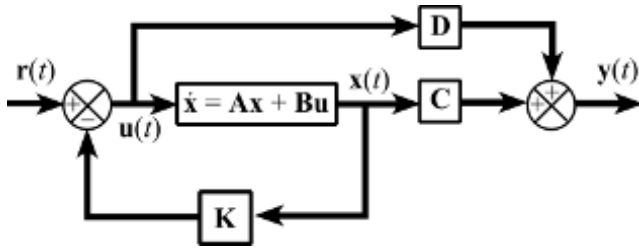


Figure 10.28: State feedback for MIMO system.

Substituting into the state and output equations gives

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{y}(t) + \mathbf{B}\mathbf{r}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) - \mathbf{D}\mathbf{K}\mathbf{y}(t) + \mathbf{D}\mathbf{r}(t).$$

Solving the output equation for $\mathbf{y}(t)$ and substituting into the state equation gives

$$\begin{aligned} \dot{\mathbf{x}}(t) = & \left[\mathbf{A} - \mathbf{B}\mathbf{K}(\mathbf{I} + \mathbf{D}\mathbf{K})^{-1}\mathbf{C} \right] \mathbf{x}(t) \\ & + \mathbf{B} \left[\mathbf{I} - \mathbf{K}(\mathbf{I} + \mathbf{D}\mathbf{K})^{-1}\mathbf{D} \right] \mathbf{r}(t). \end{aligned}$$

Similarly, for the output equation

$$\mathbf{y}(t) = (\mathbf{I} + \mathbf{D}\mathbf{K})^{-1}\mathbf{C}\mathbf{x}(t) + (\mathbf{I} + \mathbf{D}\mathbf{K})^{-1}\mathbf{D}\mathbf{r}(t).$$

In many systems, there is no direct transmission of the input to the output, meaning matrix $\mathbf{D}$ is absent. By removing $\mathbf{D}$ the equations reduce to the simpler form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= [\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}] \mathbf{x}(t) + \mathbf{B}\mathbf{r}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned}$$

Another method for feedback, called *state feedback*, is to multiply the state vector $\mathbf{x}(t)$ by a matrix $\mathbf{K}$, negatively feed the resulting vector back, and add it to the reference input. This is shown in the block diagram of Figure 10.28.

10.9 Stochastic Control

Stochastic control is a specialty area of controls engineering that addresses the analysis and design of control systems in which the input, state, and/or output variables change randomly. A stochastic control problem must account for the probability of uncertainty in the control system variables. It may also include random noise and disturbances in the closed-loop system to determine their effects on the model and the controller.

If the randomness can be characterized, for example, it can be represented by a Gaussian distribution, it is often possible to extend the results of deterministic closed-loop control to stochastic control. In linear, stationary systems, the time-domain relations for stochastic variables can be generalized and the input-output relation in terms

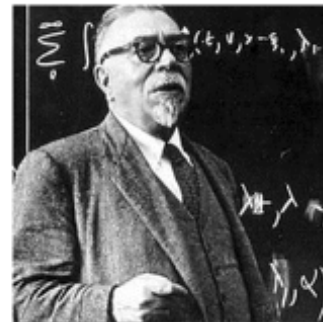
of a transfer function can be expressed in a form that closely parallels those of deterministic signals. In both cases, the derivations are mathematically challenging.

The subject of stochastic control requires a thorough understanding of the theory of random processes. An interesting development in this area has been the theory of optimal filtering and prediction, proposed independently by Wiener in the U.S. and Kolmogorov in the U.S.S.R. in the 1940s. It was followed in the 1960s by the Kálmán filter theory, which can be used to determine optimal estimates of the states of a system from noise-contaminated measurements of its inputs and outputs.

Although the derivations and discussion are beyond the scope of this textbook, we wanted to introduce the topic of stochastic control. The interested reader can find references in more advanced controls textbooks.⁴⁹

NORBERT WIENER

(November 26, 1894 – March 18, 1964)



Contribution: Norbert Wiener was an American theoretical and applied mathematician who founded *cybernetics*, a field whose subject is the notion of feedback. Cybernetics has important implications for engineering, computer science, biology, philosophy, and the organization of society. He was a pioneer in the study of stochastic and noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems.

Life Sketch: Wiener was born in Columbia, Missouri. He was the first child of Leo Wiener and Bertha Kahn. He was educated at home. Thanks to his father's tutelage and his own abilities, Wiener was a child prodigy.

He graduated from Ayer High School in Ayer, Massachusetts in 1906 at age 11, and then entered Tufts College. He was awarded a BA degree in mathematics in 1909 at the age of 14, and then began graduate studies in zoology at Harvard. In 1910 he transferred to Cornell to study philosophy.

The next year he returned to Harvard, while still continuing his studies in philosophy. Wiener earned a PhD from

⁴⁹Three recommended books are: **Introduction to Stochastic Control Theory**, K.J. Aström, Dover, 2006; **Optimal Control and Estimation**, R.F. Stengel, Dover, 1994; and **Control and Dynamic Systems**, Y. Takahashi, M.J. Rabins, and D.M. Auslander, Addison-Wesley, 1970.

Harvard in 1912, when he was 18 years old, for a dissertation on mathematical logic.

In 1914, Wiener traveled to Europe, to study under Bertrand Russell and G. H. Hardy at Cambridge University, and under David Hilbert and Edmund Landau at the University of Göttingen. In 1915-16, he taught philosophy at Harvard, then worked for General Electric and wrote for the *Encyclopedia Americana*. When World War I broke out, he was invited to work on ballistics at the Aberdeen Proving Ground in Maryland, where Wiener, an eventual pacifist, worked from 1917-18. Living and working with other mathematicians deepened his interest in mathematics.

After the war, Wiener sought an academic position at Harvard but was denied (because he was Jewish). He did secure an instructor position in mathematics at MIT, where he spent the remainder of his career, rising to rank of Professor.

In 1926, Wiener returned to Europe as a Guggenheim scholar. He spent most of his time at Göttingen and at Cambridge (with Hardy), working on Brownian motion, the Fourier integral, Dirichlet's problem, harmonic analysis, and the Tauberian theorems.

In 1926, Wiener's parents arranged his marriage to a German immigrant, Margaret Engemann. They had two daughters. Wiener died in Stockholm, Sweden.

Noteworthy: During World War II, Wiener's work on the automatic aiming and firing of anti-aircraft guns led Wiener to communication theory and eventually to formulate cybernetics. After the war, his prominence helped MIT recruit a crack research team in cognitive science, made up of researchers in neuropsychology and the mathematics and biophysics of the nervous system, that made pioneering contributions to computer science and artificial intelligence. Shortly after the group was formed, Wiener broke off all contact with its members. It is not clear why this split occurred.

Wiener went on to break new ground in cybernetics, robotics, computer control, and automation. He shared his theories and findings with others, including Soviet researchers. Wiener's connections with them placed him under suspicion during the Cold War. He was a strong advocate of automation to improve the standard of living and to overcome economic underdevelopment. His ideas became influential in India, whose government he advised during the 1950s.

Wiener declined an invitation to join the Manhattan Project. After the war, he became increasingly concerned with what he saw as political interference in scientific research, and the militarization of science. His article "A Scientist Rebels" in the January 1947 issue of *The Atlantic Monthly* urged scientists to consider the ethical implications of their work. After the war, he refused to accept any government funding or to work on military projects.

Wiener won the Bôcher Prize in 1933 and the National Medal of Science in 1963 (presented by President Johnson at a White House Ceremony in January 1964), shortly before his death.

10.10 Concepts Summary

The ultimate objective of any control system is to improve the performance of a dynamic system. This chapter is meant to be an introduction to the topic of feedback control and show *connections* between the subjects of engineering vibration and control. Some basic concepts regarding vibration control have been discussed, the most important of these being performance, feedback control, sensitivity, and stability, to varying degrees of detail.

A large portion of control analysis and design is accomplished in the frequency domain. However, within the context of a vibration text, we wanted to convey that control analysis and design can be carried out in the time domain, the purview of modern control.

10.11 Quotes

- "If everything seems under control, you're just not going fast enough." (Mario Andretti)
- "Though no one can go back and make a brand new start, anyone can start from now and make a brand new ending."
- Student: "What is a common filter?"
Professor: "I said Kálmán filter."
- "I want to reiterate that we need to iterate again." (Mark Nagurka)
- "Engineers ... are not superhuman. They make mistakes in their assumptions, in their calculations, in their conclusions. That they make mistakes is forgivable; that they catch them is imperative. Thus it is the essence of modern engineering not only to be able to check one's own work but also to have one's work checked and to be able to check the work of others." (Henry Petroski)
- "Wonder rather than doubt is the root of all knowledge." (Abraham Joshua Heschel)
- "Pay attention. It's all about paying attention. Attention is vitality." (Susan Sontag)
- "The best way to predict the future is to create it." (Peter Drucker)
- "You can get the right answer if you make an even number of mistakes, but not if you make an odd number of mistakes." (Milton Stecher, 1972, Cooper Union Physics Professor)

10.12 Problems

Problems for Section 10.1 – Introduction

1. In each of the following systems, identify the control component(s), if possible. Describe qualitatively how the control system affects system behavior: (a) airplane wing, (b) elevator, (c) standard transmission in car, (d) automatic transmission in car, (e) hand holding a pen and writing, (f) computer.
2. Identify advantages and disadvantages of passive and active control.

Problems for Section 10.3 – Feedback Control

3. Derive the transfer functions for the following governing equations of motion:
 - (a) $\ddot{x} + 2\dot{x} + 4x = \cos \omega t$
 - (b) $m\ddot{x} + c\dot{x} + kx = A \cos \omega t$
 - (c) $m\ddot{x} + c\dot{x} + kx = A_1 \cos \omega t + A_2 \sin \omega t$
 - (d) $m\ddot{x} + kx = u(t)$, $u(t) = 1$, $t \geq 0$
 - (e) $m\ddot{x} + c\dot{x} + kx = e^{-\omega t} \cos \Omega t$.
4. For each of the equations in the previous problem, discuss how appropriate choices of m , c , and k can be made to either maximize or minimize the effect of the transfer function. Can you think of applications where either maximum or minimum effects are desirable?

Problems for Section 10.4 – Performance of Control Systems

5. For the step response $x_s(t)$ governed by the equation

$$\ddot{x}_s + 2\zeta\omega_n\dot{x}_s + \omega_n^2 x_s = u(t),$$

where $u(t)$ is the unit step function, evaluate rise time t_r , peak time t_p , overshoot, and settling time where $\delta = 0.02$ for the following cases:

- (a) $\zeta = 0.01$, $\omega_n = 1$ rad/s, (b) $\zeta = 0.01$, $\omega_n = 2$ rad/s, (c) $\zeta = 0.01$, $\omega_n = 0.5$ rad/s, (d) $\zeta = 0.1$, $\omega_n = 1$ rad/s, (e) $\zeta = 0.1$, $\omega_n = 2$ rad/s, (f) $\zeta = 0.1$, $\omega_n = 0.5$ rad/s, (g) $\zeta = 0.5$, $\omega_n = 1$ rad/s, (h) $\zeta = 0.5$, $\omega_n = 2$ rad/s, (i) $\zeta = 0.5$, $\omega_n = 0.5$ rad/s.
6. For the equation of motion $m\ddot{x} + c\dot{x} + kx = F(t)$, with $m = 9$ kg, $c = 4$ N-s/m, $k = 4$ N/m, and $F(t)$ is the unit step load, find the response. If the response is oscillatory, determine the possible modifications that need to be implemented to make the system critically damped. Plot both original and modified step response.

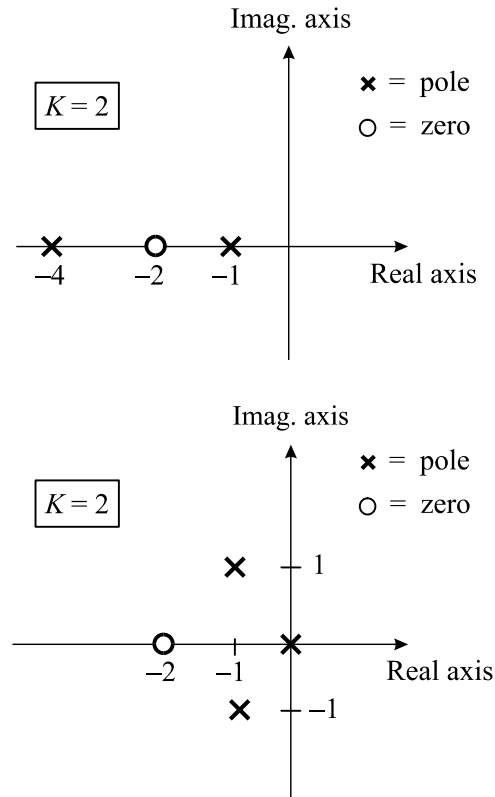


Figure 10.29: Pole-zero locations in the complex plane.

7. What applications exist where speed of response is more important than accuracy of response? Are there applications where the opposite is true? For what applications are both of critical importance?
8. What is the maximum value of the step response as a function of ζ ?
9. Identify the poles and zeros for the following second-order systems, and discuss system stability:
 - (a) $\ddot{y} + 2\dot{y} + y = A \cos 3t$
 - (b) $\ddot{y} + \dot{y} + 0.1y = A \cos 3t$
 - (c) $\ddot{y} + 300\dot{y} + 10y = A \cos 3t$
 - (d) $\ddot{y} + 2\dot{y} + y = A_1 x + A_2 \dot{x}$
 - (e) $\ddot{y} + y = A \cos \omega t$
 - (f) $\ddot{y} + y = B(x(t) + \dot{y})$.
10. For the given transfer functions, determine the governing differential equation of motion:
 - (a) $T(s) = (2s + 2)/(s^2 + 2s + 5)$
 - (b) $T(s) = (s - 1)/(3s^2 + 4)$
 - (c) $T(s) = (s + 1)/(s^2 + s + 1)$.
11. For a gain factor $K = 2$, use the pole-zero maps in Figure 10.29 to find each transfer function.

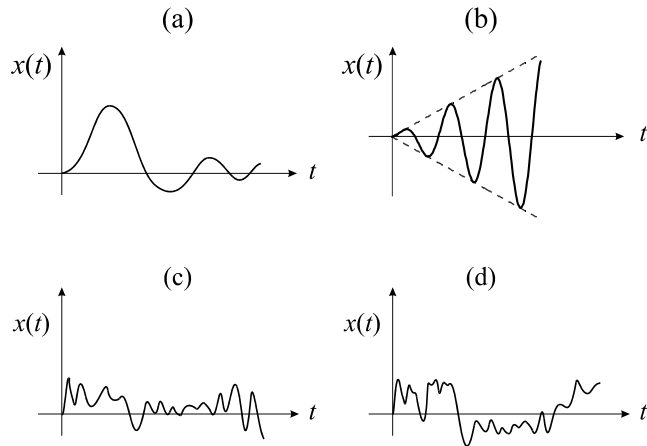


Figure 10.30: Four possible time-histories.

12. For responses (a) to (d) in Figure 10.30, explain why the response appears stable or unstable.

Problems for Section 10.5 – Control of Response

13. For the transfer function

$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{K_p}{Js^2 + K_p},$$

solve for $\theta_o(t)$ in terms of J , K_p , and $\theta_i(t)$. How do variations in the values of parameters J and K_p affect the behavior of $\theta_o(t)$? Discuss and show if a bounded $\theta_i(t)$ can be selected to destabilize response $\theta_o(t)$.

14. For a rotating element with PD controller, the transfer function is given by Equation 10.14, repeated here,

$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{K_p(1 + T_d s)}{Js^2 + K_p T_d s + K_p}.$$

Explain how to select control parameters K_p and T_d that meet performance specifications, for example, the shortest settling time for a step input. What is the effect of parameter J on the response?

Problems for Section 10.6 – Sensitivity to Parameter Variations

15. A system transfer function is given by

$$T(s) = \frac{Ks}{Ks^2 + 1}.$$

Derive the sensitivity function and determine the value(s) of K that minimize sensitivity. Plot $T(s)$ for $K = 1, 10, 100$ on one set of axes and draw conclusions from the comparison. Plot the sensitivity function for each of the K values, also on one set of axes. Which value of K reduces sensitivity?

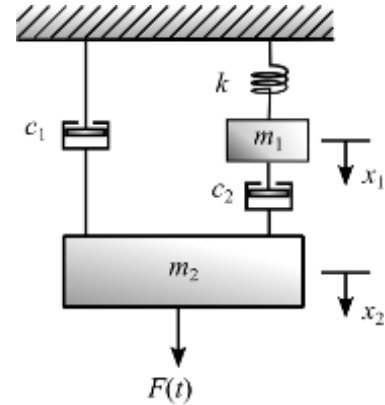


Figure 10.31: Mechanical system with two masses, two dampers, and one spring acted on by an external force.

16. Repeat Problem 15 for the transfer function

$$T(s) = \frac{Ks^2}{Ks^2 + s + 1}.$$

17. For the general transfer function

$$T(s) = \frac{C_1(s) + pC_2(s)}{C_3(s) + pC_4(s) + p^2C_5(s)},$$

where $C_i(s)$ are polynomials in s , find the general sensitivity as a function of parameter p .

Problems for Section 10.7 – State Variable Models

18. For the mechanical system shown in Figure 10.31, determine the differential equations of motion. Then, determine the state-space model where the input is the force and the output is the displacement of mass acted upon by the force.

19. Calculate the transfer function of the linear system described by the following state and output equations,

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{Bmatrix} &= \begin{bmatrix} -2 & -5 & -4 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} \\ &+ \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix}. \end{aligned}$$

Is the transfer function proper, strictly proper, or not proper?

20. Determine the transfer function of state and output equations in Example Problem 10.10 if $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ are unchanged but $D = 1$. Is the transfer function proper, strictly proper, or not proper?
21. Determine the controllability and observability of the system

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \\ &\quad + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}. \end{aligned}$$

If the system is uncontrollable or unobservable, explain why.

Chapter 11

Nonlinear Vibration

“Solutions no longer add.”

11.1 Introduction

So far we have focused primarily on linear systems. We considered briefly a nonlinear system with Coulomb friction.¹ In practice, however, all physical systems have some nonlinearities. As we will see in this chapter, even a small nonlinearity can significantly change the behavior of a system. Accounting for nonlinearities is often necessary to model real-world behavior accurately.

Nonlinear systems differ from linear systems in several ways. Perhaps the most important differences are that the principle of linear superposition and the property of homogeneity are not valid for nonlinear systems. The response of a nonlinear system can be highly dependent on the magnitude of the input, making it hard to predict even if the response to standard inputs is known.

Similarly, the stability of a nonlinear system is not just a characteristic of the system, as it is for a linear system. It may depend on the magnitude of the input and the initial conditions. In a linear system, if the input is sinusoidal then the output in steady-state is sinusoidal with, in general, a different amplitude and phase than that of the input. In contrast, in a nonlinear system, the steady-state output may contain harmonics, sub- and superharmonics of the input.

There are a number of phenomena that are unique to nonlinear systems, such as limit cycles and jump resonances. In limit cycle behavior, the system may produce oscillations of a certain period and amplitude, not necessarily sinusoidal and independent of the magnitude of the input or the initial conditions. In jump resonances, jumps occur in the magnitude and the phase as the input frequency is changed near resonance. (This is in contrast to smooth frequency response plots for linear systems.) The jumps or discontinuities can occur as the input frequency is increased or decreased.

¹Systems with Coulomb friction were converted to piecewise linear systems.

In this chapter we will investigate how such nonlinear characteristics arise and introduce some analytical methods for studying nonlinear systems. We will see that linear systems are comparatively simpler and much more tractable than nonlinear systems. The world of nonlinear systems has a much richer range of characteristics than that of linear systems, and more completely reflects real-world behavior.

Nonlinear vibration occurs in systems modeled by nonlinear governing equations of motion. The equations are nonlinear due to nonlinear models, nonlinear functions of materials, or nonlinear geometric properties. If the forcing is *parametric*,² this is considered to be a nonlinear effect in part because the stability of the system depends on the values of the parametric function.

Consider as an example the following differential equation for a simple pendulum,

$$ml^2\ddot{\theta} + c\dot{\theta}^2 + mgl \sin \theta = 0,$$

where there are two types of nonlinearities. The damping term is nonlinear because $\dot{\theta}$ is raised to a power other than one; the stiffness term is nonlinear because $\sin \theta$ is a nonlinear trigonometric function. Mathematically, the model of pendulum motion is nonlinear due to the presence of these two nonlinear terms, both related to “system properties” that appear on the left-hand side of the equation of motion.

Since the governing equation is nonlinear, its solution violates the property of homogeneity. If the solution for the initial conditions, say $\theta(0) = \theta_{10}$ and $\dot{\theta}(0) = 0$, is denoted $\theta_1(t)$, and the solution for the initial conditions, say $\theta(0) = \theta_{20} = 2\theta_{10}$ and $\dot{\theta}(0) = 0$, is denoted $\theta_2(t)$, then $\theta_2(t) \neq 2\theta_1(t)$. In addition, the solution violates the principle of linear superposition, that is, if the initial

²A parametric term is one that appears on the left-hand or system side of the equation of motion. It is generally considered to be a nonlinear input to the system. A generic example is

$$m\ddot{x} + c(t)\dot{x} + kx = 0,$$

where $c(t)$ is a parametric damping function of time. Depending on the value of $c(t)$, the system may exhibit stable or unstable behavior as it moves through time.

conditions are $\theta(0) = \theta_{10} + \theta_{20}$ and $\dot{\theta}(0) = 0$, the solution is not $\theta_1(t) + \theta_2(t)$.

These concepts can be generalized for the forced case in which the pendulum is acted upon by an external torque, $T(t)$. For the forced equation of motion,

$$ml^2\ddot{\theta} + c\dot{\theta}^2 + mgl\sin\theta = T_1(t) + T_2(t),$$

the complete response $\theta(t)$ due to both torques is not equal to $\theta_1(t) + \theta_2(t)$, where $\theta_1(t)$ is the response due only to $T_1(t)$ and $\theta_2(t)$ is the response due only to $T_2(t)$.

Because of the complexity of nonlinear behavior, there is no single overarching principle that governs the solution of nonlinear equations. There are, however, general approaches that can be utilized for solving certain classes of nonlinear differential equations.³

The study of nonlinear system behavior is broadly categorized as either *qualitative* or *quantitative*. Qualitative approaches are concerned less with response time histories and more with the stability characteristics of the system in the neighborhood of an equilibrium point or condition. Quantitative approaches, on the other hand, are devoted to the derivation of (usually approximate) solutions to the governing nonlinear equations of motion.

Perturbation methods^{4,5} are quantitative methods used to approximate the response of systems with small nonlinearities. If the nonlinearity is large, numerical methods are most often needed to determine the response time history. Of particular interest in solving nonlinear vibration problems^{6,7} are systems that have periodic solutions. A nonlinear equation may have periodic solutions as well as nonperiodic solutions.

One of the interesting and challenging aspects of solving nonlinear equations is that they generally have numerous solutions. Depending on what the analyst needs to find out about the behavior, different techniques may be appropriate. In this chapter we are interested in periodic solutions of slightly nonlinear oscillators. Here, the focus is on nonlinear single degree-of-freedom oscillators that are simple enough so the key concepts can be introduced. Although higher-order nonlinear systems and nonlinear continuous systems are important, they are beyond our scope.

Autonomous Equations

We can define two classes of equations:

Definition 1 An *autonomous* equation is one in which time t does not appear explicitly.

Definition 2 A *non-autonomous* equation is one in which t appears explicitly, for example, in the forcing term.

The difference between non-autonomous and autonomous systems is that the solutions of the former have the period of the external excitation, or, more generally, are in a rational ratio of this period, whereas the period of the autonomous system is determined by the parameters of the differential equation itself.

In an autonomous differential equation, t can be replaced by $t + t_0$ and still have the same solution. This means that the time axis can be translated arbitrarily and that the origin can be selected so that the initial velocity $dx/dt = 0$. This is not true in a non-autonomous equation.

Sometimes for non-autonomous equations, additional conditions such as a zero initial velocity are imposed. Although the resulting analysis is less general, it may still be useful. On occasion we add such conditions in this chapter.

 **NOISE FROM LIGHTS.** Incandescent, fluorescent, and LED lights sometime hum or buzz. The source of this audible annoyance usually stems from vibration caused by the voltage. The vibration is not always due to the bulb. Devices on the same electrical line can lead to noisy lines and bulbs.

Incandescent light bulbs are normally very quiet. When they hum it is usually because of a dimmer switch. TRIAC dimmers, the most common type of incandescent dimmer switch, remove part of the voltage from the line to decrease the amount of energy reaching the bulb. This leads to "on-off" switching of the bulb that dims the light output. The on-off voltage excites and cools the filament, causing vibration, which leads to an annoying buzzing or humming noise called "filament sing."

In addition to flickering, fluorescent lights sometimes buzz. While this can be caused by a bad bulb or starter, it is usually caused by an aging ballast (the device used to regulate voltage for a fluorescent lamp). Most buzzing comes from re-striking issues in the fluorescent arc when the lamp is cycled on and off. Constant re-striking happens when a ballast or lamp ages and can no longer regulate voltage properly. It can also be caused by extremely cold conditions.

LEDs lack a filament or firing arc so there are no "moving parts" to cause humming. LEDs still suffer from electromagnetic (EM) hum caused by incorrect dimming or EM interference by other devices. If LED lights hum only when the microwave is on, it is due to EM interference.

³**Nonlinear Vibrations in Mechanical and Electrical Systems**, J.J. Stoker, original edition 1950, Wiley Classics Library Edition, reprinted 1992.

⁴**Perturbation Techniques in Mathematics, Physics, and Engineering**, R. Bellman, Holt, Rinehart and Winston, 1966.

⁵The historical origin of quantitative methods lies in the early history of astronomical calculations, where it received the name *perturbation method*.

⁶**Nonlinear Oscillations**, A.H. Nayfeh, D.T. Mook, Wiley-Interscience, New York, 1979.

⁷**Nonlinear Ordinary Differential Equations**, D.W. Jordan, P. Smith, Oxford University Press, 2nd ed., 1988.

11.2 Physical Examples

To help gain an appreciation of physical systems modeled by nonlinear equations, we present some examples.

- *Motion in a gravitational field:* A pendulum that oscillates in a vertical plane in a gravitational field, assuming no dissipation, is governed by the equation of motion,

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0,$$

where θ is the angular displacement measured from the local horizontal. For small oscillations θ is small, and we can approximate $\sin \theta \simeq \theta$ about $\theta = 0$, resulting in the equation of the simple harmonic oscillator, $\ddot{\theta} + (g/l)\theta = 0$. Figure 11.1 shows the range over which the linear approximation reasonably matches the nonlinear term $\sin \theta$. Figure 2.8 is another view of this linearization.

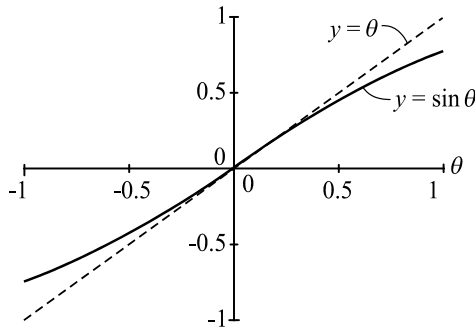


Figure 11.1: The straight line is a plot of θ vs. θ and the curved line is a plot of $\sin \theta$ where it is observed that for approximately $\theta < 0.4$ rad the linear approximation matches reasonably well.

- *Restoring moments for floating bodies:* A floating body, such as a ship, oscillates in response to the wave, current, and wind loads that act on it. In general, the motion of a floating body is governed by a nonlinear equation. As an example,⁸ consider the *rolling* motions of a ship that can lead to capsizing. Rolling motion is a rotation about the longitudinal axis of the ship. Although it is difficult to establish the nonlinear properties, these determine the stability conditions of the ship. An example of a simplified governing equation for the roll angle is

$$\begin{aligned} \ddot{\phi} + c_L \dot{\phi} + c_N |\dot{\phi}| \dot{\phi} + \omega_\phi^2 \phi + k_3 \phi^3 + k_5 \phi^5 \\ = \lambda \omega_e \alpha_m \cos \omega_e t, \end{aligned}$$

where c_L and c_N are linear and nonlinear damping coefficients, respectively, ω_ϕ is the natural frequency of rolling, k_3 and k_5 are nonlinear “stiffness”

⁸S. Surendran, S.K. Lee, J.V.R. Reddy, and G. Lee, “Nonlinear roll dynamics of a Ro-Ro ship in waves,” *Ocean Engineering*, Vol.32, 2005, pp.1818-1828.

constants, λ is an inertia term, ω_e is the loading frequency, and α_m is the maximum wave slope. This equation is called a Duffing-type equation, which we will study later in this chapter.

- *Elastic restoring forces:* The restoring force for a moored body, such as a vessel or an aircraft secured with lines or anchors, is nonlinear. An example of a mooring force⁹ for a system of two opposing cables with symmetry of configuration and environmental conditions is

$$\begin{aligned} F \simeq & (a_1 + a_6 \dot{x}^2 + a_8 \ddot{x}^2)x + (a_3 + a_5 x^2 + a_{11} \ddot{x}^2)\dot{x} \\ & + (a_4 + a_7 x^2 + a_{10} \dot{x}^2)\ddot{x} + a_2 x^3 \\ & + a_9 \ddot{x}^3 + a_{12} \dot{x}^3 + a_{13} x \dot{x} \ddot{x}, \end{aligned}$$

where the coefficients a_i are fit to a particular sea state.

- *Geometric nonlinearities:* It is possible that as a body moves, additional forces come into play at discrete locations and times. For example, heat exchanger tubes encounter stops; ships with mooring lines and fenders have “on-off” constraints, that is, they can move a certain distance until they come into contact with physical restraints. Such nonlinearities can be solved in a piecewise linear sequence; each linear segment is solved separately and the system states between linear portions are matched.

Restoring forces can also be due to nonlinear springs. A case considered below is the nonlinear restoring moment, $mgl \sin \theta$, that acts in a simple pendulum, and studied in Example 11.1.

Whereas nonlinear restoring forces exchange energy between the system and the environment, nonlinear dissipative forces generally, but not always, remove energy from a system. Consider the following classes of nonlinear dissipative forces.

- *Internal damping:* Vibrating structures dissipate energy due to internal friction, cracking, and plastic deformation. As a structure cycles through inelastic ranges of its constitutive relation, energy is lost through permanent deformation and the structure is said to be hysteretic. For example,¹⁰ the nonlinear restoring force known as Reid’s model is given by

$$H(x) = kx[1 + g \operatorname{sgn}(x\dot{x})],$$

⁹R. Pascoal, S. Huang, N. Barltrop, and C. Guedes Soares, “Equivalent force model for the effect of mooring systems on the horizontal motions,” *Applied Ocean Research*, Vol.27, 2005, pp.165-172.

¹⁰Y. Zhang and W.D. Iwan, “Some observations on two piecewise-linear dynamic systems with induced hysteretic damping,” *International Journal of Nonlinear Mechanics*, Vol.38, 2003, pp.753-765.

where k and g are constants and sgn is the *signum function*, defined as

$$\text{sgn}(v) = \begin{cases} 1 & \text{if } v > 0 \\ 0 & \text{if } v = 0 \\ -1 & \text{if } v < 0. \end{cases}$$

The nonlinear equation of motion is therefore

$$m\ddot{x} + kx[1 + g \text{sgn}(\dot{x})] = 0.$$

- **Interface damping or friction:** A body sliding on a surface experiences Coulomb friction, discussed in [Section 3.3.2](#).
- **Flow-induced forces:** Drag between a fluid and a structure results in forces that are proportional to the product $v|v|$, where v is the relative velocity. For example, the drag force is often written as

$$F_D = \frac{1}{2}\rho AC_D v |v|,$$

where ρ is the fluid density, A is the cross-sectional area, and C_D is the drag coefficient (usually determined experimentally).

Example 11.1 Nonlinear Pendulum Model

For the simple pendulum of [Figure 11.2](#), the nonlinear equation of motion is

$$ml^2\ddot{\theta} + mgl \sin \theta = 0.$$

Derive an approximate equation of motion that includes nonlinear properties.

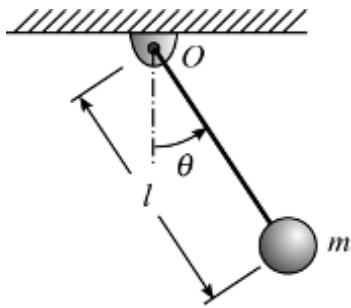
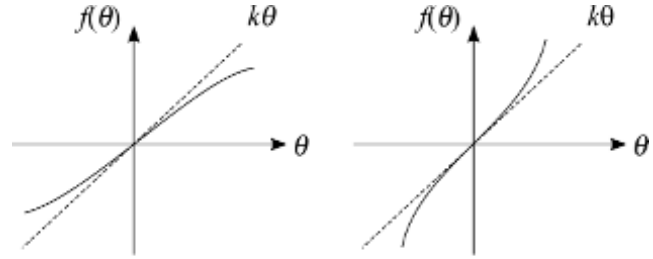


Figure 11.2: Simple pendulum suspended by a string of length l and mass m with its position given by angle θ .

Solution: Drawing on the idea of [Figure 11.1](#), it is relatively straightforward to approximate the nonlinear equation of motion by a linear equation of motion assuming small angular deflections. In this example, we develop an approximate nonlinear equation of motion by retaining the first two terms of the sine series,

$$\sin \theta \simeq \theta - \frac{\theta^3}{6}.$$



(a) Softening spring

(b) Hardening spring

Figure 11.3: Torque vs. angular displacement spring characteristics: (a) a softening spring experiences larger displacements for the same torque increment, (b) a hardening spring requires greater torque for the same displacement increment.

The equation of motion is then

$$ml^2\ddot{\theta} + mgl \left(\theta - \frac{\theta^3}{6} \right) = 0,$$

or

$$\ddot{\theta} + \omega_n^2 \left(\theta - \frac{\theta^3}{6} \right) = 0. \quad (11.1)$$

This class of oscillator, which also occurs where there is a torsional spring, can be written in general form,

$$\ddot{\theta} + f(\theta) = 0,$$

where $f(\theta)$ is a nonlinear function of θ . A linear spring is a special case that obeys Hooke's law in rotational form, $k\theta$. A nonlinear spring can be categorized as a *softening spring*, for which the slope decreases with increasing θ , or a *hardening spring*, for which the slope increases with θ , as depicted in [Figure 11.3](#). The slope k is the stiffness. Mathematically:

If $\frac{df}{d\theta} = k$ (constant), the spring is linear.

If $\frac{df}{d\theta} \neq k$, the spring is nonlinear

with local stiffness $k = \left. \frac{df}{d\theta} \right|_{\theta=\theta_0}$.

If $\frac{df}{d\theta} > 0$ and $\frac{d^2f}{d\theta^2} > 0$, the spring is hardening.

If $\frac{df}{d\theta} > 0$ and $\frac{d^2f}{d\theta^2} < 0$, the spring is softening.

These conditions define spring characteristics for different ranges of behavior. ■



Figure 11.4: A common playground swing is an example of a pendulum undergoing large motion in free vibration. If someone is pushing, the swing is being forced periodically. (Photograph by M. Nagurka)

11.2.1 Simple Pendulum: Approximate Solution

From [Example 11.1](#), the equation of motion of a simple pendulum is

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0. \quad (11.2)$$

For oscillations of small amplitude, the equation can be linearized by approximating $\sin \theta \simeq \theta$ (radians). The equation of motion for small oscillations is then

$$\ddot{\theta} + \frac{g}{l} \theta = 0. \quad (11.3)$$

The general solution of [Equation 11.3](#) is

$$\theta(t) = \theta_m \sin(\omega_n t + \phi),$$

where θ_m is the amplitude of the oscillations, ϕ is a phase angle, and $\omega_n = \sqrt{g/l}$ is the natural frequency. The period of small oscillations of the pendulum is

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{l}{g}}. \quad (11.4)$$

11.2.2 Simple Pendulum: Exact Solution

By assuming small angles, we approximated the nonlinear equation of motion (11.2) by a linear [Equation \(11.3\)](#), which we solved in closed form. Here we relax the small angle assumption and seek an *exact* equation for the period of the pendulum.

Multiplying [Equation 11.2](#) by $2\dot{\theta}$ and integrating from an initial position corresponding to the maximum deflection, that is, $\theta = \theta_m$ and $\dot{\theta} = 0$, we find

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l} (\cos \theta - \cos \theta_m).$$

Substituting the identities $\cos \theta = 1 - 2\sin^2(\theta/2)$ and $\cos \theta_m = 1 - 2\sin^2(\theta_m/2)$, solving for dt , and integrating over a quarter period from $t = 0$, $\theta = 0$ to $t = T/4$, $\theta = \theta_m$, we have

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\theta_m} \left[\frac{1}{\sqrt{\sin^2(\theta_m/2) - \sin^2(\theta/2)}} \right] d\theta.$$

The integral, known as an *elliptical integral*, cannot be solved in closed form. However, by writing

$$\sin(\theta/2) = \sin(\theta_m/2) \sin \psi,$$

we can express T as

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left[\frac{1}{\sqrt{1 - \sin^2(\theta_m/2) \sin^2(\psi)}} \right] d\psi$$

or

$$T = \frac{2K}{\pi} \left(2\pi \sqrt{\frac{l}{g}} \right), \quad (11.5)$$

where

$$K = \int_0^{\pi/2} \left[\frac{1}{\sqrt{1 - \sin^2(\theta_m/2) \sin^2(\psi)}} \right] d\psi.$$

The integral defined by K can be calculated numerically or found in tables of elliptic integrals for a given value of θ_m . Formally, it is called a *complete elliptic integral of the first kind*, and is written generally as

$$K(\sigma) = \int_0^{\pi/2} \left[\frac{1}{\sqrt{1 - \sigma^2 \sin^2(\psi)}} \right] d\psi,$$

where in our case $\sigma = \sin(\theta_m/2)$. Although it cannot be solved analytically, K can be expressed in terms of a series

$$K(\sigma) = \frac{\pi}{2} \left[1 + \left(\frac{1}{2}\right)^2 \sigma^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \sigma^4 + \dots \right].$$

For small amplitudes $\sigma = \sin(\theta_m/2) \simeq \theta_m/2$ and we can approximate K as

$$K(\sigma) \simeq \frac{\pi}{2} \left[1 + \frac{1}{16} \theta_m^2 \right],$$

where we have retained the second-order term. This corresponds to the period

$$T \simeq \left(1 + \frac{1}{16} \theta_m^2 \right) \left(2\pi \sqrt{\frac{l}{g}} \right)$$

Table 11.1: Correction Factors for Nonlinear Pendulum.

θ_m	K	$2K/\pi$
0°	1.571	1.000
10°	1.574	1.002
20°	1.583	1.008
30°	1.598	1.017
60°	1.686	1.073
90°	1.854	1.180
120°	2.157	1.373
150°	2.768	1.762
180°	∞	∞

and the natural frequency

$$\omega_n \simeq \left(1 - \frac{1}{16}\theta_m^2\right) \sqrt{\frac{g}{l}}.$$

For very small amplitudes, the second-order term vanishes in these expressions and we recover the equations for the linear approximate solution.

From Equation 11.5 the actual value of the period of a simple pendulum is equal to the approximate value $2\pi\sqrt{l/g}$ given in Equation 11.4 multiplied by the correction factor $2K/\pi$. Values of the correction factor are given in Table 11.1 for various values of the amplitude θ_m . For small angles the correction factor is close to unity, again validating the approximate solution.

11.2.3 Duffing and van der Pol Equations

Next, we consider a single degree-of-freedom oscillator with a nonlinear spring force, $h(x)$, and a nonlinear damping force, $g(\dot{x})$. By Newton's second law of motion, the sum of the external forces equals the mass times the acceleration,

$$-g(\dot{x}) - h(x) + F(t) = m\ddot{x},$$

where the external force is the function $F(t)$. In standard form, we have

$$m\ddot{x} + g(\dot{x}) + h(x) = F(t).$$

There are two special cases of this equation, one where only the stiffness force is nonlinear, and the other where only the damping force is nonlinear. The nonlinear stiffness case is governed by the equation

$$m\ddot{x} + c\dot{x} + h(x) = F(t).$$

The most important equation of this form is known as the *Duffing* equation.

A possible equation of motion with nonlinear damping is given by

$$m\ddot{x} + g(\dot{x}) + kx = F(t),$$

where $\begin{cases} g(\dot{x}) < 0 \text{ for small } \dot{x} \\ g(\dot{x}) > 0 \text{ for large } \dot{x}. \end{cases}$

The damping force $g(\dot{x})$ is a nonlinear function of the velocity $\dot{x}$ and can be positive or negative. In the usual case of positive damping, energy is dissipated. Negative damping adds energy to the system so that the state of rest is unstable and motion develops from the rest position under the slightest disturbance, even if the external force is zero. An example of this is the *van der Pol* equation. We will study the Duffing and the van der Pol equations in more detail later in this chapter.

11.3 The Phase Plane

The phase¹¹ plane has been introduced as an alternate view of the oscillatory system in Section 3.4.5. An advantage of the phase plane is that the *trajectory* paths indicate the nature of the oscillation. In a *qualitative* way, one can follow the trajectories and observe whether the system is stable, oscillatory, periodic, or otherwise. In a qualitative method, the differential equation does not have to be solved and yet it is possible to ascertain much about the behavior of the system.

The *configuration space* or *plane* is the geometric domain of the governing equations. A single degree-of-freedom system is governed by a second-order differential equation that is transformed into two first-order differential equations in two generalized coordinates. For example, the second-order linear differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$$

is rewritten as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega_n^2x - 2\zeta\omega_n y.\end{aligned}$$

In the phase plane (x, y) , the velocity y can be plotted as a function of position x , creating a plot of velocity versus displacement. For an autonomous system, phase plots do not cross because there is only one value for its slope dx/dy at a given point (x, y) . Therefore, the initial position and velocity are enough to determine the behavior of the system. For a nonautonomous system, the behavior of the system depends on time in addition to the initial position and velocity. To uniquely determine the behavior of a nonautonomous system, the slope of the phase plot as well as the position and velocity are needed, as shown in Figure 11.5.

¹¹The word "phase" used in phase plane is distinct from the phase between input and output of a dynamic system.

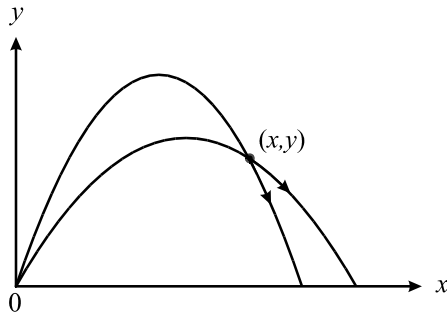


Figure 11.5: Phase plane plots showing two trajectories for a nonautonomous system. Time is changing along the trajectories. At the intersection, we cannot determine on which curve (x, y) is located unless we know the slope dy/dx at that point.

A system with n degrees of freedom has $2n$ state equations. This leads to the generalization of the two-dimensional phase plane to the $2n$ -dimensional phase, or configuration, space. In this space there are $2n$ generalized coordinates. (In Lagrangian dynamics the generalized coordinates are interpreted to mean the minimum set of coordinates that defines the system configuration.)

The phase plane cannot be drawn for a non-autonomous equation since the vector field at a point changes in time. The system may, however, be made autonomous by increasing its dimension by one. For a single non-autonomous equation then, the phase plane becomes three-dimensional, thereby including the variation with time.

Example 11.2 Phase Plane for an Undamped Nonlinear Pendulum

Derive the trajectories of an undamped, nonlinear pendulum. Plot the trajectories in a phase plane for a variety of energy levels.

Solution: The equation of motion is

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

or

$$\ddot{\theta} + \omega_n^2 \sin \theta = 0,$$

where $\omega_n^2 = g/l$. By defining $x = \theta$ and $y = \dot{x} = \dot{\theta}$, this second-order equation can be written as a system of two first-order equations,

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega_n^2 \sin x.\end{aligned}$$

These two equations can be combined into an equation for the trajectory, where time is implicit, by taking the ratio,

$$\frac{dy/dt}{dx/dt} = \frac{-\omega_n^2 \sin x}{y}$$

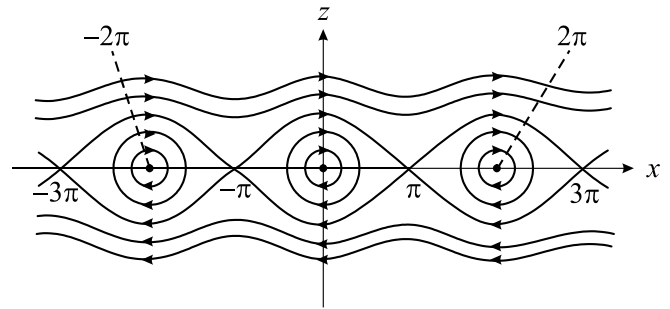


Figure 11.6: Trajectories for an undamped nonlinear pendulum where $z = y/\omega_n = \dot{\theta}/\omega_n$. Closed trajectories correspond to periodic motion and approximately represent simple harmonic motion.

and eliminating dt ,

$$\frac{dy}{dx} = -\frac{\omega_n^2 \sin x}{y},$$

to find

$$y dy = -\omega_n^2 \sin x dx. \quad (11.6)$$

Integrating both sides of Equation 11.6, and assuming that at the end of a swing cycle $\dot{x} = 0$ and $x = x_0$, results in the relation

$$y^2 = 2\omega_n^2 (\cos x - \cos x_0),$$

which can be simplified to

$$z^2 = 2 (\cos x - \cos x_0)$$

by letting $z = y/\omega_n$.

Figure 11.6 shows a few representative trajectories. Trajectories that intersect at $x = \pm n\pi$, and those within, approximately represent the case of simple harmonic motion. They are circles in the x - z plane. To see why, for small x , after integration, Equation 11.6 becomes

$$y^2 + \omega_n^2 x^2 = C,$$

or

$$z^2 + x^2 = C_1,$$

which is the equation of a circle. Trajectories on the outside are not oscillatory. They may represent a pendulum that rotates like a propeller rather than oscillates about the static equilibrium. ■

Example 11.3 Phase Plane for an Undamped Nonlinear Oscillator

Consider the nonlinear equation

$$\ddot{x} + \omega_n^2 (x - 2\alpha x^3) = 0.$$

Find the equations for the trajectories in the phase plane.

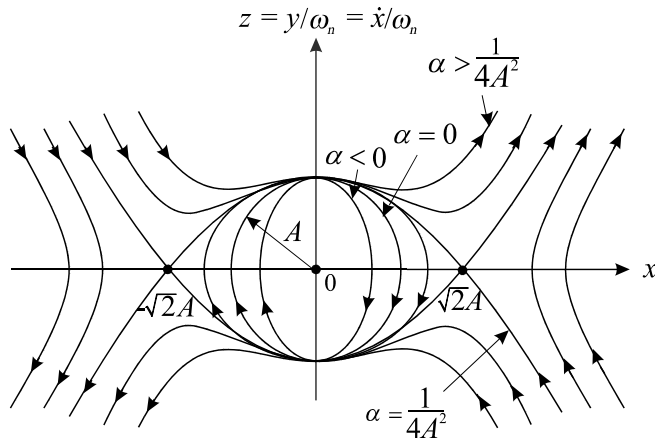


Figure 11.7: Trajectories for a nonlinear oscillator for different values of α and x_0 .

Solution: This is the linear oscillator with the added nonlinear restoring term $-2\alpha\omega_n^2 x^3$. We observe that for small values of x , the term $(x - 2\alpha x^3)$ is greater than zero, and for larger values of x , $(x - 2\alpha x^3)$ is less than zero. We will see that the value of this term determines whether the motion is oscillatory.

If we define $y = \dot{x}$, then

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega_n^2 (x - 2\alpha x^3),\end{aligned}$$

and

$$\frac{dy}{dx} = -\frac{\omega_n^2 (x - 2\alpha x^3)}{y},$$

or

$$y dy = -\omega_n^2 (x - 2\alpha x^3) dx.$$

Integrating both sides, and assuming that at the end of a swing cycle $\dot{x} = 0$ and $x = x_0$, yields the equation

$$z^2 + x^2 - \alpha x^4 = A^2, \quad (11.7)$$

where $z = y/\omega_n$ and $A^2 = x_0^2 (1 - \alpha x_0^2)$ is a constant. The intersection points of the curves that separate oscillatory behavior from non-oscillatory and unstable behavior are given by solving Equation 11.7 for α with $z = 0$.

Figure 11.7 depicts trajectories for several values of α . For $\alpha < 1/(4A^2)$, the motion is periodic. The circle for $\alpha = 0$ represents simple harmonic motion. For $\alpha = 1/(4A^2)$, the trajectories are given by the parabolas,

$$\frac{y}{\omega_n} = \pm \left(A - \frac{x^2}{2A} \right),$$

where the points $(x, y) = (\pm\sqrt{2}A, 0)$ are unstable equilibria. For $\alpha > 1/(4A^2)$, the motion is unstable. (We note that the vertical axis is normalized by ω_n .) ■

The governing equation for the two examples above can be generalized to the nonlinear equation

$$\ddot{x} + f(x, \dot{x}) = 0.$$

If we define $\dot{x} = y$, then $\dot{y} = -f(x, y)$. The slope of a trajectory is given by

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = -\frac{f(x, y)}{y} \equiv \varphi(x, y). \quad (11.8)$$

If $\varphi(x, y)$ is not indeterminate (due to a zero denominator), then there is a unique slope of the trajectory at every point (x, y) in the phase plane. If $y = 0$ and $f(x, 0) \neq 0$, the point lies on the x axis and the slope of the trajectory is infinite. This means that all trajectories cross the x axis at a right angle. If $y = 0$ and $f(x, 0) = 0$, the point is called a *singular point* and the slope is indeterminate. Such a point represents an equilibrium position where x equals a constant and the velocity equals zero, $y = \dot{x} = 0$. Therefore, $\dot{y} = -f(x, y) = 0$ and there is no force on the system. In Figure 11.7 the points at $x = \pm\sqrt{2}A$ are unstable singular points. The stability of singular points is discussed in the next section.

11.3.1 Stability of Equilibria

As an introduction to the study of dynamic stability, a single degree-of-freedom nonlinear dynamic system is studied. It can be represented by the two first-order differential equations

$$\frac{dx}{dt} = g_1(x, y) \quad (11.9)$$

$$\frac{dy}{dt} = g_2(x, y), \quad (11.10)$$

where g_1 and g_2 are nonlinear functions of x and y . The slope of the trajectories in the phase plane is given by

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g_2(x, y)}{g_1(x, y)}. \quad (11.11)$$

Equation 11.11 is a more general version of Equation 11.8. For the singular or equilibrium point (x_0, y_0) , that is, where

$$g_1(x_0, y_0) = g_2(x_0, y_0) = 0,$$

the slope has the indeterminate value $0/0$.

The simple phase plane discussed earlier, with Equation 11.8 for the slope, is useful for certain mechanical systems. More complex mechanical problems may need to be modeled using the more general first-order system of Equations 11.9 and 11.10. Because the slope Equation 11.11 is more general, the trajectories do not necessarily cross the x axis perpendicularly. In the following example we consider a particular set of functions $g_1(x, y)$ and $g_2(x, y)$.

Example 11.4 The Slope from Phase Plane

For the system given in terms of two first-order equations,

$$\begin{aligned}\dot{x} &= 3x + 2y \\ \dot{y} &= -2x - 2y,\end{aligned}$$

determine the slope where the trajectory crosses the x axis.

Solution: Equation 11.11 can be used to find the slope. These trajectories cross the x axis at a slope other than $\pi/2$ rad. The slope is given by

$$\frac{dy}{dx} = \frac{-2x - 2y}{3x + 2y}.$$

When crossing the x axis, the value of y is zero. Then,

$$\frac{dy}{dx} = -\frac{2x}{3x} = -\frac{2}{3}$$

is the slope at that point. ■

Trajectories in the Neighborhood of Singular Points

It is useful to examine the behavior of the trajectories in the neighborhood of the singular points to answer questions about the stability of the equilibrium. We can accomplish this by expanding the nonlinear functions g_1 and g_2 in Taylor series about a singular point. There may be many singular points and each has to be examined individually.

It is convenient to assume that $(x_0, y_0) = (0, 0)$ is such a singular point, since the slope of the trajectories does not vary with translation, as shown here,

$$\begin{aligned}x_1 &= x - x_0 \\ y_1 &= y - y_0 \\ \frac{dy_1}{dx_1} &= \frac{dy}{dx}.\end{aligned}$$

Then, via the Taylor series expansion of a function of two variables, we have

$$\begin{aligned}\dot{x} &= g_1(x, y) = \left. \frac{\partial g_1}{\partial x} \right|_{(0,0)} x + \left. \frac{\partial g_1}{\partial y} \right|_{(0,0)} y \\ &+ \text{higher-order terms}\end{aligned}\quad (11.12)$$

$$\begin{aligned}\dot{y} &= g_2(x, y) = \left. \frac{\partial g_2}{\partial x} \right|_{(0,0)} x + \left. \frac{\partial g_2}{\partial y} \right|_{(0,0)} y \\ &+ \text{higher-order terms}.\end{aligned}\quad (11.13)$$

In the neighborhood of the singular point, the higher-order terms can be neglected because of their relatively

small magnitudes, and the equations can be approximated (about zero) in matrix-vector form,

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} \simeq \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}, \quad (11.14)$$

where d_{ij} represents the respective partial derivatives in Equations 11.12 and 11.13.

The solution of the linearized matrix Equation 11.14 is locally *geometrically similar*¹² to the solution of nonlinear Equations 11.9 and 11.10. Assuming the solution

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} \exp(\lambda t), \quad (11.15)$$

where X , Y , and λ are constants, and substituting Equation 11.15 into Equation 11.14, leads to the eigenvalue problem,

$$\begin{bmatrix} d_{11} - \lambda & d_{12} \\ d_{21} & d_{22} - \lambda \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The eigenvalues $\lambda_{1,2}$ can be found by evaluating the characteristic equation arising from the determinant of the characteristic matrix,

$$\begin{vmatrix} d_{11} - \lambda & d_{12} \\ d_{21} & d_{22} - \lambda \end{vmatrix} = 0,$$

that is,

$$\begin{aligned}\lambda^2 - p\lambda + q &= 0 \\ p &= (d_{11} + d_{22}) \\ q &= d_{11}d_{22} - d_{12}d_{21}.\end{aligned}$$

Then, $\lambda_{1,2} = (p \pm \sqrt{p^2 - 4q})/2$, with corresponding eigenvectors

$$\begin{Bmatrix} X \\ Y \end{Bmatrix}_1, \begin{Bmatrix} X \\ Y \end{Bmatrix}_2.$$

For the case $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, the general solution is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = C_1 \begin{Bmatrix} X \\ Y \end{Bmatrix}_1 \exp(\lambda_1 t) + C_2 \begin{Bmatrix} X \\ Y \end{Bmatrix}_2 \exp(\lambda_2 t), \quad (11.16)$$

where the constants C_1 and C_2 depend on the initial conditions. We note that these results are identical to

¹²Geometrically similar means that the geometrical properties are similar, just as two triangles can be similar even if the dimensions of the respective sides of the triangle are different. The solution of Equation 11.14 could be topologically different than the solution of Equations 11.9 and 11.10. They are “similar” only “locally” (in the neighborhood of a non-singular point). Thus, a general nonlinear system cannot be represented by a linearized system “globally,” that is, for large motion.

those derived when studying the second-order differential equation governing a damped harmonic oscillator. Classes of behavior can be determined from the sign of the discriminant:

if $(p^2 - 4q) < 0$, the motion is oscillatory

if $(p^2 - 4q) \geq 0$, the motion is not oscillatory.

The stability characteristics of a system are related to the robustness of a given outcome to small changes in initial conditions or small random fluctuations. Considering the exponential for the first eigenvalue when the discriminant is negative,

$$\exp(\lambda_1 t) = \exp(p/2) \exp\left(i \frac{1}{2} \sqrt{4q - p^2}\right),$$

then

if $p > 0$, the system is unstable

if $p \leq 0$, the system is stable.

Therefore, the character of the trajectory of the system depends on the values of p and q , as well as the relative values of p^2 and $4q$. The possible cases are described next:

Case (i): λ_1 and λ_2 are real and distinct; $p^2 > 4q$.

1. If λ_1 and λ_2 are of the same sign and $q > 0$, the equilibrium point is called a *node*. If $\lambda_2 < \lambda_1 < 0$ and $p < 0$, all the trajectories tend to the origin as $t \rightarrow \infty$, and the origin is called a *stable node* (shown in Figure 11.8(a)). If $\lambda_2 > \lambda_1 > 0$ and $p > 0$, all the trajectories tend in the opposite direction as $t \rightarrow \infty$, and the origin is called an *unstable node* (shown in Figure 11.8(b)).
2. If λ_2 and λ_1 are of opposite signs and $q < 0$, with any sign for p , one solution tends to the origin and the other tends to infinity. The origin is called a *saddle point* and it corresponds to an *unstable equilibrium* (shown in Figure 11.8(c)).

Case (ii): λ_1 and λ_2 are real and equal; $p^2 = 4q$.

The phase plane trajectories are straight lines and asymptotic lines passing to the origin, which is the equilibrium point. The origin is a *stable node* if $\lambda_{1,2} < 0$ and an *unstable node* if $\lambda_{1,2} > 0$, as shown in Figure 11.9.

Case (iii): λ_1 and λ_2 are complex conjugates; $p^2 < 4q$.

The phase plane trajectories are *logarithmic spirals*. The equilibrium point is called a *focus* or a *spiral point*.

1. If $p < 0$ and $q > 0$, the motion is asymptotically stable and thus the focus is stable (shown in Figure 11.10(a)).

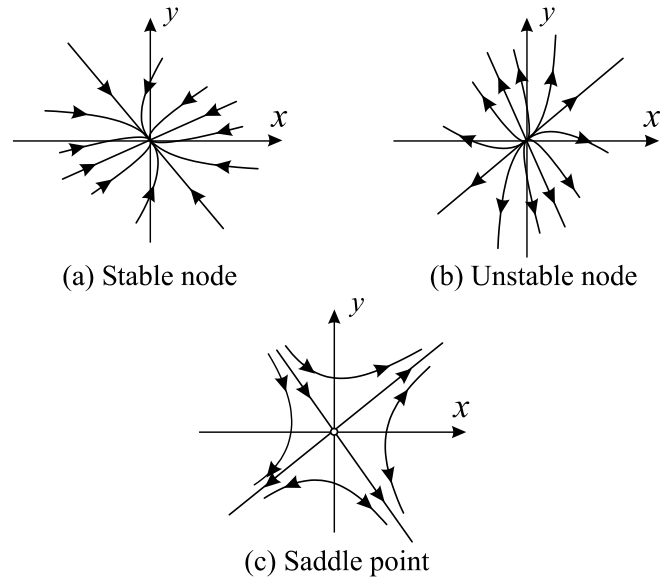


Figure 11.8: Trajectories near equilibrium points in the phase plane for Case (i).

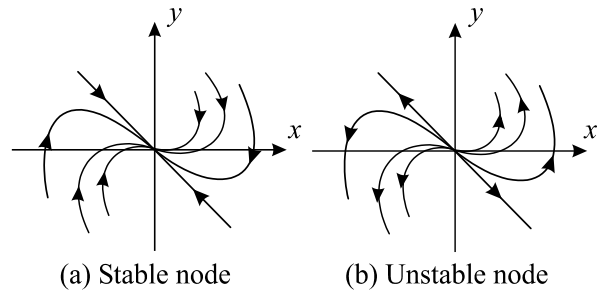


Figure 11.9: Trajectories near equilibrium points in the phase plane for Case (ii).

2. If $p > 0$ and $q > 0$, the motion is unstable and thus the focus is unstable (shown in Figure 11.10(b)).
3. If $p = 0$, the trajectories reduce to circles if the vertical axis is normalized. Otherwise, the trajectories are ellipses. The equilibrium point is called a *center* or a *vertex point*, and the motion is periodic and thus stable (shown in Figure 11.10(c)).

Figure 11.11 summarizes the stability characteristics for various values of p and q .

If the linearized system is found to be asymptotically stable or unstable (as t increases), it is said to possess *significant behavior*. In this case, the stability characteristics of the linearized system are the same as those of the nonlinear system. On the other hand, if the linearized system is found to be stable, the linearized system possesses *critical behavior*, and the stability conclusions about it do not necessarily extend to the full nonlinear system. In this case, an analysis of the nonlinear system is required.

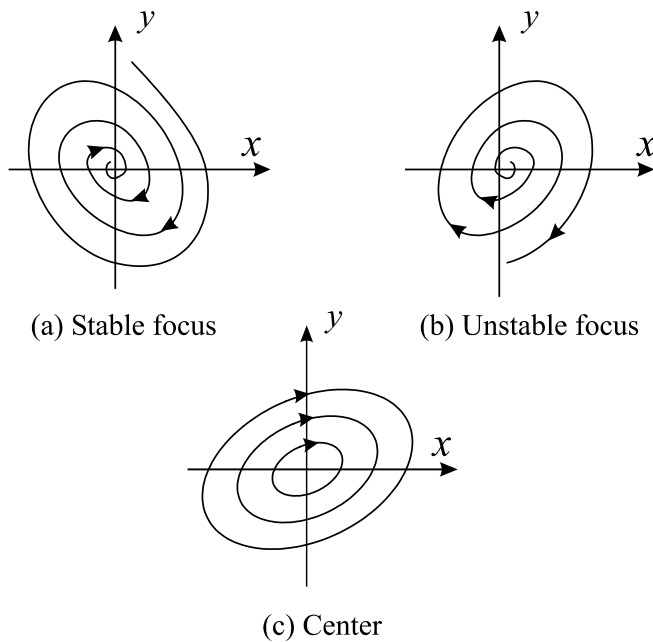


Figure 11.10: Trajectories near equilibrium points in the phase plane for Case (iii).

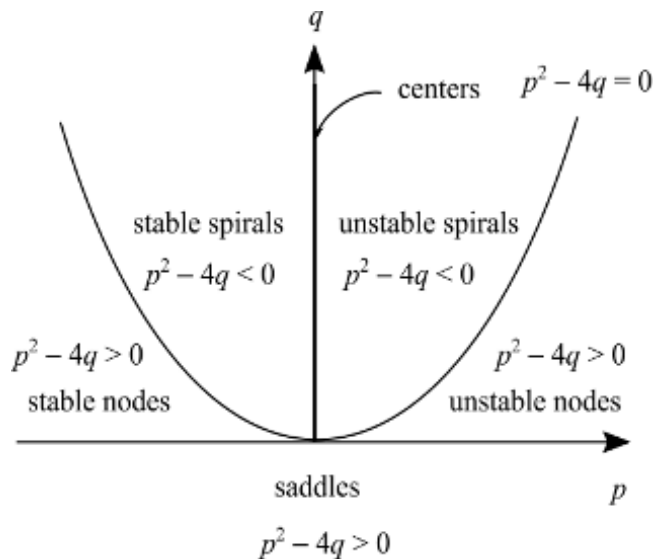


Figure 11.11: General stability characteristics near an equilibrium position.

Example 11.5 Behavior Near Equilibrium from Characteristic Roots

Identify the behavior of the equilibrium positions for the characteristic equations or characteristic values,

- (a) $\lambda^2 - \lambda + 4 = 0$
- (b) $\lambda^2 + \lambda + 4 = 0$
- (c) $\lambda = 1, -2$
- (d) $\lambda = -1, -1$.

Solution: (a) The characteristic roots are $\lambda = 0.50 \pm i1.94$, meaning they are complex conjugates with positive real parts. The system is unstable about the equilibrium point. The equilibrium point is an unstable spiral whose behavior is shown in Figure 11.10(b).

(b) The characteristic roots are $\lambda = -0.50 \pm i1.94$, meaning they are complex with negative real parts. The system is stable about the equilibrium point. The equilibrium point is a stable spiral whose behavior is shown in Figure 11.10(a).

(c) The characteristic roots are real and distinct. One of the roots is positive, and, therefore, the system is unstable about the equilibrium point. The equilibrium point is a saddle whose behavior is shown in Figure 11.8(c).

(d) The characteristic roots are repeated and negative real. The system is stable about the equilibrium point. The equilibrium point is a stable node whose behavior is shown in Figure 11.9(a). ■

Example 11.6 Behavior Near Equilibrium of a Linear System

Describe the trajectories near the equilibrium point of the linear system given in Example 11.4.

Solution: The system is given by

$$\begin{aligned}\dot{x} &= 3x + 2y \\ \dot{y} &= -2x - 2y.\end{aligned}$$

The equilibrium point is obtained by setting $\dot{x} = 0$ and $\dot{y} = 0$,

$$\begin{aligned}3x + 2y &= 0 \\ -2x - 2y &= 0.\end{aligned}$$

Thus, the equilibrium point is at $(0, 0)$. The eigenvalues are obtained by solving the eigenvalue problem,

$$\begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \lambda \begin{Bmatrix} X \\ Y \end{Bmatrix}. \quad (11.17)$$

Setting the determinant to zero and solving,

$$\begin{aligned}\det \begin{bmatrix} 3 - \lambda & 2 \\ -2 & -2 - \lambda \end{bmatrix} &= 0 \\ \lambda^2 - \lambda - 2 &= 0 \\ \lambda &= -1 \text{ or } 2.\end{aligned}$$

The eigenvalues are real and have opposite signs. Therefore, the equilibrium point is a saddle point and is unstable. ■

Example 11.7 Behavior Near Equilibrium of a Nonlinear System

Find the equilibrium position and describe the trajectory near the equilibrium points of the system governed by the equation of motion

$$\ddot{\theta} + \dot{\theta} + \theta + \sin \theta = 0.6.$$

Solution: The equilibrium point can be obtained by setting the time derivatives to zero so that we are left with the static equilibrium equation,

$$\theta_{eq} + \sin \theta_{eq} = 0.6.$$

The solution is obtained numerically, resulting in the equilibrium position $\theta_{eq} = 0.302$. We rewrite the equation of motion in state-space form using

$$x = \theta \text{ and } y = \dot{\theta},$$

to find

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 0.6 - x - y - \sin x.\end{aligned}$$

The right-hand sides are linearized using a Taylor series expansion about the equilibrium position, $(x = 0.302, y = 0)$, using Equations 11.12 and 11.13. The first equation is already linear. The right-hand side of the second equation is linearized as

$$\begin{aligned}g_2(x, y) &= 0.6 - x - \sin x \\ &\simeq g_2(0.302, 0) + \left. \frac{\partial g_2}{\partial x} \right|_{(0.302, 0)} (x - 0.302) \\ &\quad + \left. \frac{\partial g_2}{\partial y} \right|_{(0.302, 0)} (y - 0) \\ &= (-1 - \cos 0.302)(x - 0.302) - y \\ &= -1.95x - y - 0.590.\end{aligned}$$

The linearized equation for $\dot{y}$ is then

$$\begin{aligned}\dot{y} &= 0.6 - x - y - \sin x \\ &\simeq -1.95x - y - 0.590,\end{aligned}$$

and the linearized equations of motion are

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ -1.95 & -1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ -0.590 \end{Bmatrix}.$$

We may redefine our coordinates to eliminate the nonhomogeneous term by subtracting the equilibrium position such that

$$\begin{aligned}x_{new} &= x - 0.302 \\ y_{new} &= y.\end{aligned}$$

The linearized equations of motion are rewritten as

$$\begin{Bmatrix} \dot{x}_{new} \\ \dot{y}_{new} \end{Bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ -1.95 & -1 \end{bmatrix} \begin{Bmatrix} x_{new} \\ y_{new} \end{Bmatrix}.$$

The characteristic roots are the eigenvalues,

$$\begin{aligned}\det \begin{bmatrix} 0 - \lambda & 1 \\ -1.95 & -1 - \lambda \end{bmatrix} &= 0 \\ \lambda^2 - \lambda + 1.95 &= 0 \\ \lambda &= 0.5 \pm 1.30i.\end{aligned}$$

The characteristic roots are complex conjugates with positive real parts. Therefore, the equilibrium point is an unstable spiral. ■

Example 11.8 Behavior Near Equilibrium of Example 11.3

Classify the type of equilibrium point for the cases of Example 11.3 by linearizing the equations of motion near the equilibrium points.

Solution: In Example 11.3, the stabilities of the equilibrium points were identified from the phase plot. In this example, the equations of motion are linearized about the equilibrium points, and the characteristic roots are obtained to identify the behavior near the equilibria. The system model is given by

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega_n^2 (x - 2\alpha x^3).\end{aligned}$$

The equilibrium equation is given by

$$-\omega_n^2 (x_{eq} - 2\alpha x_{eq}^3) = 0.$$

There are three equilibrium points and they are $(0, 0)$, $(1/\sqrt{2\alpha}, 0)$, and $(-1/\sqrt{2\alpha}, 0)$. From Figure 11.7, we already have found that the equilibrium point at $(0, 0)$ is a center and the other two are saddle points. We show this again by examining the characteristic roots of the linearized equations of motion about these three equilibrium points. The first equation is already linear, and the second equation must be linearized.

(a) Equilibrium point at $(0, 0)$

The right-hand side of the second equation is expanded about $x = 0$ and is given by

$$\begin{aligned}\dot{y} &= g_2(x) = -\omega_n^2(x - 2\alpha x^3) \\ &\simeq g_2(0) + \left. \frac{dg_2(x)}{dx} \right|_{x=0} \cdot x \\ &= -\omega_n^2 x.\end{aligned}$$

The equations of motion are then

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}.$$

The characteristic roots are obtained from

$$\det \begin{bmatrix} 0 - \lambda & 1 \\ -\omega_n^2 & 0 - \lambda \end{bmatrix} = 0,$$

giving the eigenvalues, $\lambda = \pm i\omega_n$. Therefore, the equilibrium point at $(0, 0)$ is a center.

(b) Equilibrium point at $(1/\sqrt{2\alpha}, 0)$

The linearized $g_2(x)$ about $x = 1/\sqrt{2\alpha}$ is given by

$$\begin{aligned}g_2(x) &\sim g_2\left(1/\sqrt{2\alpha}\right) + \left. \frac{dg_2(x)}{dx} \right|_{x=1/\sqrt{2\alpha}} \left(x - \frac{1}{\sqrt{2\alpha}}\right) \\ &= 2\omega_n^2 \left(x - \frac{1}{\sqrt{2\alpha}}\right).\end{aligned}$$

The equations of motion are rewritten as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 2\omega_n^2 \left(x - \frac{1}{\sqrt{2\alpha}}\right),\end{aligned}$$

or equivalently,

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ 2\omega_n^2 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ -2\omega_n^2/\sqrt{2\alpha} \end{Bmatrix}.$$

The characteristic roots are obtained from

$$\det \begin{bmatrix} 0 - \lambda & 1 \\ 2\omega_n^2 & 0 - \lambda \end{bmatrix} = 0,$$

and the eigenvalues are $\pm\sqrt{2}\omega_n$. They are real and have opposite signs. Therefore, the equilibrium point is a saddle point.

(c) Equilibrium point at $(-1/\sqrt{2\alpha}, 0)$

The linearized $g_2(x)$ about $x = -1/\sqrt{2\alpha}$ is given by

$$\begin{aligned}g_2(x) &\sim g_2\left(-1/\sqrt{2\alpha}\right) + \left. \frac{dg_2(x)}{dx} \right|_{x=-1/\sqrt{2\alpha}} \left(x + 1/\sqrt{2\alpha}\right) \\ &= 2\omega_n^2 \left(x + 1/\sqrt{2\alpha}\right).\end{aligned}$$

The equations of motion are rewritten as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 2\omega_n^2 \left(x - \frac{1}{\sqrt{2\alpha}}\right),\end{aligned}$$

or in matrix form,

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ 2\omega_n^2 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ 2\omega_n^2/\sqrt{2\alpha} \end{Bmatrix}.$$

The eigenvalues are $\pm\sqrt{2}\omega_n$. They are real and have the opposite signs. Therefore, the equilibrium point is a saddle point. ■

11.4 Perturbation or Expansion Methods

Approximation techniques are available to model and solve nonlinear governing equations. One approach is a series expansion method known as a *perturbation* of the system. Inherent in this approach, as in all approximate analytical approaches, is that the nonlinearity is “small.” Perturbation methods¹³ are useful as the first steps in a nonlinear analysis to help in understanding the general behavior of the system. The method becomes cumbersome for more realistic and complicated models. Once the general behavior is understood, other methods, such as numerical methods, are useful.

The essence of the perturbation method, as it is used in locating periodic solutions of nonlinear oscillators, is the following. Given a linear harmonic oscillator, there exists an infinite number of periodic solutions depending on the two constants of integration. The solutions are in the form of a series expansion.

The question arises as to what will happen if the linear system $\ddot{x} + x = 0$ is perturbed by adding the small term $\varepsilon f(t, x, \dot{x})$ to the left-hand side, where f is a nonlinear function. The answer depends on the function $f(t, x, \dot{x})$ and the smallness of ε . For example, if $\varepsilon f(t, x, \dot{x}) = \varepsilon c\dot{x}$, then the trajectories become logarithmic spirals, stable if $c > 0$ and unstable if $c < 0$. The value of ε must be sufficiently small to ensure that a series solution for $x(t)$ converges.

The perturbation method attempts to ascertain under what conditions the perturbed equation has periodic solutions. This is sometimes called the *problem of Poincaré*, who studied such problems rigorously. The

¹³**Perturbations, Theory and Methods**, J.A. Murdock, Wiley-Interscience, 1991. This book provides a rigorous discussion of perturbation methods, their use, and their mathematical basis.

following unforced nonlinear pendulum example demonstrates the method.

Example 11.9 Unforced Nonlinear Pendulum

The equation that approximately governs an undamped nonlinear pendulum, Equation 11.1,

$$\ddot{\theta} + \omega_n^2 \left(\theta - \frac{\theta^3}{6} \right) = 0,$$

can be written in the more general form

$$\ddot{\theta} + \omega_n^2 \theta + \varepsilon \theta^3 = 0, \quad (11.18)$$

where $\omega_n^2 = \sqrt{g/l}$ and $\varepsilon = -\omega_n^2/6$. Equation 11.18 is known as an unforced *Duffing* equation. Assume that the nonlinearity is weak, that is, ε is small compared to 1. Use a perturbation (series) solution for $\theta(t)$, and solve to order one, that is, retain terms to the first power in ε .

Solution: The perturbation solution is given by the infinite series,

$$\theta(t, \varepsilon) = \theta_0(t) + \varepsilon \theta_1(t) + \varepsilon^2 \theta_2(t) + \dots \quad (11.19)$$

Using a two-term approximation for $\theta(t)$ and substituting this into the equation of motion, we find

$$(\ddot{\theta}_0 + \varepsilon \ddot{\theta}_1) + \omega_n^2 (\theta_0 + \varepsilon \theta_1) + \varepsilon (\theta_0 + \varepsilon \theta_1)^3 = 0.$$

Expanding and grouping terms according to the power, or order, of the perturbation parameter ε , we have

$$\begin{aligned} (\ddot{\theta}_0 + \omega_n^2 \theta_0) + \varepsilon (\ddot{\theta}_1 + \omega_n^2 \theta_1 + \theta_0^3) + \varepsilon^2 (3\theta_0^2 \theta_1) \\ + \varepsilon^3 (3\theta_0 \theta_1^2) + \varepsilon^4 \theta_1^3 = 0. \end{aligned}$$

Since ε is assumed to be small, terms of order ε^2 , ε^3 , and ε^4 are of higher order than the order of the assumed solution and are therefore neglected. The solution obtained in this way is said to be correct to order ε . Each order of the solution is satisfied independently,¹⁴ as follows,

$$\varepsilon^0: \quad \ddot{\theta}_0 + \omega_n^2 \theta_0 = 0 \quad (11.20)$$

$$\varepsilon^1: \quad \ddot{\theta}_1 + \omega_n^2 \theta_1 = -\theta_0^3. \quad (11.21)$$

θ_0 is known as the *generating solution* for the sequence of equations and is taken to satisfy the initial conditions.

The Duffing equation requires two initial conditions. These can be specified in full generality, but we assume the values $\theta(0) = C$ and $\dot{\theta}(0) = 0$. If the initial angular velocity is other than zero, only the phase is changed and not the character of the solution for the autonomous

Equation 11.18. It is easiest to stipulate that the ε^0 order equation satisfies the initial conditions. Then the remaining equations of higher order will satisfy zero initial conditions. The sum of solutions, as per Equation 11.19, will satisfy the initial conditions of the problem.

The equations are solved in sequence. Equation 11.20 is solved for θ_0 , which is then substituted as needed into the right-hand side of Equation 11.21 so that the solution θ_1 can be obtained. From Equation 11.20,

$$\theta_0(t) = A \sin(\omega_n t + \phi),$$

and satisfying the initial conditions results in $A = C$ and $\phi = \pi/2$ rad. Equation 11.21 then becomes

$$\begin{aligned} \ddot{\theta}_1 + \omega_n^2 \theta_1 &= -C^3 \sin^3 \left(\omega_n t + \frac{\pi}{2} \right) \\ &= -C^3 \left[\frac{3}{4} \sin \left(\omega_n t + \frac{\pi}{2} \right) \right. \\ &\quad \left. - \frac{1}{4} \sin 3 \left(\omega_n t + \frac{\pi}{2} \right) \right], \end{aligned}$$

where a trigonometric identity has been used to transform the cubic sine term. The solution of this equation is

$$\begin{aligned} \theta_1(t) &= \frac{3}{8\omega_n} t C^3 \cos(\omega_n t + \pi/2) \\ &\quad - \frac{C^3}{32\omega_n^2} \sin 3(\omega_n t + \pi/2). \end{aligned} \quad (11.22)$$

The two-term approximate solution is then $\theta(t) = \theta_0(t) + \varepsilon \theta_1(t)$. We immediately see a problem with the first term on the right-hand side of Equation 11.22. It is called a *secular term*, that is, an expression that grows without bound due to the factor t multiplying the cosine. We know that the solution to Equation 11.18 should be periodic for small ε , and therefore this unbounded term should not be part of the solution.

The problem with this approach is that the truncation has removed terms that would balance the secular term and lead to a periodic solution. This effect is demonstrated by expanding the following sine function,

$$\begin{aligned} \sin(\omega_n + \varepsilon)t &= \sin \omega_n t \cos \varepsilon t + \cos \omega_n t \sin \varepsilon t \\ &= \left(1 - \frac{1}{2!} \varepsilon^2 t^2 + \frac{1}{4!} \varepsilon^4 t^4 - \dots \right) \sin \omega_n t \\ &\quad + \left(\varepsilon t - \frac{1}{3!} \varepsilon^3 t^3 + \frac{1}{5!} \varepsilon^5 t^5 - \dots \right) \cos \omega_n t. \end{aligned}$$

Approximating the expression $\sin(\omega_n + \varepsilon)t$ by two terms of the expansion on the right-hand side, we obtain

$$\begin{aligned} \sin(\omega_n + \varepsilon)t &= \left(1 - \frac{1}{2!} \varepsilon^2 t^2 \right) \sin \omega_n t \\ &\quad + \left(\varepsilon t - \frac{1}{3!} \varepsilon^3 t^3 \right) \cos \omega_n t. \end{aligned}$$

¹⁴ When substituting the expansion into the differential equation, we obtain a power series in ε that must vanish *identically* in ε . This is why each order in ε results in a differential equation that must be satisfied.

Since the sine function is harmonic, the secular terms should not be present and must be removed in a logical way. This defect in the direct method above is corrected with the use of the Lindstedt-Poincaré method, derived in the next section. ■

JULES HENRI POINCARÉ

(April 29, 1854 – July 17, 1912)



Contribution: Poincaré was a French mathematician, theoretical physicist, and a philosopher of science. He made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. His work spanned multiple disciplines: fluid mechanics, the special theory of relativity, and the philosophy of science.

In the field of applied mathematics he studied optics, electricity, telegraphy, capillary action, elasticity, thermodynamics, potential theory, quantum theory, theory of relativity, and cosmology. In the field of celestial mechanics he studied the three-body-problem, as well as the theories of light and of electromagnetic waves.

He was responsible for formulating the Poincaré conjecture, a famous problem in mathematics. In his research on the three-body problem, Poincaré became the first person to discover a chaotic deterministic system which laid the foundations of modern chaos theory. He is considered to be one of the founders of the field of topology.

Poincaré introduced the modern principle of relativity and was the first to present the Lorentz transformations in their modern symmetrical form. Poincaré discovered the remaining relativistic velocity transformations and recorded them in a letter to Lorentz in 1905. Thus he obtained perfect invariance of all of Maxwell's equations, an important step in the formulation of the theory of special relativity. In so doing, he became a cofounder, along with Einstein and Lorentz, of the special theory of relativity.

Life Sketch: Jules Henri Poincaré was born in Nancy, Lorraine, France into an influential family. His father was a professor of medicine at the University of Nancy. During his

childhood he was seriously ill with diphtheria and was tutored by his mother.

In 1862 Poincaré entered the Lycée in Nancy (now renamed the Lycée Henri Poincaré in his honor, along with the University of Nancy). He spent eleven years at the Lycée during which he was one of the top students. His mathematics teacher described him as a “monster of mathematics” and he won first prize in a competition between the top students from all the Lycées across France. (His weakest subjects were music and physical education. Some later attributed this in part to his poor eyesight.) He graduated from the Lycée in 1871 with a bachelor's degree in letters and sciences.

During the Franco-Prussian War of 1870 he served alongside his father in the Ambulance Corps.

Poincaré entered the École Polytechnique in 1873. There he studied mathematics as a student of Charles Hermite. He graduated in 1875 or 1876. He then enrolled at the École des Mines, continuing to study mathematics in addition to the mining engineering syllabus and received the degree of ordinary engineer in March 1879.

As a graduate of the École des Mines he joined the Corps des Mines as an inspector for the Vesoul region in northeast France. He was on the scene of a mining disaster at Magny in August 1879 in which 18 miners died. He carried out the official accident investigation in a characteristically thorough and humane way.

At the same time, Poincaré was preparing for his doctorate in mathematics under the supervision of Charles Hermite. His doctoral thesis was in the field of differential equations. Poincaré realized that they could be used to model the behavior of multiple bodies in free motion within the solar system. Poincaré graduated from the University of Paris in 1879.

Soon after, he was offered a post as junior lecturer in mathematics at Caen University, but he never fully abandoned his mining career to mathematics. He worked at the Ministry of Public Services as an engineer in charge of northern railway development from 1881 to 1885. He eventually became chief engineer of the Corps de Mines in 1893 and inspector general in 1910.

Beginning in 1881 and for the rest of his career, he taught at the University of Paris (the Sorbonne). He held the chairs of Physical and Experimental Mechanics, Mathematical Physics and Theory of Probability, and Celestial Mechanics and Astronomy. In 1881 Poincaré married. He and his wife had four children.

In 1912 Poincaré underwent surgery for a prostate problem and subsequently died from an embolism at 58 years of age. He is buried in the Poincaré family vault in the Cemetery of Montparnasse, Paris.

Noteworthy: Poincaré achieved the highest honors. He was elected to the French Academy of Sciences in 1887 (at age 32) and became the President of the Academy in 1906. The breadth of his research allowed him to be the only member elected to every one of the five sections of the Academy: geometry, mechanics, physics, geography, and navigation.

The problem of finding the general solution to the motion of more than two orbiting bodies in the solar system had eluded mathematicians since Newton's time. This was known originally as the three-body problem and later the n -body problem. In 1887, in honor of his 60th birthday, Oscar II, King of Sweden established a prize for anyone who could find the solution.

Although he did not succeed in solving the problem the prize was awarded to Poincaré as acknowledgement of his ingenious attempt. One of the judges, Karl Weierstrass (a distinguished German mathematician considered the father of "modern analysis"), remarked: "This work cannot indeed be considered as furnishing the complete solution of the question proposed, but that it is nevertheless of such importance that its publication will inaugurate a new era in the history of celestial mechanics." (The problem was solved by Karl Sundman for $n = 3$ in 1912 and was generalized to the case of $n > 3$ by Qindong Wang in 1991.)

In 1908, Poincaré was elected to the French Academy; he was elected director in the year of his death. He was also made knight of the French Legion of Honor. He was honored by a large number of learned societies around the world and received numerous prizes, medals, and awards for his work.

Poincaré had a remarkable memory. He retained much from the texts he read and was able to link the ideas in a visual way. His ability to visualize what he heard proved particularly useful when he attended lectures. His eyesight was very poor and it is questionable whether he could see what lecturers were writing on the blackboard.

Poincaré maintained very precise working hours. He undertook mathematical research for four hours a day, between 10 am and noon then again from 5 pm to 7 pm. He would read articles in journals later in the evening.

Poincaré tended to develop his results from first principles. Although many mathematicians refer back to previous work, this was not Poincaré's practice. Not in only his research, but also in his lectures and books he developed everything carefully from basics.

In 1893 Poincaré joined the French Bureau des Longitudes, which engaged him in the synchronization of time around the world. In 1897 Poincaré backed an unsuccessful proposal for the decimalization of circular measure. It was this post which led him to consider the question of establishing international time zones and the synchronization of time between bodies in relative motion.

In 1899, and again more successfully in 1904, he intervened in the trials of Alfred Dreyfus. He attacked the spurious claims of the evidence brought against Dreyfus, a Jewish officer in the French army framed by a military court and baselessly charged with treason by anti-Semitic colleagues.

After Poincaré achieved prominence as a mathematician, he turned his superb literary gifts to the task of describing for the general public the meaning and importance of science and mathematics. Poincaré's popular works include *Science and Hypothesis* (1901), *The Value of Science* (1905), and *Science*

and Method (1908).

Poincaré was interested in the way his mind worked. He studied his habits and gave a talk about his observations in 1908 at the Institute of General Psychology in Paris. He linked his way of thinking to how he made several discoveries. He acknowledged that his success was primarily the result of visual representation. He believed that logic was not a way to invent but a way to structure ideas. He disliked logic believing it limited ideas. Poincaré claimed that intuition was the life of mathematics.

The Poincaré group used in physics and mathematics was named after him.

As an aside, his cousin, Raymond Poincaré, was the President of France from 1913 to 1920, and a fellow member of the French Academy.

In 2004 the French Minister of Education, Claude Allegre, proposed that Poincaré be reburied in the Panthéon in Paris, which is reserved for French citizens of the highest honor.

11.4.1 Lindstedt-Poincaré Method

An examination of the expansion in [Example 11.9](#) shows that the solution is constrained to oscillate at the frequency ω_n . Linear harmonic systems are characterized by a constant period of oscillation, regardless of the initial conditions. The linear response ($\varepsilon = 0$) is harmonic with a period $T_n = 2\pi/\omega_n$. Nonlinear quasi-harmonic systems, however, have periods, and thus frequencies, that are functions of the nonlinearity and the initial conditions. The breakdown of the above method, leading to secular terms, is due to ignoring this nonlinearity. In the presence of nonlinear terms the response needs to be periodic with period $T = 2\pi/\omega$, where ω is an unknown fundamental frequency that is a function of ε and the initial conditions.

The *Lindstedt-Poincaré* method circumvents the secularity problem by expanding the response frequency ω , as well as θ , in powers of a small parameter. We recall that the initial conditions of linear and nonlinear systems are coupled to the solution through the constants of integration. We therefore expect that the expansion in ω will be a function of an integration constant. Since ω does not appear in the equation of motion, only ω_n appears, the transformation $\tau = \omega t$ is first introduced.¹⁵ Then, with $dt = d\tau/\omega$, we have

$$\begin{aligned}\frac{d}{dt} &= \omega \frac{d}{d\tau} \\ \frac{d}{dt} \left(\frac{d}{dt} \right) &= \omega \frac{d}{d\tau} \left(\omega \frac{d}{d\tau} \right)\end{aligned}$$

or

$$\frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2}.$$

¹⁵ The radian is a derived SI unit and is considered dimensionless. Therefore, τ can be considered here to be a dimensionless time.

The essential feature of this method (emphasized by Minorsky¹⁶) is that there appears to be an arbitrariness in the approximations since two expansions have been introduced in the same differential equation. This arbitrariness enables us to dispose of the available constants so as to gradually eliminate the secular terms in the subsequent approximations.

Equation 11.18, $\ddot{\theta} + \omega_n^2 \theta + \varepsilon \theta^3 = 0$, becomes

$$\omega^2 \theta'' + \omega_n^2 \theta + \varepsilon \theta^3 = 0, \quad (11.23)$$

after the transformation, where primes denote differentiation with respect to τ , $\omega_n^2 = \sqrt{g/l}$, and $\varepsilon = -\omega_n^2/6$. For $\varepsilon < 0$, the spring is softening and for $\varepsilon > 0$, the spring is hardening. To solve, we assume the initial conditions are $\theta(0, \varepsilon) = a_0$ and $\theta'(0, \varepsilon) = 0$. For an autonomous system, we can allow the first-order solution $\theta_0(\tau)$ to satisfy these conditions, $\theta_0(0) = a_0$ and $\theta'_0(0) = 0$, with the remaining orders satisfying zero initial conditions,

$$\theta_i(0) = 0 \quad \text{and} \quad \theta'_i(0) = 0, \quad i = 1, 2, \dots$$

For non-autonomous systems, the constants of integration must be certain values so that periodic solutions exist. Not all initial conditions lead to periodic solutions.

Next, we expand θ and ω , and consider a linear approximation (that is, only ε^1 terms are retained in addition to the linear terms),

$$\begin{aligned} \theta(\tau, \varepsilon) &= \theta_0(\tau) + \varepsilon \theta_1(\tau) + \dots \\ \omega &= \omega_n + \varepsilon \omega_1 + \dots \end{aligned}$$

We note that the oscillation frequency reduces to ω_n for $\varepsilon = 0$. Substituting these expansions into Equation 11.23, we find

$$\begin{aligned} (\omega_n + \varepsilon \omega_1)^2 (\theta_0 + \varepsilon \theta_1)'' \\ + \omega_n^2 (\theta_0 + \varepsilon \theta_1) + \varepsilon (\theta_0 + \varepsilon \theta_1)^3 = 0. \end{aligned}$$

This equation is expanded and grouped according to powers of ε ,

$$\varepsilon^0: \quad \omega_n^2 \theta_0'' + \omega_n^2 \theta_0 = 0 \quad (11.24)$$

$$\varepsilon^1: \quad \omega_n^2 \theta_1'' + \omega_n^2 \theta_1 = -\theta_0^3 - 2\omega_n \omega_1 \theta_0'' \quad (11.25)$$

⋮

We retain only the first two equations in the series since a two-term expansion is utilized. The equations are solved iteratively. Equation 11.24 is solved for $\theta_0(\tau)$ and its powers and derivatives are substituted into Equation 11.25, which is then solved for $\theta_1(\tau)$. Also, each equation is linear with the nonlinear effects shifted to the right-hand side, acting as inputs to the system, allowing

¹⁶**Nonlinear Oscillations**, N. Minorsky, Krieger Publishing Company, 1987. Reprint of original 1962 edition.

the use of linear theory to solve each equation in this sequence. We can therefore rewrite Equations 11.24 and 11.25 as

$$\begin{aligned} \theta_0'' + \theta_0 &= 0 \\ \theta_1'' + \theta_1 &= -\frac{1}{\omega_n^2} \theta_0^3 - 2\frac{\omega_1}{\omega_n} \theta_0'' \end{aligned}$$

The solution to the first equation is $\theta_0(\tau) = a_0 \cos \tau$, after satisfying the initial conditions. The second equation becomes¹⁷

$$\theta_1'' + \theta_1 = \left(-\frac{3}{4} \frac{a_0^3}{\omega_n^2} + 2\frac{\omega_1 a_0}{\omega_n} \right) \cos \tau - \frac{a_0^3}{4\omega_n^2} \cos 3\tau.$$

To remove secular terms,¹⁸ the coefficient of the resonant loading function $\cos \tau$ is set to zero. The resonant loading function is the one with the same frequency as that of the system on the left-hand side. This leads to an equation governing the possible value of ω_1 ,

$$\omega_1 = \frac{3}{8} \frac{a_0^2}{\omega_n}.$$

Therefore,

$$\theta_1(\tau) = a_1 \cos \tau + b_1 \sin \tau + \frac{a_0^3}{32\omega_n^2} \cos 3\tau.$$

The coefficients a_1 and b_1 are found by applying zero initial conditions for $\theta_1(\tau)$, leading to the values $a_1 = -a_0^3/32\omega_n^2$ and $b_1 = 0$. The approximate solution for $\theta(\varepsilon, \tau)$ is then¹⁹

$$\theta(\tau, \varepsilon) = a_0 \cos \tau - \varepsilon \frac{a_0^3}{32\omega_n^2} (\cos \tau - \cos 3\tau) + O(\varepsilon^2),$$

where $O(\varepsilon^2)$ represents terms of minimum order ε not retained. Transforming back to the t -domain, we find

$$\theta(t, \varepsilon) = a_0 \cos \omega t - \varepsilon \frac{a_0^3}{32\omega_n^2} (\cos \omega t - \cos 3\omega t) + O(\varepsilon^2)$$

$$\omega = \omega_n + \varepsilon \frac{3}{8} \frac{a_0^2}{\omega_n} + O(\varepsilon^2).$$

These equations provide a measure of the effect of system nonlinearity on the system response and frequency of response.

¹⁷To solve we use the trigonometric identity

$$\cos^3 \tau = \frac{3}{4} \cos \tau + \frac{1}{4} \cos 3\tau.$$

¹⁸The removal of secular terms is justified by physical evidence that a slightly perturbed system will still oscillate, but in a near-harmonic way. Therefore, terms in the solution that grow without bound need to be removed in a rational way. This is the Lindstedt procedure.

¹⁹Orders of approximation are used not only to make approximations with various degrees of accuracy, but also to indicate relative accuracy. In scientific and engineering disciplines, approximations can be classified based on the order of magnitude of the rounding error.

When to Expand ω

There may be some ambiguity regarding when the expansion of a second parameter is needed. Sometimes we see the expansion in ω and other times not. Essentially, one expands a parameter when its value is unknown. Of course, the oscillator response amplitude is unknown and expanded. For the autonomous oscillator the period and thus frequency are unknown, and therefore ω needs to also be expanded.

In the following sections, forced nonlinear oscillators are studied with the purpose of finding periodic oscillations at the forcing period or frequency. For these cases, therefore, there is no need to expand ω .

11.4.2 Forced Oscillations of Quasi-Harmonic Systems

Next, we consider the forced Duffing equation,

$$\ddot{x} + \omega_n^2 x = \varepsilon [-\omega_n^2 (\alpha x + \beta x^3) + F \cos \Omega t] \quad (11.26)$$

$$\varepsilon \ll 1,$$

where α and β are given constant parameters, $\omega_n^2 = k/m$, and the harmonic forcing function is $\varepsilon F \cos \Omega t$. Here, the harmonic excitation is of small magnitude and the response will be almost harmonic, or *quasi-harmonic*. It is of interest to determine the circumstances under which the response $x(t)$ is periodic with period $T = 2\pi/\Omega$ and, therefore, an expansion in the frequency is not necessary.

Expansion for $x(t)$

This problem is approached using the perturbation method with the following change of variables,

$$\Omega t = \tau + \phi$$

$$\frac{d}{dt} = \Omega \frac{d}{d\tau},$$

where τ is a dimensionless time and ϕ is a phase angle, which is introduced for generality. The variable τ has a period of 2π .

With these transformations, Equation 11.26 becomes

$$\Omega^2 x'' + \omega_n^2 x = \varepsilon [-\omega_n^2 (\alpha x + \beta x^3) + F \cos (\tau + \phi)]$$

$$\varepsilon \ll 1, \quad (11.27)$$

where the primes denote differentiation with respect to τ . Potential secular terms must be removed since they are non-physical and we require that the solution to Equation 11.27 be periodic, $x(\tau + 2\pi) = x(\tau)$.

We also assume that $x(0) = C$ and, for convenience, $x'(0) = 0$. The value of C_0 is not arbitrary, and is determined in terms of other system parameters below. The

perturbation solution is based on the expansions of $x(\tau)$ and ϕ ,

$$x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots$$

$$\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \dots,$$

where $x_i(\tau + 2\pi) = x_i(\tau)$, $x'_i(0) = 0$, for $i = 1, 2, \dots$, and it is assumed that the first term in the expansion for x will satisfy the initial condition, $x_0(0) = C_0$. The expansions are substituted into Equation 11.27, and equating the coefficients of like powers of ε , we find

$$\Omega^2 x''_0 + \omega_n^2 x_0 = 0 \quad (11.28)$$

$$\Omega^2 x''_1 + \omega_n^2 x_1 = -\omega_n^2 (\alpha x_0 + \beta x_0^3) + F \cos (\tau + \phi_0) \quad (11.29)$$

$$\Omega^2 x''_2 + \omega_n^2 x_2 = -\omega_n^2 (\alpha x_1 + 3\beta x_0^2 x_1) + F \cos (\tau + \phi_1)$$

$$\vdots \quad (11.30)$$

We solve for $x_i(\tau)$ in sequence for all i , applying the periodicity and initial conditions. The solution of Equation 11.28 is

$$x_0(\tau) = C_0 \cos \frac{\omega_n}{\Omega} \tau,$$

where C_0 is a constant amplitude. Satisfying the 2π -periodicity in τ is only possible if $\Omega = \omega_n$. This substitution is made in Equations 11.29 and 11.30. Substituting for $x_0(\tau)$ in Equation 11.29, we find

$$x''_1 + x_1 = -(\alpha C_0 \cos \tau + \beta C_0^3 \cos^3 \tau) + \frac{F}{\omega_n^2} \cos (\tau + \phi_0).$$

This expression can be simplified using the trigonometric relation $\cos^3 \tau = (3 \cos \tau + \cos 3\tau)/4$, resulting in

$$x''_1 + x_1 = -\alpha C_0 \cos \tau - \beta C_0^3 \frac{1}{4} (3 \cos \tau + \cos 3\tau)$$

$$+ \frac{F}{\omega_n^2} (\cos \tau \cos \phi_0 - \sin \tau \sin \phi_0)$$

$$= \left(-\frac{F}{\omega_n^2} \sin \phi_0 \right) \sin \tau$$

$$- \left(\alpha C_0 + \frac{3}{4} \beta C_0^3 - \frac{F}{\omega_n^2} \cos \phi_0 \right) \cos \tau$$

$$- \frac{1}{4} \beta C_0^3 \cos 3\tau. \quad (11.31)$$

To avoid secular terms, the coefficients of the $\sin \tau$ and $\cos \tau$ terms are set equal to zero. From the coefficient of the $\sin \tau$ term, we have $\sin \phi_0 = 0$, providing two possible values for ϕ_0 . These, in conjunction with the coefficient of the $\cos \tau$ term, yield the conditions that enforce periodicity:

$$\alpha C_0 + \frac{3}{4} \beta C_0^3 - \frac{F}{\omega_n^2} = 0, \quad \text{for } \phi_0 = 0 \quad (11.32)$$

$$\alpha C_0 + \frac{3}{4} \beta C_0^3 + \frac{F}{\omega_n^2} = 0, \quad \text{for } \phi_0 = \pi. \quad (11.33)$$

The parameter C_0 can now be determined since α , β , and F are all known. (C_0 can have three values, as we will see below, since the equation is cubic.) Equation 11.33 does not offer additional information beyond that of Equation 11.32. (Equation 11.33 tells us that for the phase $\phi_0 = \pi$ rad, the response and the forcing function are 180° out of phase. This is the same as being in phase with a response of negative amplitude.)

With Equation 11.32 satisfied, we solve Equation 11.31 for $x_1(\tau)$,

$$x_1(\tau) = C_1 \cos \tau + \frac{1}{32} \beta C_0^3 \cos 3\tau. \quad (11.34)$$

The value of C_1 is determined based on the required periodicity of $x_2(\tau)$, just as C_0 was determined based on the periodicity of $x_1(\tau)$. Equation 11.34 is substituted into Equation 11.30, and utilizing a number of trigonometric identities, expressions are derived that must be examined for the possibility of secular responses.

Since only $x_1(\tau)$ is retained, then to order ε , or $O(\varepsilon)$,

$$\begin{aligned} x(\tau, \varepsilon) &\simeq x_0(\tau) + \varepsilon x_1(\tau) \\ &= C_0 \cos \tau + \varepsilon \left(C_1 \cos \tau + \frac{1}{32} \beta C_0^3 \cos 3\tau \right), \end{aligned}$$

where $\phi \simeq \phi_0 + \varepsilon \phi_1 = 0$ since the ε^2 solution yields $\phi_1 = 0$. It turns out that all the phase angle terms equal zero since there is no damping in the system. This makes sense from our studies of linear undamped oscillators where there is no phase lag. Thus, we could have avoided the ϕ expansion. For systems with damping ϕ will have a nonzero value.

Solution for C_0 vs. ω_n

Equation 11.32 is reminiscent of the frequency response function $G(i\omega)$, Equation 3.42, as both are relations between the response amplitude and the forcing amplitude. Equation 11.32 is the analogous nonlinear relation. We are interested in studying the relations between the response amplitude C_0 and the linear natural frequency ω_n for various values of εF .

Introducing the relation

$$\omega_0^2 \equiv (1 + \varepsilon \alpha) \omega_n^2, \quad (11.35)$$

Equation 11.26 becomes

$$\ddot{x} + \omega_0^2 x + \varepsilon \omega_n^2 \beta x^3 = \varepsilon F \cos \Omega t, \quad \varepsilon \ll 1.$$

Solving Equation 11.35 for α , substituting the result into Equation 11.32, and solving this equation for ω_n^2 , we find

$$\omega_n^2 = \omega_0^2 \left(1 + \frac{3}{4} \varepsilon \beta C_0^2 \right) - \varepsilon \frac{F}{C_0}, \quad (11.36)$$

where the following approximation has been made,

$$\left(1 - \frac{3}{4} \varepsilon \beta C_0^2 \right)^{-1} \simeq \left(1 + \frac{3}{4} \varepsilon \beta C_0^2 \right), \quad (11.37)$$

based on the Taylor series expansion for $1/(1+x) \simeq 1-x$ for small x and where the term of order ε^2 has been assumed negligible.

Equation 11.36 can be plotted as $|C_0|$ versus ω_n , where ω_n is measured in units of ω_0 . The term $\varepsilon \beta$ is given and the term εF is a parameter. Representative curves are shown in Figure 11.12 for $\varepsilon \beta = \pm 0.1$. Note that the parametric curves for a single case appear in pairs, on either side of the $F = 0$ curve.

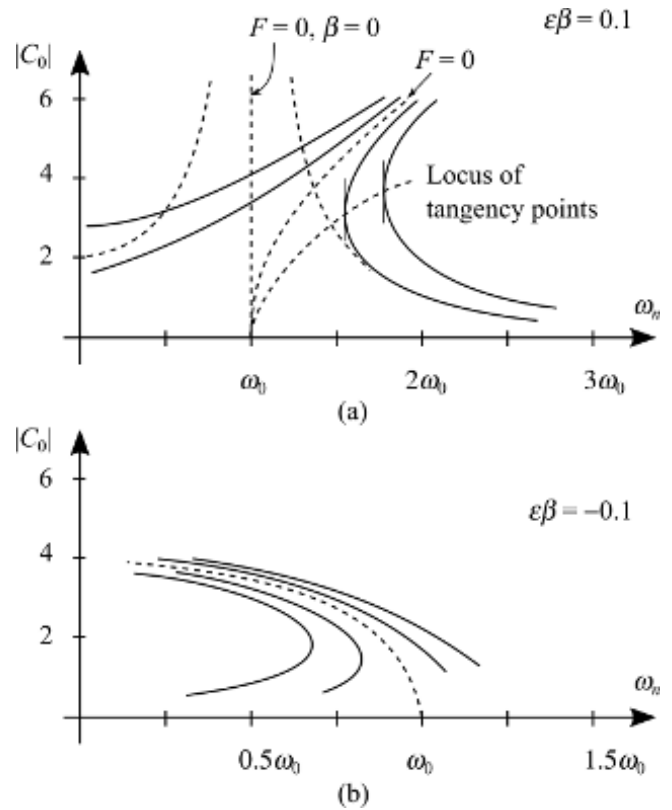


Figure 11.12: Amplitude curves $|C_0|$ vs. ω_n for the Duffing equation for two values of $\varepsilon\beta$, with multiple values of εF . In (a), the dotted lines for $\beta = 0$ represent the linear response. The dotted lines that slant to the right represent the nonlinear case for $F = 0$ as well as the locus of tangency points. Two tangent lines are drawn. In (b), the curves slant to the left because of the negative parameter $\varepsilon\beta$.

Significant information can be discerned from this set of curves. The sign of $\varepsilon\beta$ determines whether the curves slant to the right ($\varepsilon\beta > 0$ for a hardening spring) or to the left ($\varepsilon\beta < 0$ for a softening spring). Unlike the linear case, there is no resonance condition where, for an undamped oscillator, the amplitudes grow larger without bounds around the natural frequency of the system.

For each pair of amplitude curves, if we draw a vertical line to the right of one of the *tangent* lines, it will intersect the curves at two locations on one branch and at one location on the mirror branch on the other side of the $F = 0$ curve. Any intersection represents a real root, and so this vertical line indicates that there are three possible amplitudes corresponding to a given excitation force amplitude. A vertical line to the left of one of the *tangent* lines has only one real root, and also two complex roots. Following a representative curve as ω_0 increases results in ever larger amplitudes.

11.4.3 Jump Phenomenon

Since all systems have some damping, it is interesting to consider a lightly damped Duffing oscillator to examine the changes in predicted behavior. We consider the governing equation

$$\ddot{x} + \omega_n^2 x = \varepsilon [-2\zeta\omega_n \dot{x} - \omega_n^2 (\alpha x + \beta x^3) + F \cos \Omega t], \quad \varepsilon \ll 1,$$

where the small damping term $-2\varepsilon\zeta\omega_n \dot{x}$ has been added. (Note that α is dimensionless and β has units of x^{-2} .) Following the same procedure as before, the analogous equation to Equation 11.36 is found to be

$$\left[\omega_0^2 \left(1 + \frac{3}{4}\varepsilon\beta C_0^2 \right) - \omega_n^2 \right]^2 + (2\varepsilon\zeta\omega_0^2)^2 = \left(\frac{\varepsilon F}{C_0} \right)^2,$$

and is plotted in Figure 11.13. The primary difference between this figure and Figure 11.12 for the undamped case is that the two branches connect at some location.

This connection is significant because it means that an increase in the frequency does not always lead to an increase in amplitude. At some value of frequency, there will be a drop in amplitude to the other solution branch. This can be observed by following the path from point 4 for increasing frequency, with a drop at point 1 to point 2. Similarly, for decreasing frequency, moving from location 2 to location 3 leads to a jump in amplitude at point 3 to point 4. The path between point 1 to point 3 is unstable.

This behavior, where there are drops and jumps in amplitude, is called the *jump phenomenon*. It exists for nonlinear systems with damping. There are practical implications. For example, as a motor accelerates to its operating speed, nonlinear effects may result in jumps in amplitude that need to be considered in the design. Such a phenomenon occurs in vortex-induced oscillations, and is mentioned in Section 11.6.3.

11.4.4 Periodic Solutions of Nonautonomous Systems

Consider the system governed by the equation

$$\ddot{u} + \omega_n^2 u = F_0 \cos t + \varepsilon f(u, \dot{u}, t, \varepsilon),$$

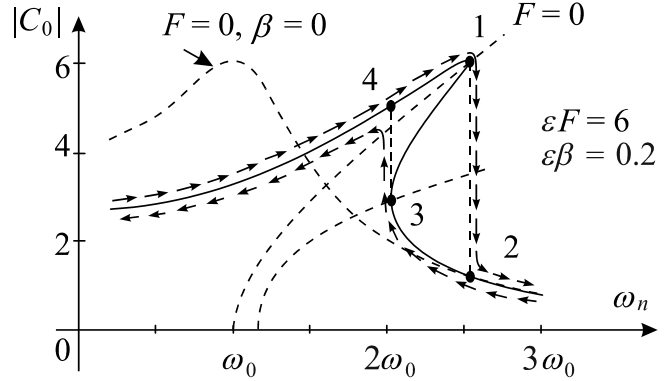


Figure 11.13: Jump phenomenon. The solid line path between points 1 and 3 is unstable.

where the forcing function F_0 is large compared to the nonlinearities that are of order ε . When $\varepsilon = 0$ and $F_0 = 0$, the nonlinear equation reduces to the simple harmonic oscillator with natural frequency ω_n and period $T = 2\pi/\omega_n$.

For the case where $\varepsilon = 0$, with the forcing frequency set to 1 rad/s, meaning a period of 2π s, the linear solution u_0 is

$$u_0 = C_0 \cos \omega_n (t - t_0) + \frac{F_0}{\omega_n^2 - 1} \cos t, \quad \omega_n^2 \neq 1,$$

where C_0 and t_0 are the constants of integration. There are two ways the solution can have the same 2π -period as the forcing term. One way is for ω_n to be an integer. If ω_n is an irrational number, then the second way the solution u_0 can be periodic is if $C_0 = 0$, in which case the period is automatically 2π , like the forcing.

Forced Pendulum

In nonlinear problems a coupling exists between the free and forced responses and the initial conditions. We cannot solve for the free and forced vibrations separately and then add the solutions. The constants of integration do not only specify the allowable initial conditions for periodic solutions, but are a result of enforcing periodicity.

Consider the forced pendulum undergoing large motion, governed by the equation

$$\ddot{\theta} + \omega_n^2 \sin \theta = F_0 \cos \omega t.$$

An approximate nonlinear equation that is representative of the fully nonlinear pendulum equation is

$$\ddot{\theta} + \omega_n^2 \left(\theta - \frac{\theta^3}{6} \right) = F_0 \cos \omega t.$$

Next, consider a nondimensionalized version using the transformation $\tau = \omega t$,

$$\omega^2 \theta'' + \omega_n^2 \theta - \frac{1}{6} \omega_n^2 \theta^3 = \Gamma \omega^2 \cos \tau$$

or

$$\theta'' + \Omega^2 \theta - \frac{1}{6} \Omega^2 \theta^3 = \Gamma \cos \tau,$$

where $\Gamma = F_0/\omega^2$ and $\Omega = \omega_n/\omega$. To further generalize,²⁰ we define $\varepsilon \equiv -\Omega^2/6$ and apply the perturbation expansion to the equation

$$\theta'' + \Omega^2 \theta + \varepsilon \theta^3 = \Gamma \cos \tau, \quad (11.38)$$

which reduces to the linear problem for $\varepsilon = 0$. We look for solutions of the same period as the driving force, which has 2π -periodicity.

The expansion is taken to be a function of ε and τ in the following way,

$$\theta(\tau, \varepsilon) \equiv \theta_0(\tau) + \varepsilon \theta_1(\tau) + \varepsilon^2 \theta_2(\tau) + \cdots, \quad (11.39)$$

where the periodicity condition $\theta_i(\tau + 2\pi) = \theta_i(\tau)$, $i = 0, 1, 2, \dots$, is used to remove any secular terms from the solution. If this approach is to work in a practical sense, it is necessary that $\varepsilon \ll 1$ such that $\theta_0(\tau) \gg \varepsilon \theta_1(\tau) \gg \varepsilon^2 \theta_2(\tau) \cdots$. Then we can retain the expansion to order ε or ε^2 and obtain an accurate approximation. What has been discussed so far makes no assumptions about the character of the forcing function. An expansion in ω is not necessary here since oscillations with known period are sought.

The procedure is to substitute the expression for $\theta(\tau, \varepsilon)$ and its second derivative with respect to τ , $\theta''(\tau, \varepsilon)$, into Equation 11.38, with the result

$$\begin{aligned} &(\theta_0'' + \varepsilon \theta_1'' + \varepsilon^2 \theta_2'' + \cdots) + \Omega^2 (\theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \cdots) \\ &+ \varepsilon (\theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \cdots)^3 = \Gamma \cos \tau. \end{aligned}$$

When the cubic term is expanded, expressions of order ε^3 , ε^4 , and higher are found.²¹ Retaining a solution to order ε^2 and dropping all higher order terms, we obtain the approximation

$$\varepsilon (\theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2)^3 \simeq \varepsilon (\theta_0^3 + 3\theta_0^2 \varepsilon \theta_1).$$

Since the θ_i are independent of ε , and the approximation must be satisfied for all ε , the expanded differential equation is actually a sequence of differential equations that can be found by equating terms of the same order

²⁰ We can define ε as a positive quantity and retain the negative sign in the equation of motion. The results are the same. Sometimes retaining the physical meaning of ε as a frequency ratio is desirable and in that case we set $\varepsilon > 0$.

²¹ The expansion of this cubic term is

$$\begin{aligned} &(\theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2)^3 = \theta_0^3 + 3\theta_0^2 \varepsilon \theta_1 + 3\theta_0^2 \varepsilon^2 \theta_2 + 3\theta_0 \varepsilon^2 \theta_1^2 \\ &+ 6\theta_0 \varepsilon^3 \theta_1 \theta_2 + 3\theta_0 \varepsilon^4 \theta_2^2 + \varepsilon^3 \theta_1^3 + 3\varepsilon^4 \theta_1^2 \theta_2 + 3\varepsilon^5 \theta_1 \theta_2^2 + \varepsilon^6 \theta_2^3. \end{aligned}$$

in ε ,

$$\varepsilon^0: \quad \theta_0'' + \Omega^2 \theta_0 = \Gamma \cos \tau \quad (11.40)$$

$$\varepsilon^1: \quad \theta_1'' + \Omega^2 \theta_1 = -\theta_0^3 \quad (11.41)$$

$$\varepsilon^2: \quad \theta_2'' + \Omega^2 \theta_2 = -3\theta_0^2 \theta_1. \quad (11.42)$$

The approximate solution is then given by Equation 11.39, with an error of $O(\varepsilon^3)$.

We solve Equations 11.40 to 11.42 in sequence. The solution of the ε^0 order governing equation is

$$\theta_{0h}(\tau) = a_0 \cos \Omega \tau + b_0 \sin \Omega \tau,$$

where the subscript h denotes the homogeneous solution and a_0 and b_0 are constants, the values of which are established (as done below) by enforcing the required periodicity. The particular solution is given by

$$\theta_{0p}(\tau) = \frac{\Gamma}{\Omega^2 - 1} \cos \tau.$$

The complete solution is the sum of the homogeneous and particular solutions. The particular solution is not valid at or near the resonant condition $\Omega = 1$. This case is solved separately.²² We are also interested here, for the sake of brevity, only in solutions that have the period of the loading, that is, $T = 2\pi/1$ s. Other solutions exist, some of which are examined in the next section.

For the response to have the period 2π , it follows that $a_0 = b_0 = 0$, and therefore $\theta_0(\tau) = \theta_{0p}(\tau)$. This function is cubed,

$$\begin{aligned} \theta_0^3 &= \left(\frac{\Gamma}{\Omega^2 - 1} \cos \tau \right)^3 \\ &= \left(\frac{\Gamma}{\Omega^2 - 1} \right)^3 \left(\frac{3}{4} \cos \tau + \frac{1}{4} \cos 3\tau \right), \end{aligned}$$

and becomes the forcing function for the equation governing $\theta_1(\tau)$. The expansion of $\cos^3 \tau$ is important because it helps identify the various harmonic components contained within the cubed cosine. Since $\theta_0(\tau)$ satisfies the initial conditions, we need only consider the particular solutions for the higher order terms. For $\theta_{1p}(\tau)$, the solution is

$$\theta_{1p}(\tau) = a_1 \cos \tau + b_1 \cos 3\tau; \quad \Omega \neq 1, 3$$

$$\begin{aligned} a_1 &= \frac{3\Gamma^3}{4(\Omega^2 - 1)^4} \\ b_1 &= \frac{\Gamma^3}{4(\Omega^2 - 1)(\Omega^2 - 9)}. \end{aligned}$$

²² For the case $\Omega \simeq 1$,

$$\Omega^2 = 1 + \varepsilon \beta, \quad (11.43)$$

with $\Gamma = \varepsilon \gamma$, and we expand as before.

For the next term in the series, it is necessary to evaluate the input function $3\theta_0^2\theta_1$. This step is omitted here, resulting in the truncated solution correct to order ε ,

$$\theta(\varepsilon, \tau) \simeq \theta_0(\tau) + \varepsilon\theta_1(\tau) + O(\varepsilon^2),$$

where expressions for $\theta_0(\tau)$ and $\theta_1(\tau)$ are known, and $\varepsilon = -\Omega^2/6$. The variables can now be transformed back to physical ones.

Arbitrary Forcing Function

We begin with the nonlinear governing equation,

$$\ddot{\theta} + \alpha\dot{\theta} + \omega_n^2\theta + \varepsilon g(\theta, \dot{\theta}) = f(t), \quad (11.44)$$

where the forcing function $f(t)$ is general. We expand θ as

$$\theta(t, \varepsilon) = \theta_0(t) + \varepsilon\theta_1(t) + \varepsilon^2\theta_2(t) + \dots,$$

where $\theta_0(t)$ satisfies the linear differential equation. Before substituting the expansion into the governing equation, the nonlinear function $g(\theta, \dot{\theta})$ is first expanded about the linear solution $(\theta_0, \dot{\theta}_0)$ to keep track of the various orders of ε ,

$$\begin{aligned} g(\theta, \dot{\theta}) &\equiv g(\theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \dots, \dot{\theta}_0 + \varepsilon\dot{\theta}_1 + \varepsilon^2\dot{\theta}_2 + \dots) \\ &= g(\theta_0, \dot{\theta}_0) + (\varepsilon\theta_1 + \varepsilon^2\theta_2 + \dots) \frac{\partial}{\partial\theta} g(\theta_0, \dot{\theta}_0) \\ &\quad + (\varepsilon\dot{\theta}_1 + \varepsilon^2\dot{\theta}_2 + \dots) \frac{\partial}{\partial\dot{\theta}} g(\theta_0, \dot{\theta}_0) \\ &\quad + \text{higher-order terms.} \end{aligned}$$

Retaining terms to order ε^2 and equating terms of equal power in ε upon substituting into Equation 11.44, the following sequence of equations is generated,

$$\begin{aligned} \varepsilon^0: \quad &\ddot{\theta}_0 + \alpha\dot{\theta}_0 + \omega_n^2\theta_0 = f(t) \\ \varepsilon^1: \quad &\ddot{\theta}_1 + \alpha\dot{\theta}_1 + \omega_n^2\theta_1 = -g(\theta_0, \dot{\theta}_0) \\ \varepsilon^2: \quad &\ddot{\theta}_2 + \alpha\dot{\theta}_2 + \omega_n^2\theta_2 \\ &= -\theta_1 \frac{\partial}{\partial\theta} g(\theta_0, \dot{\theta}_0) - \dot{\theta}_1 \frac{\partial}{\partial\dot{\theta}} g(\theta_0, \dot{\theta}_0). \end{aligned}$$

As a practical matter, it becomes very difficult to solve more than a two or three term approximation. The solutions for θ_0 and θ_1 are now obtained in sequence using the convolution integral.

Proceeding for a general nonlinear function $g(\theta, \dot{\theta})$, we find

$$\begin{aligned} \theta_0(t) &= \int_{-\infty}^{\infty} f(t-\tau) h(\tau) d\tau \\ \theta_1(t) &= - \int_{-\infty}^{\infty} g[\theta_0(t-\tau), \dot{\theta}_0(t-\tau)] h(\tau) d\tau \\ &\vdots \end{aligned}$$

and then, with $\theta(t, \varepsilon) \simeq \theta_0(t) + \varepsilon\theta_1(t)$, the approximate response is given by

$$\begin{aligned} \theta(t) &\simeq \int_{-\infty}^{\infty} f(t-\tau) h(\tau) d\tau \\ &\quad - \varepsilon \int_{-\infty}^{\infty} g[\theta_0(t-\tau), \dot{\theta}_0(t-\tau)] h(\tau) d\tau. \end{aligned}$$

The above development identifies oscillations at the forcing frequency. One can also transform the differential equation and introduce a response frequency that is also expanded as in the Lindstedt method.

11.4.5 Subharmonic and Superharmonic Oscillations

Linearly oscillating systems respond at the frequency of the forcing function. A nonlinear system responds at *subharmonic* and *superharmonic* frequencies as well.

A subharmonic response involves oscillations of frequencies ω_m that are related to the forcing frequency Ω by the equation

$$\omega_m = \frac{\Omega}{m}, \quad m = 2, 3, \dots$$

Similarly, a superharmonic response involves oscillations of frequencies ω_n that are related to the forcing frequency Ω by the equation

$$\omega_n = n\Omega, \quad n = 2, 3, \dots$$

Note that ω_m and ω_n are not natural frequencies. The subscripts m and n are used to identify specific frequencies. One may ask why these sub- and superharmonics exist. As Stoker²³ states, "... it is not an entirely simple matter to give a plausible physical explanation for their occurrence."

In addition, a nonlinear system loaded by a combination of forces at two distinct frequencies, say Ω_1 and Ω_2 , responds at various combinations of those frequencies.

²³**Nonlinear Vibrations in Mechanical and Electrical Systems**, J.J. Stoker, 1992, p.103: "... it is not an entirely simple matter to give a plausible physical explanation for their occurrence. Let us recall the behavior of linear systems. If the frequency of the free oscillation of a linear system is ω_n/m , where m is an integer, say, then a periodic external force of frequency ω_n can excite the free oscillation in addition to the forced oscillation of frequency ω_n . Why should the situation be different for a nonlinear system? The explanation usually offered is as follows: Any free oscillation of a nonlinear system contains the higher harmonics in profusion, and hence it is possible that an external force with a frequency the same as one of these might be able to excite and sustain the harmonic of lower frequency. Of course that this actually should occur probably requires that the damping be not too great and that proper precautions of various kinds be taken."

One may add that a similar argument can be made for harmonics of higher frequency, but in this case, the force may have to be of larger amplitude.

Several cases are studied in this section²⁴ to demonstrate possible behavior.

It is noteworthy that additional periodic solutions appear for forcing frequencies *nearly equal* to a rational number, not only for forcing frequencies exactly equal to a rational number. Furthermore, not every rational number produces this effect, but only rational numbers that are ratios of small integers, for example, 1, 1/2, 1/3, 2/3.

Subharmonics

When a nonlinear equation is forced, the generation of “alien” harmonics due to the nonlinear terms may cause a stable subharmonic to appear for a range of parameters and over a range of frequencies. The forcing amplitude also plays a part in generating and sustaining the stable subharmonic even in the presence of damping.

We consider the undamped Duffing oscillator that has a periodic solution with the fundamental frequency equal to one-third the driving frequency. We begin with Equation 11.26,

$$\ddot{x} + \omega_n^2 x = -\varepsilon \omega_n^2 (\alpha x + \beta x^3) + F \cos \Omega t, \quad \varepsilon \ll 1,$$

except where the forcing amplitude is not necessarily small and is therefore not multiplied by ε . Let $\omega_n = \Omega/3$, and assume the following expansion in t ,

$$x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$

There is no need to expand the phase for the undamped oscillator, as we have already shown that it equals zero for an undamped system. Following the earlier procedure, we have the sequence of equations,

$$\ddot{x}_0 + \left(\frac{\Omega}{3}\right)^2 x_0 = F \cos \Omega t \quad (11.45)$$

$$\ddot{x}_1 + \left(\frac{\Omega}{3}\right)^2 x_1 = -\left(\frac{\Omega}{3}\right)^2 (\alpha x_0 + \beta x_0^3) \quad (11.46)$$

$$\ddot{x}_2 + \left(\frac{\Omega}{3}\right)^2 x_2 = -\left(\frac{\Omega}{3}\right)^2 (\alpha x_1 + 3\beta x_0^2 x_1) \quad (11.47)$$

$\vdots$

Equations 11.45 to 11.47 are solved in sequence subject to the required periodicity conditions,

$$x_i\left(\frac{\Omega}{3}t + 2\pi\right) = x_i\left(\frac{\Omega}{3}t\right),$$

and the assumed initial conditions, $x'_i(0) = 0$, for all i . The solution of Equation 11.45 is

$$x_0(t) = C_0 \cos \frac{\Omega}{3}t - \frac{9F}{8\Omega^2} \cos \Omega t. \quad (11.48)$$

This equation is substituted into the governing equation for $x_1(t)$. After expanding the x_0^3 term and simplifying to ensure the periodic solution, the factor multiplying the term $\cos(\Omega t/3)$ in the solution for $x_1(t)$ is set equal to zero, resulting in the equation

$$C_0^2 - \frac{9F}{8\Omega^2} C_0 + 2\left(\frac{9F}{8\Omega^2}\right)^2 + \frac{4\alpha}{3\beta} = 0. \quad (11.49)$$

This is a quadratic equation with roots

$$C_0 = \frac{1}{2} \frac{9F}{8\Omega^2} \pm \frac{1}{2} \sqrt{-7\left(\frac{9F}{8\Omega^2}\right)^2 - \frac{16\alpha}{3\beta}},$$

where C_0 must be a real number, meaning that the expression under the radical must be greater than or equal to zero. Additionally, defining ω_0^2 by the following expression,

$$\omega_0^2 \equiv (1 + \varepsilon\alpha) \omega_n^2 = (1 + \varepsilon\alpha) \frac{\Omega^2}{9}, \quad (11.50)$$

and solving for Ω^2 , we find the approximation

$$\Omega^2 \simeq 9\omega_0^2 (1 - \varepsilon\alpha),$$

where $(1 + \varepsilon\alpha)^{-1} \simeq (1 - \varepsilon\alpha)$, as per the discussion for Equation 11.35, here with $\omega_n = \Omega/3$.

Such oscillations are called subharmonic of order 3. The order coincides with the power of the nonlinearity in the spring restoring force.

Combination Harmonics

In linear vibration, if an oscillator is subjected to two forces of distinct frequencies, the response can be found by the superposition of the responses, one for each loading. The nonlinear counterpart is examined here. In addition to the uncoupled responses, there are responses that are integer multiples of the driving frequencies as well as linear combinations of the driving frequencies.

We consider the following Duffing equation,

$$\ddot{x} + \omega_n^2 x = -\varepsilon \beta \omega_n^2 x^3 + F_1 \cos \Omega_1 t + F_2 \cos \Omega_2 t, \quad \varepsilon \ll 1.$$

Assuming the expansion solution and equating terms of equal order, we find the following sequence of equations,

$$\ddot{x}_0 + \omega_n^2 x_0 = F_1 \cos \Omega_1 t + F_2 \cos \Omega_2 t \quad (11.51)$$

$$\ddot{x}_1 + \omega_n^2 x_1 = -\beta \omega_n^2 x_0^3 \quad (11.52)$$

$$\ddot{x}_2 + \omega_n^2 x_2 = -3\beta \omega_n^2 x_0^2 x_1$$

$\vdots$

Considering only the steady-state response, the solution of Equation 11.51 is

$$x_0(t) = G_1 \cos \Omega_1 t + G_2 \cos \Omega_2 t, \quad (11.53)$$

²⁴We follow **Fundamentals of Vibrations**, L. Meirovitch, McGraw-Hill, 2001.

where

$$G_1 = \frac{F_1}{\omega_n^2 - \Omega_1^2}, \quad G_2 = \frac{F_2}{\omega_n^2 - \Omega_2^2}.$$

Inserting Equation 11.53 into Equation 11.52, we obtain²⁵

$$\begin{aligned} \ddot{x}_1 + \omega_n^2 x_1 = & C_1 \cos \Omega_1 t + C_2 \cos \Omega_2 t \\ & + C_3 [\cos (2\Omega_1 + \Omega_2) t + \cos (2\Omega_1 - \Omega_2) t] \\ & + C_4 [\cos (\Omega_1 + 2\Omega_2) t + \cos (\Omega_1 - 2\Omega_2) t] \\ & + C_5 \cos 3\Omega_1 t + C_6 \cos 3\Omega_2 t, \end{aligned} \quad (11.54)$$

where

$$\begin{aligned} C_1 = & -\frac{3}{4}\beta\omega_n^2 G_1 (G_1^2 + 2G_2^2), \\ C_2 = & -\frac{3}{4}\beta\omega_n^2 G_2 (2G_1^2 + G_2^2) \\ C_3 = & -\frac{3}{4}\beta\omega_n^2 G_1^2 G_2, \quad C_4 = -\frac{3}{4}\beta\omega_n^2 G_1 G_2^2 \\ C_5 = & -\frac{1}{4}\beta\omega_n^2 G_1^3, \quad C_6 = -\frac{1}{4}\beta\omega_n^2 G_2^3. \end{aligned}$$

It is clear from these results that the solution of the nonlinear equation consists of various linear combinations of the harmonic components Ω_1 and Ω_2 . Since these *combination harmonics* occur in the solution to the ε^1 order term, they will generally be one order of magnitude smaller than the zero-order solution $x_0(t)$. If, however, any of the combination frequencies are near the frequency ω_n , then higher resonance-type amplitudes are possible. The expressions for G_1 and G_2 show how amplification can occur when $\Omega_1 \simeq \omega_n$ or $\Omega_2 \simeq \omega_n$.²⁶

11.5 Mathieu Equation

We return to the equations that have time-dependent coefficients. In Section 11.11, the oscillation of a pendulum suspended from a moving base was studied as an example of the stability of motion. Figure 6.60 is reproduced here as Figure 11.14. This simplified model has been used to represent fluid sloshing in a container, such as fuel in an aircraft wing or in a rocket. The reason for the equivalence is that sloshing fluid motion at its fundamental frequencies is analogous to pendulum oscillations at those fundamental frequencies.

The equations of motion were derived as

$$\ddot{y} + L\ddot{\theta} \sin \theta + L\dot{\theta}^2 \cos \theta + g = \frac{F}{m} \quad (11.55)$$

$$mL^2\ddot{\theta} + mL\ddot{y} \sin \theta + mgL \sin \theta = 0. \quad (11.56)$$

²⁵We use the trigonometric identity

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)].$$

²⁶A more general solution can be found in Jordan and Smith, pp.196-200.

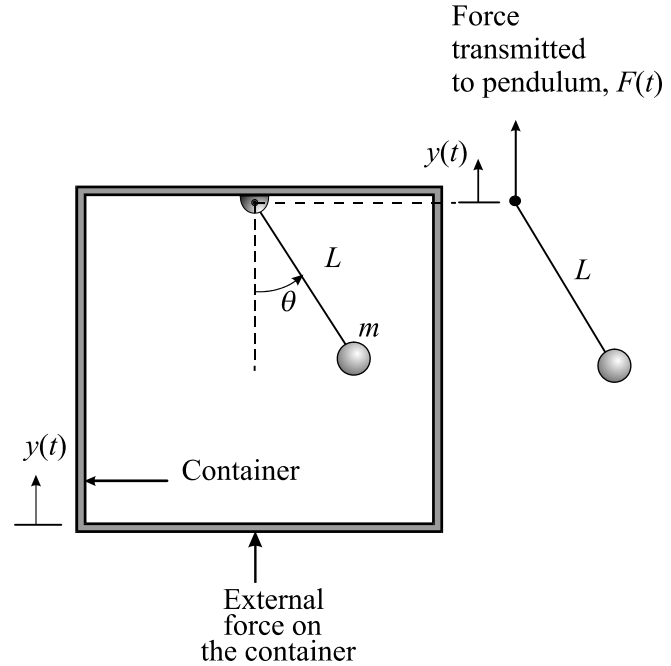


Figure 11.14: Pendulum oscillating about a vertically oscillating support. The complete system (left) and the isolated pendulum (right) are shown.

We examine the stability of motion about the equilibrium position $\theta = 0$. In the region about small θ , Equation 11.56 can be written as

$$\ddot{\theta} + \frac{1}{L}(\ddot{y} + g)\theta = 0, \quad (11.57)$$

which is nonlinear due to the coupling between θ and the acceleration in the y direction, and Equation 11.55 can be written as

$$\ddot{y} + L\ddot{\theta} + L\dot{\theta}^2 + g = \frac{F}{m}, \quad (11.58)$$

which is simplified but still nonlinear. If θ and $\dot{\theta}$ are small, higher order terms such as $\dot{\theta}^2$ and $\ddot{\theta}\theta$ may be neglected. In this case, Equation 11.58 becomes

$$\ddot{y} + g = \frac{F}{m}, \quad (11.59)$$

which is linear and decoupled from the θ motion. If the support motion is harmonic, $y = A \cos \omega t$, then, from Equation 11.59, the force required is $F(t) = -mA\omega^2 \cos \omega t + mg$. Substituting $\ddot{y}$ into Equation 11.57, gives

$$\ddot{\theta} + \frac{1}{L}(g - A\omega^2 \cos \omega t)\theta = 0, \quad (11.60)$$

which is an equation governing θ with a time-varying coefficient.

An equation of this form is called a *Mathieu equation*.²⁷ We note that $(g - A\omega^2 \cos \omega t)$ is a harmonic function that can have negative or positive values depending on the relative magnitudes of A and ω . If $g > A\omega^2$, then $g > A\omega^2 \cos \omega t$ and the coefficient of θ is always positive and the response $\theta(t)$ is always stable. If $g < A\omega^2$, we obtain solutions with coefficients of the form $\exp(+\omega T)$ that grow without bound.

The Mathieu equation has applications in many disciplines in engineering and physics. For example, it can be used to answer questions about the stability of floating bodies. Because of the possibility of instabilities, we look to delineate parameter ranges for which the solution is periodic and stable and to define the boundaries between stability and instability. The perturbation method can be used to locate the periodic solutions as well as the stability boundaries. A general form of the Mathieu equation is

$$\ddot{\theta} + (\delta + 2\varepsilon \cos 2t)\theta = 0, \quad (11.61)$$

where $\varepsilon \ll 1$, and the transformation from Equation 11.60 to Equation 11.61 is clear. This equation represents a quasi-harmonic system. For $\varepsilon = 0$, the linear harmonic oscillator is recovered where parameter δ represents the square of the natural frequency.

The character of the oscillation depends on the relative magnitudes of δ and ε and the stability regions can be conveniently identified graphically in the δ - ε plane. This plane is divided into regions of stability and instability by *boundary* or *transition* curves. Any point on such a curve is characterized by periodic solutions. (Figure 11.15, which is discussed subsequently, is an example of such a graph. The shaded regions are sometimes called *tongues* of instability.)

Lindstedt's method can be used to obtain periodic solutions of Mathieu's equation. To this end, we assume an expansion solution for $\theta(t)$ and expand the "frequency" parameter δ as follows,

$$\begin{aligned} \theta(t, \varepsilon) &= \theta_0(t) + \varepsilon \theta_1(t) + \varepsilon^2 \theta_2(t) + \cdots \\ \delta &= n^2 + \varepsilon \delta_1 + \varepsilon^2 \delta_2 + \cdots, \quad n = 0, 1, 2, \dots \end{aligned}$$

The implication in the expansion for δ is that it differs from an integer n squared by a small quantity. Again, δ represents a frequency-squared term, and since we know that the behavior is quasi-oscillatory, it is found that δ does not vary much from n^2 . Other expansion possibilities exist for δ , including non-integer expansions.²⁸

Substituting these expansions into Equation 11.61 and equating coefficients of like powers of ε , we obtain the

²⁷The Mathieu equation is a special case of *Hill's equation*. The texts cited earlier on the perturbation method provide good introductions to this class of equations. Another useful reference is **Nonlinear Ordinary Differential Equations**, R. Grimshaw, CRC Press, 1993, which includes a chapter on linear equations with periodic coefficients.

²⁸There are other expansions for δ that lead to other tongues of

following sequence of equations,

$$\ddot{\theta}_0 + n^2 \theta_0 = 0 \quad (11.62)$$

$$\ddot{\theta}_1 + n^2 \theta_1 = -(\delta_1 + 2 \cos 2t) \theta_0 \quad (11.63)$$

$$\ddot{\theta}_2 + n^2 \theta_2 = -(\delta_1 + 2 \cos 2t) \theta_1 - \delta_2 \theta_0, \quad (11.64)$$

⋮

where $n = 0, 1, 2, \dots$, with one sequence of equations for every value of n . Equations 11.62 to 11.64 are solved sequentially for each n .

The first equation in the set provides the zero-order approximation,

$$\theta_0(t) = \begin{cases} \cos nt \\ \sin nt \end{cases}, \quad n = 0, 1, 2, \dots \quad (11.65)$$

The transition curves are obtained by substituting Equations 11.65 into Equations 11.62 to 11.64 and selecting parameters so that the solutions $\theta_i(t)$, $i = 1, 2, \dots$, are periodic. This procedure leads to an infinite number of solution *pairs*, one pair for each value of n , with the exception that for $n = 0$ there is only one solution.

Case $n = 0$

First, consider the case $n = 0$ where $\theta_0 = 1$. (We neglect $\theta_0 = 0$ as a trivial solution that does not lead to any further information.) The equation for θ_1 becomes

$$\ddot{\theta}_1 = -\delta_1 - 2 \cos 2t.$$

For θ_1 to be periodic, δ_1 must equal zero, giving

$$\theta_1 = \frac{1}{2} \cos 2t.$$

The equation governing θ_2 for $n = 0$ becomes

$$\begin{aligned} \ddot{\theta}_2 &= -2 \cos 2t \left(\frac{1}{2} \cos 2t \right) - \delta_2 \\ &= -\left(\frac{1}{2} + \delta_2 \right) - \frac{1}{2} \cos 4t, \end{aligned}$$

where use is made of the trigonometric relation $\cos^2 2t = (1 + \cos 4t)/2$. For θ_2 to be periodic, the constant term on the right-hand side must be set to zero, thus, $\delta_2 = -1/2$. Hence, for $n = 0$ there is only one transition curve,

$$\delta = -\frac{1}{2} \varepsilon^2 + \cdots,$$

which, to a second-order approximation, is a parabola passing through the origin of the δ - ε plane, as depicted in Figure 11.15.

instability, for example, where the tongues emanate from

$$\delta = \frac{(2n+1)^2}{4}, \quad n = 0, 1, 2, \dots$$

The other tongues of instability emerge at higher order truncations in the various perturbation methods.

Case $n = 1$

Next, consider the case $n = 1$ with two zero-order solutions,

$$\theta_0 = \begin{cases} \cos t \\ \sin t. \end{cases}$$

We derive the transition curve for the first of these solutions, $\theta_0 = \cos t$, for which the governing equation for θ_1 becomes

$$\begin{aligned} \ddot{\theta}_1 + \theta_1 &= -(\delta_1 + 2 \cos 2t) \cos t \\ &= -(\delta_1 + 1) \cos t - \cos 3t, \end{aligned}$$

where $2 \cos 2t \cos t = \cos 3t + \cos t$ is used. We know from our earlier vibration studies that if an undamped oscillator is forced at its natural frequency it will resonate. Here, the “natural frequency” equals 1 and the “loading” $-(\delta_1 + 1) \cos t$ is also at a frequency equal to 1. Therefore, to prevent secular terms, we must set $\delta_1 = -1$. The solution of the reduced differential equation is then

$$\theta_1 = \frac{1}{8} \cos 3t.$$

Inserting θ_0 , θ_1 , and δ_1 into the governing equation for θ_2 results in the equation

$$\begin{aligned} \ddot{\theta}_2 + \theta_2 &= -\frac{1}{8}(-1 + 2 \cos 2t) \cos 3t - \delta_2 \cos t \\ &= -\left(\frac{1}{8} + \delta_2\right) \cos t + \frac{1}{8} \cos 3t - \frac{1}{8} \cos 5t, \end{aligned}$$

where $2 \cos 2t \cos 3t = \cos 5t + \cos t$. Using the same argument as before with θ_1 , for θ_2 to be periodic the coefficient of $\cos t$ must be set equal to zero, yielding $\delta_2 = -1/8$.

Truncating the calculations at second-order, the transition curve corresponding to $\theta_0 = \cos t$ is

$$\delta = 1 - \varepsilon - \frac{1}{8} \varepsilon^2 + \dots$$

Proceeding with the derivation of the transition curve corresponding to $\theta_0 = \sin t$, the equation governing θ_1 becomes

$$\begin{aligned} \ddot{\theta}_1 + \theta_1 &= -(\delta_1 + 2 \cos 2t) \sin t \\ &= -(\delta_1 - 1) \sin t - \sin 3t, \end{aligned} \quad (11.66)$$

where use is made of the relation $2 \cos 2t \sin t = \sin 3t - \sin t$. The solution of Equation 11.66 is periodic for $\delta_1 = 1$ and is

$$\theta_1 = \frac{1}{8} \sin 3t.$$

The equation governing θ_2 becomes

$$\begin{aligned} \ddot{\theta}_2 + \theta_2 &= -\frac{1}{8}(-1 + 2 \cos 2t) \sin 3t - \delta_2 \sin t \\ &= -\left(\frac{1}{8} + \delta_2\right) \sin t - \frac{1}{8} \sin 3t + \frac{1}{8} \sin 5t, \end{aligned}$$

where $2 \cos 2t \sin 3t = \sin 5t + \sin t$. For θ_2 to be periodic, it must be that $\delta_2 = -1/8$, and the transition curve corresponding to $\theta_0 = \sin t$ is

$$\delta = 1 + \varepsilon - \frac{1}{8} \varepsilon^2 + \dots$$

Case $n \geq 2$

For the cases $n = 2, 3, 4, \dots$, transition curves can be obtained in the same fashion as above. The transition curves corresponding to $n = 2$ are

$$\delta = 4 + \frac{5}{12} \varepsilon^2 + \dots, \quad (11.67)$$

for $\theta_0 = \cos 2t$, and

$$\delta = 4 - \frac{1}{12} \varepsilon^2 + \dots, \quad (11.68)$$

for $\theta_0 = \sin 2t$.

The transition curves can be plotted in the δ - ε plane defining regions of stability and instability. Figure 11.15 shows the curves generated for $n = 0, 1$, and 2. This figure is known as a *Strutt diagram*, named after Lord Rayleigh.²⁹ The region terminating at $(\delta, \varepsilon) = (1, 0)$ is known as the principal instability region. It is wider than the regions terminating at $\delta = n^2$, $n = 2, 3, \dots$. The instability regions are shaded and are symmetrical about the δ axis. The stable regions are connected at the points $\delta = n^2$, $\varepsilon = 0$, $n = 0, 1, 2, \dots$.

Such graphs can be used to determine whether the parameters used in a design are in or near an instability region. While δ and ε can be any values in a physical system, since

$$\delta = \frac{g}{L} \quad \text{and} \quad \varepsilon = -\frac{A\omega^2}{2L},$$

²⁹As noted in his biography, his birth name was John William Strutt.

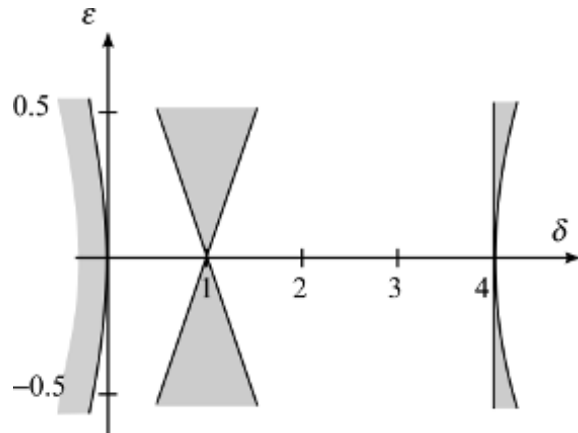


Figure 11.15: Parameter plane where shaded regions are unstable. The bounding curves are generated for $n = 0, 1$, and 2.

only certain values result in periodic solutions. The Mathieu equation with damping leads to similar results except that the shaded regions are detached from the δ axis for damping constants $c > 0$.



TACOMA NARROWS BRIDGE. On July 1, 1940, the Tacoma Narrows Bridge, connecting Seattle and Tacoma with the nearby Puget Sound Navy Yard, opened to the public after two years of design and construction. Its 853-m (2800 ft) main span connected two 128-m (420 ft) towers from which suspension cables were draped.^{30,31}

As the third longest span in the world, the Tacoma Narrows Bridge was much narrower, lighter, and more flexible than any other bridge of its time. It accommodated two lanes of traffic with a sleek appearance, which was important to the bridge's designer, Leon Moisseiff. He designed the bridge without the use of stiffening trusses, replacing them with shallower plate girders of half the weight. This modification left the Tacoma Narrows Bridge with one-third the stiffness of the Golden Gate and George Washington bridges. The bridge earned the nickname "Galloping Gertie" as large vertical oscillations occurred in even the most moderate of winds.

The bridge was designed with a 1:350 span-to-depth ratio, about one-tenth of the 1:40 ratio of the Williamsburg Bridge, and a 1:72 width-to-span ratio, much less than the 1:47 of the Golden Gate Bridge.³² The slenderness and narrowness represented a dramatic change from previous designs.

Oscillations had been observed during construction. A movie camera recorded vertical waves up to 1.2 m (4 ft) peak-to-peak at up to 25 cycles per min. Wind speeds as low as 13-17 km/h (8-10.5 mi/h) generated motions of up to 760 mm (2.5 ft). Measures taken to stop its movement proved ineffective.

On the morning of November 7, 1940 the bridge was undulating noticeably, and the bridge was closed to all traffic. The motion is captured in Figure 11.16. With a wind speed measured at 68 km/h (42 mi/h), the bridge was twisting at 38 oscillations per min with an amplitude of about 1 m. Suddenly, the bridge began twisting violently in two parts. The bridge rotated more than 45° , causing vertical movements of 8.5 m (28 ft) at the edges of the deck.

The torsional movement changed from smaller waves to two dominant twisting waves. Sidewalks and curbs began to crumble, and lampposts started coming loose. A few minutes



Figure 11.16: The 1940 Tacoma Narrows Bridge spanned the Tacoma Narrows strait of Puget Sound. It opened to traffic on July 1, 1940, and collapsed into Puget Sound on November 7. It twisted and vibrated violently on the day of the collapse. (James Bashford Press Photos, PH Coll. 290.30 University of Washington Libraries. Special Collections Division)



Figure 11.17: In many physics textbooks, the Tacoma Narrows Bridge collapse is presented as an example of forced resonance, with the wind providing an external periodic frequency that matched the bridge's natural structural frequency. The cause of failure was more complicated and involved aeroelastic flutter. (University of Washington Libraries, Special Collections, UW21422)

³⁰The history of the Tacoma Narrows Bridge design, collapse, and investigation is described in detail in **Beyond Failure**, Norbert J. Delatte Jr., ASCE Press, 2009. The summary here is based significantly on this reference.

³¹The longest span of the Brooklyn Bridge, which opened to traffic in 1883, was 486 m (1595 ft). The George Washington Bridge (1931) more than doubled that distance to 1070 m (3500 ft). The Golden Gate Bridge (1937) went even farther with a middle span of 1280 m (4200 ft).

³²To help understand the width-to-span ratio, think of the Tacoma Narrows bridge as being 72 cm long and 1 cm wide (or in U.S. Customary units 6 ft long and 1 in wide).

later, the stiffening girders in the middle of the bridge buckled, initiating the collapse shown in Figure 11.17. The suspender cables broke, and large sections of the main span dropped progressively, from the center outward, into the river below.³³

A board of engineers was assembled to investigate the collapse. They concluded that as a result of the Tacoma Narrows Bridge's extreme flexibility, narrowness, and lightness, the random force of the wind that day caused the torsional oscillations that destroyed the bridge. The wind-induced oscillations approached the natural frequencies of the structure, causing resonance.

This explanation overlooks the important question as to how wind, random in nature, could produce periodic forcing. A different explanation, proposed by von Kármán, attributed the motion of the bridge to the periodic shedding of air vortices, which created a wake known as a von Kármán's street. This wake reinforced the structural oscillations, eventually causing the bridge to collapse.³⁴

Another explanation is that a different kind of vortex, one associated with the structural oscillation itself, was created with the same frequency as the bridge. The resonance between the bridge and these vortices caused excessive motion, destroying the bridge.

Current thinking is that the bridge was destroyed by aeroelastic flutter resulting in negative damping and increasing amplitudes leading to failure. The equation that governs aeroelastic flutter is a differential equation with time-dependent coefficients like the Mathieu equation and van der Pol equation.

Although the theories differ, they all agree that the extreme flexibility, slenderness, and lightness of the Tacoma Narrows Bridge allowed the oscillations to grow until structural failure occurred. Through the years, the collapse has been studied and revisited frequently.³⁵

11.6 Van der Pol Equation

The van der Pol equation has been used for over a century to model a very special class of nonlinear behavior. Such behavior is known as *self-excited*, exhibited by a system always in motion, at times dissipating energy and at other times drawing in energy.

One example is *flow-oscillator models* of vortex-induced structural oscillations. Bishop and Hassan³⁶ are credited with first suggesting the idea of using a van der Pol type oscillator to represent the time-varying forces on a cylinder due to vortex shedding. Hartlen and Currie³⁷ formulated the most noteworthy of the early oscillator models. In their model, a van der Pol soft³⁸ nonlinear oscillator for the lift force is coupled to the cylinder motion by a linear dependence on the cylinder velocity. The cylinder motion is restricted to pure translation in the transverse direction, perpendicular to both the flow direction and the cylinder axis. The cylinder is restrained by linear springs and is linearly damped.

The mathematical model is given by the pair of coupled nondimensional differential equations,

$$x_r'' + 2\zeta x_r' + x_r = a\omega_o^2 c_L \quad (11.69)$$

$$c_L'' - \alpha\omega_o c_L' + \frac{\gamma}{\omega_o}(c_L')^3 + \omega_o^2 c_L = bx_r', \quad (11.70)$$

where primes denote derivatives with respect to the nondimensional time $\tau = \omega_n t$, x_r is the dimensionless cylinder displacement, ζ is the material damping ratio, a is a known dimensionless constant, c_L is the lift coefficient, which is the amplitude of the response of the cylinder displacement, and ω_o is the ratio of the Strouhal shedding frequency to the natural frequency of the cylinder, $\omega_o = f/f_n$. The dimensionless Strouhal number is defined as $St = fD/V$, where D is the cylinder diameter, V is the flow velocity, and f is the frequency of shedding.

Of the undetermined parameters, α , γ , and b , only two must be chosen to provide the best fit to experimental data. This follows from the fact that α and γ are related to each other by the expression $C_{Lo} = (4\alpha/3\gamma)^{1/2}$, where C_{Lo} is the amplitude of the fluctuation of c_L for a fixed cylinder. The second term on the left-hand side of Equation 11.70 provides the growth of the lift coefficient c_L , and the third term on the left-hand side prevents its unlimited growth. These terms are important to the success of the model because the large amplitude oscillations characteristic of vortex induced vibration are accompanied by a significant (yet finite) increase in the lift coefficient.

With the appropriate choice of parameters, the Hartlen and Currie model qualitatively captures many of the features seen in experimental results. For example, a large cylinder oscillation amplitude resonance region occurs when the vortex shedding frequency is near the

³³The only fatality was a black cocker spaniel named Tubby.

³⁴A problem with this theory is that the calculated frequency of a vortex caused by a 68 km/h wind is 1 Hz, whereas the frequency of the torsional oscillations of the bridge was measured at 0.2 Hz.

³⁵The collapse influenced the design of the Verrazano-Narrows Bridge, which opened in 1964 to link Brooklyn and Staten Island. Its longest span was 1300 m (4260 ft) making it, at the time, the world's longest suspension bridge. To minimize vibration, the designers made the roadway very stiff and added a second deck, even though the volume of traffic was insufficient to warrant it. The lower deck opened for traffic in 1969.

³⁶R.E.D. Bishop and A.Y. Hassan, "The Lift and Drag Forces on a Circular Cylinder in a Flowing Fluid," *Proceedings of the Royal Society. Series A, Mathematical and Physical Sciences*, Vol.277, 1963, pp.32-50.

³⁷R.T. Hartlen and I. G. Currie, "Lift-Oscillator Model of Vortex Induced Vibration," *Journal of the Engineering Mechanics*, Vol.96, No.5, 1970, pp.577-591.

³⁸The concept of a soft van der Pol oscillator is introduced in Subsection 11.6.3.

natural frequency of the cylinder. The frequency of oscillation in this region is nearly constant at a value close to the cylinder natural frequency.

11.6.1 Unforced van der Pol Equation

The unforced van der Pol equation is given by

$$\ddot{x} - \alpha(1 - x^2)\dot{x} + x = 0, \quad \alpha > 0. \quad (11.71)$$

If α is small, then the ε notation can be used and the van der Pol equation becomes

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}. \quad (11.72)$$

Expanding x to order ε^2 ,

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2,$$

the term $(1 - x^2)\dot{x}$ becomes

$$\begin{aligned} & (1 - x^2)\dot{x} \\ & \simeq \left[1 - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^2\right] (\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2) \\ & \simeq (1 - x_0^2)\dot{x}_0 + \varepsilon [-2x_0x_1\dot{x}_0 + (1 - x_0^2)\dot{x}_1] \\ & \quad + \varepsilon^2 [-x_1^2\dot{x}_0 - 2x_0x_2\dot{x}_0 - 2x_0x_1\dot{x}_1 + (1 - x_0^2)\dot{x}_2]. \end{aligned}$$

Inserting this result into Equation 11.72, and equating terms of the same order in ε , gives the following sequence of equations,

$$\begin{aligned} \varepsilon^0 : \ddot{x}_0 + x_0 &= 0 \\ \varepsilon^1 : \ddot{x}_1 + x_1 &= (1 - x_0^2)\dot{x}_0 \\ \varepsilon^2 : \ddot{x}_2 + x_2 &= -2x_0x_1\dot{x}_0 + (1 - x_0^2)\dot{x}_1 \\ \varepsilon^3 : \ddot{x}_3 + x_3 &= -x_1^2\dot{x}_0 - 2x_0x_2\dot{x}_0 - 2x_0x_1\dot{x}_1 + (1 - x_0^2)\dot{x}_2 \\ & \vdots \end{aligned}$$

The equation of $O(\varepsilon^3)$ is ignored since the expansion is to order ε^2 .

Each equation is now solved sequentially and the approximate solution for $x(t)$ is given by

$$x(t, \varepsilon) \simeq x_0 + \varepsilon x_1 + \varepsilon^2 x_2.$$

The validity of this approximate solution is established by comparisons with experimental data. We will solve the forced van der Pol equation subsequently, but first we discuss its limit cycles.

11.6.2 Limit Cycles

In Section 11.3.1, we introduced the stability of equilibria and found that all motion either tended toward the equilibrium point or became unstable. It is also possible, however, that instead of an equilibrium point there is a

closed trajectory in the phase plane called an *equilibrium path*. This trajectory can be approached from either the inside or the outside of the path, as we will see next.

For such damped vibration problems where there is an equilibrium path, the trajectories can start with initial conditions either close to the origin or far away from the origin, and all approach the same closed curve about the origin. This curve represents a periodic, but not harmonic, solution of the governing equation and is called a *limit cycle*. The classical equation that has such a limit cycle is the *van der Pol* oscillator, one form of which is Equation 11.71.

This equation governs systems from a number of disciplines, including nonlinear mechanical systems. Physically, for vibratory systems, the form of the damping force, $\alpha(1 - x^2)\dot{x}$, is such that the damping is negative for small values of the amplitude x and positive for larger amplitudes, depending on the sign of $(1 - x^2)$.

The van der Pol equation can be plotted in the phase plane. Starting with the state-space form of the van der Pol equation,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \alpha(1 - x^2)y - x, \end{aligned}$$

the trajectories are defined by

$$\frac{dy}{dx} = \frac{\alpha(1 - x^2)y - x}{y}.$$

Regardless of initial conditions, the trajectories approach the limit cycle asymptotically. An initial point inside the limit cycle follows an outwardly spiraling trajectory. An initial point outside the limit cycle follows an inwardly spiraling trajectory. An infinite number of *isoclines*³⁹ pass through the origin, which is a singularity. The limit cycle has the interesting property that the maximum value of x is always close to 2 regardless of the value of α .

Figure 11.18 shows the trajectories for $\alpha = 0.3$ and $\alpha = 3$. Figure 11.19 shows the time histories of this oscillator for the same two values of α , for several initial conditions. The van der Pol equation can only be solved numerically or approximately, as in the next section via the perturbation method.

11.6.3 Forced van der Pol Equation

The forced van der Pol equation is a model for a system that is capable of self-excitation while acted upon

³⁹ An isocline is defined as the locus of points at which the trajectories passing through them have a constant slope. The method of isoclines is used to construct the trajectories of dynamic systems with one degree of freedom.

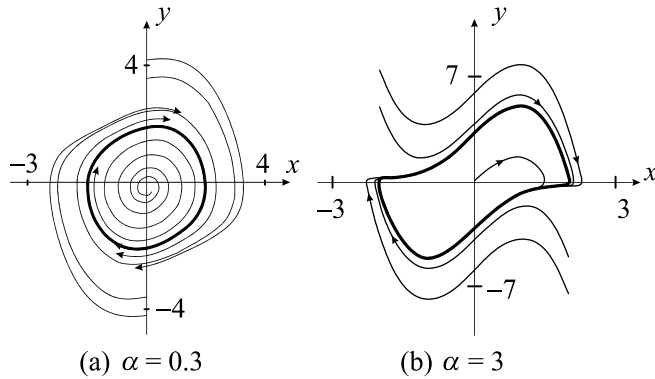


Figure 11.18: Trajectories for the van der Pol equation from a number of initial conditions for (a) $\alpha = 0.3$ and (b) $\alpha = 3$. The limit cycles are the closed dark lines.

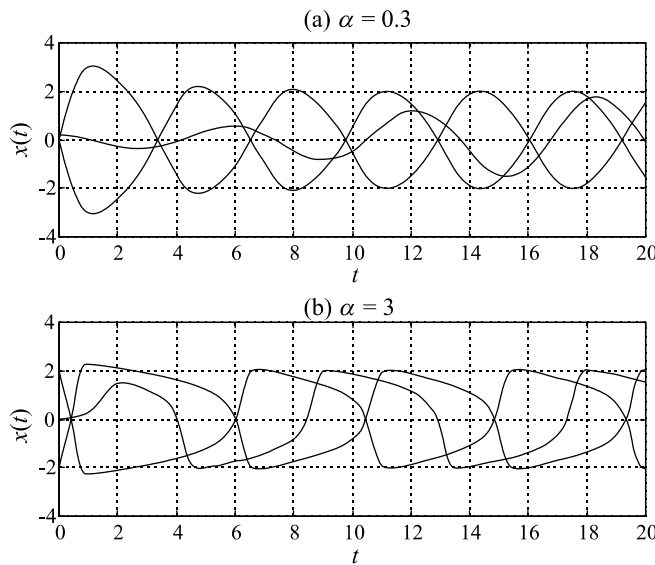


Figure 11.19: Responses of van der Pol equation for $\alpha = 0.3$ and 3 for three sets of initial conditions for each α .

by another oscillator. Rand⁴⁰ points to a biological application involving the human sleep-wake cycle in which a person's biological clock is modeled by a van der Pol oscillator and the daily night-day cycle caused by the Earth's rotation is modeled as a periodic forcing term.

Entrainment

The phenomenon where the period of a forced oscillation is an integer multiple of the period of the forcing is called *frequency entrainment*. The frequency of the response, say ν , is said to be entrained by the frequency of the forcing, say ω , so that $\nu = \omega/l$ for some positive integer l called the *entrainment index*. The van der Pol equation has such characteristics.

⁴⁰Lecture Notes on Nonlinear Vibrations, R.H. Rand. The latest version of the lecture notes can be found at his Cornell University website.

As an example of entrainment, in vortex-induced vibration, as the flow speed increases, the shedding frequency of the vortices increases as well. But when the shedding frequency is approximately equal to the natural frequency of the structure, even with increased flow speed, the shedding frequency locks onto the structural frequency for a range of speeds up to a point where it suddenly resumes its upward trend. This is the jump phenomenon we discussed earlier and is shown in Figure 11.13.

Hard and Soft Excitation

The unforced van der Pol solution has a limit cycle with a radius of approximately 2 and a period of approximately 2π . The limit cycle is generated by the balance between the internal energy loss and energy generation. A forcing term will alter this balance.

Consider the following forced van der Pol equation,

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F \cos \omega t. \quad (11.73)$$

If F is of small magnitude, the excitation is called *weak* or *soft*, and its effect depends on whether or not ω is close to the natural frequency. If it is, an oscillation will be generated that is only a perturbation of the unforced limit cycle. If F is not of small amplitude, the excitation is called *hard*. If the natural and imposed frequencies are not close, we expect that the free oscillation will be damped out as happens with the corresponding linear equation.

To further study this equation, we scale time according to $\omega t = \tau$, for which Equation 11.73 becomes

$$\omega^2 x'' + \varepsilon \omega (x^2 - 1) x' + x = F \cos \tau.$$

A number of cases require further examination. For example, if the hard excitation is far from resonance and ω is not close to one, expanding x as usual,

$$x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \dots, \quad (11.74)$$

we find

$$\begin{aligned} \omega^2 x_0'' + x_0 &= F \cos \tau \\ \omega^2 x_1'' + x_1 &= -\omega(x_0^2 - 1)x_0', \end{aligned}$$

where $x_0(\tau)$ and $x_1(\tau)$ have a period 2π . Therefore,

$$x_0(\tau) = \frac{F}{1 - \omega^2} \cos \tau.$$

Since the solution of the $x_1(\tau)$ equation is of $O(\varepsilon)$, the approximate solution is

$$x(\tau, \varepsilon) = \frac{F}{1 - \omega^2} \cos \tau + O(\varepsilon).$$

This is a perturbation of the ordinary linear response and the limit cycle is suppressed as expected.

If the excitation is soft and far from resonance, the procedure is similar to that for the hard excitation. For this case, the response is usually unstable. Instead, consider a soft excitation near resonance. Letting $F = \varepsilon\gamma$, assuming the system is near resonance with $\omega = 1 + \varepsilon\omega_1$, and using the expansion Equation 11.74, we obtain the sequence of equations

$$\begin{aligned} x_0'' + x_0 &= 0 \\ x_1'' + x_1 &= -2\omega_1 x_0' - (x_0^2 - 1)x_0' + \gamma \cos \tau \\ &\vdots \end{aligned} \quad (11.75)$$

where the forcing term is the last term in the second equation on the right-hand side. As usual, since solutions are sought with a period 2π , the solution for $x_0(\tau)$ is given by

$$x_0(\tau) = a_0 \cos \tau + b_0 \sin \tau,$$

and Equation 11.75 becomes

$$\begin{aligned} x_1'' + x_1 &= \left[\gamma + 2\omega_1 a_0 - b_0 \left(\frac{1}{4}r_0^2 - 1 \right) \right] \cos \tau \\ &+ \left[2\omega_1 b_0 + a_0 \left(\frac{1}{4}r_0^2 - 1 \right) \right] \sin \tau \\ &+ \dots \\ r_0 &= +\sqrt{a_0^2 + b_0^2}, \end{aligned}$$

where r_0 is the response amplitude for $x_0(\tau)$. For a periodic solution to exist, the coefficients of the harmonic functions must be set equal to zero, giving

$$\begin{aligned} 2\omega_1 a_0 - b_0 \left(\frac{1}{4}r_0^2 - 1 \right) &= -\gamma \\ 2\omega_1 b_0 + a_0 \left(\frac{1}{4}r_0^2 - 1 \right) &= 0. \end{aligned}$$

These two equations can be combined,

$$r_0^2 \left[4\omega_1^2 + \left(\frac{1}{4}r_0^2 - 1 \right)^2 \right] = \gamma^2,$$

which can be solved to give possible values of r_0 . Note that there may be as many as 3 real solutions for $r_0 > 0$. These are the possible amplitudes of the x_0 solution. More analysis is needed to determine which solution is physically viable, and it is then used to determine the solution $x_1(\tau)$.

BALTHASAR VAN DER POL

(January 27, 1889 – October 6, 1959)



Contribution: Van der Pol was a Dutch electrical engineer. His main interests were in radio wave propagation, theory of electrical circuits, and mathematical physics. The equation named after him is an ordinary differential equation describing self-sustaining oscillations in which energy is fed into small oscillations and removed from large oscillations.

He initiated modern experimental dynamics during the 1920s and 1930s. He investigated electrical circuits with vacuum tubes and showed they have stable oscillations called limit cycles. When these circuits are driven with a signal whose frequency is near that of the limit cycle, the periodic response shifts its frequency to that of the driving signal. The circuit is said to become “entrained” to the driving signal. The waveform, or signal shape, can be quite complicated and contains harmonics and subharmonics.

Van der Pol built a number of circuit models of the human heart to study the range of stability of heart dynamics. His investigations with an external driving signal were analogous to the situation in which a heart is driven by a pacemaker. He was interested in finding out, using his entrainment work, how to stabilize a heart’s irregular beating called arrhythmias.

Life Sketch: Balthasar van der Pol was born in Utrecht in the Netherlands. He was talented at a young age. He attended the HBS in Utrecht and graduated from the gymnasium in 1911. He later attended the University of Utrecht and studied mathematics and physics until 1916. He graduated with a degree in physics and received an award of the highest distinction.

In 1916, he studied with John Ambrose Fleming, an English electrical engineer and physicist. After a year of working with Fleming, van der Pol moved to Cambridge to study with John Joseph Thompson at the Cavendish Laboratory. Van der Pol spent two years investigating the diffraction of radio waves around a conducting earth at Cavendish Laboratory. He made a direct comparison between signal strength predicted and signal strength received in a practical case of

radio transmission.

In 1919, van der Pol returned to the Netherlands and was appointed to the Teylers Museum in Haarlem. There, he completed his doctoral thesis on the effect of ionized gas on electromagnetic wave propagation and its application to radio. Van der Pol was awarded the degree of doctor of science with distinction in 1920 from the University of Utrecht.

In 1922, van der Pol left Teylers Museum to take up an appointment as head physicist at Philips Physical Laboratory in Eindhoven.

After his retirement in 1949, he became Director of Comité Consultatif International des Radiocommunications in Geneva. He continued in this position until 1956.

Van der Pol died in Wassenaar, a town in the western Netherlands, at age 70.

Noteworthy: Van der Pol made a number of advances in propagation of radio waves, nonlinear circuits, relaxation oscillations, transient phenomena, and operational calculus.

In 1927, he was made Knight of the Order of Oranje Nassau for establishing the first radio-telephonic communication between the Netherlands and the Dutch East Indies.

In the September 1927 issue of the British journal *Nature*, he and his colleague van der Mark reported that an “irregular noise” was heard at certain driving frequencies between the natural entrainment frequencies. By reconstructing his electronic tube circuit, we now know that they had discovered deterministic chaos. Their paper is probably one of the first experimental reports of chaos.

The van der Pol oscillator was named after him. He was awarded the Institute of Radio Engineers (now the IEEE) Medal of Honor in 1935. The asteroid 10443 van der Pol was named after him.

11.7 Motion in the Large

Motion in the large refers to the behavior of the complete nonlinear governing equations, rather than system behavior about the equilibria or perturbation expansions that assume small values in some of the parameters. Nonlinear equations have no general solutions and usually require numerical solutions. However, for single degree-of-freedom nonlinear differential equations a somewhat general discussion is possible that sheds light on large behavior.

Consider the conservative nonlinear equation

$$\ddot{x} = f(x),$$

where $f(x)$ is a nonlinear conservative force per unit mass. It is useful to consider the energy possessed by

the system. The left-hand side can be written as follows,

$$\begin{aligned}\ddot{x} &= \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \frac{dx}{dt} \\ &= \frac{d\dot{x}}{dx} \dot{x}.\end{aligned}$$

Then, $\ddot{x}dx = \dot{x}d\dot{x} = f(x)dx$, and

$$\begin{aligned}\int_0^x \ddot{x}dx &= \int_0^{\dot{x}} \dot{x}d\dot{x} \\ &= \int_0^x f(\xi)d\xi + c,\end{aligned}\quad (11.76)$$

where

$$\int_0^{\dot{x}} \dot{x}d\dot{x} = \frac{1}{2}\dot{x}^2$$

is the kinetic energy per unit mass,

$$\int_0^x f(\xi)d\xi = -V(x),$$

is the negative of the potential energy per unit mass, and c is a constant of integration. Equation 11.76 can be written as

$$\frac{1}{2}\dot{x}^2 + V(x) = c.$$

This equation is a statement that the total system energy is constant, $E = c$.

We can plot the energy function in the state plane. If $y = \dot{x}$, then

$$\frac{1}{2}y^2 + V(x) = E. \quad (11.77)$$

Equation 11.77 can be plotted as E vs. x and y , and for a particular energy level, the projection plot of x vs. y is also useful. For the former, a three-dimensional surface can be created, one projection of which is shown at the bottom of Figure 11.20. For lines of constant energy (top of Figure 11.20), it is possible to visualize the possible trajectories (bottom of Figure 11.20).

Example 11.10 Trajectories and Stability of a Planar Pendulum

For the pendulum of Example 11.2 identify the equilibrium points and determine the nature of their stability for the energy levels $E = \omega_n^2, 2\omega_n^2, 3\omega_n^2$.

Solution: From Example 11.2 the equation of motion is

$$\ddot{\theta} + \omega_n^2 \sin \theta = 0,$$

and the equations of the trajectories, with $\theta = x$, are

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega_n^2 \sin x.\end{aligned}$$

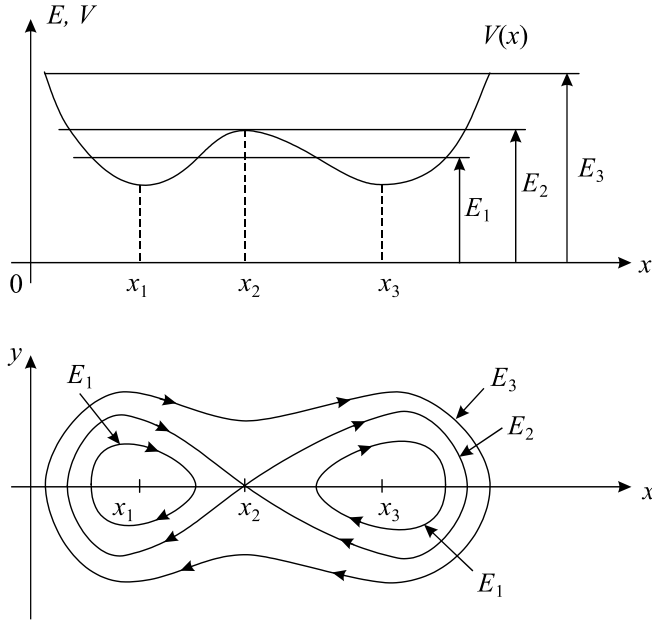


Figure 11.20: Energy curves shown in profile (top) and projection (bottom). The projection curves correspond to the energy “slices” in the top figure. In the bottom graph the energy axis E points out of the page.

The equilibrium equations are obtained for zero velocity $\dot{x} = 0$ and acceleration $\ddot{y} = 0$, and thus,

$$y = 0, \quad \sin x = 0.$$

The equilibrium points are then given by

$$\begin{aligned} y &= 0 \\ x &= \pm j\pi, \quad j = 0, 1, 2, \dots \end{aligned}$$

Physically, there are only two equilibrium points since the pendulum can only trace a circle,

$$y = 0 \text{ with } x = 0 \quad (11.78)$$

$$y = 0 \text{ with } x = \pi \text{ rad.} \quad (11.79)$$

The first equilibrium point, Equation 11.78, is the one usually considered in which the pendulum hangs at rest. The other equilibrium point, Equation 11.79, is where the pendulum is at rest in the upright position. If the pendulum is composed of a rigid massless rod with a point mass at one end, then it might be possible, if perfectly balanced, that the upright position can be a rest equilibrium point. However, it is unstable under the slightest disturbance.

Using the method of Section 11.3.1, the coefficient matrix in Equation 11.14 for the linearized state equations for the first equilibrium point $x = 0$, with $\sin x \simeq x$, becomes

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}.$$

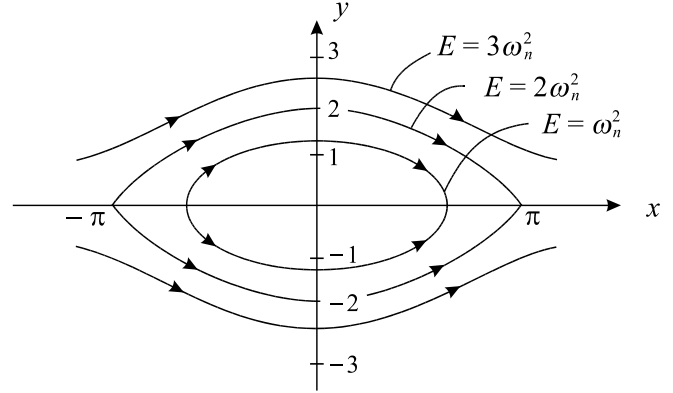


Figure 11.21: Trajectories at three energy levels for Example 11.10. The trajectory for energy level $2\omega_n^2$ is known as the *separatrix* since it separates oscillatory trajectories from non-oscillatory trajectories.

The eigenvalues of the coefficient matrix are found by setting the following determinant to zero,

$$\begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -\lambda \end{vmatrix} = 0,$$

which yields the eigenvalues $\lambda_{1,2} = \pm i\omega_n$. Both eigenvalues are purely imaginary and the equilibrium point is a center and stable.

In the neighborhood of the second equilibrium point, $x = \pi$ rad, we expand $\sin(x + \pi)$ for small values of x to find the approximation $\sin(x + \pi) \simeq -x$, with the coefficient matrix

$$\begin{bmatrix} 0 & 1 \\ \omega_n^2 & 0 \end{bmatrix},$$

from which the eigenvalues are $\lambda_{1,2} = \pm\omega_n$. Both eigenvalues are real, with one positive and the other negative, implying that the point is a saddle and unstable.

The trajectories are shown in Figure 11.21 for three energy levels. For $E = \omega_n^2$ the trajectory is closed, indicating periodic motion. For small values of E where $E < \omega_n^2$, the motion is approximately harmonic. As E increases such that $\omega_n^2 < E < 2\omega_n^2$, the motion ceases to be approximately harmonic but remains periodic. For $E \geq 2\omega_n^2$, the trajectory is open and the motion is rotary, like a propeller, as the mass rotates about the contact point. $E = 2\omega_n^2$ is a *separatrix*, signifying the boundary between two types of motion. The separatrix also describes rotary motion. ■

11.8 Nonlinear Control

In a nonlinear system, the general form of a state-space model can be written as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t),\end{aligned}$$

where the first equation is the state equation and the second equation is the output equation. The equations have been written as nonlinear functions of the state vector $\mathbf{x}(t)$, the input vector $\mathbf{u}(t)$, and possibly time t . For a single-input, single-output system, these equations can be written as

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= h(x(t), u(t), t),\end{aligned}$$

reflecting the fact that there is a single input $u(t)$, a single output $y(t)$, and a state vector $\mathbf{x}(t)$ whose elements are the state variables.

For vibration studies perhaps the most important characteristic of nonlinear system behavior that is not part of linear theory is the *limit cycle*. Self-sustaining oscillations due to stable limit cycles are often exhibited in nonlinear control systems.

Consider a feedback system that contains a nonlinear controller and a linear plant. This case is depicted in the unity feedback system of Figure 11.22, which shows the block diagram in the frequency-domain, and of Figure 11.23, which shows the block diagram in the time-domain. (The double box notation is used to represent a nonlinear element.) We will assume that there is no reference input (that is, the system is autonomous) and that the input to the nonlinear controller – which is the same as the output of the plant – can be approximated as a sinusoidal signal $e(t) = e_0 \sin \omega t$ where e_0 is the amplitude of the input sinusoid. This latter assumption reflects the condition that the nonlinear system is oscillating. The output of the nonlinear controller is, in general, not sinusoidal.

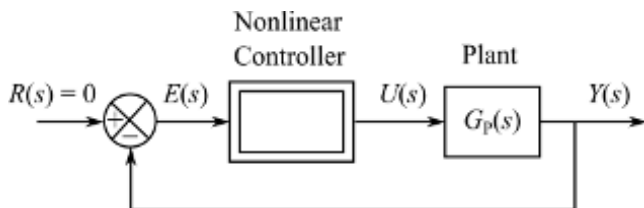


Figure 11.22: Block diagram of nonlinear feedback control system using frequency-domain representation.

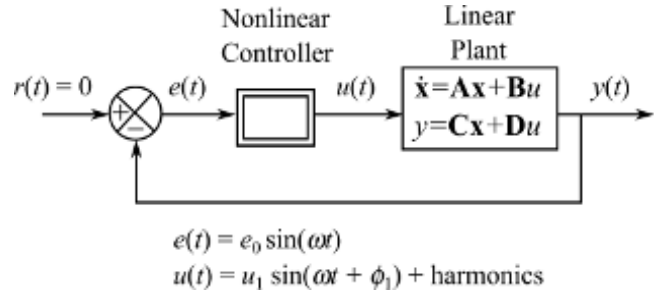


Figure 11.23: Block diagram of nonlinear feedback control system using time-domain representation.

If we assume that the output of the nonlinear controller is periodic with the same period as its input, then it may be expanded in a Fourier series,

$$\begin{aligned}u(t) &= \frac{a_0}{2} + \sum_{p=1}^{\infty} (a_p \cos p\omega t + b_p \sin p\omega t) \\ &= \frac{a_0}{2} + \sum_{p=1}^{\infty} u_p \sin(p\omega t + \phi_p).\end{aligned}$$

We can say that $u(t)$ consists of the sum of a fundamental component

$$u_1 \sin(\omega t + \phi_1),$$

a mean value $a_0/2$, and harmonic components at frequencies $2\omega, 3\omega, 4\omega, \dots$

One method to proceed is called the *describing function* method. It assumes that only the fundamental harmonic is significant. This assumption is based on the *filtering hypothesis* requiring that the linear part of the system modeled by the transfer function $G(s)$ be a low-pass filter. In a low-pass filter the magnitude of the transfer function or frequency response function is attenuated as the frequency increases. Mathematically, it can be expressed as

$$|G(i\omega)| \gg |G(ip\omega)| \quad \text{for } p = 2, 3, 4, \dots$$

The filtering hypothesis is valid in general since most real systems have low-pass frequency characteristics, meaning the higher harmonics are attenuated compared to the fundamental component. As a consequence, although the input to the plant (that is, the output of the nonlinear controller) may be highly non-sinusoidal – rich in harmonics – the output of the plant is likely to be nearly sinusoidal. This is true since the harmonics in the input have been attenuated (“filtered out”) by the plant. Thus, the input $e(t)$ to the nonlinear controller, which is the negative of the plant output, is nearly sinusoidal. It is for this reason that $e(t)$ is assumed to be a pure sinusoid.

The nonlinear controller is then approximated by its describing function,

$$N(e_0) = \frac{u_1}{e_0} \angle \phi_1.$$

The describing function consists of both magnitude and phase, where the magnitude is the ratio of the fundamental component of the output to the input amplitude. In general, it is a complex-valued function of the input amplitude e_0 and frequency. Using the describing function method we can approximate the nonlinear control system as a linear control system shown in Figure 11.24.

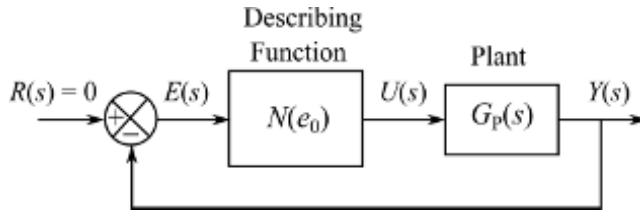


Figure 11.24: Block diagram of the linear system approximating the nonlinear system using the describing function method.

Assuming that the poles of $G_P(s)$ are in the left-half complex plane, the closed-loop system is asymptotically stable. For a limit cycle to exist, there must be an intersection between the loci of $G_P(i\omega)$ and $-1/N$ in the complex plane. We explore these ideas in the following example.

Example 11.11 Relay Control of a Linear System

Use the describing function method to analyze the nonlinear control system shown in Figure 11.25 with the controller characteristic given in Figure 11.26.

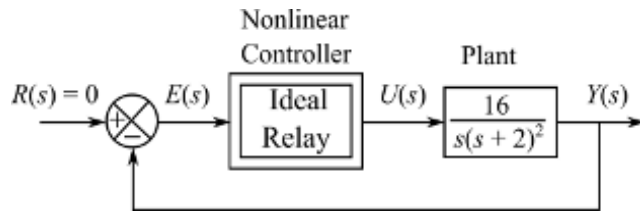


Figure 11.25: Block diagram of nonlinear feedback control system.

Solution: The nonlinear controller is a “bang-bang” controller. In control parlance, the controller is called an *ideal relay*: it is either “full on” in one direction or “full on” in the other direction. This kind of control action can occur with a very high gain saturating amplifier.

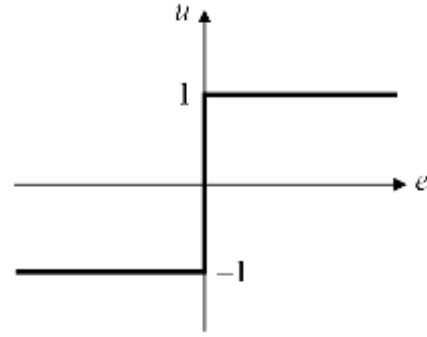


Figure 11.26: Characteristic of nonlinear controller: ideal relay.

When the input of the ideal relay is greater than zero, its output is 1. When the input is less than zero, its output is -1 . Thus, the operation of the relay can be expressed mathematically as

$$u(t) = \begin{cases} 1 & \text{for } e(t) > 0 \\ -1 & \text{for } e(t) < 0. \end{cases}$$

If the input to the relay is a sinusoid $e(t) = e_0 \sin \omega t$, the output is a square wave with the amplitude shifting between 1 and -1 every $t = \pi p / \omega$, $p = 1, 2, \dots$. The period of the square wave is $2\pi / \omega$.

Since a square wave is an odd function, the Fourier coefficients $a_p = 0$, $p = 0, 1, 2, \dots$ and thus the Fourier series of the output of the relay can be written as

$$u(t) = \sum_{p=1}^{\infty} b_p \sin p\omega t.$$

The fundamental harmonic of $u(t)$ is $b_1 \sin \omega t = u_1 \sin \omega t$, where

$$\begin{aligned} b_1 &= \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} y(t) \sin \omega t dt \\ &= \frac{2\omega}{\pi} \int_0^{\pi/\omega} \sin \omega t dt = -\frac{2}{\pi} \cos \omega t \Big|_0^{\pi/\omega} \\ &= \frac{4}{\pi}. \end{aligned}$$

Hence, the describing function of the ideal relay is

$$N(e_0) = \frac{u_1}{e_0} \angle 0 = \frac{4}{\pi e_0}.$$

The transfer function of the linear plant model is

$$G(s) = \frac{16}{s(s+2)^2},$$

which has two poles at $s = -2$ and one pole on the imaginary axis. The plant is marginally stable. Setting $s = i\omega$,

$$G(i\omega) = \frac{16}{i\omega(i\omega+2)^2} = \frac{-64\omega + 16(\omega^2 - 4)i}{\omega(\omega^2 + 4)^2}.$$

The imaginary term vanishes at $\omega = 2$ making $G(i\omega)$ real with the value $-1/16$. Setting this equal to $-1/N = -\pi e_0/4$ gives the amplitude of the limit cycle as $e_0 = 4/\pi$. The amplitude of the fundamental harmonic of the plant output is

$$\frac{4}{\pi} |G(i2)| = \frac{4}{\pi}.$$

The filtering hypothesis is satisfied in this example. The Fourier series of the output of the relay is

$$\frac{4}{\pi} \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \cdots \right).$$

Thus, for example, the amplitude of the third harmonic of the plant output is

$$\frac{4}{3\pi} |G(i6)| = \frac{1}{45} \frac{4}{\pi}.$$

The higher harmonics are highly attenuated compared with the fundamental harmonic. ■

In summary, the describing function method allows us to apply frequency-domain techniques used in linear system analysis to the analysis of nonlinear systems. The method can be used to predict the presence of limit cycles in nonlinear systems. The describing function method is based on the Fourier series and can be viewed as “harmonic linearization” of a nonlinear element. The method provides a linear approximation to the nonlinear element based on the assumption that the input to the nonlinear element is a sinusoid of known, constant amplitude.

IRMGARD FLÜGGE-LOTZ

(July 16, 1903 – May 22, 1974)



Contribution: Irmgard Flügge-Lotz was a German mathematician and engineer, known for her work on the

mathematics of aerodynamics and for being the first female engineering professor at Stanford University.

Life Sketch: Irmgard Flügge-Lotz was born in Hamelin, Germany on July 16, 1903. Lotz was raised by Osark and Dora (Grupe) Lotz. While her father was a traveling journalist, her mother contributed to the construction business established by the Grupe family. By visiting building sites, Lotz developed a fascination with the art of construction.

In 1914, Lotz’s father accepted a position writing for one of Hanover’s city newspapers. At that time, Lotz was enrolled in a high school for girls, there being no coed education. The day her father was to report to his new job, World War I began, during which he was assigned to occupied Belgium for several years. With the loss of the breadwinner, Irmgard Lotz felt obligated to contribute to the family budget. Irmgard tutored students in mathematics and Latin, providing most of the family’s income. With the return of her father in poor health, Lotz continued to financially support the family.

Lotz graduated from high school in 1923 and enrolled in the male-dominated Technical University in Hanover where she studied applied mathematics. She was the only woman in many of her classes. In 1927 she earned the degree of Diplom-Ingenieur, and in 1929 she became a Doktor-Ingenieur (that is, earned her doctorate in engineering) writing a thesis on the mathematical theory of heat conduction in circular cylinders. During this time she held a full-time job as a teaching assistant for practical mathematics and descriptive geometry.

Though it was difficult for a female to find a job in engineering, Lotz received two offers, one from the steel industry and one from a research institute, the Aerodynamische Versuchsanstalt (AVA) in Göttingen. She chose the latter, and was asked to devote half of her time to clerical work, leaving the other half for research.

She worked closely with the leading German aerodynamicists of the time, Ludwig Prandtl and Albert Betz, director of the institute. Before Lotz’s arrival, Prandtl had been working on the equation for his lifting line theory for the spanwise lift distribution of an airplane wing. Applying her mathematical skills, Lotz solved the equation, and developed a relatively convenient method for practical use. Lotz was then promoted to head of this dominant group dealing with aerodynamics.

In 1938, Irmgard married Dr. Wilhelm Flügge, a civil engineer, and the pair moved first to Berlin and later to the small town of Saulgau. He had accepted a position as a department head at the Deutsche Versuchsanstalt für Luftfahrt (DVL) in Berlin.

The leaders of the DVL quickly became aware of the talent possessed by his wife and offered her a position as consultant in aerodynamics and flight dynamics. There, she began her career in automatic control theory, developing the theory of discontinuous, or on-off, control systems. In 1944, the Flügges moved their DLV activities to Saulgau. When Germany surrendered in 1945, the Flügges did not need to undergo a denazification investigation and moved to the French zone of occupation.

In 1947, they both accepted offers to join the newly established ONERA (French National Office for Aeronautical Research) in Paris. Flügge-Lotz served as chief of a research group in aerodynamics and published papers in automatic control theory and aerodynamics. Her papers addressed the problems arising from increased aircraft speeds.

In 1948, the Flügges left France and came to the United States to teach at Stanford University. Wilhelm was hired as a full-time professor, and Irmgard was hired as a lecturer in engineering mechanics and research supervisor. She undertook the guidance of PhD dissertation research in aerodynamic theory. In 1949, Flügge-Lotz taught her first course at Stanford and later introduced a year-long sequence of courses in mathematical hydro- and aerodynamics for first-year graduate students. Flügge-Lotz continued to show strong interest in fluid mechanics, numerical methods, and automatic controls.

She guided the research of a succession of PhD candidates at Stanford, and co-authored research reports with them. She worked on what she called “discontinuous automatic control,” which laid the foundation for automatic on-off control systems in jets. The number of her students kept growing and by 1951 she established a weekly fluid mechanics seminar at which faculty and students met to discuss ideas.

In 1960 Flügge-Lotz was the only female delegate from the United States at the first Congress of the International Federation of Automatic Control in Moscow. It became evident that she was shouldering the duties of a full-time professor without official recognition. Before the fall 1960 quarter at Stanford, she was appointed full professor in both engineering mechanics, and aeronautics and astronautics. During the 1960s she was one of Stanford’s most distinguished professors.

Despite retiring in 1968 and suffering debilitating arthritis, Flügge-Lotz continued her research on satellite control, heat transfer, and drag of high-speed vehicles. After a fulfilling career of teaching and research, she had published over fifty technical papers and made exciting contributions to the fields of engineering and mathematics.

After a long illness, Flügge-Lotz died in the Stanford Hospital in 1974; she was 70 years old.

Noteworthy: Irmgard Flügge-Lotz was Stanford University’s first female full professor in engineering.

In 1970, she was awarded the Achievement award by the Society of Women Engineers. She was the first woman elected to be a Fellow of the American Institute of Aeronautics and Astronautics in 1970, and she was the first woman to be selected by the American Institute of Aeronautics and Astronautics (AIAA) in 1971 to give the prestigious von Kármán Lecture.

In 1973, she received an honorary Doctor of Science degree from the University of Maryland. A summary of her career is provided by the citation from her honorary doctorate: “Professor Flügge-Lotz has acted in a central role in the development of the aircraft industry in the Western world. Her contributions have spanned a lifetime during which she

demonstrated, in a field dominated by men, the value and quality of a woman’s intuitive approach in searching for and discovering solutions to complex engineering problems. Her work manifests unusual personal dedication and native intelligence.”

She achieved her dream, once writing: “I wanted a life which would never be boring - a life in which new things would always occur.”

11.9 Random Duffing Oscillator

We use the expansion method for the Duffing equation subjected to a random force. We can apply the perturbation technique to derive estimates of the response correlation and spectral density, drawing on the probabilistic ideas introduced in Chapter 9. The governing equation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2(x + rx^3) = f(t),$$

where $c/m = 2\zeta\omega_n$ and $k/m = \omega_n^2$. The excitation per unit mass $f(t)$ is taken to be a stationary Gaussian random process with power spectrum $S_{ff}(\omega)$.

The sequence of equations that governs the terms in the approximation $x \simeq x_0 + \varepsilon x_1 + \varepsilon^2 x_2$, where $\varepsilon = \omega_n^2 r$, are found to be

$$\ddot{x}_0 + 2\zeta\omega_n\dot{x}_0 + \omega_n^2 x_0 = f(t) \quad (11.80)$$

$$\ddot{x}_1 + 2\zeta\omega_n\dot{x}_1 + \omega_n^2 x_1 = -x_0^3 \quad (11.81)$$

$$\ddot{x}_2 + 2\zeta\omega_n\dot{x}_2 + \omega_n^2 x_2 = -3x_0^2 x_1. \quad (11.82)$$

Equation 11.80 is a linear oscillator driven by the random function $f(t)$. This procedure takes a nonlinear oscillator and converts it into a sequence of linear oscillators. The nonlinearities have been moved to the right-hand side of each equation. The system equation, that is the left-hand side, is always linear and amenable to solution via the convolution integral regardless of the form of the function driving it on the right-hand side.

The first three terms in the infinite sequence of solutions are given by

$$x_0(t) = \int_{-\infty}^{\infty} g(t) f(t - \tau) d\tau \quad (11.83)$$

$$x_1(t) = - \int_{-\infty}^{\infty} g(t) x_0^3(t - \tau) d\tau \quad (11.84)$$

$$x_2(t) = -3 \int_{-\infty}^{\infty} g(t) x_0^2(t - \tau) x_1(t - \tau) d\tau, \quad (11.85)$$

where

$$g(t) = (1/\omega_d) \exp(-\zeta\omega_d t),$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}.$$

The solution of Equation 11.83 is then cubed, substituted into Equation 11.84, and that equation is integrated to find the expression for $x_1(t)$. This expression and the one for $x_0^2(t)$ are then substituted into Equation 11.85 for integration. This is a tedious process, and the accuracy depends on the value of ε .

Since $f(t)$ is a random function of time, only the statistics of the response can be estimated. For a three-term expansion, the mean value of the response is approximated by

$$E\{x\} \simeq E\{x_0\} + \varepsilon E\{x_1\} + \varepsilon^2 E\{x_2\},$$

where

$$\begin{aligned} E\{x_0\} &= \int_{-\infty}^{\infty} g(t) E\{f(t-\tau)\} d\tau \\ E\{x_1\} &= - \int_{-\infty}^{\infty} g(t) E\{x_0^3(t-\tau)\} d\tau \\ E\{x_2\} &= -3 \int_{-\infty}^{\infty} g(t) E\{x_0^2(t-\tau) x_1(t-\tau)\} d\tau. \end{aligned}$$

Evaluating these expectations can be quite difficult analytically and requires assumptions regarding the statistical properties of the variables.

Even more intricate are the estimations of the output correlation function $R_{xx}(\tau)$ and spectral density $S_{xx}(\omega)$,

$$\begin{aligned} R_{xx}(\tau) &= E\{x(t)x(t+\tau)\} \\ &\simeq E\{[x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)] \\ &\quad \times [x_0(t+\tau) + \varepsilon x_1(t+\tau) + \varepsilon^2 x_2(t+\tau)]\} \\ &= E\{x_0(t)x_0(t+\tau) + \varepsilon x_0(t)x_1(t+\tau) \\ &\quad + \varepsilon^2 x_0(t)x_2(t+\tau) + \varepsilon x_1(t)x_0(t+\tau) \\ &\quad + \varepsilon^2 x_1(t)x_1(t+\tau) + \varepsilon^3 x_1(t)x_2(t+\tau) \\ &\quad + \varepsilon^2 x_2(t)x_0(t+\tau) + \varepsilon^3 x_2(t)x_1(t+\tau) \\ &\quad + \varepsilon^4 x_2(t)x_2(t+\tau)\}. \end{aligned}$$

To evaluate this expression we require data for numerous correlations and cross correlations. It is reasonable to drop terms of order greater than ε^2 for consistency with the expansion for $x(t, \varepsilon)$, which is to order ε^2 , giving

$$\begin{aligned} R_{xx}(\tau) &\simeq R_{x_0 x_0}(\tau) + \varepsilon [R_{x_0 x_1}(\tau) + R_{x_1 x_0}(\tau)] \\ &\quad + \varepsilon^2 [R_{x_1 x_1}(\tau) + R_{x_0 x_2}(\tau) + R_{x_2 x_0}(\tau)]. \end{aligned}$$

The approximate response spectral density is the Fourier transform of $R_{xx}(\tau)$,

$$\begin{aligned} S_{xx}(\omega) &\simeq S_{x_0 x_0}(\omega) + \varepsilon [S_{x_0 x_1}(\omega) + S_{x_1 x_0}(\omega)] \\ &\quad + \varepsilon^2 [S_{x_1 x_1}(\omega) + S_{x_0 x_2}(\omega) + S_{x_2 x_0}(\omega)]. \end{aligned}$$

While the algebra is relatively straightforward, the reality of these kinds of equations is that the necessary

data are rarely available to perform the needed computations. Usually, for an approximation we might only retain terms to order ε , and even then, major simplifications are needed.

Another approach to this problem is to apply the fundamental theorem,

$$S_{x_0 x_0}(\omega) = |G(i\omega)|^2 S_{ff}(\omega),$$

to Equation 11.80. Given $S_{x_0 x_0}(\omega)$, it is possible to find $S_{x_0^3 x_0^3}(\omega)$ for a Gaussian process, and then use the fundamental theorem again with Equation 11.81, and then with Equation 11.82. For non-Gaussian processes, this is more difficult.

11.10 Nonlinear Pendulum: Galerkin Method

By way of example, a Galerkin method⁴¹ is used in conjunction with a perturbation approximation to analyze a nonlinear pendulum using a one-term approximation. Denoting the governing equation by the shorthand function $F[\cdot]$, where θ is the exact solution and Θ is the approximate solution, the nonlinear equation is

$$F(\theta) \equiv \ddot{\theta} + \omega_n^2 \left(\theta - \frac{\theta^3}{6} \right) = 0,$$

and the assumed solution is the linear solution,

$$\Theta(t) = A \sin \omega t.$$

We are looking for estimates of A and ω .

Substituting the assumed solution into the governing equation, we find

$$\begin{aligned} F(\Theta) &= -\omega^2 A \sin \omega t + \omega_n^2 \left(A \sin \omega t - \frac{1}{6} A^3 \sin^3 \omega t \right) \\ &= \left(\omega_n^2 - \omega^2 - \frac{1}{8} \omega_n^2 A^2 \right) A \sin \omega t + \frac{\omega_n^2}{24} A^3 \sin 3\omega t. \end{aligned}$$

The Galerkin method minimizes the integral

$$\int_0^T F^2(\Theta) dt, \quad (11.86)$$

with respect to A , where T is the period of the oscillation. Therefore,

$$\frac{\partial}{\partial A} \int_0^T F^2(\Theta) dt = 0,$$

and

$$\int_0^T F(\Theta) \frac{\partial F(\Theta)}{\partial A} dt = 0.$$

⁴¹ See Section 8.3.3.

When $F(\Theta)$ and its derivative are substituted into this equation, we find

$$\int_0^T \left[\left(\omega_n^2 - \omega^2 - \frac{1}{8} \omega_n^2 A^2 \right) A \sin \omega t + \frac{\omega_n^2}{24} A^3 \sin 3\omega t \right] \\ \cdot \left[\left(\omega_n^2 - \omega^2 - \frac{3}{8} A^2 \omega_n^2 \right) \sin \omega t + \frac{1}{8} A^2 \omega_n^2 \sin 3\omega t \right] dt = 0.$$

Expanding the product yields

$$A \left(\omega_n^2 - \omega^2 - \frac{1}{8} \omega_n^2 A^2 \right) \\ \times \left(\omega_n^2 - \omega^2 - \frac{3}{8} A^2 \omega_n^2 \right) \int_0^T \sin^2 \omega t dt \\ + \frac{\omega_n^2}{24} A^3 \left(\omega_n^2 - \omega^2 - \frac{3}{8} A^2 \omega_n^2 \right) \int_0^T \sin \omega t \sin 3\omega t dt \\ + \frac{1}{8} A^3 \omega_n^2 \left(\omega_n^2 - \omega^2 - \frac{1}{8} \omega_n^2 A^2 \right) \int_0^T \sin \omega t \sin 3\omega t dt \\ + \frac{\omega_n^4}{192} A^5 \int_0^T \sin^2 3\omega t dt = 0.$$

Using trigonometric identities,⁴² we are left with the first and last terms,

$$A \left(\omega_n^2 - \omega^2 - \frac{1}{8} \omega_n^2 A^2 \right) \left(\omega_n^2 - \omega^2 - \frac{3}{8} A^2 \omega_n^2 \right) \\ + \frac{\omega_n^4}{192} A^5 = 0.$$

Assuming a nontrivial solution, that is $A \neq 0$, the product of the expressions in the parentheses must equal zero. Expanding and combining like terms results in

$$\omega^4 + \omega^2 \omega_n^2 \left(\frac{1}{2} A^2 - 2 \right) + \omega_n^4 \left(1 - \frac{1}{2} A^2 + \frac{5}{96} A^4 \right) = 0.$$

This equation is a quadratic in ω^2 that can be solved to find the two natural frequencies,

$$\omega_{1,2}^2 = \begin{cases} \omega_n^2 (1 - 0.145 A^2) \\ \omega_n^2 (1 - 0.352 A^2) \end{cases}.$$

The first root ω_1^2 is the value that minimizes [Equation 11.86](#). Lindstedt's solution is $\omega_n^2 (1 - 0.375 A^2)$, whereas

⁴²The integrals are over the period $T = 2\pi/\omega$. Replacing the functions under the integrals, many terms integrate to zero,

$$\int_0^T \sin^2 \omega t dt = \int_0^T \left(\frac{1}{2} - \frac{1}{2} \cos 2\omega t \right) dt \\ \int_0^T \sin \omega t \sin 3\omega t dt = \int_0^T \left(\frac{1}{2} \cos 2\omega t - \frac{1}{2} \cos 4\omega t \right) dt \\ \int_0^T \sin^2 3\omega t dt = \int_0^T \left(\frac{1}{2} - \frac{1}{2} \cos 6\omega t \right) dt.$$

Replacing T by $2\pi/\omega$ in each trig function, the only nonzero integrals result from the integrals of $1/2$.

the one-term Galerkin approximate solution is

$$\Theta(t) = A \sin \omega_n^2 (1 - 0.145 A^2) t.$$

Since a one-term solution was assumed, only one initial condition is needed. The initial angle $\Theta(0)$ does not affect this solution, but the initial velocity $\dot{\Theta}(0)$ and the value of A are related by

$$A (1 - 0.145 A^2) = \frac{\dot{\Theta}(0)}{\omega_n^2}.$$

Clearly, $\dot{\Theta}(0) \neq 0$, and we have the option of solving this equation for ω_n^2 and substituting it into the sine term in the solution $\Theta(t)$.

11.11 Concept Summary

This chapter presented a basic introduction to the kinds of analysis and behavior that one can expect with nonlinearities in an oscillating system. Even small nonlinearities can result in behaviors very different from those of linear systems. The phase plane graphically represents the oscillatory character of autonomous systems. The way in which the stability characteristics of nonlinear oscillators can be quantified has been introduced. The perturbation method was shown to be a quantitative way to approximate the effects of small nonlinearities, whether in damping or stiffness, on the response and the period of oscillation.

11.12 Quotes

- “Although this may seem a paradox, all exact science is based on the idea of approximation. If a man tells you he knows a thing exactly, then you can be safe in inferring that you are speaking to an inexact man.” (Bertrand Russell)
- “The test of success is not what you do when you're on top. Success is how high you bounce when you hit bottom.” (General George S. Patton Jr.)
- “Aeroplanes are not designed by science, but by art in spite of some pretence and humbug to the contrary. I do not mean to suggest that engineering can do without science, on the contrary, it stands on scientific foundations, but there is a big gap between scientific research and the engineering product which has to be bridged by the art of the engineer.” (British Engineer to the Royal Aeronautical Society, 1922. Quoted by Walter G. Vincenti)
- “Tell me, Mr. Hoover, what are your interests?” “Madam, I am an Engineer.” “Really? I took you for a gentleman.” (Herbert Hoover, conversation on making the acquaintance of a lady on a steamship)

- “People only see what they are prepared to see.” (Ralph Waldo Emerson)
- “Art should disturb the comfortable and comfort the disturbed.”
- “I attack ideas. I don’t attack people. And some very good people have some very bad ideas. And if you can’t separate the two, you gotta get another day job.” (Antonin Gregory Scalia)
- “Millions saw the apple fall but Newton was the one who asked why.” (Bernard Baruch)
- “If you want to find the secrets of the universe, think in terms of energy, frequency and vibration.” (Nikola Tesla)

11.13 Problems

Problems for Section 11.3 – The Phase Plane

1. Construct the phase diagram for the simple harmonic oscillator

$$\ddot{x} + \omega_n^2 x = 0.$$

2. Construct the phase diagram for the equation

$$\ddot{x} - \omega_n^2 x = 0.$$

3. Plot the equations in [Example 11.4](#).
4. Write the first-order equations in [Example 11.4](#) as a second-order differential equation. Solve this differential equation for $x(t)$, then find the equation for the slope and relate this result to the final result in the example problem.

Problems for Section 11.4 – Perturbation Methods

5. Using $\dot{x} = y$, investigate the equation

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$$

for how damping adds and removes energy from the system depending on the values of x and $\dot{x}$.

6. For the equation

$$\ddot{x} + 2\varepsilon\omega_0\dot{x} + \omega_0^2 x = 0,$$

$x(0) = \bar{x}_0$ and $\dot{x}(0) = \bar{v}_0$, compare the standard expansion solution with the exact solution.

7. Obtain an approximation to the forced response of period 2π for the equation

$$x'' + \frac{1}{4}x + 0.1x^3 = \cos \tau.$$

8. Obtain an approximation to the forced response of period 2π for the equation

$$x'' + \frac{1}{2}x + 0.1x^3 = \cos \tau.$$

9. For the forced, quasi-harmonic system, derive (a) [Equation 11.36](#), and (b) [Equation 11.37](#).
10. Provide a detailed discussion of the amplitude curves shown in [Figure 11.12](#). Discuss the variation of the amplitude as a sweep is made of the frequency from small values to larger values.
11. Provide a detailed discussion of the amplitude curves shown in [Figure 11.13](#). Discuss the variation of the amplitude as a sweep is made of the frequency from small values to larger values.

12. Solve the following equation to order ε^2 ,

$$\ddot{x} + x = \varepsilon x^2, \quad \varepsilon \ll 1,$$

where $x(0) = A_0$, and $\dot{x}(0) = 0$.

13. Obtain the approximate solution of period 2π for the equation⁴³

$$\ddot{x} + \Omega^2 x - \varepsilon x^2 = \Gamma \cos t, \quad \varepsilon > 0.$$

14. Derive the forced periodic response of the equation

$$x'' + (9 + \varepsilon\beta)x - \varepsilon x^3 = \Gamma \cos \tau.$$

15. Solve [Equation 11.38](#) for $\Omega \simeq 1$ using the approach suggested in the footnoted [Equation 11.43](#).

16. For the subharmonic response, derive (a) [Equations 11.45 to 11.47](#), (b) [Equation 11.48](#), (c) [Equation 11.49](#), and (d) [Equation 11.50](#).

17. Find the subharmonic response of order $1/n$ for the linear equation⁴⁴

$$\ddot{x} + \frac{1}{n^2}x = \Gamma \cos t.$$

18. Consider the following Duffing equation,

$$\ddot{x} + \alpha x + \varepsilon x^3 = \Gamma \cos \omega t,$$

with $\alpha, \Gamma, \omega > 0$. The values of ω for which the subharmonics occur are unknown. Expand x and ω in a perturbation solution.

⁴³ Jordan and Smith, pp.132.

⁴⁴ Jordan and Smith, pp.195.

19. For the combination harmonic response, derive Equation 11.54.

Problems for Section 11.5 – The Mathieu Equation

20. Referring to Figure 11.14, describe physically the exchange of energy between the pendulum, the moving container, and the external force. Discuss in terms of energies and motion.
21. Suppose in Equation 11.60 that $L = 1$, $A = 1$, and ω , representing the harmonic motion of the support, is variable. What limitations, if any, are there on stable motion?
22. Derive Equation 11.67.
23. Derive Equation 11.68.

Problems for Section 11.6 – The van der Pol Equation

24. Considering Equations 11.69 and 11.70, describe the physical coupling between the two equations. Describe the energy exchange between the systems, with reference to individual terms in the equations.
25. Derive Equation 11.75.



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Appendix A

Mathematical Concepts for Vibration

“We are reminded of the essential rules.”

This appendix presents mathematical concepts that may be helpful to those studying vibration. It is not meant to be a substitute for courses covering linear algebra, calculus, and differential equations. We assume that the reader has had some experience with the topics.

A.1 Complex Numbers

A *complex number* $z = x + iy$ is the sum of a real part x and an imaginary part iy , with x and y both real and $i = \sqrt{-1}$. The *complex conjugate* of z is $x - iy$. Complex numbers, which arise in many scientific disciplines, are needed in the general solution polynomial equations, including of quadratic equations. Before the 18th century, complex numbers and the square root of minus one were in regular use.¹ The equation²

$$e^{i\theta} = \cos \theta + i \sin \theta$$

is credited to Euler and is known as *Euler’s formula*.

A.1.1 Complex Number Operations

Two complex numbers are added or subtracted by simply adding or subtracting their corresponding real and

¹In the 16th century, a few mathematicians began to work with numbers whose squares are negative. The word “imaginary” gradually came to be used to describe numbers such as $\sqrt{-1}$. The use of the word imaginary reflects some of the original uneasiness many mathematicians had with nonreal numbers. Today, the phrase “imaginary number” seems a little unfortunate since these numbers are firmly established in mathematics.

By the end of the 18th century, complex numbers of the form $x + iy$ were commonly represented by points in the plane with the horizontal axis representing the real part and the vertical axis as the imaginary part.

²Euler’s identity sets “the gold standard for mathematical beauty.” (**Dr. Euler’s Fabulous Formula**, P. Nahin, Princeton University Press, 2006.)

imaginary parts. For example, the sum of $8 + 3i$ and $6 + 3i$ is $14 + 6i$.

Complex number multiplication is similar to real number multiplication. For example,

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu).$$

In division of complex numbers, the convention is to express the fraction such that there is no imaginary part in the denominator. This can be accomplished by multiplying the numerator and denominator of the complex fraction by the complex conjugate of the denominator $u - iv$ to find

$$\frac{x + iy}{u + iv} = \frac{(xu + yv) + i(-xv + yu)}{u^2 + v^2}. \quad (\text{A.1})$$

Example A.1 Rewrite the complex fraction

$$\frac{1 - 2i}{7 + 3i}$$

into its real and imaginary parts.

Solution: To express the fraction in terms of its real and imaginary parts, multiply its numerator and denominator by the complex conjugate of the denominator and expand as follows,

$$\begin{aligned} \frac{1 - 2i}{7 + 3i} \cdot \frac{7 - 3i}{7 - 3i} &= \frac{7 - 3i - 14i + 6i^2}{49 - 21i + 21i - 9i^2} \\ &= \frac{7 - 17i - 6}{49 + 9} \\ &= \frac{1}{58} - \frac{17}{58}i. \end{aligned}$$

Alternatively, Equation A.1 can be used directly. ■

The reciprocal of a complex number is given by

$$\frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \left(\frac{x}{x^2 + y^2} \right) + i \left(\frac{-y}{x^2 + y^2} \right).$$

This result is obtained by multiplying the numerator and denominator of the left-hand side by the complex conjugate $x - iy$ and carrying out the multiplication.

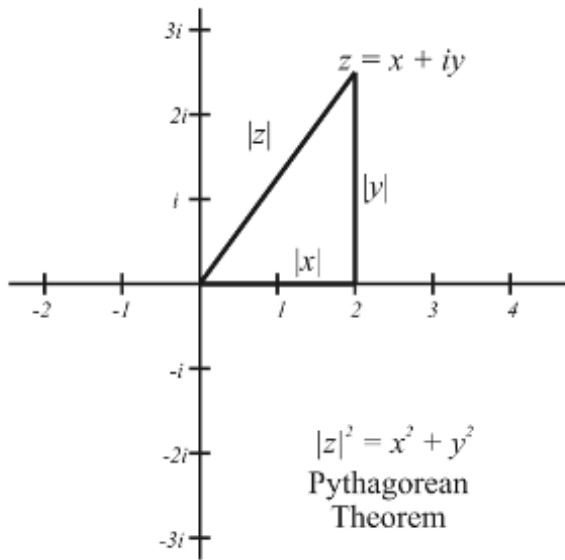


Figure A.1: A complex number shown in the complex plane.

Example A.2 Find the reciprocal of the complex number $3 + 6i$.

Solution: To find the reciprocal, multiply the numerator and denominator of the inverse by the complex conjugate of the complex number and expand,

$$\begin{aligned} \frac{1}{3 + 6i} \cdot \frac{3 - 6i}{3 - 6i} &= \frac{3 - 6i}{(3 + 6i)(3 - 6i)} \\ &= \frac{3 - 6i}{45} \\ &= \frac{1}{15} - \frac{2}{15}i. \end{aligned}$$

Thus, the reciprocal of a complex number can be expressed in terms of a real and imaginary part. ■

A.1.2 Absolute Value

The absolute value $|x|$ of a real number x is that number if it is positive or zero, and its corresponding positive value if x is negative. The absolute value function strips a real number of its sign. For a complex number $z = x + iy$, the absolute value $|z|$ is a magnitude and defined as the distance from z to the origin in the complex plane. This definition applies to real numbers; the absolute value $|x|$ of a real number x can be interpreted as the distance from x to 0 on the real number line.

The distance $|z|$ can be found using the Pythagorean theorem. Consider the right triangle with vertices $(0, 0)$, (x, y) , and $(x, 0)$, as shown in Figure A.1. The horizontal side of the triangle has length $|x|$, the vertical side has length $|y|$, and the diagonal side has length $|z|$. Therefore, $|z|^2 = x^2 + y^2$. The direction of the line from the

origin to z by angle θ is called the argument of z , sometimes denoted $\arg(z)$.

A.1.3 Equivalent Representation

A point z can be specified by either the pair of rectangular coordinates, x and y , or the pair of polar coordinates, $r = |z|$, and $\theta = \arg(z)$.

From the following three relations from the right triangle,

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad \tan \theta = \frac{y}{x},$$

the complex number $z = x + iy$ can be expressed alternatively as

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta) \\ &= |z|(\cos \theta + i \sin \theta). \end{aligned}$$

The absolute value of the complex number $\cos \theta + i \sin \theta$ is unity since $\cos^2 \theta + \sin^2 \theta = 1$ for any θ ,

$$|\cos \theta + i \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta}.$$

Thus, every complex number z is the product of a real number magnitude $|z|$ and a complex number $\cos \theta + i \sin \theta$. The magnitude is $|z| = \sqrt{x^2 + y^2}$ and the angle θ is $\theta = \arctan(y/x)$.

Example A.3 Express the complex number $z = 6 + 3i$ in terms of its magnitude and angle.

Solution: The magnitude and angle are

$$|z| = \sqrt{6^2 + 3^2} = 6.71,$$

$$\theta = \arctan(3/6) = 0.464 \text{ rad.}$$

Thus, $z = 6 + 3i$ can equivalently be written as $z = 6.71 \angle 0.464 \text{ rad}$. ■

A.2 Matrices

A *matrix* **A** is a rectangular array of numbers generally arising as short-hand notation that is useful in expressing a set of algebraic equations.³ If the *elements* of the matrix are *real* numbers, then the matrix is called a *real matrix*. The number of columns in a matrix is not necessarily the same as the number of rows. However, for special matrices in our vibration applications, the number of rows n equals the number of columns, resulting in a *square* $n \times n$ matrix of *order* n .

³A boldface font is used to indicate a matrix.

A general matrix $\mathbf{A}$ (with n rows and m columns) is written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad (\text{A.2})$$

where the matrix is $n \times m$.

A square $n \times n$ matrix having nonzero elements along its main diagonal and a zero in all other locations is called a *diagonal* matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

The *identity* matrix $\mathbf{I}$ is a diagonal matrix where each element along the main diagonal is unity,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

For an $n \times m$ matrix $\mathbf{A}$, if the rows and columns are interchanged, the resulting $m \times n$ matrix $\mathbf{A}^T$ is called the *matrix transpose* of $\mathbf{A}$,

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}.$$

Properties using the transpose operation include

$$\begin{aligned} (\mathbf{A}^T)^T &= \mathbf{A} \\ (\mathbf{A} + \mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\ (\mathbf{AB})^T &= \mathbf{B}^T \mathbf{A}^T. \end{aligned}$$

A *symmetric* matrix is one which is equal to its transpose,

$$\mathbf{A}^T = \mathbf{A},$$

that is, the elements $a_{ji} = a_{ij}$ ($i, j = 1, \dots, n$).

A.2.1 Matrix Operations

The *sum* of two $n \times m$ matrices $\mathbf{A}$ and $\mathbf{B}$ is a matrix with each element equal to the sum of the corresponding

elements of the original matrices,

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{bmatrix}.$$

This operation is valid only if both matrices have the same dimension. One can define the *difference of two matrices* in the same manner by replacing all sums by differences.

The *multiplication of a matrix by a scalar* results in a new matrix, $k\mathbf{A}$, where each element a_{ij} is multiplied by scalar k .

The *multiplication of two matrices* is a more intricate operation. Consider multiplying the $n \times m$ matrix $\mathbf{A}$ by the $m \times r$ matrix $\mathbf{B}$,

$$\begin{aligned} \mathbf{C} &= \mathbf{AB} \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{bmatrix}. \end{aligned}$$

The operations required to obtain the element in row 1, column 1 of the new matrix $\mathbf{C}$ are:

$$\begin{aligned} c_{11} &= (a_{11} \times b_{11}) + (a_{12} \times b_{21}) \\ &\quad + (a_{13} \times b_{31}) + \cdots + (a_{1m} \times b_{m1}). \end{aligned} \quad (\text{A.3})$$

In order for such a set of operations to work, the matrices must be *commensurate*, i.e., the first matrix must have as many *columns* as the second matrix has *rows*. The elements of matrix $\mathbf{C}$ can be obtained from the general relation⁴

$$c_{ik} = \sum_{j=1}^m a_{ij} b_{jk}.$$

Matrix $\mathbf{C}$ will be of size $(n \times m) \times (m \times r) = n \times r$, and

$$\mathbf{AB} \neq \mathbf{BA}.$$

A.2.2 Determinant and Matrix Inverse

The determinant of the square matrix $\mathbf{A}$

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (\text{A.4})$$

⁴In general, the matrix product of $\mathbf{A}$ of dimension $n \times m$ and $\mathbf{B}$ of dimension $m \times r$ is matrix $\mathbf{C}$ whose dimensions are $n \times r$ and whose element in the i^{th} row and j^{th} column is the sum of the products of the corresponding elements of the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$. From this definition we can see that matrices $\mathbf{A}$ and $\mathbf{B}$ can be multiplied only if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.

is defined as a number obtained as the sum of all possible products in each of which there appears one and only one element from each row and each column, each such product being prefixed by a plus or minus sign.⁵ The rules given below show the steps for evaluating the determinant of a matrix. We note that the determinant gives a unique value and $|\mathbf{A}|$ is said to be of order n .

We denote $|M_{ij}|$ as the *minor determinant* of $\mathbf{A}$, obtained by taking the determinant of $\mathbf{A}$ after removing row i and column j . The order of $|M_{ij}|$ is $(n-1)$. Attaching a sign (plus or minus) to this minor determinant,

$$|C_{ij}| = (-1)^{i+j} |M_{ij}|,$$

results in the term known as the *signed minor*. This corresponds to the matrix element a_{ij} , otherwise known as the *cofactor* of a_{ij} . With these definitions, the value of the determinant can be obtained by “expanding” the determinant in terms of the cofactors for any row r ,

$$|\mathbf{A}| = \sum_{s=1}^n a_{rs} |C_{rs}|.$$

A similar expansion is possible for any column. These expansions are called *Laplace expansions*. For example, for the case $n = 3$, Equation A.4 can be evaluated as follows by expanding across row 1:

$$\begin{aligned} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ & \quad + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (\text{A.5}) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ & \quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}). \end{aligned}$$

Computations can be reduced significantly by choosing the row containing the most zeros. Looking carefully at these operations, it can be seen that

$$|\mathbf{A}| = |\mathbf{A}^T|.$$

If the value of $|\mathbf{A}|$ is equal to zero, that is, its determinant is zero, then matrix $\mathbf{A}$ is said to be *singular*. The following example shows an interesting fact about the

cancellation of matrices. Consider two *singular* matrices,

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \neq \mathbf{0}, \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \neq \mathbf{0},$$

where the zero matrix is a matrix consisting of all zeros,

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Taking the matrix product,

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Therefore, we see that $\mathbf{AB} = \mathbf{0}$ implies neither $\mathbf{A} = \mathbf{0}$ nor $\mathbf{B} = \mathbf{0}$. It is possible that either matrix is zero, but if they are not zero, then both must be singular.

By definition, the *adjoint* of square matrix $\mathbf{A}$ is the transpose of the matrix obtained from $\mathbf{A}$ by replacing each element of $\mathbf{A}$ by its cofactor. It can be shown that

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A})\mathbf{A} = |\mathbf{A}|\mathbf{I}. \quad (\text{A.6})$$

For example, the 2×2 determinants in Equation A.5 are the *first column* of the adjoint of $\mathbf{A}$. We need to go through the same procedure for rows two and three of $\mathbf{A}$ in order to derive the second and third *columns* of the adjoint matrix. For matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix},$$

$$\text{adj } \mathbf{A} = \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix},$$

and

$$\mathbf{A}(\text{adj } \mathbf{A}) = 17 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $|\mathbf{A}| = 17$.

The *inverse* of matrix $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, can be interpreted as the matrix counterpart to dividing by a number. Without proof, using Equation A.6, the expression for the inverse of a non-singular matrix is

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}.$$

⁵Methods of Applied Mathematics, F.B. Hildebrand, Prentice Hall, 1965, p.10.

Linear Algebraic Equations

Consider Equations A.7-A.9, which are a set of n linear algebraic equations with m unknowns, x_i , $i = 1, \dots, m$,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = y_1 \quad (\text{A.7})$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m = y_2 \quad (\text{A.8})$$

$$\dots \dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m = y_n. \quad (\text{A.9})$$

These equations can be written compactly in matrix-vector form as $\mathbf{Ax} = \mathbf{y}$. If the number of equations equals the number of unknowns, then matrix $\mathbf{A}$ is square and a unique solution is possible. Taking the inverse, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. Many numerical approaches exist to solve this matrix equation. *Cramer's rule* is one example, although there are many computationally more efficient numerical approaches.

A.2.3 Eigenvalues and Eigenvectors

Eigenvalues and *eigenvectors* are concepts developed formally as part of matrix theory. For the $n \times n$ matrix $\mathbf{A}$, the determinant

$$|\mathbf{A} - \lambda\mathbf{I}|$$

is called the *characteristic polynomial* of $\mathbf{A}$, and is an n^{th} order polynomial in terms of λ . The *characteristic equation* is obtained by setting this polynomial equal to zero,

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} \\ &= (-1)^n (\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots \\ &\quad + c_{n-2}\lambda^2 + c_{n-1}\lambda + c_n) = 0. \end{aligned} \quad (\text{A.10})$$

This equation has n *characteristic roots* that are the eigenvalues of the square matrix $\mathbf{A}$. Even though the matrix has real elements, polynomial Equation A.10 with real coefficients c_i may have *complex conjugate* pairs of roots $\alpha \pm i\beta$. These complex conjugate roots are of primary interest in a vibration analysis.

For a diagonal matrix, the n diagonal elements are the n eigenvalues of the matrix. Also, *the eigenvalues of matrix $\mathbf{A}$ are equal to the eigenvalues of the inverse matrix $\mathbf{A}^{-1}$.*

Once the eigenvalues λ_i are found, then any nonzero vector $\mathbf{v}_i$ satisfying

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad (\text{A.11})$$

is an eigenvector. There is generally one eigenvector for each eigenvalue, but there are degenerate cases where

there is not a one-to-one relationship. From Equation A.11, we see that if $\mathbf{v}_i$ is an eigenvector, then for any nonzero scalar α , $\alpha\mathbf{v}_i$ is also an eigenvector since $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}$. If $\mathbf{A}$ is symmetric, λ_i are real and non-negative, the eigenvectors are orthogonal $\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$, and the eigenvectors can be made perpendicular, even for repeated eigenvalues.

A.2.4 Matrix Derivatives and Integrals

The study of large scale systems requires the differentiation and integration of matrices. The *time derivative* of matrix $\mathbf{A}(t)$ is given by the matrix whose elements are the respective time derivatives of the original matrix,

$$\frac{d}{dt}\mathbf{A}(t) = \left(\frac{d}{dt}a_{ij}(t) \right),$$

where each element is differentiated. Implicit is the assumption that all elements have derivatives.

Similarly, the *time integral* of matrix $\mathbf{A}(t)$ is given by the matrix whose elements are the respective time integrals of the original matrix,

$$\int \mathbf{A}(t)dt = \left(\int a_{ij}(t)dt \right).$$

Some rules of matrix differentiation follow from the scalar rules, but multiplication of matrices (and of their derivatives) is not commutative,

$$\begin{aligned} \frac{d}{dt}(\mathbf{AB}) &= \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt} \\ \frac{d}{dt}(\mathbf{A} + \mathbf{B}) &= \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \\ \frac{d}{dt}(\mathbf{A}\alpha) &= \frac{d\mathbf{A}}{dt}\alpha + \mathbf{A}\frac{d\alpha}{dt}. \end{aligned}$$

The time derivative of the inverse of matrix $\mathbf{A}$ is an important rule that is derived in the following steps,

$$\begin{aligned} \frac{d}{dt}(\mathbf{AA}^{-1}) &= \frac{d\mathbf{A}}{dt}\mathbf{A}^{-1} + \mathbf{A}\frac{d\mathbf{A}^{-1}}{dt} = \frac{d}{dt}\mathbf{I} = \mathbf{0} \\ \frac{d\mathbf{A}}{dt}\mathbf{A}^{-1} &= -\mathbf{A}\frac{d\mathbf{A}^{-1}}{dt}. \end{aligned}$$

Therefore,

$$\frac{d\mathbf{A}^{-1}}{dt} = -\mathbf{A}^{-1}\frac{d\mathbf{A}}{dt}\mathbf{A}^{-1},$$

which we notice is the matrix counterpart of the scalar derivative,

$$\frac{d}{dt}(x^{-1}) = -x^{-2}\frac{dx}{dt}.$$

A.3 Taylor Series & Linearization

The Taylor series expansion^{6,7} of a (real or complex) function is a power series expansion that is useful when simplifying nonlinear functions. For a function of one variable, the expansion about the (real or complex) number a is given by

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \dots, \quad (\text{A.12})$$

where the primes denote differentiation with respect to variable x . For a function of two variables, the expansion about the point (a, b) is given by

$$\begin{aligned} f(x, y) = & f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b) \\ & + \frac{1}{2!}\{(x-a)^2 f_{xx}(a, b) \\ & + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\} \\ & + \dots, \end{aligned}$$

where the subscripts denote partial differentiation with respect to the subscripted variables. For example, $f_{xy}(a, b)$ is the second partial derivative of $f(x, y)$ with respect to x and y evaluated at point (a, b) .

Taylor series are useful when we wish to *linearize* nonlinear functions. For example, consider a simple pendulum whose length equals the acceleration due to gravity g . The equation of motion is

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad (\text{A.13})$$

where θ represents the angle of the pendulum measured counterclockwise with respect to a vertical axis, as shown in Figure A.2. Equation A.13 is a nonlinear equation⁸ due to the sine function, and represents the governing behavior of the pendulum for any initial conditions.

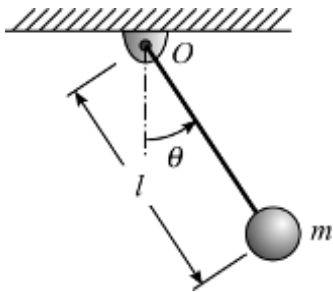


Figure A.2: A simple pendulum.

⁶The concept of a Taylor series was formulated by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715.

⁷A highly recommended general reference covering many topics in calculus, including Taylor series, is **Advanced Calculus for Application**, F.B.Hildebrand, Prentice Hall, 1976.

⁸Formally, this nonlinear equation is called a *transcendental equation* since it involves a trigonometric function.

For small angles (in units of radians), Equation A.13 can be *linearized* about an *operating point*. Suppose $\sin\theta$ is expanded about the angle $\theta = \theta_0$ via a Taylor series. Then, by using Equation A.12, we find the terms in the series to be

$$\begin{aligned} \sin\theta = & \sin\theta_0 + (\theta - \theta_0)\cos\theta_0 - \frac{(\theta - \theta_0)^2}{2!}\sin\theta_0 \\ & - \frac{(\theta - \theta_0)^3}{3!}\cos\theta_0 + \dots \end{aligned} \quad (\text{A.14})$$

In general, we are interested in the behavior of a vibrating system about its equilibrium position. In this case, the equilibrium position is $\theta_0 = 0$, and Equation A.14 becomes

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \dots$$

If oscillations about the equilibrium position are small, that is $\theta \ll 1$ rad, then the approximation,

$$\sin\theta \simeq \theta,$$

is reasonable. The approximation⁹ can be verified numerically and the angle beyond which the assumption breaks down can be identified. With this approximation, the equation of motion, Equation A.13, becomes

$$\frac{d^2\theta}{dt^2} + \theta = 0, \quad (\text{A.15})$$

which is an equation for a linear harmonic oscillator. Equation A.15 is valid only for small oscillations about the equilibrium position, $\theta = 0$. To summarize, for small oscillations about the equilibrium, we set $\theta \ll 1$ and retain the first term in the Taylor series of the nonlinear function. This first term of the series is always linear. Some useful series about $\theta_0 = 0$ are listed in Table A.1. The Taylor series centered at zero is also called a Maclaurin series.¹⁰

A.4 Ordinary Differential Equations

Any equation involving derivatives is called a *differential equation*. If there is only one independent variable the equation is termed an *ordinary differential equation*. If the dependent variable is a function of two or more parameters, for example time and spatial coordinate $y(t, x)$, then the differential equation is called a *partial differential equation*.

The study of vibration depends heavily on the solution of linear ordinary differential equations with time as

⁹The symbol $\simeq$ means “approximately equal to.”

¹⁰The Maclaurin series is named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series in the 18th century.

Table A.1: Taylor Series Expansion about 0.

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$	$-\infty < x < \infty$
$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$	$-1 < x < \infty$
$\sin \theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \dots$	$-\infty < \theta < \infty$
$\cos \theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \dots$	$-\infty < \theta < \infty$
$\tan \theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \frac{17}{315}\theta^7 + \dots$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sinh \theta = \theta + \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \frac{1}{7!}\theta^7 + \dots$	$-\infty < \theta < \infty$
$\cosh \theta = 1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \frac{1}{6!}\theta^6 + \dots$	$-\infty < \theta < \infty$

the independent variable. A common form in vibration studies is

$$a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = f(t), \quad (\text{A.16})$$

subject to the initial conditions $y(0)$ and $\dot{y}(0)$. The coefficients a_2, a_1 , and a_0 are constants,¹¹ $f(t)$ is a given function of time, and overdots denote differentiation with respect to time. If $f(t) \neq 0$, the differential equation is called *non-homogeneous*; if $f(t) = 0$, the differential equation is called *homogeneous*. Equation A.16 is known as an *initial-value problem* since its solution depends on the given initial values (or conditions) $y(0)$ and $\dot{y}(0)$. If the governing equation represents the behavior of a vibrating mass, then $y(t)$ is the displacement and $\dot{y}(t)$ is the velocity.

The *order* of a differential equation is equal to the highest derivative. For example, Equation A.16 is second order. To fully solve a differential equation, the number of initial conditions must equal the order of the equation. Otherwise, there will be arbitrary constants in the solution.

A differential equation is called *linear* if the dependent variable and its derivatives are all to the first power. The following theorem is of great value in solving differential equations.

Theorem 1 (Uniqueness of Solution) *If a_i and $f(t)$ are continuous over a time interval Δt and the initial*

¹¹If the coefficients are constant, the differential equation is known as *time-invariant*. Coefficients a_2, a_1 , and a_0 may be functions of t in more complex applications. Then, the differential equation is called *time-varying*.

time t_0 is a point in that interval, then the solution $y(t)$ for the linear initial-value problem is unique.

In other words, if *any* solution to linear Equation A.16 is found, *it is the only solution*. Therefore, when we solve the differential equation governing a linear vibration, we will know that we have the only possible solution. This is not true for nonlinear problems, where there may be multiple valid solutions.

Example A.4 Verify that

$$y(t) = 3e^{2t} + 3e^{-2t} - 3t$$

is the solution to

$$\ddot{y}(t) - 4y(t) = 12t, \quad y(0) = 6, \quad \dot{y}(0) = -3.$$

Solution: To verify, substitute the solution $y(t)$ and its second derivative $\ddot{y}(t)$ into the differential equation. If the solution is correct, it will satisfy the differential equation. In addition, the initial conditions $y(0)$ and $\dot{y}(0)$ should be equal to their respective given values,

$$y(0) = 6 = 3e^{2(0)} + 3e^{-2(0)} - 3(0)$$

$$\dot{y}(0) = -3 = 3 \cdot 2e^{2(0)} - 3 \cdot 2e^{-2(0)} - 3.$$

Thus, the solution satisfies the differential equation and its initial conditions. ■

Example A.5 Verify that $y(t) = ct^2 + t + 3$ is a solution of the linear differential equation

$$t^2\ddot{y}(t) - 2t\dot{y}(t) + 2y = 6, \quad y(0) = 3, \quad \dot{y}(0) = 1,$$

for any c .

Solution: Proceed as in the previous example. Note here that the coefficients are functions of time, not constants. ■

A.4.1 Solution of Linear Equations

When solving linear differential equations, we will take advantage of the *principle of linear superposition*. This principle allows complicated problems to be solved by breaking them into smaller and simpler components. The complete solution is then the sum of these components, as the following theorem below explains.

Theorem 2 (Principle of Superposition) *If $y_1(t), y_2(t), \dots, y_k(t)$ are solutions of a k^{th} order linear differential equation over an interval, then the linear combination*

$$y(t) = c_1y_1(t) + c_2y_2(t) + \dots + c_ky_k(t)$$

is also a solution, and is the complete solution.

Consider as an example a second-order differential equation, meaning $k = 2$. Let $y_1(t)$ and $y_2(t)$ be solutions to $a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = 0$. The combination $y(t) = c_1y_1(t) + c_2y_2(t)$ is also a solution. The following sequence of equations,

$$\begin{aligned} a_2\ddot{y} + a_1\dot{y} + a_0y &= 0 \\ a_2(c_1\ddot{y}_1 + c_2\ddot{y}_2) + a_1(c_1\dot{y}_1 + c_2\dot{y}_2) \\ + a_0(c_1y_1 + c_2y_2) &= 0 \\ c_1(a_2\ddot{y}_1 + a_1\dot{y}_1 + a_0y_1) + c_2(a_2\ddot{y}_2 + a_1\dot{y}_2 + a_0y_2) &= 0, \end{aligned} \quad (\text{A.17})$$

shows that since the sum within each pair of parentheses in Equation A.17 is zero, the principle of superposition is confirmed.

Superposition has an important extension when the right-hand side of the differential equation is non-homogeneous, that is $f(t) \neq 0$. In this case, the complete solution is equal to the sum of the solution to the homogeneous equation, $y_h(t)$, and the solution to the non-homogeneous equation, $y_p(t)$,

$$y(t) = y_h(t) + y_p(t).$$

The non-homogeneous solution is also called the *particular* solution since it is particular to the specific function on the right-hand side of the differential equation.

A.4.2 Homogeneous Solution

Consider the solution of the linear second-order homogeneous equation

$$a\ddot{y}(t) + b\dot{y}(t) + cy(t) = 0.$$

We assume the solution to be of the form $y(t) = e^{rt}$, where the values of r are determined by having the assumed solution satisfy the governing equation. Substituting $y(t)$, $\dot{y}(t)$, and $\ddot{y}(t)$, we find

$$\begin{aligned} ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ e^{rt}(ar^2 + br + c) &= 0. \end{aligned}$$

Since $e^{rt} \neq 0$, it must be that $ar^2 + br + c = 0$. This quadratic equation for r is called the *characteristic equation*. The values of r are found using the quadratic formula,

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three possible cases for the values of $r_{1,2}$:

- Case I: There are two real and unequal values, r_1 and r_2 . In this case the two solutions are $y_1 = e^{r_1t}$ and $y_2 = e^{r_2t}$, with the complete solution,

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}.$$

- Case II: There are two real and equal values, $r_1 = r_2$. In this case the two (independent) solutions are $y_1 = e^{r_1t}$ and $y_2 = te^{r_2t}$, with the complete solution,

$$y(t) = c_1e^{r_1t} + c_2te^{r_2t}.$$

- Case III: The two values are a *complex conjugate pair*, $r_{1,2} = \alpha \pm i\beta$, with the complete solution,

$$\begin{aligned} y(t) &= c_1e^{(\alpha+i\beta)t} + c_2e^{(\alpha-i\beta)t} \\ &= e^{\alpha t}(c_1e^{i\beta t} + c_2e^{-i\beta t}). \end{aligned} \quad (\text{A.18})$$

Using Euler's identity, $e^{\pm i\theta} = \cos \theta \pm i \sin \theta$, Equation A.18 becomes

$$y(t) = e^{\alpha t}(d_1 \cos \beta t + d_2 \sin \beta t). \quad (\text{A.19})$$

Here c_1 and c_2 are complex numbers, and d_1 and d_2 are real numbers. In both Equations A.18 and A.19, $y(t)$ is real-valued.

Case III is the most interesting case for vibration studies. Another way of writing Equation A.18 that is convenient in vibration studies is

$$y(t) = Ae^{\alpha t} \sin(\beta t + \phi),$$

where $A = \sqrt{d_1^2 + d_2^2}$, and $\phi = \arctan(d_1/d_2)$. A is called the vibration *amplitude* and ϕ is the *phase angle*. This form is convenient from a physical perspective because it separates the exponential part (usually a decaying envelope) from the purely oscillatory part.

All constants of integration, whether A and ϕ or d_1 and d_2 , are determined by satisfying the initial conditions, as demonstrated in the examples that follow.

Example A.6 Solve the homogeneous equation

$$2\ddot{y}(t) - 5\dot{y}(t) - 3y(t) = 0, \quad y(0) = 1, \quad \dot{y}(0) = 10.$$

Solution: We start with the assumed solution $y(t) = e^{rt}$. Differentiating and substituting into the governing equation leads to the characteristic equation $2r^2 - 5r - 3 = 0$, with roots $r_{1,2} = 3, -1/2$. The solution is then

$$y(t) = c_1e^{3t} + c_2e^{-t/2}.$$

Constants c_1 and c_2 are evaluated by satisfying the initial conditions, resulting in the two algebraic equations,

$$\begin{aligned} y(0) &= 1 = c_1 + c_2 \\ \dot{y}(0) &= 10 = 3c_1 - \frac{c_2}{2}. \end{aligned}$$

Solving these equations leads to $c_1 = 3$ and $c_2 = -2$, giving the solution

$$y(t) = 3e^{3t} - 2e^{-t/2}.$$

We note that the solution becomes unbounded as t increases due to the $3e^{3t}$ term. ■

Example A.7 Solve the homogeneous equation

$$\ddot{y}(t) - 10\dot{y}(t) + 25y(t) = 0.$$

Solution: The characteristic equation is found to be $r^2 - 10r + 25 = 0$, with two equal roots $r_{1,2} = 5$. The solution is then

$$y(t) = c_1 e^{5t} + c_2 t e^{5t},$$

which is unbounded since $y(t)$ increases without bound. (No initial conditions were provided, meaning the specific values of c_1 and c_2 cannot be found. However, since the solution is unbounded, it would not be physically meaningful.) ■

Example A.8 Solve the homogeneous equation

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Solution: The characteristic equation is $r^2 + r + 1 = 0$, with two complex conjugate roots $r_{1,2} = (-1 \pm i\sqrt{3})/2$. The complete solution is then

$$\begin{aligned} y(t) &= c_1 \exp\left(-0.5 + i\frac{\sqrt{3}}{2}\right)t + c_2 \exp\left(-0.5 - i\frac{\sqrt{3}}{2}\right)t \\ &= e^{-0.5t} \left[\bar{c}_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + \bar{c}_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right]. \end{aligned}$$

The constants of integration can be found by satisfying the initial conditions,

$$\begin{aligned} y(0) = 1 &= e^0 [d_1 \cos(0) + d_2 \sin(0)] \rightarrow d_1 = 1 \\ \dot{y}(0) = 0 &= -0.5d_1 + \frac{\sqrt{3}}{2}d_2 \rightarrow d_2 = \frac{\sqrt{3}}{3}. \end{aligned}$$

Substituting into Equation A.19 gives the complete homogeneous solution,

$$y(t) = e^{-0.5t} \left[\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \right].$$

The bracketed term could alternatively be expressed as a cosine term with a phase lag or a sine term with a phase lead. ■

The Assumed Homogeneous Solution

Before finding the particular solution, we ask: Where did the assumption come from that *sines*, *cosines*, or the *exponential* are valid solutions to harmonic equations? The key to remember is that these functions are shorthand notation for *power series expansions*. Consider the differential equation

$$\ddot{x}(t) + n^2 x(t) = 0.$$

Not knowing the solution, one may generally assume that $x(t)$ can be written as a power series in t ,

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

Then,

$$\ddot{x}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + \dots$$

By substituting these into the differential equation, we can derive expressions for the constants a_i ,

$$\begin{aligned} (2a_2 + 6a_3 t + 12a_4 t^2 + \dots) \\ + n^2(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots) &= 0, \end{aligned}$$

where the equality must be valid for all t . This equality can hold only if the coefficient of every power in t equals zero,

$$\begin{aligned} t^0 : 2a_2 + n^2 a_0 &= 0 \\ t^1 : 6a_3 + n^2 a_1 &= 0 \\ &\vdots \end{aligned}$$

The constants defining the power series solution are found to be

$$\begin{aligned} a_2 &= -n^2 a_0 / 2 \\ a_3 &= -n^2 a_1 / 6 \\ a_4 &= -n^2 a_2 / 12 = n^4 a_0 / 24, \end{aligned}$$

etc., with as many terms as necessary. Note that the only undetermined coefficients are a_0 and a_1 , and combining like terms, we find the general solution

$$\begin{aligned} x(t) &= a_0 \left(1 - \frac{n^2 t^2}{2} + \frac{n^4 t^4}{24} - \frac{n^6 t^6}{720} + \dots \right) \\ &+ a_1 \left(t - \frac{n^2 t^3}{6} + \frac{n^4 t^5}{120} - \dots \right), \end{aligned}$$

where a_0 and a_1 are the arbitrary constants required for a second-order equation. The series in the first parenthesis is called $\cos nt$, and the series in the second parenthesis is $\sin nt$. Thus, the homogeneous solution is the sum of a sine and a cosine term. Similarly, the series expansion called the *exponential* is

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \dots,$$

which is equal to the sum of the sine and cosine expansions, where $z = nt$.

The power series is the assumed solution to many important differential equations.

A.4.3 Particular Solution

Consider the following non-homogeneous equation,

$$a\ddot{y}(t) + b\dot{y}(t) + cy(t) = f(t),$$

where $f(t)$ may be any of the following functions:

- a constant, k
- a polynomial in t , for example, $c_n t^n + \cdots + c_1 t + c_0$
- an exponential function, $\exp(at)$
- a trigonometric function, $\sin \beta t$, $\cos \beta t$,

or a finite sum and/or product of these functions. As a general rule, for these cases the particular solution $y_p(t)$ has the same form as the forcing function:

- If $f(t) = \text{constant}$, then $y_p(t) = A$, a constant.
- If $f(t) = \text{polynomial}$, then $y_p(t)$ is a polynomial of the same order.
- If $f(t) = \exp(at)$, then $y_p(t) = c \exp(at)$.
- If $f(t) = \sin \beta t$ or $\cos \beta t$, then $y_p(t) = d_1 \sin \beta t + d_2 \cos \beta t$.

Example A.9 Find the particular solution of the ordinary differential equation,

$$y''(x) + 3y'(x) + 2y = 4x^2.$$

In this equation, the independent variable is x , the dependent variable is y , and the primes denote differentiation with respect to x .

Solution: The particular solution is taken to be of the form $y_p(x) = ax^2 + bx + c$. To evaluate constants a , b , and c , substitute $y_p(x)$, $y'_p(x)$, and $y''_p(x)$ into the governing equation,

$$\begin{aligned} 2a + 3(2ax + b) + 2(ax^2 + bx + c) &= 4x^2 \\ \text{or } (2a + 2c + 3b) + (6a + 2b)x + (2a)x^2 &= 4x^2. \end{aligned}$$

By equating the coefficients of the same powers of x , we have the three equations that need to be solved for the three unknowns,

$$\begin{aligned} x^0 : 2a + 2c + 3b &= 0 \\ x^1 : 6a + 2b &= 0 \\ x^2 : 2a &= 4. \end{aligned}$$

Solving these equations yields $a = 2$, $b = -6$, and $c = 7$. The particular solution is then $y_p(x) = 2x^2 - 6x + 7$. ■

Example A.10 Find the particular solution of the ordinary differential equation,

$$y''(x) - 3y'(x) = 4 \sin x.$$

Solution: To solve, we assume a particular solution of the form $y_p(x) = a \sin x + b \cos x$. By differentiating and substituting as before, and then equating coefficients of sine and cosine terms, we find

$$\begin{aligned} -3a - b &= 0 \\ -a + 3b &= 4. \end{aligned}$$

Solving these equations gives

$$a = -2/5 \text{ and } b = 6/5.$$

The particular solution is then $y_p(x) = -(2/5) \sin x + (6/5) \cos x$. ■

Example A.11 Find the solution of the differential equation that has two terms on the right-hand side,

$$y''(x) + 8y(x) = 5 + 2e^{-x}.$$

Solution: The forcing function consists of a constant plus an exponential function. For this linear equation, proceed using superposition by assuming a similar particular solution $y_p(x) = A + Be^{-x}$. Following the previous procedure, the two equations governing the values of A and B are $9B = 2$, and $8A = 5$, with the result

$$y_p(x) = \frac{5}{8} + \frac{2}{9}e^{-x}.$$

Next, solve for the homogeneous solution. Then, the complete solution is the sum $y_h(x) + y_p(x)$. ■

Example A.12 Find the solution of the following differential equation that has a product non-homogeneous term,

$$y''(x) - 2y'(x) + y(x) = 10e^{-2x} \cos x.$$

Solution: Assume the solution

$$\begin{aligned} y_p(x) &= Ae^{-2x}(B \cos x + C \sin x) \\ &= e^{-2x}(a_1 \cos x + a_2 \sin x). \end{aligned}$$

The steps of substituting this assumed solution into the governing equation and solving are identical to those done previously. ■

Example A.13 Find the particular solution of the differential equation

$$y''(x) + y(x) = \cos x.$$

Solution: It would seem that the previous solution procedures would be directly applicable. However, if we try

$$y_p(x) = A \sin x + B \cos x,$$

and differentiate and substitute as before, we find

$$\begin{aligned} -A \sin x - B \cos x + A \sin x + B \cos x &= \cos x \\ 0 &= \cos x, \end{aligned}$$

which is not valid over all x . The problem lies with the fact that the chosen particular solution is not linearly independent of the homogeneous solution. This is similar to Case II in Section A.4.2, and, by analogy, an appropriate choice for the particular solution here is

$$y_p(x) = Ax \sin x + Bx \cos x.$$

When substituted and solved, the coefficients are found to be $A = 1/2$ and $B = 0$, giving

$$y_p(x) = \frac{1}{2}x \sin x.$$

This example considers the very important case in which the question of linearly independent solutions arises. The approach is to first solve the homogeneous equation, and then find the particular solution. If the assumed particular solution contains a term that is present in the homogeneous solution, $y_p(x)$ is then multiplied by x until no similar terms exist. This problem is encountered when resonance is studied in Chapter 2. ■

A.5 Laplace Transforms

The *Laplace transform* is one of a powerful group of techniques known as *transform methods for linear systems* that transform one class of functions into another. Given a real function $x(t)$ that satisfies the condition

$$\int_0^{\infty} |x(t)e^{-\sigma t}| dt < \infty$$

for some real σ , the Laplace transform of $x(t)$ is defined as

$$\mathcal{L}[x(t)] \equiv X(s) = \int_0^{\infty} x(t)e^{-st} dt, \quad (\text{A.20})$$

where $s = \sigma + i\omega$ is a complex number, sometimes referred to as the *Laplace operator*, where σ and ω are real numbers. Notationally, any variable with s as its argument is a Laplace-transformed variable.

The defining equation, Equation A.20, is also known as the *one-sided Laplace transform*,¹² as the integration is evaluated from $t = 0$ to ∞ . It reflects the fact that all information contained in $x(t)$ prior to $t = 0$ is ignored or considered to be zero. This assumption does not impose any limitation on the applications of the Laplace transform for our vibration studies, since the time reference is usually chosen at $t = 0$. For a physical system with an input applied at $t = 0$, the response of the system does not start before this time, that is, the response of a physical system does not precede the excitation.¹³

The essential purpose in using the Laplace transform is to convert a differential equation in the t domain into an equivalent *algebraic* expression in the s domain that can be solved in a simpler way. The forward process, transforming from t to s , is relatively straightforward. In the s domain, the algebraic relation is solved for the transform of the response, $X(s)$. Then it is necessary to transform back to the time domain to obtain the desired solution, $x(t)$. The inverse transform can be intricate to solve analytically, requiring *complex variable methods*, but tabulated transforms ease the process.

In addition to the algebraic nature of the Laplace transform technique, the solution automatically satisfies the initial conditions. For example, a differential equation that governs the vibration of a mass-spring-damper system can be written as $\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = F(t)/m$, with initial conditions $x(0)$ and $\dot{x}(0)$, where $x(t)$ is the displacement, $F(t)$ is the force, m is the mass, ζ is the damping ratio, and ω_n is the natural frequency. We can transform this time-domain differential equation to the s -domain as follows. First, we need to take Laplace transforms of the first and second derivatives. From the definition of the Laplace transform, the following relations can be found,

$$\mathcal{L}[x(t)] = X(s)$$

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$$

$$\mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0)$$

$$\mathcal{L}[F(t)/m] = F(s)/m,$$

where the initial conditions are included. Generally, capital letters are used to represent the transformed variables. Since some variables, such as force, $F(t)$, are customarily capitalized in the time domain, it is necessary

¹²The one-sided Laplace transform is more strictly defined by integration from $t = 0^-$ to ∞ in the defining equation. The symbol 0^- implies that the limit of $t \rightarrow 0$ is taken from the left side of $t = 0$. This limit process will include situations for which the function $x(t)$ has a jump discontinuity or an impulse at $t = 0$.

¹³Such a system is also known as being *causal*.

to distinguish between the original time domain and the transformed s domain by the argument (t) or (s) .

Substituting these transformed variables into the governing equation, and solving for the transformed variable, $X(s)$, we find

$$X(s) = \frac{F(s)/m + \dot{x}(0) + (s + 2\zeta\omega_n)x(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad (\text{A.21})$$

where the initial conditions are explicit. For any force $F(t)$, Equation A.21 must be inverted to solve for displacement $x(t)$.

A.5.1 Borel's Theorem

Borel's theorem states that the inverse Laplace transform of the product of two transforms is equal to the convolution of their inverse transforms,

$$\int_0^t x(t-\tau)h(\tau)d\tau = \mathcal{L}^{-1}[X(s)H(s)].$$

Using Borel's theorem and a table of Laplace transform pairs, $X(s)$ of Equation A.21 can be inverted term by term as follows,

$$\begin{aligned} & \mathcal{L}^{-1}\left\{\frac{F(s)/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \\ &= \frac{1}{m\omega_d} \int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)d\tau \\ & \mathcal{L}^{-1}\left\{\frac{\dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \\ &= \frac{\dot{x}(0)}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \\ & \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \\ &= \frac{x(0)}{(1-\zeta^2)^{1/2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \psi), \end{aligned}$$

where

$$\begin{aligned} \omega_d &= \omega_n(1-\zeta^2)^{1/2} \\ \psi &= \tan^{-1} \frac{\zeta}{(1-\zeta^2)^{1/2}}. \end{aligned}$$

This leads to the inverse transform of Equation A.21,

$$\begin{aligned} x(t) &= \frac{1}{m\omega_d} \int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)d\tau \\ &+ \frac{\dot{x}(0)}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t + \frac{x(0)}{(1-\zeta^2)^{1/2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \psi). \end{aligned}$$

The integral

$$\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)d\tau$$

is known as a *convolution* integral.

A.5.2 Partial Fraction Expansion

Partial fraction expansion is performed when we want to represent a complicated fraction as a sum of simpler fractions to ease the inversion from $X(s)$ to $x(t)$. We consider the case where the fraction is of the form

$$D(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+a_1)(s+a_2)\dots(s+a_n)},$$

where all the roots a_i are distinct, the denominator $D(s)$ is a polynomial of order n , and the numerator $N(s)$ is assumed to be of order less than n . The partial fraction expansion is performed by the solution of the equality,

$$\begin{aligned} & \frac{N(s)}{(s+a_1)(s+a_2)\dots(s+a_n)} \\ &= \frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \dots + \frac{A_n}{s+a_n}, \end{aligned}$$

where

$$A_i = (s+a_i)D(s)|_{s=-a_i}.$$

Example A.14 Express the fraction $D(s)$,

$$D(s) = \frac{s+3}{s^3+7s^2+10s},$$

in terms of its partial fraction expansion.

Solution: To solve, factor the denominator and write out the factors in terms of their distinct roots.

$$\begin{aligned} D(s) &= \frac{s+3}{s^3+7s^2+10s} = \frac{s+3}{s(s+2)(s+5)} \\ &= \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+5}. \end{aligned}$$

The problem then becomes one of finding A_1 and A_2 . For A_1 ,

$$\begin{aligned} A_1 &= (s+a_1)D(s)|_{s=-a_1} \\ &= sD(s)|_{s=0} = \frac{s+3}{s(s+2)(s+5)} \Big|_{s=0} \\ &= 0.3. \end{aligned}$$

For A_2 ,

$$\begin{aligned} A_2 &= (s+a_2)D(s)|_{s=-a_2} \\ &= (s+2)D(s)|_{s=-2} = \frac{s+3}{s(s+5)} \Big|_{s=-2} \\ &= -1/6. \end{aligned}$$

A_3 can be found to be $-2/15$. ■

There are more complicated fractions for which this simple technique does not work, such as when the order of the numerator polynomial is not less than that of the denominator, when the roots are repeated, when the roots are complex, and when there is an exponential or other function in the numerator.

Table A.2: A Short List of Laplace Transform Pairs.

Function of Time	Laplace Transform
$x(t)$	$X(s)$
$\delta(t)$ unit impulse	1
$\delta(t-a)$	$\exp(-as)$
$u(t)$ step	$1/s$
t ramp	$1/s^2$
$\exp(-at)$	$1/(s+a)$
$\sin \omega t$	$\omega/(s^2 + \omega^2)$
$\cos \omega t$	$s/(s^2 + \omega^2)$
$\exp(-at) \sin \omega t$	$\omega/((s+a)^2 + \omega^2)$
$\exp(-at) \cos \omega t$	$(s+a)/((s+a)^2 + \omega^2)$
$\exp(-at)f(t)$	$F(s-a)$
$t \exp(-at)$	$1/(s+a)^2$
$\frac{\exp(-at) - \exp(-bt)}{(b-a)}$	$1/(s+a)(s+b)$
$f(t-a)u(t-a), a > 0$	$\exp(-as)F(s)$
$\int_0^t x(t-\tau)h(\tau)d\tau$	$X(s)H(s)$
$\frac{1}{\omega_d} \exp(-\zeta\omega_n t) \sin \omega_d t, \zeta < 1$	$1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$

A.5.3 Laplace Transform Table

Table A.2 provides a short list of useful Laplace transform pairs, where the time-domain function is shown in the left column and the corresponding Laplace transform is shown in the right column.

A.5.4 Initial-Value, Final-Value Theorem

The initial-value theorem is

$$x(0) = \lim_{t \rightarrow 0} x(t) = \lim_{s \rightarrow \infty} sX(s),$$

if the time limit exists.

The final-value theorem is

$$x(t \rightarrow \infty) = \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s).$$

The final-value theorem is very useful for the analysis and design of control systems, since it gives the final value of a function of time by knowing the value of its Laplace transform at $s = 0$. The final-value theorem does have some restrictions. For example, the final-value theorem is not valid if $sX(s)$ contains any pole whose real part is

zero or positive.¹⁴

Example A.15 Determine the final value of $y(t)$ from the Laplace transform

$$Y(s) = \frac{8}{s(s^2 + 3s + 2)}.$$

Solution: First test whether the final-value theorem can be applied by determining the poles of $sY(s)$,

$$\begin{aligned} sY(s) &= \frac{8}{(s^2 + 3s + 2)} \\ &= \frac{8}{(s+1)(s+2)}. \end{aligned}$$

Since the poles are negative, at $s = -1$ and $s = -2$, the final-value theorem may be applied,

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} sY(s) \\ &= \lim_{s \rightarrow 0} \left[\frac{8}{(s+1)(s+2)} \right] \\ &= 4. \end{aligned}$$

Thus, the steady-state value is $y = 4$. ■

A.6 Fourier Series & Transforms

The Fourier series and Fourier transform are used widely in solving engineering and science problems.¹⁵

A.6.1 Fourier Series

Fourier series are representations of periodic functions in terms of the sum of harmonic functions. In other words, the Fourier series is the sum of sines and cosines with appropriate amplitudes and frequencies that give an equivalent representation of a periodic function.

The Fourier series of an arbitrary periodic function,¹⁶ $F(t)$, is given by

$$F(t) = \frac{a_0}{2} + \sum_{p=1}^{\infty} (a_p \cos p\omega_T t + b_p \sin p\omega_T t),$$

where $p = 1, 2, 3, \dots$ and $\omega_T = 2\pi/T$ with T being the period of the function. The *Fourier coefficients* a_p and b_p

¹⁴Equivalently, in mathematical jargon, the final-value theorem can only be applied if $sX(s)$ is analytic on the imaginary axis and in the right-half of the s -plane.

¹⁵**Fourier Series and Boundary Value Problems**, J.W. Brown and R.V. Churchill, McGraw-Hill, 2008.

¹⁶If this function meets the *Dirichlet conditions* (it must be single-valued, be finite, and have a finite number of discontinuities and maxima and minima in one cycle), it may be represented by a Fourier series.

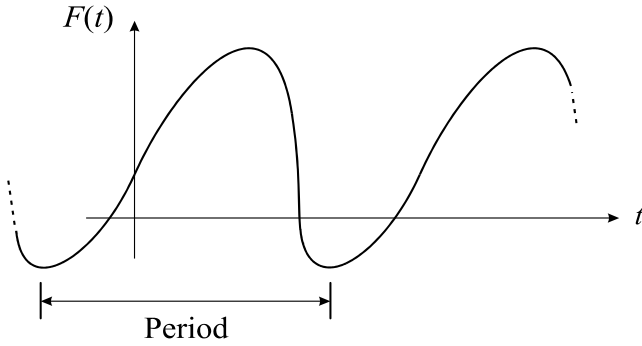


Figure A.3: Periodic but not harmonic function.

are derivable using the orthogonality properties of *sine* and *cosine* functions:

$$a_p = \frac{2}{T} \int_{-T/2}^{T/2} F(t) \cos p\omega_T t \, dt \quad p = 0, 1, 2, \dots$$

$$b_p = \frac{2}{T} \int_{-T/2}^{T/2} F(t) \sin p\omega_T t \, dt \quad p = 1, 2, \dots$$

The Fourier coefficients are a measure of participation of each harmonic component in the final waveform. We note that $a_0/2$ is the average (mean) value of the waveform. (It can also be seen from the equation for a_p with $p = 0$.)

In principle, an infinite number of harmonic components is required to exactly duplicate the waveform of $F(t)$. In engineering practice, often only a few terms are needed for a reasonably accurate representation.

Some books present equivalent definitions of Fourier series where the integrals defining the coefficients a_p and b_p have shifted intervals or different factors.

Example A.16 Derive the equation for coefficient b_p .

Solution: We begin with the Fourier series expression for $F(t)$, multiply both sides by $\sin p\omega_T t \, dt$, and integrate over the period, from $-T/2$ to $T/2$,

$$\begin{aligned} \int_{-T/2}^{T/2} F(t) \sin p\omega_T t \, dt &= \int_{-T/2}^{T/2} \frac{a_0}{2} \sin p\omega_T t \, dt \\ &+ \sum_{p=1}^{\infty} \int_{-T/2}^{T/2} (a_p \cos p\omega_T t \sin p\omega_T t \\ &+ b_p \sin p\omega_T t \sin p\omega_T t) \, dt \\ &= \frac{1}{2} b_p T. \end{aligned}$$

The integrals of the a_0 and a_p terms equal zero, and thus the integral of the b_p term simplifies as shown. ■

A non-periodic function may be interpreted to have a periodicity of infinite time. In these cases, Fourier transforms are used rather than Fourier series.

A.6.2 Fourier Transforms

Fourier transforms¹⁷ are linear transforms that are used in many branches of science and engineering. A physical waveform and its power spectrum are Fourier transforms of each other. An example of a physical waveform is a forcing function. Its Fourier transform, known as spectrum, power spectrum, or spectral density, is a measure of how the waveform energy is distributed across the frequency energy spectrum.

The Fourier transform of $q(t)$ is defined as

$$\mathcal{F}\{q(t)\} = Q(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(t) e^{-i\omega t} dt.$$

The inverse Fourier transform of $Q(\omega)$ recovers $q(t)$,

$$\mathcal{F}^{-1}\{Q(\omega)\} = q(t) = \int_{-\infty}^{\infty} Q(\omega) e^{i\omega t} d\omega.$$

$Q(\omega)$ and $q(t)$ are known as a Fourier transform pair. In some books, the coefficient $(1/2\pi)$ appears as a factor in the inverse transform, and in some books $(1/\sqrt{2\pi})$ appears in both transforms. In all definitions, the product of the two coefficients is $1/2\pi$. Figure A.4 graphically depicts some important Fourier transform pairs.

A.7 Partial Differential Equations

Partial differential equations are the foundation of much of mathematical physics and engineering. Some calculus books provide introductions to partial differential equations, and there are numerous books that specialize in this class of equations. They are more difficult to solve than ordinary differential equations because the functions involve two or more independent variables. In this text, we are introduced to partial differential equations when we consider systems that have distributed parameters. The displacement of a point on the structure is a function of location as well as time.

An example of an important partial differential equation is the wave equation,¹⁸

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2},$$

where $y(x, t)$ is a function of two independent variables, the spatial variable x and the time variable t . The most elementary solution of a partial differential equation is the *separable* solution,

$$y(x, t) = X(x)T(t),$$

¹⁷A recommended text is **The Fourier Transform and Its Applications**, R. Bracewell, McGraw-Hill, 1965.

¹⁸The wave equation represents one of three common types of partial differential equations. It is sometimes known as a *hyperbolic equation*, as distinguished from *elliptic* and *parabolic* partial differential equations.

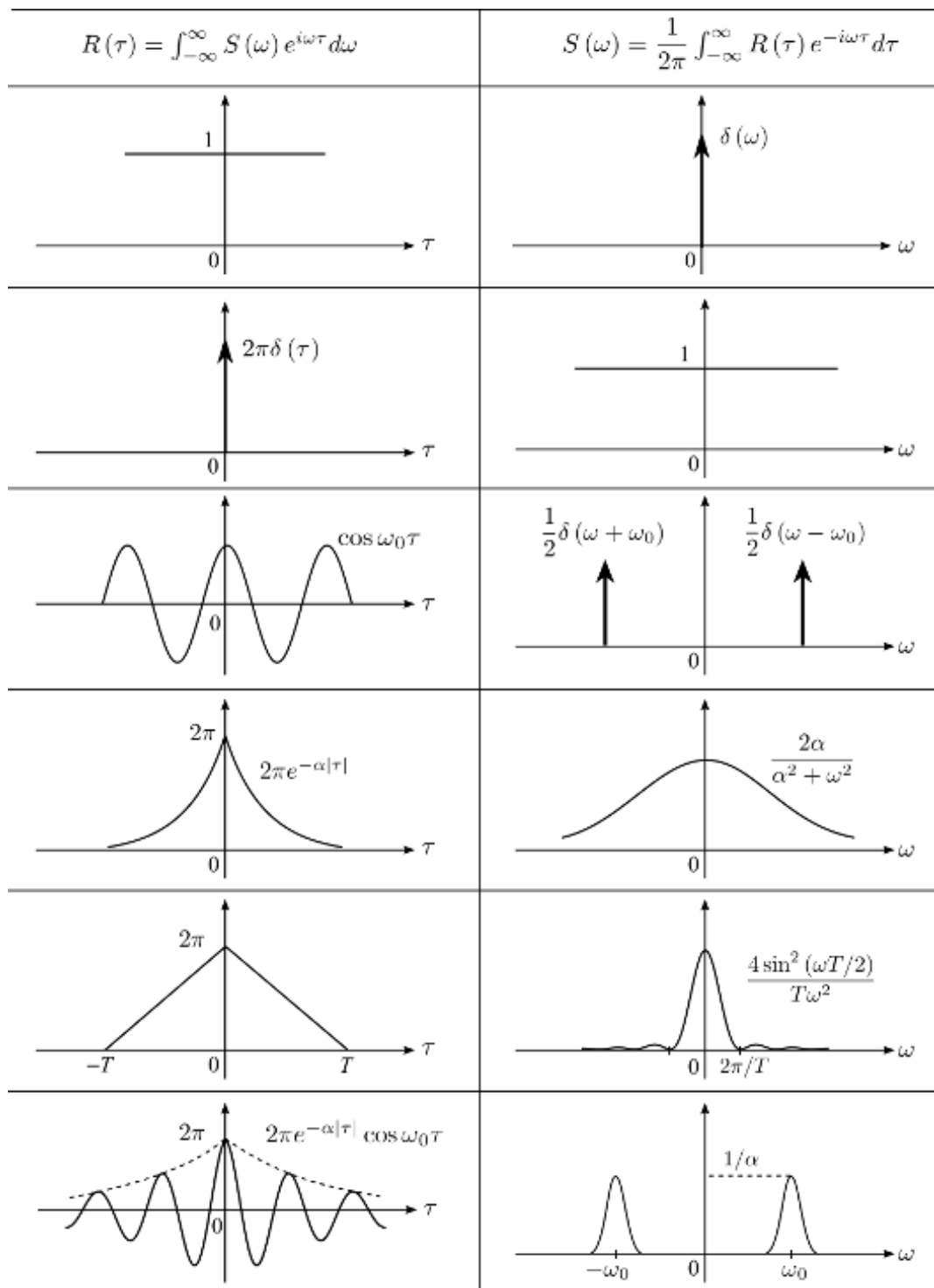


Figure A.4: Sketches of important Fourier transform pairs where $R(\tau)$ is a function known as a correlation function and $S(\omega)$ is its Fourier transform known as the spectral density.

which implies that behavior in time and space are independent. Since the assumed solution must satisfy the wave equation, we can take the appropriate derivatives, $y'' = X''T$, $\ddot{y} = X\ddot{T}$, and substitute to find

$$\begin{aligned} X''T &= \frac{1}{c^2} X\ddot{T} \\ \underbrace{c^2 \frac{X''}{X}}_{\text{I}} &= \underbrace{\frac{\ddot{T}}{T}}_{\text{II}} = -\omega^2. \end{aligned}$$

In the last equation, the time-dependent variables and space-dependent variables have been placed on opposite sides of the equal sign. Since part I is only a function of x and part II is only a function of t , it must be that each is equal to the same constant, say $-\omega^2$. Separation of variables has led to two ordinary differential equations, one for each independent variable.

The equation in time is $\ddot{T} + \omega^2 T = 0$. When it is solved the result is simple harmonic motion with frequency ω . The negative sign is assigned to the constant ω^2 to ensure that the solution in time is harmonic rather than an ever-growing exponential. Therefore, if the separation of variables is valid, it implies that the solution is harmonic in time. The equation for $T(t)$ can be solved given two initial conditions, $T(0)$ and $\dot{T}(0)$.

The equation in space is

$$X'' + \beta^2 X = 0, \quad \beta^2 = \frac{\omega^2}{c^2},$$

which must be satisfied in the domain $0 \leq x \leq L$, the spatial domain of X . This equation can be solved given two boundary conditions, $X(0)$ and $X(L)$. Whereas $T(t)$ is harmonic in time, $X(x)$ is harmonic in the spatial domain.

Example A.17 Solve the wave equation. First find $T(t)$ and $X(x)$ in the above formulation.

Solution: The solution to $\ddot{T} + \omega^2 T = 0$ is of the form $T(t) = A \cos \omega t + B \sin \omega t$. Note that ω will be determined from the solution of the differential equation for $X(x)$. The initial conditions are used to evaluate the integration constants A and B only after the complete solution is determined.

The solution to $X'' + \beta^2 X = 0$ is $X(x) = C \cos \beta x + D \sin \beta x$. The boundary conditions are used to evaluate the integration constants C and D ,

$$\begin{aligned} X(0) &= C \\ X(L) &= C \cos \beta L + D \sin \beta L. \end{aligned}$$

Solving the second equation for D results in

$$D = \frac{X(L) - X(0) \cos \beta L}{\sin \beta L}.$$

For specificity, suppose $X(0) = 0$ and $X(L) = 0$. Then $C = 0$, and it must be that $D \neq 0$ so that $X(x) \neq 0$ to avoid the trivial solution. It follows that $\sin \beta L = 0$, from which we find

$$\beta_n L = n\pi \quad \text{or} \quad \beta_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, \dots,$$

or, using the above definition for β ,

$$\frac{\omega_n}{c} = \frac{n\pi}{L} \quad \text{or} \quad \omega_n = \frac{n\pi c}{L}, \quad n = 0, 1, 2, \dots,$$

giving the values of ω_n . The complete solution is the sum of all solutions,

$$\begin{aligned} y(x, t) &= \sum_{n=0}^{\infty} X_n(x) T_n(t) \\ &= \sum_{n=0}^{\infty} D_n \sin \beta_n x (A_n \cos \omega_n t + B_n \sin \omega_n t), \end{aligned}$$

where the values of D_n are generally obtained in ways customary to the application. In vibration, we will see that amplitude D is found via a normalization procedure and coefficients A_n and B_n are found by satisfying the initial conditions. ■

Appendix B

Viscoelastic Damping

“The mechanisms of energy loss.”

This appendix provides an introduction to viscoelastic damping models. It generalizes the classic viscous damping model discussed in [Chapter 3](#).

B.1 Viscoelastic Materials

Viscoelastic materials have history-dependent properties or memory. They are commonly found in practice, and often used to remove vibrational energy. Viscoelastic material models represent material damping at very small scales where differential motions due to lattice dislocations and imperfections result in energy loss through friction-induced thermal processes. Viscoelastic energy dissipation is primarily due to shear deformation.

Where significant energy dissipation is required, composites and viscoelastic (rubberlike) materials are used. Their hysteresis loops have much larger enclosed areas than do typical materials such as steel and aluminum. A difficulty with using high damping alloys is that they have inferior stiffness properties, are not as strong as needed, and have corrosion and thermal properties that make them unsuitable for structural applications except for specialized purposes.¹ Fiber-reinforced composites, on the other hand, have found extensive usage in the aerospace and automotive industries.

Viscoelastic materials are generally characterized by the results of experimental testing. An example would be performing uniaxial tensile tests similar to those used for elastic solids, except these tests are modified in order to highlight the time-dependent material response. The tests are used to demonstrate *creep*, *stress relaxation*, and *sinusoidal loading*.²

The creep test measures time-dependent strain that results from the application of a constant uniaxial stress.

For a linear material, when the stress is doubled, for example, the resulting strain is doubled over the full range of time required to achieve the complete displacement. Stress relaxation is the reverse of creep. Here, a time-dependent stress corresponds to a constant strain. Generally, the relaxation response moves more quickly to its equilibrium value than does the creep response.

Of interest is how a viscoelastic structure responds to a dynamic force. A viscoelastic material driven by a harmonic force will respond in steady state at the same frequency with a phase lag. The origin of the strain-stress ϵ - σ graph can be chosen so that

$$\begin{aligned}\epsilon &= \epsilon_0 \cos \omega t \\ \sigma &= \sigma_0 \cos(\omega t + \phi),\end{aligned}$$

where ϵ_0 is the peak strain, σ_0 is the peak stress, ω is the frequency of the harmonic force, and ϕ is the phase angle.

Following Roylance, it is convenient to write the stress as a complex quantity σ^* with a real part that is in phase with the strain and an imaginary part that is $\pi/2$ rad out of phase with it,

$$\sigma^* = \sigma'_0 \cos \omega t + i\sigma''_0 \sin \omega t,$$

where σ^* can be plotted in the complex plane. From this rotating vector representation we can find the following relations,

$$\begin{aligned}\tan \phi &= \sigma''_0 / \sigma'_0 \\ |\sigma^*| &= \sigma_0 = \sqrt{(\sigma'_0)^2 + (\sigma''_0)^2} \\ \sigma'_0 &= \sigma_0 \cos \phi \\ \sigma''_0 &= \sigma_0 \sin \phi.\end{aligned}$$

Note that primes and double primes are part of the variable name rather than being differentiations of any kind.

Based on these definitions, two dynamic moduli can be defined. The first is the *storage modulus*, E' ,

$$E' = \frac{\sigma'_0}{\epsilon_0},$$

¹ **Vibration Damping of Structural Elements**, C.T. Sun and Y.P. Lu, Prentice Hall 1995.

² D. Roylance, Engineering Viscoelasticity, MIT Course Notes, 24 October 2001.

which is the ratio of the in-phase stress to strain. The second is the *loss modulus*, E'' ,

$$E'' = \frac{\sigma_0''}{\epsilon_0},$$

which is the ratio of the out-of-phase stress to strain. Note that we can define a complex modulus E^* such that

$$E^* = \sqrt{[E']^2 + [E'']^2}$$

where

$$\begin{aligned} E' &= |E^*| \cos \phi \\ E'' &= |E^*| \sin \phi. \end{aligned}$$

The phase angle ϕ is the phase difference between the dynamic stress and the dynamic strain in a viscoelastic material subject to a sinusoidal oscillation. As per Graesser,³ for the case where strain energy is the measure of stored energy in the material, the following relation exists between the loss factor, η , and the phase angle, ϕ ,

$$\eta = \tan \phi = \frac{E''}{E'}.$$

The loss factor is a dimensionless ratio that is a measure of energy loss, and represents mechanical damping or internal friction in a viscoelastic system. A high value indicates a material with a high nonelastic strain component, while a low value indicates one that is more elastic. (For the Kelvin-Voigt model discussed below, $\eta = \tan \phi = c\omega/k = 2\zeta\omega/\omega_n$.)

B.1.1 Work Done Per Cycle

Following Roylance again, we pursue a physical interpretation of the two moduli defined above. Consider the work done per loading cycle using the strain energy per unit volume,

$$W = \oint \sigma^* d\epsilon = \oint \sigma^* \frac{d\epsilon}{dt} dt,$$

where we integrate the in-phase and out-of-phase components separately,

$$\begin{aligned} W_{\text{in-phase}} &= \int_0^{2\pi/\omega} (\sigma_0' \cos \omega t) (-\epsilon_0 \omega \sin \omega t) dt = 0 \\ W_{\text{out-of-phase}} &= \int_0^{2\pi/\omega} (\sigma_0'' \sin \omega t) (-\epsilon_0 \omega \sin \omega t) dt \\ &= -\pi \sigma_0'' \epsilon_0. \end{aligned}$$

The in-phase work done, or the strain energy associated with the in-phase stress, is *reversible* and can be

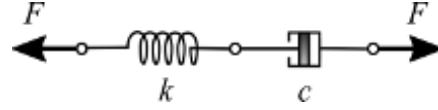


Figure B.1: The Maxwell model.

recovered during a loading cycle without loss. On the other hand, the out-of-phase work done, or the energy supplied to the material by the out-of-phase stress, is *irreversibly lost as heat*.

An alternative form for the stress and strain is via complex exponentials,

$$\begin{aligned} \sigma &= \sigma_0^* \exp(i\omega t) \\ \epsilon &= \epsilon_0^* \exp(i\omega t). \end{aligned}$$

The complex modulus is now defined as

$$E^* = \frac{\sigma_0^*}{\epsilon_0^*}.$$

B.2 Viscoelastic Material Models

Viscoelastic models, as the name implies, are generalized models of elasticity and viscosity. They are modeled mechanically using hybrid combinations of linear springs and dashpots.

Before proceeding, we distinguish between two applications of spring-dashpot models. One application, material modeling, uses the spring-dashpot system to model material behavior using such elements. Another application, vibration system modeling, uses the spring-dashpot system to model the boundary conditions of a solid or structure. The springs and dashpots are elements that connect the structures or solids to each other and to any ground.

Three basic viscoelastic material models are the *Maxwell* model, the *Voigt* model (sometimes referred to as *Kelvin-Voigt*), and the *standard linear* model. We introduce these next.

B.2.1 Maxwell Model

The Maxwell model in Figure B.1 consists of a spring of constant stiffness k and dashpot of constant viscous damping c connected in series. The total displacement u , due to the elongation of the spring and the dashpot, of the Maxwell model subject to a force $F(t)$ is given by the first-order differential equation for $u(t)$,

$$\frac{du}{dt} = \frac{1}{k} \frac{dF}{dt} + \frac{F}{c}, \quad (\text{B.1})$$

where u represents the total elongation resulting from the sum of the elongations of the spring and dashpot

³Graesser and Wong, p.16, op. cit.

elements. Alternatively,

$$\frac{dF}{dt} + \frac{kF}{c} = k \frac{du}{dt}. \quad (\text{B.2})$$

This first-order differential equation can be integrated between time 0 and t , to yield an equation for the force as a function of displacement u .

JAMES CLERK MAXWELL

(June 13, 1831 – November 5, 1879)



Contribution: Maxwell was a world-famous Scottish mathematical physicist, whose most notable achievement was to formulate the classical theory of electromagnetic radiation, bringing together electricity, magnetism, and light as manifestations of the same phenomenon. Maxwell's equations for electromagnetism have been called the "second great unification in physics" after the first one realized by Isaac Newton.

Maxwell demonstrated that electric and magnetic fields travel through space as waves moving at the speed of light. Maxwell proposed that light is an undulation in the same medium that is the cause of electric and magnetic phenomena. The unification of light and electrical phenomena led to the prediction of the existence of radio waves.

Maxwell had many major achievements. He helped develop the Maxwell-Boltzmann distribution, a statistical means of describing aspects of the kinetic theory of gases. He proved that the Rings of Saturn were made of numerous small particles. Maxwell's paper on the centrifugal governor used to regulate steam engines is the basis of feedback control theory. Maxwell is credited with discovering photoelasticity as well as dimensional analysis. He presented the first durable color photograph. He developed fundamental results on the rigidity of rod-and-joint frameworks (trusses) used in bridges.

His discoveries helped usher in the era of modern physics, laying the foundation for such fields as special relativity and quantum mechanics.

Life Sketch: James Clerk Maxwell was born in Edinburgh, Scotland. Maxwell's parents met and married when

they were well into their thirties; his mother was nearly 40 when he was born. They had had one earlier child, a daughter named Elizabeth, who died in infancy. His father, John Clerk Maxwell, was an advocate and a man of comfortable means.

When he was young his family moved to the Galloway area of southwestern Scotland. There he enjoyed a country upbringing and his natural curiosity displayed itself at an early age. His mother taught him scripture and his knowledge was very detailed; he could give chapter and verse for almost any quotation from the psalms.

When Maxwell was eight his mother was taken ill with abdominal cancer and, after an unsuccessful operation, died in December 1839. His parents' plan to educate him at home until he was 13 and for him to go the Edinburgh University fell through. A tutor was hired but the arrangement was not successful and it was decided that he should attend the prestigious Edinburgh Academy.

In 1841 he began Edinburgh Academy where he had the nickname "Dafty" (attributed to arriving on his first day of school wearing a pair of homemade shoes and a tunic). Having been raised in isolation on his father's countryside estate, he did not fit in well at school. He was at first regarded as shy and dull, made no friends, and spent his occasional free time reading old ballads, drawing curious diagrams, and making mechanical models.

Despite winning the school's scripture biography prize in his second year, Maxwell's academic work remained unnoticed until, at the age of 13, he won the school's mathematical medal and first prize for both English and poetry. At about this time, social isolation ended when he met Lewis Campbell and Peter Guthrie Tait, two boys of a similar age who became notable scholars later in life and remained his lifelong friends. Maxwell became one of the most brilliant students at the Academy, gaining the highest prizes for scholarship, mathematics, and English verse.

In early 1846 at the age of 14, Maxwell wrote a paper on ovals. In this work he generalized the definition of an ellipse by defining the locus of a point where the sum of m times the distance from one fixed point plus n times the distance from a second fixed point is constant. If $m = n = 1$ then the curve is an ellipse. Maxwell also defined curves where there were more than two foci. This became his first paper, *On the description of oval curves, and those having a plurality of foci*, which was read to the Royal Society of Edinburgh in April 1846. (Maxwell was deemed too young to present the work himself.) Although these ideas were not entirely new as René Descartes had also examined the properties of such multifocal ellipses in the seventeenth century, he had simplified their construction, remarkable for a 14 year old.

Maxwell left the Academy in 1847 at age 16 and began attending classes at the University of Edinburgh. The academic staff included some highly regarded names; his first year tutors included Sir William Hamilton, who lectured him on logic and metaphysics, Philip Kelland on mathematics,

and James Forbes on natural philosophy. He did not find his classes at Edinburgh University very demanding, and was able to immerse himself in private study during free time at the university and particularly when back home. There he would experiment with improvised chemical, electric, and magnetic apparatus, with special interest in the properties of polarized light. He constructed shaped blocks of gelatin, subjected them to various stresses, and with a pair of polarizing prisms, viewed the colored fringes that developed within the jelly. Through this, he discovered photoelasticity, an experimental means of determining stress distributions within physical structures.

At age 18, Maxwell wrote two papers that were published in the Transactions of the Royal Society of Edinburgh. One, *On the Equilibrium of Elastic Solids*, laid the foundation for an important discovery later in his life, which was the temporary double refraction produced in viscous liquids by shear stress. His other paper was *The Theory of Rolling Curves*.

In October 1850, already an accomplished mathematician, Maxwell left Scotland for the University of Cambridge. At Trinity College he was elected to the elite secret society known as the Cambridge Apostles. Maxwell's intellectual understanding of his Christian faith and of science grew rapidly during his Cambridge years.

For example, he wrote: "Now my great plan, which was conceived of old, ... is to let nothing be wilfully left unexamined. Nothing is to be holy ground consecrated to Stationary Faith, whether positive or negative. All fallow land is to be ploughed up and a regular system of rotation followed. ... Never hide anything, be it weed or no, nor seem to wish it hidden. ... Again I assert the Right of Trespass on any plot of Holy Ground which any man has set apart." In this same essay, he continued: "The Old Testament and the Mosaic Law and Judaism are commonly supposed to be 'Tabooed' by the orthodox. Sceptics pretend to have read them, and have found certain witty objections ... which too many of the orthodox unread admit, and shut up the subject as haunted. But a Candle is coming to drive out all Ghosts and Bugbears. Let us follow the light."

The extent to which Maxwell "ploughed up" his Christian beliefs and put them to the intellectual test can be judged only incompletely from his writings. But, there is plenty of evidence, especially from his undergraduate days, that he deeply examined his faith. His knowledge of the Bible was remarkable; his confidence in the Scriptures was not based on ignorance.

In 1854, Maxwell graduated from Trinity with a degree in mathematics. He scored second highest in the final examination, coming behind Edward Routh and earning himself the title of Second Wrangler. He was later declared equal with Routh in the more exacting ordeal of the Smith's Prize examination. Immediately after earning his degree, Maxwell read his paper, *On the Transformation of Surfaces by Bending*, to the Cambridge Philosophical Society. This is one of the few purely mathematical papers he had written, demonstrating

Maxwell's growing stature as a mathematician.

Maxwell decided to remain at Trinity after graduating and applied for a fellowship. Buoyed by his success as a research student, he would be free, apart from some tutoring and examining duties, to pursue scientific interests at his own leisure.

The nature and perception of color was one such interest which he had begun at Edinburgh University. With colored spinning tops, Maxwell was able to demonstrate that white light would result from a mixture of red, green, and blue light. His paper, *Experiments on Colour*, laid out the principles of color combination and was presented to the Royal Society of Edinburgh in March 1855.

Maxwell was made a fellow of Trinity in October 1855, and was asked to prepare lectures on hydrostatics and optics and to set examination papers. The following February he was urged by Forbes to apply for the newly vacant Chair of Natural Philosophy at Marischal College, Aberdeen. He was selected, and Maxwell accepted the professorship at Aberdeen, leaving Cambridge in November 1856.

The 25-year-old Maxwell was 15 years younger than any other professor at Marischal. He engaged himself with his new responsibilities as head of a department, and lectured 15 hours a week, including a weekly pro bono lecture to the local working men's college. He lived in Aberdeen during the six months of the academic year and spent the summers at Glenlair, the home he had inherited from his father.

He focused his attention on the nature of Saturn's rings, a problem that had eluded scientists. It was unknown how the rings could remain stable without breaking up, drifting away, or crashing into Saturn. The problem took on particular interest because St John's College, Cambridge had chosen it as the topic for the 1857 Adams Prize. Maxwell devoted two years studying the problem, proving that a regular solid ring could not be stable, while a fluid ring would be forced by wave action to break up into blobs. Since neither was observed, Maxwell concluded that the rings must be composed of numerous small particles he called "brick-bats," each independently orbiting Saturn.

Maxwell was awarded the Adams Prize in 1859 for his essay, *On the Stability of the Motion of Saturn's Rings*; he was the only entrant to have made enough headway to submit an entry. His work was so detailed and convincing that when George Biddell Airy (an English mathematician and royal astronomer) read it he commented, "It is one of the most remarkable applications of mathematics to physics that I have ever seen." Direct observations by the Voyager flybys of the 1980s confirmed Maxwell's prediction.

In 1857 Maxwell befriended the Reverend Daniel Dewar, who was then the Principal of Marischal. Through him Maxwell met Dewar's daughter, Katherine Mary Dewar, who he married in June 1858. On the marriage record, Maxwell is listed as Professor of Natural Philosophy in Marischal College, Aberdeen. Comparatively little is known of Katherine, although she was seven years his senior, helped in his lab, and

worked on experiments in viscosity.

In 1860 Marischal College merged with the neighboring King's College to form the University of Aberdeen. With no room for two professors of Natural Philosophy, Maxwell, despite his scientific reputation, found himself laid off. He was unsuccessful in applying for Forbes's recently vacated chair at Edinburgh. Maxwell was granted the Chair of Natural Philosophy at King's College, London. After recovering from a near-fatal bout of smallpox in 1860, Maxwell moved with his wife to London.

Maxwell's time at King's was probably the most productive of his career. He was awarded the Royal Society's Rumford Medal in 1860 for his work on color and was elected to the Society in 1861. This period of his life would see him display the world's first light-fast color photograph, further develop his ideas on the viscosity of gases, and propose a system of defining physical quantities — now known as dimensional analysis.

Maxwell would often attend lectures at the Royal Institution, where he came into regular contact with Michael Faraday. Although Faraday was 40 years Maxwell's senior and showed signs of senility, they maintained a strong respect for each other's talents.

This time is especially noteworthy for the significant advances Maxwell made in the fields of electricity and magnetism. He examined the nature of both electric and magnetic fields in his two-part paper, *On Physical Lines of Force*, which was published in 1861. In it he provided a conceptual model for electromagnetic induction, consisting of tiny spinning cells of magnetic flux. Two more parts were later added to and published in that same paper in early 1862. In the first additional part he discussed the nature of electrostatics and displacement current. In the second additional part, he dealt with the rotation of the plane of the polarization of light in a magnetic field, a phenomenon that had been discovered by Faraday and is now known as the Faraday effect.

In 1865 Maxwell resigned the chair at King's College, London, and returned to Glenlair with Katherine. In his paper, *On Reciprocal Figures, Frames and Diagrams of Forces* (1870), he discussed the rigidity of various designs of lattices. He wrote the textbook *Theory of Heat* (1871) and the treatise *Matter and Motion* (1876). Maxwell made explicit use of dimensional analysis in 1871, and is given credit for it.

Maxwell published a paper, *On Governors*, in the *Proceedings of Royal Society* (1867–1868). This paper is considered a cornerstone paper of the early days of control theory. Here “governors” refers to the centrifugal governor used to regulate steam engines through negative feedback.

In 1871 he became the first Cavendish Professor of Physics at Cambridge. Maxwell was put in charge of the development of the Cavendish Laboratory, supervising every step in the progress of the building and the purchase of the equipment. One of Maxwell's last great contributions to science was editing the original research notes of Henry Cavendish. Cavendish studied, among many topics, questions such as the density of the Earth and the composition of water.

Maxwell died in Cambridge of abdominal cancer at the age of 48, at the same age his mother had died and of the same type of cancer.

Maxwell was buried at Parton Kirk, near Castle Douglas in Galloway close to where he grew up.

Noteworthy: Many physicists regard Maxwell as the 19th century scientist having the greatest influence on 20th century physics. His contributions to science are considered to be of the same significance as those of Isaac Newton and Albert Einstein. In the millennium poll (a survey of the 100 most prominent physicists), Maxwell was voted the third greatest physicist of all time, behind Newton and Einstein. On the centenary of Maxwell's birthday, Einstein described Maxwell's work as the “most profound and the most fruitful that physics has experienced since the time of Newton.”

The minister who regularly visited Maxwell in his last weeks of his life was astonished at his lucidity and the immense power and scope of his memory. More particularly, he commented: “... his illness drew out the whole heart and soul and spirit of the man: his firm and undoubting faith in the Incarnation and all its results; in the full sufficiency of the Atonement; in the work of the Holy Spirit. He had gauged and fathomed all the schemes and systems of philosophy, and had found them utterly empty and unsatisfying — ‘unworkable’ was his own word about them — and he turned with simple faith to the Gospel of the Saviour.”

As death approached, Maxwell told a Cambridge colleague: “I have been thinking how very gently I have always been dealt with. I have never had a violent shove all my life. The only desire which I can have is like David to serve my own generation by the will of God, and then fall asleep.”

The extended biography *The Life of James Clerk Maxwell*, by his former schoolfellow and lifelong friend Professor Lewis Campbell, was published in 1882. His collected works were issued in two volumes by the Cambridge University Press in 1890.

His name is honored in many ways. The maxwell (Mx) is a derived CGS unit for magnetic flux. The IEEE has a Maxwell Award. The Maxwell Montes is a mountain range on Venus. There is the Maxwell Gap in the Rings of Saturn. The James Clerk Maxwell Telescope is the largest submillimeter-wavelength telescope in the world, with a diameter of 15 m.

B.2.2 Voigt Model

The Voigt model in Figure B.2 consists of a spring and a dashpot connected in parallel. The relation between displacement and force is

$$\frac{du}{dt} + \frac{1}{c}ku = \frac{1}{c}F, \quad (\text{B.3})$$

or, alternatively,

$$F = ku + c\frac{du}{dt}. \quad (\text{B.4})$$

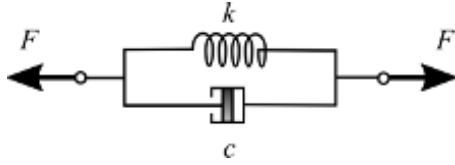


Figure B.2: The Voigt model.

We note that a harmonic force $F(t) = \exp(i\omega t)$ in Equation B.3 results in harmonic motion $u(t) = X \exp(i\omega t)$. Then, Equation B.4 becomes

$$\begin{aligned} F(t) &= kX e^{i\omega t} + i\omega cX e^{i\omega t} \\ &= (k + i\omega c)X e^{i\omega t} \\ &= (k + i\omega c)u(t), \end{aligned}$$

where all terms in this equation are complex quantities.

This last equation can be the motivation for the complex stiffness,

$$\begin{aligned} (k + i\omega c) &= k \left(1 + i \frac{c\omega}{k} \right) \\ &= k(1 + i\eta) \equiv k^*, \end{aligned}$$

where k^* is the complex stiffness and η is the loss factor, given by Equation 3.4. The real part of k^* represents the stiffness and the imaginary part represents the dissipation.

We emphasize that since the complex stiffness has been derived for harmonic forcing, it cannot be used as a viscoelastic model in free vibration. The equation of motion needs to be of the form

$$m\ddot{x} + k^*x = \exp(i\omega t) \quad (\text{B.5})$$

$$\text{or } \ddot{x} + \omega_n^2(1 + i\eta)x = \frac{1}{m} \exp(i\omega t), \quad (\text{B.6})$$

since the use of this model for cases of non-periodic motion results in non-causal behavior. To avoid this, k^* should be used only in the frequency-domain form of the equation of motion. (See the discussion in Section B.3.)

WOLDEMAR VOIGT

(September 2, 1850 – December 13, 1919)



Contribution: Voigt⁴ was a German physicist who worked on crystal physics, thermodynamics, and electro-optics. He discovered the Voigt effect, a magnetic birefringence of a material with an index of refraction parallel and perpendicular to an applied magnetic field. A viscoelastic material having the properties of both elasticity and viscosity is named after him, and is called a Voigt material, or Kelvin–Voigt material. A Voigt model, also called the Kelvin–Voigt model, is a mechanical model of a Voigt material represented by the parallel connection of a purely viscous damper and purely elastic spring.

Life Sketch: Voigt was born in Leipzig, Germany. He graduated from the Nikolaischule in Leipzig in 1868 and then entered the University of Leipzig. In 1870 his studies were interrupted by service in the Franco-Prussian War. He resumed his studies in 1871, this time at Königsberg.

Voigt was undecided between a career in physics and a career in music. He finally decided on a career in physics, on the grounds that, unlike music, in physics there is a reasonable mean, not simply highs and lows.

While at Königsberg, Voigt was a student of Franz Ernst Neumann, the famous German mineralogist, physicist, and mathematician. His deep respect and love for Neumann largely determined his career, in terms of subject matter, the style of his research, and the manner in which he presented his work to the physics community.

His dissertation on the elastic constants of rock salt was completed in 1874. He then returned to Leipzig, where he taught at the Nikolaischule, and in 1875 was called back to Königsberg as professor of physics. In 1883 Voigt was appointed professor of theoretical physics at Göttingen, with the promise that he (and Eduard Reike) would have a new physical institute (which was not ready until 1905). His chief

⁴Image source: A brief history of theoretical physics in Göttingen, University of Göttingen Institute for Theoretical Physics.

research interests centered on the understanding of crystals, but near the turn of the century he became more and more concerned with the Zeeman effect and electron theory.

Voigt's interest in crystals was closely related to Neumann's work. At Königsberg, Neumann was in both the physics department and the department of mineralogy, and pursued extensive work on the optical properties of crystals. Neumann had developed a mechanical theory of light propagation that assumed that light oscillations had a mechanical-elastic nature. The oscillations were transmitted through an ether conceived of as an elastic solid. His students, including Voigt, spent many hours in his laboratories studying the properties of crystals.

Voigt brought this tradition of theoretical and experimental work to Göttingen. He pursued theoretical studies of the properties of crystals and, although he was hampered by lack of adequate facilities for many years, conducted delicate experiments measuring the physical properties of many crystalline substances.

According to the theories of Poisson and Cauchy, certain relationships must exist between the constants of a crystal regardless of its classification. Voigt determined the elastic constants for a wide variety of crystals and showed that the predicted relationships were not satisfied. While some felt that his work vindicated those who objected to forming special hypotheses about the nature of crystals, Voigt did not accept this point of view and in many of his publications indicated the direction that must be taken in amending the molecular hypothesis.

In 1887, in a paper on the Doppler effect in which he analyzed the differential equations for oscillations in an incompressible elastic medium, Voigt established a set of transformation equations that later became known as the Lorentz transformations.

In 1898 he discovered the Voigt effect, a magneto-optical phenomenon in which birefringence is produced when a constant magnetic field is applied to a transparent medium perpendicular to the direction of propagation. Unlike many other magneto-optical effects, such as the Kerr or Faraday effect which are linearly proportional to the magnetization (or to the applied magnetic field for a non-magnetized material), the Voigt effect is proportional to the square of the magnetization (or square of the magnetic field) and can be seen experimentally at normal incidence.

Voigt's extensive theoretical and physical research on the nature of crystals were summarized in *Magneto-und Elektro-Optik* (1908) and *Kristallphysik* (1910). These books reveal the elegance of his mathematical treatments and the great orderliness that has research had brought to the understanding of crystals. The elastic, thermal, electric, and magnetic properties of crystals were ordered in magnitudes of three types: scalar, vector, and tensor. Voigt introduced the term "tensor" into the vocabulary of mathematical physics.

Even though Voigt devoted considerable time to his research and his students, and had administrative responsibilities

at Göttingen, he never gave up an active interest in music and musicology. He was recognized as an expert on Bach's vocal works and in 1911 published a book on Bach's church cantatas. Voigt often referred to the study of physics in musical terms. To him the region of science that represented the highest degree of orchestration and that possessed the utmost in rhythm and melody was crystal physics.

Voigt died in Göttingen, Germany. A Bach chorale accompanied his funeral bier as it was carried from his house to its final resting place.

Noteworthy: Voigt wrote more than 200 publications, between his papers and books.

Voigt headed the Mathematical Physics Department at Göttingen and was succeeded in 1914 by Peter Debye, who in 1921 was succeeded by Max Born.

The word tensor in its current meaning was introduced by Voigt in 1898. The Voigt profile and Voigt notation are named after him.

In spectroscopy, the Voigt profile is a line profile resulting from the convolution of a Lorentz profile and a Gaussian profile, considered two broadening mechanisms. Voigt profiles are common in many branches of spectroscopy and diffraction.

The pseudo-Voigt profile (or pseudo-Voigt function) is an approximation of the Voigt profile using a linear combination of a Gaussian curve and a Lorentzian curve instead of their convolution. The pseudo-Voigt function is often used for calculations of experimental spectral line shapes.

In mathematics, the Voigt notation or Voigt form in multilinear algebra is a way to represent a symmetric tensor by reducing its order. There are a few variants and associated names for this idea: Mandel notation, Mandel-Voigt notation, Nye notation, and others.

Music played a large role in Voigt's life. He was exposed to music at an early age: Felix Mendelssohn and Robert Schumann had been frequent visitors to his parents' house. His musical ear was highly trained. While in the army he would often pass the time while marching by reciting, note for note, the complete orchestration of entire symphonic pieces. Voigt gained fame as an amateur musician and as a Bach expert.

B.2.3 Maxwell Standard Linear Model

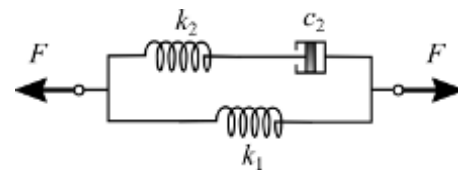


Figure B.3: The Maxwell Standard Linear model.

The standard linear model in Figure B.3 consists of a Maxwell model (k_2, c_2) in parallel with a linear spring

(k_1). The differential equation that relates the force to displacement is

$$\frac{dF}{dt} + \frac{k_2}{c_2}F = (k_1 + k_2) \frac{du}{dt} + \frac{k_1 k_2}{c_2}u. \quad (\text{B.7})$$

This is a more complex first-order differential equation relating force and displacement than Equation B.2.

B.2.4 Stress-Strain Equivalent Model

Analogous models in terms of stress and strain are created by replacing the Hookean spring equation $F = kx$ by $\sigma = E_0\epsilon$. With this change, the Maxwell model of Equation B.1 becomes

$$\begin{aligned} \frac{d\epsilon}{dt} &= \frac{1}{E_0} \frac{d\sigma}{dt} + \frac{\sigma}{c} \\ \text{or } E_0\dot{\epsilon} &= \dot{\sigma} + \frac{E_0\sigma}{c}, \end{aligned} \quad (\text{B.8})$$

where possible units are N/m² for σ and E_0 , and ϵ is dimensionless. This constitutive equation relates the stress to the strain but there is no “modulus,” that is, there is no direct ratio of stress to strain as in the Hookean elastic material. Equation B.8 is solvable once either σ or ϵ is given.

The ratio,

$$\tau = c/E_0,$$

is a useful parameter that is a measure of the viscoelastic response time of the material. Following the discussion in Roylance, for a stress relaxation test a constant strain ϵ_0 is the known input and the output is $\sigma(t)$. Then, $\dot{\epsilon} = 0$, and Equation B.8 becomes

$$\frac{d\sigma}{dt} = -\frac{1}{\tau}\sigma.$$

Separating variables and integrating as shown,

$$\begin{aligned} \int_{\sigma_0}^{\sigma} \frac{d\sigma}{\sigma} &= -\frac{1}{\tau} \int_0^t dt \\ \ln \sigma - \ln \sigma_0 &= -\frac{t}{\tau}, \end{aligned}$$

we find

$$\sigma(t) = \sigma_0 \exp(-t/\tau),$$

where the relaxation time τ is the time needed for the stress to fall to $1/e$ of its initial value. At $t = 0$, the initial modulus is $E_0 = \sigma_0/\epsilon_0$, and then a time-dependent relaxation modulus $E_{\text{rel}}(t)$ can be defined,

$$\begin{aligned} E_{\text{rel}}(t) &= \frac{\sigma(t)}{\epsilon_0} = \frac{\sigma_0}{\epsilon_0} \exp(-t/\tau) \\ &= E_0 \exp(-t/\tau). \end{aligned}$$

$E_{\text{rel}}(t)$ decays with time at a rate that is a function of $1/\tau$. This model has two parameters, E_0 and τ , that can

be fit from or determined by experimental data. As per Roylance, “the relaxation time τ is strongly dependent on temperature and other factors that affect the mobility of the material, and is roughly inverse to the rate of molecular motion.”

The Maxwell model allows unrestricted “flow” that is generally unrealistic for most materials of engineering interest. The standard linear model removes this possibility by placing a spring in parallel with the Maxwell unit. This spring has a constant E_{eq} , signifying that it provides an equilibrium stress that remains after the Maxwell stresses dissipate. Following the same procedure as above, Roylance derives the respective relaxation modulus to be

$$\frac{\sigma(t)}{\epsilon_0} = E_{\text{rel}}(t) = E_{\text{eq}} + E_0 \exp(-t/\tau).$$

The above formulation can be generalized, yielding the stress-strain equation by using differential operators,

$$\sum_{m=0}^M a_m \frac{d^m \sigma}{dt^m} = \sum_{n=0}^N b_n \frac{d^n \epsilon}{dt^n}. \quad (\text{B.9})$$

Other models are available, and the paper by Park⁵ is an excellent reference for this summary.

B.2.5 Boltzmann Superposition Model

In addition to the above differential models, there are integral viscoelastic models. The standard integral mechanical model is the Boltzmann model,⁶ that applies to uniaxial, isothermal stress-strain for a non-aging, linear, viscoelastic material. It models creep as a function of the entire loading history - this is the material memory. Also, the total deformation is the sum of independent deformations that are a result of loading increments - this is the linearity. Conversely, if loading is removed, an identical deformation will occur in the opposite direction and the recovery will take as much time as the original deformation.

Consider a series of loadings resulting in respective changes in stress $\Delta\sigma_i$. The total strain $\epsilon(t)$ equals the increments of individual strains $\epsilon_i(t) = \Delta\sigma_i D(t - \tau_i)$, where $D(t)$ is the creep compliance and the load occurs at τ_i ,

$$\epsilon(t) = \sum_i \Delta\sigma_i D(t - \tau_i).$$

For a continuous load, this expression can be generalized,

$$\epsilon(t) = \int_{-\infty}^t D(t - \tau_n) d\sigma_n.$$

⁵S.W. Park, “Analytical modeling of viscoelastic dampers for structural and vibration control,” *International Journal of Solids and Structures*, Vol.38, 2001, pp.8065-8092.

⁶**Structural Biomaterials**, J. Vincent, 3rd ed., Princeton University Press, 2012.

This equation is sometimes written in the form

$$\epsilon(t) = \frac{\sigma}{E} + \int_{-\infty}^t D(t - \tau_n) \frac{d\sigma_n}{d\tau_n} d\tau_n,$$

where E is the immediate or unrelaxed stiffness. Thus, the strain is comprised of a time-independent and a time-dependent component. Similarly, the stress can be expressed as

$$\sigma(t) = E\epsilon + \int_{-\infty}^t E(t - \tau_n) \frac{d\epsilon_n}{d\tau_n} d\tau_n,$$

where $E(t)$ is the relaxation modulus. If the loads are applied at $t = 0$, then the lower limit on the integrals can be set to zero, and then $E(t) = \epsilon(t) = 0$ for $-\infty < t \leq 0$.

B.2.6 General Nonviscous Damping

Woodhouse⁷ and Adhikari⁸ discussed an integral formulation for nonviscous damping. For a single degree-of-freedom system, the damping force is given by the convolution equation

$$\int_{-\infty}^t g(t - \tau) \dot{x}(\tau) d\tau, \quad (\text{B.10})$$

where, for a viscously damped system, $g(t) = c\delta(t)$ and the damping force becomes $c\dot{x}(t)$. This formulation can be generalized for a system of N degrees of freedom and then the convolution integral is a matrix-vector product. This damping force formulation results in the integro-differential governing equation,

$$m\ddot{x} + \int_{-\infty}^t g(t - \tau) \dot{x}(\tau) d\tau + kx = F(t).$$

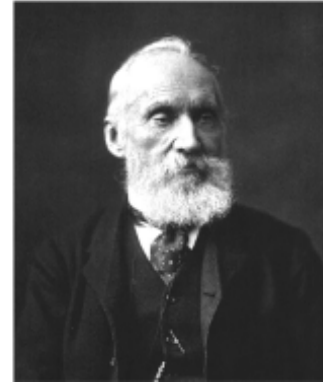
Note that if this equation is differentiated with respect to time, it becomes a third-order differential equation, also known as a “jerk” equation. There is a large body of literature on such equations, but these are outside our scope.

⁷Woodhouse, op. cit.

⁸A. Srikantha Phani and S. Adhikari, “Rayleigh Quotient and Dissipative Systems,” *Journal of Applied Mechanics*, Vol.75, 2008.

WILLIAM THOMSON, 1st Baron KELVIN

(June 26, 1824 – December 17, 1907)



Contribution: William Thomson, 1st Baron Kelvin was a Scotch-Irish mathematical physicist, and engineer who made significant contributions in the mathematical analysis of electricity, formulated the first and second laws of thermodynamics, and did much to unify the emerging discipline of physics in its modern form. He also had a career as an electric telegraph engineer and inventor, which propelled him into the public eye and earned him wealth, fame, and honor. For his work on the transatlantic telegraph project he was knighted by Queen Victoria, becoming Sir William Thomson. He had extensive maritime interests and was noted for his work on the mariner’s compass, which had been limited in reliability.

Life Sketch: Kelvin was born in Belfast in 1824. William Thomson’s father, James Thomson, had originally intended to become a minister of the Presbyterian Church, but had opted for an academic career as a mathematician. William’s mother died when he was six years old and from that time he was brought up by his father. James Thomson was a professor of engineering in Belfast at the time of William’s birth and, when William was eight years old, his father James was appointed to the chair of mathematics at the University of Glasgow. James Thomson was a dominant father who brought his family up in a strict Presbyterian fashion.

Despite his father’s strictness he had a very close relationship with William and taught him mathematics. At a very young age William became an accomplished mathematician with knowledge of the latest developments in the subject.

William attended Glasgow University from the age of 10. This early age was not unusual, for at that time the universities in Scotland to some extent competed for the most able junior pupils. Thomson began what we would consider university level work in 1838 when he was 14 years old. In the academic year 1838-39 he studied astronomy and chemistry. The following year he took natural philosophy courses (today called physics) which included the study of heat, electricity, and magnetism. His *Essay on the Figure of the Earth* won him a gold medal from the University of Glasgow when he

was 15 years old and it was a truly remarkable work containing important ideas which Thomson returned to throughout his life.

At the end of the 1839-40 academic year, Thomson read Fourier's *The Analytical Theory of Heat*, a work on the application of abstract mathematics to heat flow.

There was strong interest among the lecturers in Glasgow at that time in the French mathematical approach to physical science. In particular the works of Lagrange, Laplace, Legendre, Fresnel, and Fourier were treated with "reverence" to use a word which Thomson himself would later use to describe the attitude that his lecturers had towards these French mathematicians. Thomson also read Laplace's *Mécanique céleste* in that academic year and visited Paris during that time.

In 1841 Thomson entered Cambridge and in the same year his first paper was published. This paper, *Fourier's expansions of functions in trigonometrical series*, was written to defend Fourier's mathematics against criticism from a professor of mathematics at the University of Edinburgh. A more important paper, *On the uniform motion of heat and its connection with the mathematical theory of electricity*, was published in 1842 while Thomson was studying for the mathematical tripos examinations at Cambridge.

At Cambridge Thomson was coached by William Hopkins, a famous Cambridge coach who played a more important role than the lecturers. Despite the efforts of Babbage, Peacock, and Herschel to introduce the new French mathematics into Cambridge, the style of the Mathematical Tripos taken by Thomson still left much to be desired. Herschel and Babbage had conducted some experiments on magnetism in 1825, developing methods introduced by Arago, but nothing on heat, electricity, or magnetism had entered the syllabus.

Thomson took the final part of the Mathematical Tripos examinations in 1845. He graduated with a BA and he was Second Wrangler (ranked second among those obtaining a First Class degree). Further examinations saw him become first Smith's prizeman and he was elected a fellow of Peterhouse.

Also in 1845 Thomson read George Green's work which was to have a major influence on the direction of his research. His interest in the French approach, and advice from his father, meant that after taking his degree Thomson went to Paris. There he worked in the physical laboratory of Henri-Victor Regnault and he was soon taking part in deep discussions with Biot, Cauchy, Liouville, Dumas, and Sturm.

Perhaps the most profitable discussions that Thomson had in Paris were with Liouville. It was at Liouville's request that Thomson began to try to bring together the ideas of Faraday, Coulomb, and Poisson on electrical theory. Ideas of "action at a distance" or properties of the "ether," and ideas of an "electrical fluid" were difficult to unify. There were problems of whether or not an "electrical fluid" was an actual physical entity with the properties of a fluid. Thomson was led to study the whole methodology of a physical science, distinguishing "physical" parts of a theory from "mathematical"

parts.

In 1846 the chair of natural philosophy at Glasgow became vacant. Thomson's father used his influence in the University to help his son become the leading candidate for the post. Thomson returned from Paris to Glasgow and, in 1846, was unanimously elected professor of natural philosophy at the University. In 1847-49 he collaborated with Stokes on hydrodynamic studies, which Thomson applied to electrical and atomic theory. His collaboration lasted over fifty years with frequent letters on scientific matters being exchanged.

Many of these letters have survived, for example, copies of 407 letters from Thomson to Stokes and 249 letters from Stokes to Thomson have been published. Many of these letters discuss the mathematical similarities in the theory of heat and the theory of fluids.

Thomson's thermodynamic studies led him to propose an absolute scale of temperature in 1848. The absolute scale was based on his studies of the theory of heat, in particular, the theory proposed by Sadi Carnot. The Kelvin absolute temperature scale, as it is now known, was defined much later after conservation of energy had become better understood. It derives its name from the title, Baron Kelvin of Largs, that Thomson received from the British government in 1892, and named after Thomson based on his proposal in 1848.

Between 1849 and 1852 Thomson published three influential papers on the theory of heat. Historians of science have called attention to Thomson's difficulties in reconciling a principle formulated by James Prescott Joule with another principle formulated by Nicolas Leonard Sadi Carnot, and to errors Thomson made in his calculations.

In 1852 Thomson observed what is now called the Joule-Thomson effect, namely the decrease in temperature of a gas when it expands in a vacuum. Joule's ideas on heat were to change Thomson's views over the years. Thomson came to believe in a dynamic theory of heat.

The dynamic theory of heat led Thomson to also think of a dynamic theory for electricity and magnetism. In 1856 he sent a paper on this subject to the Royal Society of London entitled *Dynamical illustrations of the magnetic and helicoidal rotary effects of transparent bodies on polarised light*. He explained his notion of electricity in these words:

"... we can conceive that electricity itself is to be understood as not an accident, but an essence of matter. Whatever electricity is, it seems quite certain that electricity in motion is heat; and that a certain alignment of axes of revolution in this motion is magnetism."

This work by Thomson in 1856 on electricity and magnetism is important for it was these ideas which led Maxwell to develop his remarkable new theory of electromagnetism. Thomson did not eagerly support Maxwell's theory, which his own work had helped to create. Thomson hoped his ideas would lead him to a unifying theory, and that took him further and further from accepting those of Maxwell.

Thomson's initial hope that his theory could explain electromagnetism, light, gravity, and chemical processes slowly

faded. However, Thomson contributed directly and indirectly to the theory of the electromagnetic field. Thomson was the first who tried to treat mathematically Faraday's conception of lines of force, and he introduced Maxwell to the problems of the electromagnetic field by his work and his personal initiative.

Like many scientists, Thomson made some mistakes in accepting new theories and predicting the future of technology. He refused to accept the concept of atoms, he opposed Darwin's theories, he incorrectly estimated the age of the Earth and the Sun, and he opposed Rutherford's ideas of radioactivity, among many other missteps. These views put him on the losing side of many arguments later in his career.

In addition to Thomson's research contributions, he is credited with the innovations in teaching at the University of Glasgow. He introduced laboratory work into the degree courses, keeping this part of the work distinct from the mathematical side. He encouraged the best students by offering prizes. Some prizes were awarded to the best student based on a vote of the students. Other prizes were given by Thomson to the student that he considered most deserving. Not only did Thomson take a unified view of the physical world in his research, but he carried this into his teaching.

Another of Thomson's famous pieces of work was his joint project with Tait to produce their famous text *Treatise on Natural Philosophy*, which they began working on in the early 1860s. They worked by posting a notebook back and forward to each other on this huge project which Thomson envisaged covering all physical theories. Many volumes were intended, but only the first two covering kinematics and dynamics were ever written. These were remarkable volumes which became the standard texts for many generations of scientists.

Thomson was always greatly interested in the improvement of physical instrumentation, and Thomson designed and implemented many new devices, including the mirror-galvanometer that was used in the first successful sustained telegraph transmissions via a transatlantic submarine cable. Thomson had joined a group of industrialists in the mid-1850s on a project to lay a submarine cable between Ireland and Newfoundland. He played several roles in this project, being on the board of directors and also being an advisor on theoretical electrical matters.

The electrician who was in charge of the practical side of the operation was E.O.W. Whitehouse, who insisted on using his own system against Thomson's advice. The cable was successfully laid in 1858, an attempt having failed the previous year when the cable broke. After initial difficulties with transmitting a signal, there was a sudden marked improvement and Whitehouse claimed success for his system. However, it was soon discovered that he had substituted Thomson's mirror-galvanometer for his own instruments and there was a furious row between Whitehouse, Thomson, and the other directors. Thomson's instruments were fully used for the third attempt at laying a cable in 1865 and this proved highly successful with rapid transmission of signals possible.

For his work on the transatlantic cable Thomson was knighted in 1866 and made Baron Kelvin of Largs in 1892. (The Kelvin river runs through the grounds of Glasgow University and Largs is the town on the Scottish coast where Thomson built his house.) As well as fame, his participation in the telegraph cable project led to a large personal fortune brought about by his cable patents and consulting. He was able to buy a 126-ton yacht as well as a fine house with surrounding estate.

In the winter of 1860–1861 Kelvin slipped on some ice and fractured his leg, causing him to limp thereafter. He remained a celebrity on both sides of the Atlantic until his death.

In November 1907 he caught a chill and his condition deteriorated until he died at his Scottish residence, Netherhall, in Largs on December 17.

Lord Kelvin was an elder of St Columba's Parish Church (Church of Scotland) in Largs for many years. It was to that church that his remains were taken after his death. Following the funeral service, the body was taken to his beloved University of Glasgow for a service of remembrance before the body was taken to London for interment at Westminster Abbey, near the final resting place of Sir Isaac Newton.

Noteworthy: Thomson published more than 600 papers. He was elected to the Royal Society in 1851, received its Royal Medal in 1856, received its Copley Medal in 1883, and served as its president from 1890 to 1895. In addition to his activities with the Royal Society, he was active with the Royal Society of Edinburgh for many years. He served three terms as president of this Society, first from 1873 to 1878, for the second time from 1886 to 1890, and for the third time from 1895 until his death in 1907. Thomson served as president of yet a third society when he was elected as president of the British Association for the Advancement of Science in 1871.

He was ennobled in 1892 in recognition of his achievements in thermodynamics, and of his opposition to Irish Home Rule, becoming Baron Kelvin, of Largs in the County of Ayr. He was the first British scientist to be elevated to the House of Lords.

Despite offers of elevated posts from several world-renowned universities Lord Kelvin refused to leave Glasgow, remaining Professor of Natural Philosophy for over 50 years, until his eventual retirement from that post.

Always active in industrial research and development, around 1899 he was recruited by George Eastman to serve as vice-chairman of the board of the British company Kodak Limited, affiliated with Eastman Kodak.

The Hunterian Museum at the University of Glasgow has a permanent exhibit of the work of Lord Kelvin that includes many of his original papers, instruments, and other artifacts such as his smoking pipe.

Absolute temperatures are stated in units of kelvin in his honor. While the existence of a lower limit to temperature (absolute zero) was known prior to his work, Kelvin is widely known for determining its correct value as approximately -273.15 degree Celsius or -459.67 degree Fahrenheit.

Lord Kelvin was commemorated on the £20 note issued by the Clydesdale Bank in 1971; his image appears on the current issue of the £100 banknotes. He is shown holding his adjustable compass and in the background is a map of the transatlantic cable.

Thomson was articulate although not known as an impressive lecturer. Unable to meet his class one day, he posted a note on the door of his lecture room: "Professor Thomson," it said, "will not meet his classes today." As a joke, some of his mischievous students erased the "c," leaving a message reading: "Professor Thomson will not meet his lasses today." The following day when the pranksters assembled in anticipation of the effect of their joke, they were chagrined to find that the professor had outwitted them. The note was now found to read: "Professor Thomson will not meet his asses today." [This story was told by Cyrus Northrup, University of Washington Address, November 2, 1908.]

Here are a few of his quotes, some of which were disproven:

"When you measure what you are speaking about and express it in numbers, you know something about it, but when you cannot express it in numbers your knowledge about it is of a meagre and unsatisfactory kind."

"I am never content until I have constructed a mechanical model of what I am studying. If I succeed in making one, I understand; otherwise I do not."

"Scientific wealth tends to accumulate according to the law of compound interest. Every addition to knowledge of the properties of matter supplies [the physical scientist] with new instrumental means for discovering and interpreting phenomena of nature, which in their turn afford foundations of fresh generalizations, bringing gains of permanent value into the great storehouse of [natural] philosophy." (Presidential address to British Association 1871)

"Heavier-than-air flying machines are impossible." (Speaking in 1895)

"I have not the smallest molecule of faith in aerial navigation other than ballooning, or of the expectation of good results from any of the trials we hear of." (Letter, 1896)

"Radio has no future." (Speaking in 1897)

"I have not had a moment's peace or happiness in respect to electromagnetic theory since November 28, 1846. All this time I have been liable to fits of ether dipsomania, kept away at intervals only by rigorous abstention from thought on the subject." (April 9, 1896)

"There is at present in the material world a universal tendency to the dissipation of mechanical energy."

"Although mechanical energy is indestructible, there is a universal tendency to its dissipation, which produces throughout the system a gradual augmentation and diffusion of heat, cessation of motion and exhaustion of the potential energy of the material Universe."

"Any restoration of mechanical energy, without more than an equivalent of dissipation, is impossible in inanimate material processes, and is probably never effected by means of

organized matter, either endowed with vegetable life, or subjected to the will of an animated creature."

"There cannot be a greater mistake than that of looking superciliously upon practical applications of science. The life and soul of science is its practical application." (May 3, 1883)

"When you are face to face with a difficulty, you are up against a discovery."

"The more you understand what is wrong with a figure, the more valuable that figure becomes."

"There is one thing I feel strongly in respect to investigation in physical or chemical laboratories—it leaves no room for shady, doubtful distinctions between truth, half-truth, whole falsehood. In the laboratory everything tested or tried is found true or not. Every result is true. Nothing not proved true is a result; there is no such thing as doubtfulness. The search for absolute and unmistakable truth is promoted by laboratory work in a manner beyond all conception."

"There is nothing new to be discovered in physics now. All that remains is more and more precise measurement."

"X-rays will prove to be a hoax."

"I accept no theory of gravitation. Present science has no right to attempt to explain gravitation. We know nothing about it. We simply know nothing about it." (1900)

"Blow a soap bubble and observe it. You may study it all your life and draw one lesson after another in physics from it."

"Vortices of pure energy can exist and, if my theories are right, can compose the bodily form of an intelligent species."

"I firmly believe in an electromagnetic theory of light, and that when we understand electricity and magnetism and light we shall see them all together as parts of a whole. But I want to understand light as well as I can, without introducing things that we understand even less of. That is why I take plain dynamics. I can get a model in plain dynamics; I cannot in electromagnetics."

"We know that light is propagated like sound through pressure and motion."

"We can conceive that electricity itself is to be understood as not an accident, but an essence of matter. Whatever electricity is, it seems quite certain that electricity in motion is heat; and that a certain alignment of axes of revolution in this motion is magnetism."

"If you can not measure it, you can not improve it."

"You, in this country, are subjected to the British insularity in weights and measures; you use the foot, inch and yard. I am obliged to use that system, but must apologize to you for doing so, because it is so inconvenient, and I hope Americans will do everything in their power to introduce the French metrical system. ... I look upon our English system as a wickedly, brain-destroying system of bondage under which we suffer. The reason why we continue to use it, is the imaginary difficulty of making a change, and nothing else; but I do not think in America that any such difficulty should stand in the way of adopting so splendidly useful a reform."

“The true measure of a man is what he would do if he knew he would never be caught.”

“Nothing can be more fatal to progress than a too confident reliance on mathematical symbols; for the student is only too apt to take the easier course, and consider the formula not the fact as the physical reality.”

“The fact that mathematics does such a good job of describing the Universe is a mystery that we don’t understand. And a debt that we will probably never be able to repay.”

“[The vector] has never been of the slightest use to any creature.”

“With three parameters, I can fit an elephant.”

“I have no satisfaction in formulas unless I feel their numerical magnitude.”

“The telephone is one of the most interesting inventions that has ever been made in the history of science.”

“The steam engine has given more to science than science has given to the steam engine.”

“The steam engine is passing away.”

“Trust you will avoid the gigantic mistake of alternating current.” (Writing to Niagara Falls Power Company)

“Photography in natural colors will soon be an established fact, although it will necessitate a lot of study to get it perfected.” (May 11, 1902, *The New York Times*)

“It seems, therefore, on the whole most probable that the sun has not illuminated the earth for 100,000,000 years, and almost certain that he has not done so for 500,000,000 years. As for the future, we may say, with equal certainty that inhabitants of the earth cannot continue to enjoy the light and heat essential to their life, for many million years longer, unless sources now unknown to us are prepared in the great storehouse of creation.”

“Creative Power is the only feasible answer to the origin of life from a scientific perspective.”

“Overwhelming strong proofs of intelligent and benevolent design lie around us.”

“I believe that the more thoroughly science is studied, the further does it take us from anything comparable to atheism.”

“The more thoroughly I conduct scientific research, the more I believe that science excludes atheism.”

“Do not be afraid of being free thinkers. If you think strongly enough you will be forced by science to the belief in God, which is the foundation of all religion. You will find science not antagonistic but helpful to religion.”

“To live among friends is the primary essential of happiness.”

“War is a relic of barbarism probably destined to become as obsolete as duelling.”

“Large increases in cost with questionable increases in performance can be tolerated only in race horses and fancy women.”

“If only we had thirty hours in a day instead of twenty-four, we might get some of our work done.” [Quoted in his obituary in the *New York Times*.]

B.3 Causality Issues in Damping Models

Crandall⁹ and others discussed the concept of the frequency dependent dashpot, which is often used as a replacement for the viscous damper in equations of motion for oscillating systems. A key point is that since loss factors are frequency dependent in a complicated way, a challenge is how to mathematically model the damping force. Such functions must satisfy certain criteria in order to be physically viable.

When damping is light, then an equivalent dashpot can be defined with a loss value that equals the actual loss value at the resonant frequency. With light damping, any errors introduced at frequencies away from resonant are filtered by the structure and the resulting response is still reasonably accurate.

Where this approach is not adequate, some frequency dependence must be retained, and the question arises as to which functions for the damping force result in causal responses.

Discussions of frequency dependent functions naturally gravitate to operations in the frequency domain. In particular, we assume that the response $x(t)$ and its Fourier transform $X(\omega)$ exist. Then, the ideal dashpot with constant parameter c defines the time-domain relation between velocity and damping force $f_d = c\dot{x}(t)$. In the frequency domain, we have

$$F_d(\omega) = i\omega c X(\omega).$$

For the case where c needs to retain its frequency dependence, the loss factor, Equation 3.4, solved for c with a frequency dependence, provides an equation that can be used to define $c(\omega)$ for any loss mechanism,

$$c(\omega) = \frac{k\eta(\omega)}{|\omega|}.$$

Then, for cyclic or near-cyclic oscillations, we can define the damping force in the frequency domain as

$$\begin{aligned} F_d(\omega) &= i\omega \frac{k\eta(\omega)}{|\omega|} X(\omega) \\ &= ik\eta(\omega) \operatorname{sgn}(\omega) X(\omega), \end{aligned}$$

where $\operatorname{sgn}(\omega)$ takes on the value ± 1 for $\pm\omega$, and 0 for $\omega = 0$. In the time domain, we have

$$f_d(t) = \int_{-\infty}^{\infty} i\omega c(\omega) e^{i\omega t} d\omega \int_{-\infty}^{\infty} x(\tau) e^{-i\omega\tau} d\tau.$$

⁹S.H. Crandall, “The Role of Damping in Vibration Theory,” *Journal of Sound and Vibration*, Vol.11, No.1, 1970, pp.3-18.

When considering a governing equation of motion for a damped oscillator,

$$m\ddot{x} + c\dot{x} + kx = f(t),$$

we can replace c by the frequency-dependent dashpot in the transformed equation,

$$(-m\omega^2 + i\omega c(\omega) + k) X(\omega) = F(\omega).$$

These equations are appropriate and useful in frequency domain calculations. But their commonly used “time domain” equivalents,

$$\begin{aligned} m\ddot{x} + c(\omega)\dot{x} + kx &= f(t) \\ \text{or } m\ddot{x} + kx[1 + i\eta(\omega)\operatorname{sgn}(\omega)] &= f(t), \end{aligned} \quad (\text{B.11})$$

are, to use Crandall’s term, “non-equations.” Such models have led to non-causal behavior, where the response depends on events from the future.

Differential Equation B.11 is the same as Equation B.6, used to model a viscoelastic material with a Voigt representation. This can be expected by comparing the system in Figure 3.6(a) with Figure B.2. These are similar in the sense that both have a spring in parallel to a dashpot. In the first case the spring-dashpot represent a boundary condition, and in the second case they represent material behavior. Therefore, we can expect that the conclusions drawn here on the causal nature of the solution of Equation B.11 also apply to the use of the Voigt model.

More recent work on the issue of frequency dependence has also been concerned about causality of damping models. For linear viscoelastic models, force-displacement $F(t)$ - $x(t)$ relations in the time domain have used the complex stiffness representation,¹⁰

$$F(t) = (k_1 + ik_2)x(t), \quad (\text{B.12})$$

where k_1 and k_2 are constants independent of frequency. The only way this equation makes physical sense is if $x(t)$ is a complex quantity so that $F(t)$ is also complex. Otherwise, a real displacement results in a complex valued force.

In the frequency domain, the force displacement relation can be written as^{11,12}

$$F(\omega) = [K_1(\omega) + iK_2(\omega)] X(\omega),$$

¹⁰N. Makris, J.A. Inaudi, and J.M. Kelly, “Macroscopic Models with Complex Coefficients and Causality,” *Journal of Engineering Mechanics*, 1996, Vol.122, No.6, 566-573.

¹¹N. Makris, “Stiffness, Flexibility, Impedance, Mobility, and Hidden Delta Function,” *Journal of Engineering Mechanics*, Vol.123, No.11, 1997, pp.1202-1208.

¹²N. Makris, “Causal Hysteretic Element,” *Journal of Engineering Mechanics*, Vol.123, No.11, 1997, pp.1209-1214.

where the sum in square brackets is the dynamic stiffness $K(\omega)$. Using Hilbert transforms, Makris has derived a causal-appropriate dynamic stiffness function,

$$\mathbf{K}(\omega) = k \left[1 + \frac{2}{\pi} \delta \ln \left| \frac{\omega}{\gamma} \right| + i \delta \operatorname{sgn} \left(\frac{\omega}{\gamma} \right) \right], \quad \omega \neq 0,$$

where δ is a positive real number called the hysteretic damping coefficient, and γ is an arbitrary constant that can be set as small as desired so that the real part of $K(\omega)$ matches a realistic stiffness value measured at any finite frequency.

These discussions on causality are beyond the scope of this book, as are many deeper discussions of damping. Hopefully, the reader appreciates that there are many complexities in the modeling of damping.

B.4 Concepts Summary

This appendix has introduced the physical concept of damping, its sources, and the most commonly used mathematical models of viscoelastic damping. Damping can be generated within a material, or due to relative motion between components of a larger structure, or due to the effects of the environment on a structure. We use stiffness and dashpot elements to model material damping as well as to model how a structure is “connected” to its environment. An important phenomenon introduced by the inclusion of damping is that a phase lag is introduced between the loading and the response.

B.5 Quotes

- “A gem cannot be polished without friction, nor a man perfected without trials.” (Lucius Annaeus Seneca)
- “The world is wide, and I will not waste my life in friction when it could be turned into momentum.” (Frances E. Willard)
- “Heat can also be produced by the impact of imperfectly elastic bodies as well as by friction. This is the case, for instance, when we produce fire by striking flint against steel, or when an iron bar is worked for some time by powerful blows of the hammer.” (Hermann von Helmholtz)
- “It’s not the work which kills people, it’s the worry. It’s not the revolution that destroys machinery it’s the friction.” (Henry Ward Beecher)
- “We were then satisfied that, with proper lubrication and better adjustments, a little more power could be expected. The completion of the motor according to drawing was, therefore, proceeded with at once.” (Orville Wright)

- “It is not our sexual preferences, the color of our skin, the language we speak, nor the religion we practice that creates friction, hatred and wars amongst in society. It is our words and the words of our leaders that can create that disparity.” (Yehuda Berg)
- “A raised weight can produce work, but in doing so it must necessarily sink from its height, and, when it has fallen as deep as it can fall, its gravity remains as before, but it can no longer do work.” (Hermann von Helmholtz)
- “The first principle is that you must not fool yourself and you are the easiest person to fool.” (Richard P. Feynman)
- “It is better to solve one problem five different ways, than to solve five problems one way.” (George Polya)
- “Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice.” (George Polya)
- “Trust is the lubrication that makes it possible for organizations to work.” (Warren Bennis)
- “Difference between school and life: School teaches you lessons and then gives you a test. Life gives you a test and you learn the lessons.”
- “Math teachers: putting the fun in functions since $t = 0$.”



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Appendix C

Solving Vibration Problems with MATLAB

C.1 Introduction

In this Appendix,¹ we show how MATLAB² can be used to aid in the solution of some of the vibration problems studied in this book. Hopefully, in the process, we offer both the novice and experienced MATLAB programmer some useful strategies in attacking vibration problems of their own interest.

MATLAB (MATrix LABoratory) was born from the LINPACK routines written for use with C and Fortran.³ MATLAB provides both command-line and programming language interfaces, allowing the user to test simple statements at the command prompt, or to run complicated codes by calling a function.

MATLAB is designed to handle matrix quantities; it assumes every variable is an array. Thus, calling on the multiplication operator alone causes MATLAB to attempt matrix, not scalar, multiplication. Also, MATLAB includes special “array operators” that allow multiplication of arrays on an element-by-element basis. For example, if we set the variable $\mathbf{a} = [1 \ 2 \ 3]$ and $\mathbf{b} = [4 \ 5 \ 6]$, we can perform the matrix multiplications,

$$\mathbf{c} = \mathbf{a} * \mathbf{b}' \quad (\text{C.1})$$

$$\mathbf{d} = \mathbf{a}' * \mathbf{b} \quad (\text{C.2})$$

(The apostrophe is the transpose operator in MATLAB.) The result $\mathbf{c}$ would be a scalar (specifically, 32). The variable $\mathbf{d}$ would contain a 3-by-3 matrix, whose rows would be scalar multiples of $\mathbf{b}$.

What if the values stored in $\mathbf{a}$ are three masses and those in $\mathbf{b}$ their corresponding accelerations? If we wish to find the force on each mass, we would need to multiply the first element of $\mathbf{a}$ by the first element of $\mathbf{b}$, and so on for the second and third elements. MATLAB provides

“array multiplication” for such an instance,

$$\mathbf{e} = \mathbf{a} .* \mathbf{b} \quad (\text{C.3})$$

where the period between $\mathbf{a}$ and the asterisk tells MATLAB to ignore its usual matrix multiplication rules, and instead create $\mathbf{e}$ by multiplying the corresponding elements of $\mathbf{a}$ and $\mathbf{b}$. The result would be $\mathbf{e} = [4 \ 10 \ 18]$. This specialized command in MATLAB is one we will use frequently. Other commands will be discussed as they arise.

The general format in this Appendix is to introduce a problem, with reference to the text where applicable, and to show the analytic solution (if derivable). The MATLAB code is presented, with comments explaining what was done, why it was done, and sometimes other ways it could have been done in MATLAB. Graphs follow the code, showing simulation results of the vibration problem with different sets of parameters. The code for the majority of the problems assumes MATLAB is used in its programming mode, and all plots are generated using MATLAB’s integrated graphics capabilities.

One distinction to make with MATLAB programming is between *script* m-files and *function* m-files. Both are sets of commands that are saved in a file. Variables used in a script file are retained in the MATLAB workspace and can be called upon after the script completes. Variables within a function m-file cannot be used outside of the function unless they are returned (much like C functions or Fortran subroutines). A function begins with the line `function output = function_name(var1,var2,...varN)` and the file is interpreted by MATLAB as a function separate from the workspace.

C.2 SDOF Undamped System

The simplest model of vibration we can study is the single degree-of-freedom system without damping or external forcing. A lumped-parameter model of the system with mass m and stiffness k is shown in Figure C.1. Applying

¹An earlier version of this Appendix was prepared by Dr. S. Kuchnicki for the first edition of this textbook.

²MATLAB is a registered trademark of The MathWorks, Inc.

³Fortran (formerly FORTRAN, derived from FORMula TRANslation) is a general-purpose, programming language designed for numerical computation and scientific computing. It was originally developed by IBM in the 1950s for scientific and engineering applications.

Newton's second law, the equation of motion is

$$m\ddot{x} + kx = 0. \quad (\text{C.4})$$

To solve, we need the initial displacement x_o and initial velocity v_o of the mass. Dividing by m , and introducing $\omega_n = \sqrt{k/m}$, we obtain the equation $\ddot{x} + \omega_n^2 x = 0$ which has a solution of the form,

$$x(t) = A \sin(\omega_n t + \phi). \quad (\text{C.5})$$

In terms of known initial conditions, the solution is

$$x(t) = \frac{\sqrt{\omega_n^2 x_o^2 + v_o^2}}{\omega_n} \cos\left(\omega_n t - \tan^{-1} \frac{v_o}{\omega_n x_o}\right). \quad (\text{C.6})$$

This is Equation 2.27 with expressions for amplitude A and phase ϕ substituted. Thus, the complete response of an undamped, unforced, one degree-of-freedom oscillator depends on three physical parameters: ω_n , x_o , and v_o .

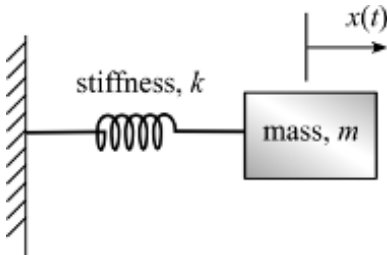


Figure C.1: Single degree-of-freedom undamped free oscillator.

From its definition, the natural frequency is inversely proportional to $\sqrt{m}$ and directly proportional to $\sqrt{k}$. Variation of mass or stiffness, then, will cause a variation in the frequency of vibration. The graphs that follow demonstrate these sensitivities. Figure C.2 shows the variation of the vibrational response for an increasing mass ($m = 2, 4, 12$) with stiffness constant ($k = 8$). Figure C.3 shows how the oscillatory behavior changes for increasing stiffness ($k = 2, 8, 13$) with mass constant ($m = 5$). For both Figures C.2 and C.3, the initial displacement $x_0 = 3$ and the initial velocity $v_o = 1$. We note that the frequency decreases with increasing mass and increases with increasing stiffness, as expected.

From Equation C.6, it is evident that the amplitude and phase angle are functions of the natural frequency. Again, referring to Figure C.2, the amplitude decreases with increasing mass, due to the corresponding reduction in natural frequency. As a result, the phase shift diminishes, with the peak of oscillation becoming nearer to $t = 0$. We note that the maximum displacement would occur at $t = 0$ if the initial velocity were zero. It is easily verified that, for this case, the parameter A (from Equation C.5, above) reduces to x_o , and the phase angle becomes $\tan^{-1}(0/x_o^2)$, or 0° .

In Figure C.3, the amplitude increases with increasing stiffness, due to the increase in natural frequency. The phase angle also increases with the stiffness, so the amplitude of oscillation occurs at an increasingly later time with increasing stiffness.

The MATLAB code used to produce the graphs follows. The **input** statements in the first few lines demonstrate one type of interactive programming available in MATLAB. The portion of the statement inside the quotation marks will appear on the screen when the program is run, so user prompts are easily added and explained. The initialization and use of the mass matrix is demonstrated in two ways. If the variable **matflag** is set to zero, then the masses are each given scalar values. If this flag is set to one, then the masses are initialized as part of an array. This is done to demonstrate the use of MATLAB array variables and to show how they can help streamline your code. We note the **if..then** structure in MATLAB (similar to Fortran); these statements must be closed with an **end** statement, as must any other loop (**for** or **do**, for example). We also demonstrate MATLAB's plotting routines using the **subplot** command, allowing plots for all three masses to show together. The second figure below was produced by modifying the code to take three stiffnesses and one mass. The modifications necessary are left as an exercise for the reader.

By default, MATLAB prints the results of each operation to the screen. Placing a semicolon at the end of a line suppresses this output. Since printing to screen takes time and memory for MATLAB to perform, suppressing screen output of intermediate results increases computational speed. (The reader is invited to remove the semicolons from the program to see the difference!) This feature is also a useful debugging tool; the programmer can delete semicolons to check intermediate calculations.

Program 1: varym.m

```
% { The first few lines obtain the desired
parameter values. This could also be done by
using assignment statements in the program,
but then the program would need to be edited
to change the parameters. As written,
parameters can be changed by re-running the
program.
%}
clear all
matflag=0; % Set to 1 to activate mass array,
zero for mass scalars.
k=input('Enter the stiffness value. ');
x0=input('Enter the initial displacement. ');
v0=input('Enter the initial velocity. ');
tf=input('Enter the time duration to test, in
seconds. ');
```



```

if (matflag)
    for i=1:3
        m(i)=input(['Enter mass value ',
num2str(i),'. ']);
        % More about the 'num2str' command below.
    end
else
    m1=input('Enter the first mass value. ');
    m2=input('Enter the second mass value. ');
    m3=input('Enter the third mass value. ');
end
%[This loop initializes the natural frequency
values. This is more streamline by making a
mass matrix [m(1) instead of m1, etc.], as
shown. Note that the natural frequency is
stored in a matrix.
%]
if (matflag)
    wn=sqrt(k./m);
%[Array division, akin to the array
multiplication described above. This one
line produces a three-element array of natural
frequencies. Compare to the machinations
below.
%]
else
    for i=1:3
        switch i
            case 1
                m=m1;
            case 2
                m=m2;
            case 3
                m=m3;
        end
        wn(i)=sqrt(k/m);
    end
end
%[Now, the values for A and phi in the
expression x(t)=Asin(wn*t + phi) are
evaluated. Notice that, in order for the
cosine function to evaluate properly in
MATLAB, the wn vector is used element by
element (thus, the loop).
%]
t=0:tf/1000:tf; % We need only initialize the
time increment once.
for j=1:3
    a=sqrt(wn(j)^2*x0^2+v0^2)/wn(j);
% The caret is the power operator.
    phi=atan2(v0,wn(j)*x0);
% atan2 is the four-quadrant arctangent.
    x(j,:)=a*cos(wn(j)*t-phi);
end

% {Since this program was made to compare
different parameters, a subplot format makes
sense. If the number of varied masses is
changed, the subplot statement must also be.
%}
subplot(3,1,1)
plot(t,x(1,:))
% {This line demonstrates the use of the
num2str command. This command allows the
value of a variable to be used in a title or
other text. Note that the command doesn't
work unless the text intended as a title is
enclosed in brackets as well as parentheses.
%}
if (matflag)
    title(['Response for m=',num2str(m(1)), ',
k=', num2str(k)])
else
    title(['Response for m=',num2str(m1), ',
k=', num2str(k)])
end
ylabel('Response x')
grid
subplot(3,1,2)
plot(t,x(2,:))
if (matflag)
    title(['Response for m=',num2str(m(2)), ',
k=', num2str(k)])
else
    title(['Response for m=',num2str(m2), ',
k=', num2str(k)])
end
ylabel('Response x')
grid subplot(3,1,3)
plot(t,x(3,:))
if (matflag)
    title(['Response for m=',num2str(m(3)), ',
k=', num2str(k)])
else
    title(['Response for m=',num2str(m3), ',
k=', num2str(k)])
end
ylabel('Response x')
xlabel('Time, seconds')
grid

```

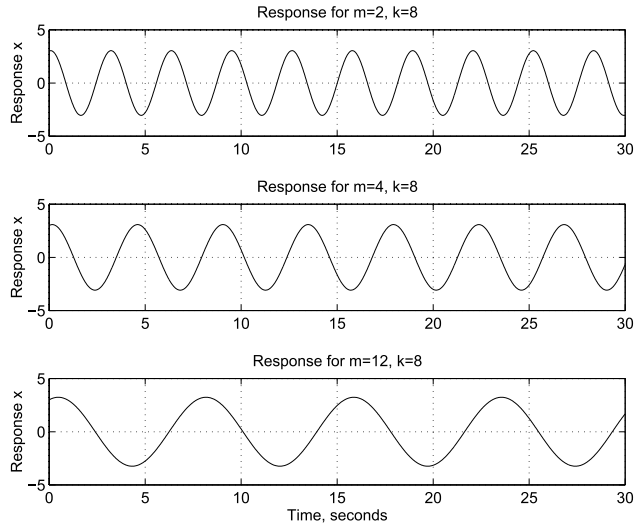


Figure C.2: Responses for three different masses.

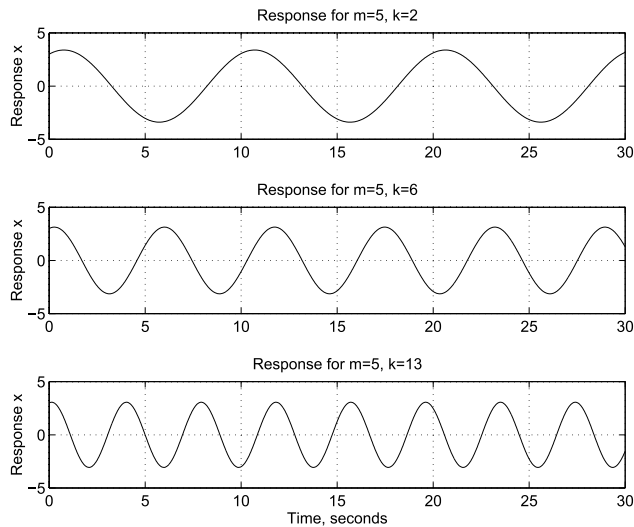


Figure C.3: Responses for three different stiffnesses.

C.3 SDOF Damped System

In the previous section, we examined the response of a single degree-of-freedom undamped oscillator to changes in mass and stiffness values. Here, we investigate the response of a single degree-of-freedom damped system to changes in damping.

We begin with the equation of motion for the mass-spring-damper model of Figure C.4,

$$m\ddot{x} + c\dot{x} + kx = 0, \quad (\text{C.7})$$

where c is the viscous damping constant. If we divide by m , we can rewrite the equation in terms of the dimensionless parameters ω and ζ ,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \quad (\text{C.8})$$

where ω_n represents the undamped natural frequency and ζ is the viscous damping ratio. For vibration studies, we are most interested in the underdamped case ($\zeta < 1$). The solution to this equation is

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi), \quad (\text{C.9})$$

where ω_d is the damped natural frequency, equal to $\omega_n \sqrt{1 - \zeta^2}$.

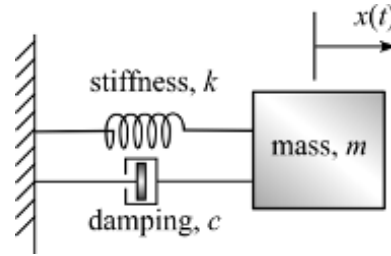


Figure C.4: Single degree-of-freedom damped free oscillator.

This equation can be written in terms of parameters ω_n and ζ and the initial conditions,

$$x(t) = \sqrt{\frac{(x_o\omega_n\sqrt{1-\zeta^2})^2 + (v_o + \zeta\omega_n x_o)^2}{(\omega_n\sqrt{1-\zeta^2})^2}} e^{(-\zeta\omega_n t)} \times \sin \left[\left(\omega_n\sqrt{1-\zeta^2} \right) t + \tan^{-1} \left(\frac{x_o\omega_n\sqrt{1-\zeta^2}}{v_o + \zeta\omega_n x_o} \right) \right], \quad (\text{C.10})$$

where $\omega_n\sqrt{1-\zeta^2}$ has been substituted for ω_d .

While this equation looks intimidating, it depends only on four quantities: x_o , v_o , ω_n , and ζ . We have already seen the effects of changing the mass and stiffness parameters. Figure C.5 shows the variation of the response with increasing viscous damping coefficient for

$x_o = 3$, $v_o = 1$, and $\omega_n = 7$. Note how quickly the response decays to virtually zero; this occurs within ten seconds for a damping ratio as small as 0.05, and much sooner for higher damping ratios.

The MATLAB code used to generate the figure follows. This code is similar to the code for the mass-spring case. However, since both the exponential term and the sine term in our solution (Equation C.9) depend on time, they are both vector quantities. To multiply them properly, we must use the array multiplication operator, `.*`. The code again makes use of `input` statements to initialize the parameters. This time, however, the damping ratios are entered in array form only.

The code to produce the plots works only for the underdamped case; since we have the term $\omega_d = (\omega_n \sqrt{1 - \zeta^2})$ in the denominator of our response equation, $\zeta = 1$ will cause division by zero, and $\zeta > 1$ will give an imaginary damped natural frequency. We employ some defensive programming to ensure allowable values for the damping ratio are entered. This gives us a reason to introduce the `while` loop. The code checks each value `zeta(zi)` to see if it lies between zero and one. If not, a reminder is sent to the screen, and the loop asks for a new entry. Otherwise, it continues to the next damping ratio value. The logical and relational operators in MATLAB are for the most part intuitive. Thus, if `a>0` simply means “if a is greater than zero.” A full list can be found by typing `help ops` at the MATLAB prompt; the operators we use are explained in the comments of the code.

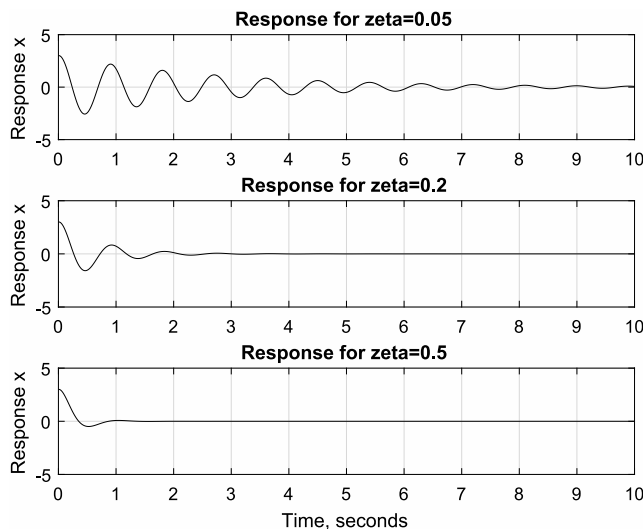


Figure C.5: Response for various damping ratios.

Program 1: varyzeta.m

%{Initial value entry. The while loop in the zeta initialization section prevents certain

values for zeta from being entered, since such values would crash the program.

```
%}
clear all
wn=input('Enter the natural frequency. ');
x0=input('Enter the initial displacement. ');
v0=input('Enter the initial velocity. ');
tf=input('Enter the time duration to test, in seconds. ');
for zi=1:3
    zeta(zi)=12;
    while(zeta(zi)<0 | zeta(zi)>=1)
        % The pipe (|) means "or".
        zeta(zi)=input(['Enter damping coefficient value ', num2str(zi), '. ']);
        if (zeta(zi)>=1 | zeta(zi)<0)
            fprintf('Zeta must be between 0 and 1!');
            zeta(zi)=12;
        end
    end
end
%Now, having  $\omega_n$  and  $\zeta$ , the  $\omega_d$  values can be found.
for i=1:3
    wd(i)=wn*sqrt(1-zeta(i)^2);
end
%{Solving for the response. Note the use of the array multiplication command (.* in the expression for  $x(t)$ . This command is necessary, else the program gives a multiplication error.
%}
t=0:tf/1000:tf;
for j=1:3
    a=sqrt((wn*x0*zeta(j)+v0)^2+(x0*wd(j))^2)/wd(j);
    phi=atan2(wd(j)*x0,v0+zeta(j)*wn*x0);
    x(j,:)=a*exp(-zeta(j)*wn*t).
    *sin(wd(j)*t+phi);
end
%Now, the program plots the results in a subplot format.
subplot(3,1,1)
plot(t,x(1,:))
title(['Response for zeta=', num2str(zeta(1))])
ylabel('Response x')
grid
subplot(3,1,2)
plot(t,x(2,:))
title(['Response for zeta=', num2str(zeta(2))])
ylabel('Response x')
grid
```

```
subplot(3,1,3)
plot(t,x(3,:))
title(['Response for zeta=',
num2str(zeta(3))])
ylabel('Response x')
xlabel('Time, seconds')
grid
```

C.4 SDOF Overdamped System

The previous section addressed the underdamped case, $\zeta < 1$. When this condition is not met, there are two cases to explore: $\zeta > 1$ (the overdamped case), and $\zeta = 1$ (the critically damped case).

The equation of motion of a single degree-of-freedom damped oscillator is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0. \quad (\text{C.11})$$

We assume a solution of the form $Ae^{\lambda t}$, substitute it and its derivatives into Equation C.11, obtain a quadratic algebraic equation, and solve using the quadratic formula to find two possible values for λ ,

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \quad (\text{C.12})$$

For the $\zeta > 1$ case, the quantity under the radical is always greater than zero, and the solution of the equation of motion is

$$x(t) = e^{-\zeta\omega_n t} (A_1 e^{\omega_n t \sqrt{\zeta^2 - 1}} + A_2 e^{-\omega_n t \sqrt{\zeta^2 - 1}}). \quad (\text{C.13})$$

In terms of initial displacement x_o and initial velocity v_o , the constants A_1 and A_2 are

$$\begin{aligned} A_1 &= \frac{-v_o + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_o}{2\omega_n \sqrt{\zeta^2 - 1}} \\ A_2 &= \frac{v_o + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_o}{2\omega_n \sqrt{\zeta^2 - 1}}. \end{aligned} \quad (\text{C.14})$$

Equation C.13 is a decaying exponential and the system will simply return to its initial position and not oscillate about the equilibrium. This is shown in Figure C.6.

If $\zeta = 1$, a singularity exists in the constants; a second independent solution must be found. From our knowledge of ordinary differential equations, we can write

$$x(t) = (A_1 + A_2 t)e^{-\omega_n t}, \quad (\text{C.15})$$

where $A_1 = x_o$ and $A_2 = v_o + \omega_n x_o$.

Figure C.6 shows responses for $\omega_n = 7$, $x_o = 3$, and $v_o = 1$. Notice how the critically damped response returns to equilibrium faster than the others. For the plots in the figure, the motion with critical damping stops in less than two seconds, while the others do not reach equilibrium until approximately eight seconds. This is a distinguishing characteristic of the critically damped case. Note also that the motion of the masses is, as expected, purely exponential; there is no oscillation, only a decay of the response to equilibrium.

The MATLAB code is presented below. The reader may ask why the codes for underdamped, critically damped, and overdamped vibration could not be combined into a single code. Certainly they can be; the modifications are left to the reader. The resulting code should

reject negative values for the damping ratio. (Hint: You may find the `switch` command to be useful, especially when combined with its `otherwise` case.)

Program 1: overdamp.m

```
%[Initial value entry. The while loop in the
zeta initialization section prevents illegal
values for zeta from being entered, since such
values would crash the program.
%}
wn=input('Enter the natural frequency. ');
x0=input('Enter the initial displacement. ');
v0=input('Enter the initial velocity. ');
tf=input('Enter the time duration to test, in
seconds. ');
for zi=1:3
    zeta(zi)=0.12;
    while(zeta(zi)<1)
        zeta(zi)=input('Enter a damping
coefficient value. ');
        if zeta(zi)<1
            fprintf('Zeta must be greater than 1!');
            zeta(zi)=0.12;
        end
    end
end
%[Solving for the response. Notice that the
variable den is used for the denominator
of the constants, as well as the exponent
values. If a variable can be created that
will serve several purposes, it will save time
and typing. (imagine typing the expression
for den three times and you'll understand).
%}
t=0:tf/1000:tf;
for j=1:3
    if zeta(j)>1
        a1n=-v0+(-zeta(j)+(zeta(j)^2-1)^0.5)*wn*x0;
        a2n=v0+(zeta(j)+(zeta(j)^2-1)^0.5)*wn*x0;
        den=wn*(zeta(j)^2-1)^0.5;
        a1=a1n/(2*den);
        a2=a2n/(2*den);
        x(j,:)=(a1*exp(-den*t)+a2*exp(den*t)).
*exp(-zeta(j)*wn*t);
    elseif zeta(j)==1
        a1=x0;
        a2=v0+wn*x0;
        x(j,:)=(a1+a2*t).*exp(-wn*t);
    end
end
%Now, the program plots the results in a
subplot format.
subplot(3,1,1)
```

```
plot(t,x(1,:))
title(['Response for zeta=',num2str(zeta(1))])
ylabel('Response x')
grid
subplot(3,1,2)
plot(t,x(2,:))
title(['Response for zeta=',
num2str(zeta(2))])
ylabel('Response x')
grid
subplot(3,1,3)
plot(t,x(3,:))
title(['Response for zeta=',
num2str(zeta(3))])
ylabel('Response x')
xlabel('Time, seconds')
grid
```

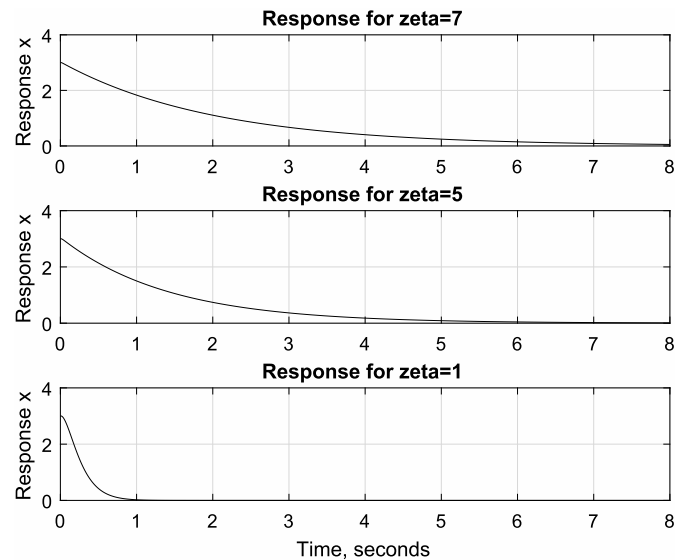


Figure C.6: Response of three overdamped systems.

C.5 SDOF Undamped System with Harmonic Excitation

In the previous sections, we examined the responses of single degree-of-freedom systems that vibrated as a result of initial conditions. They were not subjected to an external force. Now, we will examine the effects of an external force on the system. We begin with the simplest form of external forcing: the harmonic load.

The general system under consideration is shown in the model of Figure C.7. The forcing function is assumed to be of the form $F(t) = F_o \cos \omega t$, where ω is the driving frequency. For the case with no damping, Newton's Second Law gives us the equation of motion $m\ddot{x} + kx = F_o \cos \omega t$, or

$$\ddot{x} + \omega_n^2 x = f_o \cos \omega t, \quad (\text{C.16})$$

where $f_o = F_o/m$. The solution of the differential equation is the response $x(t)$,

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t + \frac{f_o}{\omega_n^2 - \omega^2} \cos \omega t, \quad (\text{C.17})$$

where

$$A_1 = \frac{v_o}{\omega_n}, \quad A_2 = x_o - \frac{f_o}{\omega_n^2 - \omega^2}. \quad (\text{C.18})$$

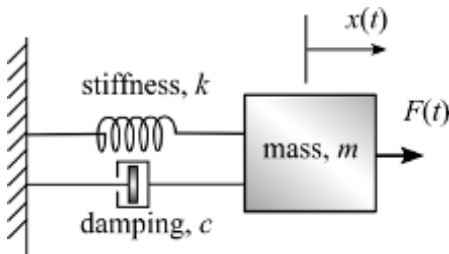


Figure C.7: Single degree-of-freedom damped oscillator subject to an external force.

The key parameters that define the character of the response are the natural and driving frequencies, or, more precisely, their ratio ω/ω_n . Figure C.8 shows the effect of varying driving frequency ω for a given natural frequency, and Figure C.9 does the same for various natural frequencies ω_n . In these simulations, $f_o = 6$ and $x_o = 0, v_o = 0$. Making the initial conditions zero means there is only a particular solution and no transient response, that is, the response is due solely to the harmonic excitation.

Note that two of the three constants in the expression for $x(t)$ involve the differences of the squares of the natural and driving frequencies. This gives rise to two interesting phenomena: *beats* and *resonance*. Beats occur when the natural frequency and the driving frequency are close but not equal. The result is then a rapid oscillation with a slowly varying amplitude, as shown in

Figure C.10. Note how the the rapid oscillation and the slow change of the amplitude both vary along a sinusoid.

When the driving and natural frequencies are equal, $\omega = \omega_n$, resonance is the result. The third term in Equation C.17 is not valid as a particular solution of the governing equation of motion. Instead, the particular solution is

$$x_p(t) = \frac{f_o}{2\omega_n} t \sin \omega_n t. \quad (\text{C.19})$$

This term is the heart of the resonance problem; the amplitude of oscillation will increase without limit. In a real system, there is always damping and even if it is small the stiffness element has a certain yield point that will be met and exceeded by a resonant vibration. Figure C.11 shows resonant vibration.

The MATLAB code uses the `subplot` command more elegantly than had been used in earlier examples. Instead of typing out three separate sets of plotting commands, the subplots are done in a loop. A simple (and natural) modification to this program would be to change the number of plots in the subplot. This could be done by introducing an integer variable `nplot` and changing the loops to run from one to `nplot` instead of one to three. From a practical point of view, though, introducing too many plots would make them unreadable. Also, the parameter values are changed from dynamic inputs to static values. They can easily be changed back to user inputs.

Program 1 - varywdr.m

```
%{This program is very straightforward. The
program takes the necessary input values
to solve Equation C.18 for three different
driving frequencies, checks for resonance
(since we don't want to handle that problem
yet), and plots.
%}
```

```
%{This file varies driving frequency of a
single degree-of-freedom undamped oscillator
at a set natural frequency. The initial
displacement is 3, the velocity is 1, the
force magnitude per unit mass is 6, and the
natural frequency is 7.
%}
```

```
wn=7;
x0=0;
v0=0;
f0=6;
tf=10;
wdr=zeros(3,1);
x=zeros(3,1001);
for i=1:3;
```

```

    wdr(i)=wn; % This is how we initialize our
while loop.
    while wdr(i)==wn;
        wdr(i)=input('Enter the driving
frequency. ');
        if wdr(i)==wn;
            fprintf('This will produce resonance!!')
        end
    end
end
t=0:tf/1000:tf;
for j=1:3
    A1=v0/wn;
    A2=x0-(f0/(wn^2-wdr(j)^2));
    A3=f0/(wn^2-wdr(j)^2);
    x(j,:)=A1*sin(wn*t)+A2*cos(wn*t)
+A3*cos(wdr*j*t);
end
for k=1:3 % We could have used subplot this
way all along.
    subplot(3,1,k)
    plot(t,x(k,:))
    title(['Response for wdr=', num2str(wdr(k)),
'wn=', num2str(wn)])
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')

```

Program 2: varywn.m

```

%{This program is similar in form to
varywdr.m, except that now the natural
frequency is the variable. This file
varies natural frequency of a single
degree-of-freedom undamped oscillator at a set
driving frequency. The initial displacement
is 3, the velocity is 1, the force magnitude
per unit mass is 6, and the driving frequency
is 7.
%}
wdr=7;
x0=0;
v0=0;
f0=6;
tf=10;
wn=zeros(3,1);
x=zeros(3,1001);
for i=1:3;
    wn(i)=wdr; % Analogous to the initialization
above.
    while wn(i)==wdr;
        wn(i)=input('Enter the natural frequency.
');
    end
end

```

```

        if wn(i)==wdr;
            fprintf('This will produce resonance!!')
        end
    end
end
t=0:tf/1000:tf;
for j=1:3
    A1=v0/wn(j);
    A2=x0-(f0/(wn(j)^2-wdr^2));
    A3=f0/(wn(j)^2-wdr^2);
    x(j,:)=A1*sin(wn(j)*t)+A2*cos(wn(j)*t)
+A3*cos(wdr*t);
end
for k=1:3
    subplot(3,1,k)
    plot(t,x(k,:))
    title(['Response for wdr=', num2str(wdr),
'wn=', num2str(wn(k))])
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')

```

Program 3: beatres.m

%{The main feature of note for this code is the use of the if-then-else protocol in MATLAB to allow solutions other than "Inf" for the resonant case. By changing the values of the initial conditions, wn, and f0 to input statements, this program would become a general solver for the single degree-of-freedom undamped oscillator, subject to harmonic forcing.

This program shows beats and resonance in undamped systems. Again, in order to better see the effects, the initial velocity and displacement are zero.

```

%}
x0=0;
v0=0;
wn=3;
wdr=input('Enter the driving frequency. ');
f0=6;
tf=120;
%{This section chooses the proper response
formula for the given situation.
%}
t=0:tf/1000:tf;
if wdr==wn
    A1=v0/wn;
    A2=x0;
    A3=f0/2*wn;
end

```

```

x=A1*sin(wn*t)+A2*cos(wn*t)
+A3*t.*cos(wdr*t);
else
    A1=v0/wn;
    A2=x0-(f0/(wn^2-wdr^2));
    A3=f0/(wn^2-wdr^2);
    x=A1*sin(wn*t)+A2*cos(wn*t) +A3*cos(wdr*t);
end
plot(t,x);
if wdr==wn
    title('Example of Resonance Phenomenon');
else
    title('Example of Beat Phenomenon');
end
xlabel('Time, seconds')
ylabel('Response x')
grid

```

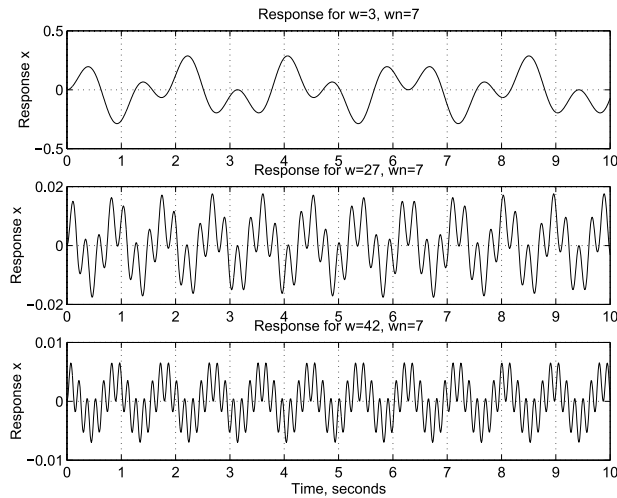


Figure C.8: Responses for different driving frequencies.

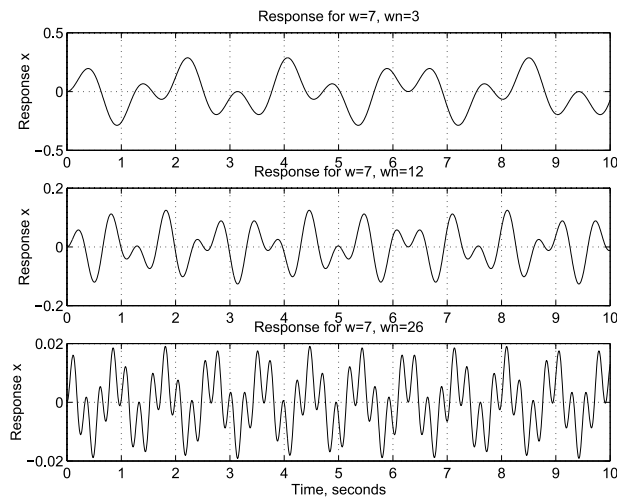


Figure C.9: Responses for different natural frequencies.

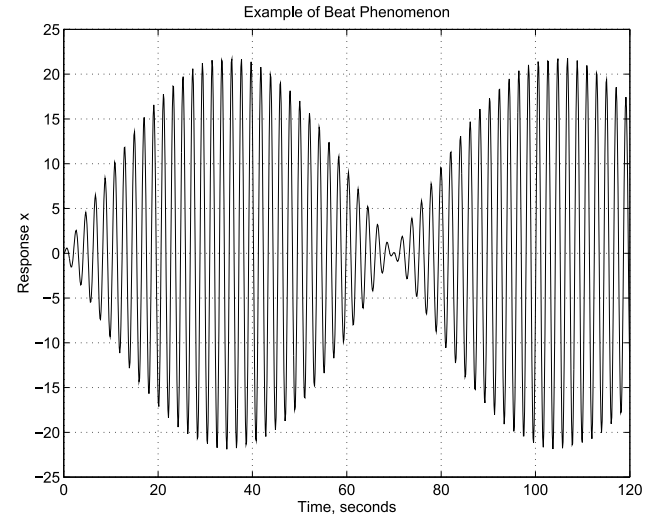


Figure C.10: Example of the beating phenomenon.

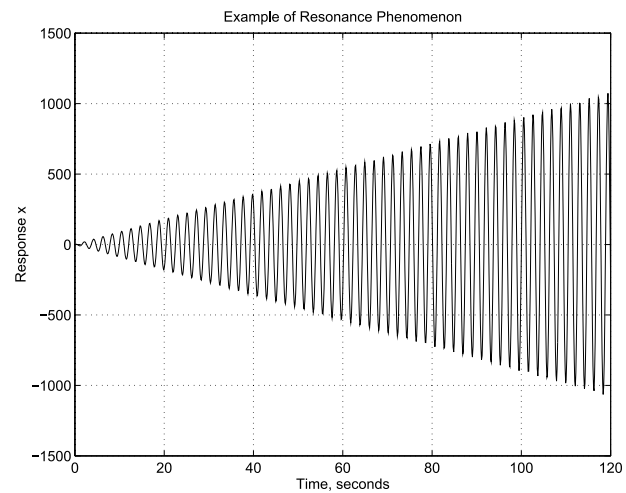


Figure C.11: Example of resonant vibration.

C.6 SDOF Damped System with Harmonic Excitation

Building on the previous case, we examine how the behavior of the system changes when the model includes damping. The equation of motion with damping is

$$m\ddot{x} + c\dot{x} + kx = F_o \cos(\omega t), \quad (\text{C.20})$$

or, with the damping ratio $\zeta = c/2m\omega_n$,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_o \cos(\omega t), \quad (\text{C.21})$$

where $f_o = F_o/m$. The homogeneous solution of this equation is of the form

$$x_h(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \theta), \quad (\text{C.22})$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, and constants A and θ depend on the initial conditions. The particular solution due to the external force is

$$x_p(t) = A_0 \cos(\omega t - \phi), \quad (\text{C.23})$$

where

$$A_0 = \frac{f_o}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \\ \phi = \tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}. \quad (\text{C.24})$$

The complete solution, $x(t) = x_h(t) + x_p(t)$, is then used to evaluate constants A and θ . These constants were found for zero initial conditions using a symbolic program, such as Maple; the solution is reflected in the code that follows.

The addition of damping causes the response of the system to differ slightly, as shown in [Figures C.12](#) and [C.13](#). In [Figure C.12](#), the damping ratio ζ was varied. The plots show that the duration of the transient effect varies inversely with the damping ratio. The duration of the transient varies from about 4.5 seconds for $\zeta = 0.05$ to about 2 seconds for $\zeta = 0.3$, suggesting that, in many cases, the transient response can be ignored since its effect is short lived. However, for some cases, the transient duration may be much longer or may have a very large amplitude, so it is always important to examine the transient effects of a system before neglecting them. Notice also that the damping ratio affects the amplitude of the steady-state vibration, again in an inverse relationship. That is, the amplitude of the response for $\zeta = 0.05$ is almost 2, whereas that for $\zeta = 0.3$ is less than 1.

[Figure C.13](#) shows the effects of changing the natural frequency. Notice how, for the two frequencies that are near the driving frequency, the transient duration is quite long, almost 10 seconds. However, for the largest natural

frequency tested, the transient duration is less than 4 seconds, which shows that the duration of the transient effect also depends on the natural frequency.

In the damped system, resonance takes on a different meaning. Notice how, for $\omega = \omega_n$, the amplitude does not become infinite; due to damping in the model there is a term that keeps the denominator of the steady-state amplitude from becoming zero. At this frequency, the phase angle becomes 90° . For a damped system, this condition defines resonance, since it is also corresponds to the denominator of the amplitude being minimum. To help understand this last assertion, consider the denominator of the amplitude constant, A_0 . The amplitude will be maximized when the denominator is minimized. Both terms are never negative, so the minimum will occur when the two frequencies are equal (making the first term of the denominator zero). Also, as the driving frequency increases greatly, the amplitude nears zero.

The MATLAB code below was used to produce the figures. Instead of inserting a **while** statement to control the range of damping ratios, we use an **if** statement, in tandem with an **error** statement. If the damping ratio is outside the allowed range, MATLAB will halt execution of the program, printing the statement found inside the quotation marks to the screen. Controlling input in this manner is more drastic than the method used in the previous programs. However, those programs had many more values to input.

This program introduces MATLAB's method for continuing lines. To continue an expression onto the next line, simply type three periods in succession (ellipsis) and move to the next line. This is helpful for long mathematical expressions, like the constants of integration below. This ellipsis can be placed anywhere a space would be allowed, so continuing in the middle of a variable name, for example, is not recommended.

Program 1: harmzeta.m

```
%{This program solves for the response of a
damped single degree-of-freedom system subject
to a harmonic external force. The expressions
used for the constants were found by using
Maple.
%}
wdr=3;
wn=3.5;
fo=4;
tf=10;
t=0:tf/1000:tf;
for k=1:3
    zeta(k)=input('Enter a damping ratio (zeta).
');
    if (zeta(k)<0 | zeta(k)>=1)
```

```

    error('Zeta out of range for this
program!')
end
end
for k=1:3
    wd=wn*sqrt(1-zeta(k)^2);
    Ao=fo/sqrt((wn^2-wdr^2)^2
+(2*zeta(k)*wn*wdr)^2);
    phi=atan2(2*zeta(k)*wn*wdr,(wn^2-wdr^2));
    Z1=-zeta(k)*wn-wdr*tan(phi);
    Z2=sqrt((zeta(k)*wn)^2+2*zeta(k)*wn...
*wdr*tan(phi)+(wdr*tan(phi))^2+wd^2);
%{ If the code extends beyond a single line,
use ... to continue writing code on the next
line.
%}
    Z=(Z1+Z2)/wd;
    Anum=Ao*((zeta(k)*wn*Z-wd)*cos(phi)
+wdr*Z*sin(phi));
    Aden=Z*wd;
    A=Anum/Aden;
    theta=2*atan(Z);
    x(k,:)=A*exp(-zeta(k)*wn*t).
*sin(wd*t+theta) +Ao*cos(wdr*t-phi);
end
for k=1:3
    subplot(3,1,k)
    plot(t,x(k,:))
    title(['Response for
zeta=',num2str(zeta(k)), ' ...
, wn=',num2str(wn), ', and wdr=',
num2str(wdr)])
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')

```

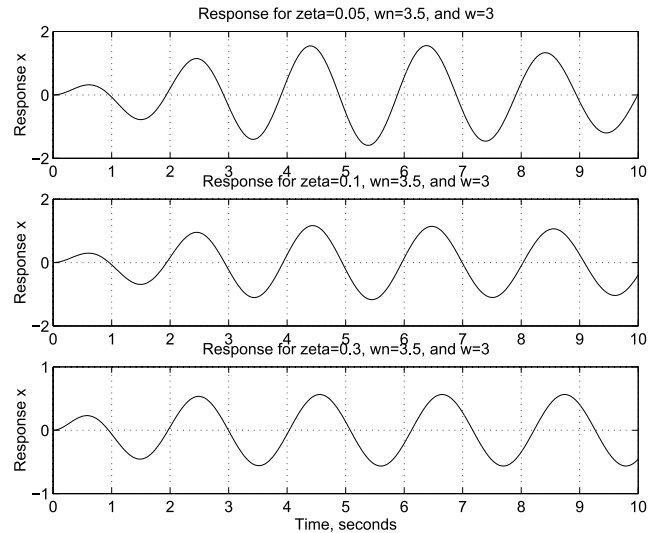


Figure C.12: Responses for different damping ratios.

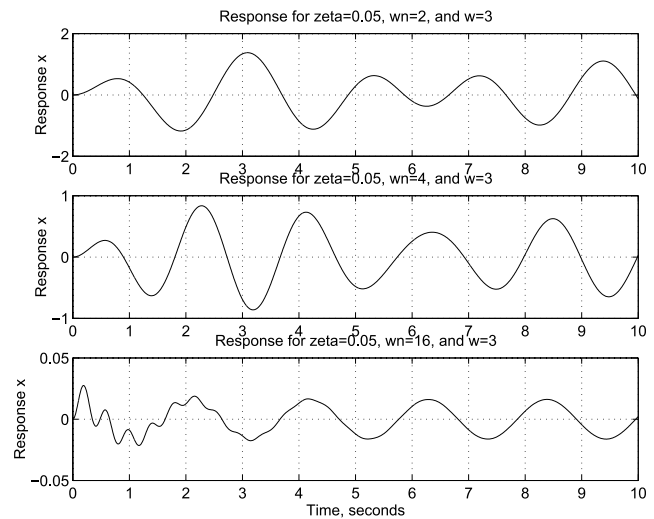


Figure C.13: Responses for different natural frequencies.

C.7 SDOF Damped System with Base Excitation

The base-excitation problem is illustrated in Figure C.14. We let the motion of the base be denoted by $y(t)$ and the response of the mass by $x(t)$, and assume that the base has harmonic motion of the form $y(t) = Y \sin(\omega_b t)$. The equation of motion for this system is

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0. \quad (\text{C.25})$$

Using the assumed form for the base motion, we can substitute for y and its derivative, resulting in

$$m\ddot{x} + c\dot{x} + kx = cY\omega_b \cos \omega_b t + kY \sin \omega_b t, \quad (\text{C.26})$$

which, when divided by the mass, yields

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 2\zeta\omega\omega_b \cos \omega_b t + \omega^2 Y \sin \omega_b t. \quad (\text{C.27})$$

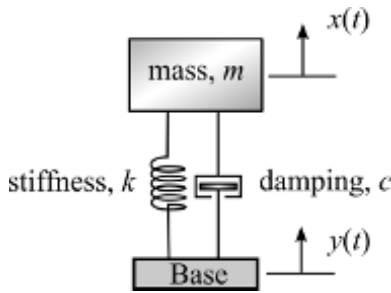


Figure C.14: Single degree-of-freedom damped system subject to base excitation.

The homogeneous solution is of the form,

$$x_h = Ae^{-\zeta\omega t} \sin(\omega_d t + \theta). \quad (\text{C.28})$$

The expression for each part of the particular solution is similar to that for the general sinusoidal forcing function; the sine term produces a sine solution, and the cosine term produces a cosine solution. If we find these solutions and combine their sum into a single sinusoid, we obtain

$$x_p = A_o \cos(\omega_b t - \phi_1 - \phi_2), \quad (\text{C.29})$$

where

$$A_o = \omega Y \sqrt{\frac{\omega^2 + (2\zeta\omega_b)^2}{(\omega^2 - \omega_b^2)^2 + (2\zeta\omega\omega_b)^2}}$$

$$\phi_1 = \tan^{-1} \frac{2\zeta\omega\omega_b}{\omega^2 - \omega_b^2}, \quad \phi_2 = \tan^{-1} \frac{\omega}{2\zeta\omega_b}.$$

The complete solution is the sum of the homogeneous and particular solutions, or

$$x(t) = Ae^{-\zeta\omega t} \sin(\omega_d t + \theta) + A_o \cos(\omega_b t - \phi_1 - \phi_2). \quad (\text{C.30})$$

This equation tells us a great deal about the motion of the mass. First, we can see that the particular solution represents the steady-state response, while the homogeneous solution is the transient response, since the particular solution is independent of the initial displacement and velocity. It is very difficult to solve the initial value problem (with initial displacement and velocity not zero) to find the expressions for the constants A and θ , in general, even using a symbolic package, such as Maple. So, for the sake of programming in MATLAB, the initial velocity and displacement were both assumed to be zero. This assumption yielded simpler equations, which were used in the base excitation programs that follow.

Figure C.15 shows the effects of changing the excitation frequency while holding all other parameters constant. In steady state, from about three seconds forward, the frequency of vibration increases with the base frequency. This is expected, since the base excitation portion dominates the steady state. Of particular note is the bottom plot, with $\omega_b = 20$. In the transient portion, the response has the shape of a sum of two sinusoids; these are, of course, the transient and steady-state functions. Since the base excitation is of such high frequency, this graph shows best what is happening between the transient and steady responses. Note that drawing an envelope line through the upper or lower peaks of the motion would result in a curve similar to that exhibited by a damped free response. The midpoint of the oscillation caused by the steady response becomes exponentially closer to zero with increasing time, as the transient response diminishes.

Figure C.16 shows plots for three different vibration amplitudes. The differences caused by changing the amplitude is what would be expected; the maximum amplitude of the overall vibration and of the steady-state response both increase with increasing input amplitude.

The plots in Figure C.17 for various damping ratios show two effects. First, the change in damping ratio causes the duration of the transient period to vary; an increase in damping ratio causes the transient period to decrease. Second, the change in damping ratio causes the frequency of the transient vibration to change. An increase in damping ratio causes a decrease in the damped natural frequency, although the decrease is not entirely evident from just looking at the plots. This is because the plots also include the base excitation (steady-state) terms, whose frequency has not changed.

The MATLAB code for this situation is much more complex than the code used in previous examples. This is mainly due to the need to account for the relative displacement between the base and mass, instead of just the displacement of the mass relative to a fixed ground. The code uses a solution obtained from Maple to produce a plot of the response. Actually, Maple gives two possible solutions; the MATLAB code attempts to locate the cor-

rect one by checking which of them matches the initial displacement condition. Note that the initial displacements and initial velocities are zero for all plots. The three programs given below show the changes that need to be made to test different parameters.

Program 1: varywb.m

```
%{This program solves the base excitation
problem. The code assumes a sinusoidal base
function. Also, the program tests three
different natural frequencies.
%}
y0=input('Enter the base excitation magnitude.
');
zeta=input('Enter the damping ratio (zeta).
');
if (zeta<0 | zeta>=1)
% The test on damping ratio.
error('Damping ratio not in acceptable
range!')
end
wn=4;
tf=10;
t=0:tf/1000:tf;
for k=1:3
wb(k)=input('Enter a base excitation
frequency. ');
end
for m=1:3
%{This section solves the transient response,
using the equations obtained from Maple.
%}
wd=wn*sqrt(1-zeta^2);
phi1=atan2(2*zeta*wn*wb(m),(wn^2-wb(m)^2));
phi2=atan2(wn,2*zeta*wb(m));
xi=phi1+phi2;
%{These constants are what produces the two
possible solutions discussed above. Notice
the way by which the extraordinarily long
expressions for the constants are broken into
parts, to keep the expressions from spreading
over several lines.
%}
Z1=(-zeta*wn-wb(m)*tan(xi)
+sqrt((zeta*wn)^2 +2*zeta*wn*wb(m)*tan(xi)
+(wb(m)*tan(xi))^2+wd^2))/wd;
Z2=(-zeta*wn-wb(m)*tan(xi)
-sqrt((zeta*wn)^2 +2*zeta*wn*wb(m)*tan(xi)
+(wb(m)*tan(xi))^2+wd^2))/wd;
Anum=sqrt((wn^2 +(2*zeta*wb(m))^2) /((wn^2
-wb(m)^2)^2 +(2*zeta*wb(m)*wn)^2)) *wn*y0;
Bnum1=(-wd*cos(xi) +Z1*zeta*wn*cos(xi)
+Z1*wb(m)*sin(xi));
```

```
Bnum2=(-wd*cos(xi) +Z2*zeta*wn*cos(xi)
+Z2*wb(m)*sin(xi));
Aden1=wd*Z1;
Aden2=wd*Z2;
A1=Anum*Bnum1/Aden1;
A2=Anum*Bnum2/Aden2;
th1=2*atan(Z1);
th2=2*atan(Z2);
y1(m,:)=A1*exp(-zeta*wn*t).*sin(wd*t+th1);
y2(m,:)=A2*exp(-zeta*wn*t).*sin(wd*t+th2);
end
%This portion solves the steady-state
response.
for j=1:3
A=sqrt((wn^2+(2*zeta*wb(j))^2)
/((wn^2-wb(j)^2)^2 +(2*zeta*wn*wb(j))^2));
phi1=atan2(2*zeta*wn*wb(j),(wn^2-wb(j)^2));
phi2=atan2(wn,(2*zeta*wb(j)));
xp(j,:)=wn*y0*A*cos(wb(j)*t-phi1-phi2);
end
if (xp(1,1)+y1(1,1)==xp(2,1)+y1(2,1)
==xp(3,1)+y1(3,1)==0)
x=xp+y1;
else
x=xp+y2;
end
for i=1:3
subplot(3,1,i)
plot(t,x(i,:))
ylabel('Response x');
title(['Base Excitation with wb=',
num2str(wb(i)), ' and wn=',num2str(wn)]);
grid
end
xlabel('Time, seconds')
```

Program 2: varyyobe.m

```
%{This program solves the base excitation
problem. The code assumes a sinusoidal
base function. Also, the program tests
three different base amplitudes. Notice
that the natural frequency is now given as
a set variable, and the variable 'y0' is a
user-input matrix.
%}
wb=input('Enter the base excitation frequency.
');
zeta=input('Enter the damping ratio (zeta).
');
if (zeta<0 | zeta>=1)
% The usual test on damping ratio.
error('Damping ratio not in acceptable
range!')
```

```

end
wn=4;
tf=10;
t=0:tf/1000:tf;
for k=1:3
    y0(k)=input('Enter a base excitation
magnitude. ');
end
for m=1:3
    wd=wn*sqrt(1-zeta^2);
    phi1=atan2(2*zeta*wn*wb, (wn^2-wb^2));
    phi2=atan2(wn, 2*zeta*wb);
    xi=phi1+phi2;
    Z1=(-zeta*wn-wb*tan(xi)
+sqrt((zeta*wn)^2 +2*zeta*wn*wb*tan(xi)
+(wb*tan(xi))^2+wd^2))/wd;
    Z2=(-zeta*wn-wb*tan(xi)
-sqrt((zeta*wn)^2 +2*zeta*wn*wb*tan(xi)
+(wb*tan(xi))^2+wd^2))/wd;
    Anum=sqrt((wn^2 +(2*zeta*wb)^2) /((wn^2
-wb^2)^2 +(2 *zeta *wb *wn)^2)) *wn *y0(m);
    Bnum1=(-wd*cos(xi) +Z1*zeta*wn*cos(xi)
+Z1*wb*sin(xi));
    Bnum2=(-wd*cos(xi) +Z2*zeta*wn*cos(xi)
+Z2*wb*sin(xi));
    Aden1=wd*Z1; Aden2=wd*Z2;
    A1=Anum*Bnum1/Aden1;
    A2=Anum*Bnum2/Aden2;
    th1=2*atan(Z1);
    th2=2*atan(Z2);
    y1(m,:)=A1*exp(-zeta*wn*t).*sin(wd*t+th1);
    y2(m,:)=A2*exp(-zeta*wn*t).*sin(wd*t+th2);
end
for j=1:3
    A=sqrt((wn^2+(2*zeta*wb)^2) /((wn^2-wb^2)^2
+(2*zeta*wn*wb)^2));
    phi1=atan2(2*zeta*wn*wb, (wn^2-wb^2));
    phi2=atan2(wn, (2*zeta*wb));
    xp(j,:)=wn*y0(j)*A*cos(wb*t -phi1-phi2);
end
if (xp(1,1)+y1(1,1)==xp(2,1)+y1(2,1)
==xp(3,1)+y1(3,1)==0)
    x=xp+y1;
else
    x=xp+y2;
end
for i=1:3
    subplot(3,1,i)
    plot(t,x(i,:))
    ylabel('Response x');
    title(['Base Excitation with
wb=',num2str(wb), ', wn=',num2str(wn), ' , and
y0=',num2str(y0(i))]);
    grid
end

```

```
xlabel('Time, seconds')
```

Program 3: varyzbe.m

```

%{This program solves the base excitation
problem. The code assumes a sinusoidal base
function. Also, the program tests three
different damping ratios. Again, note the
changes between this program and the previous
one.
%}
y0=input('Enter the base excitation magnitude.
');
wb=input('Enter the base excitation frequency.
');
wn=4;
tf=10;
t=0:tf/1000:tf;
for k=1:3
    zeta(k)=input('Enter a damping ratio (zeta).
');
    if (zeta(k)<0 | zeta(k)>=1)
        error('Damping ratio not in acceptable
range!')
    end
end
for m=1:3
    wd=wn*sqrt(1-zeta(m)^2);
    phi1=atan2(2*zeta(m)*wn*wb, (wn^2-wb^2));
    phi2=atan2(wn, 2*zeta(m)*wb);
    xi=phi1+phi2;
    Z1=(-zeta(m)*wn-wb*tan(xi)
+sqrt((zeta(m)*wn)^2 +2*zeta(m)*wn*wb*tan(xi)
+(wb*tan(xi))^2+wd^2))/wd;
    Z2=(-zeta(m)*wn-wb*tan(xi)
-sqrt((zeta(m)*wn)^2 +2*zeta(m)*wn*wb*tan(xi)
+(wb*tan(xi))^2+wd^2))/wd;
    Anum=sqrt((wn^2 +(2*zeta(m)*wb)^2) /((wn^2
-wb^2)^2 +(2 *zeta(m) *wb *wn)^2)) *wn*y0;
    Bnum1=(-wd*cos(xi) +Z1*zeta(m)*wn*cos(xi)
+Z1*wb*sin(xi));
    Bnum2=(-wd*cos(xi) +Z2*zeta(m)*wn*cos(xi)
+Z2*wb*sin(xi));
    Aden1=wd*Z1;
    Aden2=wd*Z2;
    A1=Anum*Bnum1/Aden1;
    A2=Anum*Bnum2/Aden2;
    th1=2*atan(Z1);
    th2=2*atan(Z2);
    y1(m,:)=A1*exp(-zeta(m)*wn*t)
.*sin(wd*t+th1);
    y2(m,:)=A2*exp(-zeta(m)*wn*t)
.*sin(wd*t+th2);
end

```

```

for j=1:3
    A=sqrt((wn^2+(2*zeta(j)*wb)^2)
/((wn^2-wb^2)^2 +(2*zeta(j)*wn*wb)^2));
    phi1=atan2(2*zeta(j)*wn*wb,(wn^2-wb^2));
    phi2=atan2(wn,(2*zeta(j)*wb));
    xp(j,:)=wn*y0*A*cos(wb*t-phi1-phi2);
end
if (xp(1,1)+y1(1,1)==xp(2,1)+y1(2,1)
==xp(3,1)+y1(3,1)==0)
    x=xp+y1;
else
    x=xp+y2;
end
for i=1:3
    subplot(3,1,i)
    plot(t,x(i,:))
    ylabel('Response x');
    title(['Base Excitation with wb=',
num2str(wb), ' and zeta=', num2str(zeta(i))]);
    grid
end
xlabel('Time, seconds')

```

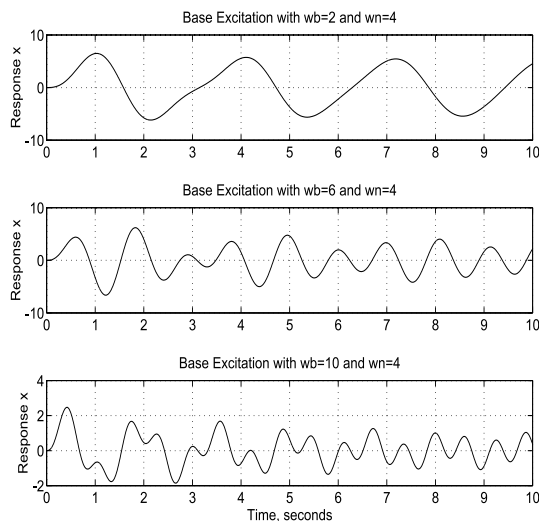


Figure C.15: Responses of a base-excited system subject to different excitation frequencies.

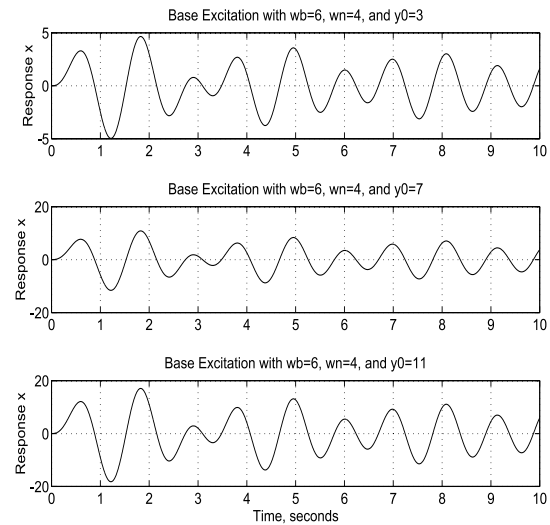


Figure C.16: Results obtained for three different base excitation magnitudes.

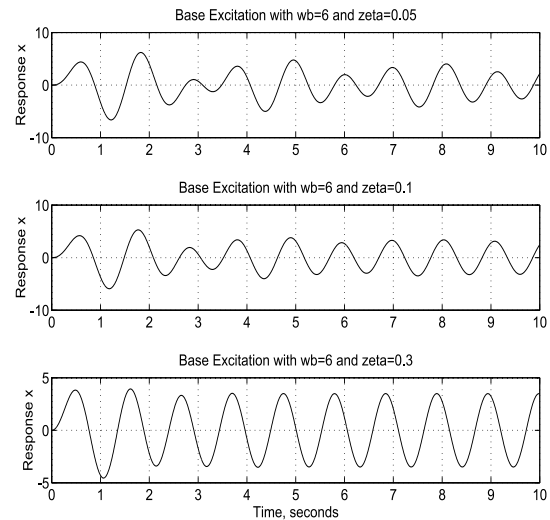


Figure C.17: Responses of a base-excited system for different damping ratios.

C.8 SDOF Damped System with Rotating Unbalance

A rotating unbalance is depicted in the model of Figure C.18. The radius e is measured from the center of mass of the mass m . To find the equation of motion, we need an expression for the motion of the rotating unbalance in terms of x . If the mass rotates with a constant angular velocity ω_r , then the circle it defines can be described parametrically as

$$x(t) = e \sin \omega_r t, \quad y(t) = e \cos \omega_r t. \quad (\text{C.31})$$

(Note that the sine defines the x coordinate because coordinate x is vertical.) The position coordinate of the rotating unbalance is $x + e \sin \omega_r t$, and the acceleration is the second derivative of this expression with respect to time. The acceleration of the mass without the unbalance is $\ddot{x}$. Adding in the effects of the stiffness and damper, the equation of motion is

$$(m - m_o) \ddot{x} + m_o \frac{d^2}{dt^2} (x + e \sin \omega_r t) = -kx - c\dot{x}, \quad (\text{C.32})$$

or

$$(m - m_o) \ddot{x} + m_o (\ddot{x} - e \omega_r^2 \sin \omega_r t) = -kx - c\dot{x}. \quad (\text{C.33})$$

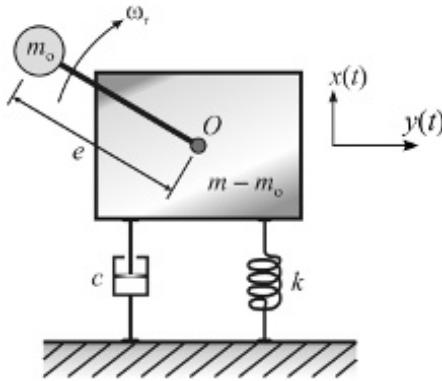


Figure C.18: Single degree-of-freedom damped system with rotating unbalance. (Note the coordinates chosen.)

Rearranging Equation C.33 and dividing by the system mass gives the equation of motion,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = m_o e \omega_r^2 \sin \omega_r t. \quad (\text{C.34})$$

Note that this is identical to the equation of motion for a single degree-of-freedom system with harmonic excitation, except that now our force is in the form of a sine rather than a cosine. The particular solution here is of the form,

$$x_p(t) = X \sin(\omega_r t - \phi), \quad (\text{C.35})$$

where, with $r = \omega_r / \omega_n$,

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \\ \phi = \tan^{-1} \frac{2\zeta r}{1 - r^2}. \quad (\text{C.36})$$

As before, the homogenous solution is

$$x_h(t) = A e^{-\zeta\omega_n t} \sin(\omega_d t + \theta), \quad (\text{C.37})$$

where A and θ are determined from the initial conditions. The final solution is then $x(t) = x_p(t) + x_h(t)$.

For the purpose of creating a program in MATLAB, the initial conditions were assumed to be zero. Then, Maple was used to obtain a solution, which is not reproduced here due to the complexity of the expression; the solutions for A and θ depend on the solution of a quadratic equation. The MATLAB code that follows contains the expression (in a few parts) for the constants in question.

Figures C.19 through C.21 show different varying parameter sets for the system. Unless otherwise specified, $m = 7$, $m_o = 3$, and $e = 0.1$. For Figure C.19, the natural frequency was varied while holding all other parameters constant. Notice that when ω_n is not a multiple of ω_r , the motion is the sum of two sinusoids; this is shown best by the top plot, where $\omega_n = 2$. For the highest natural frequency tested, the oscillation occurs along a single sinusoid. This is because the natural frequency of the system is too high to be excited by the relatively slow rotation frequencies. The first two plots have natural frequencies small enough to be excited by the slow rotation of the eccentric mass.

In Figure C.20, the system damping is varied. The result is that the transient portion (the portion with the curve that looks like the sum of sinusoids) becomes smaller, to the point where it disappears at $\zeta = 0.3$. A difference in the magnitude of oscillation, as would be predicted from the expression we have derived for the parameter X , is not present because the frequency ratio we are testing is in the range where oscillation magnitude shows little variation with damping ratio. This consideration is important in the design of machinery; if the machine can be designed to have a much higher natural frequency than the oscillating mass, then the level of damping can be made low without increasing the amplitude past acceptable levels.

Finally, Figure C.21 shows the variation of vibration with increasing system mass. Notice how the amplitude of the vibration decreases with increasing mass; this is due to the dependence of X on m_o/m . As the mass ratio decreases, so does the amplitude of vibration.

Program 1: vrywnrot.m

```
%{This program solves for the response of
a single degree-of-freedom system having a
rotating unbalance. The equations of motion
were derived using Maple and are valid only
for zero initial conditions.
%}
mo=3;
m=7;
e=0.1;
wr=4;
zeta=0.05;
tf=10;
t=0:tf/1000:tf;
for i=1:3
    wn(i)=input('Enter a natural frequency. ');
    wd(i)=wn(i)*sqrt(1-zeta^2);
end
for j=1:3
    r=wr/wn(j);
    X=mo*e/m*(r^2/sqrt((1-r^2)^2+(2*zeta*r)^2));
    phi=atan2(2*zeta*r,(1-r^2));
    Z1=(-zeta*wn(j)+wr*cot(phi))/wd(j);
    Z2=sqrt((zeta*wn(j))^2 -2*zeta*wn(j)
*wr*cot(phi) +(wr*cot(phi))^2+wd(j)^2)/wd(j);
    Z=Z1+Z2;
    theta=2*atan(Z);
    Anum=X*(wd(j)*sin(phi)
-Z*zeta*wn(j)*sin(phi) +Z*wr*cos(phi));
    Aden=Z*wd(j);
    A=Anum/Aden;
    xh(j,:)=A*exp(-zeta*wn(j)*t) .*sin(wd(j)*t
+theta);
    xp(j,:)=X*sin(wr*t-phi);
end
x=xp+xh;
for k=1:3
    subplot(3,1,k) plot(t,x(k,:))
    title(['Rotating Unbalance with wr=',
num2str(wr),' wn=', num2str(wn(k)),' and
zeta=',num2str(zeta)]);
    grid
end
xlabel('Time, seconds')
```

Program 2: vryzrot.m

```
%{This program solves for the response of
a single degree-of-freedom system having a
rotating unbalance. The equations of motion
were derived using Maple and are valid only
for zero initial conditions.
```

```
%}
mo=3;
m=7;
e=0.1;
wr=4;
wn=12;
tf=10;
t=0:tf/1000:tf;
for i=1:3
    zeta(i)=input('Enter a damping ratio (zeta).
');
end
for j=1:3
    wd=wn*sqrt(1-zeta(j)^2);
    r=wr/wn;
    X=mo*e/m*(r^2/sqrt((1-r^2)^2
+(2*zeta(j)*r)^2));
    phi=atan2(2*zeta(j)*r,(1-r^2));
    Z1=(-zeta(j)*wn+wr*cot(phi))/wd;
    Z2=sqrt((zeta(j)*wn)^2 -2*zeta(j)*wn
*wr*cot(phi) +(wr*cot(phi))^2+wd^2)/wd;
    Z=Z1+Z2;
    theta=2*atan(Z);
    Anum=X*(wd*sin(phi) -Z*zeta(j) *wn*sin(phi)
+Z*wr *cos(phi));
    Aden=Z*wd;
    A=Anum/Aden;
    xh(j,:)=A*exp(-zeta(j)*wn*t)
.*sin(wd*t+theta);
    xp(j,:)=X*sin(wr*t-phi);
end
x=xp+xh;
for k=1:3
    subplot(3,1,k)
    plot(t,x(k,:))
    title(['Rotating Unbalance with wr=',
num2str(wr),' wn=', num2str(wn), ' and
zeta=',num2str(zeta(k))]);
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')
```

Program 3: vrymrot.m

```
%{This program solves for the response of
a single degree-of-freedom system having a
rotating unbalance. The equations of motion
were derived using Maple and are valid only
for zero initial conditions.
%}
mo=3;
zeta=0.05;
e=0.1;
```

```

wr=4;
wn=12;
tf=10;
t=0:tf/1000:tf;
for i=1:3
    m(i)=input('Enter a system mass. ');
end
for j=1:3
    wd=wn*sqrt(1-zeta^2);
    r=wr/wn;
    X=mo*e/m(j)*(r^2/sqrt((1-r^2)^2
+ (2*zeta*r)^2));
    phi=atan2(2*zeta*r,(1-r^2));
    Z1=(-zeta*wn+wr*cot(phi))/wd;
    Z2=sqrt((zeta*wn)^2 -2*zeta*wn*wr*cot(phi)
+ (wr*cot(phi))^2+wd^2)/wd;
    Z=Z1+Z2;
    theta=2*atan(Z);
    Anum=X*(wd*sin(phi) -Z*zeta*wn*sin(phi)
+Z*wr*cos(phi));
    Aden=Z*wd;
    A=Anum/Aden;
    xh(j,:)=A*exp(-zeta*wn*t).*sin(wd*t+theta);
    xp(j,:)=X*sin(wr*t-phi);
end
x=xp+xh;
for k=1:3
    subplot(3,1,k)
    plot(t,x(k,:))
    title(['Rotating Unbalance with wr=',
num2str(wr),' ,wn=', num2str(wn),' , mass=',
num2str(m(k)),' , and rotating mass=',
num2str(mo)]);
    grid
end
xlabel('Time, seconds')

```

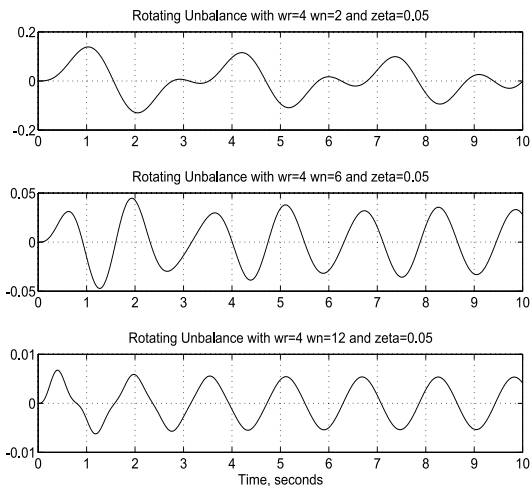


Figure C.19: Responses for different natural frequencies.

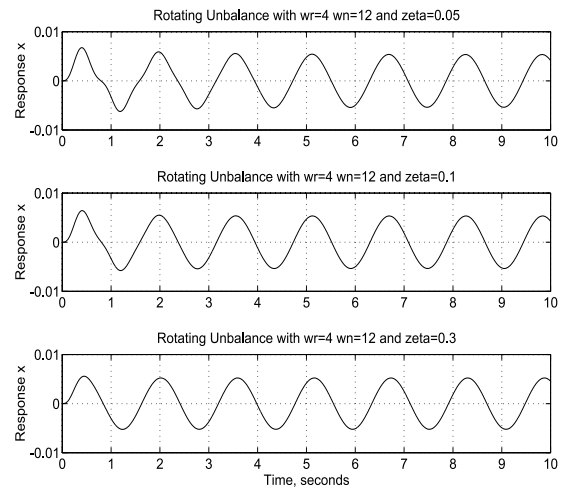


Figure C.20: Responses for different damping ratios.

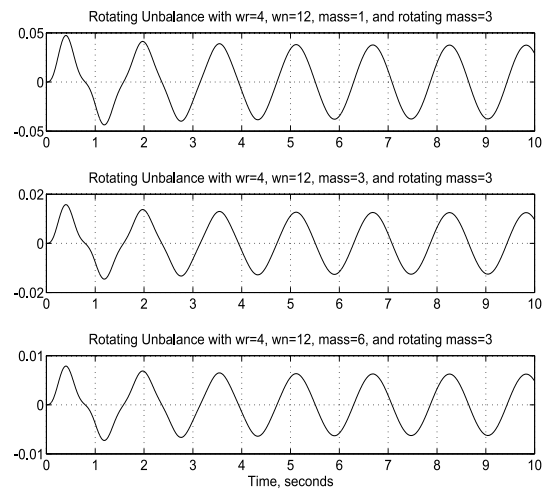


Figure C.21: Responses for different system masses.

C.9 SDOF Damped System with Impulse Input

In previous sections, we have investigated the response of single degree-of-freedom systems to different forms of sinusoidal inputs. Here and in the following sections, we will examine the effects of non-sinusoidal inputs on single degree-of-freedom systems. The simplest of these is the impulse response.

An impulse is a force that is applied over a very short time. Here, we take the duration over which the impulse is applied to be 2ϵ . If the impulse is centered about time t , then it is applied from $t - \epsilon$ to $t + \epsilon$. If the force has a total value of F_o , then the average value is $F_o/2\epsilon$. So, for all time except the interval around t , the value of the impulse is 0; within the interval, it is $F_o/2\epsilon$.

Since impulse is force multiplied by time and it is also the change in momentum, we can write

$$(F_o/2\epsilon) \cdot 2\epsilon = mv_o, \quad (\text{C.38})$$

where v_o is the initial velocity. (The velocity before the impulse is zero, and it is v_o after the impulse.) This problem reduces to a single degree-of-freedom free vibration with zero initial displacement and initial velocity equal to F_o/m . Recall that for a damped oscillator, the response is of the form,

$$x(t) = Ae^{-\zeta\omega t} \sin(\omega_d t + \phi). \quad (\text{C.39})$$

Since

$$A = \frac{\sqrt{(v_o + \zeta\omega x_o)^2 + (x_o\omega_d)^2}}{\omega_d}$$

$$\phi = \tan^{-1} \frac{x_o\omega_d}{v_o + \zeta\omega x_o},$$

where for $x_o = 0$, $A = v_o/\omega_d$, and $\phi = 0$. Thus, Equation C.39 becomes

$$x(t) = \frac{F_o}{\omega_d} e^{-\zeta\omega t} \sin(\omega_d t). \quad (\text{C.40})$$

This response, as noted above, is simply a single degree-of-freedom oscillator subject to an initial velocity. The behavior with changing F_o is similar to changing v_o in a single degree-of-freedom oscillator. For that reason, a program and plots for this situation are not included; they would be the same as those for the SDOF damped system.

Included here is a program that simulates the Dirac delta function (an impulse with $F_o = 1$), for an impulse around a given time and with a given total time interval. The program takes the entire interval and sets the function equal to zero for all times other than the one specified, and equal to one for the specified time.

Program 2 gives an example of how this delta function can be used inside of another program, by plotting

the function over a specified time interval. This program makes use of subfunctions as well. This is more handy for single-use functions, such as several of the unwieldy constants we derived for some of our earlier examples. A subfunction is called just like any other MATLAB function. The subfunction is written after the main code, and the variables used inside the subfunction stay in the subfunction; they are not introduced to the MATLAB workspace. Thus, if we tried to use the variable “int” in the main code of Program 2, we would generate an error.

One more item of note is the `\delta` in the ylabel function call. That is a LaTeX tag, which is usable inside of a MATLAB string. When the program is run, the y-axis reads “ $\delta(t_d)$,” substituting the value entered for t_d . Superscripts, subscripts, Greek characters, and other useful effects are available in this manner; typing `help latex` at the MATLAB prompt can give you a start on using these.

Program 1: delta.m

```
function delta=delta(td,tf)
%{This function simulates the delta function.
The user must input the time at which the
nonzero value is desired, td, and the ending
time, tf.
%}
int=tf*td;
%{The variable int ensures that a value in the
time vector will match the desired time. The
strategy is to subdivide the interval into
at least 500 steps (this ensures the interval
over which the function is nonzero is small).
%}
while int<500
    int=int*10;
end
t=0:tf/int:tf;
for i=1:int
    if t(i)==td
        delta(i)=1;
    else
        delta(i)=0;
    end
end
```

Program 2: subdemo.m

```
function [t,y]=subdemo(td,tf);
%
%{A simple routine to demonstrate use of
subfunctions, making use of the delta function
as an example. Inputs are the time of the
```

delta function, td, and the overall timespan, tf. Note that subfunctions cannot be used inside of script m-files. If we were to write Program 2 as a script (like most of our m-files to this point), the delta function would have to be external.

```
%}
y=subdelta(td,tf);
%{ Calling the subfunction version "subdelta"
guarantees we'll be calling the subfunction,
not the function from Program 1. Now that
we have the value of the delta function, we
need to create a time vector that goes from
zero to tf, and has the same number of points
as the vector y (else we will get a plotting
error). For this, we use the "max" and "size"
commands.
%}
len=max(size(y));
%{"size" returns the vector [nrows ncols],
corresponding to the number of rows and
columns in y. Taking the maximum will return
the longer dimension, and is a shortcut usable
for both row and column vectors.
%}
t=linspace(0,tf,len);
%{ "linspace" is much more natural a command
for this instance than the colon operator.
This will generate a vector with len
equally-spaced elements between 0 and tf.
%}
plot(t,y)
title('Delta Function Sample');
ylabel(['\delta(',num2str(td),')']);
xlabel('Time, seconds');
grid
function delta=subdelta(td,tf)
%{This function is Program 1. Note that a
subfunction is included with a "function"
call, just like an external function.
%}
int=tf*td;
while int<500
    int=int*10;
end
t=0:tf/int:tf;
for i=1:int
    if t(i)==td
        delta(i)=1;
    else
        delta(i)=0;
    end
end
```

C.10 SDOF Damped System with Step Input

The case we investigate here is that of a single degree-of-freedom damped system with a step force input. The input is assumed to be applied instantaneously, and as such the step input can be viewed as an *on-function*. The response of the system under this type of load is called the step response. We assume the system is underdamped and there are zero initial conditions. The equation of motion is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = F(t)/m, \quad (\text{C.41})$$

where

$$F(t) = \begin{cases} 0 & \text{if } 0 < t < t_o \\ F_o & \text{if } t \geq t_o \end{cases}. \quad (\text{C.42})$$

We solve the equation of motion using the convolution integral,

$$x(t) = \int_0^t F(\tau)g(t-\tau)d\tau. \quad (\text{C.43})$$

Recall that the convolution integral is derived by treating the force as an infinite series of impulse forces. The first fundamental theorem of calculus then allows the infinite series to be treated as the integral given above. The impulse response, of the previous section, is given by

$$x(t) = \frac{F_o}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t = F_o g(t), \quad (\text{C.44})$$

where

$$g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t. \quad (\text{C.45})$$

Therefore,

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t F(\tau) e^{\zeta\omega_n \tau} \sin \omega_d(t-\tau) d\tau. \quad (\text{C.46})$$

Since we have a general expression for $x(t)$, we can substitute our $F(t)$ into Equation C.46 to give

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \left\{ \int_0^{t_o} (0) e^{\zeta\omega_n \tau} \sin \omega_d(t-\tau) d\tau + \int_{t_o}^t F_o e^{\zeta\omega_n \tau} \sin \omega_d(t-\tau) d\tau \right\}. \quad (\text{C.47})$$

Since the first term inside the braces is zero, for $t < t_o$, the response of the system is zero. To find the response for other times, we must evaluate the second integral (by parts),

$$x(t) = \frac{F_o}{k} \left\{ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n(t-t_o)} \times \cos[\omega_d(t-t_o) - \phi] \right\}, \quad t \geq t_o, \quad (\text{C.48})$$

where $\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2})$. This equation is valid only for the time after the force is applied; the response is zero before the force is applied.

Figure C.22 shows the variation of the response with the force magnitude. As might be expected, the magnitude of the response is directly proportional to the magnitude of the external force. The magnitude of the external force also causes a second difference: note that when the oscillatory motion begins, it is not centered around zero. Instead, the mass oscillates around a displacement greater than zero. The value of this center point is also dependent on the magnitude of the external force (from the term 1 in the Equation C.48).

In Figure C.23, we vary the natural frequency. This causes two changes in the response. First, the rate of exponential decrease in the response (the effect of damping) is increased, that is, the response stabilizes more quickly. Second, the oscillation frequency decreases, since the natural frequency is directly related to the damped frequency.

Figure C.24 shows the changes caused by changing the damping ratio. With increasing damping ratio, the amount of time to damp out all vibration decreases. For the third ratio tested, $\zeta = 0.3$, the damping is sufficient to allow no oscillation around the new center point ($x = 1.5$). A second result, which is not immediately evident from the figure but follows from the mathematics, is that the phase angle changes with the damping ratio (note that ϕ is a function of only ζ).

The three programs used for this example follow below. The way in which the force array is created in Program 1 illustrates a convenient feature of MATLAB. In MATLAB, the greater-than operator applied to a vector will produce a vector with zeros where the relation is false, and ones where it is true. If we create a matrix having the results of $t > t_o$ in each row, and array-multiply this matrix by the three-by-*npts* force matrix, we will produce a matrix having step forcing in each row of the appropriate magnitude, activated at $t = t_o$.

The reader may then wonder if the loop calculating the response could have been removed in a similar manner. The answer is “yes,” and the programming is left to the reader. The **for** loop is one of the more time-consuming operations in MATLAB. For more complicated calculations, it is advantageous to remove these loops from the code; this process is called *vectorization* of code. Using the calculation below as an example, note first that the parameters **phi** and **wd** are calculated outside the loop. These quantities are constant for all forces. Thus, the time-dependent part of all three responses is independent of the forcing parameters. We could then create a matrix that has the time response in each row, and use array multiplication to include the force-dependent parameters. (This is the algorithm for the reader exercise

described above.)

Program 1: stepfm.m

```
%{This program calculates the step response
of a single degree-of-freedom system.
This version tests three different force
magnitudes. The program assumes a system mass
of 1.
%}
zeta=0.05;
tf=10;
npts=1000;
t=linspace(0,tf,npts);
to=2;
wn=12;
k=(wn)^2;
for i=1:3
    Fm(i,1)=input('Enter a force magnitude. ');
    % Forcing Fm to be a column vector.
end
Fint=Fm*ones(1,npts); % Fint is thus
3-by-npts.
%{ Now, we perform the logical operation t>to,
and make a 3-by-npts matrix of the result.
%}
qtest=t>to;
% Three rows of qtest.
fmult=[qtest;qtest;qtest];
Fo=Fint.*fmult; % The force matrix.
wd=wn*sqrt(1-zeta^2);
phi=atan2(zeta,sqrt(1-zeta^2));
for n=1:3
    A=Fo(n,:)/k;
    B=Fo(n,:)/(k*sqrt(1-zeta^2));
    x(n,:)=A-B.*exp(-zeta*wn*t) .*cos(wd*t-phi);
end
for l=1:3
    subplot(3,1,l)
    plot(t,x(l,:))
    title(['Response for wn=', num2str(wn),',
    , Fmax=', num2str(Fm(l)),', and time=',
    num2str(to)]);
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')
```

Program 2: stepwn.m

```
%{This program calculates the step response
of a single degree-of-freedom system. This
version tests three different natural
```



```

frequencies. The program assumes a system
mass of 1.
%}
Fm=5;
npts=1000;
zeta=0.05;
tf=10;
t=linspace(0,tf,npts);
to=2;
% Only one force vector here, so we can do
this here.
qtest=t>to;
Fo=Fm*qtest; % Scalar Fm * 1-by-npts vector.
for i=1:3
    wn(i)=input('Enter a natural frequency. ');
    k(i)=wn(i)^2;
end
% This could also be vectorized.
for n=1:3
    wd=wn(n)*sqrt(1-zeta^2);
    A=Fo/k(n);
    B=Fo/(k(n)*sqrt(1-zeta^2));
    phi=atan2(zeta,sqrt(1-zeta^2));
    x(n,:)=A-B.*exp(-zeta*wn(n)*t) .*cos(wd*t
-phi);
end
for l=1:3
    subplot(3,1,l)
    plot(t,x(1,:))
    title(['Response for wn=', num2str(wn(l)),',
Fmax=', num2str(Fm),', and time=',
num2str(to)]);
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')

k=(wn)^2;
for i=1:3
    zeta(i)=input('Enter a damping ratio (zeta).
');
end
% This could also be vectorized.
for n=1:3
    wd=wn*sqrt(1-zeta(n)^2);
    A=Fo/k;
    B=Fo/(k*sqrt(1-zeta(n)^2));
    phi=atan2(zeta(n),sqrt(1-zeta(n)^2));
    x(n,:)=A-B.*exp(-zeta(n) *wn*t) .*cos(wd*t
-phi);
end
for l=1:3
    subplot(3,1,l)
    plot(t,x(1,:))
    title(['Response for wn=', num2str(wn),',
zeta=', num2str(zeta(l)),', and time=',
num2str(to)]);
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')

```

Program 3: stepzeta.m

```

%This program calculates the step response
%of a single degree-of-freedom system. This
%version tests three different natural
frequencies.
%The program assumes a system mass of 1.
%
npts=1000;
Fm=5;
tf=10;
t=linspace(0,tf,npts);
to=2;
qtest=t>to;
Fo=Fm*qtest;
wn=12;

```

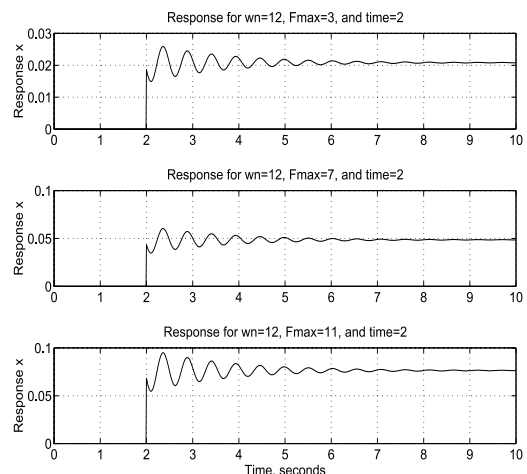


Figure C.22: Responses for different step magnitudes.

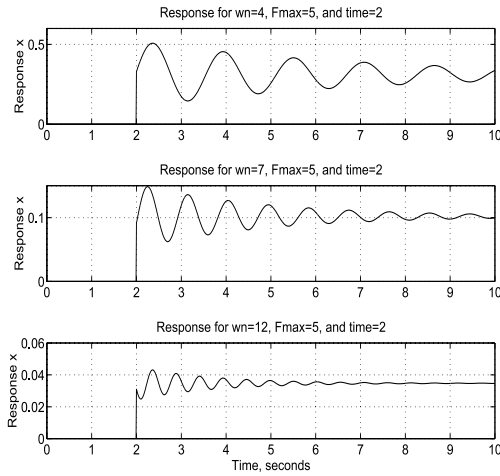


Figure C.23: Responses for different natural frequencies.

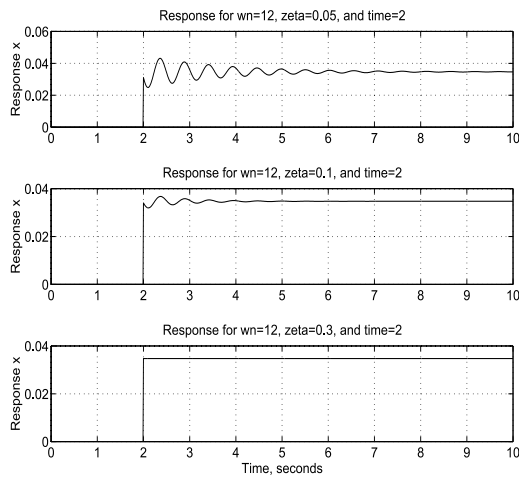


Figure C.24: Responses for different damping ratios.

C.11 SDOF Damped System with Square Pulse Input

A square pulse is a single pulse of constant magnitude and finite duration. To analyze the response of a system to a square wave input, we will treat the square wave as the sum of two equal and opposite step inputs applied at different times. The time interval between application of the step inputs is the duration of the square wave.

Let us assume that the magnitude of the square wave is F_o , and its duration is t_1 seconds. To simulate the wave using step inputs, we begin with a step input of magnitude F_o from time $t = 0$, and add to it at time t_1 a step input of magnitude $-F_o$. By superposition, the total response is the sum of the response of the system to each step input.

The response of a single degree-of-freedom system to a step input of magnitude F_m applied at time t_o is

$$x(t) = \frac{F_m}{k} \left\{ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n(t-t_o)} \times \cos(\omega_d(t-t_o) - \phi) \right\}, t \geq t_o, \quad (\text{C.49})$$

where $\phi = \arctan(\zeta/\sqrt{1-\zeta^2})$. If we now consider the two step inputs separately, denoting the response of the system to the input at time $t = 0$ as $x_1(t)$ and the response to the input at time $t = t_1$ as $x_2(t)$, we find that

$$x_1(t) = \frac{F_o}{k} \left\{ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \right\}, t \geq 0 \quad (\text{C.50})$$

and

$$x_2(t) = -\frac{F_o}{k} \left\{ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n(t-t_1)} \times \cos(\omega_d(t-t_1) - \phi) \right\}, t \geq t_1. \quad (\text{C.51})$$

The total response is then:

$$x(t) = \frac{F_o}{k} \left\{ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \times \cos(\omega_d t - \phi) \right\}, 0 \leq t < t_1, \quad (\text{C.52})$$

$$x(t) = \frac{F_o e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}} e^{\zeta\omega_n t_1} \{ \cos(\omega_d(t-t_1) - \phi) - \cos(\omega_d t - \phi) \}, t \geq t_1. \quad (\text{C.53})$$

Notice that, for the time interval after t_1 , the response no longer includes a “1-” term; the addition of the two responses has removed this term entirely. Again recalling a previous example, the “1-” term caused the oscillation to be about a new equilibrium (i.e., $x = F_o/k$). Since the

term has disappeared, the oscillation is centered around zero.

The movement of the center point of the oscillation is best shown in Figure C.25, which tests three different values of F_o . The oscillation begins about a center point at $x = F_o/k$. When the square wave ends, or, when the equal and opposite step is added, the center point returns to zero. Assume, for a moment, that the magnitude of the second step is not equal to that of the first; call it F_1 . In this case, the center point of the oscillation after adding the second step input would be at $x = (F_o - F_1)/k$. To prove this to yourself, perform the superposition used to obtain Equation C.53, above (remember that F_1 is negative!). A constant term will remain, and this term will be equal to $(F_o - F_1)/k$. Another result which is evident from Figure C.22 is that the change in F_o causes the magnitude of the oscillations to increase, as would be expected from Equation C.53.

Figure C.26 shows the effects of changing the natural frequency. Notice that a transition point occurs when the second step input is added. The sudden shift in vibration characteristics is expected, since we have a piecewise expression for $x(t)$. Although the transition becomes more abrupt as the natural frequency increases, it is never discontinuous, a physically impossible situation.

Figure C.27 demonstrates the response behavior for different damping ratios. Again, notice how the high damping ratio ($\zeta = 0.3$) causes the vibration to be damped out quickly, so that the mass is practically at rest when the second step input is applied. Again, we see that the transient period decreases with increasing damping ratio.

The MATLAB code to produce these figures demonstrates the vectorization discussed in the previous case. The logical operation $t < t_o$ is used to create a matrix of zeros and ones, where the ones correspond to times before t_o . Then the MATLAB “not” operator (the tilde, $\sim$) is used to arrive at times after t_o . The reader is encouraged to rewrite the vectorized calculation as a loop (in any of the three codes) and see the difference in time expenditure as the number of points is increased by a few orders of magnitude. While there are no fewer actual calculations performed in the vectorized code, the difference is that MATLAB’s vector/matrix calculations are highly optimized, and so the code is faster when handling one matrix calculation instead of many scalar calculations.

One more difference between these codes and previous ones is that we use the colon operator. As explained in Program 1, the statement $\mathbf{a}=\mathbf{b}(:)$; has a particular meaning in MATLAB. For a vector $\mathbf{b}$, MATLAB will return a column vector $\mathbf{a}$ that is equal to a column vector $\mathbf{b}$ or the transpose of a row vector $\mathbf{b}$. That is, if we wish to be certain whether a vector we are working with is a column vector, then we can use the colon operator in this manner to force it to be so; the output of $\mathbf{a}=\mathbf{b}(:)$ is

a column vector for *any* vector $\mathbf{b}$.

Another interesting consequence of this use of the colon operator is the result for a *matrix* $\mathbf{b}$. The colon operator would return a column vector $\mathbf{a}$ consisting of the rows of $\mathbf{b}$ stacked on top of each other. The reader is encouraged to try this, and see the results. Additionally, what would we do if we wanted to force $\mathbf{a}$ to be a row vector? Finally, what if we wished to make these scripts instead functions, taking the number of points in the time vector as inputs? How would we modify the code then? The advantage would be that all the variables from the function would not become part of our workspace. This can be a disadvantage from a debugging perspective, but is well worth having working code.

Program 1: sqrewn.m

```
%{ This program finds the response of a single
degree-of-freedom system to a square wave
input. The wave is assumed to begin at t=0,
and lasts until t=to. The system mass is
again assumed to be equal to 1.
%}
npts=1000;
Fm=5;
zeta=0.05;
to=3;
tf=10;
t=linspace(0,tf,npts);
for i=1:3
    wn(i)=input('Enter a natural frequency. ');
    k(i)=sqrt(wn(i));
end
%{ Demonstrating another use of the colon
operator. In this sense, wn(:) turns wn into
a column vector if it was a row vector, and
leaves it as a column vector if it was one
already. Either way, we know the result will
be a column vector wn.
%}
wn=wn(:);
k=k(:);
%{ To use the logical operators, we need to
create a vector that is one for t<to, and turn
it into a matrix.
%}
qtest=t<to;
qpiece=[qtest;qtest;qtest]; % 3-by-npts
matrix.
% Now, we need a similar sized matrix for the
force magnitude.
Fo=Fm*ones(3,npts); % 3-by-npts, again.
q=sqrt(1-zeta^2);
```

```
%{ In order to calculate the response in a
vectorized manner, we'll need a wn, k, and
wd for each point in time and each different
value. That can be done by multiplying the
3-by-1 matrices we get from our input and
calculation by ones(1,npts).
%}
wd=wn*q; % 3-by-1 matrix.
wnmat=wn*ones(1,npts);
wdmat=wd*ones(1,npts);
kmat=k*ones(1,npts);
phi=atan2(zeta,q);
A=Fm./kmat; % We need a stiffness for each
force/time point.
tmat=[t;t;t]; % And a 3-by-npts time matrix.
x1=A.*(1-exp(-zeta*wnmat.*tmat)
.*cos(wdmat.*tmat-phi)/q);
x2=A.*exp(-zeta*wnmat.*tmat)
.*(exp(zeta*wnmat.*to)
.*cos(wdmat.*(tmat-to)-phi)
-cos(wdmat.*tmat-phi))/q;
x=x1.*qpiece+x2.*(~qpiece); % "not" qpiece.
for l=1:3
    subplot(3,1,l)
    plot(t,x(l,:))
    title(['Square Wave Response for wn=',
num2str(wn(l)),' , to=',num2str(to),' ,and
Fm=', num2str(Fm)])
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')
```

Program 2: sqrefm.m

```
%{This program finds the response of a gsingl
dereed-of-freedom system to a square wave
input. The wave is assumed to begin at t=0,
and lasts until t=to. The system mass is
again assumed to be equal to 1.
%}
npts=1000;
wn=11;
k=(wn)^2;
zeta=0.05;
to=3;
tf=10;
t=linspace(0,tf,npts);
for i=1:3
    Fm(i)=input('Enter a force magnitude. ');
end
Fm=Fm(:); % See above.
% As with the step response example earlier,
we can use MATLAB's logical operators to
```

create a piecewise function without resorting to a loop. This can be done with the statement: $qtest=(t>t1)-(t>t2)$, when the square wave begins at $t=t1$ and ends at $t=t2$. For the case of $t1=0$, a simpler statement is valid:

```
%}
qtest=(t<to);
Fo=Fm*ones(1,npts); % Since solution is
piecewise, we need Fm throughout.
%{ This code demonstrates vectorization of
this piecewise expression. We'll be making
use of the qtest vector above to tell the code
when to activate each solution.
%}
% One row for each Fm value.
qpiece=[qtest;qtest;qtest];
q=sqrt(1-zeta^2);
wd=wn*q;
phi=atan2(zeta,q);
A=Fo/k; % 3-by-npts matrix.
tmat=[t;t;t]; % Again, 3-by-npts.
% Solution for t<to.
x1=A.*(1-exp(-zeta*wn*tmat)
.*cos(wd*tmat-phi)/q);
x2=A.*exp(-zeta*wn*tmat) .* (exp(zeta*wn*to)
.*cos(wd*(tmat-to)-phi) -cos(wd*tmat-phi))/q;
x=x1.*qpiece+x2.*(~qpiece);
% ~is the "not" operator.
for l=1:3
    subplot(3,1,l)
    plot(t,x(l,:))
    title(['Square Wave Response for wn=',
num2str(wn),' , to=',num2str(to),' ,and Fm=',
num2str(Fm(1))])
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')
```

Program 3: sqrez.m

```
%{ This program finds the response of a single
degree-of-freedom system to a square wave
input. The wave is assumed to begin at t=0,
and lasts until t=to. The system mass is
again assumed to be equal to 1.
%}
npts=1000;
wn=11;
k=(wn)^2;
Fm=7;
to=3;
tf=10;
```

```

t=linspace(0,tf,npts);
for i=1:3
    zeta(i)=input('Enter a damping ratio (zeta).
');
end
zeta=zeta(:); % Once more.

%{ Again we will vectorize the code by using
the logical operators on t and to. This
time it will be somewhat easier, since the
stiffness and force are scalar. We will need
matrices for any quantities derived from the
damping ratio, and for time.
%}

qtest=t<to;

qpiece=ones(3,1)*qtest; % Another way to
create the matrix.

tmat=ones(3,1)*t;

zmat=zeta*ones(1,npts); % 3-by-npts.

q=sqrt(1-zmat.^2); % 3-by-npts. Note the
array square operator.

wd=wn*q; % wd is 3-by-npts.

phi=atan2(zmat,q); % MATLAB will handle this
on an element-by-element basis.

A=Fm/k; % Scalar.

x1=A*(1-exp(-zmat*wn.*tmat).*cos(wd.*tmat
-phi))./q);

x2=A*exp(-zmat*wn.*tmat) .* (exp(zmat*wn*to)
.*cos(wd.*(tmat-to) -phi) -cos(wd.*tmat
-phi))./q;

x=x1.*qpiece+x2.*(~qpiece);
for l=1:3
    subplot(3,1,l)
    plot(t,x(1,:))
    title(['Square Wave Response for wn=',
num2str(wn),', to=', num2str(to),', and
zeta=', num2str(zeta(l))])
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')

```

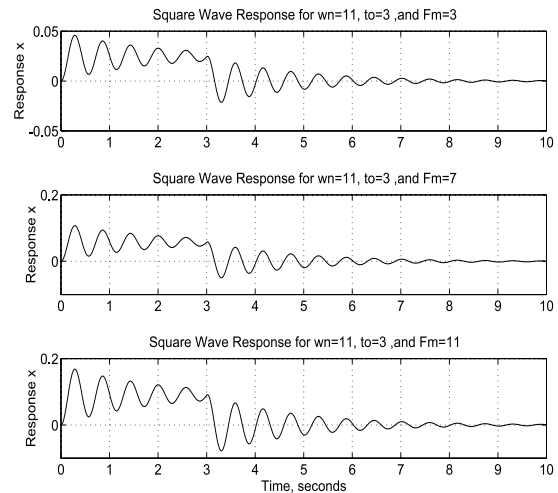


Figure C.25: Responses for different force magnitudes.

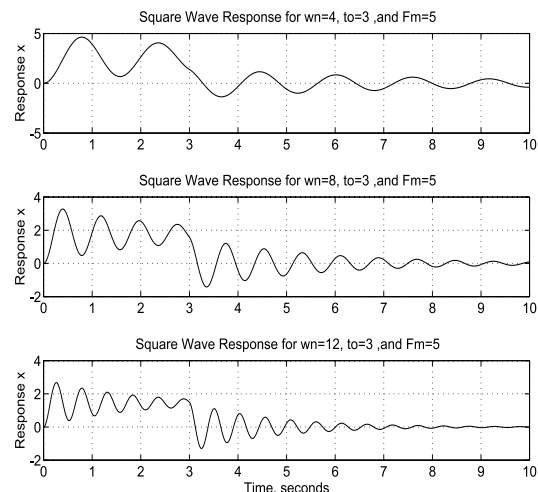


Figure C.26: Responses for different natural frequencies.

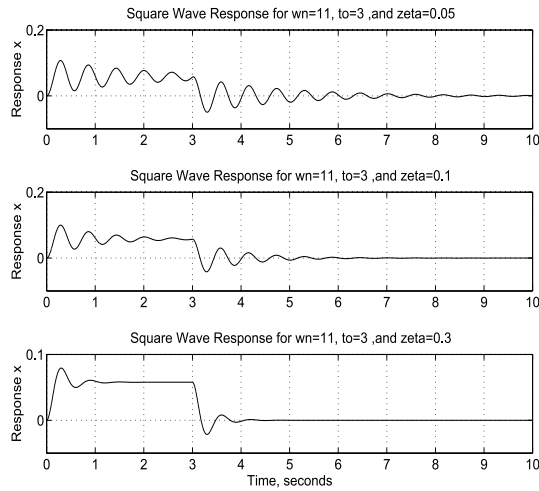


Figure C.27: Responses for different damping ratios.

C.12 SDOF Damped System with Ramp Input

To determine the response of a single degree-of-freedom damped system to a ramp input, we apply the convolution integral. Assuming that the load is increased uniformly at a rate of f_o and reaches its maximum at time t_d , then the expression for the external force is

$$F(t) = \begin{cases} f_o t & \text{for } 0 \leq t < t_d \\ f_o t_d & \text{for } t \geq t_d. \end{cases} \quad (\text{C.54})$$

Substituting the expression for $F(t)$ into the convolution integral yields

$$x(t) = \begin{cases} \frac{f_o}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \tau e^{\zeta\omega_n \tau} \sin \omega_d(t-\tau) d\tau & \text{for } 0 \leq t < t_d \\ \frac{f_o}{m\omega_d} e^{-\zeta\omega_n t} \left[\int_0^{t_d} \tau e^{\zeta\omega_n \tau} \sin \omega_d(t-\tau) d\tau + t_d \int_{t_d}^t e^{\zeta\omega_n \tau} \sin \omega_d(t-\tau) d\tau \right] & \text{for } t \geq t_d, \end{cases} \quad (\text{C.55})$$

where m is the mass, ω_n is the undamped natural frequency, ω_d is the damped natural frequency, and ζ is the damping ratio. Equation C.55 is evaluated using a symbolic manipulator, such as Maple, or a table of integrals, giving

$$\begin{aligned} x(t) &= \frac{f_o}{m\omega_d} e^{-\zeta\omega_n t} \frac{1}{(\zeta^2\omega_n^2 + \omega_d^2)^2} \\ &\quad \times \{ \omega_d e^{\zeta\omega_n t} (t\zeta^2\omega_n^2 + t\omega_d^2 - 2\zeta\omega_n) \\ &\quad + 2\zeta\omega_n\omega_d \cos(\omega_d t) + (\zeta^2\omega_n^2 - \omega_d^2) \sin(\omega_d t) \} \\ &\quad \text{for } 0 \leq t < t_d \\ \\ x(t) &= \frac{f_o}{m\omega_d} e^{-\zeta\omega_n t} \frac{1}{(\zeta^2\omega_n^2 + \omega_d^2)^2} \\ &\quad \times \{ \omega_d e^{\zeta\omega_n t} (t\zeta^2\omega_n^2 + t\omega_d^2 - 2\zeta\omega_n) \\ &\quad + 2\zeta\omega_n\omega_d \cos(\omega_d t) + (\zeta^2\omega_n^2 - \omega_d^2) \sin(\omega_d t) \} \\ &\quad + \frac{f_o t_d}{k} \\ &\quad - \frac{f_o t_d}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n(t-t_d)} \cos(\omega_d(t-t_d) - \phi) \\ &\quad \text{for } t \geq t_d, \end{aligned} \quad (\text{C.56})$$

where k represents the stiffness.

This solution is reflected in the MATLAB code that follows. We note that for this solution there is no equilibrium position, as seen for the step and square wave responses, until after the input has leveled off. This is because the constant that creates the new center point is the result of an integration that does not start at zero,

and no such integration exists in this solution until after t_d .

From Figure C.28, one might infer that the transition to the new equilibrium of vibration is discontinuous, as it was in the step and square wave responses. Notice how there seems to be no response for the first few seconds, until the load is fully applied, and then begins oscillating, as in the step response. At this point, it is reasonable to assume that the ramp response and step response of a single degree-of-freedom system are similar.

However, Figure C.29 shows that this is not the case. This figure shows only the response during the transient loading period. Notice how the system is oscillating during this period, around a constantly increasing equilibrium. That is, if a line were drawn through the identical point on each period of the sinusoid, the result would be a line of positive slope. This shows that the ramp response does have a subtle difference from the step response; the ramp response has less deflection at the point in time that the full load is applied than the step response.

The MATLAB code follows the same general structure we have seen previously. We vectorize the code, so that we apply MATLAB's strength (vector math) to the problem, avoiding the slower loops. Note that both portions of the piecewise solution for $x(t)$ are multiplied by the same constant factor. Thus, we initialize the variable `a1` in Program 1 once. Also, we use vector multiplication to construct two solution matrices over the entire time interval. Finally, we make use of MATLAB's logical operators to assemble the full piecewise solution.

Program 1: rampfo.m

```
%{This program solves the ramp response of a
single degree-of-freedom system. The external
force is assumed of the form  $F(t)=fo*t$ , and is
assumed to last for ten seconds, after which
it levels off at  $F=fo*te$  until infinity. Again,
a unit system mass is assumed.
%}
npts=1000;
wn=5;
zeta=.05;
wd=wn*sqrt(1-zeta^2);
k=wn^2;
te=4;
for kr=1:3
    fo(kr)=input('Enter a force magnitude (fo).
');
end
fo=fo(:); % Force fo to be a column vector.
tf=10;
t=linspace(0,tf,npts);
qtest1=t<te;
```

```
qtest2=t>te;
% This time, we need two conditional matrices.
amult1=[qtest1;qtest1;qtest1];
amult2=[qtest2;qtest2;qtest2];
% This constant is used for both parts.
a1=fo/(k*(zeta^2*wn^2+wd^2)^2);
% The solution for  $t < te$ .
num1=t*(zeta*wn)^2+t*wd^2-2*zeta*wn;
num2=exp(-zeta*wn*t).*(2*zeta*wd*wn*cos(wd*t)
+((zeta*wn)^2-wd^2)*sin(wd*t));
x1=a1*(num1+num2);
% (3-by-1) *(1-by-npts) = (3-by-npts).
% Now, the solution for  $t > te$ .
num3=te*(zeta*wn)^2+te*wd^2-2*zeta*wn;
num4=exp(-zeta*wn*te).*(2*zeta*wd*wn*cos(wd*te)+
...
((zeta*wn)^2-wd^2)*sin(wd*te));
a=fo*te/k;
q=sqrt(1-zeta^2);
phi=atan2(zeta,q);
r=a/q*(1-exp(-zeta*wn*(t-te)).*cos(wd*(t-te)
-phi));
s=a1*(num3+num4)*ones(1,npts);
% num3 and num4 are scalars, so we must
vectorize this.
x2=r+s;
x=x1.*amult1+x2.*amult2;
% This gives a matrix of three 1-by-npts
solution vectors.
for i=1:3
    subplot(3,1,i)
    plot(t,x(i,:))
    title(['Response for wn=',num2str(wn),',
fo=', ...
num2str(fo(i)), ' zeta=',num2str(zeta)])
    ylabel('Response x')
    grid
end
xlabel('Time, seconds')
```

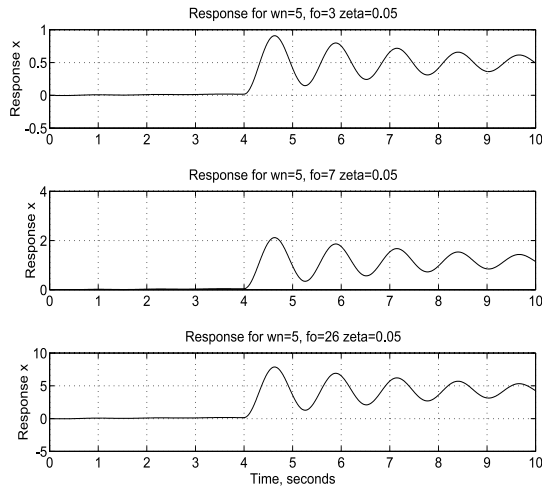


Figure C.28: Responses for different rates of loading.

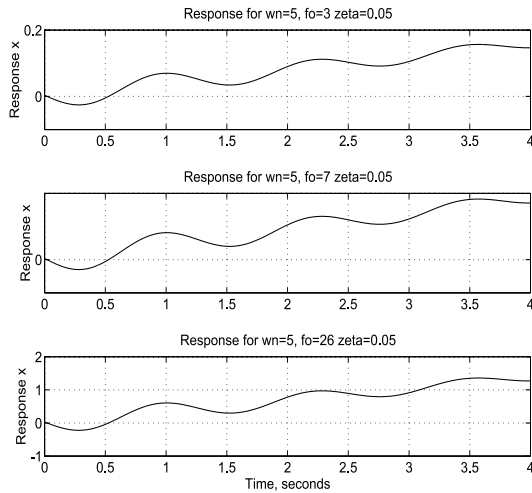


Figure C.29: Responses focusing on the first few seconds of oscillation, showing that the system does oscillate during the transient period.

C.13 SDOF System with Arbitrary Periodic Input

An arbitrary periodic input can be thought of as any periodic input that cannot be expressed as a single sine or cosine function. For example, a sum of sines and cosines is an arbitrary periodic input. The classic method to analyze the response to a periodic input is the Fourier series. A Fourier series representation of an arbitrary periodic function can take the form,

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_T t + b_n \sin n\omega_T t), \quad (\text{C.57})$$

where T is the period of the function, and $\omega_T = 2\pi/T$. The coefficients are obtained by using the orthogonality properties of harmonic functions,

$$a_n = \frac{2}{T} \int_0^T F(t) \cos n\omega_T t dt, \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{2}{T} \int_0^T F(t) \sin n\omega_T t dt, \quad n = 1, 2, \dots$$

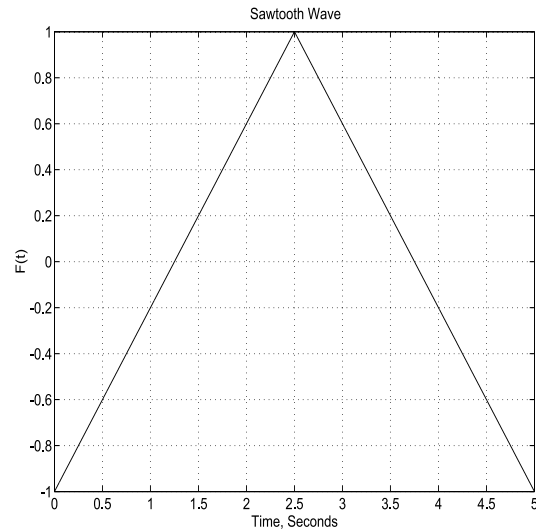


Figure C.30: Sawtooth wave to be defined by a Fourier series. One period is shown.

These coefficients can be evaluated using the well-known integrals,

$$\int_0^T \sin n\omega_T t \sin m\omega_T t dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$

$$\int_0^T \cos n\omega_T t \cos m\omega_T t dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$

$$\int_0^T \cos n\omega_T t \sin m\omega_T t dt = 0.$$

As an example, Figure C.30 shows a plot of a sawtooth wave with period $T = 5$. The wave is defined by:

$$F(t) = \begin{cases} \frac{4}{T}t - 1 & \text{for } 0 \leq t \leq \frac{T}{2} \\ 1 - \frac{4}{T}(t - \frac{T}{2}) & \text{for } \frac{T}{2} \leq t \leq T. \end{cases} \quad (\text{C.58})$$

The Fourier coefficients that define this function are $a_0 = 0$,

$$b_n = \frac{2}{T} \left\{ \int_0^{T/2} \left(\frac{4}{T}t - 1 \right) \sin n\omega_T t dt + \int_{T/2}^T \left[1 - \frac{4}{T} \left(t - \frac{T}{2} \right) \right] \sin n\omega_T t dt \right\}. \quad (\text{C.59})$$

Using Maple, we find that that this reduces to

$$b_n = \frac{4 \sin(n\pi) - 2 \sin(2n\pi) + n\pi \cos(2n\pi)}{n^2 \pi^2} - \frac{1}{n\pi},$$

where n is an integer, $\sin(n\pi) = \sin(2n\pi) = 0$, and $\cos(2n\pi) = 1$. Thus,

$$b_n = \frac{n\pi}{n^2 \pi^2} - \frac{1}{n\pi} = 0.$$

The only nonzero terms that describe this wave are the a_n terms,

$$a_n = \frac{2}{T} \left\{ \int_0^{T/2} \left(\frac{4}{T}t - 1 \right) \cos n\omega_T t dt + \int_{T/2}^T \left[1 - \frac{4}{T} \left(t - \frac{T}{2} \right) \right] \cos n\omega_T t dt \right\},$$

or

$$a_n = \frac{4 \cos n\pi - 2 \cos 2n\pi - n\pi \sin 2n\pi - 2}{n^2 \pi^2}. \quad (\text{C.60})$$

Since n is an integer, $\sin(2n\pi) = 0$ for all n . For even n , $a_n = 0$. For odd n , $a_n = -8/n^2 \pi^2$. Therefore,

$$F(t) = -\frac{8}{\pi^2} \left[\cos \frac{2\pi}{T}t + \frac{1}{9} \cos \frac{6\pi}{T}t + \frac{1}{25} \cos \frac{10\pi}{T}t + \dots \right]. \quad (\text{C.61})$$

To evaluate this expression using MATLAB, we would need to perform two simple steps. First, we would need to find a point where the terms of the series become negligible, to our desired level of accuracy. Then, we would need to solve for each term of the infinite series and superpose these results. This process is possible, but the MATLAB program needed to perform this would be long and involved. We take a different approach. The

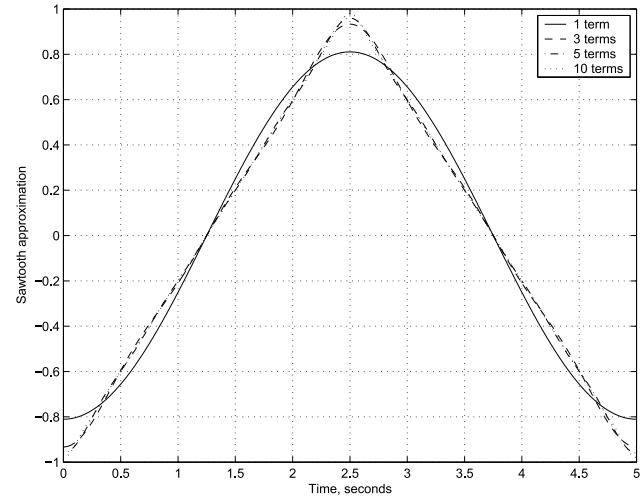


Figure C.31: Four sawtooth approximations. Note that 10 terms provide a very good approximation to the actual function.

program below shows that a few terms of the infinite series can provide a very close approximation to the total series.

This program is written as a function, taking as input the desired number of terms in the Fourier series. The time span is presumed to be 5 seconds. In Figure C.31, we plot the results for several numbers of terms to show the convergence. To get different sawtooth waves, one would have to either change T (to get a different period) or add a premultiplying factor (to change the amplitude). Such improvements are left to the reader.

Program 1: foursaw.m

```
function [t,F]=foursaw(nterm)
%{ This function returns the Fourier
representation for a sawtooth wave having
nterm number of nonzero Fourier coefficients,
as well as a corresponding time vector (for
ease in plotting).
%}
t=linspace(0,5,500);
T=5; % Period of 5 seconds.
F=zeros(size(t));
if nterm<1
    error('Number of terms must be a positive
integer!')
end
for i=1:nterm
    n=2*i-1; % Getting odd values only.
    F=F+(-8/pi^2)*cos(2*pi*n/T*t)/n^2;
end
plot(t,F)
```

C.14 MDOF Undamped System

The simplest multi degree-of-freedom system is the two degree-of-freedom undamped system in free vibration. The model of the system is shown in Figure C.32. We need two coordinates to define the motion. Applying Newton's second law to each mass results in two coupled equations of motion,

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \quad (\text{C.62})$$

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0. \quad (\text{C.63})$$

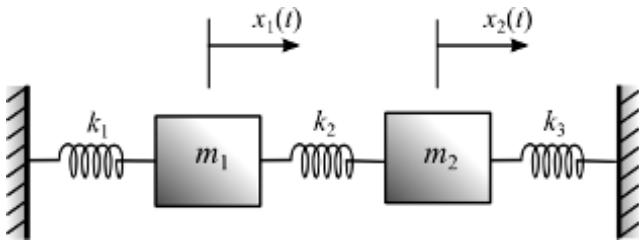


Figure C.32: Two degree-of-freedom undamped free oscillator.

To solve these equations, it would be advantageous if we could find a coordinate system in which the equations were not coupled. Then, the problem would reduce to two single degree-of-freedom systems, which we have solved already. Prior to showing the procedure of decoupling, we write the equations of motion in matrix notation,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \mathbf{x} = \mathbf{0}, \quad (\text{C.64})$$

where $\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$, and $\ddot{\mathbf{x}} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$. The modal analysis procedure may then be summarized as follows, where we denote the matrix multiplying $\ddot{\mathbf{x}}$ by M (the mass matrix) and the matrix multiplying $\mathbf{x}$ by K .

1. Find the eigenvalues and matrix of eigenvectors for the matrix $K - \lambda M$. Let this matrix of eigenvectors be U , and denote its columns (the eigenvectors) by $\{u\}_i$.

2. Normalize the eigenvectors with respect to the mass matrix through the products $\{u\}_i^T M \{u\}_i = 1$, $\{u\}_i^T K \{u\}_i = \omega_i^2 = \lambda_i$, where ω_i is the natural frequency of mode i .

3. From the solution of the motion in the decoupled modal coordinates,

$$\mathbf{x}(t) = \sum_{i=1}^2 \{u\}_i \left(\{u\}_i^T M \mathbf{x}(0) \cos \omega_i t + \frac{1}{\omega_i} \{u\}_i^T M \dot{\mathbf{x}}(0) \sin \omega_i t \right). \quad (\text{C.65})$$

The main benefit of this procedure (from a MATLAB standpoint) is that it is easily extended to n degrees of freedom by changing the limit of the summation in Equation C.65 from 2 to n . We would also have n natural frequencies and eigenvectors, but the procedures for finding them (solving the eigenvalue problem) and normalizing the eigenvectors with respect to the mass matrix are unchanged. The following program, `modfree.m`, is capable of solving n degree-of-freedom free vibration through the modal analysis procedure above.

The figures show the results for different masses, stiffnesses, and initial conditions. (Since this is free vibration, zero initial conditions will lead to zero motion.) For the first three figures, $m_1 = 1$, $m_2 = 4$, $k_1 = k_3 = 10$, $k_2 = 2$, and the initial velocity is zero. Since the masses and stiffnesses are the same, the natural frequencies for these cases are identical: $\omega_1 = 3.480$, and $\omega_2 = 1.700$. For Figure C.33, the initial displacement of each mass is 1. In this case, the second mode of vibration seems to show little evidence of coupling to the first mode; it looks like a single sinusoid, instead of a sum of sinusoids. This is due to the difference in mass values; the second mass is four times the first.

Figure C.34 shows the results for zero initial velocity and initial displacements of 3 and -2 for the masses, respectively. Again, the second mode is vibrating as if a single sinusoid; since the mass ratio is unchanged, the mode behavior is as well. If we eliminated an initial displacement from the second mass, we would expect the system to vibrate in both modes, at least in the second degree of freedom. Figure C.35 shows this is the case. The first mode vibrates only at one frequency; the displacement of the second mass (and its velocity) is too small to impart any appreciable force on the first mass.

We can further explore the effects of the mass on the mode shapes. The simulation shown in Figure C.36 was the result of interchanging the masses; the mass of 4 units is in the first degree of freedom, and the mass of 1 unit is in the second. The first mode shape is now a single sinusoid, and the second is a sum. A peculiarity of this case is that the system is symmetric, that is, the response of the first mode in Figure C.33 matches that of the second mode in Figure C.36, and vice versa. This is due to the symmetric stiffness matrix. If we change the value of either k_1 or k_3 , the symmetric behavior would no longer exist.

Figure C.37 shows the effects of adding an initial velocity to the system. As expected from a single degree-of-freedom system, the initial velocity affects the slope of the response curve at the starting point. It has no effect on the qualitative aspects of the mode shapes or on the natural frequencies. Also, the maximum amplitude of vibration is increased for both modes.

Figure C.38 shows the result of changing the masses to $m_1 = 3$, $m_2 = 9$, and as before the initial displace-

ments are both 1 and the initial velocities are 0. Now, $\omega_1 = 2.014$, and $\omega_2 = 1.131$. These reduced natural frequencies are reflected in the figure; when compared to Figure C.33, the curves of Figure C.38 have a lower frequency. The curves look a little more “spread out” than those in Figure C.33. However, the second mode shape is still a single sinusoid, not a sum like the first shape.

Figure C.39 shows the result of changing the stiffnesses to $k_1 = k_3 = 1$, $k_2 = 3$, and as before the initial displacements are both 1 and the initial velocities are 0. The natural frequencies are now $\omega_1 = 2.150$, and $\omega_2 = 0.6154$. Again, the difference between Figure C.33 and Figure C.39 is readily apparent. A peculiarity that develops is that the first mass’s vibration comes in more distinct peaks; one peak reaches 1, and the next is about 0.7. In Figure C.33, the difference is 1 to less than 0.5.

Program 1: modfree.m

```
%[This code solves n degree-of-freedom free
vibration using the modal analysis techniques
of Section 6.6.
%}
clear;
M=input('Enter the mass matrix: ');
[n,o]=size(M);
if n~=o then
    error('M matrix must be square!');
end
K=input('Enter the stiffness matrix: ');
[n,o]=size(K);
if n~=o then
    error('K matrix must be square!');
end
qu=0;
[u,l]=eig(K,M);
%[Using "eig" in this way allows us to
subtract  $M\omega^2$  from K, instead of  $I\omega^2$  (where
I is the n by n identity matrix). The output
from "eig" gives unit-length eigenvectors. We
need to scale them with respect to M.
%}
for s=1:n
    alfa=sqrt(u(:,s)'*M*u(:,s));
    u(:,s)=u(:,s)/alfa;
end
x0=input('Enter the initial displacement
column vector: ');
xd0=input('Enter the initial velocity column
vector: ');
tf=input('Enter the final time: ');
t=0:0.1:tf;
q=tf/0.1;
x=zeros(size(n,q));
```

```
% Applying Equation 7.183.
```

```
for j=1:n
```

```
    w(j)=sqrt(l(j,j));
```

```
    xt=u(:,j)*(u(:,j)'*M*x0*cos(w(j).*t)
+u(:,j)'*M*xd0/w(j) *sin(w(j).*t));
```

```
    x=x+xt;
```

```
end
```

```
%[Plotting the modes in a subplot format.
Note that, for more than 3 or 4 degrees
of freedom, the plots will become nearly
unreadable.
```

```
%}
```

```
for r=1:n
```

```
    subplot(n,1,r)
```

```
    plot(t,x(r,:))
```

```
    xlabel('Time, seconds');
```

```
    ylabel(['Response x',num2str(r)]);
```

```
end
```

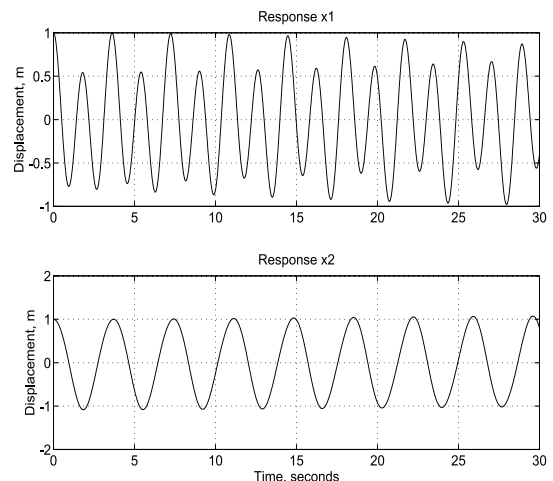


Figure C.33: Responses to initial displacements of [1 1] and zero initial velocities.

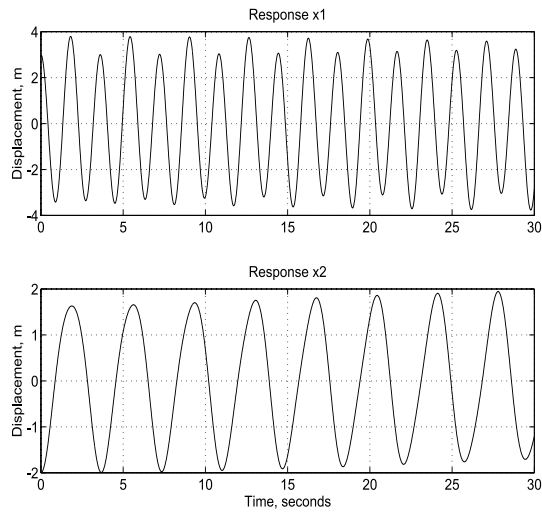


Figure C.34: Responses to initial displacements of $[3 \ -2]$ and zero initial velocities.

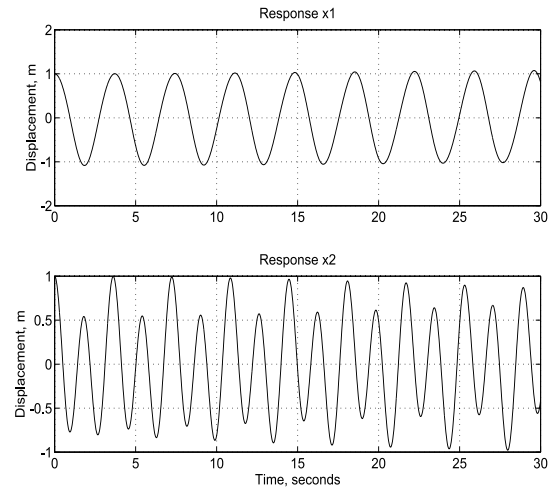


Figure C.36: Responses to initial displacements of $[1 \ 1]$ and zero initial velocities, and with values of masses interchanged.

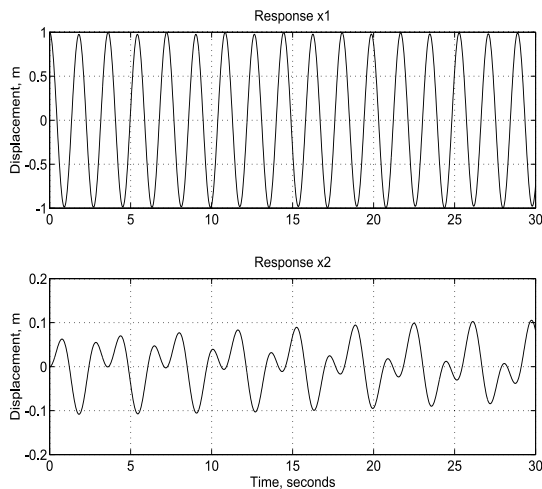


Figure C.35: Responses to initial displacements of $[1 \ 0]$ and zero initial velocities.

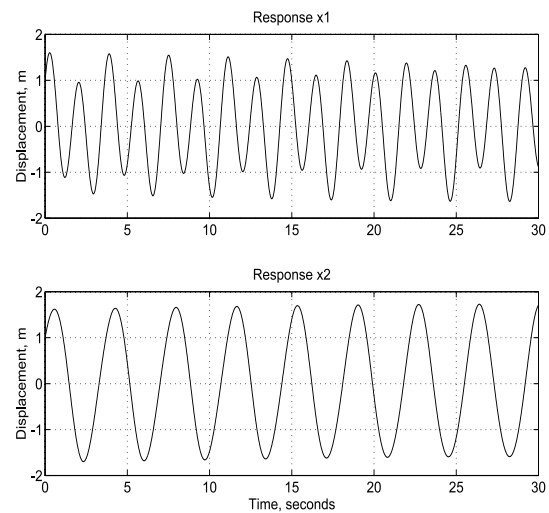


Figure C.37: Responses to initial displacements of $[1 \ 1]$ and initial velocities $[4 \ 2]$.

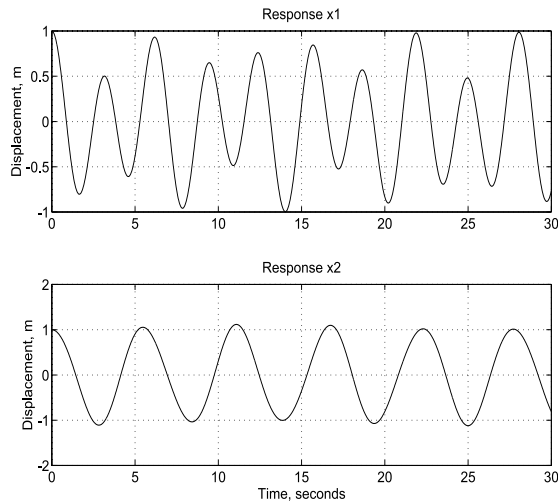


Figure C.38: Effect of increased system masses.

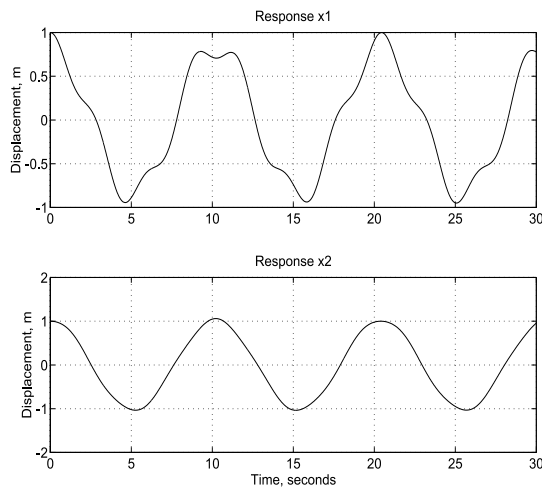


Figure C.39: Effect of reduced system stiffnesses.

C.15 MDOF Damped System

In the previous section, we explored the free response of an undamped system having more than one degree of freedom. Now, viscous damping is added to the system, as shown in the model of Figure C.40, and the equations of motion become

$$m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 + (k_2 + k_1) x_1 - k_2 x_2 = 0 \quad (\text{C.66})$$

$$m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) = 0. \quad (\text{C.67})$$

In matrix form,

$$M \ddot{\mathbf{x}} + C \dot{\mathbf{x}} + K \mathbf{x} = \mathbf{0},$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}.$$

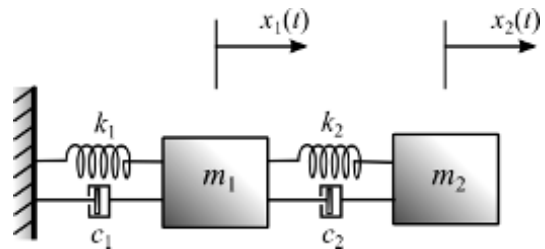


Figure C.40: Two degree-of-freedom damped oscillator.

With damping present, there is no guarantee that the equations will decouple by the procedure used earlier. However, for proportional damping, i.e., $C = C_m M + C_k K$, it is possible to follow the previous decoupling procedure. This is because both K and M are diagonalized through the matrix of eigenvectors U . Thus, a scalar multiplied by M or K will also be diagonalized, and the sum of two diagonal matrices is also diagonal.

While proportional damping is something of a savior for modal analysis, it presents a headache when programming. The only way to assure decoupled equations in a program is to specify C_m and C_k instead of entering the matrix C into the program, or to take the matrix C , go through the first two steps of modal analysis with it, and then check it for diagonality. Both of these methods have flaws; in the first case, the person using the program may not know C_m and C_k , and they may be difficult to calculate. For the second case, the program may run through significant calculations to deduce that

the given data are invalid. This wastes time and effort, especially if the data are invalid due to an error in entry of values. So, when programming, the most efficient way to handle damping is through the modal damping ratios (though the program given below retains some flexibility by allowing entry of the modal damping ratios or the factors C_m and C_k).

The modal damping ratios are simply an analog to the damping ratios we saw earlier in our study of single degree-of-freedom systems. They are called modal damping ratios because they are applied independently to each (decoupled) modal equation; the modal damping ratios are automatically decoupled. Calling the modal damping ratios ζ_i , where i is the number of the mode, the equations of motion in the modal coordinates become

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2q_i = 0, \quad i = 1, 2, \dots, \quad (\text{C.68})$$

where q_i is the modal coordinate variable and ω_i is the natural frequencies of mode i . This is the same as for the damped single degree-of-freedom oscillator, whose solution was

$$q_i(t) = A_i e^{-\zeta_i\omega_i t} \sin(\omega_{di}t + \phi_i), \quad (\text{C.69})$$

where ω_{di} is the damped natural frequency of that mode ($\omega_i\sqrt{1-\zeta_i^2}$), and

$$A_i = \left[\frac{(\dot{q}_{io} + \zeta_i\omega_{di}q_{io})^2 + (q_{io}\omega_{di})^2}{\omega_{di}^2} \right]^{1/2} \quad (\text{C.70})$$

$$\phi_i = \tan^{-1} \frac{q_{io}\omega_{di}}{\dot{q}_{io} + \zeta_i\omega_{di}q_{io}}, \quad (\text{C.71})$$

where q_{io} and $\dot{q}_{io}$ refer to the initial displacement and velocity in the modal coordinate, respectively. Having obtained the solution for $\mathbf{q}(t)$, we then use the last step of the modal analysis procedure to find $\mathbf{x}(t)$. This method uses the entire modal analysis procedure as if damping did not exist, and then adds in the damping when solving for $q_i(t)$. The program at the end of this example, used to generate the figures that follow, demonstrates this approach.

Figures C.41 through C.43 show results for $m_1 = 9$, $m_2 = 1$, $k_1 = 24$, $k_2 = 3$, $\zeta_1 = 0.05$, and $\zeta_2 = 0.1$. Figure C.41 uses an initial displacement vector of $x_0 = [1 \ 0]$ and no initial velocities. The results show the effect of the damping; the response gradually dies out. In Figure C.42, an initial velocity of $[3 \ 2]$ is added, and the result is no increase in the transient duration (as expected) and an increase in amplitude. In Figure C.43 the initial velocity is removed and the initial displacement is changed to $[2 \ 1]$, and the result is a greater amplitude, but the same transient duration as Figure C.42. In all three figures, the responses show the effects of multiple sinusoids,

even though the first mass is much greater than the second. This is because the first stiffness is very large; this stiffness keeps the mass from gaining too much velocity, and so the smaller mass can effectively transfer its momentum to the larger.

In Figure C.44 the damping ratios change to $\zeta_1 = 0.07$ and $\zeta_2 = 0.03$. The difference between this figure and Figure C.41 is that here the responses have a greater amplitude and a larger transient duration due to the smaller damping ratios.

Program 1: moddamp.m

```
%[This code solves n degree-of-freedom damped
vibration using the modal analysis techniques
of Section 7.6.3.
%]
clear;
M=input('Enter the mass matrix: ');
[n,o]=size(M);
if n~=o
    error('M matrix must be square!');
end
K=input('Enter the stiffness matrix: ');
[n,o]=size(K);
if n~=o
    error('K matrix must be square!');
end
%{ Giving the option to input damping
via modal damping ratios or through the
multiplying factors Cm and Ck.
%}
fprintf('Press 1 to enter modal damping
ratios, ');
qz=input('or anything else to enter Cm and Ck.
');
if qz==1
    for iz=1:n
        zeta(iz)=input(['Enter the damping for mode
',num2str(iz),': ']);
    end
else
    fprintf('Given that [c]=Cm[M]+Ck[K], ');
    Cm=input('Enter the factor Cm: ');
    Ck=input('Enter the factor Ck: ');
end
qu=0;
[u,1]=eig(K,M);
%{ Using "eig" in this way allows us to
subtract M*w^2 from K, instead of I*w^2 (where
I is the n by n identity matrix). The output
from "eig" gives unit-length eigenvectors. We
need to scale them with respect to M.
%}
```

```

for s=1:n
    alfa=sqrt(u(:,s)'*M*u(:,s));
    u(:,s)=u(:,s)/alfa;
end
x0=input('Enter the initial displacement
column vector: ');
xd0=input('Enter the initial velocity column
vector: ');
tf=input('Enter the final time: ');
t=0:0.1:tf; q=tf/0.1;
x=zeros(size(n,q));
for j=1:n
    w(j)=sqrt(l(j,j));
    %{ If modal damping ratios were entered, we
    already have a zeta vector. If not, we need
    to calculate from Cm and Ck.
    %}
    if qz~=1
        zeta(j)=0.5*(Cm/w(j)+Ck*w(j));
    end
    wd(j)=w(j)*sqrt(1-zeta(j)^2);
    xt=u(:,j)*(u(:,j)'*M*
x0*cos(w(j).*t) /sqrt(1-zeta(j)^2)
+u(:,j)'*M*xd0/w(j)*sin(w(j).*t)/wd(j));
    x=x+xt;
end
for i=1:n
    x(i,:)=x(i,:).*exp(-zeta(i)*w(i).*t);
end
for r=1:n
    subplot(n,1,r)
    plot(t,x(r,:))
    xlabel('Time, seconds');
    ylabel(['Response x',num2str(r)]);
end
end

```

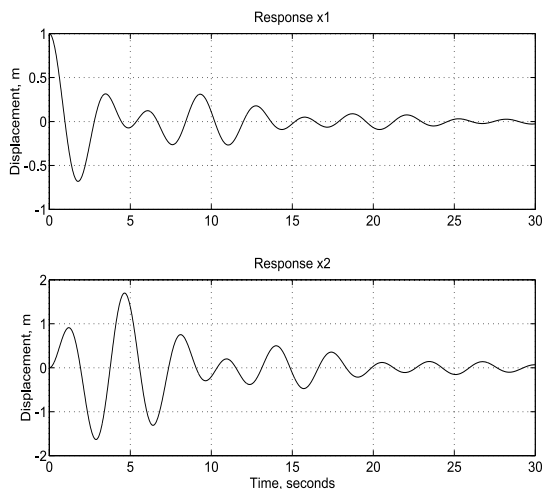


Figure C.41: Responses of a two degree-of-freedom damped system with initial displacement and no initial velocity.

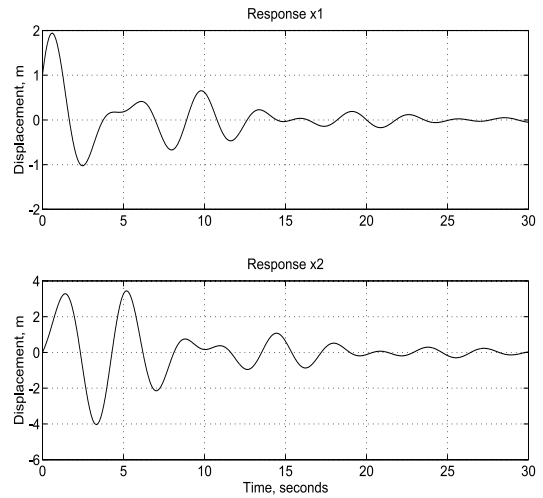


Figure C.42: Responses of a two degree-of-freedom damped system with initial displacement and initial velocity.

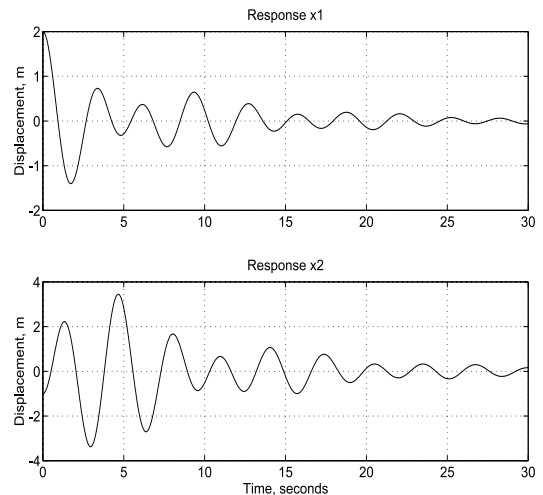


Figure C.43: Responses of a two degree-of-freedom damped system with a different initial displacement and no initial velocity.

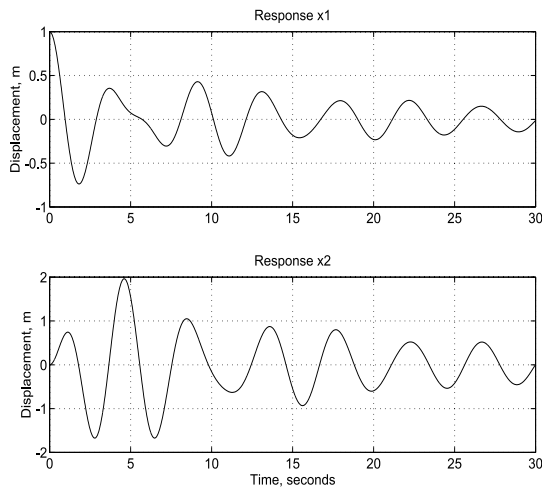


Figure C.44: Responses of a two degree-of-freedom damped system with different modal damping ratios.

C.16 General Vibration Solver

The next logical step in this sequence of programs would be to introduce forcing functions of different types acting on multi degree-of-freedom systems (as was done with the single degree-of-freedom cases), and create new programs. While we can pursue this approach and it is effective for special cases, it is not efficient.

Here, we propose the idea of creating a more general program that solves vibration cases for different forcing functions and arbitrary number of degrees of freedom. The simplest way to go about this is to begin with a general program that does not include the external force. Program 1, *ndofold.m*, demonstrates this method. Some techniques used in this program demonstrate ways to make a program modular. We seek a general program that is modular so that changes (including changes of parameters like masses and stiffnesses) can be accomplished simply. Toward achieving this goal, the first line of the code assigns a variable to the number of degrees of freedom. This way, in any loops or assignment statements made afterward, this variable can represent any number of degrees of freedom.

The next several lines initialize the variables needed later. This is done in lieu of a *clear* statement; the *clear* statement will remove some variable that will be needed later. Additionally, not using the *clear* command has the advantage of allowing the user to assign variable names to the desired mass, stiffness, and other matrices. This way, when the program prompts for these later, the user need only enter the variable name, not the entire matrix. If *clear* is used, the matrices cannot be created beforehand. The trade-off in not using *clear* is that the user must indicate to MATLAB the size of the matrices; if these sizes are not specified, MATLAB will take the variables to be the size that they were already. If we are now solving a 3-DOF problem and we had just solved a 2-DOF system, MATLAB will assume the mass matrix is 2 by 2. The fifth element entered would set off an error message. The command used fills each matrix with zeros to initialize variables.

After the initialization section, the program prompts for each matrix that will be needed. Having the necessary information, the program then solves the system by modal analysis, and prints out the output. Note how each calculation is done in loops that run from 1 to n . This maintains the generality of the program, allowing the user to specify with the first input sizes of the matrices. Notice also that the plots are presented one at a time. This is done as there is no way to guarantee a subplot of any number of columns will be readable. The *pause* command is implemented after each plot to allow the user to examine each plot and to print the plot, if desired, before the next one. The program does not automatically print the graphs. (Modifying the program to

specify printing would involve replacing *pause* with *print* in the code.) Also, a loop could be created at the beginning of the program to allow the user to specify whether output goes to the screen only, to a file, to the printer, or to a combination of the three. A sample of such a loop follows here:

```
%{This loop allows the user to specify
the nature of the output. Notice how the
loop assumes that the user will enter the
wrong response; this is to correct for any
mistakes, and to thwart those who love to
crash programs. Also, the program initializes
the variable right before the loop to the
value necessary to continue the loop.
%}
outp=8;
while outp==8
    fprintf('Enter 1 to print to the screen
only, 2 to print to a file, \n')
    outp=input('or 3 to print to the printer.
');
    %{Notice how the text was split across two
lines by using the fprintf statement. This
allows the text for input statements to be as
long as needed.
    %}
    if outp~=1&outp~=2&outp~=3
    %{Translation: "If outp isn't equal to 1, 2,
or 3." "~" means "not" for a MATLAB logical
statement, and the ampersand (&) is "and",
as usual. This is where we let the user know
he's made a mistake.
    %}
        fprintf('Enter 1, 2, or 3!!!')
    %{Direct and to the point. Now, outp is reset
to the value needed to continue the loop,
since we need the loop to continue.
    %}
        outp=8;
    end
end
%{One "end" for the "if", and one for the
"for". A common mistake is to leave one out,
and then MATLAB tells you that it came to the
end of the file in the middle of a loop. (Not
a pleasant feeling.) So, there's the loop!
It's not long (without the comments), and
it's effective. It can be expanded as much
as necessary to allow for any combination of
printing options. One more thing: to get
this to work, you need to put the following
into the printout loop (where the pause is
now):
%}
```

```
if outp==2
%{"==" in MATLAB is the Boolean equal sign
that compare the left and the right hand side.
%}
print -dps file1.ps
elseif outp==3
    print
elseif outp==1
    pause
end
```

If you type "else if" instead of "elseif", MATLAB reads the "else if" as a new if statement, needing its own end. (This is a common mistake.)

Now, we need to add forcing functions. This task, however, is nearly impossible to do in a general way with the algorithm selected. We need to find a simpler method. We realize that the differences among the several possible vibration cases occur in solving the differential equations. Every problem in vibrations begins as a set of second-order differential equations, possibly coupled. MATLAB includes the *ode45* command, which solves differential equations numerically for a given time interval and initial conditions. Unfortunately, this command requires that the differential equations exist in a function m-file which defines the system of differential equations as a set of first-order equations.

The problem of order is really not a difficulty at all; there is a method by which a second-order differential equation can be transformed into a system of 2 first-order equations. However, unless a general m-file can be created, the user would have to create a function m-file each time, which would then be used for *ode45*.

The answer lies in automating the process of creating an m-file. MATLAB has file transfer protocols, like any other programming language. We can write a program that tells MATLAB to create a specific m-file from a general procedure we write into the program, and then solve the resulting system of differential equations with *ode45*. Program 2, *ndof.m*, performs this task.

Again, the program begins by specifying the number of degrees of freedom for the run. The necessary matrices are initialized, and the inputs are obtained. After this, the program performs the first few steps of modal analysis, in order to find the natural frequencies. We can do this because the natural frequencies depend only on the mass and stiffness matrices; we know that these can be decoupled.

The next commands are initiated if a damping ratio matrix was entered. This loop changes the damping ratios into damping coefficients, by reversing the modal analysis procedure. This can be done because the damping ratio vector that the user enters is related to the diagonal of the decoupled damping coefficient matrix. Since this damping coefficient matrix is decoupled, it can be transformed from the modal coordinate system

(the decoupled coordinate) back to the original coordinate system by applying the reverse of the modal analysis procedure.

Now that the program has the original values of the mass, stiffness, and damping coefficient matrices, the m-file can be created. The program first opens the file with the *fopen* command. The strings in parentheses by the command specify the filename, and the read/write privilege desired (“w+” means “write and overwrite,” which allows a user of this program to make several runs in succession without erasing the m-file previously created; the previous m-file is lost). The *fopen* command is set equal to a variable because the command specifies an arbitrary number by which MATLAB identifies the file. The only way to get this number is by setting *fopen* equal to a variable, so that this number will be stored in an accessible place.

The reason that we need this identification number is shown by the *fprintf* statements that follow the *fopen* command. Each statement begins with the file identification number, so that MATLAB knows to write the text to the file specified earlier, not to the screen. If the file identification number was not set equal to a variable, we would have no way to access it, and no way to write to the file.

Next, the functions defining the external forces on each degree of freedom are entered individually. This is necessary because a text string entered into a MATLAB variable is stored one character per matrix element. So, if all the external forces were entered at once, there would be no way of telling how long the force variable would be, or how many elements the matrix would have. The method used in the program avoids these difficulties.

Notice the use of the *num2str* command within the *fprintf* statements. This command allows the value of a variable to be used in a text statement as text; it is possible to use the value of a variable in an *fprintf* statement, but not as text. To use the *num2str* command, the text to be printed must be within brackets as well as parentheses; this is because, if the brackets are not included, MATLAB thinks the statement ends when the quotes are closed, and will produce an error.

Finally, the program uses an *fclose* statement to tell MATLAB that nothing else is to be entered into the file. This is set equal to a variable, because MATLAB indicates a successful write transfer through the *fclose* statement; a user can tell if the file was written successfully by seeing the value of this variable.

The program takes initial values necessary for *ode45*, and solves. The plots are presented individually; a print loop similar to the one described above can be inserted into this code to allow printing to a printer or to files.

Figures C.45 and C.46 show the results of using *ndof.m*. Each figure represents a degree of freedom of the system, whose parameters are as follows: $m_1 = 9$,

$m_2 = 1$, $c_1 = 2.7$, $c_2 = 0.3$, $k_1 = 27$, $k_2 = 3$, $F_1 = 0$, $F_2 = 3 \cos 2t$. Notice how each degree of freedom shows the transient response over the first two seconds, and then settles into a steady-state response based on the single forcing function. The natural frequencies calculated by the program are $\omega_1 = 1.41$ and $\omega_2 = 2.00$. The responses of each mode can be found through manual calculation to be

$$x_1(t) = 0.2451 \cos(2t + 0.1974) - 0.6249 \sin(2t) \quad (\text{C.72})$$

$$x_2(t) = 0.7354 \cos(2t + 0.1974) + 1.8749 \sin(2t). \quad (\text{C.73})$$

Looking at the maximum magnitudes of the responses shown in Figures C.45 and C.46, we see that they are approximately equal to the sum of the coefficients of the sine and cosine terms in the expression above, as we would expect. So, from this example, we see the value of the program and, more importantly, verify that the program works.

Program 1: ndofold.m

```
%{This program calculates the damped response
of multi degree-of-freedom systems without
external forcing. The user must enter the
modal damping ratios; no other method of
describing damping is supported.
%}
n=input('How many degrees of freedom are
present? ');
m=zeros(n,n);
k=zeros(n,n);
s=zeros(n,n);
p=zeros(n,n);
l=zeros(n,n);
ac=zeros(1,n);
phi=zeros(1,n);
m=input('Enter the mass matrix. ');
ko=input('Enter the stiffness matrix. ');
zeta=input('Enter the damping ratio (zeta)
matrix. ');
x0=input('Enter the initial displacements.
');
xp0=input('Enter the initial velocities. ');
ti=input('Enter the initial time. ');
tf=input('Enter the final time. ');
t=ti:(tf-ti)/1000;tf;
[b1,b2]=size(t);
x=zeros(n,b2);
r=zeros(n,b2);
a=m^(-1/2);
kt=a*ko*a;
[p,l]=eig(kt);
s=a*p;
```



```

si=inv(s);
r0=zeros(n,1);
rp0=zeros(n,1);
r0=si*x0';
rp0=si*xp0';
w=sqrt(1);
wd=zeros(size(w));
for k=1:n
    wd(k,k)=w(k,k)*sqrt(1-zeta(k)^2);
end
%Note that the same loop control variable can
be used for
%every loop, so long as the loops are not
nested within
%each other.
for k=1:n
    ac(k)=sqrt(wd(k,k)^2*r0(k)^2 +(rp0(k)
+zeta(k)*w(k,k) *r0(k))^2)/wd(k,k);
    phi(k)=atan2(wd(k,k)*r0(k),(rp0(k)
+zeta(k)*w(k,k)*r0(k)));
    r(k,:)=ac(k)*exp(-zeta(k)*w(k,k)*t)
.*sin(wd(k,k)*t +phi(k));
end
x=s*r;
clg
for k=1:n
    plot(t,x(k,:))
    o=num2str(k);
    title(['Response x',o])
    xlabel('Time, seconds')
    ylabel('Displacement')
    grid
    figure(gcf)
    pause
end
for k=1:n
    fprintf('Natural frequency w%g=%g
\n',k,w(k,k))
end

```

Program 16-2: ndof.m

```

%This program, as stated above, writes its own
m-file to use in ode45. This algorithm allows
any external force combination to be examined,
and also allows the external force to be a
user-defined function.
%}
n=input('How many degrees of freedom are
present? ');
x=zeros(1,2*n);
ct=zeros(n,n);
m=input('Enter the mass matrix. ');
aw=2;

```

```

while aw==2
    fprintf('Press 0 to enter a damping
coefficient matrix, or')
    aw=input('press 1 to enter a damping ratio
(zeta) matrix. ');
    if aw==0
        c=input('Enter the damping coefficient
matrix. ');
    elseif aw==1
        zeta=input('Enter the damping ratio
matrix. ');
    else
        fprintf('Please enter 0 or 1.')
        aw=2;
    end
end
k=input('Enter the stiffness matrix. ');
a=m^(-0.5);
kt=a*k*a;
[p,q]=eig(kt);
omega=sqrt(q);
if aw==1
    for zt=1:n
        ct(zt,zt)=zeta(zt)*2*omega(zt,zt);
    end
    af=a^(-1);
    c=af*ct*af;
end
fid=fopen('pdx.m', 'w+');
fprintf(fid,'function pdx =pdx(t,x)\n');
for i=1:n
    fprintf(fid,['pdx(',num2str(i), ')=(-(']);
    for j=1:n
        if k(i,j)==0
            else
                fprintf(fid,['(',num2str(k(i,j)),')
*x(',num2str(j+n),')')]);
            end
            if c(i,j)==0
                else
                    fprintf(fid,['+(',num2str(c(i,j)),')
*x(',num2str(j),')')]);
                end
            end
        f=input(['Enter F',num2str(i), '(t) in
quotes, like '3*cos(2*t)'. ']);
        fprintf(fid,['+(',num2str(f),')
/(',num2str(m(i,i)),')\n']);
    end
    for i2=1:n
        fprintf(fid,['pdx(',num2str(i2+n),')
=x(',num2str(i2),')\n']);
    end
    status=fclose(fid);
end

```

```

v0=input('Enter the initial velocities as a
row vector. ');

x0=input('Enter the initial displacements as a
row vector. ');

ti=input('What is your initial time? ');
tf=input('What is your final time? ');

init=[v0,x0];

[t,ny]=ode45('pdx',ti,tf,init);

for kl=1:n

    plot(t,ny(:,kl+n),'b');

    title(['Response x',num2str(kl)]);

    xlabel('Time, seconds');

    ylabel('Displacement, m');

    grid

    figure(gcf)

    pause

end

for l=1:n

    fprintf('Natural frequency w/g=%g. \n', 1,
    omega(l,1))

end

```

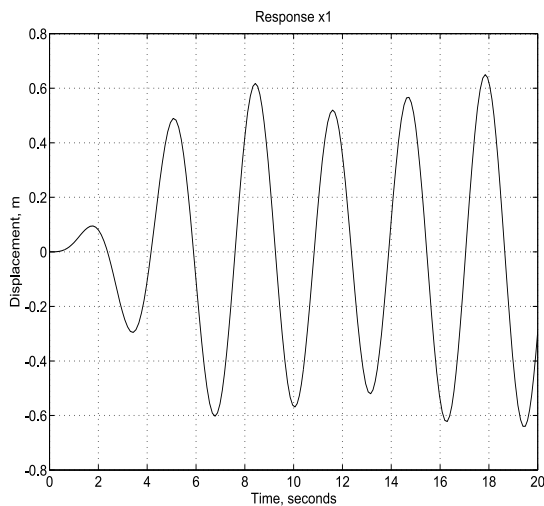


Figure C.45: Response of the first mode.

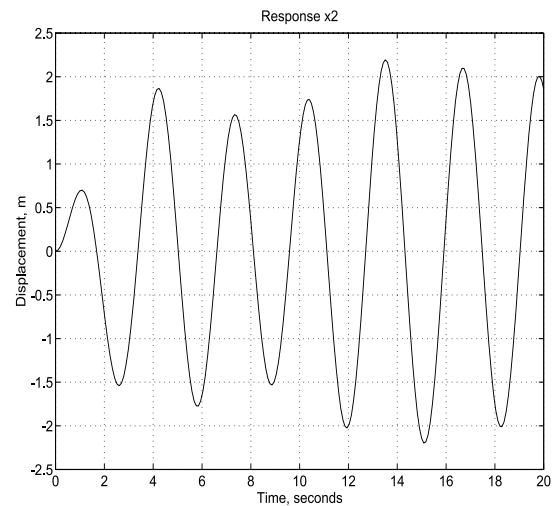


Figure C.46: Response of the second mode.

C.17 Van der Pol Oscillator

This section and the next several sections examine how we can use MATLAB to solve more real-world vibration problems. The first system we consider is the oscillatory motion of a structure surrounded by a fluid. Examples of such structures include the support pylons of offshore oil platforms and antennae attached to the exterior surfaces of aircraft. These structures exhibit vibratory motion due to the creation of vortices in the fluid by viscous interaction between the structure and the particles comprising the fluid. The van der Pol equation,

$$\ddot{x} + e(x^2 - 1)\dot{x} + x = 0, \quad e > 0, \quad (\text{C.74})$$

is sometimes used as a model for the motion of such a structure. This equation is nonlinear and includes a parameter e that has a positive value. The character of the solution depends on the magnitude of x . When $|x| < 1$ the response decays, as expected for viscous damping. However, when $|x| > 1$, the coefficient of $\dot{x}$ becomes negative. This means the damping term would *add* energy to the system, instead of removing it. This *negative damping* approximates some of the phenomena observed in such fluid-structure interactions, and so is an attractive (and necessary) feature of the model.

We will demonstrate how we can use MATLAB to help us visualize the solution to such a problem. First, we must solve the nonlinear ordinary differential equation given above. We will not arrive at an analytic solution, as we did with several of the earlier problems. What we can do, though, is numerically integrate the equation over a particular time interval.

We use a general ODE solver, `ode45.m`, that calls a function of the equation of motion case in state-space form. Here, we introduce two states, x_1 and x_2 , defined as

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x}. \end{aligned}$$

For the equation of motion, given in Equation C.74, the state variable equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -e(x_1^2 - 1)x_2 - x_1. \end{aligned}$$

These equations are entered into the function named `dxvdp`. This function can be written as a m-file, or it can be nested inside the main program. Here, we include this function inside the main program. In order to have a nested function, the main program must be a function itself.

Note that the variable e must be passed from the main program to the nested function `dxvdp` using the format `[t,x]=ode45(@dxvdp,tspan,x0,'e');`

The figures below show some results for the default case (as defined in Program 1). Figure C.47 shows displacement vs. time, and we see oscillatory behavior. The two figures that follow are *phase diagrams*,⁴ which are plots with the velocity on the y-axis and displacement on the x-axis. These diagrams are often used in studies of nonlinear and chaotic systems. They are useful to show the effects of initial conditions on the response.

The key feature of Figure C.48 is the closed loop. The initial location is denoted by a circle, and the final state by a triangle. The closed loop means that the van der Pol system eventually settles down into oscillatory behavior. We can show that a closed loop in phase space corresponds to an oscillating response by considering the function $x = \sin(t)$. If we take this (obviously oscillating) function as our displacement, then the velocity is described by $y = \cos(t)$. If we were to plot this result in phase space, we would arrive at a circle; these functions $x(t)$, $y(t)$ are parametric equations for a circle. The van der Pol oscillator's loop is not circular, so it is not periodic in the same regular way as the sine or cosine function. However, it repeats the same sets of positions and velocities in a limit cycle.

Figure C.49 is the same system with different initial conditions; we used the command `[t,x]=solvevdp(0.5,[-1,5])`. Again, this initial point $([-1, 5])$ is denoted by a circle, and the final point by a triangle. Despite the different initial conditions, the motion settles into the same limit cycle. Thus, the limit cycle is determined by the parameter e , and not by the initial conditions. This limit cycle behavior is similar to a phenomenon seen in the vibration of structures in a moving fluid, and several investigators have used the van der Pol equation to describe these systems. The reader is encouraged to try different initial conditions to investigate the limit cycle behavior.

Program 1: solvevdp.m

```
function [t,x]=solvevdp(e,x0,tspan)
%{ This function takes in a value for e
in the van der Pol oscillator expression,
x''+e(x^2-1)x'+x=0, where primes denote
differentiation in time. The function will
automatically assume x(0)=1, x'(0)=0 and a
thirty-second time interval (tspan=[0 30]) if
these values are not provided. Also, a value
of e=0.5 is assumed if none is given.
}%
if nargin<1 e=0.5; end % An "if" statement on
a single line.
if nargin<2 x0=[1 0]; end
if nargin<3 tspan=[0 30]; end
```

⁴Also called *phase portraits* by some authors.

```

%

% Off to the ode solver.

%

[t,x]=ode45(@dxvdp,tspan,x0,"",e);

%{ This is the function called on by ode45 to
provide the derivative of our state vector
x. We need to set up the second-order
differential equation as a system of two
first-order equations. Our state vector
x consists of the variable x and its time
derivative.

%}

%{ The function dxvdp is nested inside of
solvevdp.m

%}

function dx=dxvdp(t,x);

    dx(1,1)=x(2);

    dx(2,1)=-(e*(x(1)^2-1)*x(2)+x(1));

end

end

```

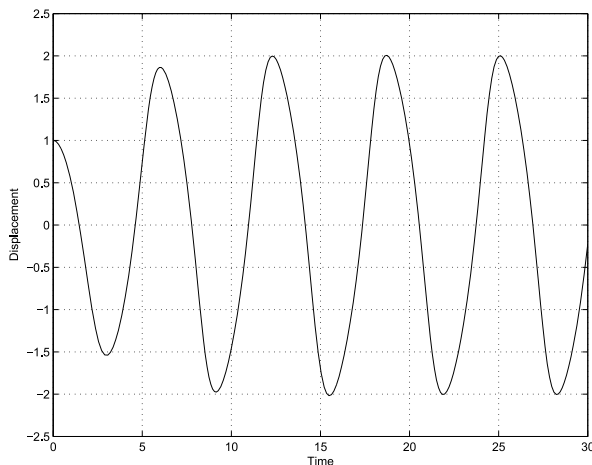


Figure C.47: Displacement vs. time curve for the van der Pol oscillator under the default conditions of Program 1.

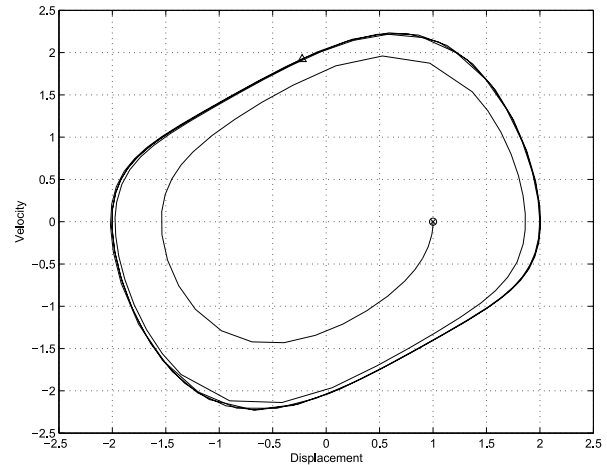


Figure C.48: Phase diagram for the van der Pol equation. Note the closed limit cycle.

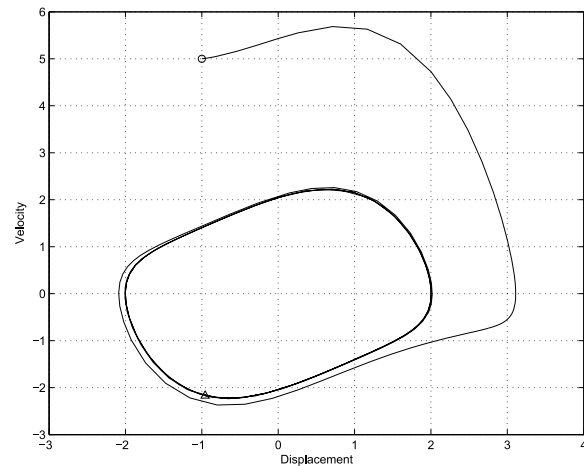


Figure C.49: Phase portrait of the same van der Pol oscillator with different initial conditions.

C.18 Random Vibration

Our interest here is to show how MATLAB can be used as an aid in modeling random processes, and thus in visualizing the response of systems subjected to random forcing.⁵ The logical starting point for this effort is to show how we can create random forcing histories to apply to a deterministic model. We will arrive at sample response realizations, which we will use to find the response statistics numerically, in contrast to an analytic approach for the response statistics.

To illustrate some of the challenges associated with a random forcing input, we start with a sinusoidal forcing input of the form $F(t) = A \cos(\omega t)$ and ask how we will introduce randomness into it. Do we want a random force amplitude, a random frequency, or both? If we are modeling a given physical system, we will find that this question often answers itself. For example, we may have derived the power spectral density of a typical input to our system, and found that the vast majority of the energy input to the system comes at one particular frequency. For this case, we could then (as a first approximation) treat the forcing frequency as deterministic, and use a random distribution for the amplitude.⁶

Having decided the parameter we will take as random, we need to select a probability density function for this parameter. Do we want to use a uniform distribution, a Gaussian distribution, or some other description? Before we decide, let us look at the distributions available in MATLAB.

The uniform and Gaussian distributions are built in through the `rand` and `randn` commands, respectively. Typing `a=rand(n)` at the MATLAB prompt will return an $n \times n$ array of randomly generated numbers in a , using a uniform distribution on the interval $[0,1]$. The `randn` command returns a random number from a Gaussian distribution having a mean of zero and standard deviation of one. Note that MATLAB's random numbers are actually "pseudorandom"; they depend on an initial state vector. For some applications, we will want to compare the results of several different systems to a random loading. Then, we can use the state of the random number generator to subject each comparison case to the same realization of the random loading process. If we want to ensure that we get a different realization each time, we can use the suggestion available in the MATLAB help for `rand`, and initialize the state to `sum(100*clock)`. This will reset the random number generator based on the current time.⁷ Both methods are useful for different cases. Any other probability distributions would have to

be derived from the two described above or created from scratch by the user.

We consider the system described by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = F(t), \quad (\text{C.75})$$

where $F(t)$ is the sinusoid above having random amplitude $A(t)$. It will be convenient to apply the condition of ergodicity to this system. Thus, we will solve for the *steady-state response* only, neglecting the transient response. (This also eliminates the need to specify initial conditions.) We can solve for the steady-state response in two ways using MATLAB. We can use the results of the analysis we did earlier, giving an analytic solution, or we can use the numerical integration routines. We will pursue the latter to show how we can transport the amplitude values back and forth where they are needed and to show how we can handle the discrete datapoint set.

We use the `margin` function to allow any number of input parameters from zero to the maximum of five. Recall from the last example that we used *less than* statements instead of a `switch` expression to define parameters. This is because expression cases of a `switch` in MATLAB do not require a `break` statement to end each expression set. Thus, we cannot allow cases for lower numbers of input arguments to "fall through" to higher numbers. We also set the length of the amplitude vector A to be the same as the length of the `tspan` vector, or 1000 elements if the input `tspan` is shorter. We do this to make interpolation of A based on values of t simpler. When MATLAB integrates the differential equation, the values of t that it will test are not necessarily the same as the values in `tspan`. Thus, we will interpolate linearly between neighboring values of A corresponding to the two elements of `tspan` that surround the current time value t .⁸ In order to ensure that the amplitude is always greater than zero, we multiply each element of A by its sign. This leads to a distribution that is not rigorously Gaussian. We could improve our odds of not getting negative values for A by changing the standard deviation such that 3σ in each direction is positive. The coding for this is left to the reader. The `error` expression in the default (`otherwise`) case of our `switch` expression will display the message to the screen in quotes, informing the user of the function that the allowed values of the flag are zero and one.

We are confronted with one more problem if we wish to use the differential equation solver. That is, how do we remove the transient response? It is not as simple as setting the initial conditions to zero, since the position and velocity imparted by the forcing function at time zero are

⁵Following the textbook, our focus is on random forcing of deterministic systems.

⁶In general, we would also have a random phase angle associated with this forcing. We omit the phase angle here for simplicity.

⁷Specifically, the `clock` command returns a six-element vector, containing the current year, month, day, hour, minute, and second.

⁸This is a coarse approximation, to be sure, but has the benefit of being simple to implement. The reader may wish to devise refinements.

not necessarily zero. What we must do is solve the differential equation at time $t = t_0$, and then substitute into the relation the values of the forcing and the forced response at that time, to solve for the initial position and velocity required to match the forced response. Recall our earlier analysis, showing that $x(t) = x_p(t) + x_h(t)$, where

$$x_h(t) = A_h e^{-\zeta \omega t} \sin(\omega_d t + \theta), \quad (\text{C.76})$$

and

$$x_p(t) = A_0(t) \cos(\omega t - \phi), \quad (\text{C.77})$$

where we found the constants A_0 and ϕ to be

$$A_0 = \frac{A(t)}{\sqrt{(\omega_n^2 - \omega^2) + (2\zeta\omega_n\omega)^2}}, \quad \phi = \tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}. \quad (\text{C.78})$$

We need to specify the initial conditions $x(t_0)$ and $\dot{x}(t_0)$ so that we can have $A_h = 0$, eliminating the transient response $x_h(t)$. This means specifying $x(t_0) = x_p(t_0)$ and $\dot{x}(t_0) = \dot{x}_p(t_0)$, or

$$\begin{aligned} x(t_0) &= A_0(t_0) \cos(\omega t_0 - \phi) \\ \dot{x}(t_0) &= -\omega A_0(t_0) \sin(\omega t_0 - \phi). \end{aligned} \quad (\text{C.79})$$

This is easily coded into MATLAB, using the initial value of the force amplitude in the random array A . We neglect the time derivative of the randomly varying amplitude A as a first approximation. Figure C.50 is the displacement vs. time curve for the oscillation. The oscillation remains periodic even with the random forcing amplitude, albeit with an irregular amplitude. The force amplitude is shown in Figure C.51. The plot shows only the first five seconds to give a flavor for the variations in the force.

Program 1: randamp.m

```
function [t,x,A]=randamp(tspan, wn, zeta,
distflag, omega);
%{This program solves a single
degree-of-freedom oscillator given an input
of the form  $F(t)=A\cos(\omega t+\phi)$ , where  $A$ 
is a random variable. distflag should be zero
for uniform, and 1 for Gaussian distribution.
%}
if nargin<1 tspan=linspace(0,30,1000); end
if nargin<2 wn=1; end
if nargin<3 zeta=0.05; end
if nargin<4 distflag=0; end
if nargin<5 omega=3.5; end
phi=0; % Zero phase angle.
%{ Since we've got to have an amplitude for
each time step (ease of interpolation), we
need to be specific about how many time steps
```

we've been given. We'll say that if tspan has fewer than 1000 elements, then we make it a 1000-element vector.

```
%}
m=max(size(tspan));
if m<1000
    tspan=linspace(t(1),t(m),1000);
end
%{ Now, we define the amplitude A, between
0 and 5 for the uniform distribution, and
as a distribution with mean of 2.5 for
the Gaussian. We will force A>=0, so the
Gaussian distribution will need a little
bit of tweaking (and it won't be rigorously
Gaussian).
%}
switch distflag
case 0
    A=5*rand(size(tspan)); % A now has a range
    from 0 to 5.
case 1
    A=2.5+randn(size(tspan));
% One standard deviation will span [1.5,3.5].
    A=A.*sign(A); % Multiplying A by the
    signum function of itself.
otherwise
    error('Distribution flag must be zero or
    one.');
```

end

%{Next, we define the initial conditions to remove transients.

```
%}
phi=atan(2*zeta*omega*wn/(wn^2-omega^2));
x0=A(1)*cos(omega*tspan(1)-phi);
xd0=-A(1)*omega*sin(omega*tspan(1)-phi);
xv0=[x0,xd0];
% Now we're ready to solve the ODE.
[t,x]=ode45('randdx',tspan,xv0, ", A, tspan,
zeta, wn, omega, phi);
function dx=randdx(t,x, A, tspan, zeta, wn,
omega, phi)
%{ Differential equation definition for the
randomly-varied amplitude case.
At=interp1(tspan,A,t);
% Finding the correct value of A.
% interp1 is a built-in function.
dx(1,1)=x(2,1);
dx(2,1)=At*cos(omega*t-phi) -(2*zeta*wn*x(2,1)
+wn^2*x(1,1));
end
end
```

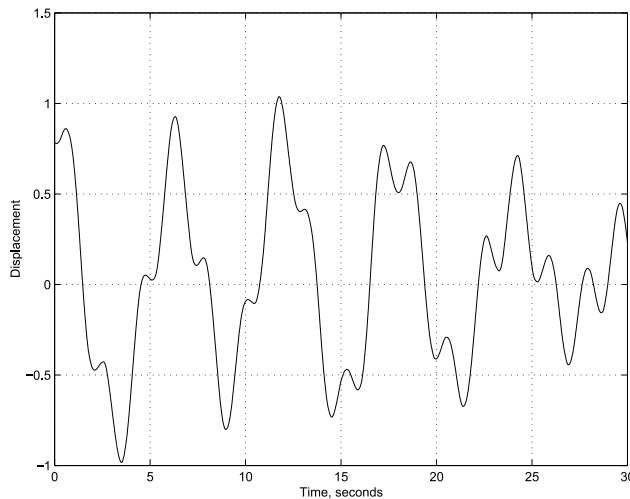



Figure C.50: Displacement vs. time plot for the random Duffing oscillator.

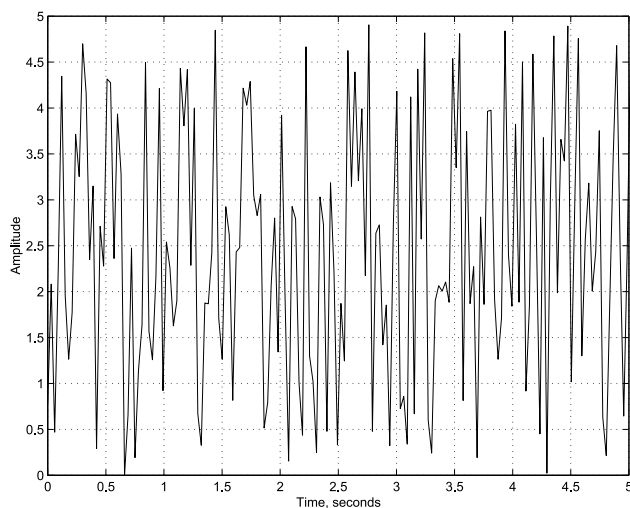


Figure C.51: Amplitude vs. time plot showing variations for five seconds.

C.19 Duffing Oscillator with Random Excitation

This section builds on ideas of the previous section. Here, we will apply a random forcing input to a nonlinear oscillator. We could revisit the van der Pol oscillator of a few examples ago, but instead we will introduce the Duffing equation,

$$\ddot{x} + c\dot{x} + kx + \varepsilon g(x) = A \cos \omega t, \quad (\text{C.80})$$

where the parameters c and k are assumed to be positive, and $|\varepsilon| \ll 1$. Note that ε is not restricted to solely positive or negative values. Equation C.80 allows us to choose the parameter ε and the function $g(x)$ to model nearly linear springs, for example. Also, this equation retains some attractive quasi-linear qualities. Specifically, if we set $A = 0$ and $c = 0$, we would expect roughly oscillatory motion for small amplitudes x , and that if we further introduce a small damping coefficient c , these small-amplitude oscillations would reduce to zero. If we assume a function $g(x)$ such that $\text{sgn}(g(x)) = \text{sgn}(x)$, then the sign of the parameter ε determines the character of the stiffness element being modeled.⁹ If $\varepsilon < 0$, then we see that the restoring force will be smaller in extension, and arrive at a *soft* spring. Conversely, if we take $\varepsilon > 0$, the spring gains stiffness in extension over the purely linear case, and is a *hard* spring.

We are not limited to the class of functions $g(x)$ described above. For example, if we choose the parameter $\varepsilon > 0$ and select $g(x) = -x^2$, we then get a net restoring force $kx + \varepsilon g(x)$ that is negative for $x < 0$ and positive for $x > 0$. In other words, the spring will be soft in extension and hard in compression, meaning that the center of oscillation (equilibrium point) will be shifted slightly away from zero, where the magnitude of the shift depends on the relative values of k and ε .

We use the Duffing oscillator as a model for response to a harmonic input. Programs 1 below uses a differential equation solver in MATLAB to numerically solve the Duffing equation, given the input values of c , k , ε , A , and ω . The function uses $g(x) = -x^2$; this can be readily changed by editing the differential equation definition function (Program 2). Also, the time span and initial conditions are fixed inside the code, to keep the number of inputs manageable. Program 1 can be edited to take a different time increment or initial conditions, or to take them as function inputs. Figure C.52 shows the results of the hard-coded default case.

Having arrived at a solution for the Duffing equation, our next step is to introduce a random forcing function. Unlike the previous example (and in preparation for the next) we will select a random forcing frequency and a

⁹The signum function $\text{sgn}(x)$ returns -1 if x is negative, 1 if x is positive, and 0 if x is zero.

random amplitude, which will remain constant for the duration of the oscillation. If we design this function properly, we will not need to write a new program to provide the differential equation; we will send the same set of global variables. Again, we leave the time interval and initial conditions defined within the function. (The reader is encouraged to rewrite the function to take the time increment, initial conditions, or both as inputs, providing default values via `nargin`.) We use the uniform distribution generator of MATLAB to provide a forcing amplitude between 0 and 5, and a forcing frequency between 0 and 2π . Note that the function returns four arrays: the time vector t , the state vector x , the random amplitude A , and the random frequency ω . This is for ease of comparison between runs, and also makes for a useful check on our random solution.

Knowing the values of A and ω from the code, we can run our deterministic solver (Program 1) using the default values for e , c , and k , and the random values for A and ω . The results for a few default runs are plotted below, along with the random frequency and amplitude used. Of special note is the third run in Figure C.54. It happened that we got lucky (or unlucky, depending on the point of view) on this run, and the random forcing frequency was very close to the system's natural frequency ($\omega_n = 1$). Hence, we see a nearly resonant condition in this undamped oscillator.

The phase diagram in Figure C.55 corresponds to this run. The response traces out arcs in the phase plane that are circular, but by no means closed. Examining the plot point by point shows that the oscillation travels back and forth through the phase plane, meaning that the mean point of the oscillation does not move one way or the other; it is stationary at zero. The interested reader is encouraged to investigate this behavior. A simple way to accomplish this is to run the Duffing oscillator function with the given parameters, and then type the following at the MATLAB prompt:

```
for i=1:max(size(x))
plot(x(i,1),x(i,2),'o')
hold on
pause
end
```

MATLAB will execute this loop as if it was inside a function or script, and you will be able to track the movement in the phase plane.¹⁰

Program 1: `simpduff.m`

```
function [t,x]=simpduff(e,c,k,A,w)
```

```
%{Function to set up and implement the ode45
solution of a Duffing oscillator. The time
increment is hard-coded to be [0,30], and the
initial conditions set to [0,0].
%}
tspan=[0 30];
xinit=[0 0]';
% The customary nargin check.
if nargin<1 e=0.01; end
if nargin<2 c=0.05; end
if nargin<3 k=1; end
if nargin<4 A=3; end
if nargin<5 w=2.7; end % w chosen to prevent
resonance.
[t,x]=ode45('duffdx',tspan,xinit,"",c, k, e, A,
w);
figure(1)
plot(t, x(:,1));
figure(2)
plot(x(:,1), x(:,2));
%{This function returns the derivative of the
state vector for a Duffing oscillator.
%}
function dx=duffdx(t,x,e,c,k,A,w)
g=-x(1,1)^2; % Explicitly stating g(x).
dx(1,1)=x(2,1);
dx(2,1)=A*cos(w*t)-(c*x(2,1)+k*x(1,1)+e*g);
%{ Since we stated g(x) above, the equation
for x'' looks like the Duffing equation. Also,
we need only change that one line of code
above to use a different function g(x).
%}
end
function [A,w]=randduff(e,c,k)
%{ This function provides random force
amplitude and frequency to a Duffing
oscillator. Both values
are taken from a uniform distribution.
%}
rand('state',sum(clock*100)); % Resetting
random generator state.
A=5*rand(1);
w=2*pi*rand(1);
end
end
```

¹⁰You may also have to sit through many pauses; the x array we computed was 2-by-457.

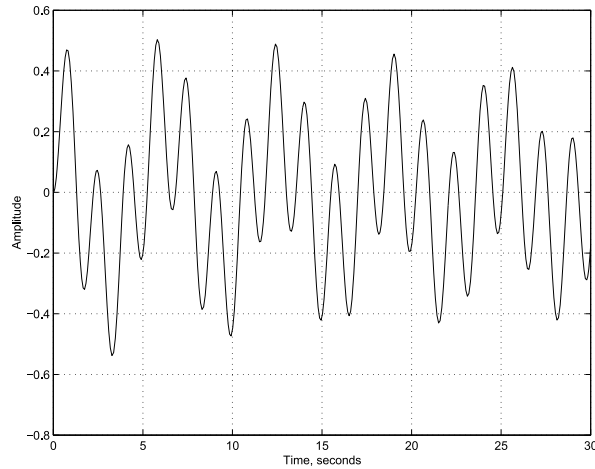


Figure C.52: Response of the randomly excited Duffing oscillator with $\omega = 3.7960$ and $A = 3.7999$.

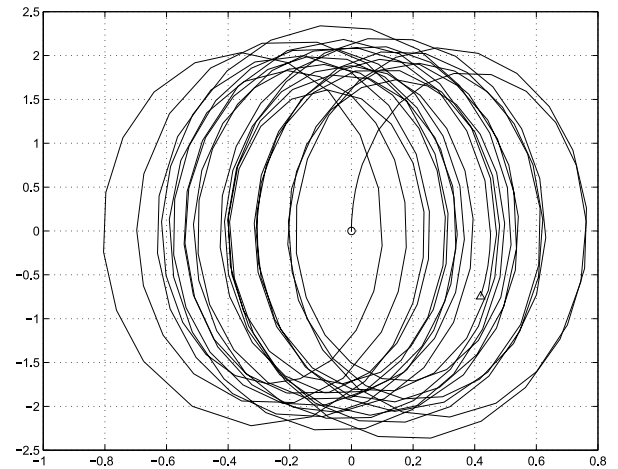


Figure C.55: Phase plot for the nearly resonant case above. The horizontal axis is $x(t)$ and the vertical axis is $\dot{x}(t)$.

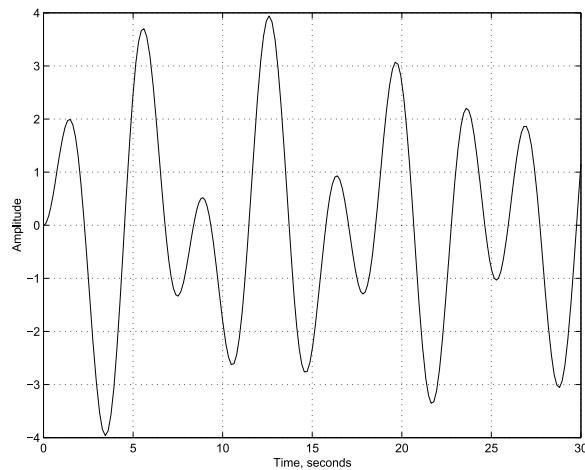


Figure C.53: Response of the Duffing oscillator to amplitude $A = 4.4531$ and forcing frequency $\omega = 1.7404$.

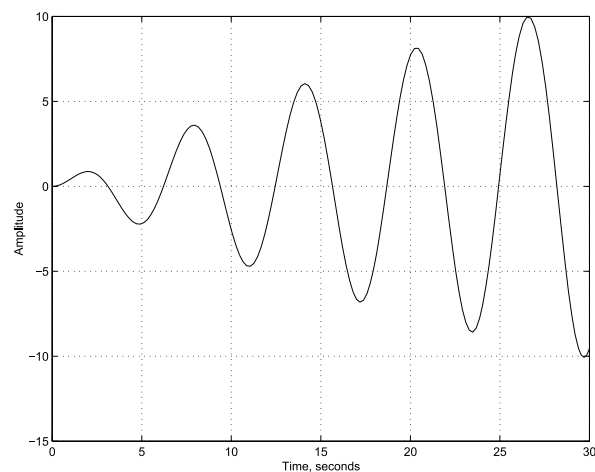


Figure C.54: Response to $A = 1.0062$ and $\omega = 1.0115$.

C.20 Monte Carlo Simulation of a Random System

In the previous section, we looked at a system under random excitation, where we modeled the amplitude of a harmonic input force as a time-varying random process, using a normal or uniform distribution. Consider now a related problem, where we again have a single degree-of-freedom system with harmonic excitation of the form,

$$F(t) = A \cos(\omega t), \quad (\text{C.81})$$

where the parameters A and ω are random variables. As with the previous analysis, we assume that the forcing will be applied in certain ranges for the amplitude and frequency, that is

$$\begin{aligned} A_1 &\leq A \leq A_2, \\ \omega_1 &\leq \omega \leq \omega_2 \end{aligned} \quad (\text{C.82})$$

where A_1 , A_2 , ω_1 , ω_2 are known. One approach would be to model the results with A and ω at mean values, which could form a possible basis to estimate the behavior of a linear system. However, let us address the more interesting (and difficult) problem of estimating the behavior of a nonlinear system, such as the Duffing oscillator from the previous example. There is no longer a guarantee that the mean values for the inputs will correspond to the mean values of the response, so we need an alternate method to arrive at some response statistics.

To approximate the statistics of our nonlinear system, we choose to employ the Monte Carlo methodology, developed systematically by Metropolis and Ulam (1949). In general, the Monte Carlo method has two forms: the probabilistic form, where actual random variable distributions are available, and the deterministic form, where such distributions are unknown. Here, we follow a deterministic algorithm, although adaptation of the method as described here to its probabilistic form is a matter of substituting the known random distributions for the ones we will presume here.

The method in both of its forms can be broken down into three steps:

1. Simulation of the random variable(s);
2. Solution of the deterministic problem for a large number of random variable realizations;
3. Statistical analysis of results.

We have already performed most of the work for the first step above. Since we have ranges over which we presume the random variables A and ω will vary, we can describe both as uniformly distributed. We could also assume a Gaussian distribution, using for the mean the

midpoint of the given ranges. Our standard deviation would then be selected so the endpoint values correspond to three standard deviations from the mean value in either direction. In this way, about 99.7% of the randomly generated values would fall between the prescribed extreme values. We will demonstrate the procedure for uniformly distributed variables, and leave the Gaussian distribution as an exercise.

In order to solve the deterministic problem, we will express the differential equations in a function as was done in a previous example. What we need to create is a separate driver for this function, a code that will run several realizations and return our desired quantities. This means that we need to determine which output quantities interest us. For the purpose of demonstration, we will track the maximum displacement and maximum velocity for each realization. If it was necessary, we could retain the entire history of oscillation for each realization, but a simple demonstration will capture the spirit of the Monte Carlo methodology. The reader is encouraged to modify the code to add complications.¹¹

We will only arrive at useful statistics if we can apply the conditions of ergodicity to the system. Thus, we will again define the system initial conditions so that the transients are removed, like we did with our first example of a random forcing input. These values can either be computed as needed, right before sending to the ODE solver, or they can be computed in advance. We choose the former for our demonstration. (The reader may be wondering if this calculation can be vectorized. It can. The programming is left as an exercise.) Also, with the manner in which we have chosen to specify global variables, we can set the value of the global amplitude and frequency right before calling for the ODE solution. We specify the random variables as row vectors (arbitrarily) and we force them to lie on the desired domains. Then, we set up a `for` loop to run the ODE solver the specified number of times, solving for the appropriate initial displacement and velocity for each. After the ODE solution is found, the maximum displacement and velocity is found and stored, as are the input amplitude and frequency, and the time span used. We may then perform statistical analyses on this data.

The four histograms below show the variations in several parameters for a set of 100 random realizations of the Duffing oscillator. Figure C.56 is the histogram of force amplitudes, and Figure C.57 is the set of forcing frequencies. Since we chose uniform distributions for both of these quantities, we expect the number of values in each range to be about the same. This is the case; the difference between the most and least common force

¹¹ A word of warning: Reduce the value of the damping coefficient c at your own risk. Our initial value for this parameter was 0.05, and the integrator did not complete any attempted run of the code at that value.

amplitude range is eight occurrences, and the difference between the most and least common frequency range is ten occurrences. For a larger sample size, we would expect these numbers to reduce on a percentage basis.¹²

Our results, however, are not uniformly distributed. Figure C.58 shows the variation of maximum displacement values. The vast majority (65%) had maximum displacements under 10. The shape of the distribution is akin to a one-sided Gaussian distribution, implying that our normally distributed inputs led to a Gaussian output. Figure C.59 also shows a one-sided Gaussian distribution, with 85% of the resulting maximum velocities falling under 10. One can readily see how this information could be applied to a design procedure; if we wanted to have 90% certainty of a displacement less than 10, we would see that the default system is inadequate.

It is also possible to pursue more complicated analyses involving other aspects of the data. The reader is invited to rewrite the routine to return all of the displacement and velocity histories. (Three-dimensional matrices will be helpful.) Also, if we had a nonlinear model for, say, a vibrating beam, we could use the displacement and velocity information to derive the maximum stress at some point in the beam for each run. The applications of this analysis are limited only by the systems we wish to study.

From the command window, run the code by typing `[xout, xdout, Aout, wout]=montduff`, and type `hist(Aout)` to plot the histogram of the amplitudes.

Program 1: montduff.m

```
function [xout, xdout, Aout,
wout]=montduff(nruns, e, tspan)
%{ This code performs a Monte Carlo simulation
of the Duffing oscillator. The random
parameters are the forcing amplitude and
frequency, presumed to be described by
uniform distributions. Also, the stiffness
is presumed to be one and the damping presumed
to be 0.1. These values can be changed by
editing the code.
%}
% Parameter check.
if nargin<1 nruns=50; end
if nargin<2 e=0.01; end
if nargin<3 tspan=[0 30]; end
% Stiffness and damping coefficient.
c=0.1; k=1;
```

```
wlo=0; whi=2*pi; % Extreme values for the
frequency.
Alo=0.1; Ahi=10; % Extreme values for the
amplitude.
A=Alo+(Ahi-Alo)*rand(1,nruns);
w=wlo+(whi-wlo)*rand(1,nruns);
%{ Initializing the t and x vectors. This is
why we want to specify the length of the time
vector.
%}
xout=zeros(1,nruns);
xdout=zeros(1,nruns);
for i=1:nruns
    AA=A(1,i); ww=w(1,i); % Set global
parameters.
    x0=AA*cos(ww*tspan(1));
    xd0=-AA*ww*sin(ww*tspan(1)); % Choose
initial conditions to remove transient.
    xinit=[x0, xd0];
    [trecv,xrecv]=ode45(@duffdx,tspan,xinit,"
c, k ,e, AA, ww); % The run.
    xout(1,i)=max(xrecv(:,1)); % Here we could
perform a more complex analysis
    xdout(1,i)=max(xrecv(:,2)); % of our data.
end
wout=w;
Aout=A;
```

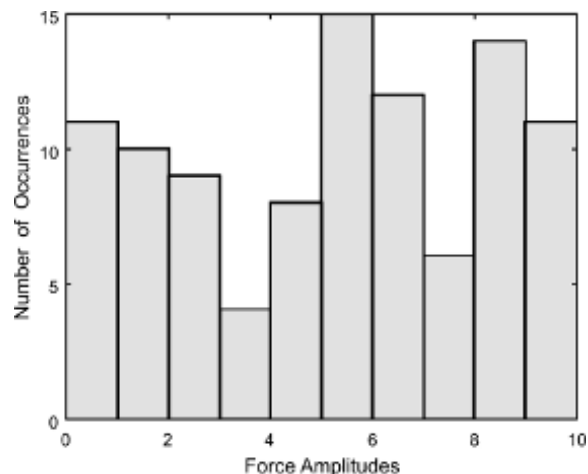


Figure C.56: Histogram showing the force magnitudes for 100 random runs.

¹² A test performed with 1000 simulations showed that the difference in amplitude occurrences was 30 of 1000 between most and least common, or 3%. The difference in number of occurrences for the most and least common frequency was also around 30 out of 1000.

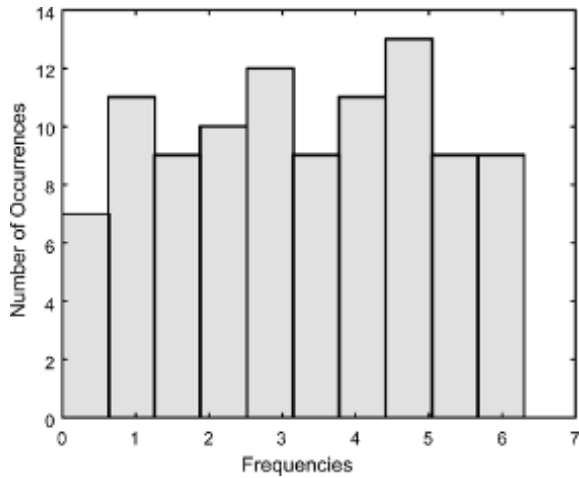


Figure C.57: Histogram showing the frequencies applied to the Duffing oscillator.

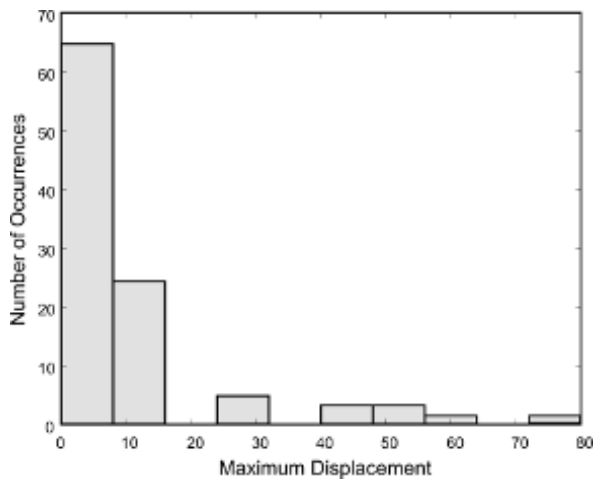


Figure C.58: Histogram of maximum displacement values.

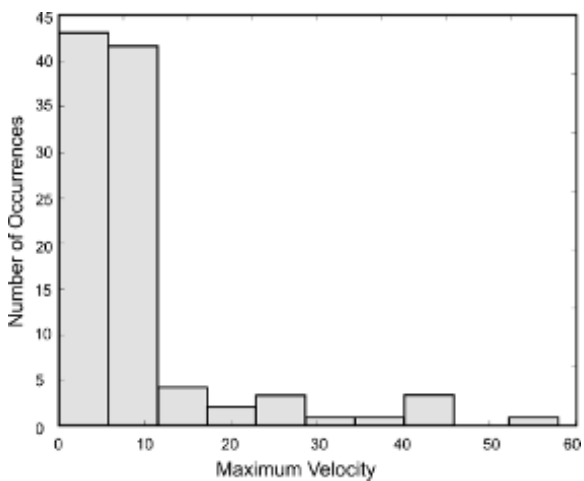


Figure C.59: Histogram showing maximum velocities of the simulated responses.

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