Bernd Bertsche

Reliability in Automotive and Mechanical Engineering

VDI

Springer

Reliability in Automotive and Mechanical Engineering

Bernd Bertsche

Reliability in Automotive and Mechanical Engineering

Determination of Component and System Reliability

In Collaboration with Alicia Schauz and Karsten Pickard

With 337 Figures and 66 Tables

Springer

Prof.Dr. Bernd Bertsche
Universität Stuttgart
Fak. 07 Maschinenbau
Inst. Maschinenelemente
Pfaffenwaldring 9
70569 Stuttgart
Germany
bertsche@ima.uni-stuttgart.de

ISBN: 978-3-540-33969-4 e-ISBN: 978-3-540-34282-3

DOI: 10.1007/978-3-540-34282-3

Library of Congress Control Number: 2008921996

Cover design: eStudio Calamar S.L., F. Steinen-Broo, Pau/Girona, Spain

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

Reliability and maintenance coupled with quality represent the three major columns of today's modern technology and life. The impact of these factors on the success and survival of companies and organisations is more important than ever before. Although these disciplines may be viewed as non-profitable, experience has shown that neglecting or omitting them can lead to severe consequences. This is underlined by the dramatically increasing number of callbacks. In fact, over the last fifteen years the number of callbacks has tripled.

Just recently a huge recall in the toy industry occurred due to lead contaminated toys. In the automotive industry callbacks arise regularly for several varying reasons. Since products are becoming ever more complex and the available time for development is continuously decreasing, the necessity for and influence of the three pillars: reliability, maintenance and quality, will only continue to increase in the future. Considering one classic example of a complex product, the passenger car, while bearing the callback statistics in mind, it is not surprising that the attributes "reliability" and "quality" are the two most important considerations for customers buying a new car.

This trend has been observed and confirmed over several years. The increasing demand on reliability methods combined with the importance of studying and understanding them led me to the decision to compose a book about reliability and maintenance. Originally, this book was only published in German, but requests from colleagues and companies all over Europe and the USA induced me to bring out the English translation as well. This book considers the basics of reliability and maintenance along with further improvements and enhancements which were found by extensive research work. In the following chapters, fundamentals are combined with practical experiences and exercises, thus allowing the reader to gain a more detailed overview of these crucial subjects.

The present book could not have originated without the help of the following persons, to whom I wish to express my appreciation. First of all,

many thanks to Prof. Gisbert Lechner, who was initiator of the German edition. I am grateful to Mrs. Alicia Schauz und Mr. Karsten Pickard for the translation from German into English. Through their editorial and organisational work accompanied by their dedication and commitment they together enabled and formed this book. I also would like to thank Ms. Andrea Dieter for editing and overworking the illustrations. My exceptional thanks goes to Mr. G.J. McNulty for his useful editorial suggestions. Finally, I would like to thank the publishing company Springer for their helpful and professional cooperation.

Stuttgart, Autumn 2007 Prof. Dr. B. Bertsche

Contents

1 Introduction

„It is impossible to avoid all faults“

„Of cause it remains our task to avoid faults if possible“

Sir Karl R. Popper

Today, the term reliability is part of our everyday language, especially when speaking about the functionality of a product. A very reliable product is a product that fulfils its function at all times and under all operating conditions. The technical definition for reliability differs only slightly by expanding this common definition by probability: reliability is the probability that a product does not fail under given functional und environmental conditions during a defined period of time (VDI guidelines 4001). The term probability takes into consideration, that various failure events can be caused by coincidental, stochastic distributed causes and that the probability can only be described quantitatively. Thus, reliability includes the failure behaviour of a product and is therefore an important criterion for product evaluation. Due to this, evaluating the reliability of a product goes beyond the pure evaluation of a product's functional attributes.

According to customers interviewed on the significance of product attributes, reliability ranks in first place as the most significant attribute, see Figure 1.1. Only costs are sometimes considered to play a more important role. Reliability, however, remains in first or second place. Because reliability is such an important topic for new products, however it does not maintain the highest priority in current development.

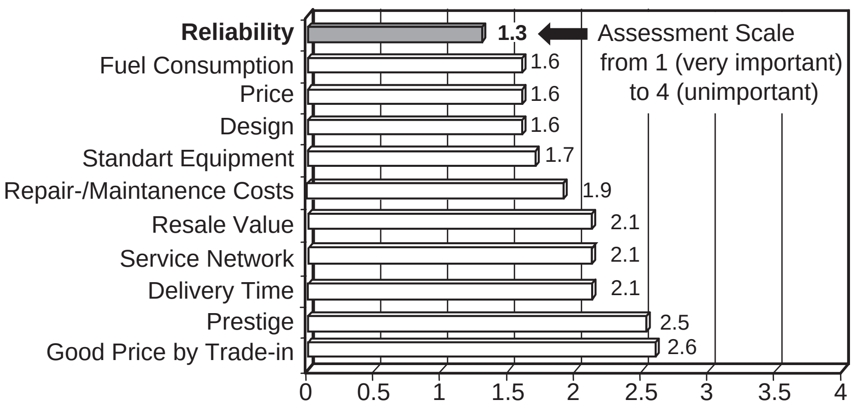


Figure 1.1. Car purchase criteria (DAT-Report 2007)

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_1, © Springer-Verlag Berlin Heidelberg 2008

Surveys show that customers desire reliable products. How does product development reflect this desire in reality? Understandably, companies protect themselves with statements concerning their product reliability. No one wants to be confronted with a lack of reliability in their product. Often, these kinds of statements are kept under strict secrecy. An interesting statistic can be found at the German Federal Bureau of Motor Vehicles and Drivers (Kraftfahrt-Bundesamt) in regards to the number of callbacks due to critical safety defects in the automotive industry: in the last ten years the amount of callbacks has tripled (55 in 1998 to 167 in 2006), see Figure 1.2. The related costs have risen by the factor of eight! It is also well known, that guarantee and warranty costs can be in the range of a company's profit (in some cases even higher) and thus make up 8 to 12 percent of their turnover. The important triangle in product development of cost, time and quality is thus no longer in equilibrium. Cost reductions on a product, the development process and the shortened development time go hand in hand with reduced reliability.

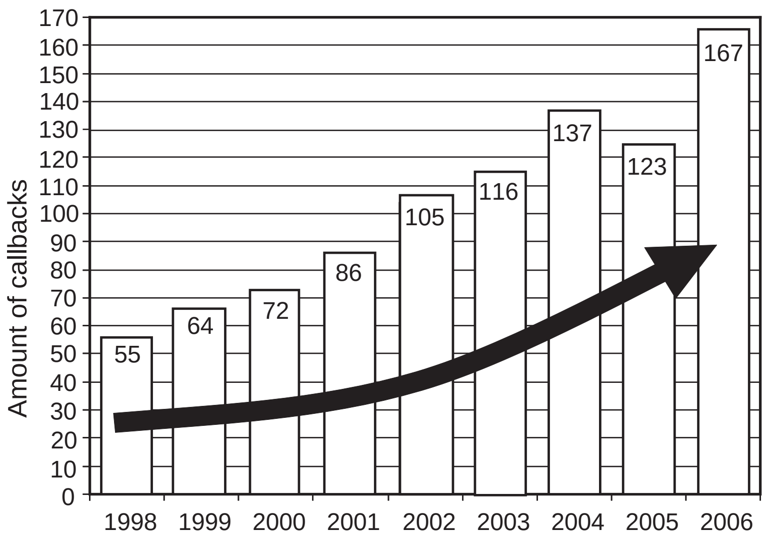


Figure 1.2. Development of callbacks in automotive industry

Today's development of modern products is confronted with rising functional requirements, higher complexity, integration of hardware, software and sensor technology and with reduced product and development costs. These, along with other influential factors on the reliability, are shown in Figure 1.3.

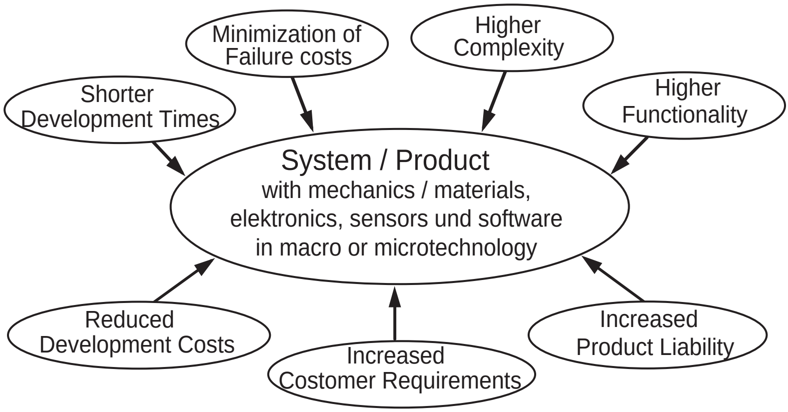


Figure 1.3. Factors which influence reliability

To achieve a high customer's satisfaction, system reliability must be examined during the complete product development cycle from the viewpoint of the customer, who treats reliability as a major topic. In order to achieve this, adequate organizational and subject related measures must be taken. It is advantageous that all departments along the development chain are integrated, since failures can occur in each development stage. Methodological reliability tools, both quantitative and qualitative, already exist in abundance and when necessary, can be corrected for a specific situation. A choice in the methods suitable to the situation along the product life cycle, to adjust them respectively to one another and to implement them consequently, see Figure 1.4, is efficacious.

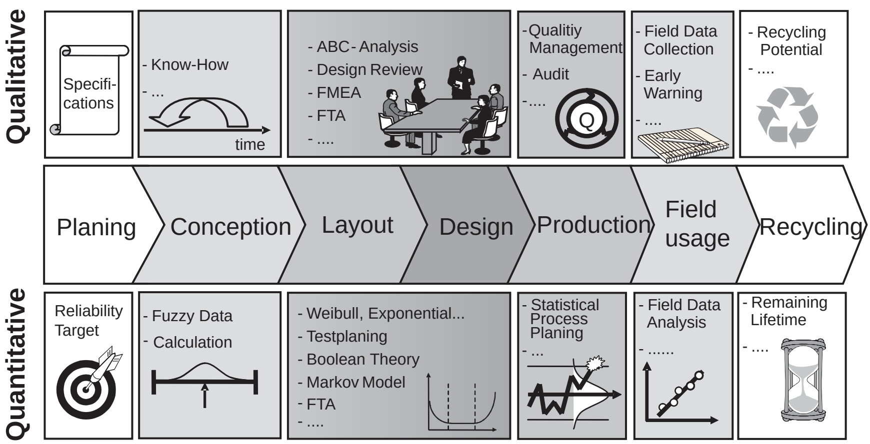


Figure 1.4. Reliability methods in the product life cycle

A number of companies have proven, even nowadays, that it is possible to achieve very high system reliability by utilizing such methods.

The earlier reliability analyses are applied, the greater the profit. The well-known “Rule of Ten” shows this quite distinctly, see Figure 1.5. In looking at the relation between failure costs and product life phase, one concludes that it is necessary to move away from reaction constraint in later phases (e.g. callbacks) and to move towards preventive measures taken in earlier stages.

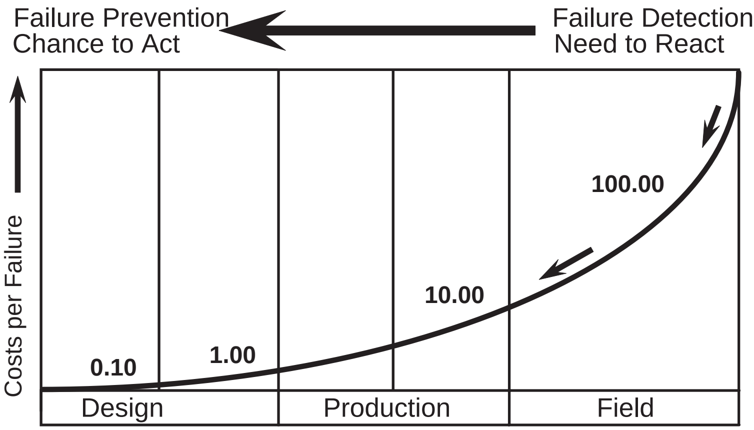


Figure 1.5. Relation between failure costs and product life phase

The easiest way to determine the reliability of a product is in hindsight, when failures have already been detected. However, this information is used for future reliability design planning. As mentioned earlier, however, the most sufficient and ever more required solution is to determine the expected reliability in the development phase. With the help of an appropriate reliability analysis, it is possible to forecast the product reliability, to identify weak spots and, if needed, comparative tests can be carried out, see Figure 1.6.

For the reliability analysis quantitative or qualitative methods can be used. The quantitative methods use terms and procedures from statistics and probability theory. In Chapter 2 the most important fundamental terms of statistics and probability theory are discussed. Furthermore, the most common lifetime distributions will be presented and explained. The Weibull distribution, which is mainly and commonly used in mechanical engineering, will be explained in detail.

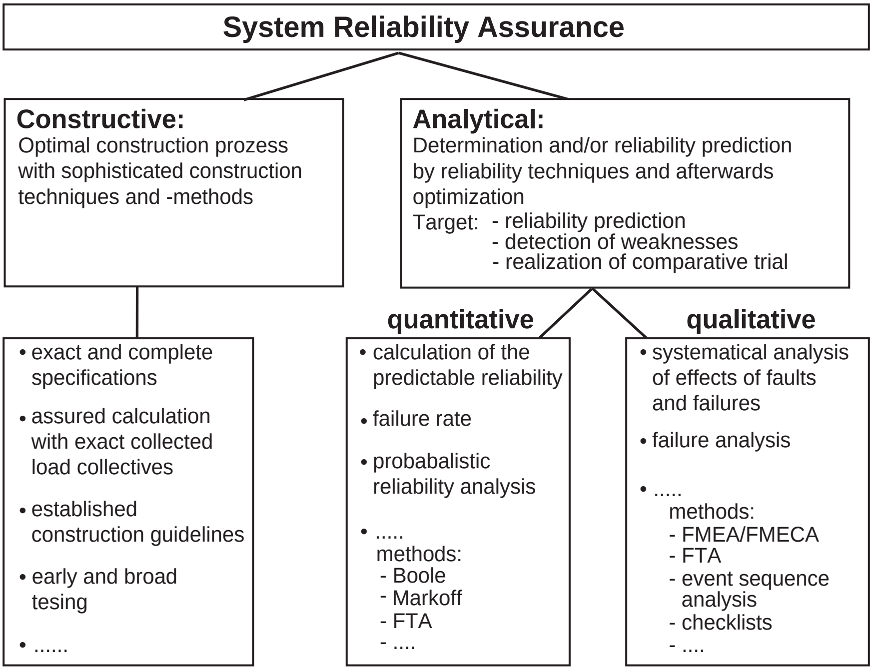


Figure 1.6. Securing of system reliability

Chapter 3 illustrates an example of a complete reliability analysis for a simple gear transmission. The described procedure is based on the fundamentals and methods described in the previous chapter.

The most well-known qualitative reliability method is the FMEA (Failure Mode and Effects Analysis). The essential contents, according to the current standard in the automotive industry (VDA 4.2), are shown in Chapter 4.

The fault tree analysis, described in Chapter 5, can be used either as a qualitative or as a quantitative reliability method.

One main focus of this book is the analysis of lifetime tests and damage statistics, which will be dealt with in Chapter 6. With these analyses general valid statements concerning failure behaviour can be made. In order to describe the lifetime distribution the Weibull distribution is used, which is the most common distribution in mechanical engineering. Next to the graphical analyses of failure times, analytical analyses and their theoretical basics will be discussed. The important terms "order statistic" and "confidence range" will be explained in detail.

There is little collected and edited information pertaining the failure behaviour of mechanical components. However, the knowledge of the failure behaviour of a component is necessary, in order to be able to predict the expected reliability under similar application conditions. With the help of system theory it is also possible to calculate the expected failure behaviour of a system. In Chapter 7 results from a reliability data base for the machine components gear wheels, axles and roller bearings will be presented. In many cases the indicated Weibull parameters can prove to serve as a first orientation.

To prove reliabilities before the start of production, it is obligatory to carry out the appropriate tests. Here, the amount of test specimens, the required test period length and the achievable confidence level may be of interest. In Chapter 8 the planning of reliability tests will be described.

Each quantitative reliability method portrays a kind of enhanced fatigue strength calculation. The basic principles of a lifetime calculation for machine components are summarized in Chapter 9.

The reliability and the availability of systems, which include repairable elements, can be determined by various calculation models. Chapter 10 describes methods in their differing complexity and their assessment for repairable elements.

In order to achieve high system reliability, an integrated process treatment is compulsory. For this, a reliability safety program has been developed. This program will be described with its basic elements in Chapter 11. In conclusion, this chapter offers a complete overview on an optimal reliability process.

For all the chapters there are problems at the end of each one and the solutions can be found at the end of chapter 11.

2 Fundamentals of Statistics and Probability Theory

A qualitative reliability analysis provides a conceptual basis for the degree of confidence placed on a particular component or system and should be capable in the incipient stages of design for alteration of these components. A quantitative reliability prognosis gives a probability assessment of the component based on well founded statistical techniques. This chapter therefore, outlines the various methods of both qualitative and quantitative methods shown in Figure 2.1.

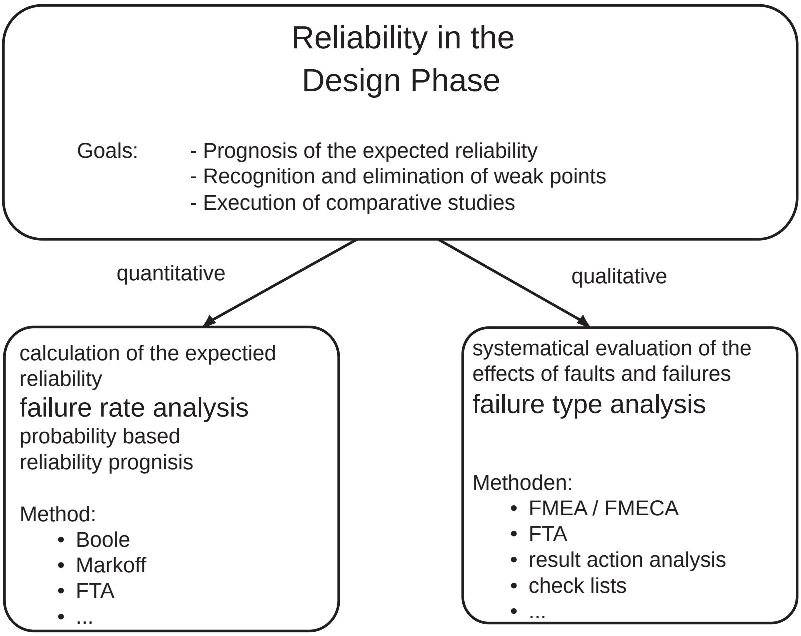


Figure 2.1. Options for reliability analysis

The results of the Wöhler tests in Figure 2.2 and Figure 2.3 show this. Despite identical conditions and loads, strongly differing down times resulted [2.15]. Out of these results it is *not* possible to assign a bearable cycles-to-failure to a component. The cycles-to-failure n_{LC} or the lifetime t

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_2, © Springer-Verlag Berlin Heidelberg 2008

can be seen as random variables, which are subject to a certain statistical spread [2.1, 2.5, 2.23, 2.29, 2.33]. When looking at reliability, the designated range of dispersion between $n_{LC,\,min}$ and $n_{LC,\,max}$ as well as which down times occur more often are of interest. For this it is necessary to know *how* the lifetime values are distributed.

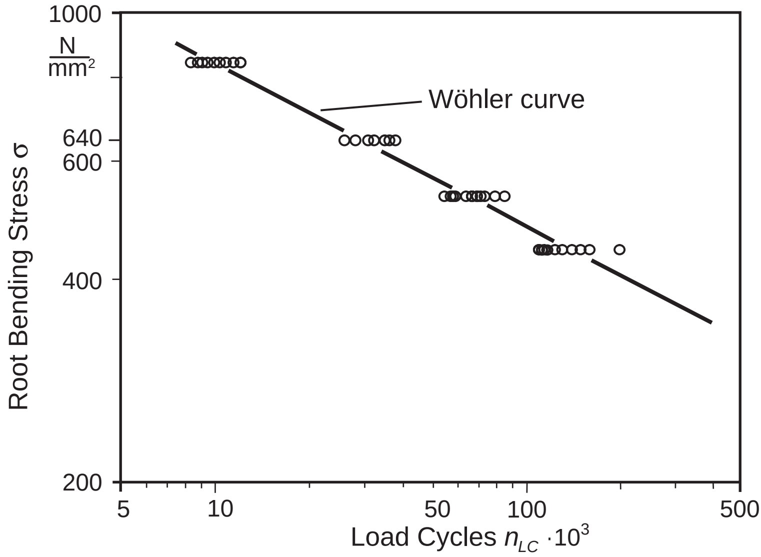


Figure 2.2. Tooth failure – Wöhler test [2.15] with the statistical spread of down times

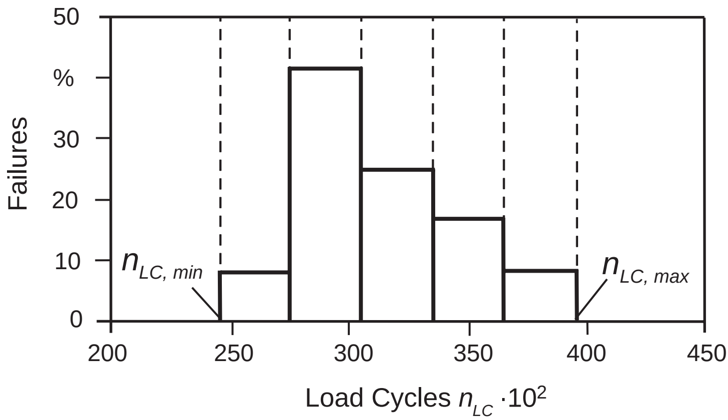


Figure 2.3. Histogram for the frequency of the load $\sigma = 640$ N/mm^2 from Figure 2.2

Terms and procedures from statistics and probability theory can be used for down times observed as random events. Therefore, the most important terms and fundamentals from statistics and probability theory will be dealt with in Section 2.1.

An introduction and explanations of generally used lifetime distributions is presented in Section 2.2. In this section the Weibull distribution, one of the most adopted in mechanical engineering will be explained.

Section 2.3 combines component reliability with system reliability with the help of Boolean theory. The Boolean theory can be understood as the fundamental system theory. Other system theories can be found in Chapter 10.

2.1 Fundamentals in Statistics and Probability Theory

The failure behaviour of components and systems can be represented graphically with various statistical procedures and functions. How this is done will be described in this chapter. Furthermore, “values” will be dealt with, with which the complete failure behaviour can be reduced to individual characteristic key figures. The result is a very compressed but also simplified description of the failure behaviour.

2.1.1 Statistical Description and Representation of the Failure Behaviour

In the following sections the four different functions for representing failure behaviour will be introduced. The individual functions stem from the observed failure times and can be carried over to one another. With each function certain statements can be made concerning the failure behaviour. The use of a certain function therefore depends on a specific question posed.

2.1.1.1 Histogram and Density Function

The simplest possibility to display failure behaviour graphically is with the histogram of the failure frequency, see Figure 2.4.

The failure times in Figure 2.4a occur at random within a certain time period. The representation in Figure 2.4b is the result after sorting the strewed failure times.

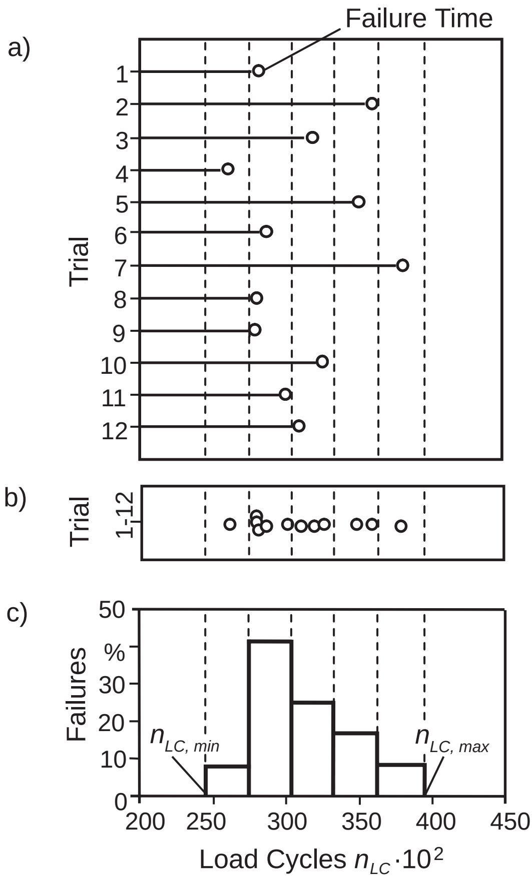


Figure 2.4. Failure times and histogram of the failure frequencies for a stress of σ = 640 N/mm^2 from Figure 2.2: a) collected failure times in trials; b) sorted failure times; c) histogram of the failure frequencies with empirical density function $f^*(t)$

The denser the data lay together in Figure 2.4b, the more "frequently" the failure times occur in that certain period. In order to show this graphically, a histogram of the failure frequencies is created, Figure 2.4c.

Therefore, the abscissa is divided into intervals of time which are denoted as classes. The quantity of failures is determined for each class. If a failure falls directly between two classes, then it is counted to both classes as half a failure. However, by assigning the intervals carefully, this can normally be avoided. The quantity of failures in each class is represented by beams with various respecting heights.

For the height or y-coordinate of each beam, the absolute frequency

$$h_{abs} = \text{number of failures in one class} = n_A \tag{2.1}$$

or the more common, relative frequency

$$h_{rel} = \frac{\text{number of failures in one class}}{\text{total number of failures}} = \frac{n_A}{n} \tag{2.2}$$

can be used. In Figure 2.4c the beam heights are determined using the relative frequency, as can be seen on the percent scale for the ordinate.

The division of the time axis into classes and the assignment of failure times to the individual classes is called classification. In this process information is lost, since a certain amount of failures is assigned to one frequency independent of the exact failure time in the interval. Through the classification, each failure within a certain class is assigned the value of that class's mean. However, a loss of information is compensated by a win in overview.

The amount of classes is not always simple to determine. If the classes are chosen to be too large, then too much information is lost. In an extreme case, there is only one beam, which of course offers little overview. If the classes are chosen to be too small, small breaks can occur along the time axis. Such breaks interrupt the continuity of the failure behaviour and are thus unfit for a correct description.

The following Equation (2.3) can be used for a rough approximation or first estimate for the amount of classes [2.30]:

$$\text{Amount of classes} \approx \sqrt{\text{Total number of failures or experimental values}}$$

$$n_c \approx \sqrt{n}\,. \tag{2.3}$$

Alternative approaches to calculate the amount of classes and the class size are given in [2.30]:

$$n_c \approx 1 + 3{,}32 \cdot \log n\,, \tag{2.4}$$

$$n_c \approx 2 \cdot \sqrt[3]{n}\,, \tag{2.5}$$

$$n_c \approx 5 \cdot \log n\,. \tag{2.6}$$

Up to a test specimen size of $n = 50$ the results are comparable, but the results differ strongly for larger test specimen sizes. A rule of thumb for estimating the class size b of a frequency distributions is based on the range R and the test specimen size n:

$$b \approx \frac{R}{1 + 3{,}32 \cdot \log n}. \tag{2.7}$$

The range R is the difference between the largest and smallest value within the test specimen.

$$R = n_{LC,max} - n_{LC,min}. \tag{2.8}$$

Instead of in a histogram, the failure behaviour can also be described with the often used "empirical density function $f^*(t)$, see Figure 2.5.

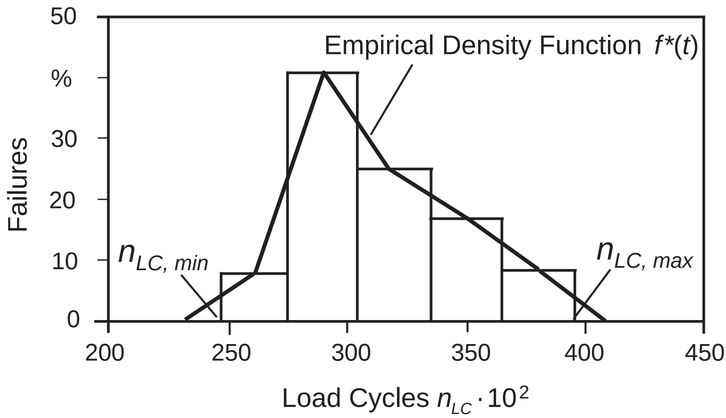


Figure 2.5. Histogram of failure frequencies and the empirical density function $f^*(t)$

In the density function the midpoints of the beams in the histogram are connected with straight lines. In this way a function between the failure time and failure frequency is represented. The term "empirical" for the density function implies, that the density function is determined based on a test specimen or a limited number of failures.

The actual "ideal" density function is reached when the amount n of tested components is increased. The amount of classes can then be raised according to the simple Equation (2.3). This means that the class size becomes continually smaller while the y-coordinate of the resulting frequencies remains relatively unchanged. For the limit $n \to \infty$ the contour of the histogram becomes an ever smoother and continuous curve, see Figure 2.6.

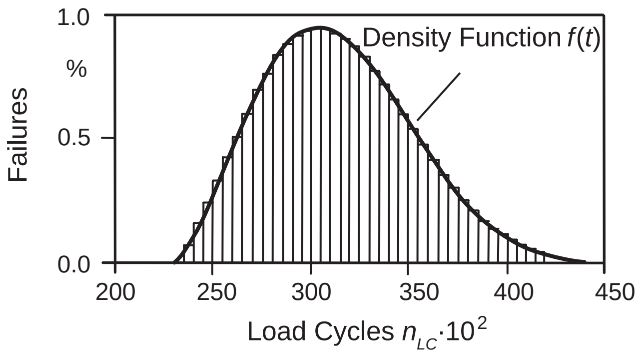


Figure 2.6. Histogram of failure frequencies and density function $f(t)$ (amount of failures $n \to \infty$)

This limit curve represents the actual density function *f(t)*. Figure 2.6 has an altered ordinate scale in comparison to Figure 2.5, since the decreased class size results in fewer failures per class.

The limit $n \to \infty$ means that all parts of a large total quantity were tested and the exact failure behaviour was determined. Thus, it is possible to shift from the experimentally determined frequencies to the theoretical probabilities. The fundamentals for this transition can be explained by the Bernoulli law for large numbers. These theoretical coherences will be described in more detail in Section 2.1.3.

The empirical density function $f^*(t)$ experiences large variations, especially for a small sample and varies considerably from the ideal density functions $f(t)$. The latter is determined from information extracted from $f(t)$, as explained in Chapter 6.

The area under the density function $f(t)$ is equal to 1 if the relative frequencies are used for the y-coordinates.

The histogram of frequencies as well as the density function describes the amount of failures as a function of time. Thus, they offer the clearest and most simple possibility to represent the failure behaviour. Along with the range of dispersion of failure times one is able to recognize in which interval the most failures occur.

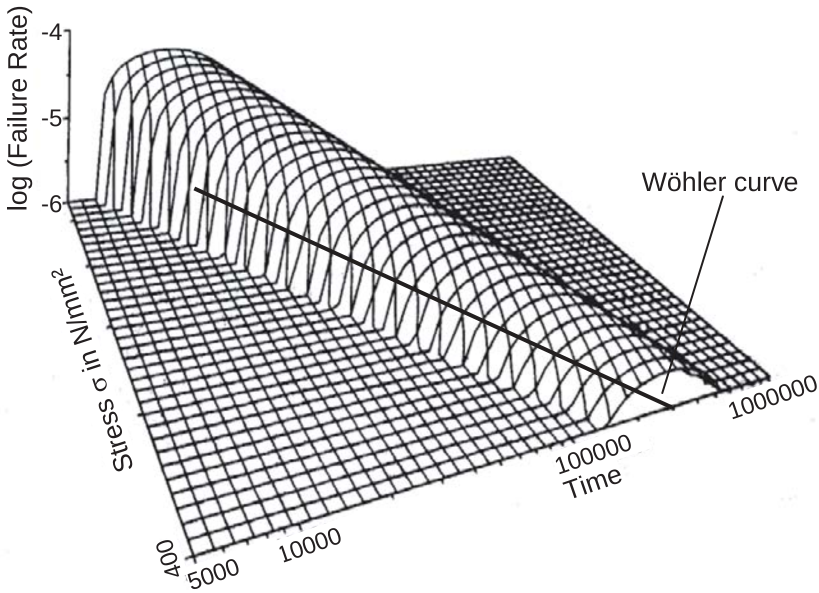


Figure 2.7. Three dimensional Wöhler curve (or SN-curve) for the tests in Figure 2.2

With the density function $f(t)$ the Wöhler curve in Figure 2.2, also referred to as the SN-curve, can be illustrated as a three dimensional "mountain range", see Figure 2.7. A failure frequency is shown for each load and corresponding time.

Figure 2.8 shows an example of a density function for a commercial vehicle transmission. Here, 2115 damaging events are observed, divided into 82 classes [2.28].

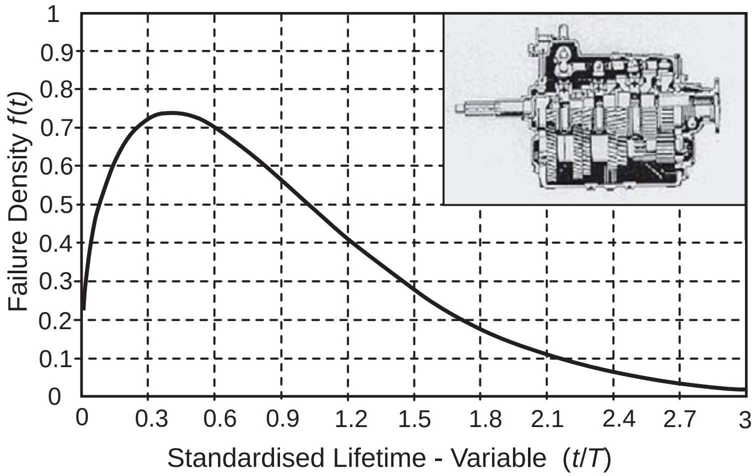


Figure 2.8. Failure density $f(t)$ of a 6 gear commercial vehicle transmission

The distribution is symmetrical on the left side. This indicates that the failures are mainly early failures. Such failures could be traced back to material or assembly failures, which are common for complex systems.

A further example of a density function is shown in Figure 2.9. Here, one sees the amount of deaths as a function of age at death. First, one is able to see a span of child deaths, then a second area with very few deaths between 15 and 40 years of age, followed by an increasing number of deaths with increasing age. For men, the most deaths occur at an age of 80, whereas for women, the most deaths occur at a later age.

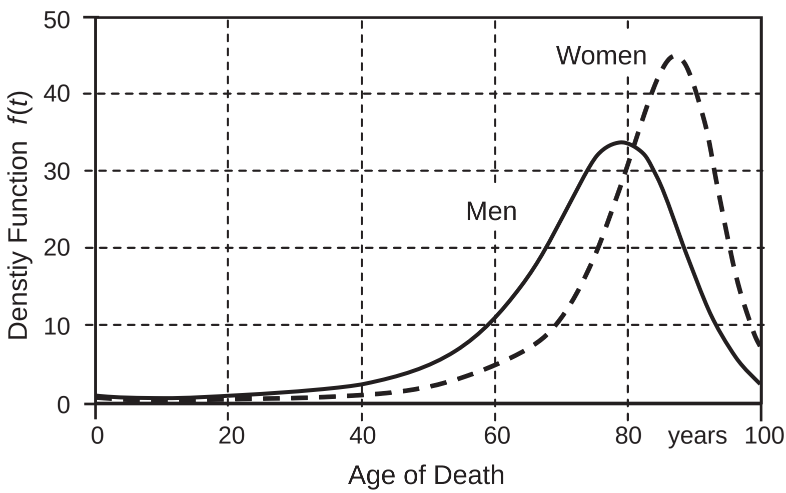


Figure 2.9. Density function $f(t)$ of human deaths

2.1.1.2 Distribution Function or Failure Probability

In many cases, the number of failures at a specific point in time or in a specific interval is not of interest but rather, how many components in *total* have failed up to a time or until a certain interval is reached. This question can be answered with a histogram of the cumulative frequency. The observed failures, see Figure 2.10a, are added together with each progressive interval. The result is the histogram of the cumulative frequency shown in Figure 2.10b.

The cumulative frequency $H(m)$ for class m can be calculated as:

$$H(m) = \sum_{i=1}^{m} h_{rel}(i)\,, \qquad i\text{: number of class.} \tag{2.9}$$

The sum of failures can be represented as a function just as the density function in Section 2.1.1.1. This function is called the "empirical distribution function $F^*(t)$", see Figure 2.10b.

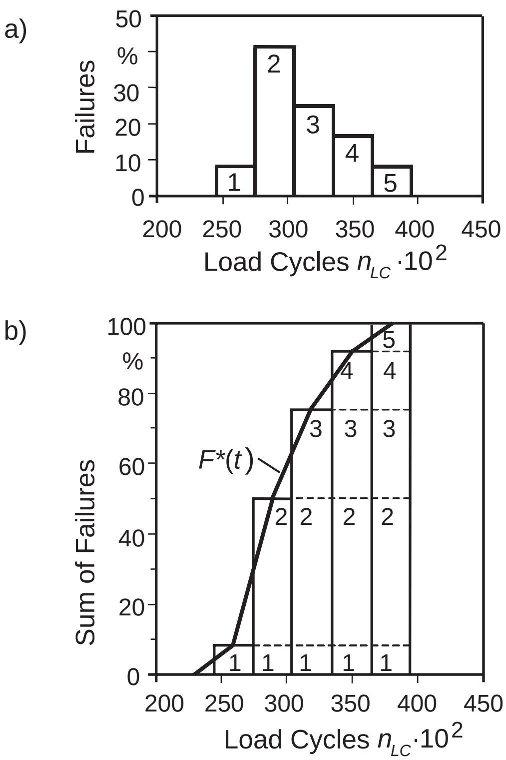


Figure 2.10. Cumulative frequency and distribution function: a) histogram of frequencies; b) histogram of the cumulative frequency and empirical distribution function $F^*(t)$

The actual distribution function $F(t)$ is determined by increasing the number of experimental values. Thus, the class size decreases continuously and the contour of the histogram becomes a smooth curve for the limit $n \to \infty$. The result is the distribution function $F(t)$, see Figure 2.11.

The distribution function always begins with $F(t) = 0$ and increases monotonically, since for each time or interval a positive value is added – the observed failure frequency. The function always ends with $F(t) = 1$ after all components have failed.

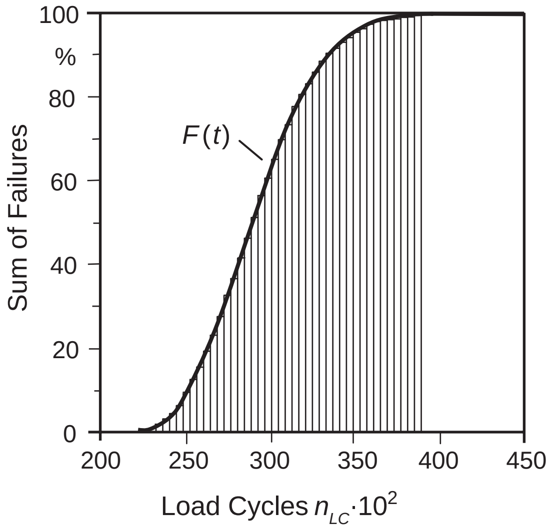


Figure 2.11. Histogram of the cumulative frequency and distribution function $F(t)$ (number of failures $n \to \infty$)

The limit of Equation (2.9) results in a distribution function as the integral of the density function:

$$F(t) = \int f(t)dt\,. \tag{2.10}$$

Thus, the density function is the derivative of the distribution function:

$$f(t) = \frac{dF(t)}{dt}\,. \tag{2.11}$$

In reliability theory the distribution function $F(t)$ is called the "failure probability $F(t)$" (F for failure). This term is adequate, since the function $F(t)$ describes the probability, with which failures occur at the time t.

Although the failure probability is visually less clear than the density function, it can be used to evaluate trials. Therefore, the failure probability is the function used most often in Chapter 6.

Here again, the 6 gear commercial vehicle transmission serves as an example of failure probability in reality, Figure 2.12. Due to the standardised lifetime it is again only possible to make a qualitative statement. It is shown that, for example, the B_{10} value corresponding to $F(t)$ = 10 % equals 0.2. This means that 10% of the transmissions are defective when the lifetime 0,2 · T has been reached.

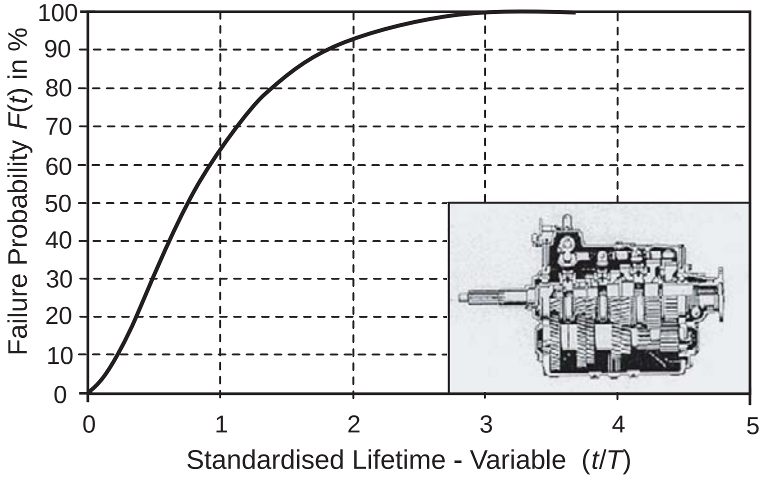


Figure 2.12. Failure probability $F(t)$ of a 6 gear commercial vehicle transmission

Figure 2.13 shows the concrete failure probability $F(t)$ corresponding to the example of human death. With this function for $F(t)$, for example, 20% of a generation has passed away by their 60th birthday.

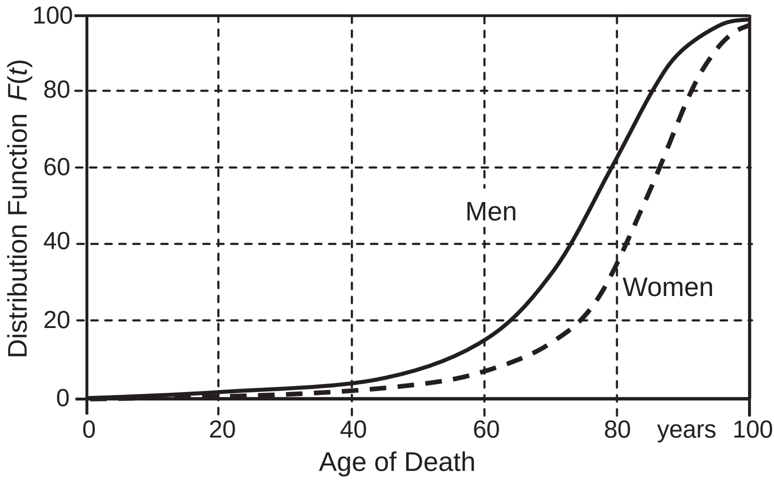


Figure 2.13. Failure probability $F(t)$ for human deaths

2.1.1.3 Survival Probability or Reliability

The failure probability in Section 2.1.1.2 described the sum of failures as a function of time. However, for many applications the sum of component parts or machines that are still intact is of interest.

This sum of functional units can be displayed with a histogram of the survival frequency, see Figure 2.14. This histogram results when the number of defect units is subtracted from the total number of components or machines. The empirical survival probability $R^*(t)$ is shown in Figure 2.14, which results by connecting the beam midpoints with straight lines.

The sum of failures and the sum of the intact units in each class i or at any point in time t always add up to 100%. The survival probability $R(t)$ is thus the complement to the failure probability $F(t)$.

$$R(t) = 1 - F(t). \tag{2.12}$$

With Equation (2.12) the histogram in Figure 2.14 can also be determined by reflecting the histogram in Figure 2.10 over the 50% axis. The survival probability $R(t)$ always begins with $R(t)$ = 100%, since no failures have occurred at $t = 0$. The function $R(t)$ decreases monotonically and ends with $R(t)$ = 0% after all units have failed.

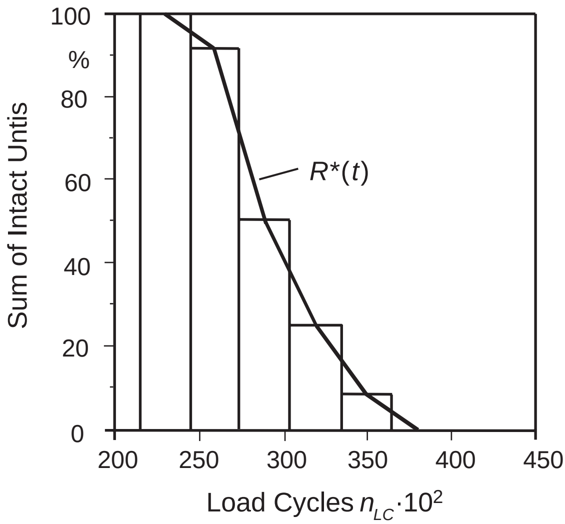


Figure 2.14. Representation of the failure behaviour from Figure 2.10 with the histogram of the survival probability or the empirical survival probability $R^*(t)$

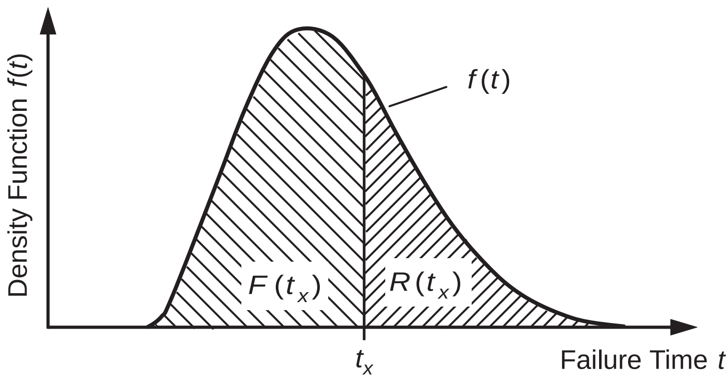


Figure 2.15. Survival probability *R*(*t*) as a complement to the failure probability *F*(*t*)

Figure 2.15 shows a visual representation of the Equation (2.12) for the failure time t_x with the help of the density function and the Equation (2.10).

In reliability theory the survival probability is called "reliability *R*(*t*)". The function *R*(*t*) corresponds to the term reliability as defined in [2.2, 2.3, 2.36, 2.38]:

RELIABILITY is the probability that a product does not fail during a defined period of time under given functional and surrounding conditions.

Thus, reliability is the time dependent probability $R(t)$ for non-failure. It should be noticed, that in order to make a statement about the reliability of a product, not only the considered time period is important but also the exact functional and surrounding conditions are especially required.

For the commercial vehicle transmission, Figure 2.16, a standardised lifetime of 0.2 results in a survival probability of $R(t)$ = 90%, which corresponds to a failure probability of $F(t)$ = 10%, see Equation (2.12). Thus, 90% of the transmissions survive a lifetime of $0.2 \cdot T$.

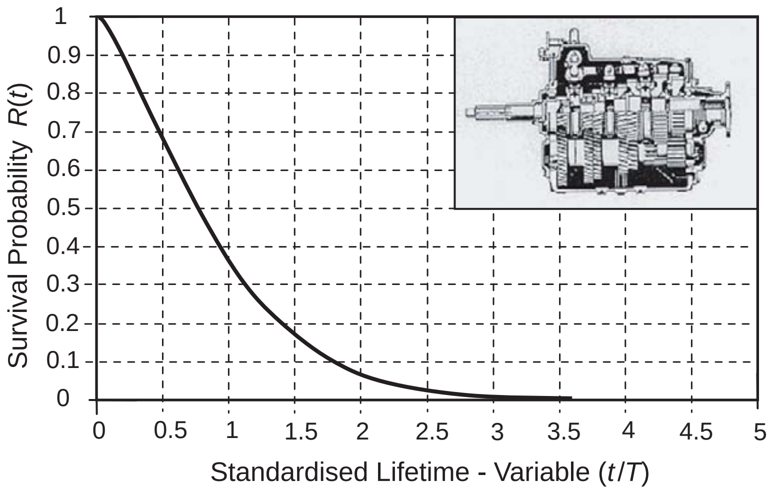


Figure 2.16. Survival probability $R(t)$ of a 6 gear commercial vehicle transmission

For the survival probability of men, see Figure 2.17, is $R(t)$ = 80% for an age of death of 60. This in turn corresponds to a failure probability of $F(t)$ = 20%, see Figure 2.13.

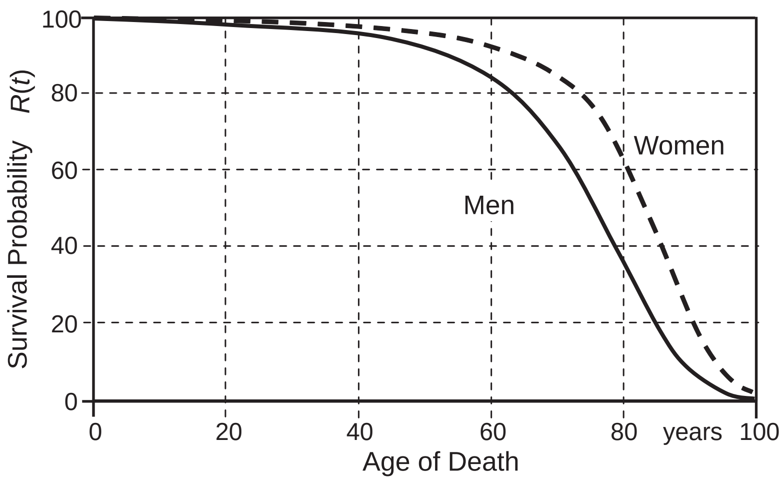


Figure 2.17. Survival probability $R(t)$ for human beings

2.1.1.4 Failure Rate

To describe the failure behaviour with the failure rate $\lambda(t)$, the failures at the point in time t or in a class i are not divided by the sum of total failures, as for the relative frequency in Section 2.1.1.1, but rather are divided by the sum of units still intact:

$$\lambda(t) = \frac{\text{Failures (at the point in time } t \text{ or in class } i\text{)}}{\text{sum of units still intact (at the point in time } t \text{ or in class } i\text{)}} \tag{2.13}$$

Figure 2.18 shows the histogram of the failure rates and the function of the empirical failure rate $\lambda^*(t)$ for the trial run in Figure 2.4. It can be seen, that the failure rate in the last class unavoidably approaches ∞, since there are no longer any intact units. Thus, the denominator in Equation (2.13) approaches zero.

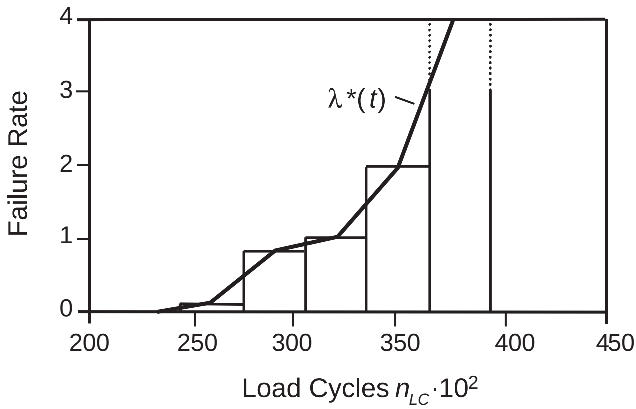


Figure 2.18. Histogram of the failure rate and the empirical failure rate $\lambda^*(t)$ for the trial run in Figure 2.4

The density function $f(t)$ describes the number of failures and the survival probability $R(t)$ describes the number of units still intact. Therefore, the failure rate $\lambda(t)$ can be calculated as the quotient of these two functions:

$$\lambda(t) = \frac{f(t)}{R(t)}. \tag{2.14}$$

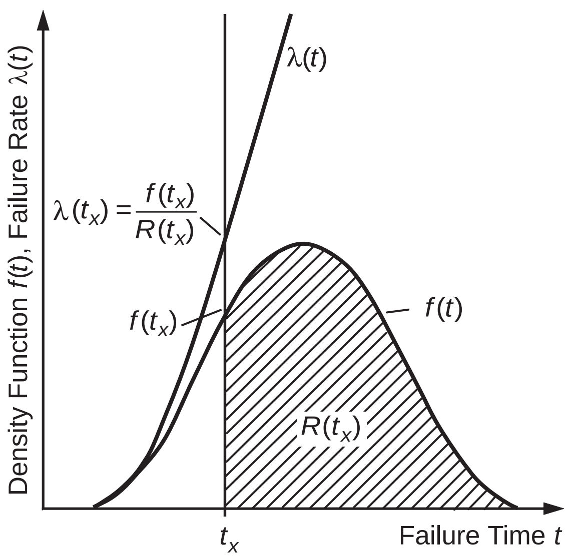


Figure 2.19. Determination of the failure rate out of the density function and survival probability

Figure 2.19 shows a graphical representation of Equation (2.14) for the failure time t_x.

The failure rate at time t can be interpreted as a measurement for the risk that a part will fail, with the prerequisite that the component has already survived up to this point in time t. The failure rate at a point in time specifies how many of the still intact parts will fail in the next unit of time.

The failure rate $\lambda(t)$ is used very often not only to describe wearout failures as in Figure 2.18, and also early and random failures. The goal is to collect the complete failure behaviour of a part or a machine. The result is always a similar and typical characteristic of the curve, see Figure 2.20.

This curve is called the "bathtub curve" based on its shape [2.29, 2.34]. The bathtub curve can be divided into three distinct sections: section 1 for early failures, section 2 for random failures, and section 3 for wearout failures.

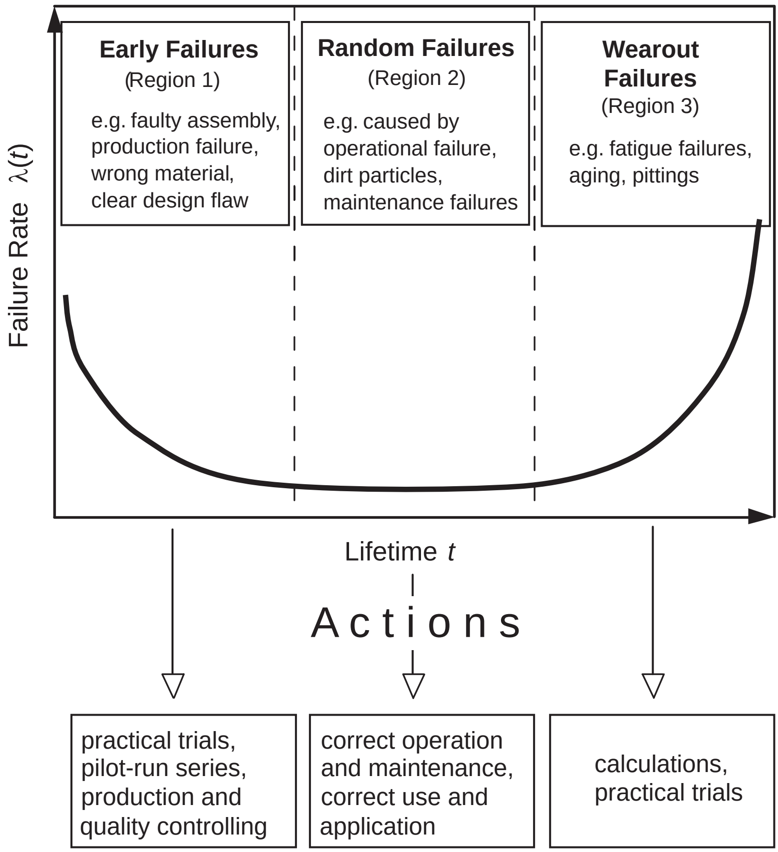


Figure 2.20. The "bathtub curve"

Section 1 is characterized by a decreasing failure rate. The risk that a part will fail decreases with increasing time. Such early failures are mainly caused by failures in the assembly, production, material or by a definite design flaw.

The failure rate is constant in section 2. Thus, the failure risk remains the same. Most of the time, this risk is also relatively low. Such failures are provoked for example by operating or maintenance failures or by dirt particles. Normally, such failures are difficult to pre-estimate.

The failure rate increases rapidly in the section for wearout failures (section 3). The risk that a part will fail increases rapidly with time. Wearout failures are caused by fatigue failures, aging, pittings, etc.

Each of the three sections corresponds to different failure causes. Accordingly, different actions must be taken for an improvement in reliability in each respective section, see Figure 2.20. For section 1 many trials and pilot-run series are recommended. The production and quality of the parts should also be controlled. In section 2, correct operation and maintenance should be considered and the established use and application of the product must be ensured. Section 3 requires either very exact calculations for components or corresponding practical trials.

The actions taken in sections 1 and 2 must be ensured by appropriate steps taken early on in the design process. The improvements in section 3, however, take place in the stage of constructive dimensioning. Thus, the designer can have a strong influence on this section. In addition to representing the most decisive section for reliability, section 3 is the only section which can be calculated. Thus, a prognosis of the expected system reliability is often limited to just this section.

These three sections can also be clearly seen in the example of man's life expectancy, see Figure 2.21. Section 1 with its decreasing failure rate is the section for child deaths. The older a child becomes, the less the risk it has to die of a children's disease. Section 2 for coincidental deaths is not distinctively formed. Deaths here can be seen as random events such as accidents for example. Section 3 shows clearly the increasing age dependent death rate with its drastically increasing failure rate.

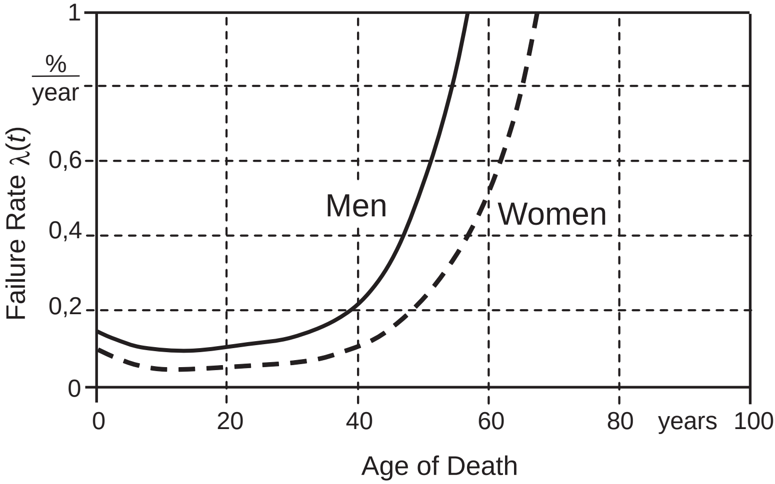


Figure 2.21. Failure rate $\lambda(t)$ for man's life expectancy

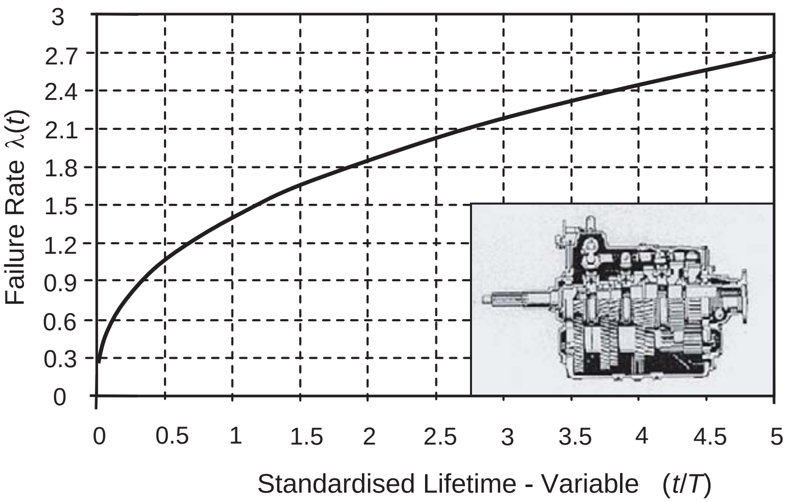


Figure 2.22. Failure rate $\lambda(t)$ of a 6 gear commercial vehicle transmission

The example of the 6 gear commercial vehicle transmission, Figure 2.22, shows that the bathtub curve is not typical for all technical systems. It is more common when only individual sections of the bathtub curve occur.

The failure behaviour for complex systems is thus not characterized alone by the bathtub curve, but much more by differing failure distributions exemplifying various behaviours in certain individual sections.

The failure behaviour "A" in Figure 2.23 shows a typical bathtub curve with its three sections – early failure, random failures, and wearout failures. No early failures are recognizable in "B"; the failure probability remains the same until wearout failures occur in section 3. The failure behaviour "C" is characterized by a continuously increasing failure probability; wearout failures cannot be distinguished. A system with a failure behaviour as in "D" has a low failure probability at the start of operation, followed by a strong increase in failures up to a constant level. A mechanism according to "E" has a constant failure probability over the entire period of time (random failure). The failure behaviour in "F" is characterized by a high failure rate in the first section for early failures (burn in) and then decreases to a constant value for the rest of the lifetime.

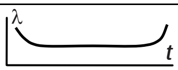
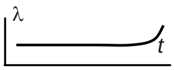
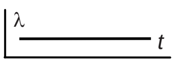
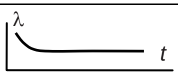
	Failure behaviour		General characteristics	Typical examples
wearout failures	A		• abnormal curve	• old steam engine (late 18th to early 19th century)
	B		• simple devices • complex machines with bad design (one single dominating type of failure)	• car water pump • shoelace • 1974 Vega engine
	C		• structures • wearout element	• car bodies • airplane and automobile tires
random failures	D		• complex machines with high-stress trials after start of operation	• high pressure relief valves
	E		• well designed complex machines	• gyro compass • multiple sealing high pressure centrifugal pump
	F		• electronic components • complex components after corrective maintenance	• computer "mother boards" • programmable controls

Figure 2.23. Various failure behaviours with examples [2.32]

The frequency at which these characterized failure behaviour curves occur is examined and summarized in [2.32], see Figure 2.24.

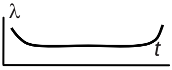
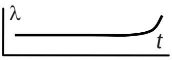
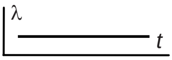
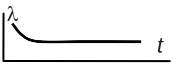
		Failure behaviour	1968 UAL	1973 Broberg	MSDP Studies	1993 SSMD
wearout failures	A		4 %	3 %	3 %	6 %
	B		2 %	1 %	17 %	
	C		5 %	4 %	3 %	
random failures	D		7 %	11 %	6 %	
	E		14 %	15 %	42 %	60 %
	F		68 %	66 %	29 %	33 %

Figure 2.24. Percent fractions according to various lifetime studies [2.32]

Studies done by civil aviation (1968 UAL) show that only 4% of all failures have a trend as in example "A", 2% have a trend as in example "B", 5% as in example "C", 7% as in example "D", 14% as in example "E" and 68% as in example "F". A constant failure behaviour trend as in example "E" should be strived for in the design phase.

2.1.2 Statistical Values

The failure behaviour can be described in detail by the functions discussed in Section 2.1.1.1 to 2.1.1.4. This requires, however, a time consuming determination and representation of the desired function. In many cases it is sufficient when the approximate "middle" of the failure function is known as well as in how much the failure times "deviate" from this mean. Here, "measures of central tendency and statistical spread" can be applied, which can easily be calculated from the failure times. The characterization of the failure behaviour with such values results in a simplified description, where it is possible that information is lost.

The most fundamental statistical values are the mean and the variance or standard deviation. These will be dealt with first.

Mean

The empirical mathematical mean, commonly just called mean, is calculated as follows for the failure times t_1, t_2, ... , t_n:

$$t_m = \frac{t_1 + t_2 + ... + t_\mathrm{n}}{n} = \frac{1}{n}\sum_{i=1}^{n} t_i \; . \tag{2.15}$$

The mean describes the location parameter where the middle of the failure times approximately lies. By viewing the represented failure times in Figure 2.4b as points of mass, the mean t_m is the centre of mass of these points. For the example in Figure 2.4 the mean is $t_m = 31200$ load cycles. The mathematical mean is sensitive to “outliers”, i.e. for extremely short or long failure times, the mean can be significantly affected.

Variance

The empirical variance s^2 describes the average quadratic deviation from the mathematical mean and is thus a measurement for the statistical spread of the failure times about the mean t_m:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (t_i - t_m)^2 \; . \tag{2.16}$$

For the calculation of the variance, the differences from the failure times to the mean are determined and the squares are summed. It is necessary to square the differences; otherwise, the positive and negative deviations would compensate each other.

Standard Deviation

The empirical standard deviation s is the square root of the variance;

$$s = \sqrt{s^2} \; . \tag{2.17}$$

The advantage of the standard deviation in comparison to the variance is, that it has the same dimension as the failure times t_i. Further important statistical values are the median and the mode.

Median

The median is the failure time which is located exactly in the middle of all failures. Therefore, the median can be most easily determined by the failure probability $F(t)$:

$$F\left(t_{median}\right) = 0{,}5 \, . \tag{2.18}$$

If the failure behaviour is represented with the density function $f(t)$, then the median divides the area underneath the function $f(t)$ into two equal sections according to Equation (2.10).

One great advantage of the median in comparison to the mean t_m, is that it is insensitive to extreme values. A short or long failure time can not shift the median.

Mode

The mode describes the failure time that occurs the most. Therefore, the mode t_{mode} can be calculated using the density function $f(t)$: t_{mode} corresponds to the failure time of the density function maximum.

$$f'(t_{modal}) = 0. \tag{2.19}$$

For example, in Figure 2.9 the mode is $t_{mode} \approx 78$ years for men. The mode plays a very important role in probability theory. If a trial is done, it is to be expected that most parts fail at the mode value. The measures of central tendency mean, median and mode are not equal to one another in the common asymmetrical distributions, see Figure 2.25.

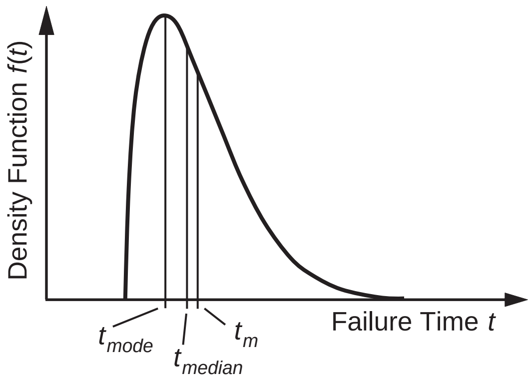


Figure 2.25. Mean, median and mode for a left symmetrical distribution

The three values are only identical when the density function possesses a perfectly symmetrical trend. This is the case for the normal distribution to be explained in Section 2.2.1.

2.1.3 Reliability Parameters

Next to the statistical values described in Section 2.1.2, further values are used in the realm of reliability engineering to characterize reliability data.

- *MTTF* (mean time to failure),
- *MTTFF* (mean time to first failure) and

MTBF (mean time between failure),
- failure rate λ und failure quota *q*,
- percent (%), per mill (‰), parts per million (ppm) and
- B_q lifetime

These variables are often used for further description of the failure or for reliability characteristics.

MTTF

There are various possibilities to specify the lifetime of a non-repairable system. The mean for the time without failures for an observed period of time is the expected value for the lifetime *t*, normally called (Mean Time To Failure). The *MTTF* can be calculated with integration as in Equation 2.20, see Figure 2.26.

$$MTTF = E(\tau) = \int_0^\infty t \cdot f(t)dt = \int_0^\infty R(t)dt \tag{2.20}$$

What happens to the components after failure is irrelevant for the *MTTF*.

The mathematical mean can serve as a good estimate for the *MTTF*, whereby t_1 to t_n are independent realizations (observations) of failure free time periods for statistically identical observation units [2.2].

MTTFF and MTBF

For the description of the lifetime of repairable components, the *MTTFF* can be used, which describes the mean lifetime of a repairable component until its first failure, see Figure 2.26.

$$\text{MTTFF} = \text{Mean Time To First Failure}, \tag{2.21}$$

Thus, *MTTFF* corresponds to the *MTTF* for non-repairable components.

Further definition of the lifetime after the first failure of a component can be described by the *MTBF*, which determines the mean lifetime of a component until its next failure and thus until repair maintenance.

$$MTBF = \text{Mean Time Between Failure} \tag{2.22}$$

Under the assumption that the element is as good as new after maintenance, then the next mean time to failure ($MTBF$) is the same as the previous mean time to first failure *MTTFF* after the end of maintenance.

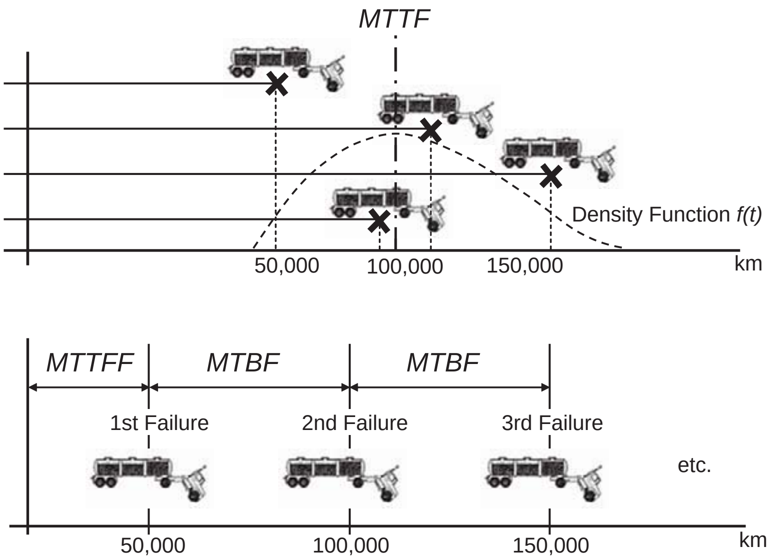


Figure 2.26. Explanation of *MTTF*, *MTTFF* and *MTBF* on behalf of an example

Failure Rate λ and Failure Quota q

The failure rate λ describes the risk that a part will fail, if it has already survived up to this point. The failure rate is determined by dividing the number of failures per time period by the sum of units still intact.

The failure quota q can serve as an estimation of the failure rate λ. In contrast to the failure rate, the failure quota specifies the relative change in an observed time interval.

$$q = \frac{\text{failures in a time interval}}{\text{initial quantity} \cdot \text{interval size}} \tag{2.23}$$

If, for example, 5 units fail out of a test specimen size of 50 units within one hour, then the failure quota is

$q = 0{,}1\frac{1}{\text{h}}$ ("10% per hour") [2.8].

Percent, Per Mill and PPM

In the realm of reliability engineering many circumstances are represented proportionally, such as the failure density, the failure probability or the reliability. The representation of these values is most commonly given in:

- percent: quantity out of 1 hundred, i.e. 1 out of 100 = 1 %,
- per mill: quantity out of 1 thousand, i.e. 1 out of 1,000 = 1 ‰ and
- ppm: quantity out of 1 million, i.e. 1 out of 1.000.000 = 1 ppm

B_x lifetime

The B_x lifetime describes the point in time at which x % of all parts have already failed. This means that a B_{10} lifetime determines the point in time at which 10% of the parts have failed, see Figure 2.27. In practice, B_1, B_{10} and B_{50} lifetime values serve as a measurement for the reliability of a product.

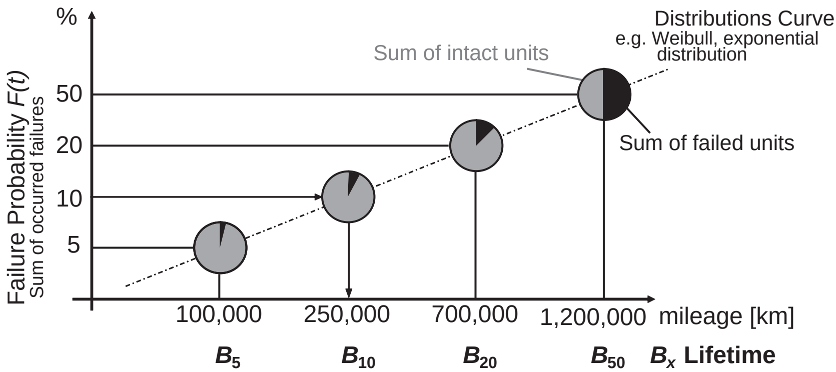


Figure 2.27. B_x lifetime

2.1.4 Definition of Probability

As described in the previous sections, the failure times of components and systems can be seen as random variables. The terms and laws of mathematical probability theory can be applied to these random events. The term probability is of particular importance and will be described in various ways in the following.

Classical Definition of Probability (Laplace 1812)

The first contemplations concerning probability were made by gamblers interested in possible odds and where it is optimal to gamble at high stakes. To answer the question “how probable” it is that a certain event A occurs in a game of gambling, Laplace and Pascal determined the following definition:

$$\text{Probability } P(A) = \frac{\text{number of cases favorable to A}}{\text{number of all possible cases}}. \tag{2.24}$$

Thus, for example, the probability of rolling a 6 with a die (event A) is:

$$P(\text{to roll a 6}) = \frac{1}{6} = 0{,}167,$$

This means that after rolling a die several times, 16.7% of the rolls would result in 6. However, the definition in Equation (2.24) is not universally valid. This equation is only applicable when it is not possible for an infinite amount of events to occur and when every possible result is equally likely. In general, this is adequate for gambling. In technical reality, however, the failure possibilities normally occur in varying amounts.

Statistical Definition of Probability (von Mises 1931)

For a random test specimen with the size n, where all elements are loaded equally in one trial, the failure of m elements is recorded.

The relative failure frequency is (compare with Section 2.1.1.1):

$$\text{relative frequency } h_{rel} = \frac{m}{n}. \tag{2.25}$$

If it is possible to conduct trials independently of one another with differing random test specimens, then different relative frequencies will result. For an increasing random test specimen size n it has been observed, that $h_{rel,n}$ is scattered less and less from a constant value h_x, see Figure 2.28.

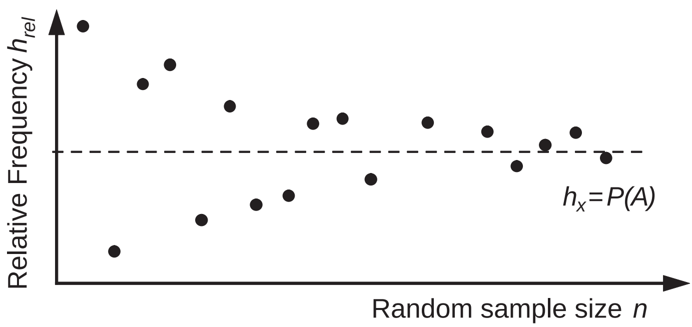


Figure 2.28. Dependency of the relative frequency to random test specimen size

Therefore, it is a good proximate to define the limit of the relative frequencies as the probability for the failure A:

$$\lim_{n\to\infty} \frac{m}{n} = P(A). \tag{2.26}$$

The exact theoretical observations can be seen in the weak and strong law of large numbers as well as in the Bernoulli law of large numbers [2.18, 2.25, 2.27].

Unfortunately, the definition of probability according to Equation (2.26) is likewise not universal because it deals with an estimation and not with a definition. Trying to develop an all inclusive probability theory on the basis of Equation (2.26) resulted in degrees of acceptance and mathematical difficulties which could not be solved.

However, for basic reliability observations and for the scope of this book, the definition out of Equation (2.26) is sufficient. This equation will be used in the following because of its clarity.

Axiomatic Definition of Probability (Kolmogoroff 1933)

In axiomatic definition "probability" is not defined in a strict sense. In modern theory, "probability" is seen much more as a basic principle that fulfils certain axioms.

The axioms of probability proposed by Kolmogoroff are as follows:

1. Each random event A is assigned to a real number $P(A)$ for $0 \le P(A) \le 1$, which is called the probability for A. (This axiom is similar to the characteristics of the relative frequency, see previous section)
2. The probability for a certain event is:
   $P(E) = 1$ (Standardization Axiom)
3. If A_1, A_2, A_3,... are random events, which are incompatible with one another, i.e. $A_i \cap A_j = 0$ for $i \neq j$, then:
   $P(A_1 \cup A_2 \cup A_3 \cup ...) = P(A_1) + P(A_2) + P(A_3) + ...$
   (Addition Axiom).

These axioms are based upon an event space for elementary events, which is also known as the Boolean quantum field or the Boolean σ-field.

The entire probability theory can be derived from the axioms 1 to 3.

2.2 Lifetime Distributions for Reliability Description

Section 2.1 showed how failure behaviour can be represented graphically with various functions. What is of interest in this section is, which curve these functions exactly have for a specific case and how to describe them mathematically. The necessary “lifetime distributions” will be dealt with in this section. The normal distribution is the most widely accepted. However, it is seldom used in reliability engineering. The exponential distribution is often used in electrical engineering, while the Weibull distribution is the most common lifetime distribution used in mechanical engineering. The Weibull distribution will be dealt with in detail in this book. The log normal distribution is occasionally used in materials science and in mechanical engineering.

2.2.1 Normal Distribution

The normal distribution features the familiar bell-curve as its density function $f(t)$, which is perfectly symmetric about the mean $\mu = t_m$, see Figure 2.29. Due to the symmetry of the density function the mean t_m, median t_{median} and mode t_{mode} are congruent.

The normal distribution includes both parameters t_m (location parameter) and σ (scale parameter), see Table 2.1. The standard deviation σ is a measurement for the statistical spread of the failure times and for the form of the failure functions. A low standard deviation results in a narrow, high bell-curve and a high standard deviation corresponds to a shallow curve for the density function, see Figure 2.29.

The principle slope of the curve of the failure functions can not be altered by the standard deviation. Most of the failures must occur around the mean and from there decrease perfectly symmetrical. Thus, it is only possible to describe one type of failure behaviour. This is the main disadvantage of the normal distribution.

In general, the normal distribution begins at $t = -\infty$. Since failure times can only have positive values, the normal distribution can only be used if the definition of failures for negative times is negligible, see Table 2.1.

The integral in Equations (2.28), (2.29) and (2.31) can not be elementary solved for the normal distribution. Thus, tables are used for the determination of the failure probability $F(t)$ and survival probability $R(t)$.

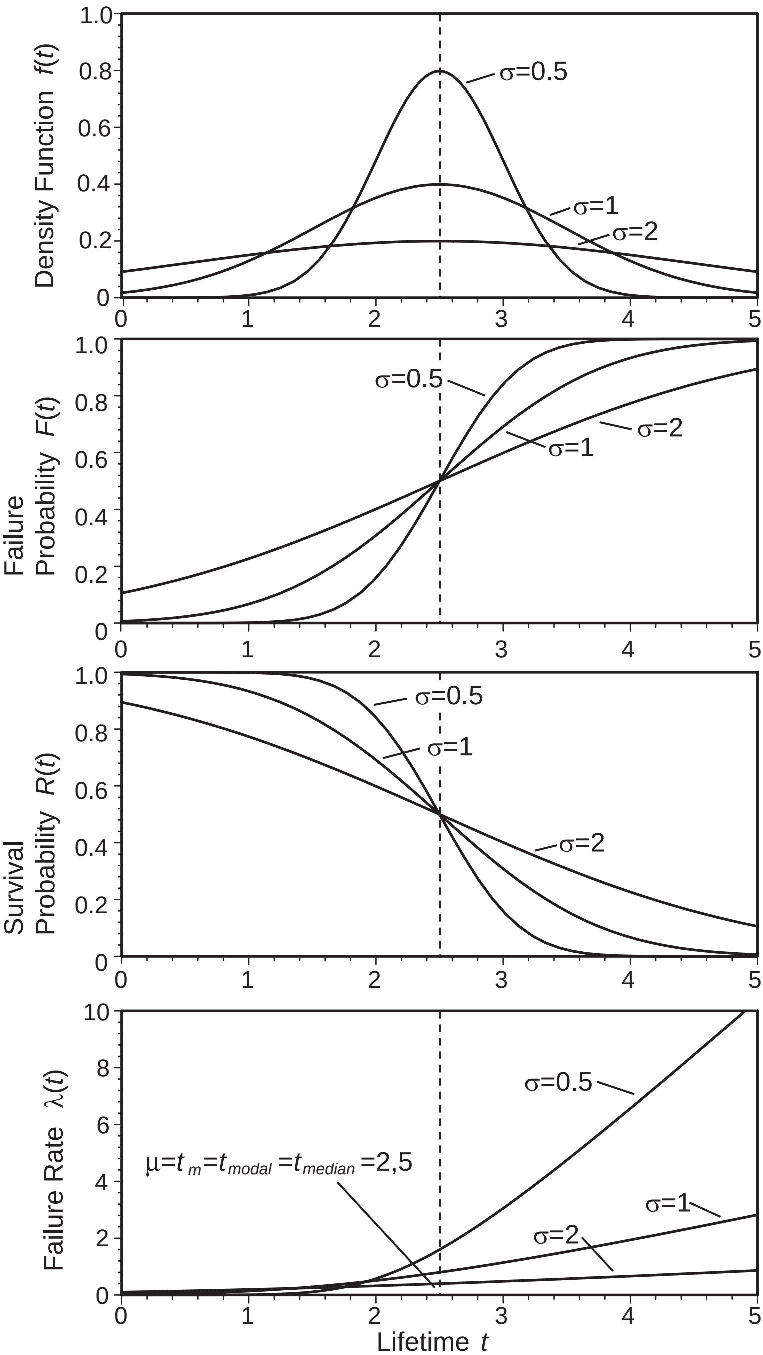


Figure 2.29. Failure function curves of the normal (Gaussian) distribution

Table 2.1. Equations for the normal (Gaussian) distribution

Density Function	$f(t) = \frac{1}{\sigma\sqrt{2 \cdot \pi}} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}}$	(2.27)
Failure Probability	$F(t) = \frac{1}{\sigma\sqrt{2 \cdot \pi}} \cdot \int_0^t e^{-\frac{(\tau-\mu)^2}{2\sigma^2}} d\tau$	(2.28)
Survival Probability	$R(t) = \frac{1}{\sigma\sqrt{2 \cdot \pi}} \cdot \int_t^\infty e^{-\frac{(\tau-\mu)^2}{2\sigma^2}} d\tau$	(2.29)
Failure Rate	$\lambda(t) = \frac{f(t)}{R(t)}$	(2.30)

Parameters:

t:	Statistical variables (load time, load cycle, number of operations, ...) > 0
μ:	Location parameter $\mu = t_m = t_{median} = t_{mode}$
σ:	scale measurement > 0

2.2.2 Exponential Distribution

The density function of the exponential distribution decreases monotonically from its starting point as an inverse exponential function, see Figure 2.30. Here, failure behaviour is described starting with a high failure frequency and from there, decreases continuously.

The equations for the exponential distribution in Table 2.2 show the simple mathematical structure of this distribution. The exponential distribution has only one parameter: the failure rate λ. This failure rate λ is the inverse of the mean t_m:

$$\lambda = \frac{1}{t_m} . \tag{2.31}$$

Out of Equations (2.33) and (2.34) the mean of the reliability is $R(t_m) = 36{,}8\%$ and for the failure probability, $F(t_m) = 63{,}2\%$.

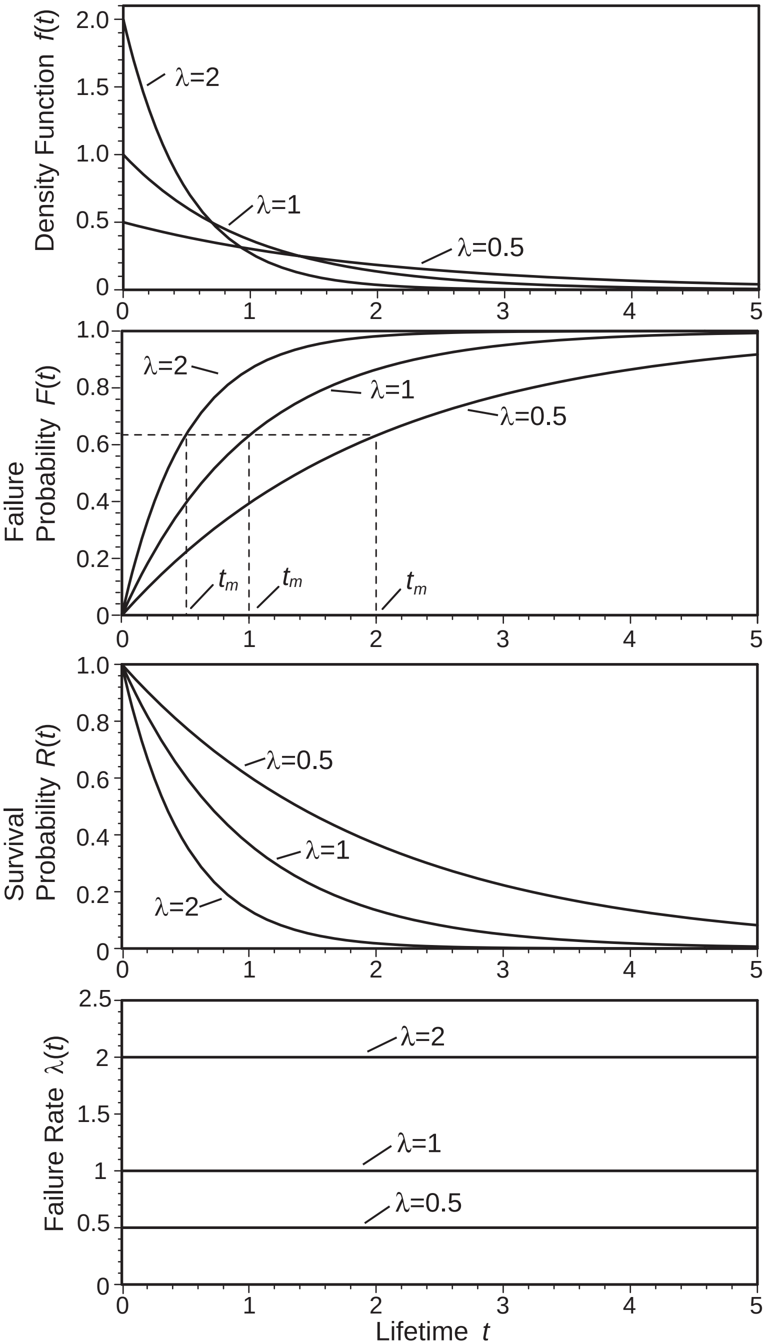


Figure 2.30. Failure functions of the exponential distribution

The constant factor λ for the failure rate is a significant characteristic of the exponential distribution. It should be observed that λ has the same value and is independent of time. The exponential distribution is used for random failures, one salient feature is that the same proportion of parts fail relative to the number of parts remaining.

Similar to the normal distribution, the exponential distribution is only suitable for the description of one certain type of failure behaviour. Such a failure behaviour begins with a high failure frequency and then becomes continuously less, which actually can only be found in mechanical engineering.

Table 2.2. Equations for the exponential distribution

Density Function	$f(t) = \lambda \cdot e^{-\lambda t}$	(2.32)
Failure Probability	$F(t) = 1 - e^{-\lambda t}$	(2.33)
Reliability	$R(t) = e^{-\lambda t}$	(2.34)
Failure Rate	$\lambda(t) = \text{const.}$	(2.35)

Parameters:

t: Statistical variables (load time, load cycle, number of operations, ...) > 0

λ: Location and shape parameter $\lambda = \dfrac{1}{t_m} > 0$

2.2.3 Weibull Distribution

2.2.3.1 Fundamental Terms and Equations

With the Weibull distribution many different failure behaviours can be described. The density functions for the Weibull distribution illustrate this, see Figure 2.31a. The density function varies in dependency upon one parameter for the distribution – the shape parameter b. For low b values ($b < 1$), the failure behaviour can be described similar as in the exponential distribution, i.e. the behaviour begins with a very high failure frequency and from there decreases continuously. An exact exponential distribution results for $b = 1$. For $b > 1$, the density function always begins at $f(t) = 0$, reaches a maximum with increasing lifetime and decreases slowly again. The maximum of the density function shifts to the right for increasing b values. The normal distribution can be approximately reproduced for a shape parameter of $b = 3.5$.

The Weibull distribution can be divided into a two parametric and a three parametric distribution, see Table 2.3.

The two parametric Weibull distribution exhibits the characteristic lifetime T (scale parameter) and the shape parameter b. The characteristic lifetime is an estimate of the mean and shows the location of the distribution. The shape parameter b is a measurement for the statistical spread of the failure times and, as mentioned earlier, for the shape of the failure density, see Figure 2.31a. A two parametric Weibull distribution always describes failures starting from time $t = 0$.

Next to the parameters T and b the three parametric Weibull distribution exhibits an additional parameter for a failure free time t_0 location parameter. With this third parameter, failures can be described that only begin to occur after a certain time t_0. The three parametric Weibull distribution can be derived from the two parametric distribution with a time transformation, where the failure time t and the characteristic lifetime T are substituted with $t - t_0$ and $T - t_0$ ($t \rightarrow t - t_0$, $T \rightarrow T - t_0$). The detailed equations are given in Table 2.3.

The reliability $R(t)$ of the Weibull distribution corresponds to an inverse exponential function. For the two parametric Weibull distribution, the exponent for this exponential function is defined as the quotient (t / T), which again can be varied by the exponent b. The equations of the remaining failure functions are listed in Table 2.3.

In this context it should be noted that for the three parametric Weibull distribution variables, different versions are known internationally. For these equations the commonly accepted scale parameter θ or η in the denominator is represented by the term $(T - t_0)$. In this book the German or European version of the three parametric Weibull distribution with $(T - t_0)$ is used. The advantage of this approach is that along with the knowledge of failure free time t_0, a direct visualization is achieved of the scale parameter T. The scale parameter T represents the characteristic lifetime starting from the origin, unlike θ or η which originates from the time t_0, see Figure 2.31.

Table 2.3. Equations and variables of the Weibull distribution

Two parametric Weibull Distribution:		
Survival Probability or Reliability	$R(t) = e^{-\left(\frac{t}{T}\right)^b}$	(2.36)
Failure Probability	$F(t) = 1 - e^{-\left(\frac{t}{T}\right)^b}$	(2.37)
Density Function	$f(t) = \frac{dF(t)}{dt} = \frac{b}{T} \cdot \left(\frac{t}{T}\right)^{b-1} e^{-\left(\frac{t}{T}\right)^b}$	(2.38)
Failure Rate	$\lambda(t) = \frac{f(t)}{R(t)} = \frac{b}{T} \cdot \left(\frac{t}{T}\right)^{b-1}$	(2.39)
Three parametric Weibull Distribution:		
Survival Probability or Reliability	$R(t) = e^{-\left(\frac{t-t_0}{T-t_0}\right)^b}$	(2.40)
Failure Probability	$F(t) = 1 - e^{-\left(\frac{t-t_0}{T-t_0}\right)^b}$	(2.41)
Density Function	$f(t) = \frac{dF(t)}{dt} = \frac{b}{T-t_0} \cdot \left(\frac{t-t_0}{T-t_0}\right)^{b-1} \cdot e^{-\left(\frac{t-t_0}{T-t_0}\right)^b}$	(2.42)
Failure Rate	$\lambda(t) = \frac{f(t)}{R(t)} = \frac{b}{T-t_0} \cdot \left(\frac{t-t_0}{T-t_0}\right)^{b-1}$	(2.43)

Parameters:

t:	Statistical variable (load time, load cycles, …) > 0
T:	Characteristic lifetime, "scale parameter". For $t = T$, $F(t) = 63{,}2\%$ or $R(t) = 36{,}8\%$. $T > t_0$
b:	Shape parameter or failure slope. Determines the shape of the curve. > 0
t_0:	Failure free time – location parameter. The parameter t_0 determines the point in time from which failures begin to occur. It corresponds to a shifting of the failure behaviour along the time axis. If $t_0 > 0$ then $t > t_0$.

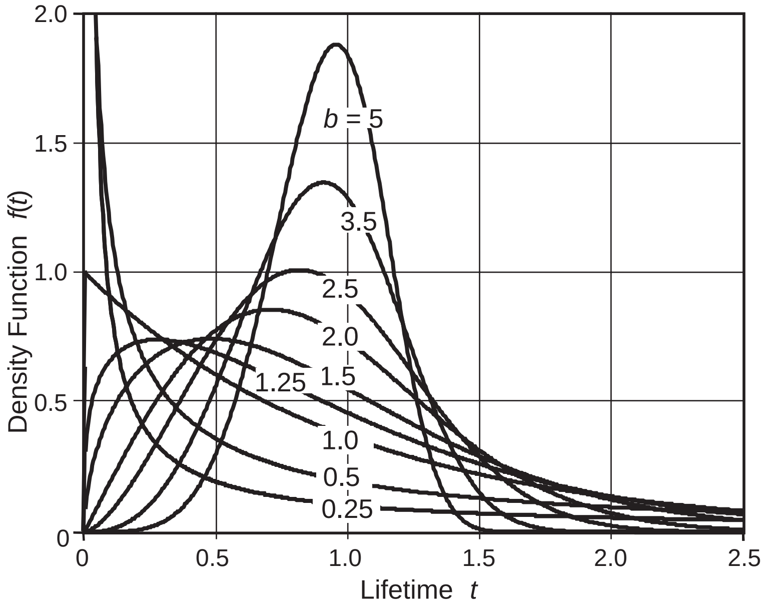


Figure 2.31a. Density function $f(t)$ of the Weibull distribution for various shape parameters b (characteristic lifetime $T = 1$, failure free time $t_0 = 0$)

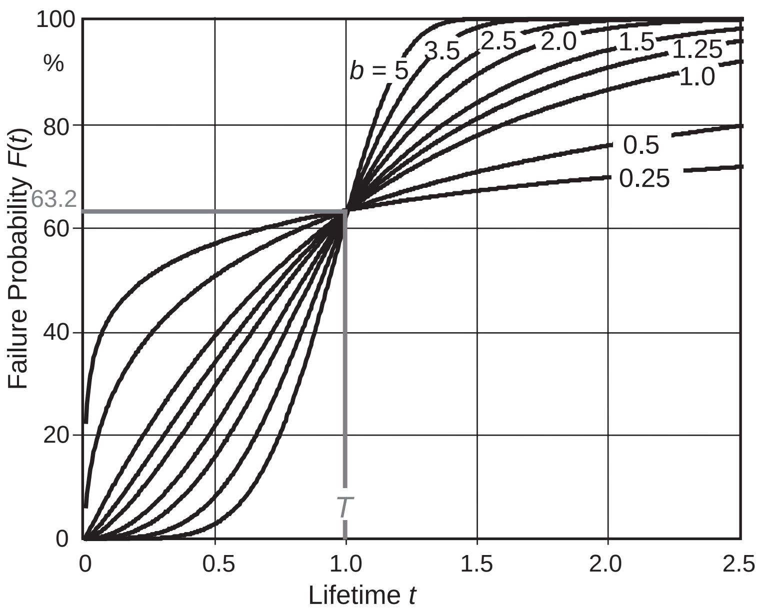


Figure 2.31b. Failure probability $F(t)$ of the Weibull distribution for various shape parameters b (characteristic lifetime $T = 1$, failure free time $t_0 = 0$)

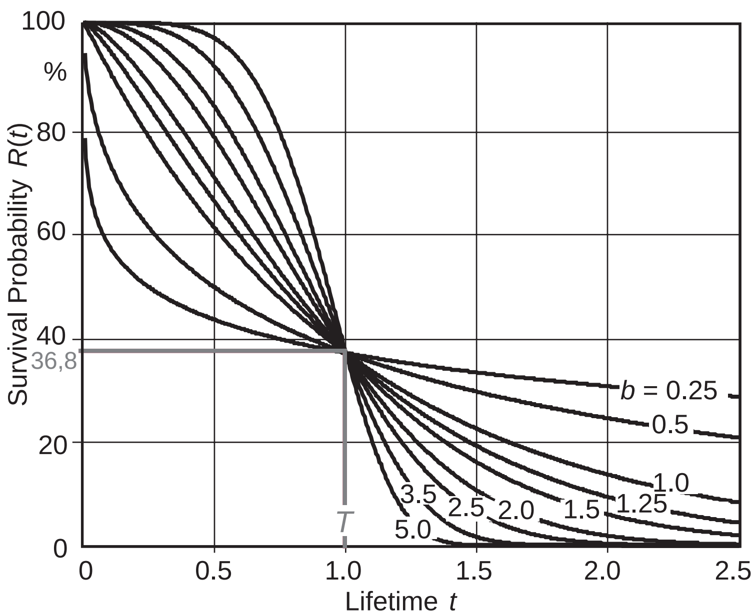


Figure 2.31c. Survival probability $R(t)$ of the Weibull distribution for various shape parameters b (characteristic lifetime $T = 1$, failure free time $t_0 = 0$)

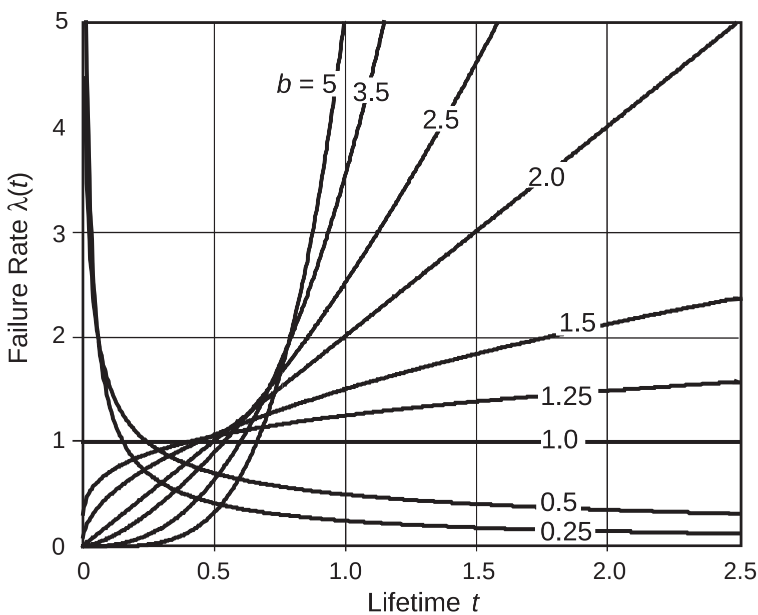


Figure 2.31d. Failure rate $\lambda(t)$ of the Weibull distribution for various shape parameters b (characteristic lifetime $T = 1$, failure free time $t_0 = 0$)

The various failure rates for the Weibull distribution as in Figure 2.31d can be divided into three sections, which are identical to the three sections of the bathtub curve in Section 2.1.1.4:

$b < 1$: Failure rates decrease with increasing lifetime: description of early failures;

$b = 1$: Failure rate is constant. The shape parameter $b = 1$ is suitable for the description of random failures in the constant failures of the bathtub curve;

$b > 1$: Failure rates increase drastically with increasing lifetime. Wearout failures can be described with b values greater than 1.

The equations for the Weibull distribution include the statistical variable t in relative form t/T or $(t - t_0)/(T - t_0)$. Thus, for the time $t = T$ this quotient is equal to 1 and the failure probability can be calculated as follows:

$$F(T) = 1 - e^{-1} = 0.632\,. \tag{2.44}$$

Therefore, the characteristic lifetime T is assigned a failure probability of $F(t) = 63{,}2\%$ which corresponds to the survival probability of $R(t) = 36{,}8\%$. Similar to the median, for which $F(t) = 50\%$, the characteristic lifetime T can be interpreted as a special mean, Figure 2.32.

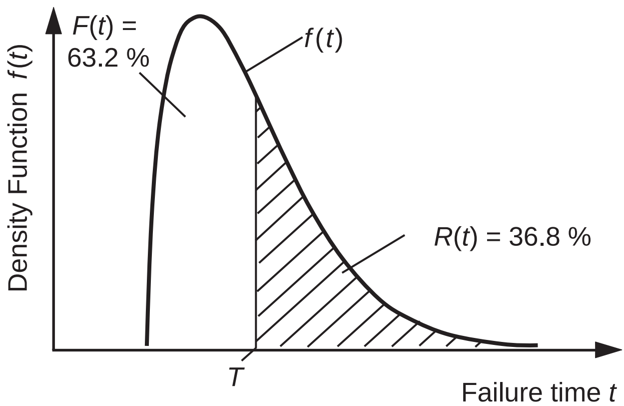


Figure 2.32. Characteristic lifetime *T* as "mean"

The mean t_m of the Weibull distribution can only be calculated with the help of the gamma function:

$$t_m = T \cdot \Gamma\left(1 + \frac{1}{b}\right) \tag{2.45}$$

$$\text{or } t_m = (T - t_0) \cdot \Gamma\left(1 + \frac{1}{b}\right) + t_0 \,. \tag{2.46}$$

The function values for the gamma function are listed for example in [2.4].

2.2.3.2 Weibull Probability Paper

The failure probabilities $F(t)$ in Figure 2.31b possess an s-shaped curve. With a special "probability paper" it is possible to plot the functions $F(t)$ of the two parametric Weibull distribution as a straight line, see Figure 2.33. Thus, the failure behaviour can be portrayed in a simple graphical way. This can prove to be useful in the evaluation of trials, since it can be found best to fit line for the entered trial data, see Chapter 6.

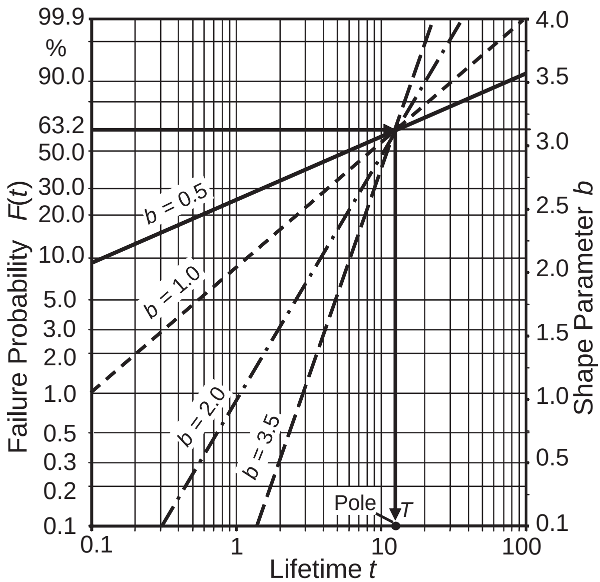


Figure 2.33. Weibull probability paper with various failure probabilities $F(t)$

Here again, the example of the commercial vehicle transmission already introduced in Section 2.1.1.2 can be used to exemplify the use of Weibull probability paper. It is easy to recognize that the once s-shaped curve in Figure 2.12 is now displayed as a straight line.

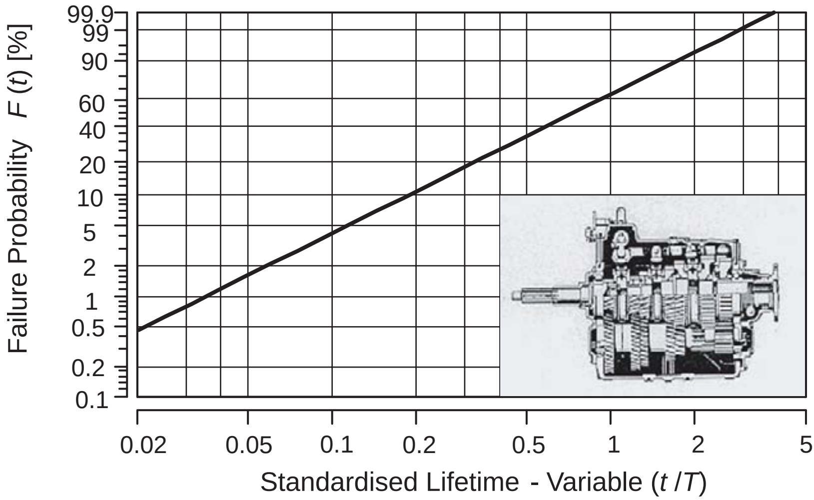


Figure 2.34. Failure probability of a 6 gear commercial vehicle transmission represented in Weibull probability paper

The transformation of the curves into straight lines is achieved with a unique scaling of the abscissa and ordinate. The abscissa is logarithmic, while the ordinate has a double logarithmic scale:

$$x = \ln t \,, \tag{2.47}$$

$$y = \ln(-\ln(1 - F(t))) \text{ or } y = \ln(-\ln(R(t))) \,. \tag{2.48}$$

The special axis scaling based on a two parametric Weibull distribution results from the following equation:

$$F(t) = 1 - e^{-\left(\frac{t}{T}\right)^b} \,, \tag{2.49}$$

$$1 - F(t) = e^{-\left(\frac{t}{T}\right)^b} \,, \tag{2.50}$$

$$\frac{1}{1 - F(t)} = e^{+\left(\frac{t}{T}\right)^b} \,. \tag{2.51}$$

Taking the logarithm twice results to:

$$\ln\left(\ln \frac{1}{1 - F(t)}\right) = b \cdot \ln\left(\frac{t}{T}\right), \tag{2.52}$$

$$\ln\left(-\ln\left(1-F(t)\right)\right)=b\cdot\ln t-b\cdot\ln T\ . \tag{2.53}$$

The Equation (2.53) corresponds to a linear equation in the form

$$y=a\cdot x+c \tag{2.54}$$

with the variables

$$a=b \quad \text{(slope),} \tag{2.55}$$

$$c=-b\cdot\ln T \quad \text{(axis intersection),} \tag{2.56}$$

$$x=\ln t \quad \text{(abscissa scaling),} \tag{2.57}$$

$$y=\ln\left(-\ln\left(1-F(t)\right)\right) \quad \text{(ordinate scaling).} \tag{2.58}$$

Thus, every two parametric Weibull distribution can be represented as a straight line in the Weibull probability chart, see Figure 2.33. The slope of the straight lines in the Weibull probability chart is a direct measurement for the shape parameter *b*. The shape parameter *b* can be read off the right ordinate in Figure 2.35 by shifting the straight lines parallel through the pole *P*.

The location of this pole and the scaling of the linear ordinate for the shape parameter *b* can be determined with Equations (2.55), (2.57) and (2.58):

$$b=\frac{\Delta y}{\Delta x}=\frac{\ln\left(-\ln\left(1-F_2(t_2)\right)\right)-\ln\left(-\ln\left(1-F_1(t_1)\right)\right)}{\ln t_2-\ln t_1}\ . \tag{2.59}$$

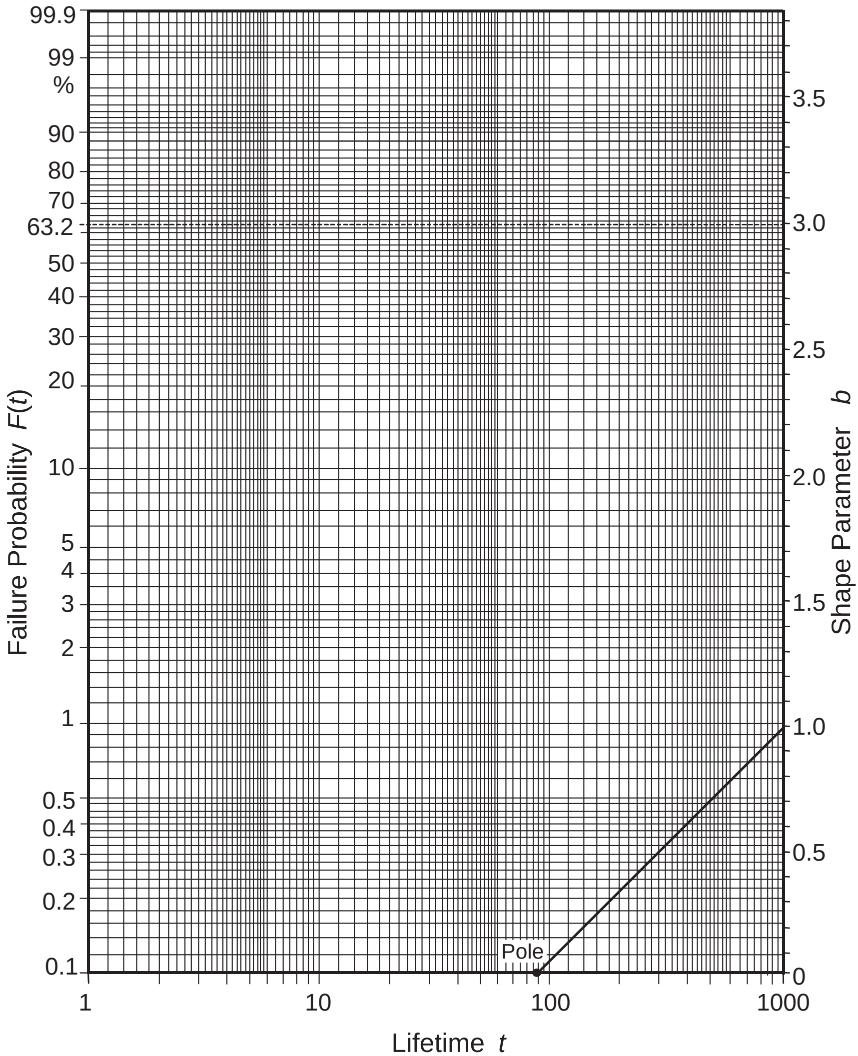


Figure 2.35. Weibull probability paper

Example:

A two parametric Weibull distribution with the shape parameter b = 1.7 and the characteristic lifetime T = 80,000 load cycles is to be drawn in Weibull probability paper. The desired function out of the given data is:

$$F(t) = 1 - e^{-\left(\frac{t}{80{,}000\ \text{LW}}\right)^{1.7}}.$$

First, an assisting straight line is sketched with slope b = 1.7 in the probability paper, see Figure 2.36. This straight line begins at the pole P and ends on the right ordinate at b = 1.7. Thus, the slope of the desired Weibull straight line is already established. The assisting straight line must then be shifted parallel until it intersects $F(t)$ = 63% for a characteristic lifetime of T = 80,000 load cycles. In conclusion, the Weibull straight line in Figure 2.36 corresponds with the desired failure probability $F(t)$.

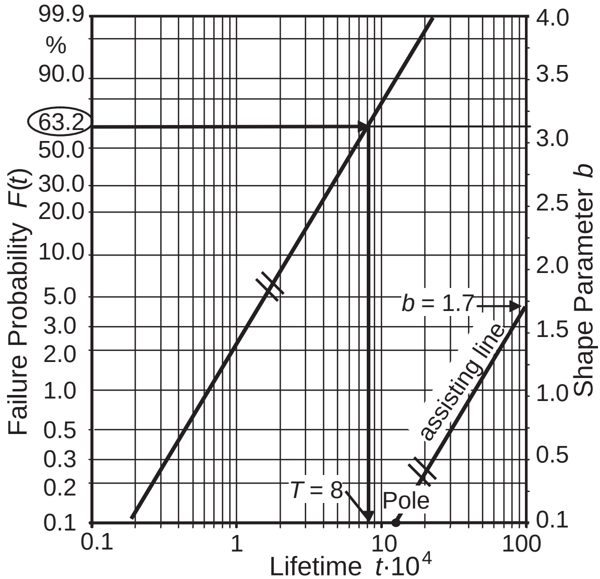


Figure 2.36. Weibull probability paper with the Weibull straight line from the above example

A three parametric Weibull distribution in Weibull probability paper is not a straight line but rather a convex curve, see Figure 2.37. Nevertheless, a three parametric Weibull distribution can also be portrayed as a straight line, if the abscissa scaling for t_0 is corrected with the failure times (t - t_0). With this transformation a three parametric Weibull distribution can be traced back to a two parametric Weibull distribution, see Figure 2.38.

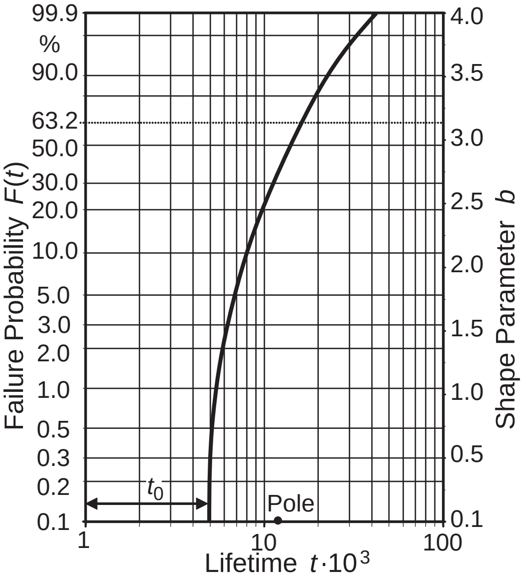


Figure 2.37. Original values of a three parametric Weibull distribution in Weibull probability paper

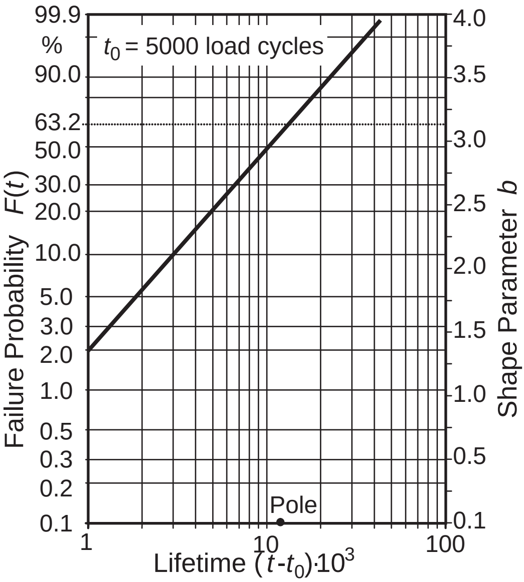


Figure 2.38. Three parametric Weibull distribution with the corrected failure times from t_0 to $(t - t_0)$

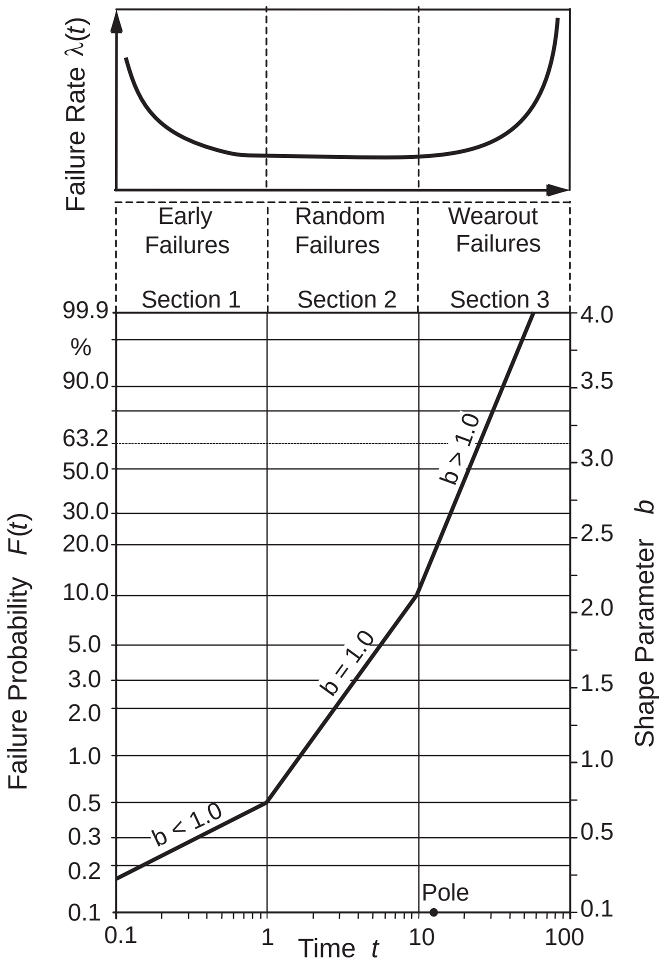


Figure 2.39. Bathtub curve realized in Weibull probability paper

As already mentioned in Section 2.1.1.4, the entire failure behaviour of components or systems with the failure rate λ(*t*) can be represented in the shape of the bathtub curve. The three sections of the bathtub curve, which describe different failure causes, can also be realized in Weibull probability paper, see Figure 2.39. Each section is then described with its own Weibull distribution and corresponding shape parameter *b* in the Weibull probability paper.

2.2.3.3 History of the Weibull Distribution

From 1930 to 1950 *W. Weibull* conducted various fatigue trials, where he realized that he could not describe the resulting failure behaviour with the common distributions known up to his time. Therefore, he himself attempted to develop a universal distribution, which he published in detail in 1951 [2.40].

W. Weibull assumed that every distribution function can be described in the form:

$$F(t) = 1 - e^{-\varphi(t)} \tag{2.60}$$

He also set the following minimal conditions for the function $\varphi(t)$:

- $\varphi(t)$ is positive and increases monotonically (thus, the principle requirement for a continuous, monotone increasing distribution function is fulfilled),
- t_0 exists a lower limit for which $\varphi(t) = 0$, so that a minimum lifetime or failure free time can be considered.

The simplest function that fulfils these conditions is:

$$\varphi(t) = \left(\frac{t - t_0}{\tilde{T}}\right)^b . \tag{2.61}$$

Before the time t_0, the argument in Equation (2.61) is negative. Thus, the function is undefined before the time t_0. Starting from t_0, $\varphi(t)$ increases monotonically.

If a value $(T - t_0)$ is substituted for the reference value $\tilde{T}$, then $F(t) = 63{,}2\%$ for all $t = T$. Thus, all conditions are fulfilled, and it becomes simpler to handle the function in calculations.

The Weibull distribution in three parametric form results by inserting Equation (2.61) in Equation (2.60):

$$F(t) = 1 - e^{-\left(\frac{t - t_0}{T - t_0}\right)^b} . \tag{2.62}$$

It could not be explained by any probability theory, why *W. Weibull's* postulate for exactly this function is suitable for the description of lifetime trials. *W. Weibull* derived the function purely empirically.

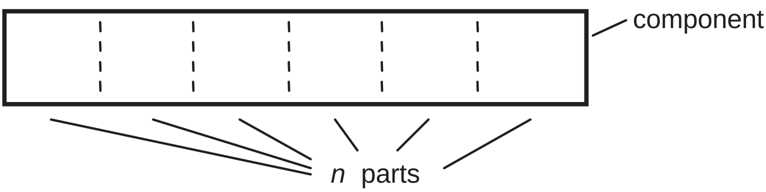
2.2.3.4 Probability Theory Justification for the Weibull Distribution

The Weibull distribution is characterized as an "asymptotical extremum distribution" in probability theory. Such a distribution had already been fundamentally researched early on by *Fischer* and *Tippett* [2.10] as well as by *Gnedenko* [2.14]. After *W. Weibull* empirically developed and introduced the distribution, it was examined further under probability theory aspects by *Freudenthal* and *Gumbel* [2.11, 2.16, 2.17] and most recently by *Galambos* [2.13]. All of these sources include for the most part the following definition:

> The Weibull distribution corresponds to an asymptotical extremum distribution of the lowest (first) order statistic for a test specimen with the size n, for large values for n ($n \to \infty$).

In order to understand this definition it is first necessary to understand the terms order statistic and order statistic distribution. If these terms are not familiar to the reader, then Section 6.2 should be dealt with first, where order statistics and their distributions are described in more detail.

Imagine a component separated into n parts:



By designating the lifetimes for the n parts as $t_1, t_2, \ldots t_n$, the lifetime of the complete component is $t_{component} = \min(t_1, t_2, \ldots t_n)$. The component fails with the failure of the weakest link. Thus, the failure time $t_{component}$ corresponds to the shortest failure time in the "random test specimen" of size n. The shortest failure time is designated as the first order statistic in the random test specimen. For another, similar component with an identical "random test specimen size" n, the $t_{component}$ or the lowest order statistic will be somewhat different. Consequently, a distribution can be assigned to this order statistic. Since the first order statistic (or also the n^{th} order statistic) represents an extreme order, it is designated as an extremum and its distribution as an extremum distribution. For the limit $n \to \infty$, the lifetime of a component is then Weibull distributed [2.13, 2.18].

The failure of a component through its weakest part corresponds to the principle of a chain's weakest link. Only for the case that a real failure cause is based on this principle, it is theoretically possible for the Weibull distribution to describe the occurrence of failures exactly. However, due to the distribution's universality, see Figure 2.31, the Weibull distribution is used in practice mainly due to purely pragmatic reasons.

2.2.4 Logarithmic Normal Distribution

The logarithmic normal distribution, commonly called log normal distribution for short, is based on the normal distribution from Section 2.2.1. The random variable *t* is substituted with the logarithmized form log *t* in Equations (2.27) to (2.30). This results that the logarithmized failure times follow the pattern of a normal distribution. The equations for the log normal distribution are summarized in Table 2.4.

Here, it must be noted that the equations of the log normal distribution can have either the logarithm to the base of 10 (lg) or the normal log (ln) in the numerator of the exponent.

Table 2.4. Equations for the log normal distribution

Density Function	$f(t)=\frac{1}{t\cdot\sigma\sqrt{2\cdot\pi}}\cdot e^{-\frac{(\lg t-\mu)^2}{2\sigma^2}}$	(2.63)
Failure Probability	$F(t)=\int_0^t\frac{1}{\tau\cdot\sigma\sqrt{2\cdot\pi}}\cdot e^{-\frac{(\lg\tau-\mu)^2}{2\sigma^2}}d\tau$	(2.64)
Survival Probability	$R(t)=1-F(t)$	(2.65)
Failure Rate	$\lambda(t)=\frac{f(t)}{R(t)}$	(2.66)

Parameters:

t: Statistical variable (load time, load cycle, number of operations, …) > 0

μ: scale parameter. The exact mean of the log normal distribution is $t_{median,\,LV}=10^{\mu}$.

σ: “shape parameter”, statistical spread > 0

Contrary to the normal distribution, it is possible to produce strongly varying density functions with the log normal distribution. Therefore, similar to the Weibull distribution, it is possible to describe many different failure behaviours with the log normal distribution.

The application of the log normal distribution is simplified by the fact that the procedure of the normal distribution, the most thoroughly researched and developed distribution, can be easily carried over for the log normal distribution. Similar to the normal distribution, the disadvantage of the log normal distribution is that the density function can only be represented with limitations, and that the other failure functions can only be determined with either laborious integration or with tables and charts.

The log normal distribution's failure rate increases with increasing lifetime and then decreases after reaching a maximum. The failure rate approaches zero for very low lifetimes. Thus, the monotonically increasing failure rate for wearout failures can only be represented by the log normal distribution with restrictions. On the other hand, the log normal distribution describes failure behaviours well that begin with a rapidly increasing failure rate followed by many robust and resistant components which can endure a long load period.

While many random factors work together in generating a normal distribution, these random factors are multiplicatively combined for the log normal distribution. Thus, the individual random factors are proportionally related to one another. This may be shown in wearout failures. Here it is required that the fracture caused by the load gradually occurs and spreads, and that a very large number of crack extensions form before final fracture. The crack growth in each gradual level can be regarded as a random variable, which is proportional to the average crack length that can be reached. With the central limit theorem [2.18, 2.25, 2.27] the log normal distribution can be used as a model for the description of wearout failures [2.24].

There is also probability paper for the log normal distribution in which the failure probability $F(t)$ can be shown as a straight line and is therefore suitable for the evaluation of trials, see Figure 2.40.

The probability network has an abscissa with a log base ten scaling and an ordinate scaled according to the normal distribution [2.18]. The median $t_{median} = 10^{\mu}$ corresponds to the intersection with the 50% failure probability line.

The standard deviation σ is:

$$\sigma = \lg \frac{t_{84\%}}{t_{50\%}} \quad \text{or}\ \sigma = \lg \frac{t_{50\%}}{t_{16\%}}\,. \tag{2.67}$$

Figure 2.40 shows a few examples of failure probabilities.

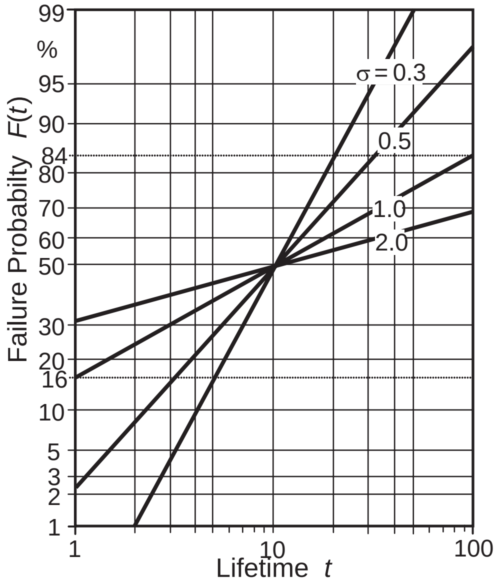


Figure 2.40. Log normal probability paper with different failure probabilities $F(t)$

Similar to the Weibull distribution, the log normal distribution also has a three parametric version with a failure free time t_0 as a third parameter [2.18]. However, this three parametric log normal distribution is applied in only very few cases.

2.2.5 Further Distributions

The distributions introduced in the following sections are seldom applied in practice. However, they offer certain advantages for individual cases and thus are mentioned here for the sake of completeness.

2.2.5.1 Gamma Distribution

Just as the Weibull distribution, the gamma distribution exists in a two and three parametric form. A generalized gamma distribution can even contain four parameters. However, the statistical analysis of data with such a flexible model can lead to such a high complexity, that this form with four parameters is not accounted here [2.19].

The gamma distribution's density for a two parametric form is

$$f(t) = \frac{a^b t^{b-1} e^{-at}}{\Gamma(b)} \tag{2.68}$$

and for a three parametric form

$$f(t) = \frac{a^b}{\Gamma(b)} (t - t_0)^{b-1} e^{-a(t-t_0)}, \tag{2.69}$$

whereby a is a scale parameter ($a \neq 0$), b a shape parameter ($b > 0$) and t_0 a location parameter. Furthermore, the complete function is, see Table A.5,

$$\Gamma(b) = \int_0^{+\infty} x^{b-1} e^{-x} dx \tag{2.70}$$

and the incomplete gamma function is

$$\Gamma(b, at) = \int_0^{at} x^{b-1} e^{-x} dx \tag{2.71}$$

which, for example, is tabulated in Bronstein [2.4].

The probability distribution of a random variable with a two parametric gamma distribution can only be described as an integral

$$F(t) = \frac{a^b}{\Gamma(b)} \int_0^t u^{b-1} e^{-au} du , \tag{2.72}$$

out of which the survival probability can be derived:

$$R(t) = 1 - \frac{a^b}{\Gamma(b)} \int_0^t u^{b-1} e^{-au} du . \tag{2.73}$$

The failure rate of the gamma distribution cannot be determined in complete form; however, it can be described by the following general equation:

$$\lambda(t) = \frac{f(t)}{1 - F(t)} . \tag{2.74}$$

The expected value and the variance of a two parametric gamma distribution are as follows:

$$E(t) = \frac{b}{a} \tag{2.75}$$

and

$$Var(t) = \frac{b}{a^2}. \tag{2.76}$$

As already mentioned, the three parametric gamma distribution also has an additional location parameter t_0 besides the parameters a and b. As in the exponential and Weibull distribution, failures can be described with this parameter that first occur after a time t_0, i.e. failures that possess a failure free time.

The expected value for a three parametric gamma distribution is

$$E(t) = t_0 + \frac{b}{a} \tag{2.77}$$

and the variance can be calculated with Equation (2.76), which means that the variance has the same value in both the two parametric and three parametric form and is thus independent from t_0.

The gamma distribution can describe various failure behaviours, just as the Weibull distribution. This may be observed especially on the density function of the gamma distribution, see Figure 2.41. The density function changes considerably in dependency upon one parameter – the shape parameter b. The gamma distribution corresponds exactly to the exponential distribution for b = 1, see Figure 2.30.

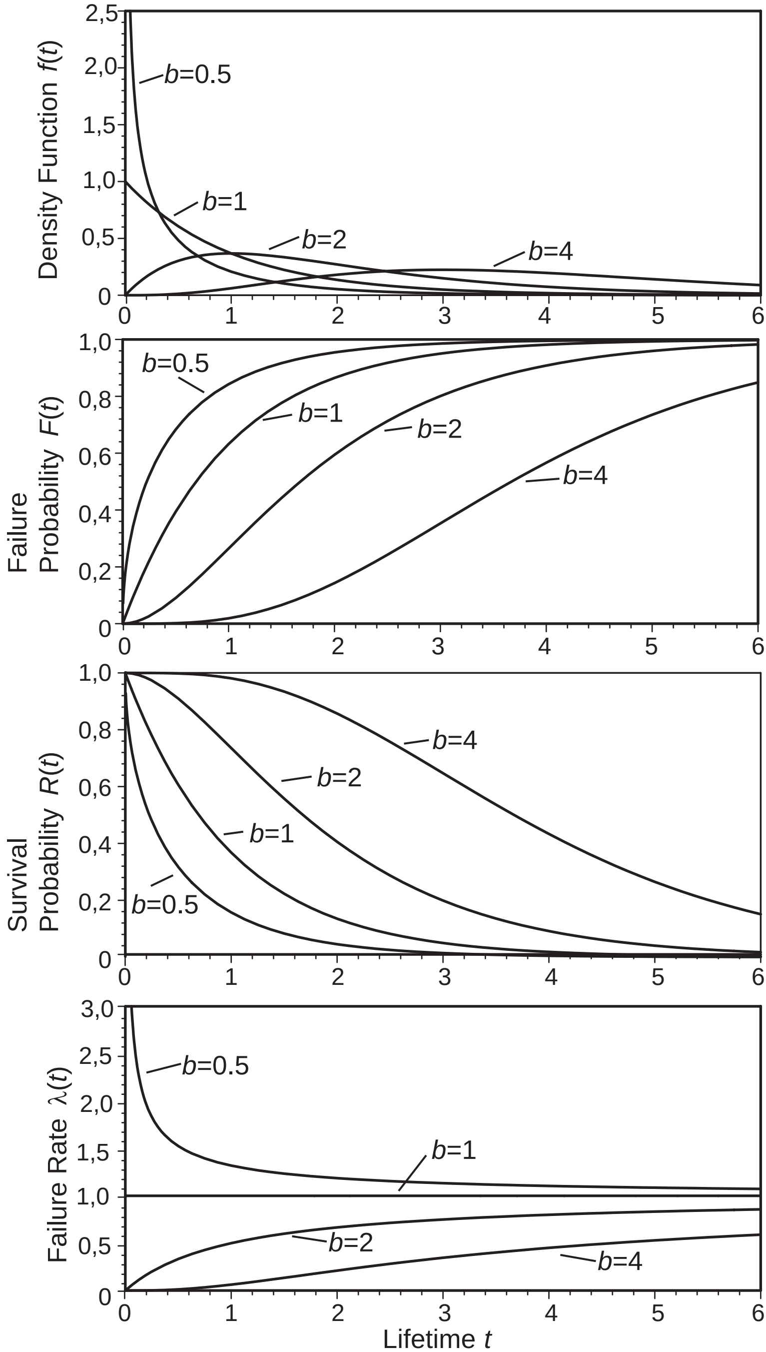


Figure 2.41. Failure functions of a two parametric gamma distribution with $a = 1$

If the shape parameter *b* has a positive whole number (*b* = 1, 2, …), then the gamma distribution turns into the Erlang distribution, described in 2.2.5.2 below.

The density function always begins at $f(t) = 0$ for $b > 1$, reaches a maximum with increasing lifetime and from there decreases continuously. The maximum of the density function shifts to the right with an increasing shape parameter, see Figure 2.41.

Figure 2.41 showed the failure probability and the survival probability for the gamma distribution. As seen in Figure 2.41, the failure rate for the gamma distribution is suitable for the description of failure behaviours with increasing, decreasing, and constant failure rates, as in the Weibull distribution. The failure rate converges to the scale parameter *a* with increasing *t*. The gamma function distinguishes itself from the Weibull distribution by the fact that the failure rate has an exponential factor of (*b* - 1) and thus changes more rapidly for increasing *t* [2.19].

2.2.5.2 Erlang Distribution

The Erlang distribution is a special case of the gamma distribution. It can be derived directly from the gamma distribution for shape parameters *b* with positive whole values. Thus, all described characteristics for the gamma distribution are applicable to the Erlang distribution. In particular, the advantage of this distribution is its simplicity and relationship to the exponential distribution for the parameter *b* = 1.

The relationship to the exponential distribution is the actual importance of the Erlang distribution. The Erlang distribution corresponds to the sum of *n* statistically independent random values $t_1, \ldots, t_b$, which possess the exact same exponential distribution. This can prove to be very practical for example when describing failures which occur in phases and the last failure occurs at the end of the phase *b*.

The density function for the Erlang distribution is:

$$f(t) = \frac{a(at)^{b-1} e^{-at}}{(b-1)!} \qquad (2.78)$$

The failure probability is calculated by integrating the density function:

$$F(t) = 1 - \sum_{r=0}^{b-1} \frac{e^{-at}(at)^r}{r!} \, . \qquad (2.79)$$

The equation for the survival probability is:

$$R(t) = \sum_{r=0}^{b-1} \frac{e^{-at}(at)^r}{r!} \tag{2.80}$$

and for the failure rate:

$$\lambda(t) = \frac{a(at)^{b-1}}{(b-1)!\sum_{r=0}^{b-1} \frac{(at)^r}{r!}} . \tag{2.81}$$

The equations according to [2.12] for the expected value and variance of the Erlang distribution are as follows:

$$E(t) = \frac{b}{a} , \tag{2.82}$$

$$Var(t) = \frac{b}{a^2} . \tag{2.83}$$

Figure 2.42 shows the graphical representation of the functions of the Equations (2.78) to (2.81). Here, it is shown that various density functions can be realized (left symmetrical, symmetrical, decreasing). As already mentioned, the characteristics are identical to those of the gamma distribution.

The failure rate for the Erlang distribution increases monotone, and it is imperative that $\lambda(0) = 0$, as well as:

$$\lim_{t \to \infty} \lambda(t) = a . \tag{2.84}$$

This means that the failure rates for the Erlang and the gamma distribution converge for, $t \to \infty$, towards a limiting value. In contrast, the Weibull distribution approaches infinity for *b* values greater than one.

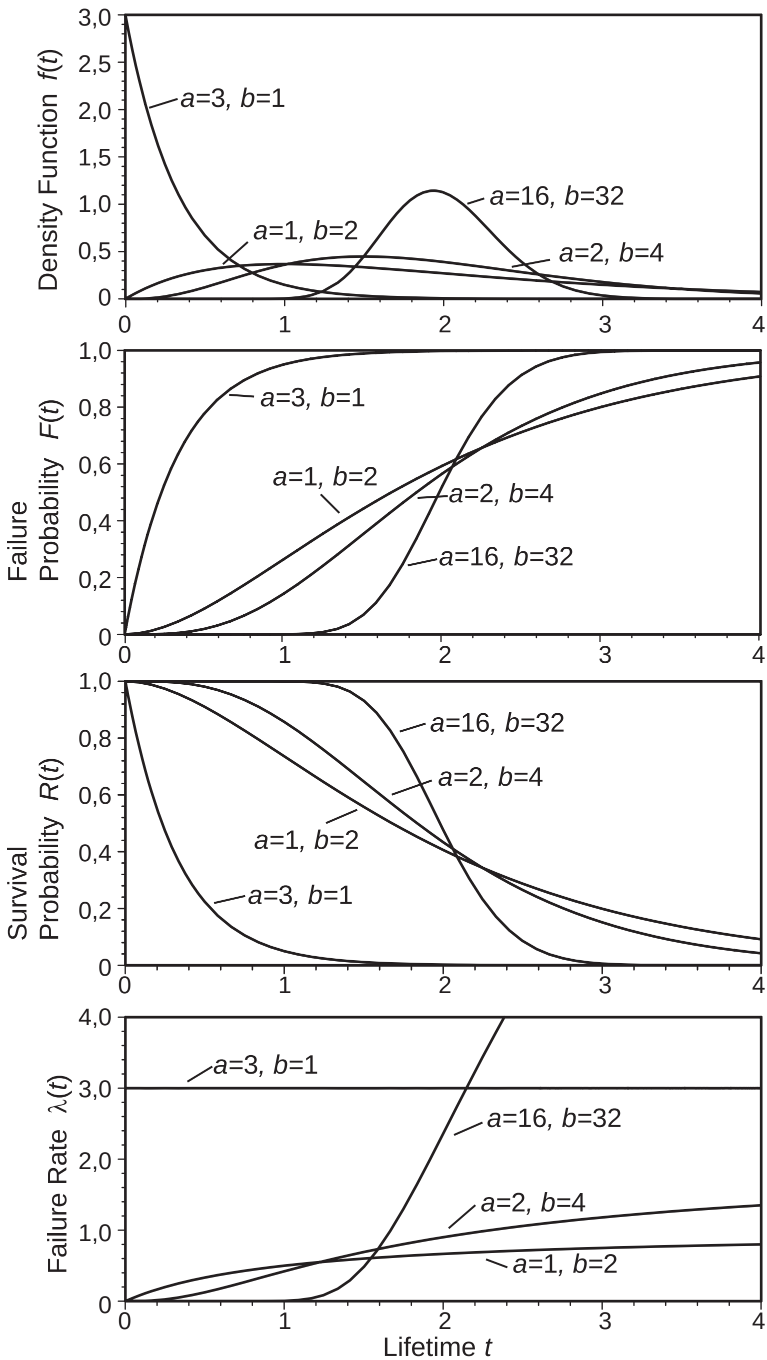


Figure 2.42. Failure functions of the Erlang distribution

2.2.5.3 Hjorth Distribution

The Hjorth distribution stems from research done by *U. Hjorth* [2.21] concerning the relationship between failure estimation and probability modelling.

Similar to the gamma and Weibull distributions, the Hjorth distribution can describe various types of failure behaviour well. Then entire bathtub curve can be described with this distribution, which means that increasing, decreasing, constant and bathtub shaped failure behaviours can be represented. The Hjorth distribution has three parameters: the scale parameter β, and two shape parameters θ and δ. Thus, in some situations the failure behaviour can be better described than by the Weibull distribution, for example, if the failure behaviour changes or the entire bathtub curve should be described with only one distribution [2.21].

The density function for the Hjorth distribution is:

$$f(t) = \frac{(1+\beta t)\delta t + \theta}{(1+\beta t)^{\frac{\theta}{\beta}+1}} e^{-\frac{\delta t^2}{2}}. \tag{2.85}$$

In this equation, $\delta \neq 0$ and $\beta \neq 0$. The respective failure probability can be calculated with integration:

$$F(t) = 1 - \frac{e^{-\frac{\delta t^2}{2}}}{(1+\beta t)^{\frac{\theta}{\beta}}}. \tag{2.86}$$

The survival probability is written as:

$$R(t) = \frac{e^{-\frac{\delta t^2}{2}}}{(1+\beta t)^{\frac{\theta}{\beta}}}. \tag{2.87}$$

The resulting failure rate is:

$$\lambda(t) = \delta t + \frac{\theta}{1+\beta t}. \tag{2.88}$$

The expected value and variance of the Hjorth distribution can only be calculated numerically. For this, the following integral must be defined:

$$I(a,b) = \int_0^\infty \frac{e^{-\frac{at^2}{2}}}{(1+t)^b} \,. \tag{2.89}$$

With Equation (2.89) the expected value can be calculated:

$$E(t) = \frac{2}{\beta^2}\left(I\left(\frac{\delta}{\beta^2}, \frac{\theta}{\beta} - 1\right) - I\left(\frac{\delta}{\beta^2}, \frac{\theta}{\beta}\right)\right) \tag{2.90}$$

and the variance:

$$Var(t) = \frac{2}{\beta^2} I\left(\frac{\delta}{\beta^2}, \frac{\theta}{\beta} - 1\right) - \frac{2}{\beta^2} I\left(\frac{\delta}{\beta^2}, \frac{\theta}{\beta}\right) - \frac{1}{\beta^2} I^2\left(\frac{\delta}{\beta^2}, \frac{\theta}{\beta}\right). \tag{2.91}$$

The graphic curves of the Hjorth distribution are shown in Figure 2.43.

A further advantage of the Hjorth distribution over the Weibull distribution can be seen by comparing the failure rate of the Hjorth distribution with that of the Weibull distribution. The Weibull failure rate reaches infinity for $b < 1$ and low t values, while the Hjorth failure rate reaches a shape parameter θ for these conditions, see Figure 2.43.

The Equation (2.88) can also be interpreted as the sum of an increasing and a decreasing term, where δt is the increasing part and $\frac{\theta}{1+\beta t}$, the decreasing part. This is advantageous because it is then possible to characterize, for example, two different failure modes.

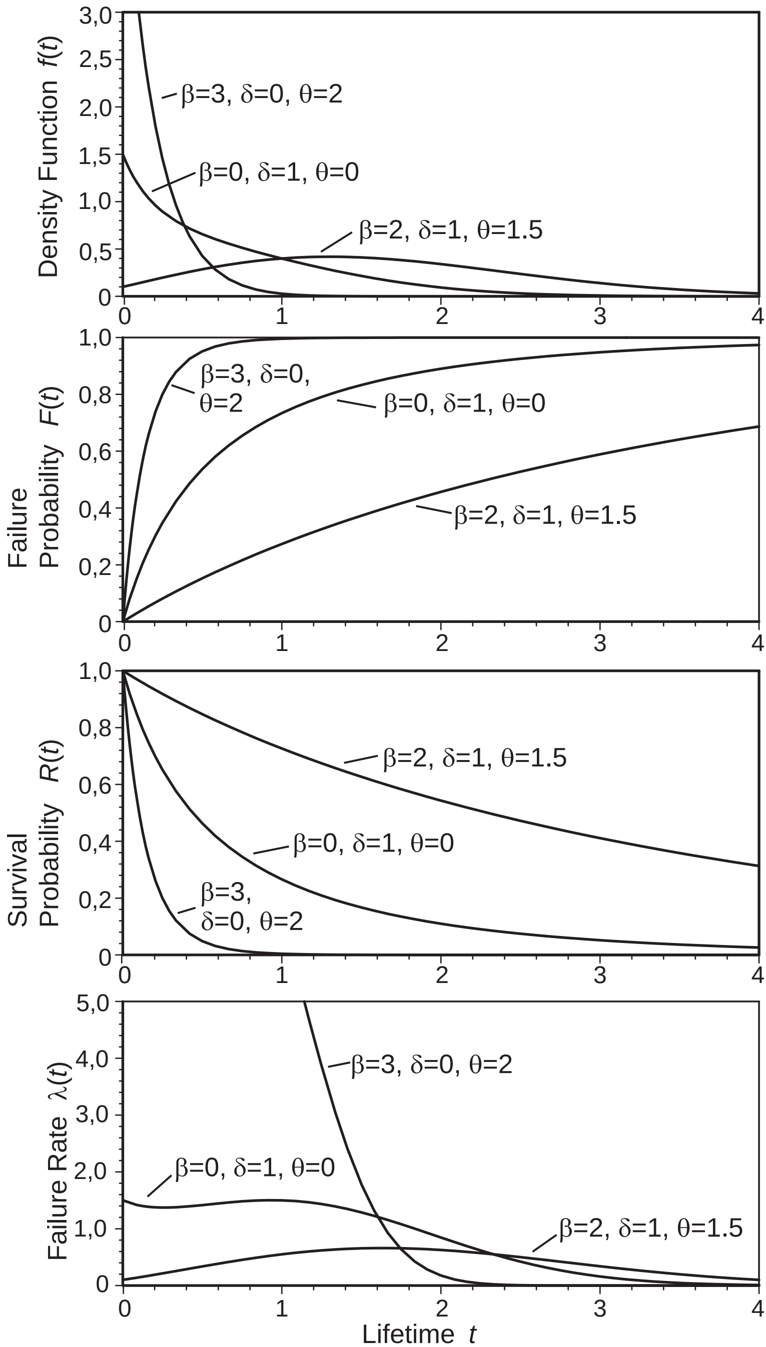


Figure 2.43. Failure functions for the Hjorth distribution

2.2.5.4 Sine Distribution

The sine distribution is derived from the $\arcsin\sqrt{P}$ transformation. P is the probability of fracture. The $\arcsin\sqrt{P}$ transformation is a simple procedure for graphical and calculative evaluation of dynamic fatigue trials. This method, which can be traced back to *R. A. Fisher*, has proven itself over extensive testing to be a simple, robust and reliable evaluation method for dynamic fatigue trials, especially in cases where the economical aspects of a trial must be kept comparably low [2.6].

The main advantage of this method is that the variance of the transformation variable $z = \arcsin\sqrt{P}$ reaches a constant asymptote for increasing z or for increasing test specimen size n, and is thus independent from n.

As already mentioned, this distribution is mainly used for the estimation of endurance strength in transition periods as well as for the estimation of the minimal lifetime in the fatigue strength zone. The coordinate (σ, z) is made up of a straight line according to the operation laws for regression calculation, in which the transformation variable $z = \arcsin\sqrt{P}$ for the observed failures pro test specimen size can be read from tables.

The coefficients in the equation $\hat{\sigma} = a + bz$ of the best fit straight lines are determined by using regression calculation.

Networks for $\arcsin\sqrt{P}$ are available for graphical evaluation, which are examined in more detail in [2.6, 2.7, 2.9].

According to [2.26], the statistical probability distribution for this transformation is:

$$F(P) = a + b\arcsin\sqrt{P}\,, \tag{2.92}$$

where P stands for the probability of fracture.

The following failure probability is found by solving the above equation for P:

$$F(t) = 1 - R(t) = sin^2\left(\frac{t-a}{b}\right). \tag{2.93}$$

The density function is found by deriving the equation by time.

$$f(t) = 2\frac{sin\left(\frac{t-a}{b}\right)cos\left(\frac{t-a}{b}\right)}{b} \tag{2.94}$$

The resulting failure rate is as follows:

$$\lambda(t) = 2\frac{\sin\left(\frac{t-a}{b}\right)\cos\left(\frac{t-a}{b}\right)}{b\left(1-\sin\left(\frac{t-a}{b}\right)^2\right)}. \tag{2.95}$$

In the sine distribution, the density function, the failure probability, as well as the survival probability are only defined for certain periods of time. Otherwise, the next sine period would begin. Furthermore, density functions can only be symmetrical, thus making this distribution impractical for general mechanical engineering applications.

2.2.5.5 Logit Distribution

The logit function from research methodology of biology from *J. Berkson* is described by the following failure probability according to [2.9]:

$$F(t) = 1 - R(t) = \frac{1}{1+e^{-(\alpha+\beta t)}} \tag{2.96}$$

The corresponding density function is:

$$f(t) = \frac{\beta e^{-(\alpha+\beta t)}}{\left(1+e^{-(\alpha+\beta t)}\right)^2} \tag{2.97}$$

and the failure rate:

$$\lambda(t) = \frac{\beta e^{-(\alpha+\beta t)}}{\left(1+e^{-(\alpha+\beta t)}\right)^2\left(1-\frac{1}{1+e^{-(\alpha+\beta t)}}\right)}. \tag{2.98}$$

For these equations it is important to remember that $\beta \neq 0$.The logit function can also be used for an approximation of a dynamic fatigue trial and can thus be easily compared with the $\arcsin\sqrt{P}$ transformation. This was done for example by *Dorff* in [2.9].

In this comparison, Equation (2.96) is transformed into a linear equation just as in the $\arcsin\sqrt{P}$ transformation.

This transformation according to [2.9] for the logit transformation is:

$$\operatorname{logit} F = \ln\frac{F}{R} = \alpha + \beta t\,, \tag{2.99}$$

whose parameters are again determined by regression. The term logit F serves as a determining characteristic of the logit distribution.

The density functions of the logit distribution show a distinct symmetrical behaviour, thus making the logit distribution an unsuitable description of failure behaviours for mechanical products.

2.2.5.6 Shifted Pareto Distribution

The Pareto distribution is used for example in aviation for the estimation of the minimal lifetime of components or in the field of reinsurance for the modelling of major damages.

According to [2.20, 2.22], the density function of the Pareto distribution is:

$$f(t) = \frac{1}{\alpha}\left(1+\frac{\xi t}{\alpha}\right)^{-\left(\frac{1}{\xi}+1\right)}, \tag{2.100}$$

where α stands for a dimensioning parameter which sets the initial value of the density function for $t = 0$, and ξ stands for a shape parameter which describes the failure slope. It is required that $\alpha > 0$ and $\xi > 0$.

Through integration the failure probability can be calculated as

$$F(t) = 1-\left(1+\frac{\xi t}{\alpha}\right)^{-\frac{1}{\xi}}, \tag{2.101}$$

resulting in the survival probability:

$$R(t) = \left(1+\frac{\xi t}{\alpha}\right)^{-\frac{1}{\xi}} \tag{2.102}$$

and the failure rate:

$$\lambda(t) = \frac{1}{\alpha\left(1+\frac{\xi t}{\alpha}\right)}. \tag{2.103}$$

According to [2.20], the expected value is:

$$E(t) = \frac{\alpha}{1-\xi} = \text{const.} \tag{2.104}$$

and the variance is:

$$Var(t) = \frac{\alpha^2}{\xi^2}\left(\frac{1}{1-2\xi} - \frac{1}{(\xi-1)^2}\right) = \text{const.} \tag{2.105}$$

2.2.5.7 S_B Johnson Distribution

With its 4 parameters, the S_B Johnson distribution is able to represent the failure behaviour of a component or system over the entire lifetime with early, random and wearout failures. Thus, this distribution is able to recreate the complete "bathtub characteristic" of the failure rate.

The density function for the S_B Johnson distribution according to [2.37] is:

$$f(t) = \frac{\eta}{\sqrt{2\pi}} \cdot \frac{\delta}{(t-\varepsilon)\cdot(\delta - t + \varepsilon)} \cdot e^{\left(-\frac{1}{2}\left(\gamma + \eta \cdot \ln\left(\frac{t-\varepsilon}{\delta - t + \varepsilon}\right)\right)^2\right)} . \tag{2.106}$$

Here, ε is the limit from the left side, δ is the dimensioning parameter and thus ε + δ is the limit from the right side of the random variable. The parameters η and γ are shape parameters. In general, the parameters must thereby fulfil the following conditions:

$$\varepsilon < x < \varepsilon + \delta\,,\ \eta > 0\,, -\infty < \gamma < \infty\,,\ \delta > 0\,.$$

The failure probability as well as the survival probability, failure rate, expected value and variance of the S_B Johnson distribution can only be calculated numerically.

2.3 Calculation of System Reliability with the Boolean Theory

Based on the components' failure behaviour it is possible to calculate the failure behaviour of a complete system using Boolean system theory [2.2, 2.33, 2.35, 2.36, 2.39]. Here, the failure behaviour for each individual component can be represented as described in Section 2.1 e.g. with a Weibull distribution with the parameters b, T and t_0.

A few important prerequisites are given for application of the Boolean theory:

- The system must be non-repairable, that is, the first system failure ends the system's lifetime. Thus, for repairable systems it is only possible to calculate the reliability up to the first system failure.
- The system elements must be either in a "functional" or "failed" state of condition.
- The system elements are "independent", that is, the failure behaviour of a component is not influenced by the failure behaviour of another component.

Under these conditions it is possible to deal with numerous mechanical engineering products using Boolean theory.

Furthermore, it is possible to build "reliability schematic diagrams" out of the system elements, out of which it is possible to recognize the reliability structure of a system. The reliability schematic diagram shows the effect of the failure of one component on the complete system. The connections between input *I* and output *O* in the diagrams in

Figure 2.44 and Figure 2.45 represent the possibilities for the functionality of the system.

The system is then functional, if there is at least one connection in the reliability schematic diagram between input and output, in which all components along the connection are intact. For a serial structure,

Figure 2.44a, the failure of one arbitrary component leads to the failure of the complete system. For a parallel structure,

Figure 2.44b, the system does not fail until all components have failed.

Figure 2.44c shows a combination of a serial and parallel structure.

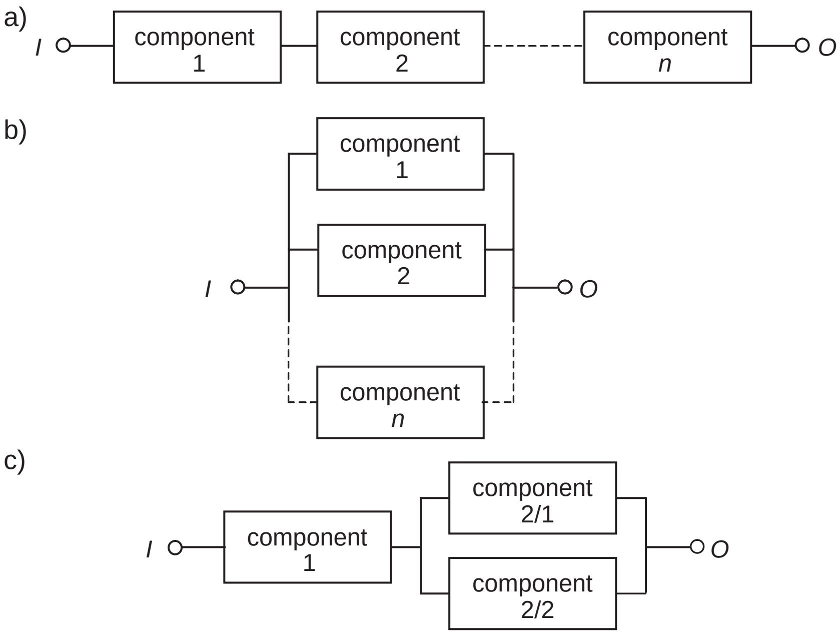


Figure 2.44. Basic structures of reliability schematic diagrams:
a) Serial structure
b) Parallel structure
c) Combination of serial and parallel structures

It should be noted, that the structure of the reliability schematic diagram does not necessarily correspond to the mechanical setup of a design. It is possible that a component appears more than once in the reliability schematic diagram.

Figure 2.45 shows an example for the creation of a reliability schematic diagram. The example system "free wheel clutch" consists of three shafts (S1, S2, S3), which are connected by 2 free wheel clutches (F1, F2), Figure 2.45a and b.

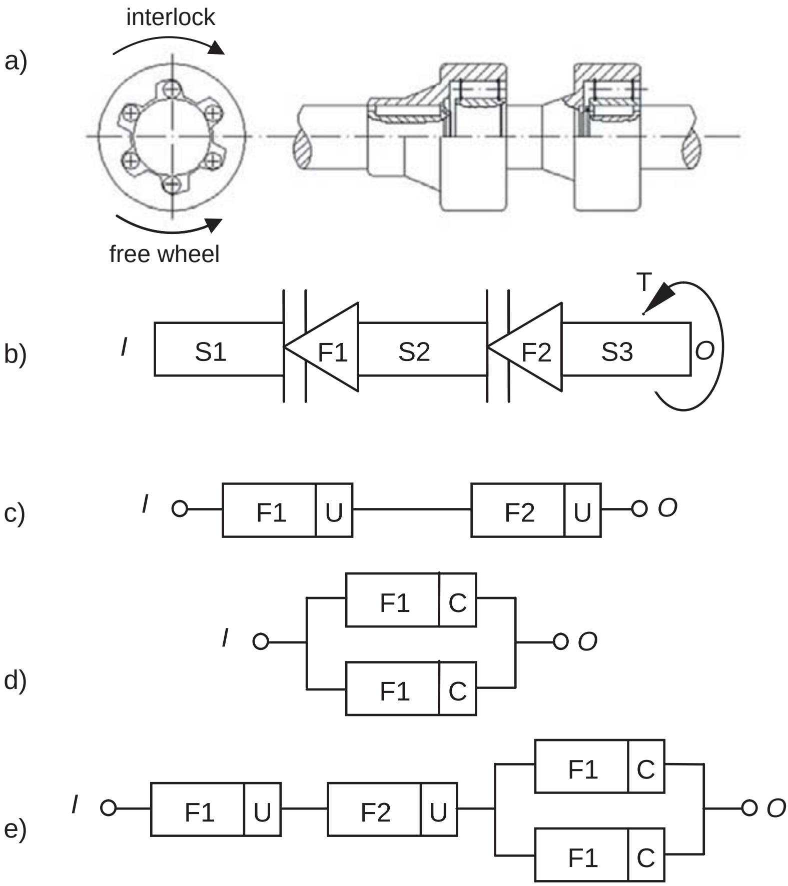


Figure 2.45. Creation of a reliability schematic diagram:
a) Drawing of the example system "free wheel clutch"
b) Principle sketch of the free wheel clutch system
c) Serial structure for the failure cause "interruption"
d) Parallel structure for the failure cause "clamping"
e) Complete reliability structure for the system free wheel clutch

The system input is indicated by *I* and the system output with *O*. The function of the system consists of transferring torque in one rotational direction and interrupting the connection between *I* and *O* in the other rotational direction through a response of the free wheel clutch, so that no further torque transfer is allowed.

The failure cause to be dealt with here for the free wheel clutch is either interruption or clamping. Interruption results in a blocking of torque transfer in both rotational directions, while clamping results in a rotational movement of the shaft in both directions. Figure 2.45c shows the reliability schematic scheme for interruption, a serial structure, since after the interruption of one free-wheel clutch, the system function is no longer fulfilled. In the case of clamping, Figure 2.45d, the reliability schematic diagram describes a parallel structure, due to the fact that when one free-wheel clutch clamps, the second free wheel clutch allows further functionality of the system. The complete block diagram, Figure 2.45e, results in a serial circuit made up of both partial structures in Figure 2.45c and Figure 2.45d.

Most mechanical products possess a serial reliability structure, since the introduction of redundancies is elaborate. This is the case especially for repetition and high volume parts. For critical components a larger dimensioning with a relatively high safety is carried out instead of a redundancy. Thus, the failure behaviour is improved in a simpler manner.

The calculation of the reliability for a serial system is calculated by the product of all survival probabilities:

$$R_S(t) = R_{C1}(t) \cdot R_{C2}(t) \cdot \ldots \cdot R_{Cn}(t) \text{ or } R_S(t) = \prod_{i=1}^{n} R_{Ci}(t). \tag{2.107}$$

With a definite reliability of each component ($R_C(t) < 1$), a value results for the system reliability that is less that the reliability of the weakest element. The system reliability becomes less with each additional component. For numerous components, the system has a low reliability, despite high individual component reliabilities, Figure 2.46.

If the component failure behaviour can be described by a three parametric Weibull distribution, then the following equation can be used for the calculation of the component reliability:

$$R_C(t) = e^{-\left(\frac{t-t_0}{T-t_0}\right)^b}. \tag{2.108}$$

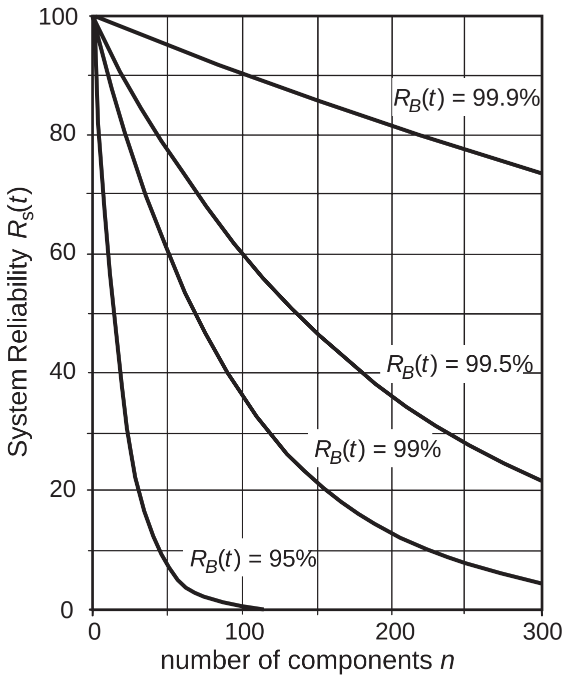


Figure 2.46. Decrease in the system reliability with an increasing amount of components with varying component reliabilities $R_C(t)$

The system reliability can be calculated with Equation (2.107):

$$R_S(t) = e^{-\left(\frac{t-t_{01}}{T_1-t_{01}}\right)^{b_1}} \cdot e^{-\left(\frac{t-t_{02}}{T_2-t_{02}}\right)^{b_2}} \cdot e^{-\left(\frac{t-t_{03}}{T_3-t_{03}}\right)^{b_3}} \quad \text{or}$$

$$-\ln R_S(t) = \left(\frac{t-t_{01}}{T_1-t_{01}}\right)^{b_1} + \left(\frac{t-t_{02}}{T_2-t_{02}}\right)^{b_2} + \left(\frac{t-t_{03}}{T_3-t_{03}}\right)^{b_3}. \tag{2.109}$$

The time t corresponding to a certain system reliability $R_S(t)$ can only be determined iteratively, except for a few exceptions. For $R_S(t) = 0{,}9$ it is possible to determine the commonly used B_{10S} lifetime of the system.

In special cases, the function $R_S(t)$ resulting from the component reliabilities represents an exact Weibull distribution. However, due to the universality of the Weibull distribution, the system reliability can be estimated nearly precisely with a certain Weibull distribution.

The reliability of a parallel system is calculated by the following equation:

$$R_S(t) = 1-(1-R_1(t))\cdot(1-R_2(t))\cdot\ldots\cdot(1-R_n(t)) \quad \text{or} \tag{2.110}$$

$$R_S(t) = 1 - \prod_{i=1}^{n} \left(1 - R_i(t)\right). \tag{2.111}$$

Here, n stands for the redundancy grade of the system.

2.4 Exercises to Lifetime Distributions

Problem 2.1

Maennig conducted dynamic fatigue trials on slightly notched shafts. The shafts were loaded with sinusoidal, pure cyclic stress-strain oscillations [2.31].

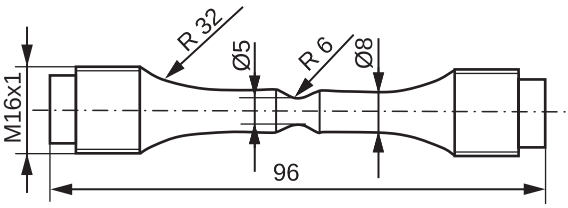


Figure 2.47. Notched shafts for dynamic fatigue trials

The following failure times resulted for a trial with n = 20 shafts with a deflection stress of 380 N/mm^2:

100,000 load cycles,	90,000 load cycles,	59,000 load cycles,
117,000 load cycles,	177,000 load cycles,	98,000 load cycles,
125,000 load cycles,	118,000 load cycles,	99,000 load cycles,
132,000 load cycles,	97,000 load cycles,	87,000 load cycles,
126,000 load cycles,	107,000 load cycles,	66,000 load cycles,
186,000 load cycles,	158,000 load cycles,	80,000 load cycles,
69,000 load cycles,	109,000 load cycles,	

a) Classify the results and create the histograms and the empirical functions,
b) the failure density,
c) the failure probability,
d) the survival probability and
e) the failure rate.

Problem 2.2

For further evaluation of the trial results in Problem 2.1 calculate the following:

a) the measures of central tendency (mean, median and mode) and
b) the statistical spread values (variance and standard deviation).

Problem 2.3

Draw the respective diagrams (linear scaling) for the following parameters of a Weibull distribution:

a) Weibull density functions:

$b = 1.0 \quad T = 2.0 \quad t_0 = 1.0$
$b = 1.5 \quad T = 2.0 \quad t_0 = 1.0$
$b = 3.5 \quad T = 2.0 \quad t_0 = 1.0$

b) Weibull failure probabilities:

$b = 1.0 \quad T = 2.0 \quad t_0 = 1.0$
$b = 1.5 \quad T = 2.0 \quad t_0 = 1.0$
$b = 3.5 \quad T = 2.0 \quad t_0 = 1.0$

Problem 2.4

The following density is given for a rectangular distribution:

$$f(t) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le t \le b \\ 0 & \text{otherwise} \end{cases}.$$

Calculate the failure probability $F(t)$, the survival probability $R(t)$ and the failure rate $\lambda(t)$ and show the results graphically.

Problem 2.5

The reliability of a technical component is given by the equation:

$$R(t) = exp\left(-(\lambda \cdot t)^2\right) \; for \; t \ge 0$$

Calculate the failure density, the failure probability and the failure rate. Show the results graphically.

Problem 2.6

The lifetime of a component can be described by a normal distribution with μ = 5,850 h and σ = 715 h.

a) Plot the distribution in a normal distribution chart.
b) What is the probability that a component does not fail before the point in time t_1 = 4,500 h?
c) What is the probability that a component fails before the point in time t_2 = 6,200 h?
d) What is the probability that a component fails between the times $\mu \pm \sigma$?
e) How long, t_3, can a component survive with a safety of 90%?

Problem 2.7

The failure behaviour of a pump can be well described with a log normal distribution with μ = 10.1 h and σ = 0.8 h.

a) Plot the distribution in a log normal chart.
b) What is the probability that the pump does not fail before the time t_1 = 10,000 h?
c) What is the probability that the pump fails before the time t_2 = 35,000 h?
d) What is the probability that the pump fails between the times t_1 and t_2?
e) How long, t_3, can the pump survive with a safety of 90%?

Problem 2.8

The lifetime (in hours) of an electrical component can be described by the exponential distribution $f(t) = \lambda \cdot \exp(-\lambda \cdot t) \quad t \geq 0; \quad \lambda = 1/(500h)$.

a) What is the probability that the component does not fail before the time t_1 = 200 h?
b) What is the probability that the component fails before t_2 = 100 h?
c) What is the probability that the component fails between the times t_3 = 200 h and t_4 = 300 h?
d) How long, t_5, can the component survive with exactly 90% safety and which range of time can the component survive with at least 90% safety?
e) What value must the parameter λ have for a lifetime distribution where the probability is 90% so that the lifetime of a component is at least 50 h?

Problem 2.9

Most failure behaviours in mechanical engineering are described by the Weibull distribution. Calculate the expected value (also called the *MTBF* value or the *MTTF* value) for a two parametric and three parametric Weibull distribution. List the actual values for the expected value for the follow parameter combinations:

a) b = 1; T = 1,000 h; t_0 = 0 h; b) b = 0.8; T = 1,000 h; t_0 = 0 h;
c) b = 4.2; T = 1,000 h; t_0 = 100 h; d) b = 0.75; T = 1,000 h; t_0 = 200 h;

Note: Use the tabulated gamma function

$$\Gamma(x) = \int_0^{\infty} e^{-t} \cdot t^{x-1} \cdot dt \, .$$

Problem 2.10

The failure behaviour of grooved ball bearings can be described very well with the Weibull distribution. The following is given: shape parameter b = 1.11, factor $f_{tB} = t_0 / B_{10} = 0.25$ and the B_{50} lifetime B_{50} = 6,000,000 load cycles.

a) What is the B_{10} lifetime?
b) Determine the Weibull parameters T and t_0 of the failure distribution.
c) What is the probability that a component fails between t_1 = 2,000,000 load cycles and t_2 = 9,000,000 load cycles?
d) How long, t_3, can a component survive with a safety of 99%?
e) For which shape parameter b (with constant T and t_0) does a lifetime distribution result with a probability of 50% that a component survives at least 5,000,000 load cycles?

Problem 2.11

Calculate the mode t_m of a three parametric Weibull distribution for $b > 1$. Check the result graphically for the following parameters: $b = 1.8$; T = 1,000 h; t_0 = 500 h.

Tip: $df(t_m) / dt = 0$.

Problem 2.12

The following information is known about the failure behaviour of an engine: The failure behaviour is described by a two parametric Weibull distribution. At time t_1 the failure probability is x_1, at time t_2 the failure probability is x_2. Conditions: $t_1 < t_2$ and $x_1 < x_2$. Calculate b and T for the failure distribution.

2.5 Exercises to System Calculations

Problem 2.13

Determine the system reliability function $R_S(t)$ for the system shown below as a function of the respective component reliabilities $R_i(t)$:

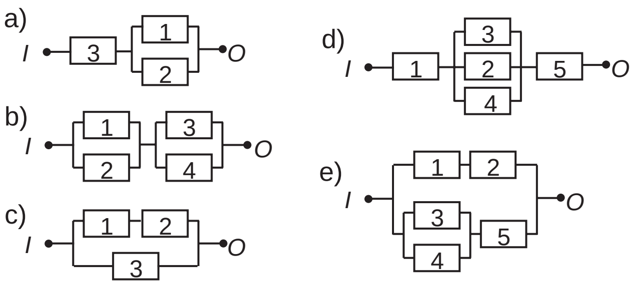


Figure 2.48. Block diagram of Problem 2.13

Problem 2.14

Describe the general relationships between the failure probability, failure density and failure rate of a serial system.

Problem 2.15

The reliability block diagram of an ABS system is given:

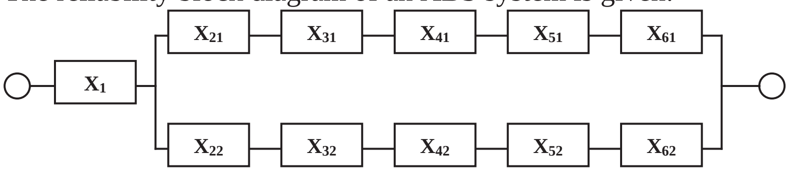


Figure 2.49. Reliability block diagram of Problem 2.15

The failure behaviour of all 11 components is described by the exponential distribution. The time independent failure rates corresponding to one year are listed in the following table:

Table 2.1. Failure rates for the system components in Problem 2.15

components:	part:	failure rate:
X_1	supply	$\lambda_1 = 4 \cdot 10^{-3}\ a^{-1}$
X_{21}, X_{22}	cables	$\lambda_{21} = \lambda_{22} = 7 \cdot 10^{-3}\ a^{-1}$
X_{31}, X_{32}	relay	$\lambda_{31} = \lambda_{32} = 5 \cdot 10^{-3}\ a^{-1}$
X_{41}, X_{42}	sensors	$\lambda_{41} = \lambda_{42} = 0{,}2 \cdot 10^{-3}\ a^{-1}$
X_{51}, X_{52}	electronics	$\lambda_{51} = \lambda_{52} = 1{,}5 \cdot 10^{-3}\ a^{-1}$
X_{61}, X_{62}	control valves	$\lambda_{61} = \lambda_{62} = 0{,}3 \cdot 10^{-3}\ a^{-1}$

a) Determine the equation for the system reliability $R_S(t)$ as a function of the component reliabilities $R_i(t)$.

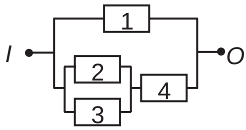
b) What is the survival probability of a useful life of 10 years? How many ABS systems out of 100 have failed after this time?

c) Determine the *MTBF* value (= expected value) of the system.

d) Determine the equation for the iterative calculation of the B_{10} lifetime of the system. Estimate a suitable initial value.
e) No system failure has occurred up the point in time $t_1 = 5$ years. With this information (condition), what is the probability for a useful life of 10 years?

Problem 2.16

The following reliability block diagram is given for a system. The failure behaviour of all parts is described by the exponential distribution. The failure rates are given as:



$\lambda_1 = 2.2 \cdot 10^{-3}\ h^{-1}$
$\lambda_2 = \lambda_3 = 4 \cdot 10^{-3}\ h^{-1}$
$\lambda_4 = 3.6 \cdot 10^{-3}\ h^{-1}$

Figure 2.50. Block diagram for Problem 2.16

a) What is the system reliability after 100 h of operation?
b) How many systems out of 250 will fail in a time period of 100 h?
c) What is the *MTBF* value of the system?
d) Determine the equation for the iterative calculation of the B_{10} lifetime of the system and estimate a suitable initial value for the calculation.

Problem 2.17

Lifetime trials are carried out for a system consisting of $n = 9$ identical gears in serial connection. The failure behaviour of one gear is described by a three parametric Weibull distribution. What is the reliability function for the system? The B_{10} lifetime of the system is $B_{10S} = 100{,}000$ load cycles. It is assumed that each gearwheel has a shape parameter of $b = 1.8$ and a factor of $f_{tB} = 0.85$ (failure due to tooth failure). What is the characteristic lifetime T of one gear?

References

[2.1] Anderson T. Theorie der Lebensdauerprüfung. Kugellagerzeitschrift 217

[2.2] Birolini A (2004) Reliability Engineering: theory and practice. Springer, Berlin, Heidelberg

[2.3] Bitter P et al (1986) Technische Zuverlässigkeit. Herausgegeben von der Messerschmitt-Bölkow-Blohm GmbH, Springer, München

[2.4] Bronstein I N, Semendjajew K A (2000) Taschenbuch der Mathematik – 5., überarb. und erw. Aufl. Verlag Harri Deutsch, Thun, Frankfurt am Main.

[2.5] Buxbaum O (1986) Betriebsfestigkeit. Verlag Stahleisen, Düsseldorf

[2.6] Dengel D (1975) Die $\arcsin\sqrt{P}$ -Transformation – ein einfaches Verfahren zur graphischen und rechnerischen Auswertung geplanter Wöhlerversuche. Zeitschrift für Werkstofftechnik, 6. Jahrgang, Heft 8, S 253-258

[2.7] Dengel D (1989) Empfehlungen für die statistische Absicherung des Zeit- und Dauerfestigkeitsverhaltens von Stahl. Materialwissenschaft und Werkstofftechnik 20, S 73-81

[2.8] Deutsche Gesellschaft für Qualität (1979) Begriffe und Formelzeichen im Bereich der Qualitätssicherung. Beuth, Berlin

[2.9] Dorff D (1966) Vergleich verschiedener statistischer Transformationsverfahren auf ihre Anwendbarkeit zur Ermittlung der Dauerschwingfestigkeit. Dissertation, TU-Berlin

[2.10] Fisher R A, Tippett L H C (1928) Limiting forms of the frequency distribution of the largest or smallest members of a sample. Proc. Cambridge Phil. Soc. 24, p 180

[2.11] Freudenthal A M, Gumbel E J (1954) Maximum Life in Fatigue. American Statistical Association Journal, Sept, pp 575-597

[2.12] Gäde K W (1977) Zuverlässigkeit – Mathematische Methoden. Hanser-Verlag, München

[2.13] Galambos J (1978) The Asymptotic Theory of Extreme Order Statistic. John Wiley & Sons Inc., New York

[2.14] Gnedenko B V (1943) Sur la distribution limite du terme maximum d'une série aléatoire. Ann. Math., 44, S 423ff

[2.15] Groß H R W (1975) Beitrag zur Lebensdauerabschätzung von Stirnrädern bei Zahnkraftkollektiven mit geringem Völligkeitsgrad. Dissertation

[2.16] Gumbel E J (1956) Statistische Theorie der Ermüdungserscheinungen bei Metallen. Mitteilungsblatt für mathematische Statistik, Jahrg 8, 13. Mittbl., S 97-129

[2.17] Gumbel E J (1958) Statistics of Extremes. Columbia University Press

[2.18] Härtler G (1983) Statistische Methoden für die Zuverlässigkeitsanalyse. Springer Wien New York

[2.19] Härtler G (1983) Statistische Methoden für die Zuverlässigkeitsanalyse. VEB Verlag Technik, Berlin

[2.20] Hipp C. Skriptum Risikotheorie 1. TH Karlsruhe http://www.quantlet.de/scripts/riskt/html/rt1htmlframe28.html

[2.21] Hjorth U (1980) A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates. Technometrics 22, S 99-10

[2.22] Jeannel D, Souris G (2001) Estimating Extremely Remote Values Of Occurrence Propability – Application To Turbojet Rotating Parts. In: Proceedings of ESREL, pp 709-716

[2.23] Joachim F J (1982) Streuungen der Grübchentragfähigkeit. Antriebstechnik 21, Nr 4, S 156-159

[2.24] Kao H K (1965) Statistical models in mechanical reliability. 11. Nat. Symp. Rel. & QC, p 240-246

[2.25] Kapur K C, Lamberson L R (1977) Reliability in Engineering Design. John Wiley & Sons Inc., New York

[2.26] Klubberg F (1999) Ermüdungsversuche statistisch auswerten. Materialprüfung 4, Heft 9

[2.27] Kreyszig E (1982) Statistische Methoden und ihre Anwendungen. Vandenhoeck & Ruprecht, Göttingen

[2.28] Lechner G, Hirschmann K H (1979) Fragen der Zuverlässigkeit von Fahrzeuggetrieben. Konstruktion 31, Heft 1, S 19-26

[2.29] Lieblein J, Zelen M (1956) Statistical Investigations of the Fatigue Life of Deep-Groove Ball Bearings. Journal of Research of the National Bureau of Standards vol 57, No 5, Nov, pp 273-316 (Research Paper 2719).

[2.30] Lienert G (1994) Testaufbau und Testanalyse - 5., völlig neubearb. und erw. Aufl. Beltz, Psychologie-Verl.-Union, Weinheim

[2.31] Maennig W-W (1967) Untersuchungen zur Planung und Auswertung von Dauerschwingversuchen an Stahl in den Bereichen der Zeit- und der Dauerfestigkeit. VDI-Fortschrittberichte, Nr 5, August

[2.32] Mercier W A (2001) Implementing RCM in a Mature Maintenance Program. Proceedings of the 2001 Annual Reliability and Maintainability Symposium (RAMS)

[2.33] Messerschmitt-Bölkow-Blohm GmbH (Hrsg.) (1971) Technische Zuverlässigkeit. Springer, Berlin

[2.34] O`Connor P D T (2001) Practical Reliability Engineering. John Wiley & Sons

[2.35] Reinschke K (1973) Zuverlässigkeit von Systemen mit endlich vielen Zuständen. Bd 1: Systeme mit endlich vielen Zuständen, VEB Verlag Technik, Berlin

[2.36] Rosemann H (1981) Zuverlässigkeit und Verfügbarkeit technischer Anlagen und Geräte. Springer, Berlin Heidelberg New York

[2.37] SAS/QC User's Guide http://www.rz.tu-clausthal.de/sashtml/qc/chap4/sect10.htm

[2.38] Verein Deutscher Ingenieure (1986) VDI 4001 Blatt 2 Grundbegriffe zum VDI-Handbuch Technische Zuverlässigkeit. VDI, Düsseldorf

[2.39] Verein Deutscher Ingenieure (1998) VDI 4008 Blatt 2 Boolesches Model. VDI, Düsseldorf

[2.40] Weibull W (1951) A Statistical Distribution Function of Wide Applicability. Journal of Applied Mechanics, September, pp 293-297

3 Reliability Analysis of a Transmission

The main target of work in reliability is to identify or to forecast the expected failure behaviour of a product as early as possible. By this, weak points in design can be determined and eliminated in early stages. In order to avoid extensive and time consuming trials one strives for calculation methods, which are based on the statistic and probabilistic basics described earlier. An unerring prognosis can only be achieved, if the failure behaviour of the components is relatively well known.

The early and random failures in sections 1 and 2 of the bathtub curve are difficult to pre-estimate, as already mentioned in Section 2.2.1.4. They are conditionally useful for probabilistic calculation methods. The following executed reliability determination is therefore limited to wearout failures (section 3 of the bathtub curve), which is, in most cases, the most dominant failure cause. The developed procedure is based on the described calculations methods mentioned in [3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.9].

The system used for this example is a single-stage transmission, which is shown in Figure 3.1. On the input shaft (IS) of the transmission is the small transmission input gear. The power is transmitted by the larger gear onto the transmission output shaft (OS). Besides the bearings for the shafts, the transmission consists of a transmission housing with a housing cover and different small bearing covers, which are sealed by sealing compounds or radial seal rings. The transmission example is therefore a manageable system due to its simple input and output devices and transmission elements.

To determine the expected system reliability it is useful to refer to the flowchart shown in Figure 3.2. The main focus of the system analysis is, to determine the components which are relevant to the reliability and to set up a reliability structure of the system. Afterwards, the system elements are considered separately and their reliability is determined. The analysis is concluded with a reliability calculation for the complete system. In the following sections, these three sequential steps will be shown in detail.

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_3, © Springer-Verlag Berlin Heidelberg 2008

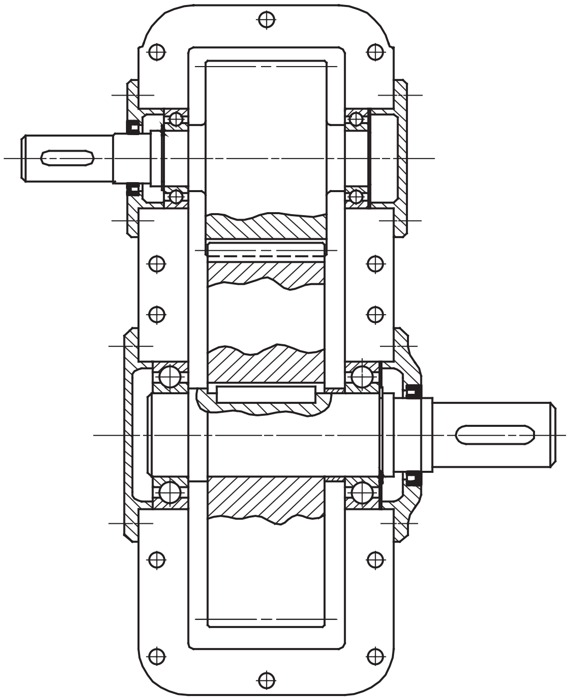


Figure 3.1. Example system “single-stage transmission”

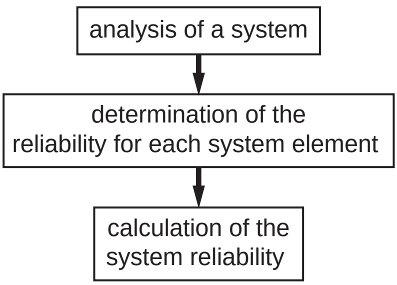


Figure 3.2. Flowchart for the determination of system reliabilities

3.1 System Analysis

3.1.1 Determination of System Components

At the beginning of the analysis, it is useful to identify all system components in order to study an overview of system, see Figure 3.3. Components and/or interfaces of components can be regarded as components. In Figure 3.4 all components of the example system "transmission" are listed. This small and manageable system already consists of 27 components. Shrink fits, welded joints, etc. are interfaces of components. Besides the components, these interfaces can also be critical elements for the system reliability. All components of the system are illustrated in the function block diagram, see Figure 3.5

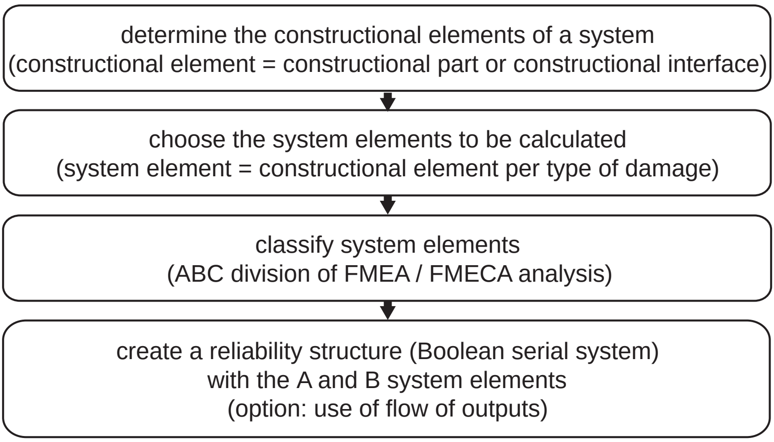


Figure 3.3. Flowchart of the system analysis

housing	roll bearing 1	bearing cover 3
housing cover	roll bearing 2	bearing cover 4
housing bolts	roll bearing 3	bearing cover sealing 1
housing cover sealing	roll bearing 4	bearing cover sealing 2
input shaft	locking washer 1	bearing cover sealing 3
output shaft	locking washer 2	bearing cover sealing 4
gearwheel 1	spacer ring	shaft seal 1
gearwheel 2	bearing cover 1	shaft seal 2
fitting key connection	bearing cover 2	hex bolt 1-12

Figure 3.4. Components of the example system "transmission"

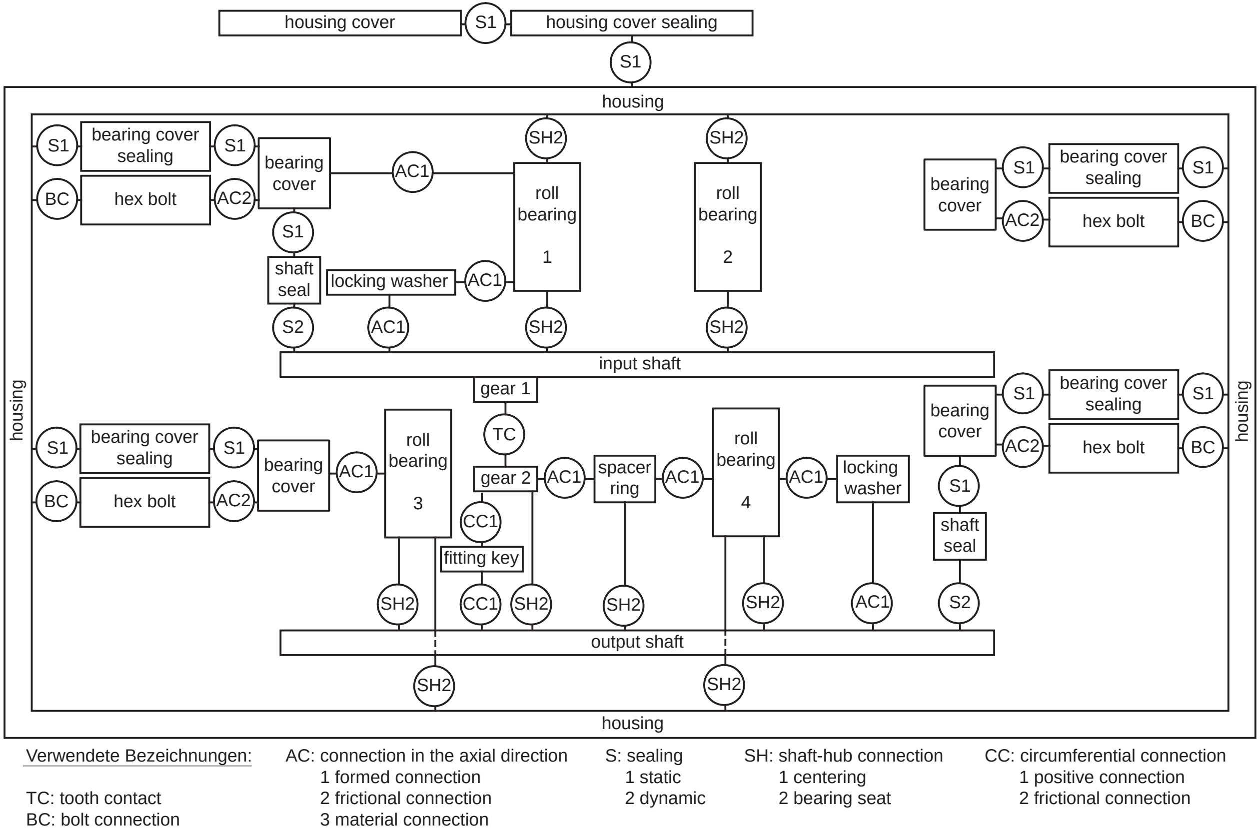


Figure 3.5. Function block diagram of the example transmission

3.1.2 Determination of System Elements

Some of the components can fail for several reasons. A gear, for example, can lose its functionality by tooth failure, pittings or scuffing. For the later calculation it is recommended that consideration to the damage potential peculiar to a specific element. Therefore, system elements are defined, which divide the components according to their kind of damage. For the example above, the system is expanded to 28 elements. The two components gear 1 and gear 2 were thereby subdivided further into two kinds of damages: “tooth failure” and “pittings”.

3.1.3 Classification of System Elements

The diverse system elements fulfil quite different functions and thereby contribute differently to the reliability of the system. Therefore, it is not reasonable and/or permitted to consider all system elements equally. A classification of the system elements will thus be executed. The elements are classified in reliability, relevant and neutral parts. Furthermore, it is necessary to differ whether the parts underlie a defined load or if their stress can only be collected inexactly. A developed ABC analysis of the system elements, which takes these aspects into account, is shown in Figure 3.6. It is possible to calculate the failure behaviour of A system elements whereas for B system elements one depends on experience and trials. The reliability neutral C system elements are not taken into account in any further calculation.

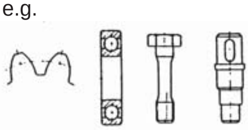
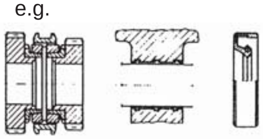
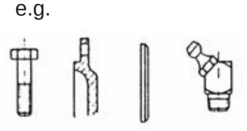
A parts (prone to risk)	B parts (prone to risk)	C parts (neutral to risk)
e.g. 	e.g. 	e.g. 
• loaded by defined static stress; load profile is given; power transmitting • lifetime calculations are possible and correspond to reality • failure behaviour is given by Wöhler experiments; shape parameter $b > 1.0$	• loaded many by friction, abrasion, extreme temperatures; dirt and corrosion • lifetime calculations are not possible or do not correspond to reality • failure behaviour must be estimated or determined by experiments; shape parameter $b \geq 1.0$	• randomly loaded by impacts, friction, abrasion, etc • calculational dimensioning only provisorily necessary or irrelevant • only random and early failures; shape paramenter $0 < b \leq 1.0$

Figure 3.6. ABC classification of system elements

The developed ABC classification is suited for small and manageable systems. For new and complex systems, the elements which pose critical influences upon the reliability should be determined by a complete FMEA analysis (see Chapter 4).

A parts	B parts	C parts
input shaft	shaft seal 1	housing
ouput shaft	shaft seal 2	housing cover
gear 1 breakage		housing bolts
gear 2 breakage		housing cover sealing
gear 1/2 pittings		locking washer 1
fitting key connection		locking washer 2
roll bearing 1		spacer ring
roll bearing 2		bearing cover 1
roll bearing 3		bearing cover 2
roll bearing 4		bearing cover 3
		bearing cover 4
		bearing cover sealing 1
		bearing cover sealing 2
		bearing cover sealing 3
		bearing cover sealing 4
		hex bolt 1-12

Figure 3.7. ABC classification of the system elements for the example system

With pre-calculations, experience from similar transmissions and technical discussions, the classification of the system example “transmission” are shown in Figure 3.7.

The complete system, which includes 28 system elements, was reduced to 12 elements that are relevant for the reliability study. The relevant elements are, apart from the radial seal rings, the power transmitting parts: input and output shafts, gears, fitting key connection and bearings.

3.1.4 Determination of the Reliability Structure

After the classification, the next step of the analysis is the structure determination of the system, see Figure 3.3. To set up the reliability schematic, it is advisable to use function block diagrams or the schematic of the power flow. Both types of diagrams show how the system elements are stressed and how their failures affect the rest of the system. Starting off

with one of these diagrams, the reliability block schematic can be created quite easily.
If one examines the function block diagram of the transmission, see Figure 3.5, it can be seen, that all system elements are necessary for a correct system function. Thus, the reliability block schematic is a pure serial structure, see Figure 3.8.

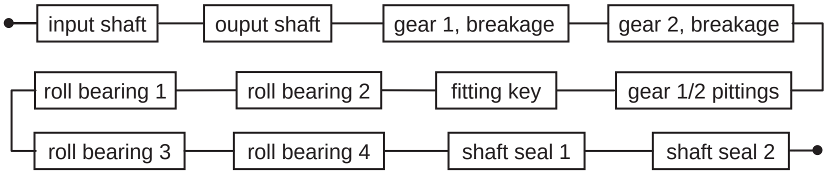


Figure 3.8. Reliability block schematic of the transmission (Boolean serial structure)

The system reliability R_s can be calculated for a Boolean serial system structure according to Section 2.3, which is the product of the reliability for all the system elements R_E:

$$\begin{aligned} R_{system} = {} & R_{IS} \cdot R_{OS} \cdot R_{gear1,tooth\ failure} \cdot R_{gear2,tooth\ failure} \\ & \cdot R_{gear1/2,pittings} \cdot R_{fitting\ key} \cdot R_{bearing1} \cdot R_{bearing2} \cdot R_{bearing3} \\ & \cdot R_{bearing4} \cdot R_{RSR1} \cdot R_{RSR2} . \end{aligned} \tag{3.1}$$

The system Equation (3.1) describes the reliability relevant system elements and their functional dependencies. Therefore, it represents the actual result of the system analysis.

3.2 Determination of the Reliability of System Elements

After the analysis of the system, it is still necessary to determine the unknown failure behaviour of the reliability critical system elements, see Figure 3.9.

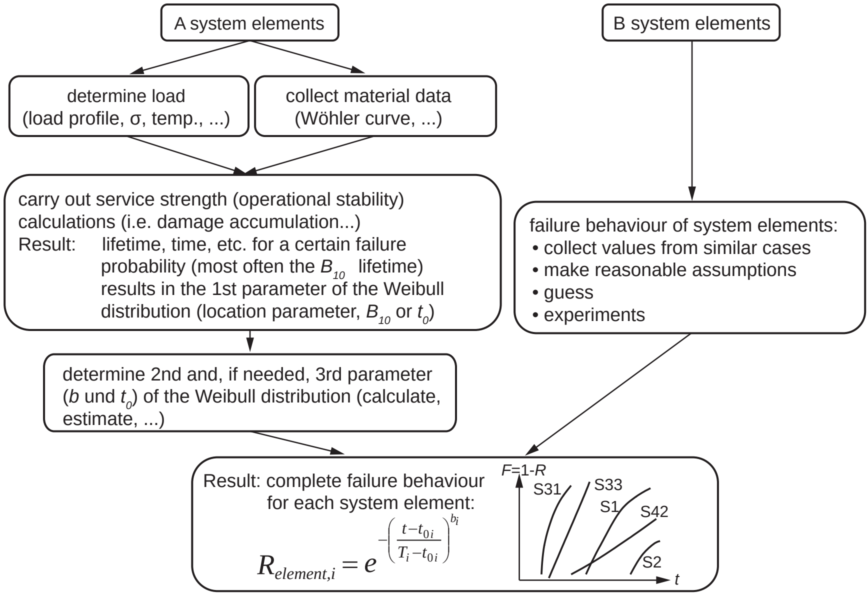


Figure 3.9. Schematic of the determination of the reliabilities of the system elements

For the A system elements relatively accurate load spectrums and Wöhler curves (SN-curves) already exist. With these data it is possible to execute operating fatigue strength calculations, with which the lifetime of the system elements can be determined. In most cases, the calculated lifetime corresponds to the B_{10} or B_1 lifetime and is therefore related to a certain failure probability. The conversion of the B_{10} and/or B_1 lifetime into the characteristic lifetime T is given in the Equations (7.1) and (7.2). In the probability net, one can determine one point and/or one parameter of the distribution: the scale parameter. With the knowledge of the remaining parameters of the distribution – shape parameter b and, if necessary, the failure free time t_0 (location parameter) – one yields the complete failure behaviour of the element.

The determination of failure behaviour for B system elements is facilitated by experience, or if not, the failure behaviour should be estimated. Trials for B system elements can prove efficacious in determination of the reliability.

Apart from the radial seal ring, all elements in the example "transmission" are A system elements, for which the failure behaviours can be calculated. With an assumed input load spectrum the important stress factors such as root bending stresses, Herzian stresses, bearing stresses, etc. were calculated for the A system elements.

The stresses, together with the Wöhler curves (SN-curves) and the bearings data, lead to the summarized lifetimes, see Figure 3.10.

input shaft	fatigue resistant
output shaft	fatigue resistant
gear 1 breakage	70,000 revolutions IS (B_1)
gear 2 breakage	120,000 revolutions IS (B_1)
gear 1/2 pittings	500,000 revolutions IS (B_1)
fitting key connection	fatigue resistant
roll bearing 1	1,500,000 revolutions IS (B_{10})
roll bearing 2	fatigue resistant
roll bearing 3	fatigue resistant
roll bearing 4	2,500,000 revolutions IS (B_{10})

Figure 3.10. Calculated B_1 and B_{10} lifetimes of the system elements

According to the definition, the B_1 and B_{10} lifetimes are related to a failure probability of $F(t) = 1\%$ and/or $F(t) = 10\%$. The B_1 and B_{10} lifetimes can be transferred into the characteristic lifetime T with the Equations (7.1) and (7.2). The result: one parameter of the failure distribution is known: the scale parameter. The two other parameters of the distribution – the shape parameter b and, if necessary, the failure free time t_0 – have been chosen according to the values given in Chapter 7. All Weibull parameters of the non endurable A system elements are given in Table 3.1.

For the two B system elements, the radial seal rings 1 and 2, the failure behaviour cannot be calculated. For those two elements, however, failure statistics of comparable transmissions are known and it can be said that such seals fail exclusively at random. Therefore, the shape parameter $b = 1$ is assigned to both system elements. For the characteristic lifetime the values of failure statistics from comparable transmissions have also been taken, see Table 3.2. It was not possible to identify a failure free time t_0 for typical random failures out of the failure statistics.

With the values given in Table 3.1 and Table 3.2 the complete failure behaviour of the system elements can be displayed, see Figure 3.11.

Table 3.1. Weibull parameters for A system elements

	b	T	t_0	f_{tB}
gear 1 tooth failure	1.4	106,600	68,600	0.9
gear 2 tooth failure	1.8	185,000	114,500	0.85
gear 1/2 pittings	1.3	2,147,300	450,700	0.6
bearing 1	1.11	9,400,000	300,000	0.2
bearing 4	1.11	15,700,000	500,000	0.2

Table 3.2. Weibull parameters for B system elements

	b	T	t_0	f_{tB}
RSR 1	1.0	66,000,000	0	0
RSR 2	1.0	66,000,000	0	0

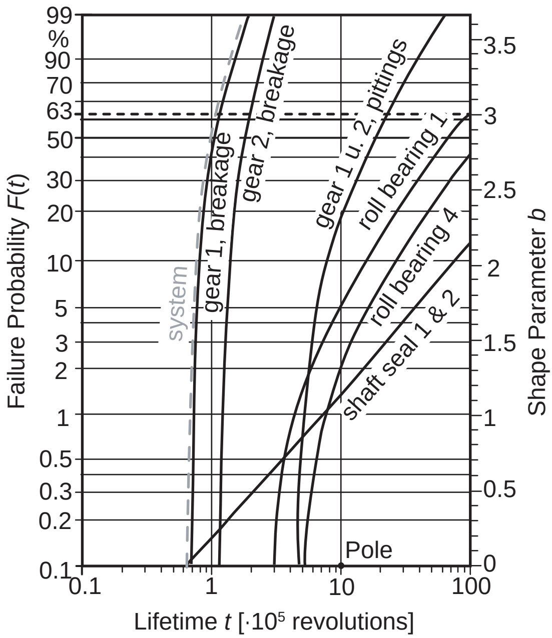


Figure 3.11. Failure behaviour of the system elements and of the system (System: dashed; $B_{10\text{-}System}$ = 76,000 rotations of the input shaft)

3.3 Calculation of the System Reliability

The calculation of the system reliability is the final step in the calculation. Here, the assigned reliabilities of the system elements are inserted in the system Equation (3.1), see Figure 3.12.

The complete system behaviour can be displayed graphically, if a curve is put through several pairs of variants $R_S(t_S)$. The system failure curve runs left of the element failure curves, see Figure 3.12. In many cases the complete system failure behaviour is not of interest, but rather which system lifetime can be achieved for a certain system reliability or which system reliability will be reached for a given system lifetime. These values can be determined out of the system equation by iteration and/or an analytical solution, see Figure 3.12.

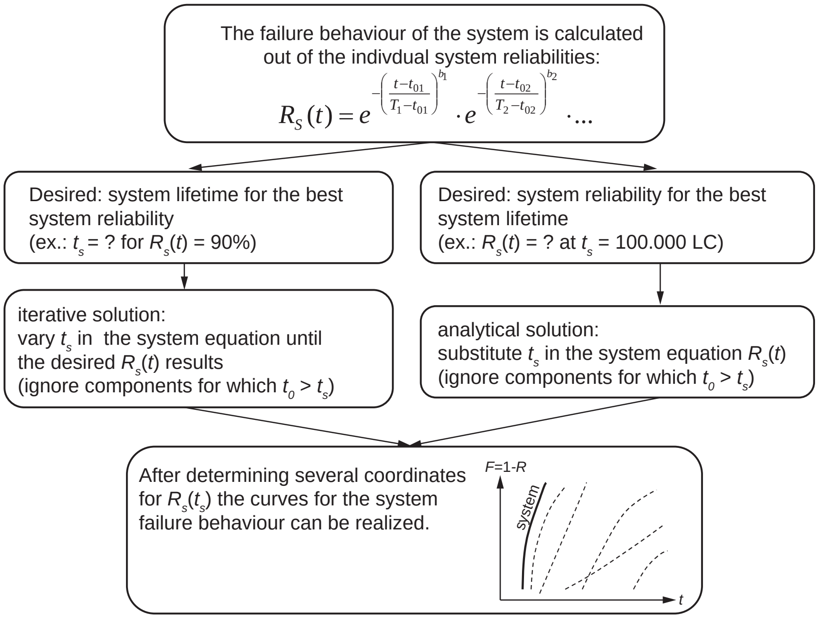


Figure 3.12. Schematic of the calculation of the system reliability

For the calculation of the system reliability, one must differentiate between the system elements with a two-parametric and with a three-parametric Weibull distribution. System elements, described by a two-parametric Weibull distribution, must always be considered in the calculation of the system reliability. Their reliability already reaches values less than 1 at the lifetime of $t = 0$. Each additional system element with a two-parametric Weibull distribution therefore decreases the system reliability directly. The statement, that further parts unavoidably decline the system reliability, is thus proven for two-parametric system elements.

System elements with a three-parametric Weibull distribution do not always have to be taken into account for the calculation of the system reliability. Only those three-parametric system elements can cause failures, whose failure free time t_0 is smaller than the regarded lifetime t. Thus, three-parametric system elements only have an influence on the system lifetime t_{xS} (or B_{xS}) if:

$$t_0 < t_{xS} \,. \tag{3.2}$$

If a system is enhanced by further three-parametric system elements, whose failure free time t_0 is greater than, for example, the B_{10S} lifetime, then these elements have no effect on the B_{10S} reliability of the system. A direct relationship between the amount of parts and system reliability is not given in these cases.

It should be noted, that for a system with two-parametric *and* three-parametric system elements, the system has a two-parametric distribution. This means that already at $t = 0$ failures of two-parametric system elements can occur.

The example system "transmission" has mainly three-parametric system elements. Only the two radial seal rings RSR 1 and RSR 2 have a two-parametric Weibull distribution. For the calculation of the system reliability of the example transmission, the failure behaviour is defined by the four system elements "gear 1 tooth failure", "gear 2 tooth failure", "RSR 1" and "RSR 2". The system equation in this case is:

$$R_{system} = R_{gear1,tooth\ failure} \cdot R_{gear2,tooth\ failure} \cdot R_{RSR1} \cdot R_{RSR2}\ . \qquad (3.3)$$

With the help of an iterative solution, the B_{10S} system lifetime of the input shaft is 76,000 rotations, see Figure 3.11.

A predominant amount of the failures is caused by the system element "gear 1 tooth failure". This system element with this type of damage represents a weak point in the system. Together with the system elements "gear 2 tooth failure", "RSR 1" and "RSR 2", the complete reliability of the transmission is defined. The remaining parts are well dimensioned and their failure is only expected to occur at a later point in time.

The four determined reliability relevant system elements are a typical example for such so called weak points of a system that constitute mainly or almost exclusively the failure behaviour. With a partial or complete description of the component failure behaviour in a three-parametric Weibull distribution, an extensive reliability analysis consequently leads to the identification of weak points [3.1].

An updated procedure can be found in [3.8, 3.10]. An overview of the modified methodology is shown in Figure 3.13.

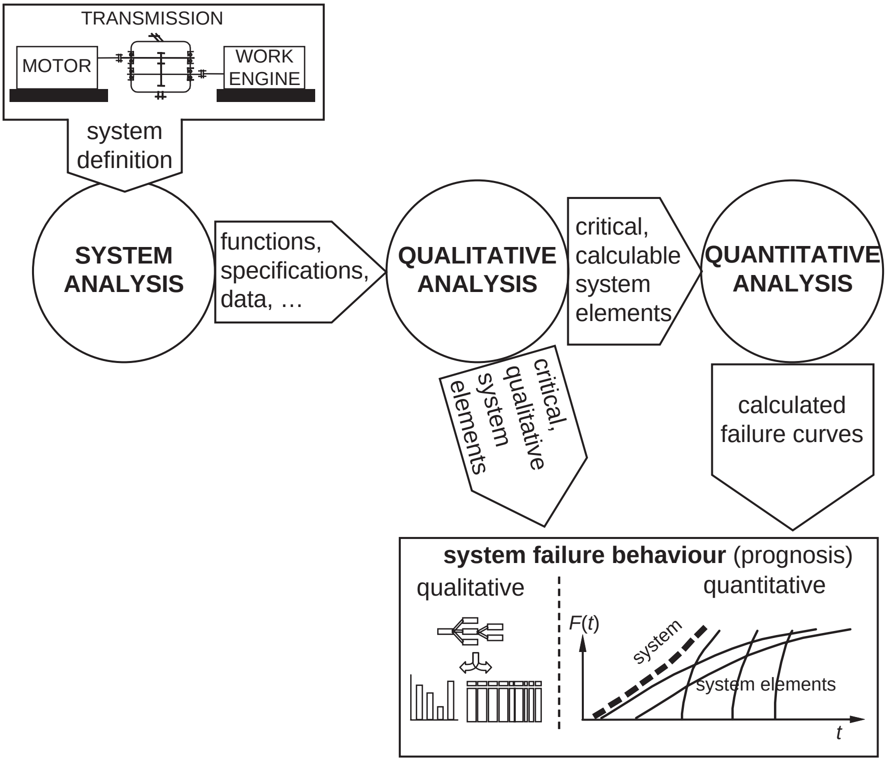


Figure 3.13. Updated procedure for the calculation of system reliability

References

[3.1] Bertsche B, Lechner G (1987) Einfluss der Teileanzahl auf die System-Zuverlässigkeit. Antriebstechnik 26, Nr 7, S 40-43

[3.2] Birolini A (2004) Reliability Engineering: theory and practice. Springer, Berlin, Heidelberg

[3.3] Heise W (2002) Praxisbuch Zuverlässigkeit und Wartungsfreundlichkeit. Hanser München Wien

[3.4] Kececioglu D (2002) Reliability engineering Handbook, Volume 2. Prentice Hall, cop. Engelwood Cliffs, N.J.

[3.5] Lechner G, Hirschmann K H (1979) Fragen der Zuverlässigkeit von Fahrzeuggetrieben. Konstruktion 1, S 19-26

[3.6] Lewicki D G, Black J D, Savage M, Coy J J (1986) Fatigue Life Analysis of a Turboprop Reduction Gearbox. Journal of Mechanisms, Transmissions and Automation in Design, June, vol. 108, pp 255-262

[3.7] O`Connor P D T (2001) Practical Reliability Engineering. John Wiley & Sons
[3.8] Rzepka B, Schröpel H, Bertsche B (2002) Studie zur Anwendung von Zuverlässigkeitsmethoden in der Industrie. Tagung TTZ 2002, 10. und 11. Oktober 2002, Stuttgart / VDI-Gesellschaft Systementwicklung und Projektgestaltung, VDI-Berichte Nr. 1713, S 279-299
[3.9] Savage M, Brikmanis C, Lewicki D G, Coy J J (1988) Life and Reliability Modeling of Bevel Gear Reductions. JK. Of Mechanisms, Transmissions and Automation in Design, June, vol. 110, pp 189-196
[3.10] Verband der Automobilindustrie (2000) VDA 3.2 Zuverlässigkeitssicherung bei Automobilherstellern und Lieferanten. VDA, Frankfurt

4 FMEA – Failure Mode and Effects Analysis

FMEA can be understood as the most commonly used and well known qualitative reliability method in the area of reliability methodology. It is a dynamic preventive reliability method used in the modification of systems and accompanies the design cycle for modification of components. The overall aim is to analyse and modify components in the light of experience to achieve an optimum criterion of reliability assessment. One significant criterion is the risk priority number (RPN) which will be discussed in the chapter.

FMEA was developed in the mid sixties in the USA by NASA (National Aeronautics and Space Administration) for the Apollo project. Afterwards, this method generally applied procedure in aerospace and aeronautical engineering. Most literary resources concerning this method stem from the American Military Standard MIL-STD-1629A [4.1], and is required as an approval standard for all parts in aerospace and aeronautical engineering. The FMEA method is elaborately detailed and involves a clearly defined procedure. Further use of the FMEA method continued in nuclear technology and in the automotive industry. The Ford Company in America was the first automotive company to integrate this method into its quality assurance concept, see Figure 4.1.

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_4, © Springer-Verlag Berlin Heidelberg 2008

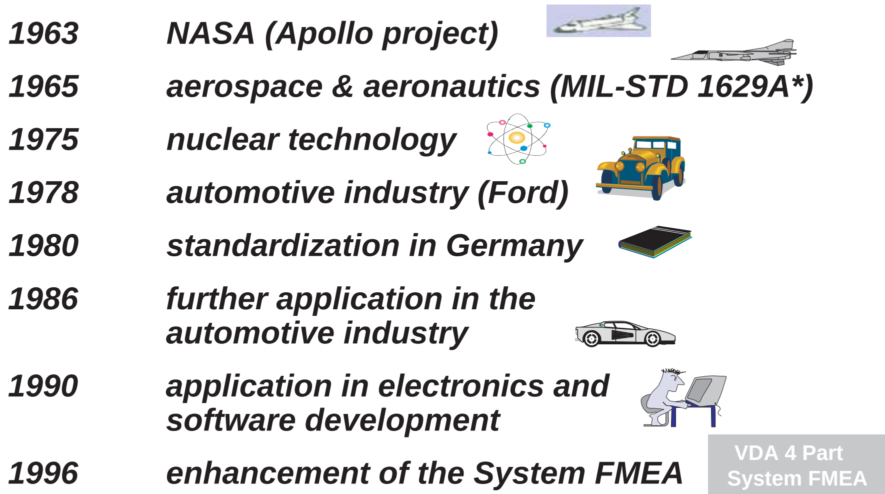


Figure 4.1. Origin of FMEA

Due to the continually increasing quality requirements made by customers, new legal restraints (production liability laws [4.5]) and norms (DIN ISO 9000 ff [4.2]), increasing product complexity, increasing cost demands, requirement for shorter development periods and lastly due to an increasing environmental awareness, FMEA has become a solid component in today's quality assurance. The FMEA procedure specified by the German Association of the Automotive Industry (VDA - Verband der Automobilindustrie) [4.7] is the prevailing standard for methodical application of an FMEA analysis in Germany.

In the following text the basic principles as well as general fundamentals concerning the FMEA methodology and the procedure of a Form FMEA according to VDA 86 will be discussed. Emphasis is put on the FMEA according to VDA 4.2, which is summarized in Section 4.4. The FMEA procedure according to VDA 4.2 is the most extensive and commonly used procedure, especially in the automotive industry, in Germany and in Europe.

4.1 Basic Principles and General Fundamentals of FMEA Methodology

The abbreviation FMEA stands for "*F*ailure *M*ode and *E*ffects *A*nalysis", see Figure 4.2. The FMEA method is specified under this name in the DIN 25 448 [4.3] since 1980.

F M E A ?

- **F**ailure **M**ode and **E**ffects **A**nalysis
- Failure Effects Analysis (DIN 25 488)
- Behaviour analysis
- Analysis of failure modes, failure effects and failure causes

Figure 4.2. Definition of the term FMEA

FMEA is a systematical method. Its fundamental idea is the determination of all possible failure modes for arbitrary systems, subsystems, or components. At the same time the possible failure effects and failure causes are presented. The procedure is concluded with a risk assessment and specification for optimization actions, see Figure 4.3. The aim of the method is to recognize the risks and weak points of a product as early as possible in order to enable execution improvements in a timely manner.

FMEA is a method to discover

- potential failure modes
- potential failure effects
- potential failure causes

for components or system parts; the risk is assessed and actions for optimization are determined.

Figure 4.3. Fundamental idea of FMEA

FMEA deals with a risk assessment integrated into the development and process planning of new products. It is an important factor in quality assurance before a new production cycle takes place. FMEA belongs to reliability analysis and must be carried out systematically, without interruption, preemptively and team-oriented.

One version of FMEA is the FMECA (Failure Mode, Effects and Criticality Analysis), which enhances the original FMEA with a separate risk characterization.

FMEA is made up of various procedures according to the type and complexity of the system to be assessed or according to the desired results. An overview of the various FMEA procedures used most often is shown in Figure 4.4.

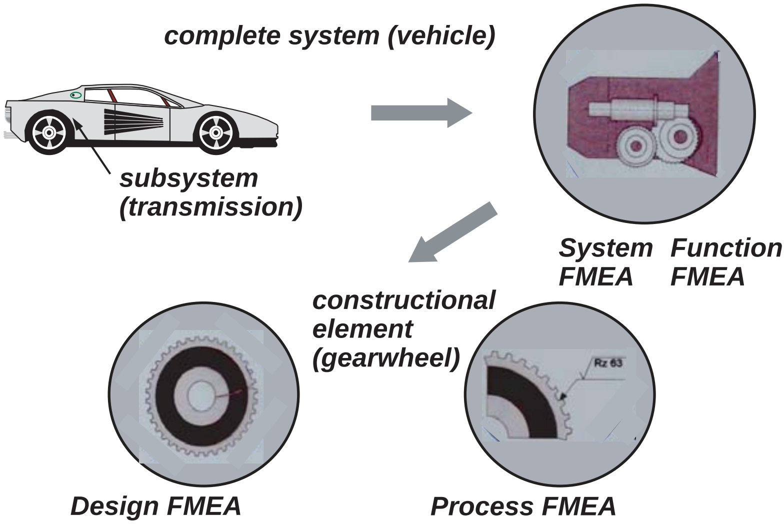


Figure 4.4. Types of FMEA

The execution of an FMEA is carried out in interdisciplinary groups, the FMEA teams. It is reasonable to execute an FMEA in teams, since is it only then possible to incorporate all operational areas affected by the analysis. In practice it has been proven to be beneficial to execute an FMEA under the direction of an FMEA moderator, who is familiar with the methodical procedure. In this way, time consuming discussions concerning the method can be avoided.

In general, the FMEA team consists of a moderator, who offers methodical knowledge, and the FMEA team, which offers technical knowledge concerning the product or process to be analyzed. The moderator, who also may possess a marginal know-how concerning the product or process, certifies that the team members acquire a basic knowledge of the FMEA methodology. A brief training at the beginning of an FMEA assignment is useful. The team for a Design FMEA should be made up of experts from various fields, see Figure 4.5, whereupon at least the fields marked with an X, design and production planning, should be covered.

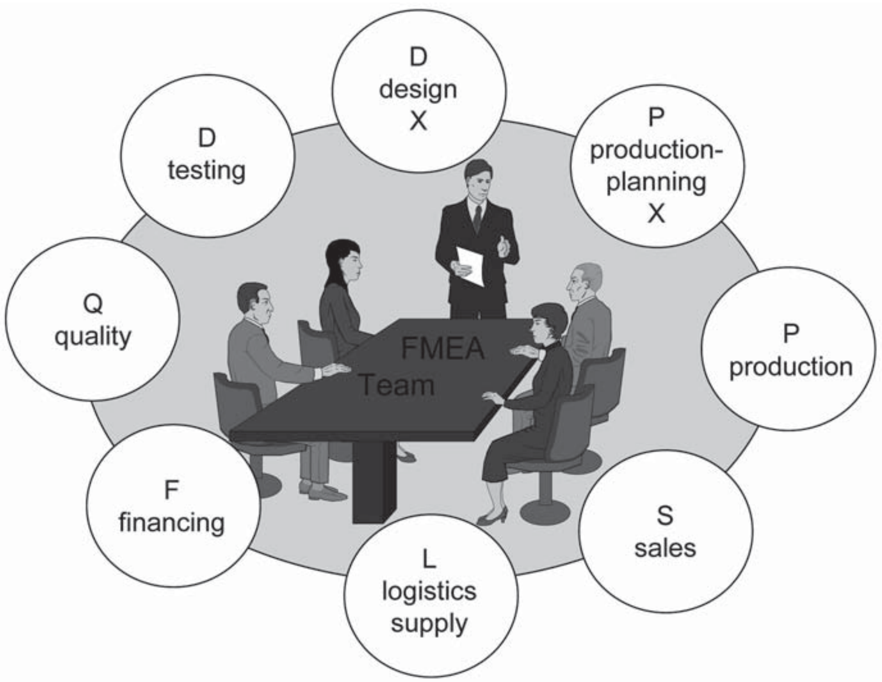


Figure 4.5. The FMEA team

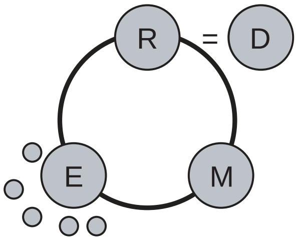
The distinction between technical knowledge in various fields and the methodology of an FMEA execution offers the advantage that the experts from the respective fields only offer their technical knowledge free of any methodical considerations. Thus, merely a basic knowledge of FMEA is adequate for the team of experts.

The team size ranges ideally between 4 – 6 members. If less than 3 – 4 team members participate in the FMEA, one runs the risk that important sub areas will be forgotten or dealt with inadequately. On the other hand, if the team consists of more than 7 – 8 members, then the dynamical group effect is significantly weakened, which could lead to team members, who do not feel integrated into the discussions, which in turn leads to an inevitable upset in FMEA meetings.

The following points are crucial for a successful FMEA:

- Supervisors who support the FMEA actions definitely and visibly
- A moderator supplying good methodical and moderating knowledge
- A small, success-oriented team consisting out of involved members closely associated with the product

A further suggestion for the organization of an FMEA is shown in Figure 4.6.



D: Department
(initiator) head project leader

R: Person responsible for the FMEA project
(designer, planer, draftsman, business)

E: Experts
(designer, draftsman, testing engineer, planer, manufacturer, laboratory assistant, resource planner, testing planer, master craftsman, machine operator, further knowledge carriers

M: Method specialist in FMEA
(may also be the same as one of the experts or responsible persons)

Figure 4.6. The FMEA team according to VDA 4.2

4.2 FMEA according to VDA 86 (Form FMEA)

The original FMEA procedure was carried out with the help of a form sheet. The workflow is oriented on given columns, which are filled out successively from left to right. The FMEA procedure can be divided into Design FMEA and Process FMEA. The first columns of the form sheet are reserved for the description of the components and their function. The next section of the form sheet deals with the risk analysis, which requires the most work out of all the sections of the form sheet. This is followed by a risk assessment in order to rank the numerous failure causes. The last step is a concept optimization derived from the analysis of the risk assessment, see Figure 4.7.

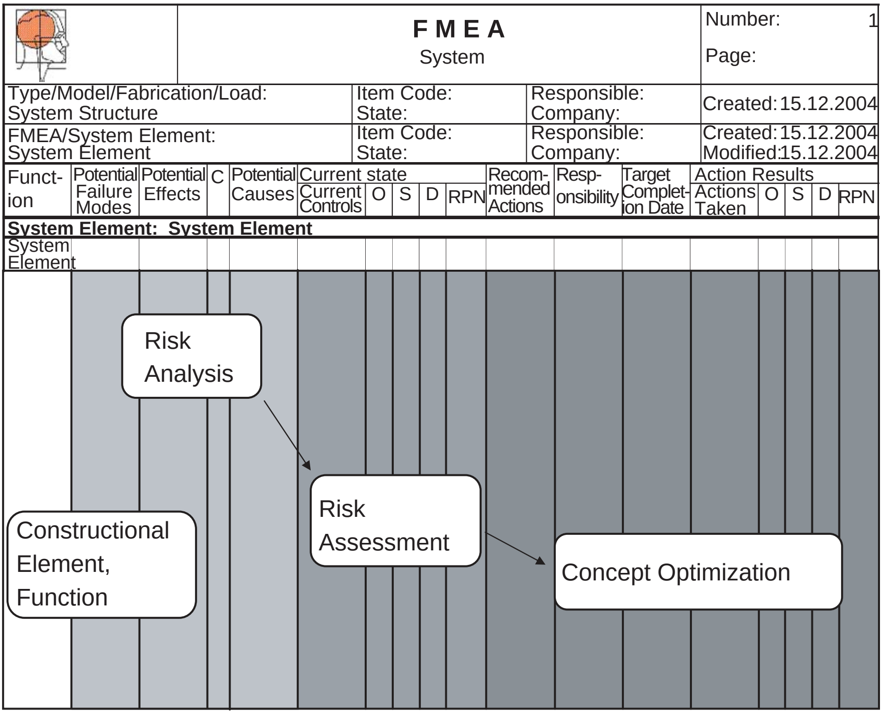


Figure 4.7. FMEA form sheet according to VDA 86

The progression of the individual sections is shown in Figure 4.8.

The fundamental step of an FMEA is the search for all conceivable failure modes (column 4). This step should be executed most carefully. Each failure mode not found can lead to dangerous failure effects and thus, later on, to drastic reliability problems.

Options available to discover failure modes are shown in Figure 4.9. An imperative principle is the observation of former arisen failures in similar cases. With the help of the experience of the FMEA participants all further failure modes can be derived. This takes place in team meetings, which are led by the FMEA moderator. Positive dynamic group effects should be taken into consideration. Very often, supplemental check lists are used in searching for failure modes. In particular dangerous cases it is helpful to use creative means to discover all failure modes. One very systematical approach is the examination of all functions along with their failure functions and failure trees.

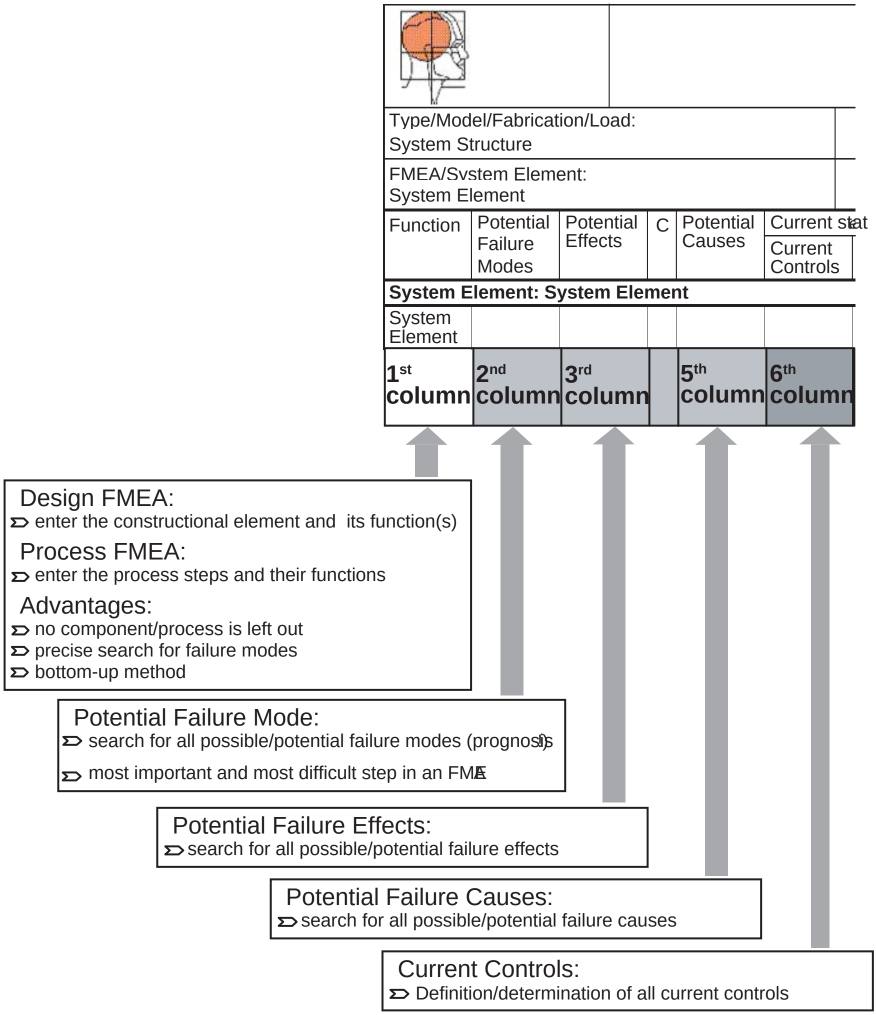


Figure 4.8. Procedure in an FMEA form sheet

- Damage statistics
- Experience of the FMEA participants
- Check lists (failure mode lists)
- Creativity methods (Brainstorming, 635, Delphi, …)
- Systematic analysis over functions or failure functions (failure trees)

Figure 4.9. Possibilities for the determination of failure modes

The completed form sheet represents a “tree structure”, see Figure 4.10. A certain component has one or more functions and normally several failure modes. Each failure mode has again various failure effects and different failure causes.

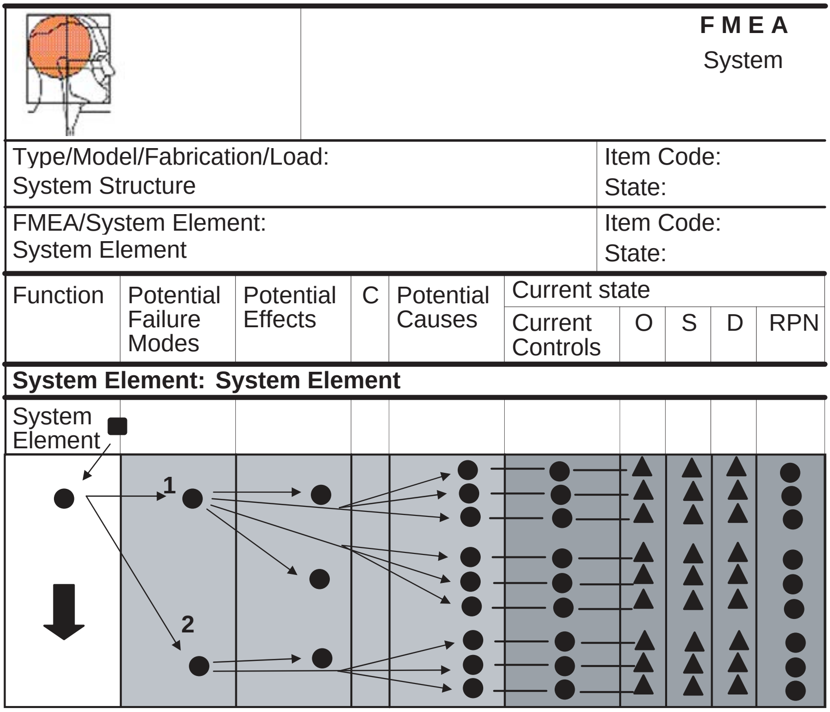


Figure 4.10. “Tree Structure” in an FMEA form sheet

A risk assessment follows the risk analysis, where out of the large amount of failure causes found, the crucial risks are determined by establishing a ranking order. The assessment is carried out under 3 criteria. With the assessment value O (= Occurrence) one can estimate how probable the occurrence of the failure cause is. This deals with the question of whether the failure is hypothetical or has already occurred often in the field. The assessment value S (= Severity) describes the severity of a failure effect. For example, if people are put into danger, the severity is evaluated higher, whereas a minimal limitation of comfort would receive a respectively lower value. With the assessment value D (=Detection) it is determined how successful the detection of the failure cause is before delivery to the customer. The ultimate measure here is the customer. However, the failure has already caused additional costs, but the customer does not receive an unreliable product. The three individual assessments are brought together in a total assessment: RPN (Risk Priority Number), which is equal to the product of O, S and D, see Figure 4.11. With the RPN a ranking of the identified failure causes and their failure connection to the failure effect can be done, i.e. a priorisation of the failure causes is enabled by the RPN.

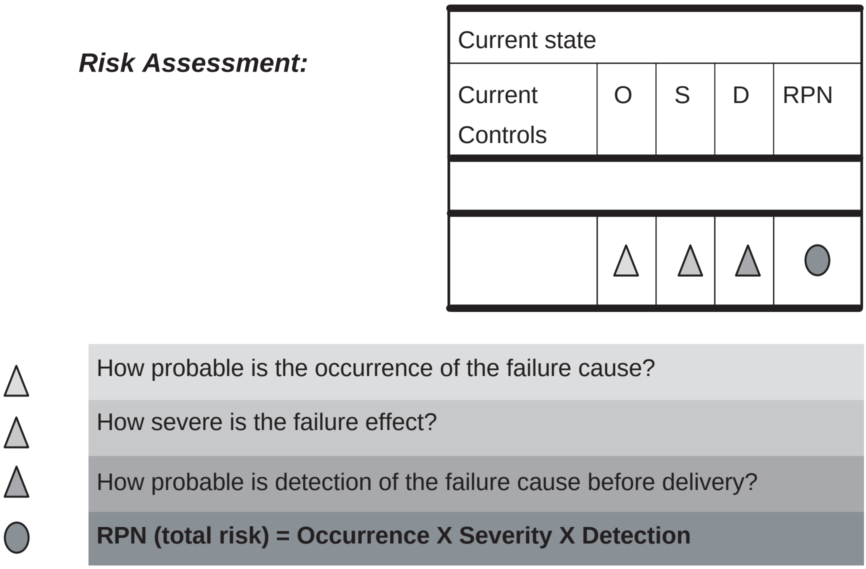


Figure 4.11. Risk assessment

The value scale for the assessment normally ranges from 1 to 10 in whole numbers. A value of 1 (very seldom occurrence, minimal severity, optimal detection) is assigned when the estimations are positive towards a reliable product. A value of 10 is assigned when the assessment tends to be extremely negative. Tables and charts are often used as helpful resources when assigning assessment values (see for example VDA tables in Section 4.4.4). The risk priority number can range from 1 (1*1*1) to 1,000 (10*10*10). The average RPN is normally 125 (5*5*5), see Figure 4.12.

Risk Assessment – Value Scale:

- Value scale from 1 to 10
  - positive, opportune = 1
  - negative, poor = 10
- Assessment values assigned with the help of tables (VDA etc.)
- Product of the individual assessment values = Risk Priority Number RPN:
  Risk of the ascertained potential failure cause

Figure 4.12. Value scale for risk assessment

The last phase of the FMEA is the optimization phase, which takes place after the risk assessment. First, the calculated Risk Priority Numbers are ordered according to their values. The optimization begins with the failure causes possessing the highest RPN value and ends, depending on the complexity of the analysis, either at a certain lower limit or according to the Pareto principle, after 20 – 30% of the RPN's have been optimized. The high individual assessment values must also be considered along with the RPN's. A value of O > 8 means that the failure occurs most often. Naturally, this must be optimized. A severity value of S > 8 points to grave functional damages as well as security risks. Such cases must also be looked at more closely. Failures can hardly be detected for values of D > 8. Thus, the danger arises that these cases are not dealt with before the product is delivered to the customers, see Figure 4.13.

- Ranking of failure causes according to their **RPN** value
- Concept optimization beginning with the failure causes with the highest **RPN**
  - ○ until a set limit RPN
    (e.g. RPN = 125) or
  - ○ until a certain amount of failure causes
    (common according to the Pareto principle: ca. 20 - 30 %)
- Failure causes with **O > 8**
  S > 8
  D > 8 observed separately
- **FMEA Result** observed separately

Figure 4.13. Procedure for concept optimization

The new optimization actions are entered on the right side of the form sheet for the optimized failure causes and the responsibility is recorded. An improved RPN is calculated for the improved state using the new assessment values assigned to D, O, and S, see Figure 4.14.

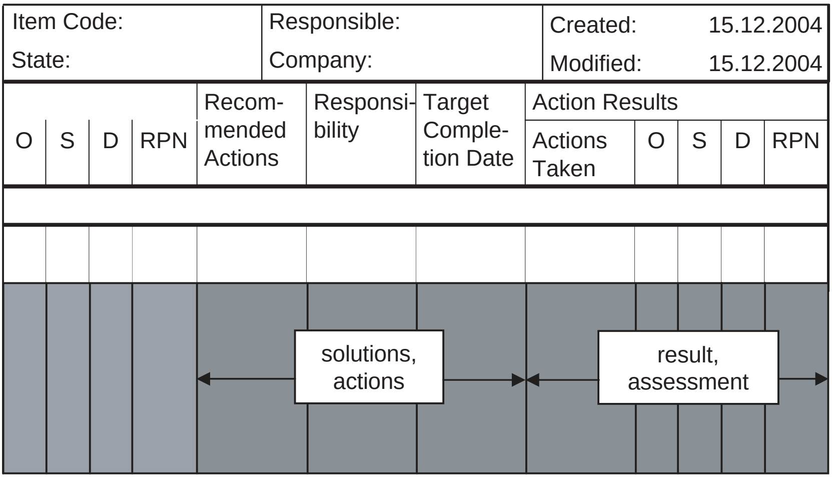


Figure 4.14. Concept optimization in a form sheet

4.3 Example of a Design FMEA according to VDA 86

With this example the procedure of a classical FMEA should be clarified. Actual occurred damage on an automatic transmission was chosen as an example for the analysis. Only this failure mode is considered and thus, the effectiveness of the FMEA is represented. A diagram of a five gear automatic transmission is shown in Figure 4.15.

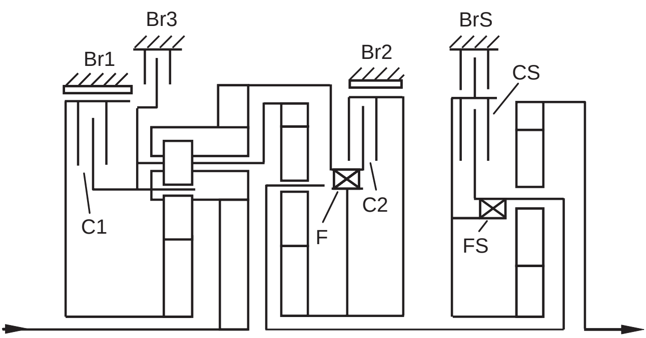


Figure 4.15. FMEA example – automatic transmission [4.4]

It is sufficient for the analysis of this particular damage case to consider a small section of the transmission: the front axial bearing, see Figure 4.16.

This bearing supports an outer rotating clutch plate carrier located across from a fixed stator shaft. The raceway of the axial bearing runs along this stator shaft. The other raceway is implemented with a running disc. The spacer also belongs to the axial bearing, in order to equalize any occurring axial play in the transmission.

The particular damage case, that is, the observed failure mode, is the interchange between the running disc and the spacer. The “interchange of components” is a standard failure mode included in every simple checklist. In the case of an automatic transmission the interchanging can lead to the destruction of the bearing and thus, to transmission breakdown. Further, a function test in the factory was completed without results reason to objection. The function test was carried out with a relatively small load, which the 0.1 mm wide and untempered spacer is able to endure. The spacer only undergoes a strong deformation under higher loads and longer running periods, thus blocking the axial bearing and the entire transmission. Thus, a relatively minimal cause can bring forth severe damage.

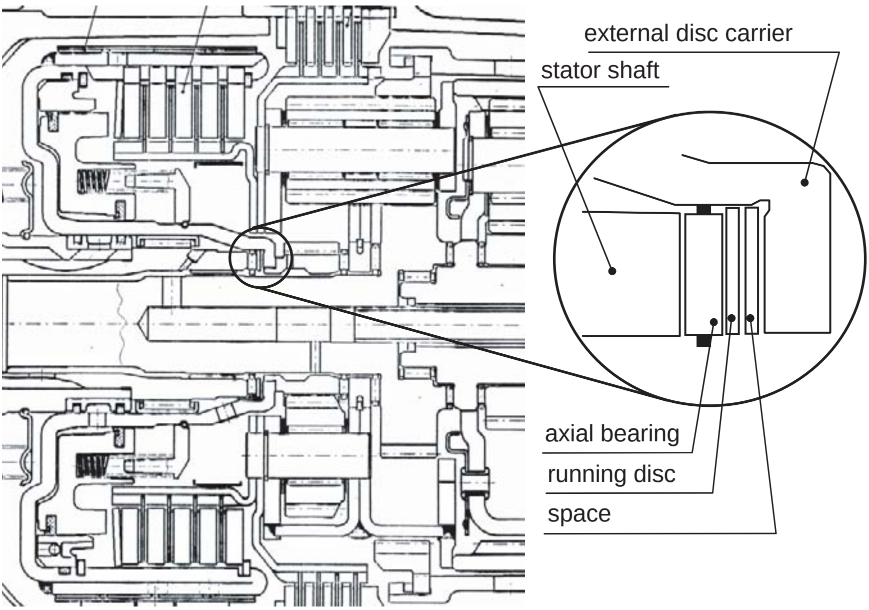


Figure 4.16. Detail of a 5-gear automatic transmission

With the FMEA the failure would be analyzed as follows. The occurrence probability of the failure would receive a 3 to a 6 (manual assembly, conceivable failure). The severity of the potential effects is rated as very critical with a value of 9 to 10, since it would mean that the vehicle comes to a standstill. The detection of the potential cause is very improbable and thus is assigned the value of 10. Out of the product of the three individual values a risk priority number is produced between 300 and 600. These values require optimization see Figure 4.17.

FMEA System			Number: 1 Page:
Type/Model/Fabrication/Load: System Structure	Item Code: State:	Responsible: Company:	Created: 15.12.2004
FMEA/System Element: System Element	Item Code: State:	Responsible: Company:	Created: 15.12.2004 Modified: 17.12.2004

Function	Potential Failure Modes	Potential Effects	Potential Causes	Current state					Recom-menden Actions	Responsi-bility	Target Comple-tion Date	Action Results				
				Current Controls	O	S	D	RPN				Actions Taken	O	S	D	RPN
System Element: spacer																
adjust axial play	spacer swapped with run-ning disk	[transmis-sion] dest-ruction of bearing -> blockes bearing -> blocked tansmission	[produc-tion] false manual as-sembly	P: visual inspection D: function test	6	10	10	600	P:	Smith	01.02.2005		3	10	5	(150)

Figure 4.17. FMEA form sheet for an automatic transmission

4.4 FMEA according to VDA 4.2

In the following section the procedure for an FMEA will be dealt with according to the VDA Guidelines 4.2 [4.7].

The previously existing FMEA has been considerably enhanced. The reason for this was the increased application of the FMEA and the awareness of a few deficiencies in the existing procedure. A new, superior term is defined as:

System FMEA.

Considerable influential factors for the increased application of FMEA are:

- increasing quality demands from customers,
- cost optimization for products and
- compulsory liability required of the producer

The aims pursued by the system FMEA are:

- increase in the function security and reliability of products,
- reduction in guarantee and warranty costs,
- shorter development processes,
- new production start ups with fewer disturbances,
- improved fulfilment of deadlines,
- economical manufacturing,
- improved services and
- improved internal communication.

Because the System FMEA is a preventive reliability method, the decision to implement the method should be made as early as possible in the product design cycle. If the application of the FMEA methodology can not be applied in the technical specification phase, then it should be executed at the latest in the development of the first design; or afterwards, a System FMEA should be executed. The execution of an FMEA accompanies the design cycle, which means that it must be continually conformed to the design process and may not be treated as a static document.

The following reasons led to the further enhancement of FMEA.

- In the Design FMEA a failure analysis is mainly carried out on the component level, which means that functional interactions between the observed components are not included.
- In the existing Process FMEA the failure analysis is carried out for individual process steps. The entire production process is not thoroughly analyzed, for example the layout of necessary tools and machines is not considered.
- The Design and Process FMEA involve the creation of the FMEA with the help of a form sheet, which means that no structured description is made of the function relationships as well as possible failure function relationships within the systems.

The new approach involves using the structure of the system to be analyzed as a starting point for a System FMEA. This led to the development of a System FMEA Product and a System FMEA Process. The old form sheet of the VDA 86 was improved and a new form sheet VDA 4.2 in 1996 was introduced, see Figure 4.18.

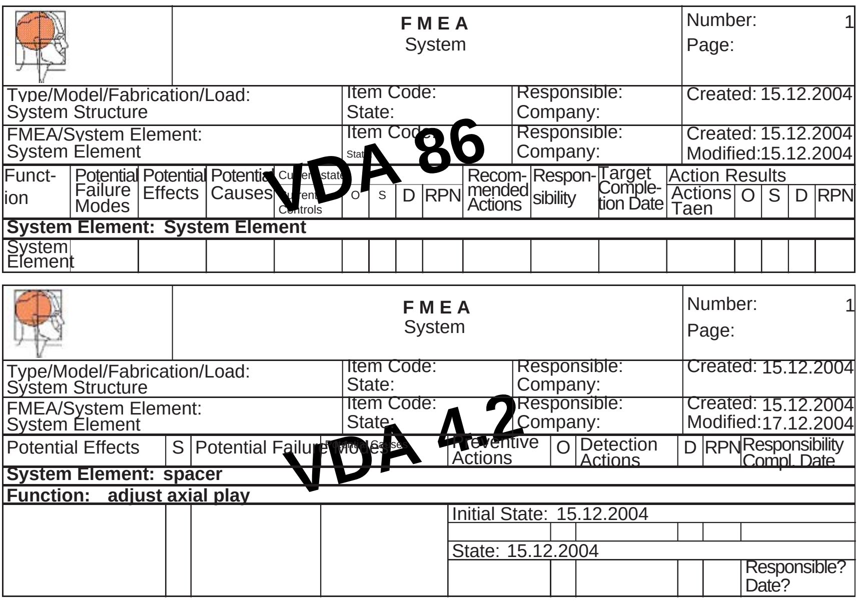


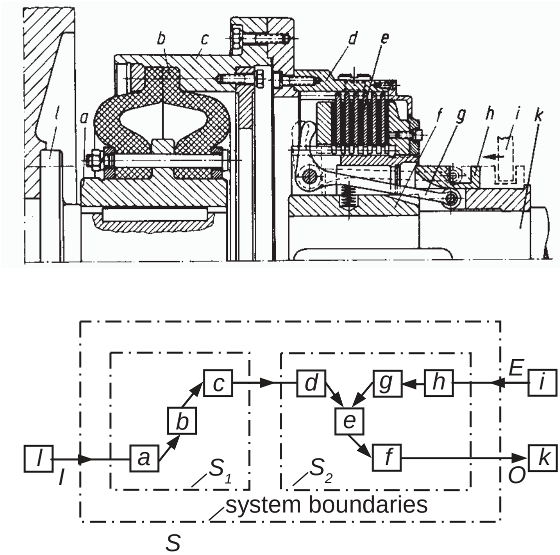
Figure 4.18. Comparison of the FMEA form sheets VDA 86 and VDA 4.2

Additional system and function observations are necessary for the new procedure. This means in detail:

- to structure the product to be analyzed as a system with system elements and to identify the functional relationships between these elements,
- to derive the conceivable failure functions (possible failures) of a system element and its described functions and
- to derive the logical connections between the failure functions of different system elements belonging together, in order to be able to describe the potential effects, failure modes and causes in the System FMEA, which could possibly be analyzed.

It would now be helpful to take a closer look at the definition of the term *"system"*: Each technical entity (equipment, machine, device, assembly, etc.) can be described as a system. A system

- excludes itself from its surroundings; thus, it possesses a system boundary; the interfaces with the system boundaries are input and output variables;
- can be divided into either partial systems or system elements;
- can be broken down into various hierarchical levels;
- can be divided into different types of systems depending on the purpose of the analysis (e.g. in assembly, in function groups, etc.),
- is an abstract product description.



System “clutch” *a...h* System elements (examples); *i...l* connection elements; *S* entire system; S_1 subsystem “elastic connection”; S_2 subsystem “separating clutch”; *I* Inputs; *O* Outputs.

Figure 4.19. “Clutch” system according to [4.6]

A clarification of the term *“system”* is shown in Figure 4.19. Here, a sectional view is transferred into a system view and thus, into another abstract level, which is profitable for the FMEA methodology.

The second important term related to the System FMEA is *“function”*. A function describes the general and specific connection between input and output variables for technical entities, systems, etc. The image of a “black box” allows for the task description of abstract and neutral solution levels, see Figure 4.20.

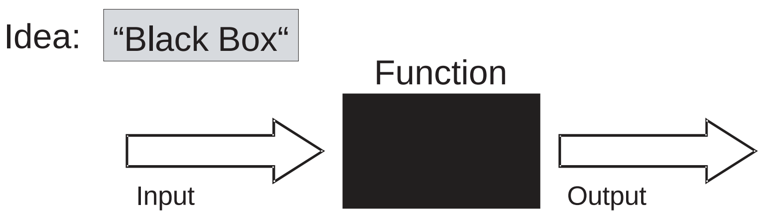


Figure 4.20. Definition of the term *“function”*

Examples of functions in technical systems:

- transmission → to convert torque / speed;
- electric engine → to convert electrical energy into mechanical energy;
- pressure relief valve → to limit pressure;
- RAM (Read Access Memory) → to save signals;

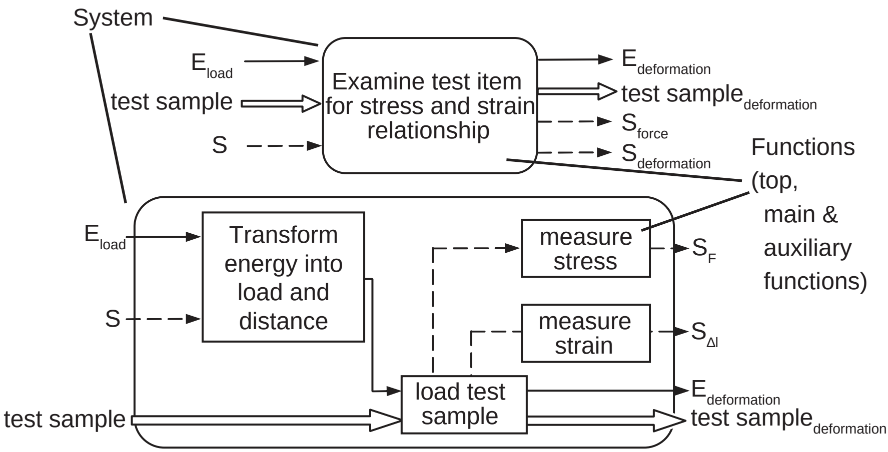


Figure 4.21. Main function of a testing machine, approximate structure [4.6]

In Figure 4.21 the procedure is elaborated by examining a testing machine. The complete system is divided step-by-step. In the first step the complete function is divided into main and auxiliary functions.

In the next step a detailed structure with further main and auxiliary functions is created, see Figure 4.22.

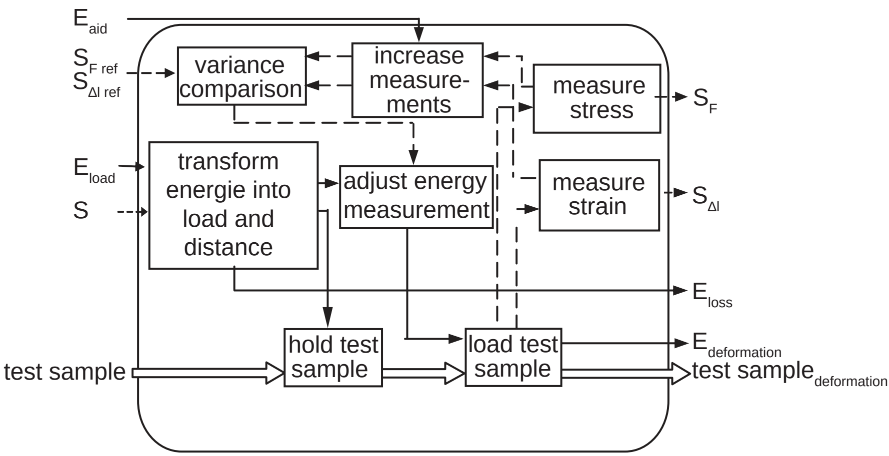


Figure 4.22. Main function of a testing machine, fine structure [4.6]

System FMEA Product (overview)

Failure functions as well as failure modes for products (machines, devices, apparatus, etc.) are analyzed in the System FMEA Product. The analysis is carried out in various hierarchical system levels all the way to the failure on the component level.

> The failure functions of a component are defined as physical failure modes such as fracture, wearout, jams, clamps, etc.

The term *"failure function"* in general stands for a failure mode, failure type or a failure. The contents of the FMEA according to VDA Guidelines 86 can be fully integrated in the new FMEA form for VDA 4.2, see Figure 4.23. Figure 4.24 shows an example of the layout of a system structure for a System FMEA Product.

FMEA System			Number: 1 Page:
Type/Model/Fabrication/Load: **transmission**	Item Code: State:	Responsible: Company:	Created: 15.12.2004
FMEA/System Element: **input shaft** system element (SE)	Item Code: State:	Responsible: Company:	Created: 15.12.2004 Modified: 15.12.2004

Function	Potential Failure Modes F(M)	Potential Effects FE	C	Potential Causes FC	Current state: Current Controls	O	S	D	RPN	Recommended Actions	Responsibility	Target Completion Date	Action Results: Actions Taken	O	S	D	RPN
System Element: System Element																	
	bearing input shaft is worn	transmission function is hindred, does not function		hardness of bearing seat is to low													
	failure function in constructional element (physica failure mode)	failure function in transmission		failure function design (e.g. dimensioning, surface, hardness, material													

Figure 4.23. Design FMEA for a transmission according to the previous procedure (VDA 86)

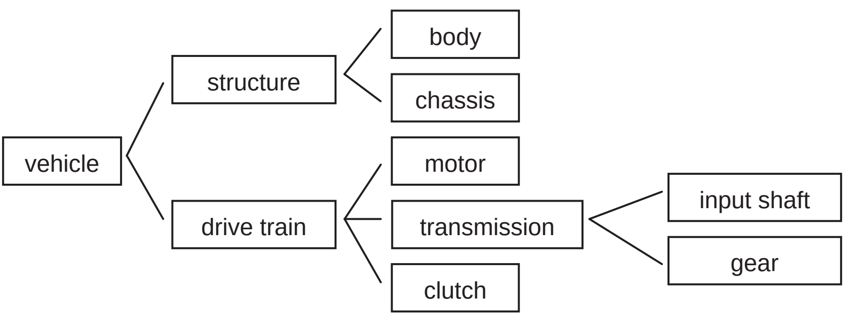


Figure 4.24. System structure of a “Complete Vehicle System” [4.7]

System FMEA Process (overview)

With the System FMEA Process all possible failure functions of a production process (manufacturing, assembly, logic, transportation, etc.) are observed. The process is structured according to a system description, where the last structure level is composed of the “4M’s” (man, machine, material, method) and “environment”, see Figure 4.25.

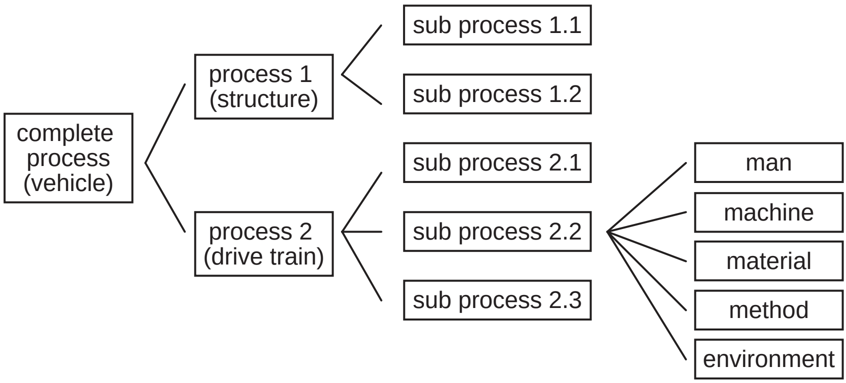


Figure 4.25. Example of a system structure for a complete process [4.7]

The procedure for the creation of a System FMEA according to VDA 4.2 is made up of 5 main steps, see Figure 4.26. These 5 steps will be dealt with in detail in the following sections.

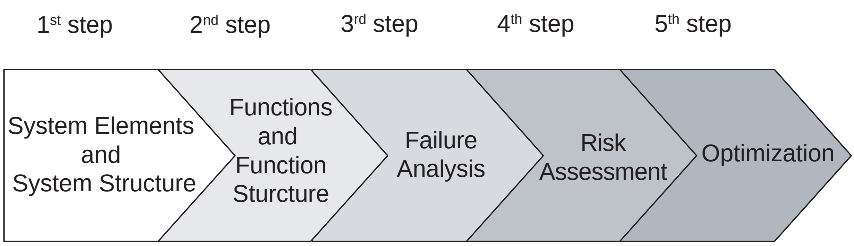


Figure 4.26. The 5 steps of the System FMEA

4.4.1 Step 1: System Elements and System Structure

The first step of an FMEA is divided into the following partial stages:

1. Definition of the system to be analyzed, see Figure 4.27. First, it must be established, how complex the system is, which should be analyzed with the help of an FMEA. This includes:
   - the definition of interfaces to the design (for System FMEA Product) or
   - the definition of the process interfaces (for System FMEA Process).

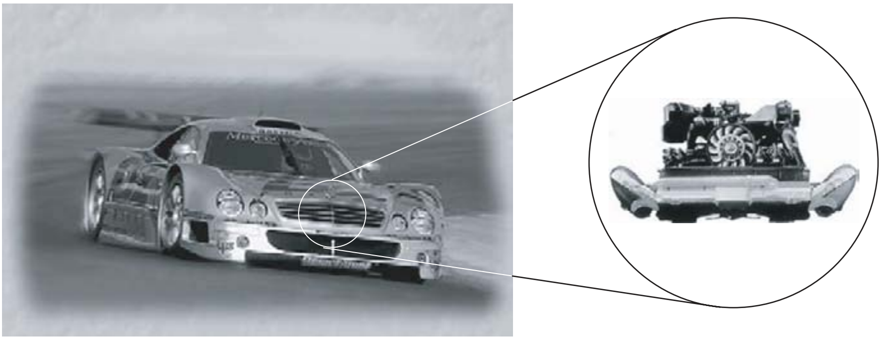


Figure 4.27. Limiting the system to be observed

2. Dividing the system into its individual system elements (SE); this partition can be carried out in:
   - assembly (subsystems)
   - function groups (subsystems)
   - components
3. Ordering the system elements hierarchically in a system element structure (structure tree), see Figure 4.28.

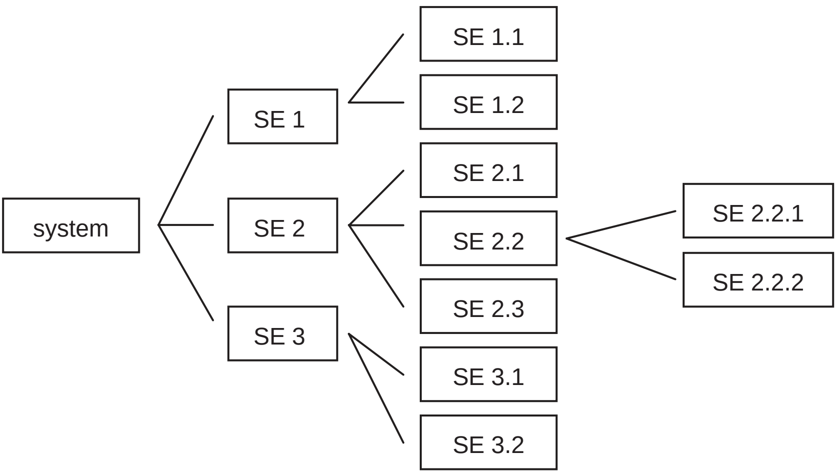


Figure 4.28. System and System Structure [4.7]

The system structure arranges the individual system elements into various hierarchical levels, beginning with the top element. Further subsystems can be arranged after each system element with a varying amount of levels. In principle, the way the system structure is set up is arbitrary. For the

FMEA product an arrangement according to assembly is common, which can be seen for example in Figure 4.29.

When creating a system structure, the following points should be taken into consideration:

- the number of hierarchical levels is arbitrary,
- each system element may appear only once (uniqueness),
- for a better overview individual system elements may be used simply for the matter of structuring (so-called "dummy system elements"); these elements are not significant for the analysis later.

A few helpful resource tools for creating the system structure are shown in Figure 4.30. An example can be found in Section 4.5.1.

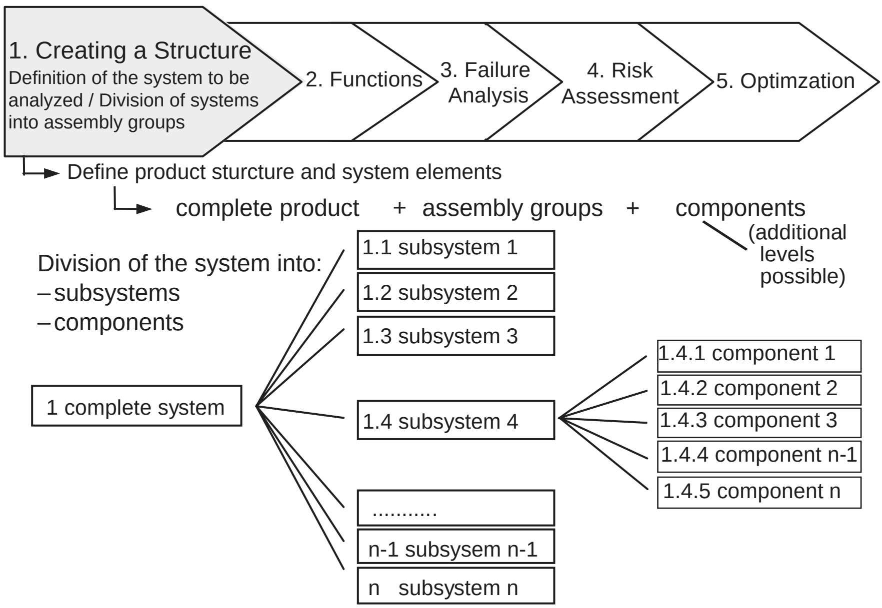


Figure 4.29. Step 1 – Creation of the Structure

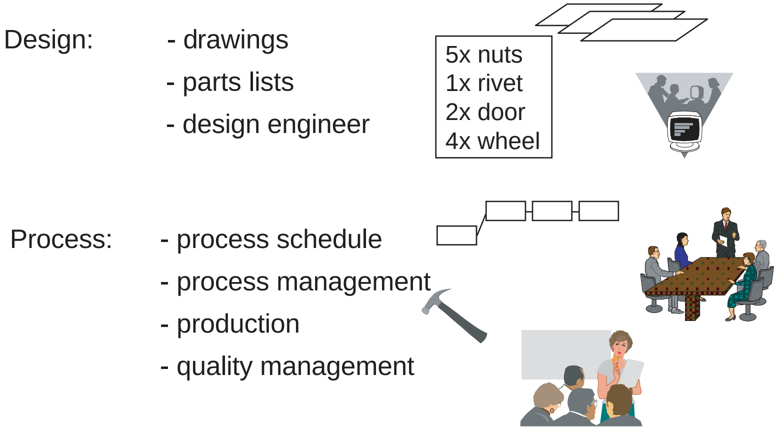


Figure 4.30. Helpful resources for creating a system structure

4.4.2 Step 2: Functions and Function Structure

The arrangement of the system elements (SE) and the set up of the system structure (structure tree) are the basis for determining the particular functions and failure functions.

The following possibilities can be used for determining the functions:

1. Creation of the functions in "top down" form, that is, the functions (functions contributing to the subordinated system elements) are created by beginning with the top function of the system, see Figure 4.31.
2. Creation of the individual functions for each system element. Here, a good know-how is required for the application conditions, e.g. with specification information such as load, hotness, coldness, dust, spray water, salt, ice, vibration, electrical malfunction, etc.

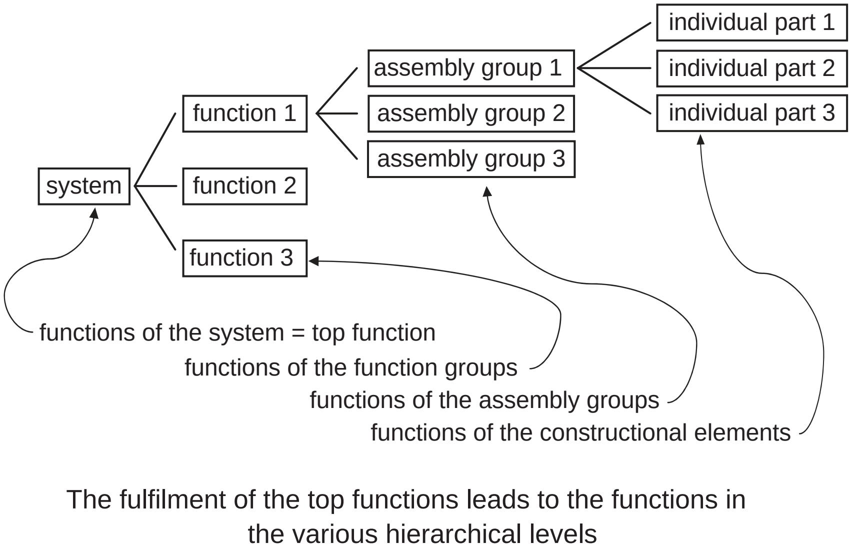
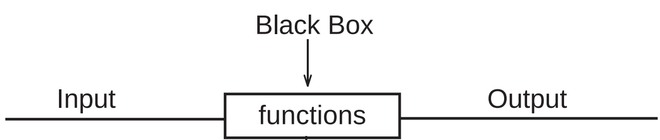


Figure 4.31. Function analysis in FMEA

Suitable helpful resource tools for both cases are:

- the "black box" observation, see Figure 4.32,
- the general "guidelines" from the design methodology, see Figure 4.33. A guideline is a search or suggestion list with higher ranking terms. It makes sure that nothing important is forgotten. Thus, it is ensured that the derived functions are complete.



Input	functions	Output	
speed hydraulics pressure signals	- transform torque and speed - change gears - choose gear	transformed speed transformed torque another gear	transmission
signal electricity	- link signals - produce light	light	display
specification of the top function	special functions	fulfils the top function	general

Figure 4.32. Black Box as a helpful resource for the determination of functions

main categories	
Geometry	dimensions, height, width, length, diameter, required space, quantity, alignment, connection, extensions and expansion
Kinematics	extension and expansion
Forces	movement type, movement direction, speed, acceleration, size of force, direction of force, frequency of force, weight, load, strain, stiffness, spring spring characteristics, stability, resonances
Energy	power, degree of efficiency, loss, friction, ventilation, state variables e.g. pressure, temperature, humidity, heating, cooling, connection energy, storage, work intake, transformation of energy
Material	physical and chemical characteristics of the input and output product, auxiliary materials, required materials (law of nourishment), material flow and transportation
Signal	input and output signals, display mode, operation and monitoring equipment, type of signal
Safety	direct safety technology, protective systems operation, work and environment safety
Ergonomics	Man-Machine relationship, operation, type of operation, lucidity, lighting, design
Manufacturing	confinement through production plants, largest producible dimensions, preferred production process, workshop facilities, possible quality and tolerances
Control	measuring and control options, specific regulations (TÜV, ASME, DIN, ISO)
Assembly	specific assembly regulations, assembly, installation, construction site assembly, foundation
Transportation	limitation through lifting gear, path profile, route of transport according to size and weight, type of dispatch
Usage	low noise level, wear rate, application / distribution area, place of installation (tropics, ..)
Maintenance	maintenance-free and/or amount and time required for maintenance, inspection, replacement and repair, painting, cleaning
Recycling	reuse, recycle, waste management, waste disposal, disposal
Costs	max. allowable production costs, tool costs, investment and amortization
Schedule	end of development, network plan for intermediate steps, time of delivery

according to Pahl/Beitz

Figure 4.33. Guidelines for specification lists according to [4.6]

The function structure refers to the cooperation of functions from several system elements for one individual, waning function. The combination of the functions to a function network or function structure is possible. Top functions are determined for the complete system function, which are essential for the fulfilment of the product goals, such as quality characteristics, design specifications or information from previous FMEA's. The top function is divided into partial system functions and subsystem functions all the way to the component functions, see Figure 4.34. An example is given in Section 4.5.2

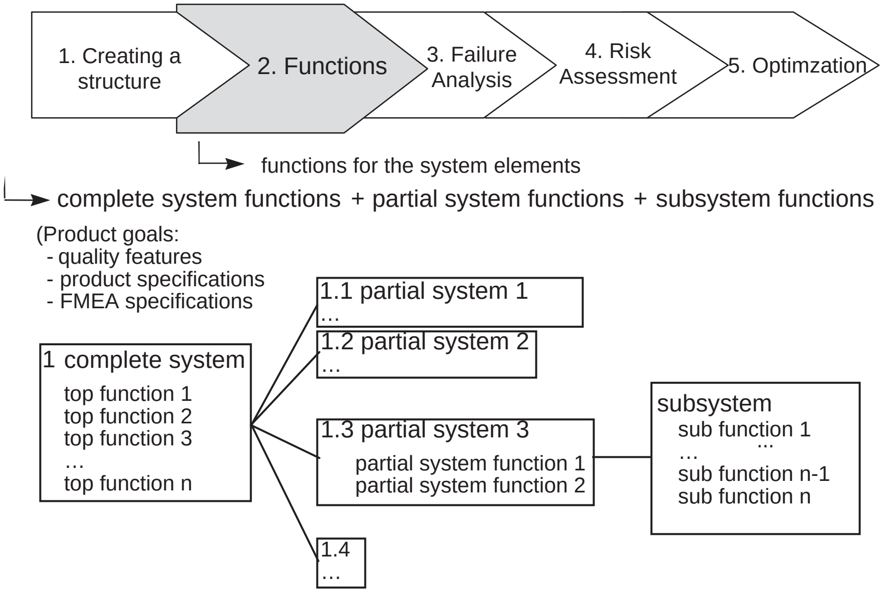


Figure 4.34. Functions of the system elements

4.4.3 Step 3: Failure Analysis

A failure analysis is carried out for each system element. However, for each individual case it must be decided for which system elements it is reasonable to carry out a failure analysis. Failure analysis means the determination of all potential failure functions. This means that the failure that leads to an unfulfilment or a limitation of a function is considered.

For abstract functions a failure function list can be created on behalf of the possibilities shown in Figure 4.35.

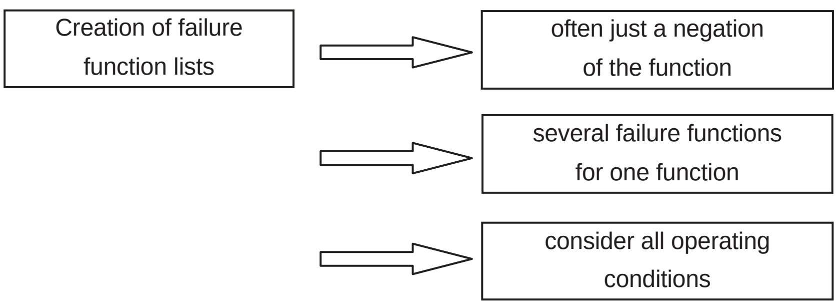


Figure 4.35. Determination of failure functions

The failure functions on the component level are physical failure modes.

Table 4.1. Typical potential failure modes

• fracture	• blocked
• crack	• overstretched
• abrasion	• bent, sagging
• rejected	• distorted, deformed, dented
• chips away	• relaxed, loose, wobbles
• wear (also bedding-in, pittings,...)	• clamps, sluggish
• insufficient time characteristics	• friction is too high of too low
• rotted, decomposed (prematurely)	• too much expanded
• damaged, prematurely worn out	• part is missing
• vibrates	• wrong part (not a safely usable constr.)
• swings	• wrong position (no constr. measurement)
• resonances	• constr. inverted assembly possible
• unpleasant sound	• interchanged (no constr. measurement)
• too loud	• location to reverse side is false
• congested	• false configuration
• contaminated	• entry of dirt and water
• leaky	• false speed
• busted	• false acceleration
• depressurized	• false spring characteristics
• false pressure	• false weight
• corroded	• poor degree of efficiency
• overheated	• too maintenance intensive
• burnt	• poorly replaceable
• charred	• not further useable

Table 4.1 is a list of typical failure modes which can be used to ensure the integrity of the fault analysis. These failure functions are classic failure modes for FMEA on the component level.

Failure functions in a structure tree are shown in Figure 4.36.

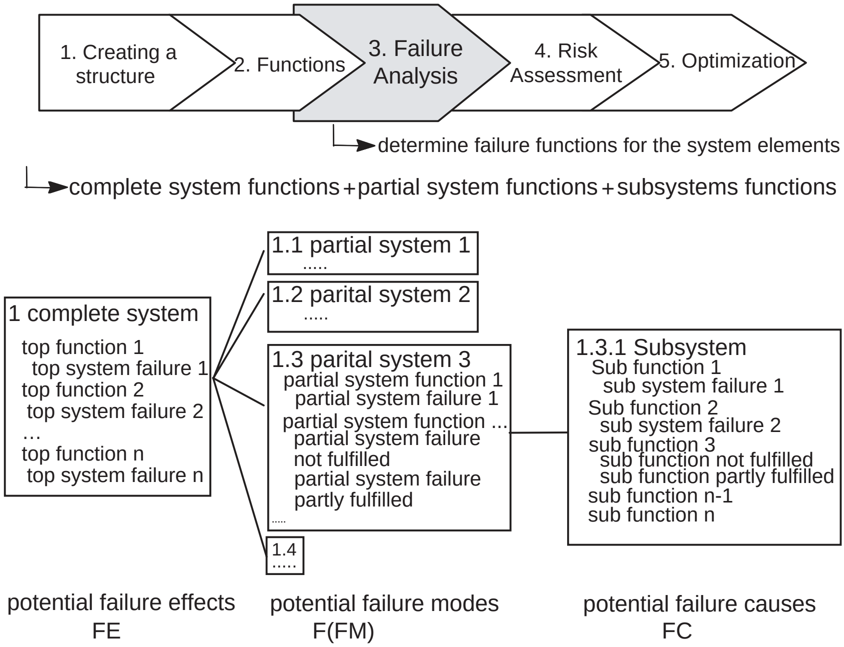


Figure 4.36. Failure functions (FE, FM, FC)

The top system failures or the top failure functions are derived from the top functions. The depth of the failure analysis is limited by the depth of the structuring levels of the system structure. If necessary, the system structure can be expanded for the determination of potential failure causes. The determination of the potential failure modes (FM) can be supported by the following methods:

- damage statistics,
- experience of the FMEA team members,
- check lists (e.g. failure modes from Table 4.1)
- creativity procedures (Brainstorming, 635, Delphi, etc.)
- systematically with the functions or failures functions / fault trees.

Check lists have proven to be very helpful in searching for failures.

The following relationships arise from the failure analysis in Step 3:

- Potential failure modes (FM) for the observed SE are failure functions derived from and described by the determined functions, e.g. failure fails to fulfil the function or limited function.

- The potential failure causes (FC) are the conceivable failure functions of the subordinated SE's in the system structure and the SE's assigned by the interfaces.
- The potential failure effects (FE) are the failure functions for the higher ranking SE's in the system structure and the SE's assigned by the interfaces.

The relationship between the different failures should be more closely observed on behalf of the following example.

- potential failure mode: sudden pressure loss in a car tire
- potential failure cause: sharp object (e.g. a nail) on the street
- potential failure effect: vehicle loses control - accident, vehicle is incapable of driving

Table 4.2 shows some typical potential failure causes (FC) on the component level. Often, it is reasonable to create company specific lists to be used again in future FMEA's.

Table 4.2. Typical potential failure causes on the component level

• dimensioning failure (geometry, stability, stiffness, ...)	• false tolerance choice (tolerance field, form & position tolerance)
• false material (material characteristics: magnetic, inhomogeneous,…)	• tolerance chains were not considered
• surface is falsely defined (hardness, form, waviness, true-running, surface roughness,…)	• confusingly constructed
• false machining process defined	• false heat treatment defined

The execution of the failure analysis can be carried out in various manners:

1. Definition of the function all the way to the component level, *from the component functions:* → component failure functions = failure modes; Question: "Which failure modes are conceivable for the observed component function?" (sleeve example, Figure 4.55).
2. Definition of the functions all the way to the assembly or the function group levels (component function = "dummy" function); *from the failure functions of the assembly and function groups:* → component

failure function = physical failure modes; Question: “Which failure modes are necessary to produces the observed assembly or function group failure functions?” (seal example, Figure 4.55).

The determined failure functions are joined together into failure trees / failure function trees or failure networks, see Figure 4.37.

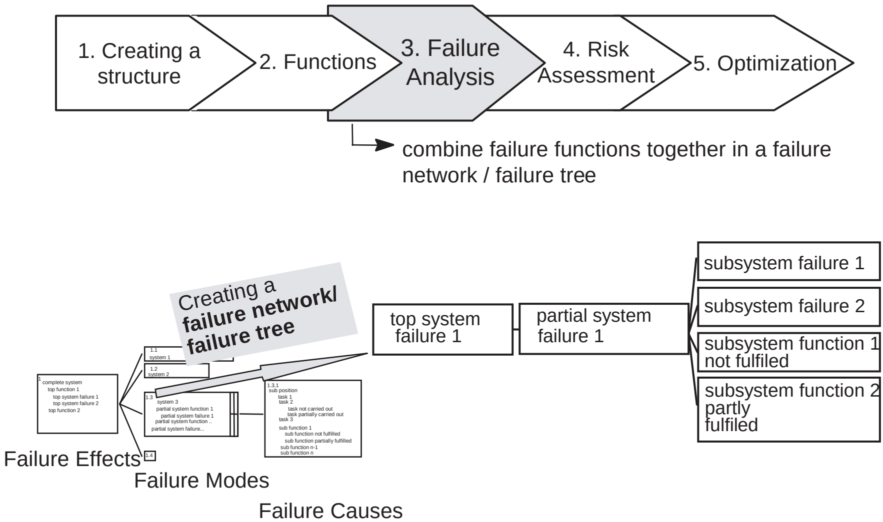


Figure 4.37. Failure network

Figure 4.38 shows another example of a failures networks for the fracture of a sleeve. Here, the relationship between potential failure causes (FC), potential failure modes (FM or F) and potential failure effects (FE) is clarified.

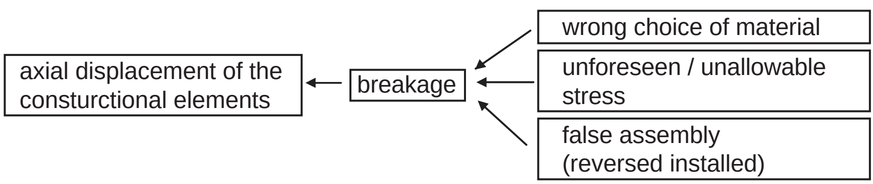


Figure 4.38. Failure network for the fracture of a sleeve

The failure function structure for a transmission is shown in Figure 4.39. In this example the System FMEA Product’s of the purchased items of supplies is also included.

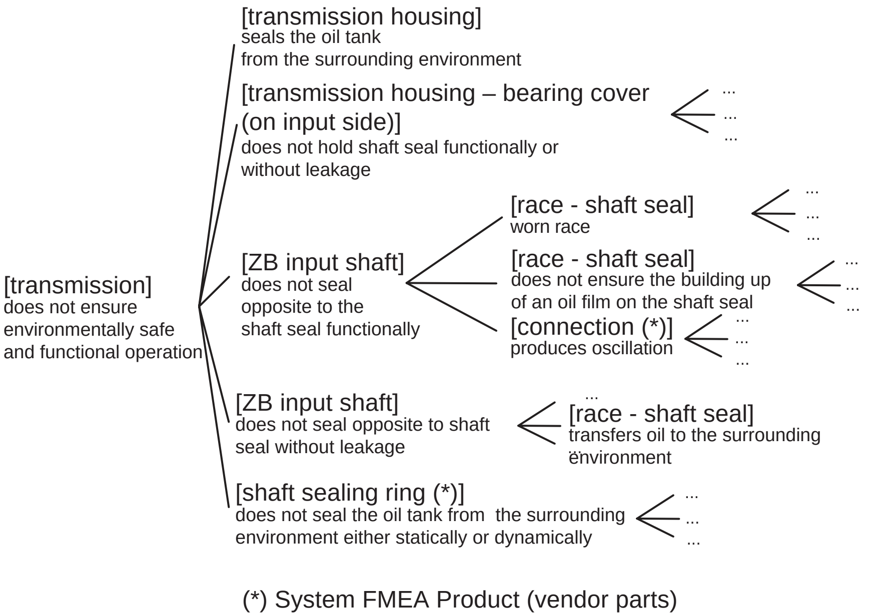


Figure 4.39. "Transmission" failure function structure [4.7]

Depending on the level chosen, the contents of the failure function structures are carried over into the FMEA form sheet according to VDA 4.2 as

- "potential failure effect" FE,
- "potential failure mode" FM and
- "potential failure cause" FC.

The FMEA's carried out on different levels overlap each other. The potential failure mode of the upper level is carried over as the potential failure effect for the FMEA of the next lower level. A potential failure cause in an upper level can be carried over as a potential failure mode in the next lower level. The overlappings are shown in Figure 4.40 and Figure 4.41.

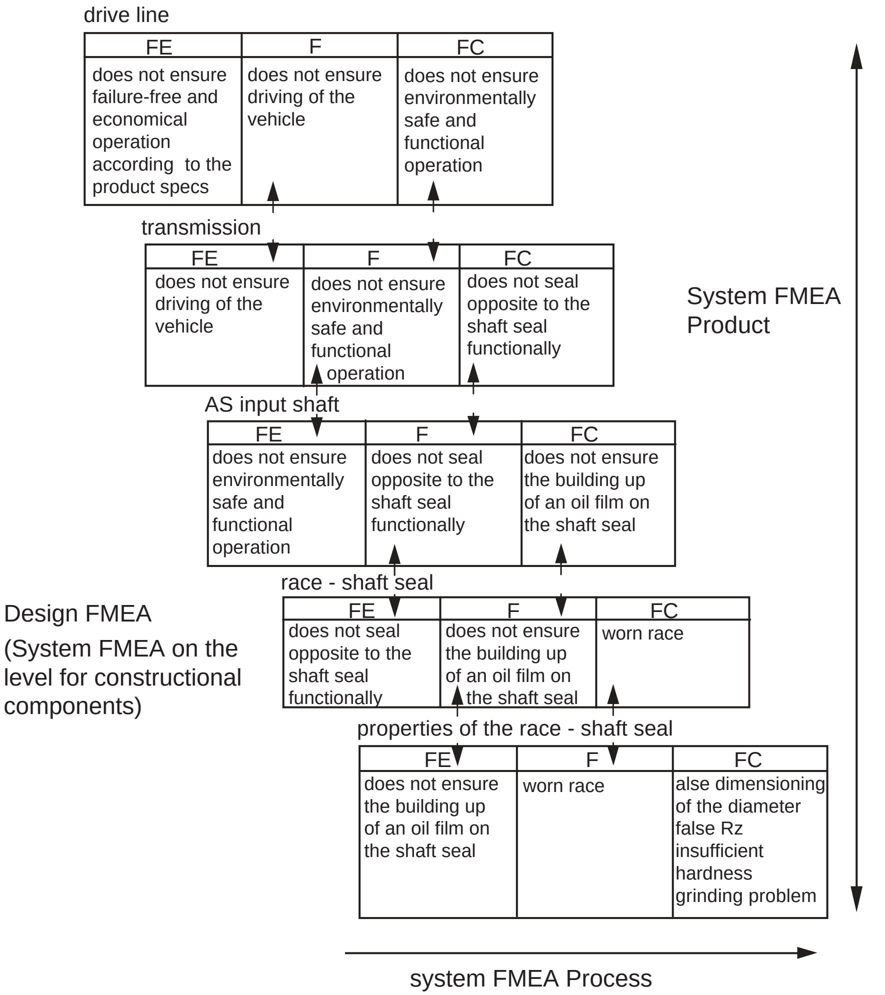
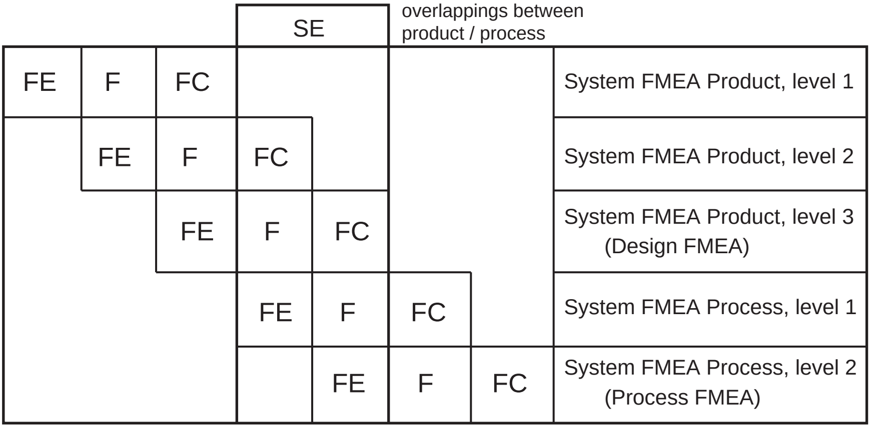


Figure 4.40. Overlapping according to [4.7]



FE: failure effect F: failures FC: failure cause

Figure 4.41. Overlapping in System FMEA Product and Process according to [4.7]

4.4.4 Step 4: Risk Assessment

The risk assessment is carried out under three evaluation criteria. These are:

- S: Severity of the potential failure effect
- O: probability for the occurrence of the failure cause and
- D: probability for the detection of the occurred failure cause.

A representation of the risk analysis in a form sheet is shown in Figure 4.42.

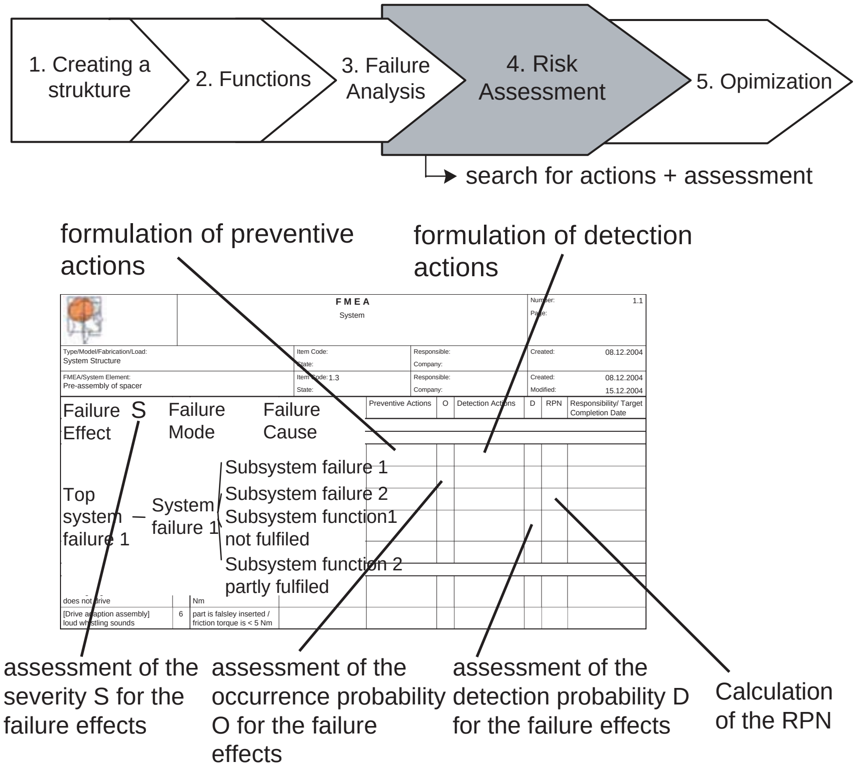


Figure 4.42. Risk assessment in a form sheet

The evaluation values range from 1 to 10 in whole numbers. The tables from VDA, see Figure 4.43 or company specific tables can be used as a guideline for the evaluation. Company specific tables can be created with the help of previous FMEA's.

Severity S

The evaluation value S evaluates the severity of the failure effect for the complete system. The assessment is always carried out from the standpoint of the final user of the product (external customer). A value of 1 stands for an extremely low severity; while a value of 10 stands for an extremely high severity (e.g. people are put into danger). Generally, similar potential failure effects should be assigned by similar values, see Figure 4.43.

assessment value for the severity S		assessment value for the occurrence O		dedicated fraction defective in ppm	assessment value for the detection D		reliability of testing procedure
very high:		very high:			improbable / very low		
10 9	Significant failure that causes total inoperability of the system and possible malfunc-tion of the safety mechanisms or non-fulfilment of legal requirements	10 9	failure cause will occur very often, inoperative, unsuitable design principle	500,000 100,000	10 9	Detection of the occurred failure cause is improbable; reliability of the constructive dimensioning was not or can not be proven. Proof procedure is uncertain, no tests.	90%
high:		high:			low:		
8 7	Operability of the system is strongly limited, immediate maintenance is obligatory, limitation of functionality of important sub systems. Safety mechanisms of the system are not impaired.	8 7	This design is only slightly different from earlier proven designs which caused prob-lems. Failure cause occurs repeatedly.	50,000 10,000	8 7	Detection of the occurred failure cause is not likely, there are probably failure causes that are not detected, uncertain testing.	98%
moderate:		moderate:			moderate:		
6 5 4	Functionality of the system is limited is limited, immediate maintenance not necessarily required, limitation of functionality of important opera-tion and comfort systems. Cus-tomer notices the malfunctioning of the system.	6 5 4	Failure cause occurs from time to time, design less accurate.	5,000 1,000 500	6 5 4	Detection of the occurred failure cause is probable, testing is relatively certain	99,7%
low:		low:			probable:		
3 2	Low limitation of system functional-ity, removal at the next regular maintenance, limitation of functionality of important opera-tion and comfort systems	3 2	Occurrence of the failure cause is low, design is correct	100 50	3 2	Detection of the occurred failure cause is very probable, testing is certain, e.g. several tests independent of one another.	99,9%
very low:		very low:			very probable:		
1	Very low limitation of functionality, can only be discovered by expert staff. Customer is not likely to notice failure.	1	It is improbable for this failure to occur.	1	1	Occurred failure cause will assuredly be detected.	99,99%

Figure 4.43. Criteria for the assessment values for the System FMEA Product according to [4.7]

Preventive Actions and Occurrence O

The assessment of the occurrence O is carried out according to the efficiency of the preventive actions taken for the respective potential failure causes. The more detailed the failure analysis is carried out in the System FMEA for the failure causes, the more differentiated the O assessment can be conducted. In a System FMEA for higher ranking systems values based on previous experience can be helpful for the O assessment of failure causes (e.g. reliability rates).

If familiar subsystems are integrated into another system, the assessment values should be re-evaluated due to the altered application conditions.

Preventive actions are all actions (most of the time preventive), which confine or avoid the occurrence of a potential failure cause. Such an action could be for example, calculations during the development phase, see Figure 4.44.

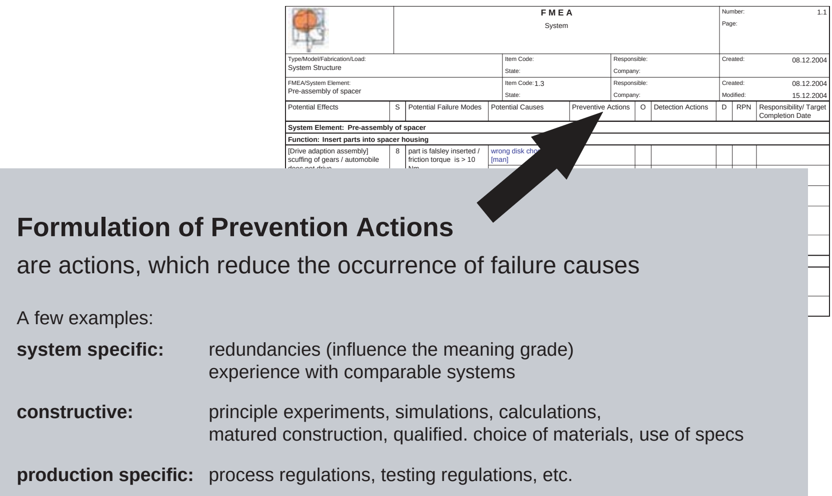


Figure 4.44. Preventive actions

The assessment of the probability for the occurrence of a potential failure cause is carried out under consideration of all listed preventive actions, see Figure 4.44. A value of 10 is assigned, if it is likely that the potential failure cause will occur. A value of 1 is assigned for a very improbable potential failure cause. Thus, the O assessment makes a statement concerning the quantity of defective components remaining in an entire batch of a certain product.

Detection Actions and Detection D

The assessment of the detection D is carried out according to the efficiency of the detection actions taken for the respective potential failure causes. The more detailed the failure analysis is carried out in the System FMEA for the failure causes, the more differentiated the D assessment can be conducted. In a System FMEA for higher ranking systems values based on previous experience can be helpful for the D assessment of failure causes.

If familiar subsystems are integrated into another system, the assessment values should be re-evaluated due to the altered application conditions.

For the detection actions one differentiates between two separate cases:

1. Detection actions in development and production:
   detection actions that are carried out during the development and production phases and allow for the visualization of possible potential failure causes in a concept or product already during development and production.

2. Detection action in operation / in the field:
   detection possibilities that the product (system) shows during its operation or that are recognized by the operator (customer). These detection actions indicate potential failures or potential failure causes that have occurred during operation and should prevent any further potential failure effects.

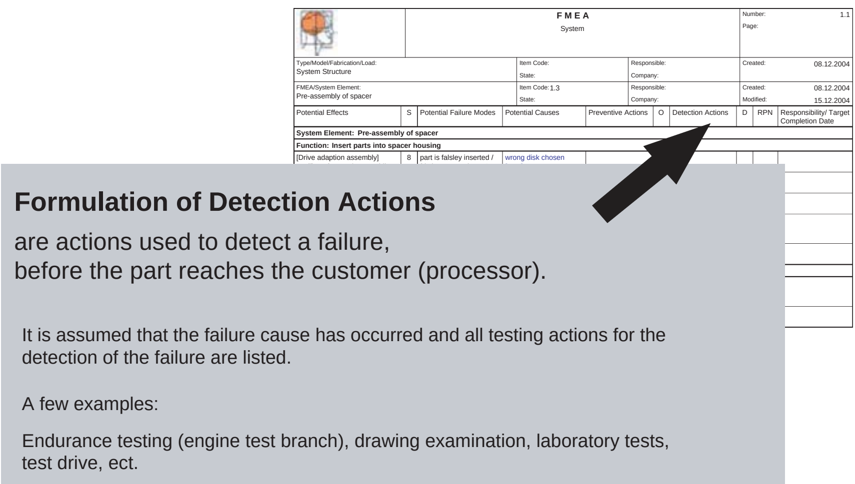


Figure 4.45. Detection actions

The assessment of the probability for detection is carried out under consideration of all detection actions listed. Detection actions that do not directly identify the potential failure cause but rather the resulting potential failure cause are taken into consideration, see Figure 4.45. A value of 10 is assigned, if no detection actions are mentioned whatsoever. A value of 1 is assigned, if the probability for the detection of the failure before the delivery to the customer is very high. Thus, the D assessment makes a statement concerning the quantity of undetected, defect components in an entire batch of a certain product.

Risk Priority Number RPN

The risk priority number RPN is calculated by multiplying the assessment values together, see Figure 4.46. The risk priority number represents the entire risk for the system user and serves as a decision criteria for the introduction of optimization actions.

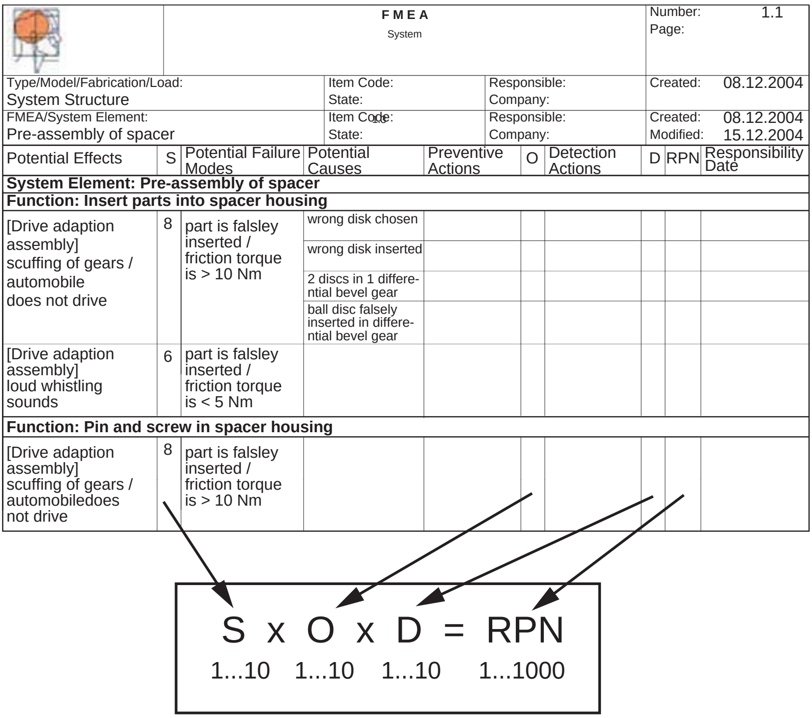


Figure 4.46. Calculation of the Risk Priority Number RPN

As a matter of principle:

- The larger the RPN is, the greater the priority that the risk is lowered with the help of design and quality assuring actions, see Figure 4.47;
- Likewise, individual values for S, O, and D that are greater than 8 should be more closely observed;
- The product O · D gives information concerning the remaining probability that defect, undetected parts will reach the hands of the customer.

The risk assessment is conducted for actions, which have already been implemented. In order to lower the risk even more, additional actions are mostly required.

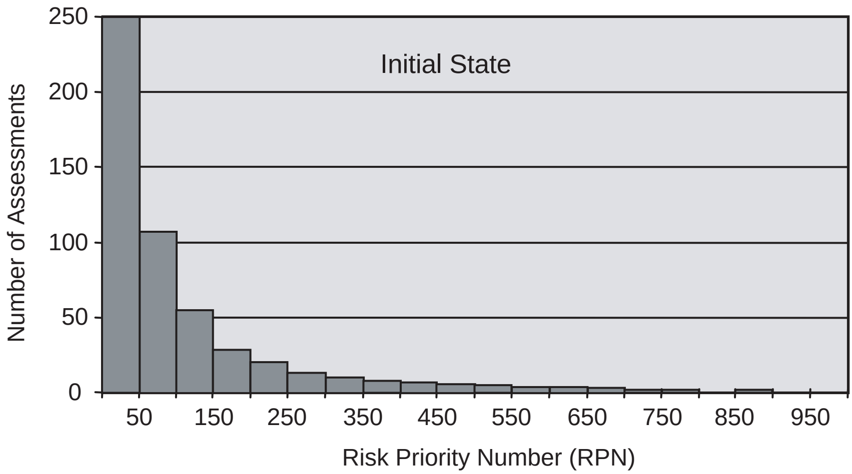


Figure 4.47. Risk assessment: RPN distribution

Analysis of the Risk Priority Numbers

The observation of the absolute value of the risk priority number (the product S · O · D) is not always sufficient in many cases to find the starting points for optimization actions. Likewise, it is not reasonable to define a "fixed RPN" as a companywide action limit (e.g. optimization conducted for all RPN ≥ 250), since the circumstances of the assessment standards may differ for each FMEA and the observation of smaller risk priority numbers could be neglected. The following example illustrates the situation and shows that by all means, the observation of smaller RPN's can prove to be a reasonable approach.

Table 4.3. Assessment examples

Example	S value	O value	D value	RPN
1	10	2	10	200
2	5	10	2	100
3	3	10	5	150
4	1	1	1	1

By analyzing the factors individually, the following results are yielded:

Example 1: An isolated potential failure cause, which has occurred, will in no way be detected after its occurrence and leads to extremely severe failure effects in the hands of the customer. Here, there is a need for a call for action, despite the relatively low absolute value for the risk priority number.

Example 2: A potential failure cause occurring very often leads to a relatively severe failure effect from the customer's point of view. The occurred failure cause is not always discovered and thus, reaches the hands of the customer from time to time. Here, it is appropriate to introduce failure preventive actions, and when appropriate, these preventive actions can replace the suggested detection actions.

Example 3: A potential failure cause, which occurs very often, is often not detected and leads to a relatively insignificant failure in the hands of the customer. However, such a condition can often lead to customer claims and should be improved with appropriate optimization actions.

Example 4: A highly improbable potential failure cause, would lead to an insignificant failure effect in the hands of the customer, if it were to occur. However, this could be easily prevented with effective detection actions. In the case of such an assessment, it is reasonable to verify the planned detection actions, and if necessary to reduce them if they are too expensive.

The (fictitious) examples mentioned above show that a "top down" analysis of the RPN's is reasonable, no matter what the absolute value of an RPN may be. At a closer look even very low risk priority numbers could offer starting points for concept optimization.

4.4.5 Step 5: Optimization

Optimization actions are to be taken for high RPN's and high individual assessment values. Firstly, the calculated risk priority numbers are ranked according to their values, see Figure 4.48. The optimization begins with the failure cause with the greatest RPN and should be ended either at a certain lower limit (e.g. RPN = 125) or according to the Pareto principle, after 20 – 30 % of the RPN's, depending on the scope of the analysis. High individual assessment values must also be observed along with the RPN. A value of O > 8 means that a failure occurs most of the time. Naturally, this must be remedied. A severity value of S > 8 points to serious function impairment or to a serious safety risk. These cases must also be looked at more closely. Failures can be very difficulty detected for values of D > 8. For this reason the danger is increased that these can reach the hands of the customer.

- Ranking of failure causes according to their **RPN** values
- Concept Optimization beginning with the failure causes with the greatest **RPN**
  - until a set RPN limit (e.g. RPN = 125) or
  - until a certain amount of failure causes (common according to the Pareto principle ca. 20 - 30 %)
- Failure causes with **O > 8**
S > 8
D > 8 observed separately
- **FMEA** result observed separately

Figure 4.48. Concept optimization procedure

Optimization actions are additional or new preventive and/or detection actions introduced based on the FMEA results.

These can be:

- actions that prevent the potential failure cause or reduce the occurrence of potential failures. Such actions are only possible by altering the design or the process.
- actions, which reduce the severity of a failure. This is attainable through conceptual alterations on the product (e.g. redundancy, error signals, etc.).
- actions taken to raise the probability of detection. Such actions could be changes in the testing procedures and/or in the design and/or in the process.

Optimization actions should be ranked according to following priority:

1. Change of concept,
   in order to eliminate the potential failure cause or to reduce the severity,
2. Increase in the concept reliability
   in order to minimize the occurrence of a failure cause.
3. Effective detection actions.
   Such actions should make up the last means of optimization, since they are generally very expensive and offer no improvement in the quality.

The optimization actions are entered into the form sheet in a revision state including the renewed assessment of O and D (so-called prognosis), responsibility (R) and deadline (D), see Figure 4.49 and Figure 4.50. For a change of concept, all 5 steps of the FMEA must be possibly repeated after the optimization, see Figure 4.50.

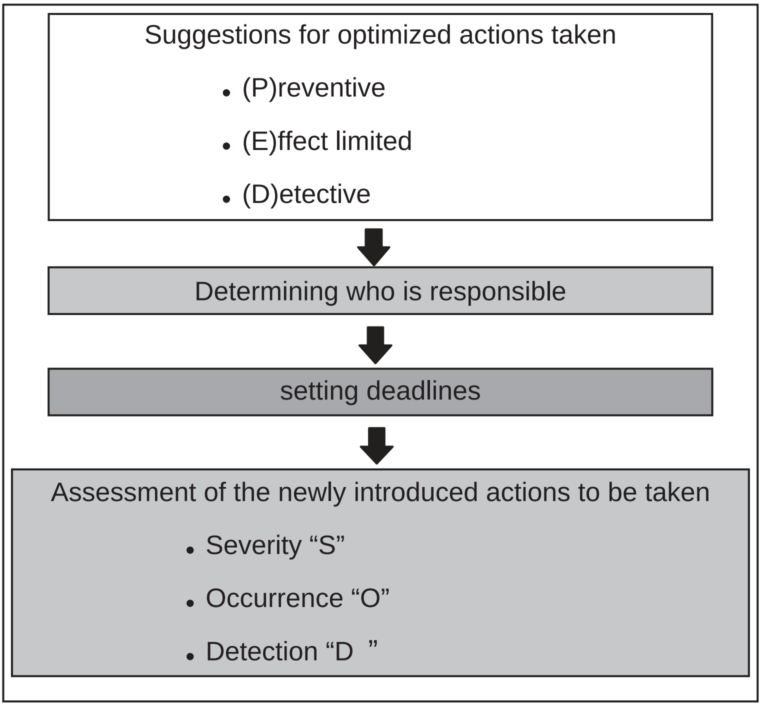
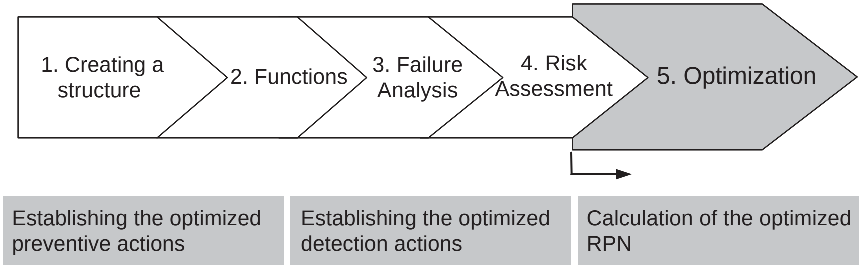


Figure 4.49. Risk minimization



FMEA
System

Number: 1.1
Page:

Type/Model/Fabrication/Load: System Structure	Item Code: State:	Responsible: Company:	Created: 08.12.2004
FMEA/System Element: Pre-assembly of spacer	Item Code: 1.3 State:	Responsible: Company:	Created: 08.12.2004 Modified: 15.12.2004

Potential Effects	S	Potential Failure Modes	Potential Causes	Preventive Actions	O	Detection Actions	D	RPN	Responsibility/ Target Completion Date
System Element: Pre-assembly of spacer									
Function: Insert parts into spacer housing									
[Drive adaption assembly] scuffing of gears / automobile does not drive	8	part is falsley inserted / friction torque is > 10 Nm	wrong disk chosen [man]						
			wrong disk inserted [man]						
			2 discs in 1 differential bevel gear [man]						
			ball disc falsely inserted in differential bevel gear [man]						
[Drive adaption assembly] loud whistling sounds	6	part is falsley inserted / friction torque is < 5 Nm							
Function: Pin and screw in spacer housing									
[Drive adaption assembly] scuffing of gears / automobile does not drive	8	part is falsley inserted / friction torque is > 10 Nm							
[Drive adaption assembly] loud whistling sounds	6	part is falsley inserted / friction torque is < 5 Nm							

Figure 4.50. Optimization

After establishing the new preventive and/or detection actions, these actions are newly assessed. This assessment represents a prognosis concerning the improvement potential to be expected. The final assessment is carried out after the new actions are implemented and tested.

A graph, such as the one shown in Figure 4.51, can be used to compare the initial and revision states by displaying both states together.

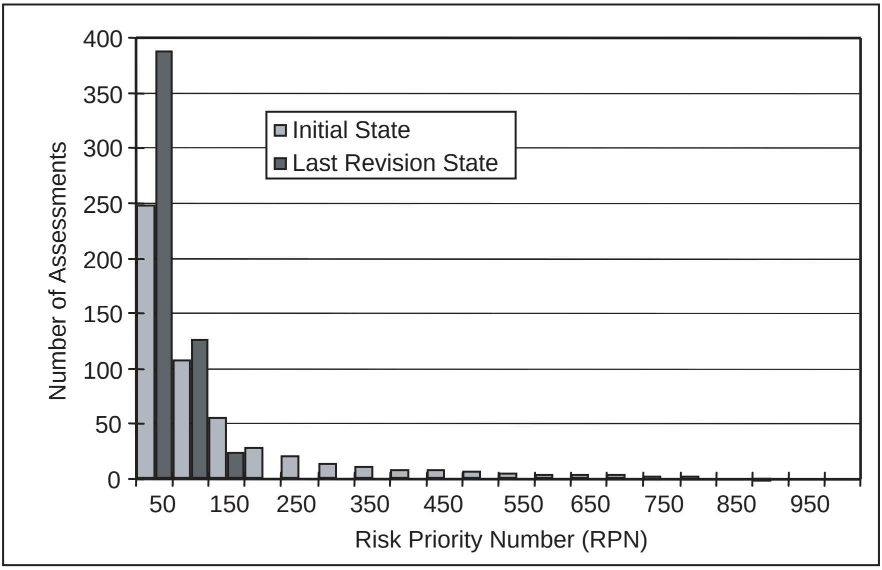


Figure 4.51. Graph of initial and revision states

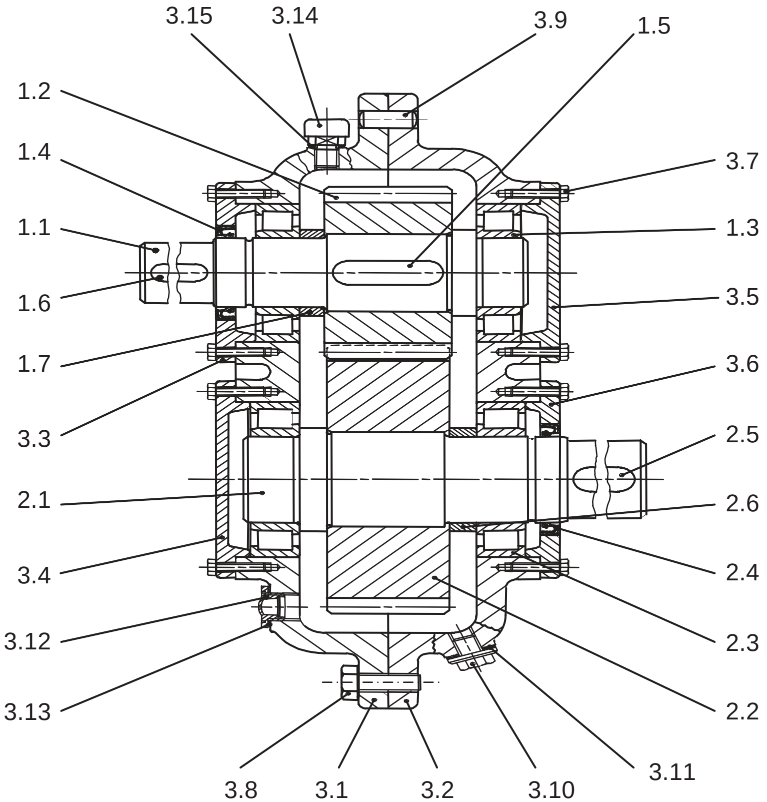
4.5 Example of a System FMEA Product according to VDA 4.2

Now, the product “adapting transmission” will be closely observed as an example in the following text.

The pinion gear (component 1.2) sits on the input shaft (component 1.1). The power is transferred through the gear (component 2.2) onto the output shaft (component 2.1). The transmission consists of bearings for the shafts and gear housing with a housing cover and various small bearing covers that are sealed with either gaskets or radial seal rings.

4.5.1 Step 1: System Elements and System Structure of the Adapting Transmission

For the first step of creating a system structure it is reasonable to consult technical documents as well as section drawings inasmuch as they are available. These can be useful in the creation of a system structure. A conventional section drawing and the transmission scheme is shown in Figure 4.52.



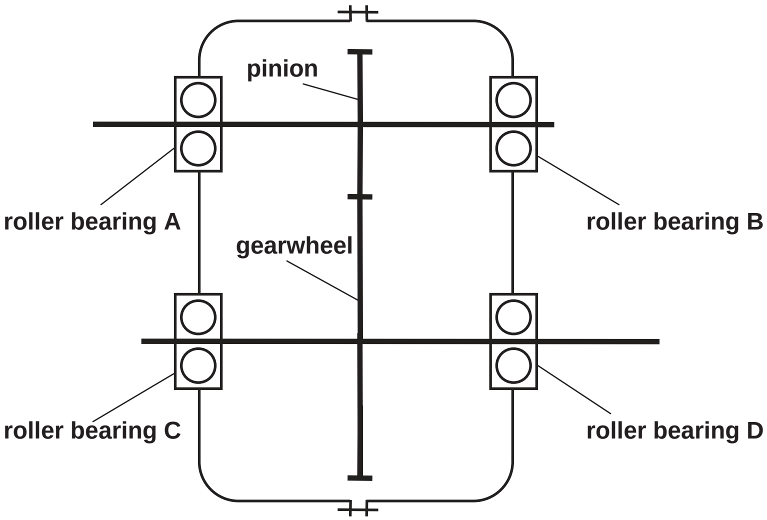


Figure 4.52. Section drawing and transmission scheme of the adapting transmission

The accompanying parts list is divided into three different assemblies, which are taken from the functionality of the adapting transmission, see Table 4.4.

Table 4.4. Assembly and part lists of the adapting transmission

Assembly	Component No.	Quantity	Component	Designation
1 Input	1.1	1	input shaft	IS
	1.2	1	Pinion gear	P
	1.3	2	roll bearing	RB1
	1.4	1	radial seal ring	RSR1
	1.5	1	fitting key for pinion gear	FK1
	1.6	1	fitting key for connection	FK2
	1.7	1	sleeve / spacer	S1
2 Output	2.1	1	output shaft	OS
	2.2	1	gear	G
	2.3	2	roll bearing	RB2
	2.4	1	radial seal ring	RSR2
	2.5	1	fitting key	FK3
	2.6	1	sleeve / spacer	S2
3 Housing	3.1	1	housing left	HL
	3.2	1	housing right	HR
	3.3	1	bearing cover	BC1
	3.4	1	bearing cover	BC2
	3.5	1	bearing cover	BC3
	3.6	1	bearing cover	BC4
	3.7	16	bolt bearing cover	BB
	3.8	8	bolt housing	BH
	3.9	2	dowel pin	DP
	3.10	1	oil drain plug	ODP
	3.11	1	seal for 3.10	S1
	3.12	1	sight glass	SG
	3.13	1	seal for 3.12	S2
	3.14	1	exhauster	E
	3.15	1	seal for 3.14	S3

The resulting system structure for the adapting transmission is shown in Figure 4.53.

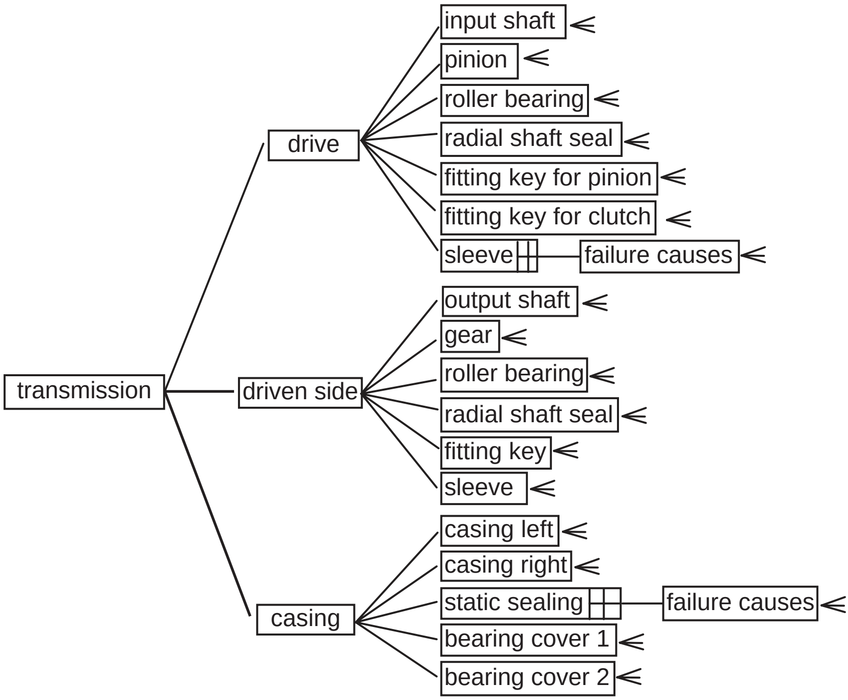


Figure 4.53. System structure of the adapting transmission

4.5.2 Step 2: Functions and Function Structure of the Adapting Transmission

The Black Box consideration and the guidelines for the specification lists according to [4.6] are used for the creation of the functions and function structure. Starting with the top element, the root element, the functions are determined for the individual assemblies and components. An extract of the resulting function structure of the "adapting transmission" is shown in Figure 4.54.

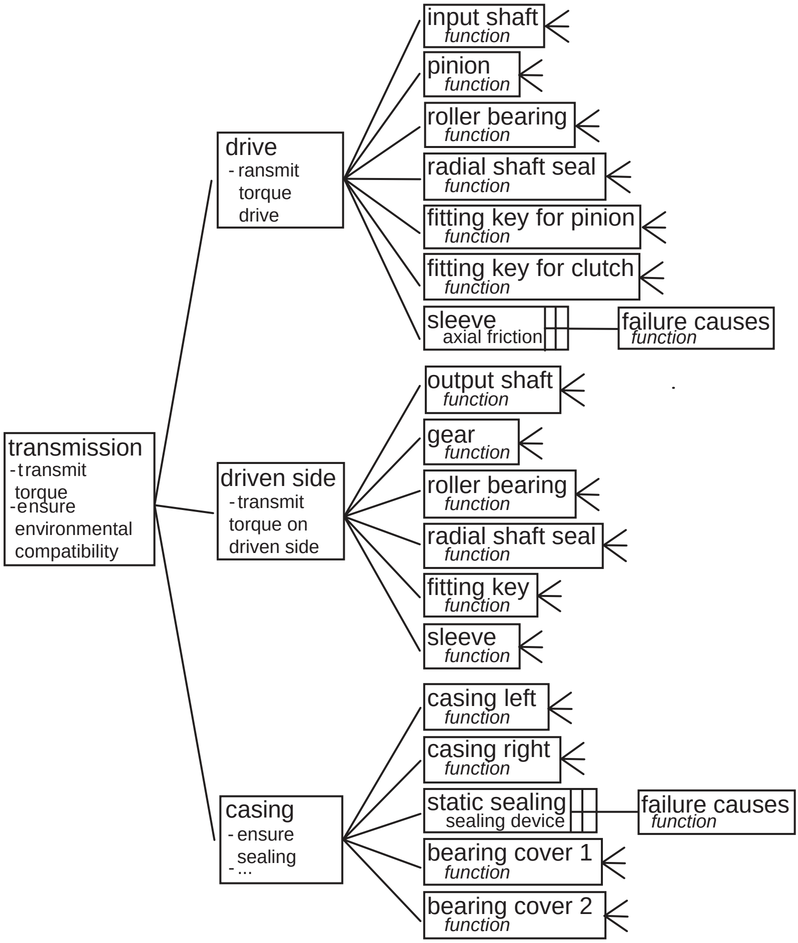


Figure 4.54. Function structure of the adapting transmission

4.5.3 Step 3: Failure Functions and Failure Function Structure of the Adapting Transmission

By negating the functions and with the determination of further failure functions under consideration of all operation conditions, the top failure functions (potential failure effects) are determined on the top level. For determining the potential failure modes the check list for physical failure modes is consulted. Likewise, for the potential failure causes the check lists mentioned in Section 2.5.3 are utilized. The acquired failure functions are represented in Figure 4.55.

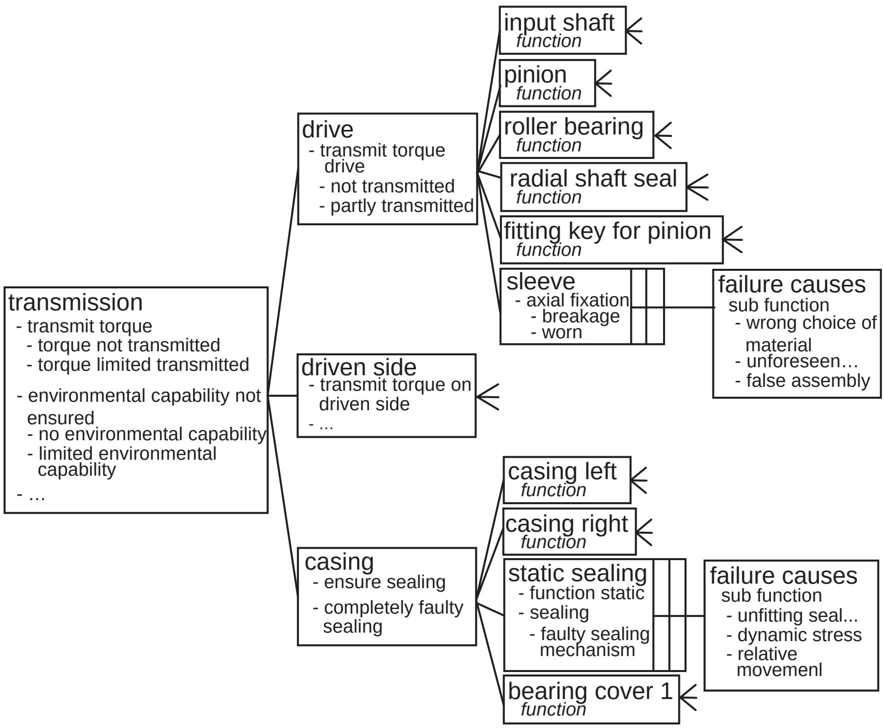


Figure 4.55. Failure functions of the adapting transmission

4.5.4 Step 4: Risk Assessment of the Adapting Transmission

An extract of the risk assessment on the adapting transmission is shown in Figure 4.56.

System: transmission Code: Team: FMEA Team Transmission Item Code: Created: 20.08.05

Potential Effects	S	Potential Failure Modes	Potential Causes	Preventive Actions	O	Detection Actions	D	RPN	Responsibility / Target Completion Date
System Elements: sleeve		Function: axial fixation							
axial movement of the construc-tional element [trans-mission]	6	breakage	wrong choice of material [sleeve]	Initial State: 20.08.05					
				calculations	2	material test	4	48	Smith
			unforeseen/ unallowable stress [sleeve]	Initial State: 20.08.05					
				calculations	2	function test	6	72	Smith
			false assembly (inverted) [sleeve]	Initial State: 20.08.05					
				construction guidelines	7	none	10	420	Bertsche
increased axial play [trans-mission]	3	worn	wrong choice of material [sleeve]	Initial State: 20.08.05					
				experience from rig testing	2	material test	7	42	Smith
			unforeseen/ unallowable stress [sleeve]	Initial State: 20.08.05					
				experience from rig testing	3	function test	7	63	Smith

risk analysis

risk assessment

Figure 4.56. Risk assessment of the adapting transmission

After the risk assessment the results of the resulting RPN's are analyzed. For this, a frequency analysis is created and the most critical 30% of the worst RPN's (according to the Pareto principle) are determined. Additionally, all individual assessment values greater than 8 are extracted. The results are summarized in the "highlights". The "highlights" concerning RPN and the very high individual values of the entire FMEA are compressed and represented in Figure 4.57. The "highlights" also serve as management information.

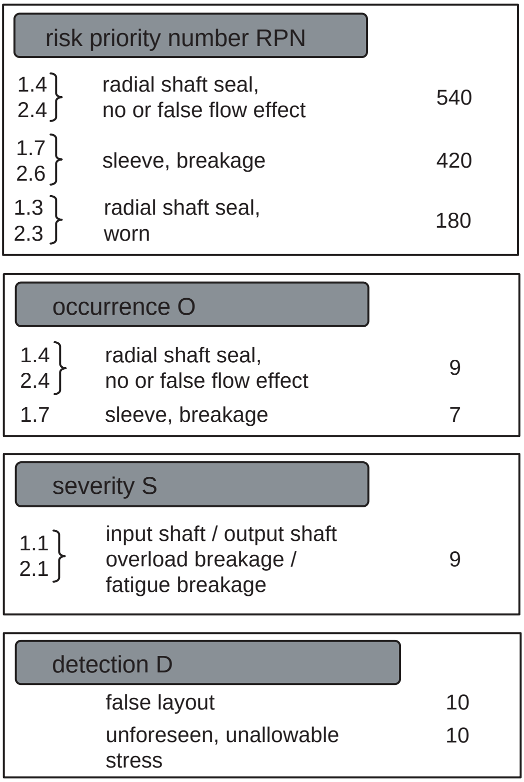


Figure 4.57. Extract from the "highlights" for the adapting transmission

4.5.5 Step 5: Optimization of the Adapting Transmission

In this step further preventive and/or detection actions are defined for the points identified to be critical in order to minimize the risk of the potential failure causes. These actions are documented in the form sheet and undergo a renewed risk assessment.

opitmization

System: transmission
Code:
Team: FMEA Team Transmission
Item Code:
Created: 20.08.05
Modified: 10.05.07

Potential Effects	S	Potential Failure Modes	Potential Causes	Preventive Actions	O	Detection Actions	D	RPN	Responsibility / Target Completion Date
System Elements: sleeve		Function: axial fixation							
axial movement of the construc-tional element [trans-mission]	6	breakage	wrong choice of material [sleeve]	Initial State: 20.08.05					
				calculations	2	material test	4	48	Smith
			unforeseen/ unallowable stress [sleeve]	Initial State: 20.08.05					
				calculations	2	function test	6	72	Smith
			false assembly (inverted) [sleeve]	Initial State: 20.08.05					
				construction guidelines	7	none	10	420	Bertsche
				State: 10.05.07					
				sleeve with double side inner chamfer	2	visual inspection	6	72	Bertsche
increased axial play [trans-mission]	3	worn	wrong choice of material [sleeve]	Initial State: 20.08.05					
				experience from rig testing	2	material test	7	42	Smith
			unforeseen/ unallowable stress [sleeve]	Initial State: 20.08.05					
				from rig testing	3	function test	7	63	Smith

opitmized risk assessment

Figure 4.58. Optimization of the adapting transmission

4.6 Example of a System FMEA Process according to VDA 4.2

The process for the manufacturing of the output shaft from the adapting transmission will be observed as an example in the following section, since this process has been identified to be a critical process. This can be determined under the consideration of the following main focus points:

- new material,
- partly new machining procedures or processes,
- high torque to be transferred.

4.6.1 Step 1: System Elements and System Structure for the Manufacturing Process of the Output Shaft

For the creation of the system structure the component drawing, see Figure 4.59, and the process flow chart, see Figure 4.60, are used. In the process flow chart all production steps and their cycle order are listed.

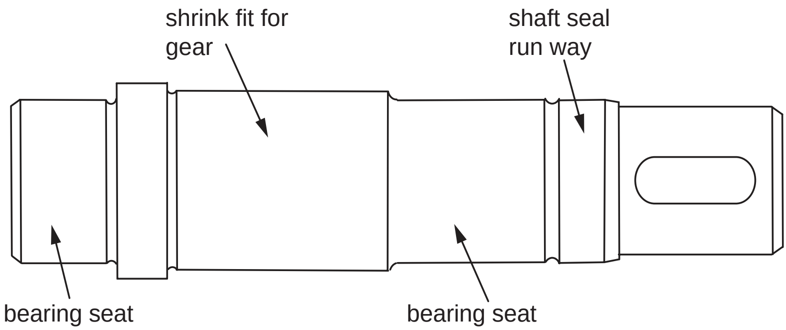


Figure 4.59. Component drawing of the output shaft

Adaptation Transmission				
Description:		output shaft adaptation transmission	Item Code: A 130.246.1	
AVO	KST	Work Process	Means of Production	Comments
Machining before heat treatment				
10	XXX	cross-cutting and centering	cross-cutting and centering m/c	
20	XXX	turning and milling (keyway)	lathe machine	outer contour and relief grooves
30	XXX	wash and dry	cycle washing machine	
40	XXX	stacking in crates	crate stacking unit	
Heat treatment				
50	XXX	case hardening	continuous furnace	
60	XXX	straighten	straightening machine	
70	XXX	annealing	annealing furnace	
80	XXX	wash and dry	cycle washing machine	
90	XXX	stacking in crates	crate stacking unit	
Machining after heat treatment				
90	XXX	hard turning of output shaft pivots	vertical lathe machine single spindle	uninterrupted cut, reception between centers and driver
100	XXX	grinding of bearing seats, seal surface (shaft seal	outer grinding machine	reception between centers and driver on output shaft pivot
110	XXX	wash and dry	cycle washing machine	
120	XXX	final inspection	inspection post	measure function related measurements (random sample)
130	XXX	stacking in crates	crate stacking unit	

Figure 4.60. Production cycle plan for the manufacturing of the output shaft

With these resources and with the help of the expertise and know-how of the participating FMEA team members a system structure for the failure functions of the transmission output shaft is set up, see Figure 4.61.

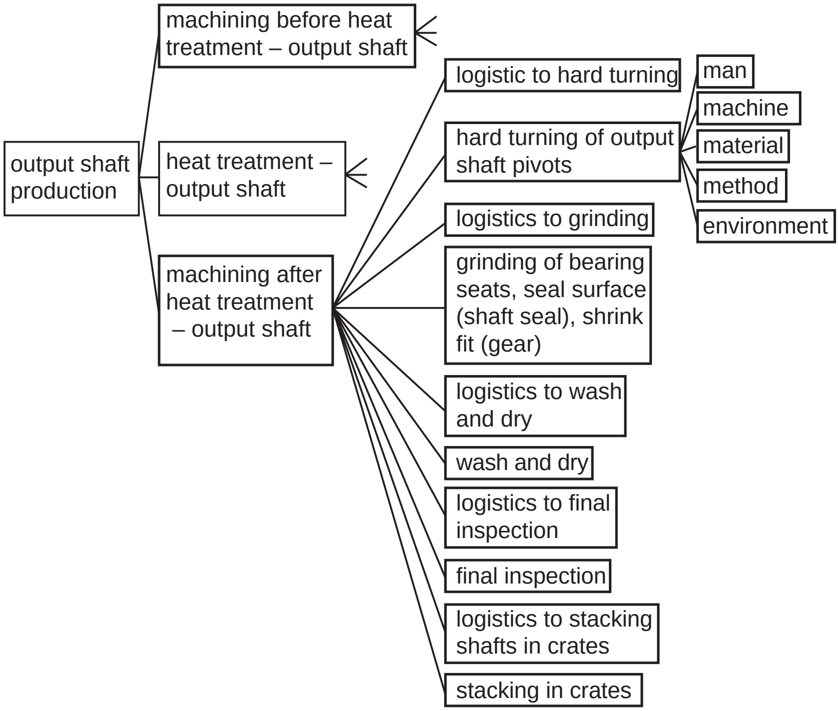


Figure 4.61. System structure for the manufacturing of the output shaft

The system structure is enhanced with the respective logic steps between the producing, testing and measuring production steps.

4.6.2 Step 2: Functions and Function Structure for the Manufacturing Process of the Output Shaft

The determination of the functions is carried out with the help of the Black Box methodology and the knowledge of the participating team members. The result is shown in the production cycle plan in Figure 4.60. An extract of the function structure is shown in Figure 4.62.

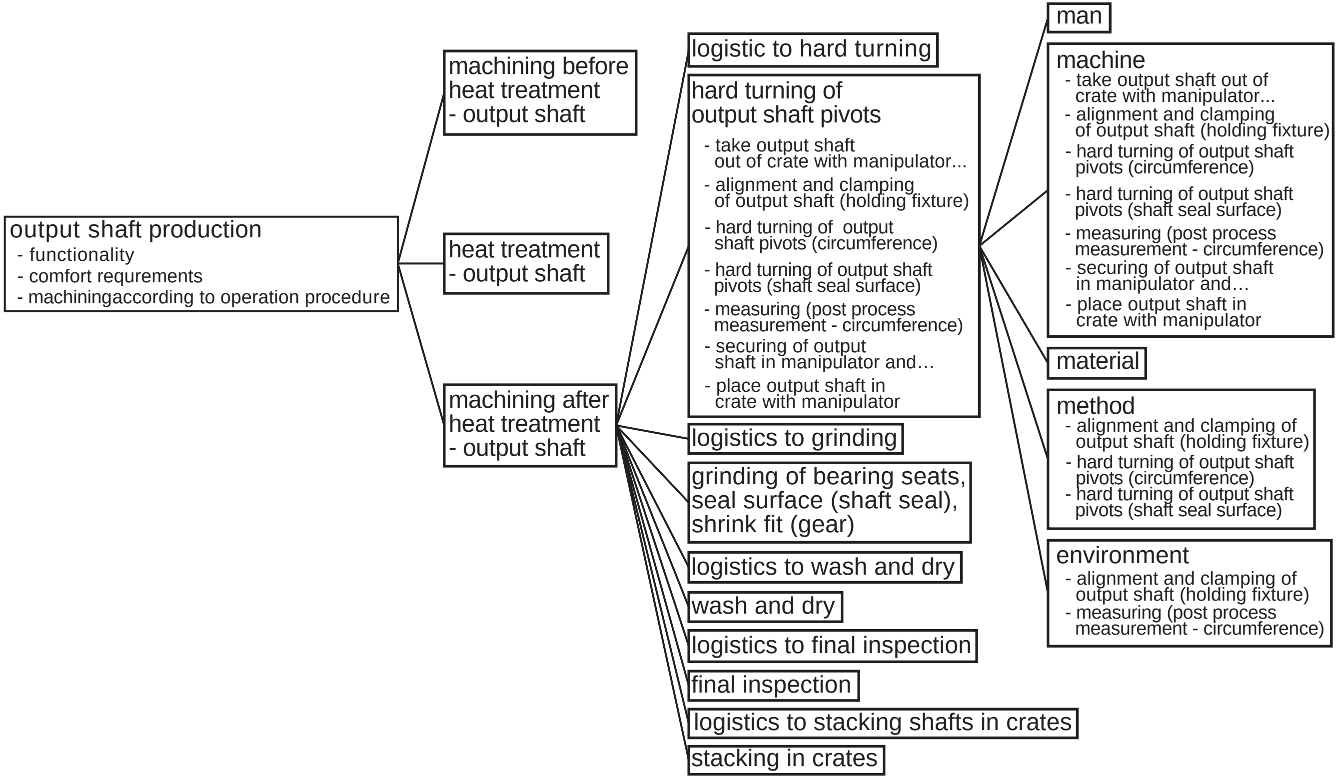


Figure 4.62. Function structure for the manufacturing process (extract)

4.6.3 Step 3: Failure Functions and Failure Function Structure for the Manufacturing Process of the Output Shaft

By negating the functions including specifications and with the determination of further failure functions under consideration of all operation conditions, the top failure functions (potential failure effects) and failure modes of the respective process steps are determined for the creation of the failure functions and the failure function structure. The failure function structure is shown in Figure 4.63.

4.6.4 Step 4: Risk Assessment of the Manufacturing Process of the Output Shaft

The is-state of the process with the preventive and detection actions already integrated into the manufacturing process are documented for the manufacturing process of the output shaft. The assessment of the occurrence and detection is carried out with the help of the assessment criteria according to VDA 4.2, see Section 4.4.4, and with knowledge about the current and comparable previous processes. An extract of the risk assessment is shown in Figure 4.64 along with the respective optimization state.

4.6.5 Step 5: Optimization of the Manufacturing Process of the Output Shaft

In this step further preventive and/or detection actions are defined for the points identified to be critical in order to minimize the risk of the potential failure causes. These actions are documented in the form sheet and undergo a renewed risk assessment, see Figure 4.64.

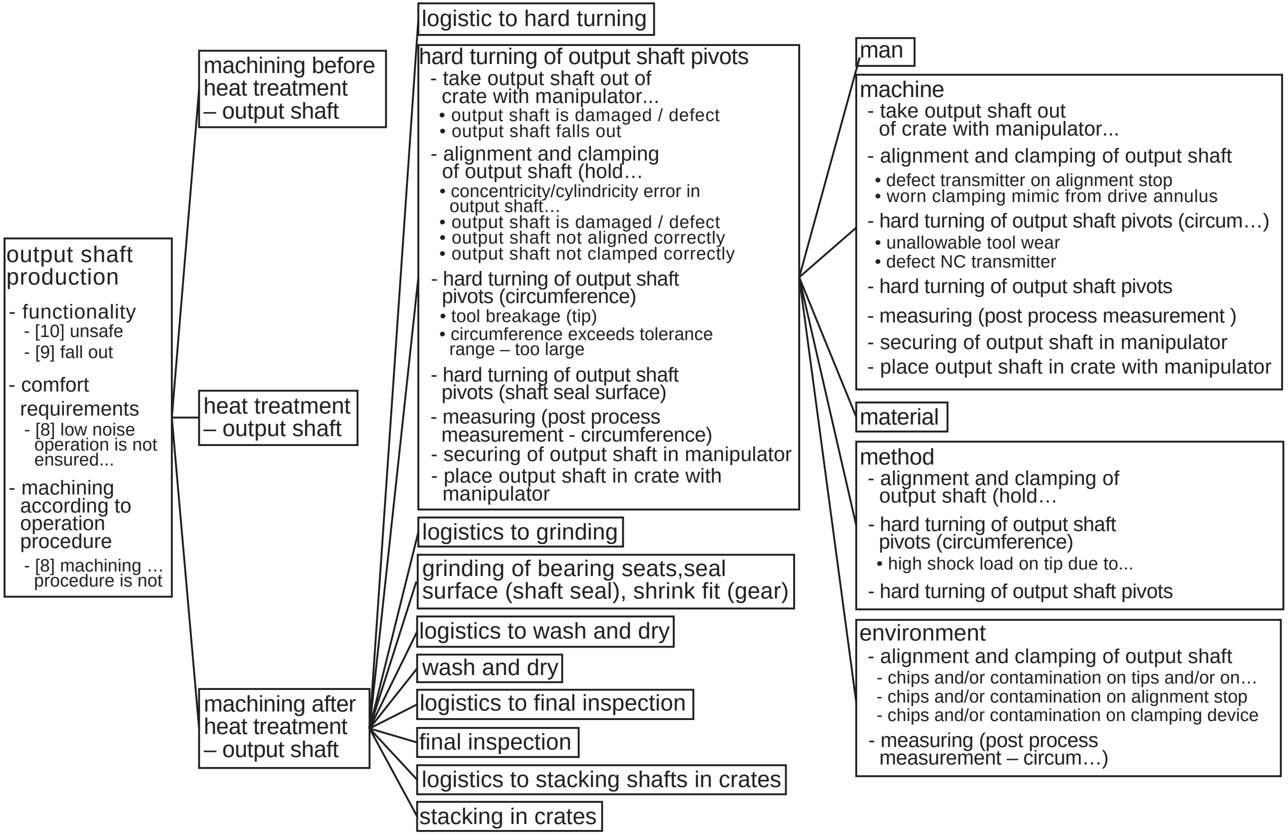


Figure 4.63. Failure function structure for the manufacturing process (extract)

Potential Effects	S	Potential Failure Mode	Potential Cause	Preventive Action	O	Detection Action	D	RPN	Responsibility Completion Date
[assembly - output shaft] machining according to operation procedure is not ensured Failure effect: possible tool breakage	8	output shaft is not correctly clamped	[environment] chips and/or contamination on clamping device	Initial State: 20.12.2004					
				Cleansing of clamping device before renewed clamping	2	operator control visual inspection	6	96	
			[machine] worn clamping mimic from drive annulus	Initial State: 20.12.2004					
				cooperation with the supplier worn-reduced, enclosed mimik	3	work sampling (1/layer)	4	96	
Function: Hard turning of output shaft pivots (circumference, chamfer)									
[assembly - output shaft] machining according to operation procedure is not ensured cycle problem, change of replace holder / tool holder	8	tool breakage (cutting tip)	[method] high shock load on tip due to non-stop cutting of hardened matherial	Initial State: 20.12.2004					
				determination of optimal process parameters use of special cutting tip and cutting tip holder	6	principle experiments determination of cutting forces	4	192	
				State: 15.01.2004					
				grinding process instead of hard turning process	2	determination of cutting forces principle experiments	4	64	Smith 01.08.2004 revised
[assembly - output shaft] machining according to operation procedure is not ensure subsequent work, part must be reentered into the assembly line	8	circumference exceeds tolerance range – too large	[machine] unallowable tool wear	Initial State: 20.12.2004					
				enclosed, waterproof unit	3	post process measurements	2	48	
			[machine] defect NC transmitter	Initial State: 20.12.2004					
				enclosed, waterproof unit protected cabeling agitation resistant encapsulation	2	principle experiments error message equipment malfunctional	2	32	

Figure 4.64. Risk assessment and optimization (extract)

References

[4.1] Department of Defense (1980) MIL-STD-1629 A, Procedures for Performing a Failure Mode, Effects and Critically Analysis. Washington DC

[4.2] Deutsches Institut für Normung (1981) DIN 9000 ff Qualitätsmanagementsysteme. Beuth, Berlin

[4.3] Deutsches Institut für Normung (1981) DIN 25448 Ausfalleffektanalyse. Beuth, Berlin

[4.4] Förster H J (1991) Automatische Fahrzeuggetriebe Grundlagen, Bauformen, Eigenschaften, Besonderheiten. Springer, Berlin

[4.5] Gesetz über die Haftung für fehlerhafte Produkte (Produkthaftungsgesetz - ProdHaftG) 15.12.1989 (BGBl. I S 2198)

[4.6] Pahl G, Beitz W (2003) Konstruktionslehre: Grundlagen erfolgreicher Produktentwicklung; Methoden und Anwendung. Springer, Heidelberg Berlin

[4.7] Verband der Automobilindustrie (1996) VDA 4.2 Sicherung der Qualität vor Serieneinsatz System FMEA. VDA, Frankfurt

5 Fault Tree Analysis, FTA

The Fault Tree Analysis (FTA) is a structured procedure for the identification of internal and external causes, which if they occur on their own or in combination, can lead to a defined product state (mostly a fault state) [5.8]. Thereby the FTA defines the system behaviour in regards to a certain event (and/or fault).

The FTA was developed in 1962 by H. A. Watson (Bell Laboratories) by order of the U.S. Air Force. Boeing was the first commercial company that saw the benefits of the method and started to use the FTA for the development of commercial aircrafts (1966). In the seventies the method was used in particular in the area of nuclear power techniques, whereupon the FTA started to spread out worldwide in the eighties. Nowadays the method is used worldwide in many different areas e.g. in the automotive industry, for communication systems and in the last couple of years in the field of robotics [5.1, 5.4].

The FTA is used to display the functionalities of a system and to quantify the system reliability. The method can be applied as a diagnosis and development tool and is especially helpful in early design stages. Thus, potential system faults can be identified and design alternatives can be evaluated. One of the major advantages of FTA is that the method provides both qualitative and quantitative results.

The FTA can be used for each kind of reliability system analysis including analysis with common mode and human failures. In these cases, the FTA provides complete results, which means that by consequent execution, all failure modes and/or failure causes are discovered because of the deductive procedure. Thus, the method is limited by both the system knowledge and the operational benefits defined by the user.

The FTA is based on Boolean algebra and probability theory and thus with a couple of simple rules and symbols it is possible to analyze complex system and complex dependencies e.g. between hardware, software and people. Due to the existing stress of competition, product design cycle with its cost optimization potentials plays an important role. Costs of failures increase with progression in the product design cycle, so that early failure detection offers the potential for massive cost reduction.

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_5, © Springer-Verlag Berlin Heidelberg 2008

In this relation the usage of the FTA as preventive quality assurance in early design stages has proven to be quite beneficial. With the execution of an FTA in the concept phase the system concept could be confirmed or fundamental failures could be found. With these analyses new requirements and adequate failure preventive actions can be introduced after the specifications are complete, see Figure 5.1.

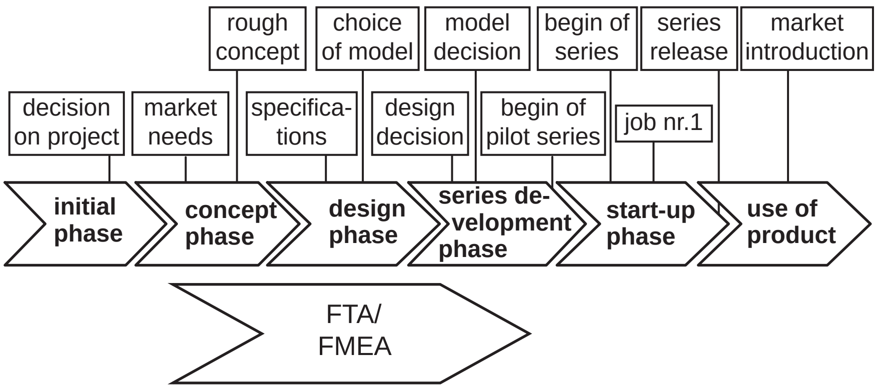


Figure 5.1. Placement in the product design cycle (car)

5.1 General Procedure of the FTA

The successful application of a Fault Tree Analysis requires a system analysis. Here, the system is subdivided exemplary into sub-systems and components.

In order to determine the failure behaviour of the system or of the system components and its interfaces, the undesired system events must first be defined. In the next step it will be examined which possible failures on the next lower system level could be expected and how they can be linked to the superior failures. This step is repeated until the lowest system level is reached on which the component failure mode is defined, so that the result of the complete failure behaviour is found.

5.1.1 Failure Modes

DIN 25424 distinguishes between three failure modes: primary, secondary and command failure, see Figure 5.2. A primary failure is a component failure under permitted conditions, whereas a secondary failure is a consecutive fault which is caused by incorrect operation conditions for a component. A command failure is caused by a wrong and missing inducement or the failure of an auxiliary source, despite the fact that the component is perfectly functional [5.3]

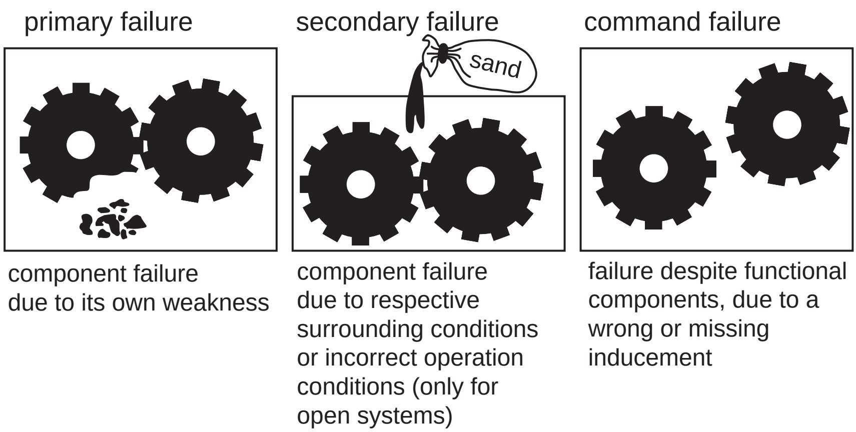


Figure 5.2. System failure modes according to DIN 25424

5.1.2 Symbolism

In order to describe a system systematically in a fault tree, the individual inputs are networked together in different ways. Diverse symbols are used to visualize these networks. The following list describes the most common symbols illustrated in Figure 5.3:

- Standard input: this symbol stands for a primary fault of a function-related element. It describes a failure cause without any further conditions. Parameters for the primary failure are assigned to the graphical symbol.
- Transfer input and output: with this symbol the fault tree is interrupted or continued at a different place.
- Commentary: these symbols are used for the description of input and output in between the network symbols.
- For AND-operations the event at the output only occurs if all events at the input occur
- For OR-operations only one event at the input has to occur for the event at the output to occur.

- The NOT-operations stands for the case of negation. Thus, for the event at the output to occur, the event at the input must occur.

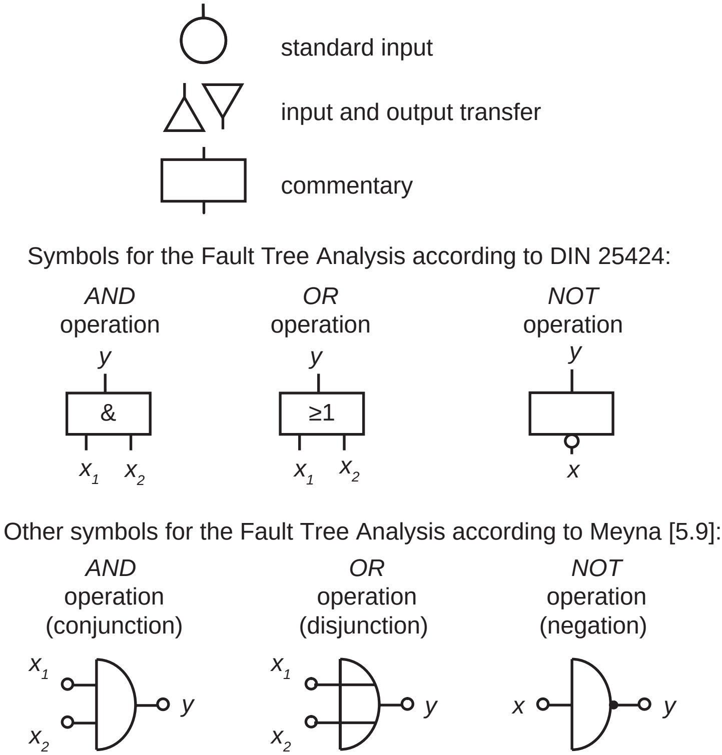


Figure 5.3. Symbolism of the Fault Tree Analysis

5.2 Qualitative Fault Tree Analysis

5.2.1 Qualitative Objectives

The qualitative fault tree analysis deals with the unwanted events, which could occur in a system. Such events (also called TOP events) are undesired system states, which can be traced back to failed single components (DOWN cause). The fault tree is a model that graphically illustrates and logically networks all combinations of undesired system states. Thereby the objectives of the FTA are:

- Systematic identification of all possible failures as well as all failure combinations and their causes, which could lead to an undesired event, the main event.
- Illustration of especially critical events and/or event combinations (e.g. failure functions that lead to an undesired event).
- To gain objective evaluation criteria of system concepts.
- A clear documentation of the failure mechanism and their functional relations.

5.2.2 Basic Procedure

In order to determine the failure behaviour (failure function, type of failure) of a system and/or of the system elements (assembly, components) together with their connections, the undesired system event (TOP-event) is firstly defined. Because of the deductive procedure (TOP-DOWN method), in the next step possible failures to be expected on the next lower system level and how they could be connected to the superior failure are analyzed. This step is repeated until the lowest system level is reached. The lowest system level corresponds to the possible failure modes, thus determining the complete failure behaviour of a system, see Figure 5.4.

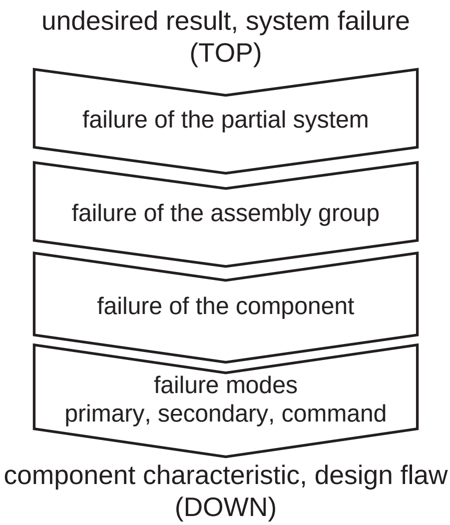


Figure 5.4. Basic procedure for the structure of a fault tree

In the standard DIN 25424 the following systematic procedure for the structure of a fault tree is described [5.3]:

1. The undesired event is determined.
2. If this event is already a failure mode of a component the procedure will be continued with step 4. Otherwise, all failures are determined which could lead to the undesired event.
3. The failures are inscribed in commentary rectangles and are logically connected with the fault tree symbols. If the failures are failures modes, then the procedure skips to step 4. Otherwise, step 2-3 are repeated.
4. In most cases the single failures are linked with OR connections, since each input event leads to the event at the output. These entries are thereby allocated among primary, secondary and command failures. Primary failures cannot be further analyzed with the fault tree analysis and thus mark a standard entry to the system; whereas, secondary and command failures do not necessarily have to be present. If, however, these are present, the failure is not a failure of a functional element and thus is partitioned further. The procedure begins again with step 2.

An example for such a qualitative fault tree is shown in Figure 5.5. Here, the TOP event, the failure of the transmission, is firstly divided into the single assembly groups whose failure could cause the failure of the complete transmission and are thereby linked with an OR connection. Afterwards, the failure of the assembly "output" is further analyzed and the elements are detected which could lead to a failure of the assembly groups on the next highest level. In this way the individual elements, in this example in particular the failure of the gear, are further partitioned in the failure modes of the components, to which tooth failure of the gear is also included. The tooth failure can possess different causes so that the failure mode must be further divided until the level of component characteristic and/or the development failure is reached. In this case, overload or incorrect calculation can lead to tooth failure of the gear. Those two failures, however, still do not represent standard inputs and therefore have to be analyzed further. The incorrect operation represents a standard input and cannot be further subdivided. Thus, the fault tree is finished at this point.

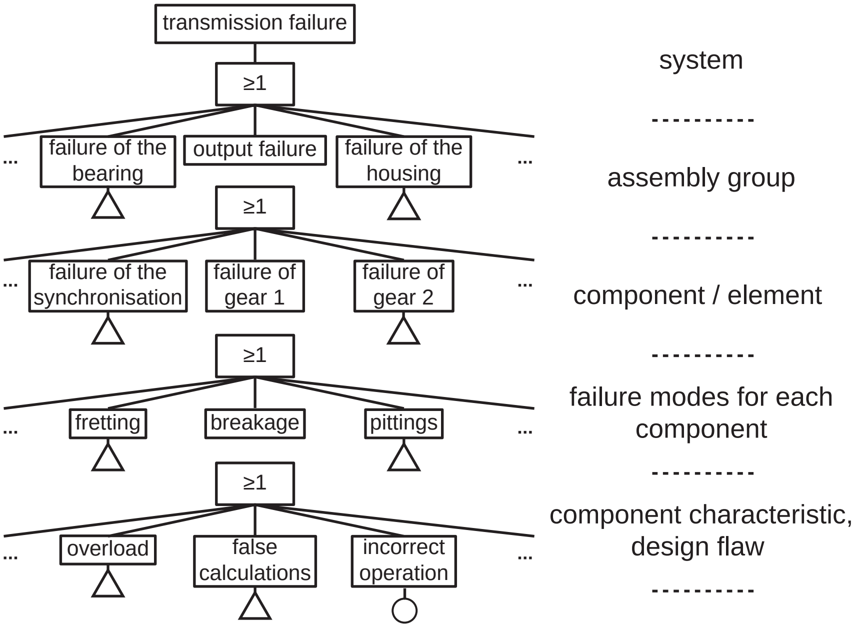


Figure 5.5. Fault tree for the example transmission

5.2.3 Comparison between FMEA and FTA

In comparison to FTA, failure combinations are not part of an FMEA. Therefore, the use of an FMEA as a basis for an FTA is limited. FMEA deals more with the evaluation of failure modes for a system und their effects on the system [5.11] thus making FMEA a good source and/or systematic catalogue of possible failure modes for the FTA. The major difference between the two methods is that the FMEA is an inductive method and the FTA is a deductive method. This means that FMEA examines the effects for a failure cause of a component on the device whereas FTA traces the failure of the device back to the failure cause of the component.

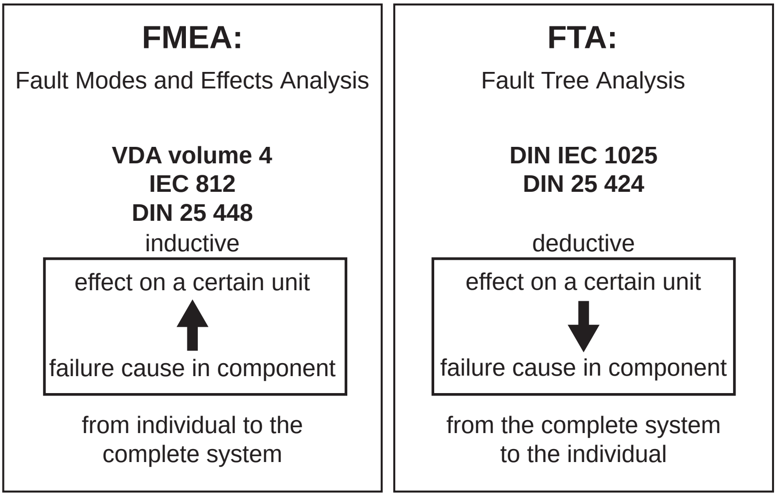


Figure 5.6. Comparison of FMEA and FTA

In comparing the two methods, the FMEA can be characterized by the following features:

- FMEA combines the two questions "What is the cause?" and "What are the effects of the failure?",
- is not as systematic as the FTA,
- assesses the risk of a failure by the combination of the two questions and defines preventive actions depending on the risk potential.

The FTA can be characterized by the following features:

- Systematic search of causes for an event and/or failure
- ETA (Event Tree Analysis) searches the effects of a failure.

Recapitulating the comparison of the two methods one determines:

- FMEA and FTA are different methods with a similar subject matter.
- The determination of failures in the FTA can be eased with precognition of FMEA.
- FMEA examines single failures and skips levels.
- FTA is more systematic
- FTA uses the combinations *AND*, *OR*, *NOT*, Maintenance / Repair.

5.3 Quantitative Fault Tree Analysis

5.3.1 Quantitative Objectives

With the help of a fault tree analysis a system can not only be describe qualitatively but also provides the possibility to make a quantitative statement concerning the failure behaviour of the system. The reliability parameters (e.g. entry probability of the undesired event or system availability) can be calculated with the system structure with help of the Boolean model if the failure probabilities of the single components are known. Thus, factors which influence the system reliability the most severely can be analyzed, such as changes for the improvement of the system reliability factors.

5.3.2 Boolean Modelling

5.3.2.1 Basic Connections of the Boolean Modelling

For the determination of the system reliability the Boolean modelling (see. Chapter 2) can be used [5.12]. Here, the symbols of the fault tree are transferred into numerical values with the help of a few simple calculation rules.

Negation

If a Boolean variable has the value of 1, than the negated variable has the value 0 and vice versa, see Table 5.1.

$$y = \overline{x}\,. \tag{5.1}$$

Disjunction

The disjunction stands for the Boolean function OR and its appliance can be found in many cases in which it is sufficient if only one event of two or more must occur at the entry in order to cause the event at the output [5.9]. For example, for two binary variables a disjunction is given if x_1 *or* x_2 equals 1 as well as if x_1 *and* x_2 equal 1. For these cases the output results to $y = 1$ and one speaks of an inclusive or (lat. vel).

Only when x_1 and x_2 equal 0 does $y = 0$, see Table 5.1.

$$y = x_1 \vee x_2\,. \tag{5.2}$$

Out of this these equations follows that

$$x \vee 1 = 1; \quad x \vee x = x,$$
$$x \vee 0 = x; \quad x \vee \bar{x} = 1 \tag{5.3}$$

as well as

$$x_1 \vee x_2 = x_2 \vee x_1 \qquad \text{(commutative law)} \tag{5.4}$$

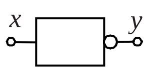
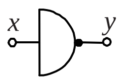
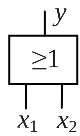
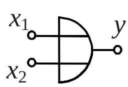
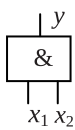
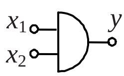
are valid condition for the disjunction of two variables. Thus, for a disjunction of n independent variables

$$y = \bigvee_{i=1}^{n} x_i \text{ with } y = \begin{cases} 0 & \text{for } x_i = 0 \\ 1 & \text{otherwise} \end{cases}. \tag{5.5}$$

Conjunction

The conjunction stands for the Boolean function AND. All events at the input must be present in order for the event at the output to occur. A conjunction of two binary variables results if x_1 *and* x_2 equal 1. Only in this case does the result yield $y = 1$, see Table 5.1.

Table 5.1. Overview of the basic connections

name	synonym	Boolean equation	operator	function table x_1	x_2	y	symbol DIN 25424	symbol acc. [5.9]
Negation	*NOT*, negator, inverter, phase turner	$y = \bar{x}$	$\bar{x}$	0	-	1	 $\bar{y} = x$	 $\bar{y} = x$
				1	-	0		
Disjunction	*OR*	$y = x_1 \vee x_2$	$\vee$	0	0	0		
		$= x_1 + x_2$	$+$	0	1	1		
				1	0	1		
				1	1	1		
Conjunction	*AND*	$y = x_1 \wedge x_2$	$\wedge$	0	0	0		
		$= x_1 \cdot x_2$	$\cdot$	0	1	0		
		$= x_1 x_2$	&	1	0	0		
		$= x_1$ & x_2		1	1	1		

With the help of Venn diagrams the described Boolean basic connections can be graphically illustrated. Thereby, all possibilities of Ω are displayed in a rectangle and the possibilities which actually occur are displayed in a hatched area, see Figure 5.7.

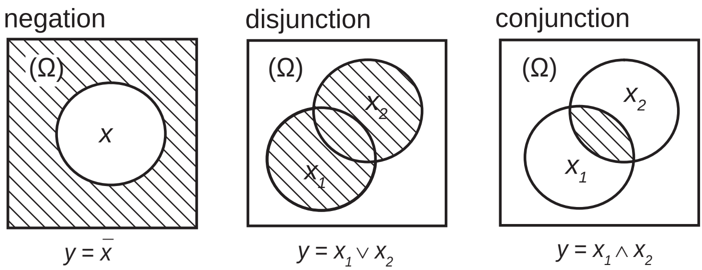


Figure 5.7. Venn diagrams of the basic connections [5.13]

5.3.2.2 Axioms and Boolean Algebra

With the help of the axioms and Boolean algebra, introduced in the following section, it is possible to change and/or simplify Boolean terms mathematically [5.6].

Commutative law

$$x_1 \wedge x_2 = x_2 \wedge x_1 , \tag{5.6}$$

$$x_1 \vee x_2 = x_2 \vee x_1 , \tag{5.7}$$

Associative law

$$x_1 \vee (x_2 \vee x_3) = (x_1 \vee x_2) \vee x_3 , \tag{5.8}$$

$$x_1 \wedge (x_2 \wedge x_3) = (x_1 \wedge x_2) \wedge x_3 , \tag{5.9}$$

Distributive law

$$x_1 \vee (x_2 \wedge x_3) = (x_1 \vee x_2) \wedge (x_1 \vee x_3) , \tag{5.10}$$

$$x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) , \tag{5.11}$$

These three laws are already well known from common algebra, so that in Boolean algebra, in order to simplify the terms, the parentheses can also be multiplied by other terms.

Postulates

Existence of 0 and 1 elements

$$x \vee 0 = x , \tag{5.12}$$

$$x \wedge 1 = x\,, \tag{5.13}$$

Existence of a complement

$$x \wedge \bar{x} = 0\,, \tag{5.14}$$

$$x \vee \bar{x} = 1\,, \tag{5.15}$$

Idempotent law

$$x \vee x = x\,, \tag{5.16}$$

$$x \wedge x = x\,, \tag{5.17}$$

Absorption law

$$x_1 \vee (x_1 \wedge x_2) = x_1\,, \tag{5.18}$$

$$x_1 \wedge (x_1 \vee x_2) = x_1\,, \tag{5.19}$$

De Morgan law

$$\overline{x_1 \vee x_2} = \overline{x_1} \wedge \overline{x_2}\,, \tag{5.20}$$

$$\overline{x_1 \wedge x_2} = \overline{x_1} \vee \overline{x_2}\,, \tag{5.21}$$

Furthermore:

$$\bar{\bar{x}} = x\,, \tag{5.22}$$

$$x \vee 1 = 1\,, \tag{5.23}$$

$$x \wedge 0 = 0\,. \tag{5.24}$$

In reliability theory the De Morgan law as well as the idempotent and absorption law are of high importance for the conversion between fault tree and function tree.

Fault tree and function tree

Principally, the method of the function tree in based on the same procedure as the FTA. In this method, instead of defining a failure mode as the main event, one defines a desired and/or a preferable event. All intermediate events as well as primary events, which secure the occurrence of the main event, are found deductively. If the logical counterpart of the top event of a fault tree is used as a main event for the function tree, the function tree can be gained as logical complement to the fault tree due to the Boolean structure. A fault tree can thus be transferred into a function tree and vice versa with the help of the negation operator. The only difference is that the function tree produces the system reliability as a result instead of the failure probability, see Figure 5.8.

The same can be achieved if the existing correlation

$$F_S(t) = 1 - R_S(t) \tag{5.25}$$

between failure probability and reliability is considered.

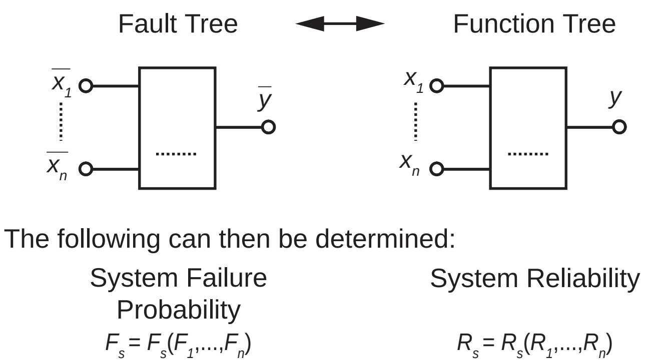


Figure 5.8. Correlation between fault tree and function tree

5.3.2.3 Transition to Probabilities

The failure behaviour of each component can be described by failure and/or reliability probabilities. By the transition of Boolean expressions to the description using probabilities, the failure and/or reliability probability for the whole system can be generated with the usage of simple transformations [5.9]. Here, the Boolean function can firstly be transferred into a term of real variables x_i, if only the real numbers 0 and 1 are used and all occurring variables are linear. Thus, the system behaviour can be described as a discrete zero-one distribution. In the second step, these discrete variables can be merged into continuous probability functions for the failure and/or survival of a component. For the most important connections, the transition from the logical to the mathematical notation can be carried out according to Table 5.2.

Table 5.2. Transition to probabilities

	logical	mathematical
negation	$y = \overline{x}$	$R_S(t) = F_K(t) = 1 - R_K(t)$ counterproductive for reliability techniques
disjunction	$y = \bigvee_{i=1}^{n} x_i$ $R_S(t) = R_1(t) \vee R_2(t) \vee \ldots = \vee_{i=1}^{n} R_i(t)$	$R_S(t) = 1 - \prod_{i=1}^{n}(1 - R_i(t))$
conjunction	$y = \bigwedge_{i=1}^{n} x_i$ $R_S(t) = R_1(t) \wedge R_2(t) \wedge \ldots = \wedge_{i=1}^{n} R_i(t)$	$R_S(t) = \prod_{i=1}^{n} R_i(t)$

5.3.3 Application to Systems

5.3.3.1 Series and Parallel Configuration

If it is possible to assign the two states "functional" and "failed" to a system and its components, a technical system can be described with the help of Boolean algebra subject to the states of its components. The positive logic forms the basis of the definition of the term system function. Here, the system reliability is determined by the reliabilities of the single components. In the application of the FTA, the rules of the negation logic are used, generally in order to determine the failure behaviour and thus the failure probability. The following tables (Table 5.3 and Table 5.4) show several typical basic structures und their formation for the system function (positive and negative logic).

Table 5.3. Positive logic

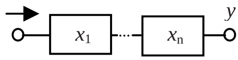
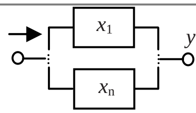
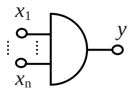
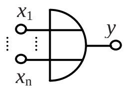
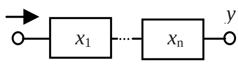
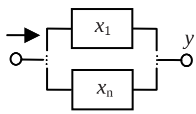
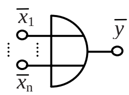
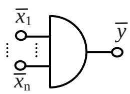
System structure	Serial configuration	Parallel configuration
Block diagram		
Function tree		
Boolean function	$y = x_1 \wedge x_2 \wedge \ldots \wedge x_n = \bigwedge_{i=1}^{n} x_i$	$y = x_1 \vee x_2 \vee \ldots \vee x_n = \bigvee_{i=1}^{n} x_i$
System reliability	$R_S(t) = \prod_{i=1}^{n} R_i(t)$	$R_S(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t))$

Table 5.4. Negative logic

System structure	Serial configuration	Parallel configuration
Block diagram		
Function tree		
Boolean function	$\overline{y} = \overline{x_1} \vee \overline{x_2} \vee \ldots \vee \overline{x_n} = \bigvee_{i=1}^{n} \overline{x}_i$	$\overline{y} = \overline{x_1} \wedge \overline{x_2} \wedge \ldots \wedge \overline{x_n} = \bigwedge_{i=1}^{n} \overline{x}_i$
System failure probability	$F_S(t) = 1 - \prod_{i=1}^{n} \left(1 - F_i(t)\right)$	$F_S(t) = \prod_{i=1}^{n} F_i(t)$

5.3.3.2 Bridge Configuration

In a bridge configuration, see Figure 5.9, the reliability cannot be calculated by the elementary equations for serial und parallel systems. For systems with a small number of elements it is still possible to use the disjunctive normal form [5.5]. If the system consists of *n* elements, the effort remarkably increases, since each system equation exists of 2^n terms. To determine the reliability and/or failure probability of a system in such a case with low effort, the following methods can be applied:

- minimal cut sets,
- minimal path sets and
- solve by separation

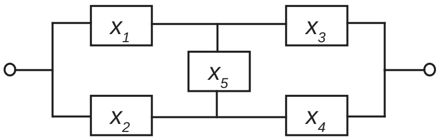


Figure 5.9. Bridge configuration

Method of Minimal Cut Sets

In the method of minimal cut sets all combinations of the components, whose failure could lead to the failure of the system, are searched for by all possible cuts in the structure. All components are negated and are connected inside of the cut sets by *and* operators and on the outside by *or* operators, resulting in a negative output – the failure probability, see Figure 5.10.

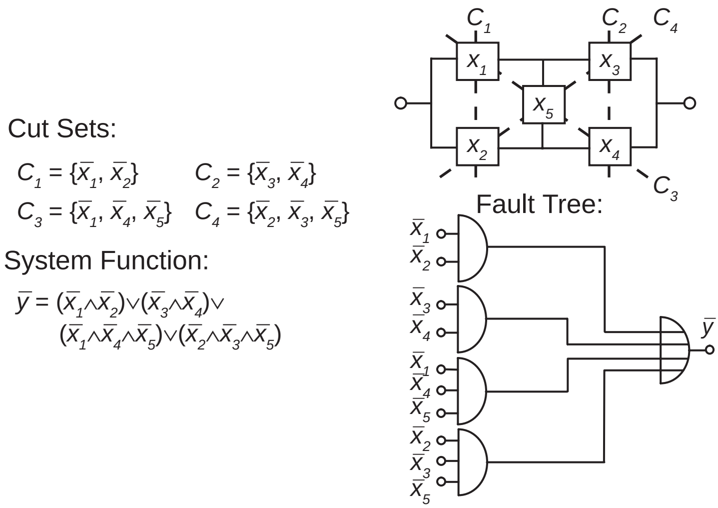


Figure 5.10. Method of minimal cut sets

The system failed if all components in one of the minimal cuts failed. Therewith, the Boolean function for the system failure can be determined as

$$C_1 = \{\bar{x}_1, \bar{x}_2\},\ C_2 = \{\bar{x}_3, \bar{x}_4\},$$
$$C_3 = \{\bar{x}_1, \bar{x}_4, \bar{x}_5\}, C_4 = \{\bar{x}_2, \bar{x}_3, \bar{x}_5\}, \tag{5.26}$$

$$\bar{y} = (\bar{x}_1 \wedge \bar{x}_2) \vee (\bar{x}_3 \wedge \bar{x}_4) \vee (\bar{x}_1 \wedge \bar{x}_4 \wedge \bar{x}_5) \vee (\bar{x}_2 \wedge \bar{x}_3 \wedge \bar{x}_5). \tag{5.27}$$

Method of Minimal Path Sets

In the method of minimal path sets all combinations of the components, whose operation ensures the function of the system, are determined by theoretical paths in the structure. All components are determined to be positive and are connected inside of the path sets by *and* operators and at the outside by *or* operators, resulting in a positive output – the system reliability, see Figure 5.11.

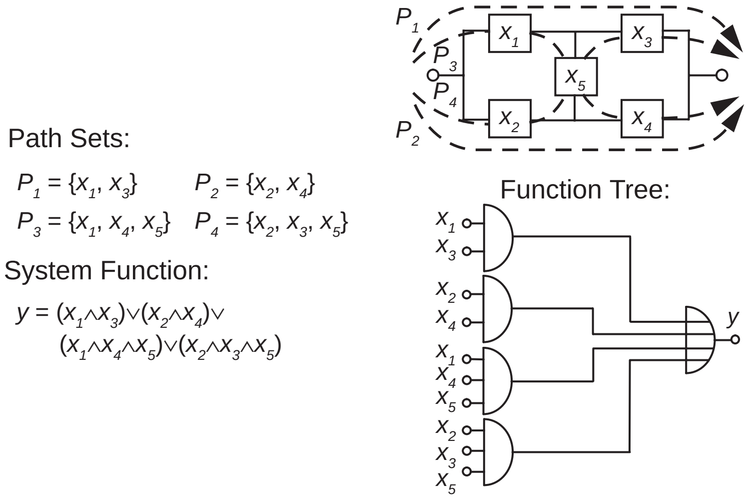


Figure 5.11. Method of minimal path sets

The system is considered to be operational if at least one path is operational. The Boolean function of the operability for the system can be determined as:

$$P_1 = \{x_1, x_3,\} \, P_2 = \{x_2, x_4\} \, P_3 = \{x_1, x_4, x_5\} \, P_4 = \{x_2, x_3, x_5\}, \tag{5.28}$$

$$y = (x_1 \wedge x_3) \vee (x_2 \wedge x_4) \vee (x_1 \wedge x_4 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_5). \tag{5.29}$$

The transition to the probabilities in these two methods can be carried out with the help of the Poincaré algorithm or the top down algorithm, which are described more detailed in [5.9].

Method of the relevant system component (Separation)

Since the system component x_5 is operational in both directions, this component plays a key role in this bridge configuration and can thus be separated, see Figure 5.12.

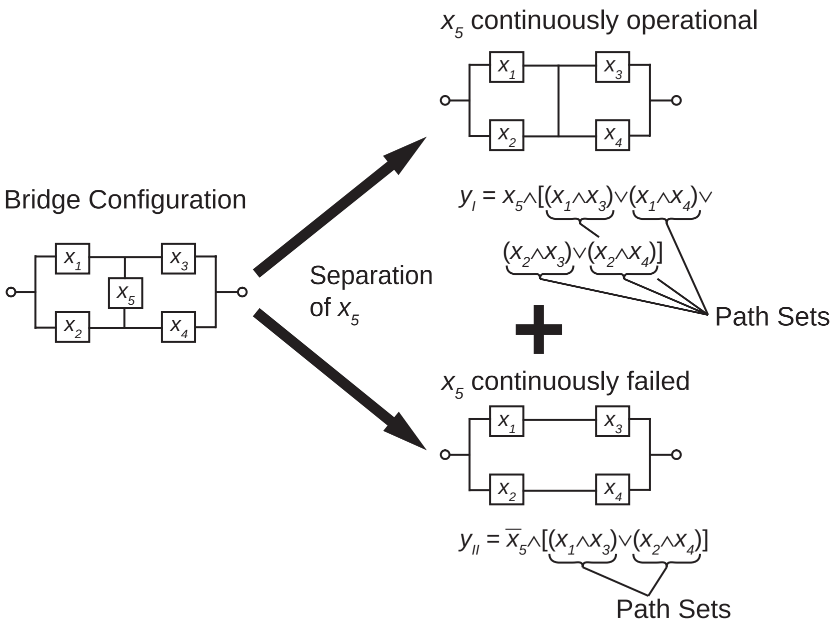


Figure 5.12. Method of the relevant system component (Separation)

For the component x_5, the two states "continuously operational" and "continuously failed" are regarded separately and afterwards reconnected with each other. In the first case, where x_5 is operational at all times, the component x_5 is determined to be positive, connected with the single success paths and connected by AND operators:

$$y_I = x_5 \wedge [(x_1 \wedge x_3) \vee (x_1 \wedge x_4) \vee (x_2 \wedge x_3) \vee (x_2 \wedge x_4)]. \tag{5.30}$$

Using the distributive law

$$y_I = x_5 \wedge [(x_1 \wedge (x_3 \vee x_4)) \vee (x_2 \wedge (x_3 \vee x_4))] \tag{5.31}$$

and the commutative law

$$y_I = x_5 \wedge [((x_3 \vee x_4) \wedge x_1) \vee ((x_3 \vee x_4) \wedge x_2)] \tag{5.32}$$

as well as substituting $(x_3 \vee x_4)$ with x^*

$$y_I = x_5 \wedge [(x^* \wedge x_1) \vee (x^* \wedge x_2)] \tag{5.33}$$

and reapplying the distributive law, one yields

$$\begin{aligned} y_I &= x_5 \wedge [x^* \wedge (x_1 \vee x_2)] \\ &= x_5 \wedge [(x_3 \vee x_4) \wedge (x_1 \vee x_2)]. \end{aligned} \tag{5.34}$$

With the transition to the probabilities the first case reliability results to

$$R_I = R_5 \cdot \left[\left(1-\left(1-R_3\right)\cdot\left(1-R_4\right)\right)\cdot\left(1-\left(1-R_1\right)\cdot\left(1-R_2\right)\right)\right]. \tag{5.35}$$

The same will be done for the second case in which x_5 fails at all times. In doing so, it is possible to jump directly to the reliability:

$$y_{II} = \overline{x_5} \wedge \left[\left(x_1 \wedge x_3\right) \vee \left(x_2 \wedge x_4\right)\right], \tag{5.36}$$

$$R_{II} = \left(1-R_5\right)\cdot\left[\left(1-\left(1-R_1R_3\right)\cdot\left(1-R_2R_4\right)\right)\right]. \tag{5.37}$$

The two probabilities can be connected because of their independence upon the events and the theorem of total probability [5.2], so that the following system reliability is yielded:

$$\begin{aligned} y = {} & x_5 \wedge \left[\left(x_1 \wedge x_3\right) \vee \left(x_1 \wedge x_4\right) \vee \left(x_2 \wedge x_3\right) \vee \left(x_2 \wedge x_4\right)\right] \vee \\ & \overline{x_5} \wedge \left[\left(x_1 \wedge x_3\right) \vee \left(x_2 \wedge x_4\right)\right], \end{aligned} \tag{5.38}$$

$$\begin{aligned} R = {} & R_5 \cdot \left[\left(1-\left(1-R_3\right)\cdot\left(1-R_4\right)\right)\cdot\left(1-\left(1-R_1\right)\cdot\left(1-R_2\right)\right)\right] \\ & + \left(1-R_5\right)\cdot\left[\left(1-\left(1-R_1R_3\right)\cdot\left(1-R_2R_4\right)\right)\right]. \end{aligned} \tag{5.39}$$

5.4 Reliability Graph

A further possibility to describe systems clearly is the reliability graph. Reliability graphs are used in particular to describe reliabilities of networks [5.7]. They consist of knots and (connection) edges. The edges are distinguished by component edges and ∞ edges. One component is imaged by a maximum of one component edge. Thus, repeated edges are not allowed. The failure of components is illustrated by the interruption of the edge. The ∞ edges and the knots do not fail. The modelled system is regarded as operational as long as at least one path with non failed edges leads from one “source knot” to a “drain knot”, see Figure 5.13.

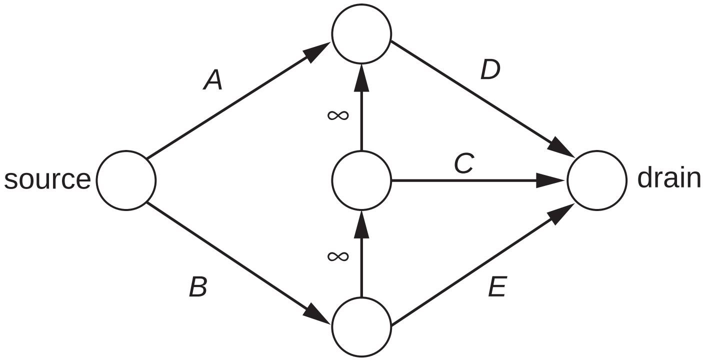


Figure 5.13. Example of a reliability graph

5.5 Examples

5.5.1 Tooth Flank Crack

The first example shows a fault tree of a tooth flank crack caused by a material failure. In the process of the analysis the causes for the occurring tooth flank crack are determined step by step. First of all, it is considered, that the gear was used under incorrect operational conditions and the crack was caused, for example, by an incorrect high operational stress. Nevertheless, a further cause for the failure behaviour could also be a damaged tooth flank, see Figure 5.14. Therefore, there are three further failure causes on one system level lower: a production failure while producing the tooth flank, a constructive failure or a material failure. If the analysis for the material failure is continued, it is possible that in principle, the wrong material was chosen, which means that it is not suitable for this kind of usage. Another possibility is an incorrect structure out of fitting material, which also represents an incorrect material.

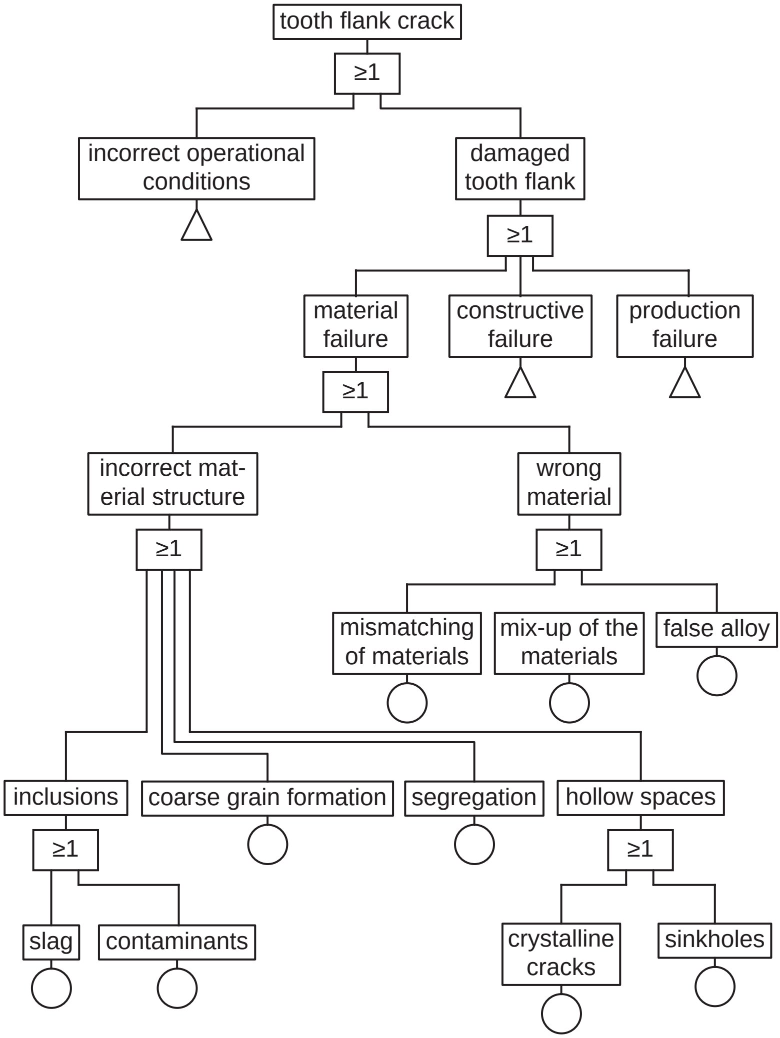


Figure 5.14. Fault tree of a tooth flank crack for a material failure

Another cause related to the usage of wrong material besides a false alloy could also be a mix-up of the materials or a mismatching of materials. Each of these three causes stands for a standard input in the fault tree. On the other hand, an incorrect material structure could be caused by segregation, coarse grain formation, inclusions or hollow spaces. Inclusions can be caused either by slag or by contaminants. Hollow spaces can be caused by sinkholes or by the formation of crystalline cracks. Because each of these points is a standard input to the fault tree, the branch of the incorrect material is completed and the fault tree analysis can be continued for further branches.

Whereas in the previous example a material failure was assumed for the tooth flank crack, the fault tree analysis shown in Figure 5.15 is continued under the assumption of a constructive failure.

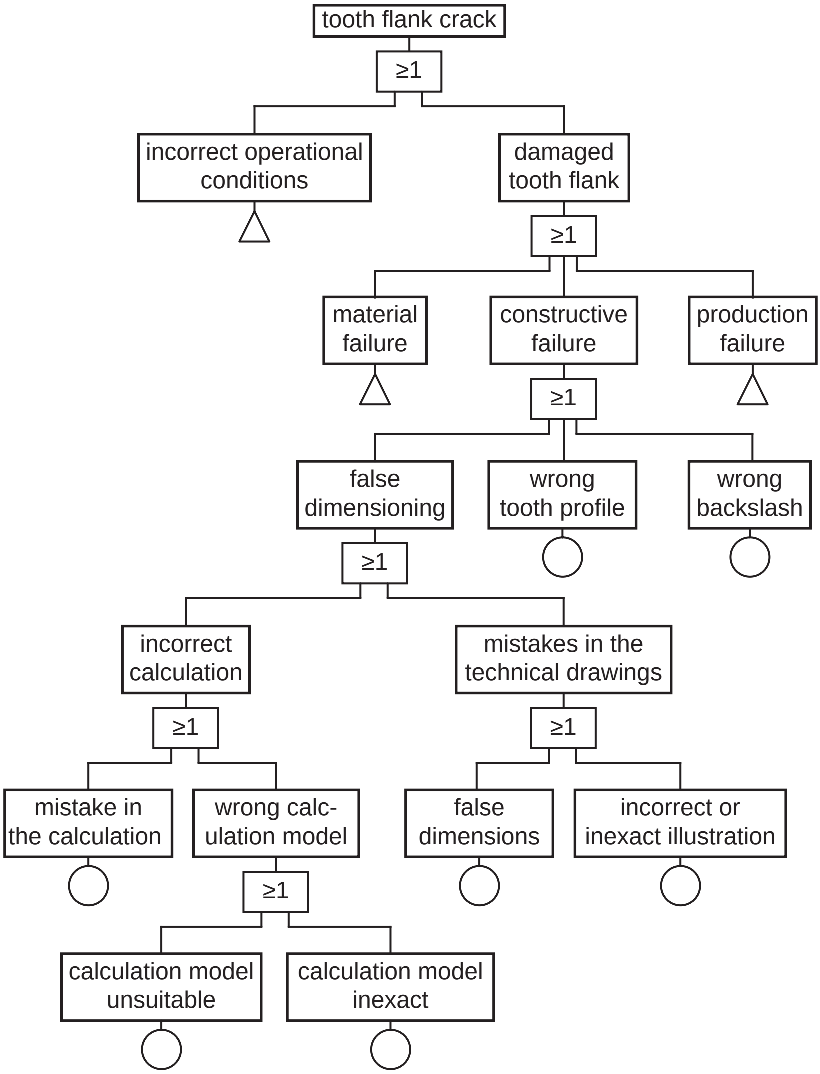


Figure 5.15. Fault tree for a tooth flank crack with a constructive failure

Here, the causes that could lead to the tooth flank crack caused by a constructive failure are determined: false dimensioning, wrong tooth profile and/or wrong backlash. The two latter failures are standard inputs to the fault tree and therefore do not have to be further considered, so that the upcoming considerations only have to be made for false dimensioning. Reasons for false dimensioning of the teeth could be incorrect calculation or mistakes in the technical drawings, which are the basics for the production of the tooth flank. Failures in the technical drawings could be false dimensions or an incorrect and/or an inexact illustration. If the incorrect dimensioning is caused by incorrect calculation, it is either possible that this is caused by a mistake in the calculation or by a wrong calculation model was used for the calculation. The calculation model used may be inexact or in principle unsuitable for this kind of calculation.

Since all inputs are now standard inputs to the fault tree, the determination of the constructive failures is finished. The fault tree analysis of the tooth flank crack is continued with either a production failure, which could lead to an incorrect tooth flank, or the determination of incorrect operational stresses.

5.5.2 Fault Tree Analysis of a Radial Seal Ring

This example [5.10] refers in particular to the design phase. The design of the radial seal ring is a packing box, see Figure 5.16, and is used for the sealing of air leakage of the cooling air under high pressure of a big generator coupled with a bulb turbine.

The pressure difference is 1.5 bar and the dimensions are substantial. The packing box runs against a “thermal protection core”. The assembly is analyzed in regards to possible failure behaviours.

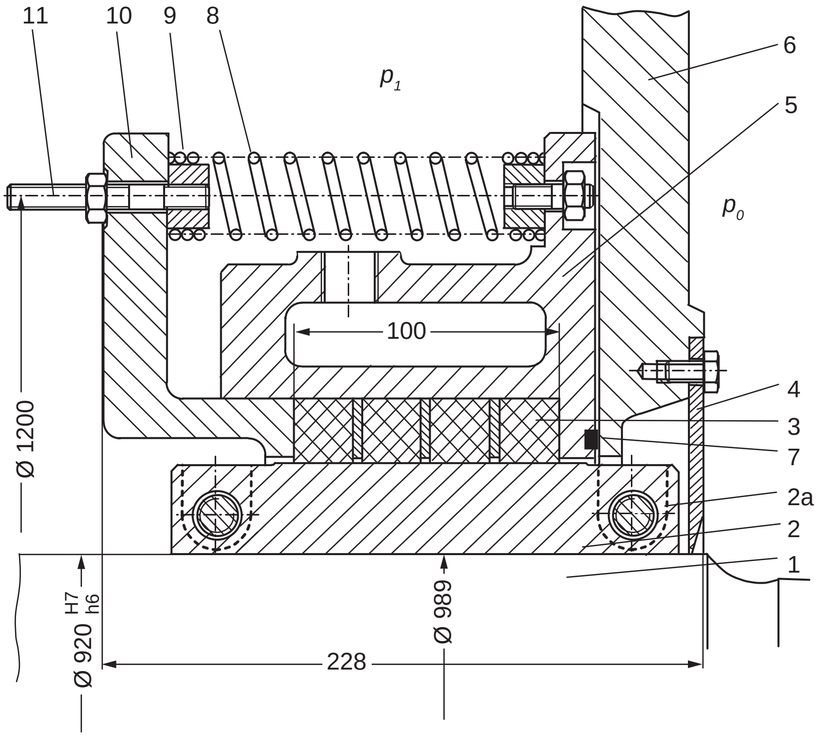


Figure 5.16. Radial seal ring of a big generator for locking cooling air [5.10]

The overall function is the "locking of cooling air". To start the analysis it is useful to determine the sub-functions, which are fulfilled by the individual components. If, for example, no function structure is given, then this can be done best with the help of a table, see Table 5.5. For the function "locking" the following sub-functions are essential:

- to apply contact force,
- to seal sliding and
- to dissipate frictional heat.

Table 5.5. Analysis of the components according to Figure 5.16 to identify functions assumed [5.10]

Nr.	part	function
1	shaft	transmit torque, carry core, dissipate frictional heat
2, 2a	core (2 parts, screwed)	provide contact and sealing surface, protect shaft, transmit frictional heat
3	package ring	seal sliding medium, accept contact force and apply sealing pressure
4	wiper ring	protect from splash oil
5	packing box	accept package ring, accept and transmit contact force
6	base frame	carry part 4 and 5
7	o-seal	seal between p_1 and p_0
8	tension spring	provide contact force
9	spring seat	transmit spring force
10	tension ring	transmit contact force, carry tension spring
11	screw	pre-stress springs adjustable

In the following process of the analysis, the sub-functions are negated and at the same time, possible causes for the failure behaviour are determined, see Figure 5.17.

The result of the fault tree primarily points the failure behaviour of the thermal protection core 2, due to heat instable behaviour: the arising friction heat on the hydroplane can practically only flow to the shaft by using the core. Thus, the core becomes warm and expands. However, if the warming continues, the friction increases and starts to lift off of the shaft. This leads to an additional leakage and damages the shaft surface by incorrect sliding of the core on the shaft. This configuration is unsuitable and requires principal constructive improvements: Either the package box is blocked with the shaft and rotates together with the shaft and the thermal protection core is left out (heat dissipation by housing 5) or usage of a radial seal ring with radial sealing surface. Further corrective actions are necessary if the configuration is retained:

- The support of housing against the base frame is unsuitable, because with pre-stressed package, the housing can be twisted with the shaft. If the sealing 7 is on the inside, the provided contact force of the pressure difference is too low to accept the friction torque by force transmission by friction. Remedy: place seal 7 at the outer diameter of housing

5. Even better would be a secure form fit for the transmission of the friction torque.

- In the illustrated position the spring 8 cannot be retightened. Remedy: provide sufficient instep way.
- Due to operational safety and to simplify the configuration, it is more favourable to use a pressure spring than a tension spring.

Basically, it is possible that the improved constructive design requires a look at other fields such as production, assembling and operation (usage and maintenance) besides the constructive action. If necessary, the corresponding test protocols must be requested, see Figure 5.17.

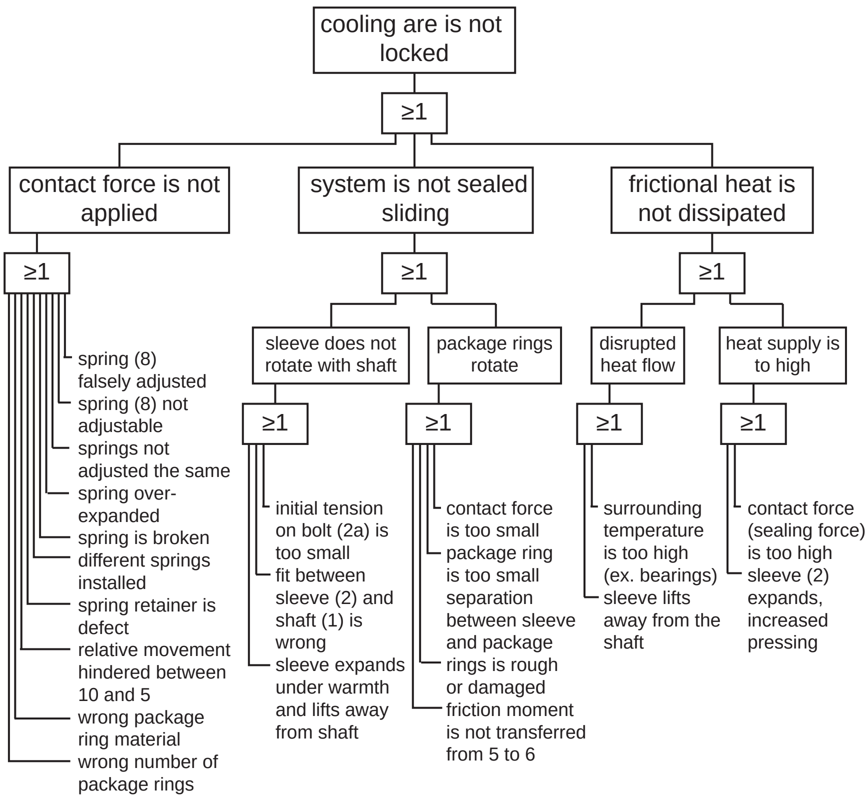


Figure 5.17. Fault tree of a radial seal ring [5.10]

Recapitulating, the following procedure for the search of failures and disturbance can be given:

- Identify the functions and negated them.
- Search for the reasons for the non-fulfilment of the functions (out of the unclear function structure, non-ideal active principle, non-ideal configuration).

5.6 Exercise Problems to the Fault Tree Analysis

Problem 5.1

Calculate the reliability of the given function tree, see Figure 5.18. Also, create the fault tree.

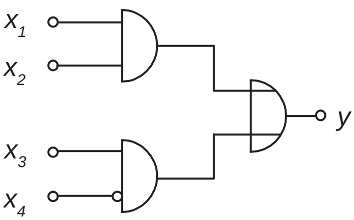


Figure 5.18. Function tree to Problem 5.1

Exercise 5.2

Create the fault and function trees for the given block diagrams, see Figure 5.19. Determine the failure probability F_s of the given systems as a function of the respective component reliabilities R_i.

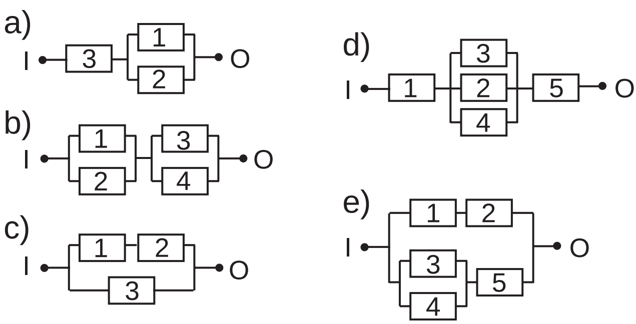


Figure 5.19. Block diagrams to Problem 5.2

Problem 5.3

The principle drawing of a jumbo jet is given below, see Figure 5.20. The system "undercarriage" fails, if the undercarriage at the front OR at the back on the right side AND on the back left side OR the wing undercarriage on the right OR on the left side fails. An undercarriage fails, if not a single wheel is available.

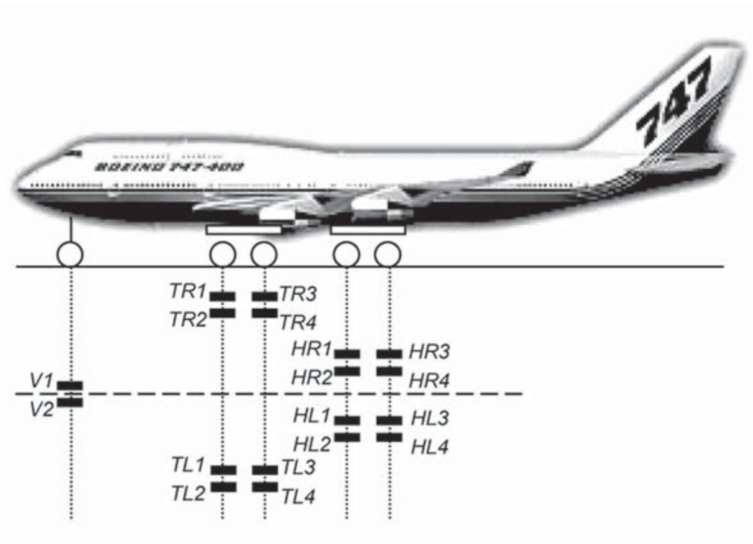


Figure 5.20. Drawing of the undercarriage of a Boeing 747

a) Create the fault tree.
b) Determine the Boolean system function for the failure of the system "undercarriage"
c) Determine the system equation for the failure probability F_S
d) Determine the Boolean system function for the operability of the system "undercarriage".
e) Determine the system equation for the reliability R_S and create the corresponding block diagram.

Problem 5.4

In order to ensure the reliability of security device a system is built up with redundancies.
It consists in three generators (in the block diagram termed with x_1, x_2, x_3) and two engines (x_4, x_5), see figure.

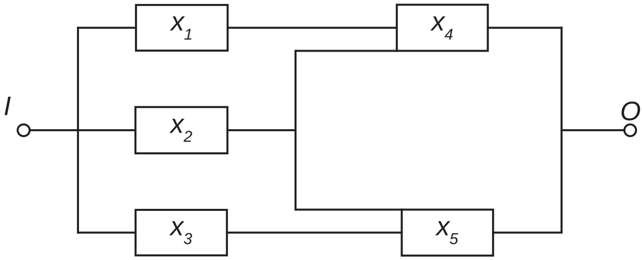


Figure 5.21. Block diagram of a security device

Determine the Reliability of the security device by separation of x_2.

Problem 5.5

A part of a fault tree for an ABS control unit is given.

a) Determine the Boolean system function for the failure of the control unit.
b) Calculate the failure probability of the system
c) Determine the system function fort he operability of the control unit.
d) Create the block diagram.

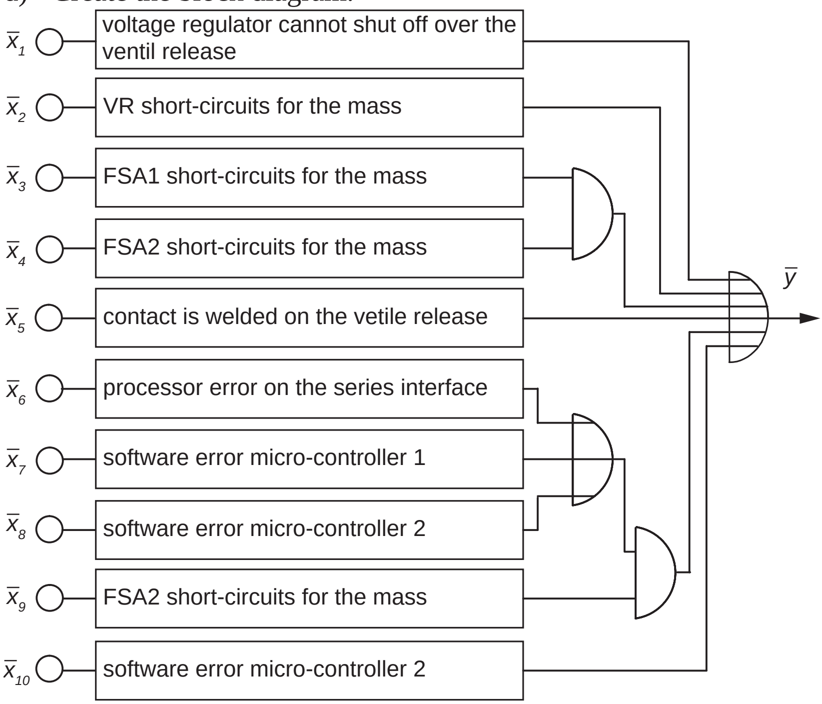


Figure 5.22. Part of a fault tree of an ABS control unit

References

[5.1] Barlow R E, Fussel J B, Singpurwalla N D (1975) Reliability and Fault Tree Analysis. Society for Industrial and Applied Mathematics, Philadelphia
[5.2] Bronstein I N, Semendjajew K A (2000) Taschenbuch der Mathematik – 5., überarb. und erw. Aufl. Thun, Frankfurt am Main
[5.3] Deutsches Institut für Normung (1981) DIN 25424 Teil 1-2 Fehlerbaumanalyse. Beuth, Berlin
[5.4] Ericson C (1999) Fault Tree Analysis – A History from the Proceeding of the 17th International System Safety Conference
[5.5] Gaede K W (1977) Zuverlässigkeit Mathematischer Modelle. Hanser, München Wien
[5.6] Grimms T (2001) Grundlagen Qualitäts- und Risikomanagement. Vieweg
[5.7] Malhotra M, Trivedi K S (1994) Power-Hierarchy of Dependability Model Types. In: IEEE Transactions on Reliability, Vol. 43, No. 3, September 1994, pp 493-502
[5.8] Masing W (1994) Deutsche Gesellschaft für Qualität DGQ-Schrift 11-19 Einführung in die Qualitätslehre. DGQ, Frankfurt am Main
[5.9] Meyna A (1994) Zuverlässigkeitsbewertung zukunftsorientierter Technologien. Vieweg, Wiesbaden
[5.10] Pahl G, Beitz W (2003) Konstruktionslehre: Grundlagen erfolgreicher Produktentwicklung; Methoden und Anwendung. Springer, Heidelberg Berlin
[5.11] Schlick G H (2001) Sicherheit, Zuverlässigkeit und Verfügbarkeit von Maschinen, Geräten und Anlagen mit Ventilen. Expert Verlag.
[5.12] Verein Deutscher Ingenieure (1998) VDI 4008 Blatt 2 Boolesches Model. VDI, Düsseldorf
[5.13] Vesely W E, Goldberg F F, Roberts N H, Haasl D F (1981) Fault tree handbook. United States Nuclear Regulatory Commission, Washington DC

6 Assessment of Lifetime Tests and Failure Statistics

In this chapter, the planning of lifetime tests and various assessment strategies will be dealt. Here, the most important fundamental principles for such procedures will be introduced.

The emphasis of this chapter is the assessment of failure times so that the failure behaviour of components and systems can be described. For this, the unknown distribution parameters are determined through various graphical and analytical methods. The Weibull distribution will be used as it is the most widely adopted in the area of mechanical engineering.

The most important "confidence levels" for the assessment will be described in detail. This is necessary since normally it is not possible to gather the lifetimes of several different parts (statistically spoken: the population or universe). Generally, it is only possible to determine the failure times of a small number of components. In statistics, this limited number of components taken as a test specimen signifies the population, see Figure 6.1. Therefore, only a statement can be made from the assessment concerning the test specimen. However, a statement concerning the entire population is desired! The failure behaviour resulting from an assessment of the test specimen can sometimes strongly deviate from the actual failure behaviour of the population itself, especially if only a few components have been tested. Here, the statistics offers a further help through the "confidence levels", with which it is possible to specify the veracity of the test specimen results. Thus, it is possible to estimate the failure behaviour of the population.

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_6, © Springer-Verlag Berlin Heidelberg 2008

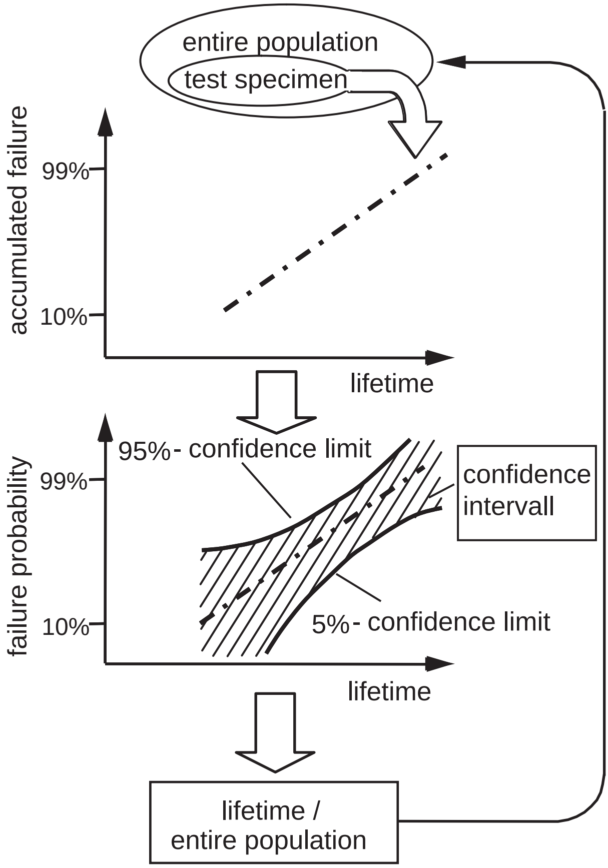


Figure 6.1. Conclusion drawn from the test specimen concerning the entire population

6.1 Planning Lifetime Tests

Planning lifetime tests can be divided into experimental-technical measurement planning and statistical test planning.

Experimental-Technical Measurement Planning

Here, the common fundamental principles for correct execution of an experiment apply. The most important of these principles are as listed:

- The boundary conditions and limits must be exactly defined and kept. For lifetime tests this is especially important for the load spectrum.

- The technical measurement process for the registration and control of the boundary conditions must be established along with their accuracy. Here, depending upon the resources, more information is acquired at the test stand than actually needed.
- If longer testing times are expected, then the use of automated and/or computer controlled measured value gathering and control equipment should be strived for.
- For a determination of the lifetime, the exact specification of a limit value is necessary, at which the nominal function is no longer fulfilled. If the damage is a continuously changing value, as for example a leak volume for a seal.
- The control equipment must be built up in such a way that the primary failure cause can even be determined after the failure effect. This is important since each failure mode is assigned its own characteristic reliability parameters.

Statistical Test Planning

In statistical test planning the first step involves determining the size of the inspection lot. The inspection lot size is in close connection with the confidence levels and the statistical spread of the measured values, see Sections 6.2 and 6.3.2. If fewer components are tested, then the result of the statistical assessment becomes more uncertain. For an accurate result it is necessary that a sufficient quantity of components is tested. This can increase the time and effort involved in a test immensely.

In statistical test planning, it must also be determined how the choice of components to be tested should be made – test specimen extraction. The test specimen should represent an actual random test specimen, which means that the components to be tested are chosen at random. Only then is the fundamental condition for a representative test specimen fulfilled.

Another important point in statistical test planning involves establishing a suitable test strategy. Possible strategies include:

- complete tests,
- incomplete (censored) tests and
- strategies for shortening test times.

The best statistical option is a complete test, in which all components of a test specimen are subjected to a lifetime test. This means that the test is run until the last element has failed. Thus, failure times for all elements are available for further assessment.

In order to reduce the time and effort involved in a test stand, it is reasonable to carry out incomplete tests, also known as censored tests. Here the test only is carried out until a certain predetermined lifetime or until a certain number of failed components has been reached. Such tests are not as meaningful as complete tests, but are often connected with a considerably lower time and effort at the test stand.

Another option for a considerably shorter test time is the Sudden Death Test and tests with an increased load. A detailed description of procedures for test planning can be found in Chapter 8.

In the following text, the fundamental assessment of complete tests will be discussed.

6.2 Order Statistics and their Distributions

The assessment of failure times, which will be dealt with in the following sections, has to do with the distribution of order statistics. In order to understand the procedure of an assessment, it is very necessary to obtain a basic knowledge of the origin and meaning of order statistic distributions. However, the derivation of order statistic distributions is quite complicated concerning probability theory. Thus, this section is for those who are interested in understanding the exact relationships of these distributions. However, this section can be skipped if the reader is only interested in the assessment of failure times.

Determining F(t) of the Failed Components

The failure times of components or systems can be acquired out of lifetime tests or damage statistics. For an assessment with a probability graph, only the abscissa values of the individual failures with these failure times are available but not the ordinate values. Therefore, each failure must be assigned a certain failure probability $F(t)$. The following example should illustrate this more clearly:

A test specimen was tested with $n = 30$ components:

component 1	component 2	. . . .	component 30

The test resulted in 30 different lifetime values t_i, which were ordered according to their respective value:

$$t_1, t_2, t_3, \ldots t_{29}, t_{30}; \quad t_i < t_{i+1};$$

for example:

t_1 = 100,000 load cycles, … t_5 = 400,000 load cycles, … t_{30} = 3,000,000 load cycles.

These ordered values are called order statistics. The index corresponds to the rank.

After the failure of the first order statistic, 1/30 of the test specimen has failed, with the second order statistic, 2/30, etc. From this point of view it is possible to assign the first order statistic a failure probability of $F(t)$ = 1/30 = 3.3%, the second order statistic, $F(t)$ = 6.7%, etc. With this method, the failure behaviour of the tested components can be represented in the form of a summation frequency or of an empirical distribution function, see Figure 2.10.

Here, it should be noted that the failure times of only *one* test specimen are taken into consideration. Of course, another test specimen of the same size returns somewhat varying result values,

for example: t_1 = 120,000 load cycles, ... t_5 = 350,000 load cycles, ...
t_{30} = 2,500,000 load cycles.

The matrix structure in Figure 6.2 is the result of m test specimens.

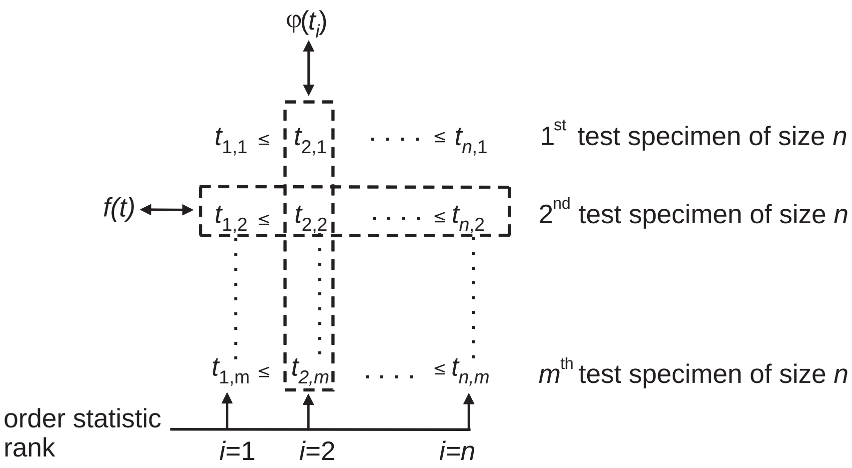


Figure 6.2. Order statistics for m test specimens of the size n

The failure time of an order statistic (any one column in Figure 6.2) varies within a certain range. An order statistic can thus be understood as a random variable, to which a distribution can be assigned. As opposed to the lifetime distributions, the density function for order statistics is signified by $\varphi(t_i)$.

The mathematical derivation of the order statistic distribution leads to a multinomial distribution (trinomial distribution), which represents an enhanced binomial distribution [6.2, 6.6, 6.7, 6.8]. The order statistic distribution can be theoretically developed similar to the development of the binomial distribution. The initial step in the derivation is the population of components with the known failure functions $f(t)$ or $F(t)$. A test specimen made up of n components is chosen out of this population. The order statistic i is observed, which lies in section 2 at the time t_i in Figure 6.3. The probability that the failure time falls in section 2 for *one* component is $f(t_i)\mathrm{d}t$, $F(t_i\text{-}0{,}5\mathrm{d}t)$ for section 1 and $(1\text{-}F(t_i\text{+}0{,}5\mathrm{d}t))$ for section 3. After all test specimen trials have run, the order statistic i lies in section 2, while (i-1) failures can be found in section 1 and (n-i) in section 3. Thus, for all test specimen trials of this one test specimen, the probability that a certain component fails during section 2 in Figure 6.3 is:

$$\varphi(t_i) = F(t_i)^{i-1} \cdot f(t_i) \cdot [1 - F(t_i)]^{n-i} \,. \tag{6.1}$$

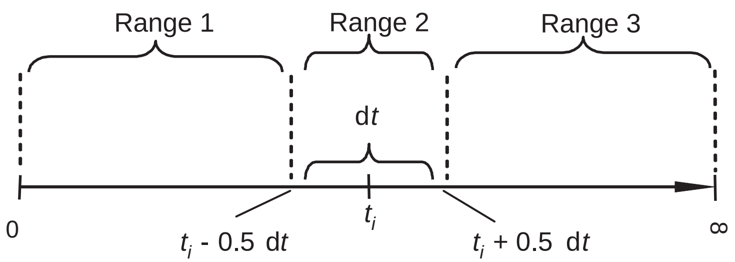


Figure 6.3. Division of the time axis into three sections for the derivation of the multinomial distribution

The limit of Equation (6.1) for $dt \rightarrow 0$ gives the density function of the order statistics. Since it is possible that each component can land in any of the three sections, several combinations must be taken into consideration.

$$\varphi(t_i) = \frac{n!}{(i-1)!\; 1!\; (n-i)!} F(t_i)^{i-1} \cdot f(t_i) \cdot [1 - F(t_i)]^{n-i} \tag{6.2}$$

As already mentioned, $f(t_i)$ and $F(t_i)$ are the density function and failure probability of the original distribution at the point t_i respectively.

Figure 6.4 shows the graphical representation of Equation (6.2) on behalf of an example. A two parametric Weibull distribution with the parameters $b = 1.5$ and $T = 1$ was used as the original distribution. In Figure 6.4 it is observed that the order statistics' failure times deviate within a certain time period with various probabilities. For example, the 5$^{\text{th}}$ order statistic ranges between 0.1 and 0.7, where the failure time 0.3 (median) occurs the most. The extreme values 0.1 and 0.7, however, only occur with a relatively low probability. Since the Weibull distribution with $b = 1.5$

assumes lower values with increasing time, the density function $\varphi(t_i)$ becomes less steep with an increasing rank.

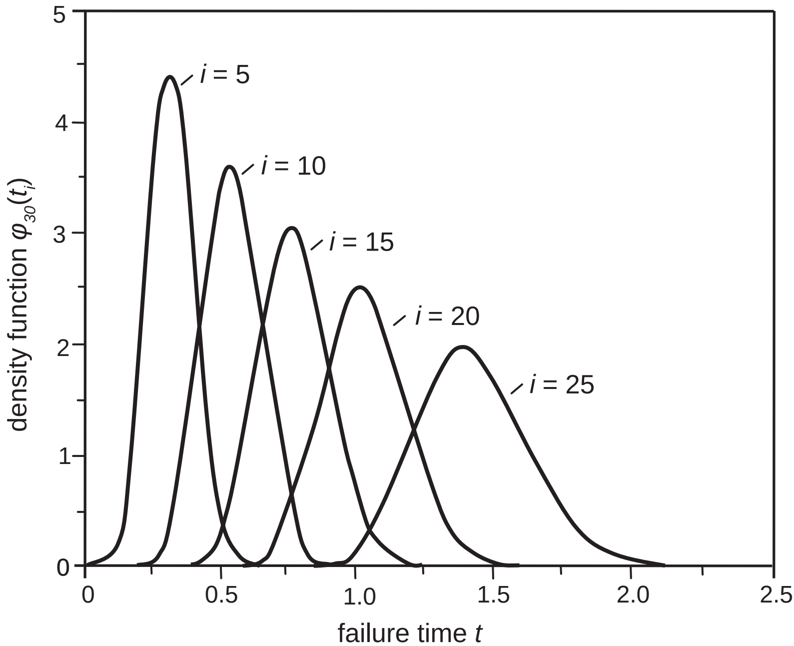


Figure 6.4. Density functions for order statistic i in a test specimen of the size n = 30 (original distribution: two parametric Weibull distribution with b = 1.5 and T = 1)

For the previous considerations, the distribution of the failure times must be known. Normally, however, this is not the case in most assessments, but rather the failure functions of the failure times must first be determined. The desired failure probabilities for the failure times assume values between 0 and 1. None of the order statistics should be favoured so that they are uniformly assigned failure probabilities from 0 to 1. Carrying out a transformation has been efficacious:

$$F(t_i) = F(u) = u, \qquad 0 < u < 1, \tag{6.3}$$

$$f(u) = 1, \qquad 0 < u < 1. \tag{6.4}$$

The Equations (6.3) and (6.4) describe a rectangular distribution, which fulfils the conditions listed: The distribution function is defined within the range of 0…1 and the order statistics can be regarded equally in the constant density function. Therefore, the order statistics are distributed equally within the interval 0…1, By substituting Equations (6.3) and (6.4) into Equation (6.2), the desired density function for the failure probabilities of orders is acquired:

$$\varphi_n(u) = \frac{n!}{(i-1)!\,(n-i)!} \cdot u^{i-1} \cdot (1-u)^{n-i} \,. \tag{6.5}$$

The Equation (6.5) corresponds to a beta distribution with the beta variable u and parameters a and b, where $a = i$ and $b = n$-i+1[6.6, 6.7].

The expressions in Equation (6.5) can be seen graphically in Figure 6.5. Figure 6.5 shows the density function with the beta variable u in a Weibull probability chart for the case represented in Figure 6.4. Because of Equation (6.3), the beta variable u can be interpreted as the failure probability $F(t_i)$. Figure 6.5 shows very clearly that the failure probability $F(t_i)$ assigned to the order statistic i deviates within a certain range with this range's given density. The 25th order statistic, for example, must be assigned a failure probability of about 60% to 98%. In most cases, the mode, 75%, would be an adequate value, whereas the extreme values would only be suitable for the 25th failure time in very seldom cases.

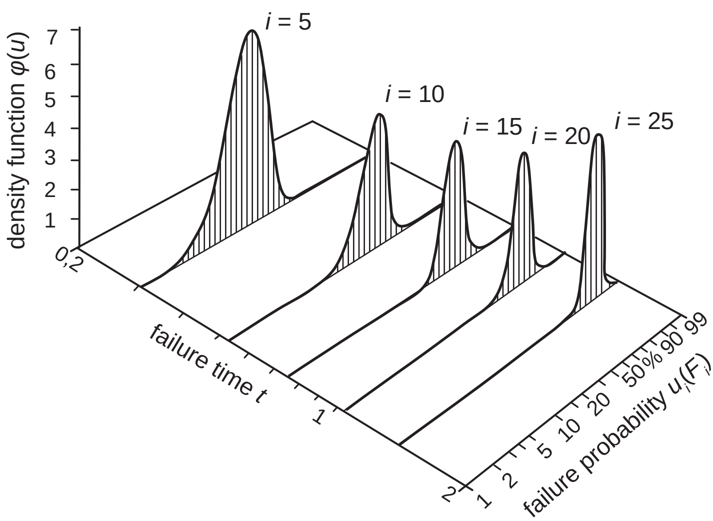


Figure 6.5. Density functions with the beta variable u for the case shown in Figure 6.4

For the analysis of failure times it is often attempted to assign a failure probability to each failure time and then to draw a straight line through the coordinates entered into the Weibull probability chart. Thus, it is necessary to choose the most adequate value from the range of dispersion for the failure probability. One of the three averages: mean, median or mode, prove to be suitable estimate values. The value of these averages can be determined from the density function φ(u) or the beta distribution:

mean: $$u_m = \frac{i}{n+1}\,; \tag{6.6}$$

median: $$u_{median} \approx \frac{i-0.3}{n+0.4}\,; \tag{6.7}$$

mode: $$u_{mode} = \frac{i-1}{n-1}\,. \tag{6.8}$$

The median possesses no close-ended solution. Thus, Equation (6.7) is an approximation. More exact values for the median can be found in the appendix in Table A.2.

Now, the question is which of the three means should be taken as an estimate for the failure probability $F(t_i)$. However, on closer examination it can be discovered that none of the three values has an advantage in comparison to the others. The values do not differ substantially for large values of n, nor for order statistics i next to 1 or n.

In practice, the median u_{median} is used the most often. In some cases the simplest option, the mean u_m, is also used. Thus, failure probabilities can be assigned to the failure times t_i.

$$F(t_i) = \frac{i}{n+1} \quad \text{(mean) or} \tag{6.9}$$

$$F(t_i) \approx \frac{i-0.3}{n+0.4} \quad \text{(median)} \tag{6.10}$$

For example, for i = 25, the median is $F(t_{25})$ = 81.3%, see Figure 6.6. It can be expected in 50% of the cases that the actual assigned failure probability is larger than 81.3%. For all other cases the values lie under 81.3%.

The ideal case is present when a straight line can be drawn through the coordinates (t_i, $F(t_i)$), also known as the Weibull line, see Figure 6.6.

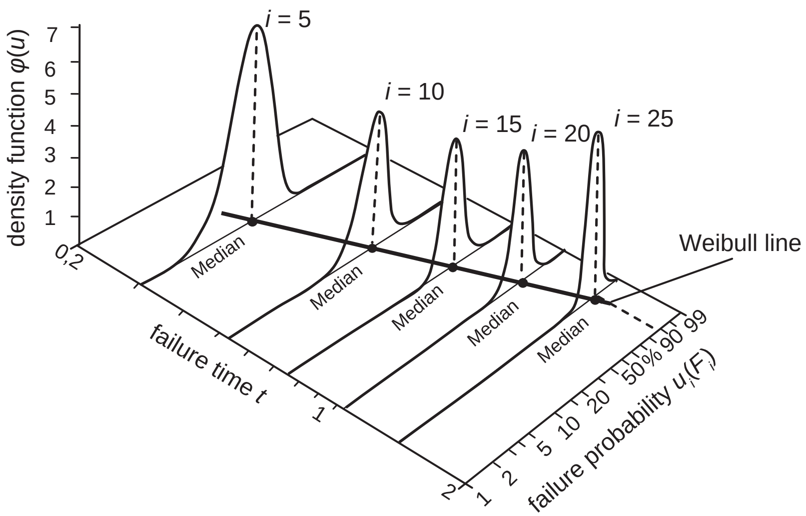


Figure 6.6. Density functions for order statistic failure probabilities with the median values and the Weibull line

Confidence Intervals

The assignment of failure times to an exact average is often not completely satisfying, since the failure probabilities of the order statistics can deviate within a certain range. The Weibull line is thus only *one* possibility to describe experimental results. If the median is used to determine $F(t_i)$, then the Weibull line represents the line for which 50% of the cases, the experimental results, lie above this line and 50% of the experimental results lie below the Weibull line. If it is necessary to know within which range the actual line can be expected to lie, that is, how much the Weibull line can be trusted, then it is necessary to determine the so-called confidence interval for the Weibull line. A confidence interval is characterized as the probability that a random value lies within a certain range. For example, a 90% confidence interval implies that in 90 out of 100 cases, the observed value falls within this certain interval. Figure 6.7 shows such 90% confidence intervals for the order statistics.

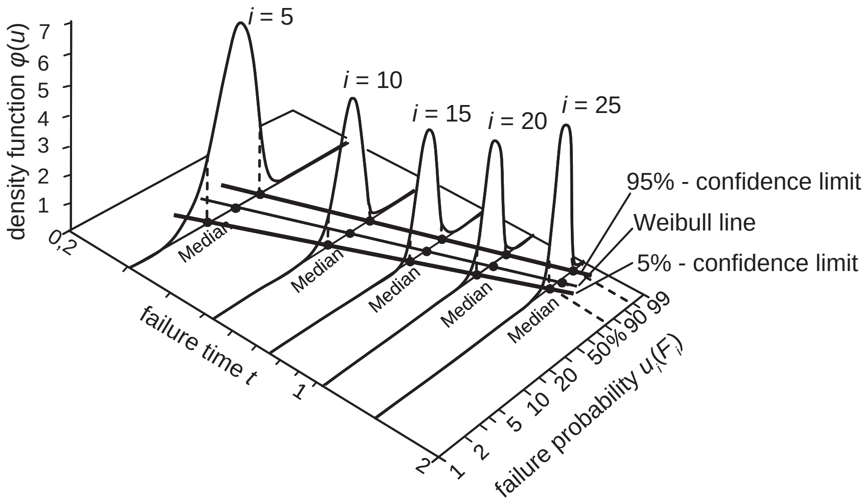


Figure 6.7. Density functions of the order statistic failure probabilities and their 90% confidence intervals

The limiting values of the confidence intervals can be calculated from the integral of the density function in Equation (6.5). An approximation equation for these limiting values can be found in [6.10]. Normally, tables are used in order to draw in the coordinates for the confidence limits in a graph. Tables A.1 and A.3 in the appendix give the values for the 5% and 95% confidence limits. The range between these confidence limits corresponds to a 90% confidence interval.

For the example shown in Figure 6.7, the limit failure probabilities are $F(t_{25})_{5\%} = 68.1\%$ and $F(t_{25})_{95\%} = 90.9\%$ for $i = 25$.

By joining the limit points of the various order statistics, the limit curves of the confidence interval over the entire failure time can be acquired, see Figure 6.7.

Figure 6.8 shows the representation in Weibull probability graph paper. The Weibull line of the median values and the confidence intervals can be interpreted as follows: Over an observation of several test specimens, the Weibull line drawn in Figure 6.8 is the most probable in the middle. The line in the middle represents the population mean – observed over several test specimens – thus 50% of the cases lie above and 50% lie below this line.

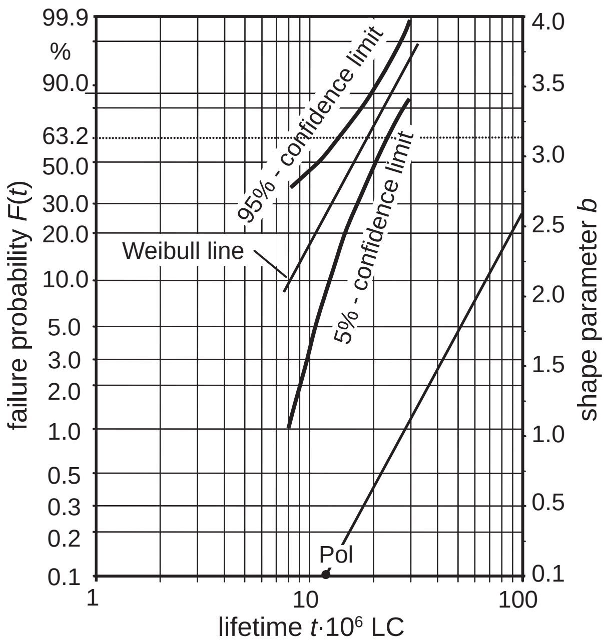


Figure 6.8. Weibull probability graph paper for the example in Figure 6.7

However, for a certain test specimen it is possible that the line assumes an arbitrary position within the confidence interval, see Figure 6.9. The probability that the failure results lie outside of the confidence intervals is only 10%. This means that the confidence interval is unreliable in only one out of ten cases.

Observing the confidence intervals is especially significant for small test specimen sizes, since then the confidence intervals can cover a very large range. With an increasing test specimen size n, the range covered by the confidence intervals becomes thinner and in certain circumstances can be completely neglected for $n > 50...100$.

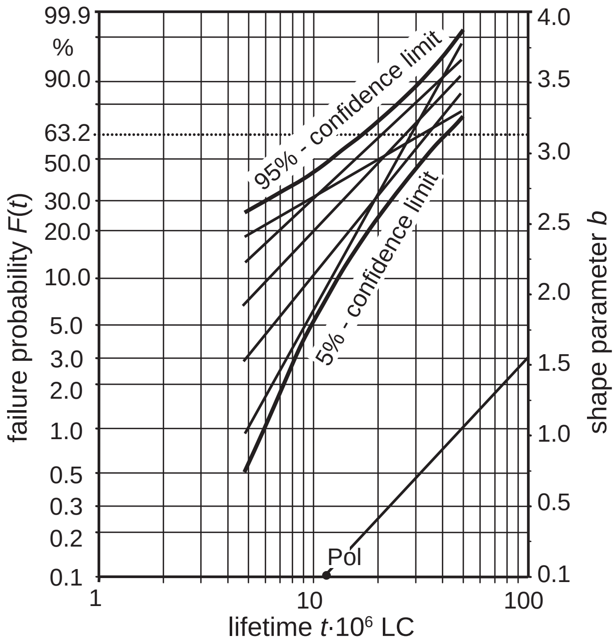


Figure 6.9. Weibull probability paper with Weibull lines for various test specimen sizes within the 90% confidence interval

6.3 Graphical Analysis of Failure Times

Explained parallel to an example the individual graphical analysis steps can be more clearly explained and the procedure can be transferred to a concrete practical example.

Gear wheel pitting or dimpling observed within the realms of a research project will serve as an example for this section [6.5]. A total of $n = 10$ gear wheels were tested under a stress of $\sigma_H = 1528\ \text{N/mm}^2$. The failure times for the gear wheels are given in one million load cycles in the following order of occurrence:

15.1; 12.2; 17.3; 14.3; 7.9; 18.2; 24.6; 13.5; 10.0; 30.5.

Knowledge concerning the order statistics and their distributions, see Section 6.2, can be useful for the graphical analysis and is helpful for an exact understanding of this analysis. The analysis steps given in the following sections however are setup and explained in such a way that the analysis can be carried out without exact knowledge of the order statistics.

6.3.1 Determination of the Weibull Lines (two parametric Weibull Distribution)

Step 1.1: Order the failure times according to increasing value

$$t_1 < t_2 \ldots < t_n \quad \text{or} \quad t_i < t_{i+1}\,; \qquad i = 1 \ldots n. \tag{6.11}$$

By ordering the failure times, an overview is won over the timely progression of the failure times. In addition, the ordered failure times are required in the next analysis step and are referred to as order statistics. Their index corresponds to their rank.

The following order statistics resulted from the test run (in one million load cycles):

$t_1 = 7.9$; $t_2 = 10.0$; $t_3 = 12.2$; $t_4 = 13.5$; $t_5 = 14.3$;

$t_6 = 15.1$; $t_7 = 17.3$; $t_8 = 18.2$; $t_9 = 24.6$; $t_{10} = 30.5$.

Step 1.2: Determine the failure probability $F(t_i)$ of the individual order statistics:

$$F(t_i) \approx \frac{i - 0.3}{n + 0.4}\,. \tag{6.12}$$

It is also possible to use the exacter values from Table A.2 (see appendix).

The order statistics t_i from step 1.1 are thus assigned to the failure probabilities $F(t_i)$. Since order statistics are seen as random variables, they possess a certain distribution. Equation (6.12) corresponds to the median of this distribution, see Section 6.2.

The calculated failure probabilities for this example are listed in the chart below:

$F(t_1) = 6.7\%$; $F(t_2) = 16.3\%$; $F(t_3) = 25.9\%$; $F(t_4) = 35.6\%$; $F(t_5) = 45.2\%$;

$F(t_6) = 54.8\%$; $F(t_7) = 64.4\%$; $F(t_8) = 74.1\%$; $F(t_9) = 83.7\%$; $F(t_{10}) = 93.3\%$.

Step 1.3: Enter the coordinates $(t_i, F(t_i))$ in the Weibull probability chart.

The failure time t_i corresponds to the x-coordinate value and the respective failure probability $F(t_i)$ corresponds to the y-coordinate value to be entered into the probability chart.

Figure 6.10 shows the coordinates for this example drawn in Weibull probability chart.

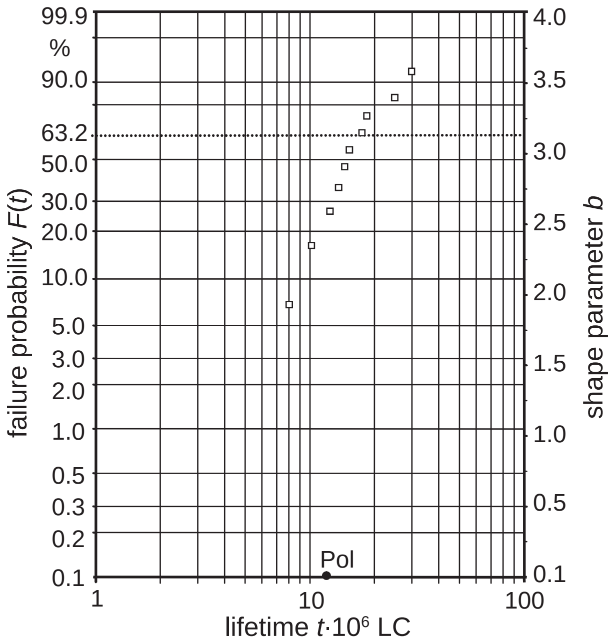


Figure 6.10. Failed gear wheels (coordinates (t_i, $F(t_i)$)) depicted in Weibull probability chart

Step 1.4: Approximately sketch the best fit straight line through the entered points and determine the Weibull parameters *T* and *b*.

Characteristic lifetime *T*: Intersection of the 63.2% line with the best fit straight line.

Shape parameter *b*: Shift the best fit straight line parallel through the pole P and read the shape parameter *b* from the right ordinate in the Weibull probability chart.

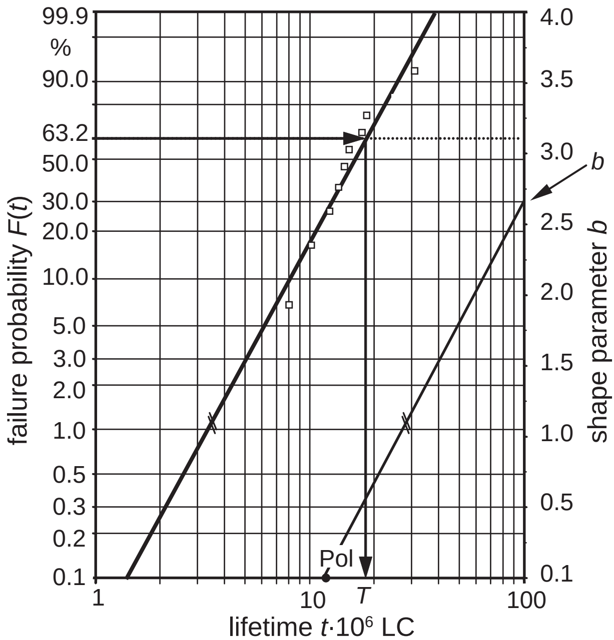


Figure 6.11. Best fit line and determination of the parameters *T* and *b*

The best fit line and the determination of the parameters *T* and *b* are depicted in Figure 6.11. The failure behaviour of the gear wheels can thus be most suitably described with the following Weibull distribution:

$$F(t) = 1 - e^{-\left(\frac{t}{18 \cdot 10^6 \text{ LW}}\right)^{2.7}} . \qquad (6.13)$$

In certain cases, the failure behaviour must be described with two or three approximation lines, see Figure 6.12. For such a mixed distribution a separate Weibull distribution must be determined for each line. The total failure behaviour is then given as a combination of the individual damage types [6.11].

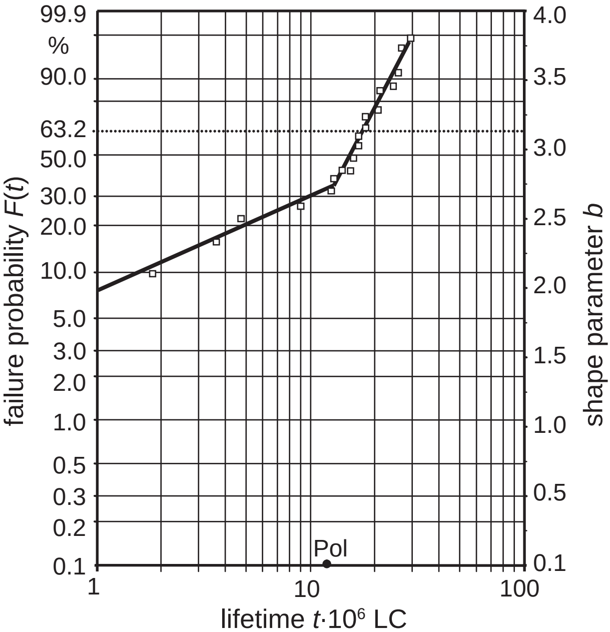


Figure 6.12. Mix distribution of a failed clutch (2 damage types: burnt clutch / worn down clutch)

6.3.2 Consideration of Confidence Intervals

If it is possible to test several test specimens of the same size from one machine element series, then the order statistic *i* will always be somewhat different. Thus, the order statistic must be seen as a random variable, which possesses a distribution, see Section 6.2.

Therefore, the Weibull line determined in Section 6.3.1 represents an "average" Weibull line, which in most cases is a mean approximation for the failure behaviour. Due to the deviating behaviour of the order statistics, it is possible that the position of the Weibull line for various test specimens can alter within a certain interval. This deviating behaviour can be taken into account with so-called "confidence intervals", see Section 6.2. With these confidence intervals it is possible to gain information concerning the entire population from just one test specimen, see Figure 6.1.

A confidence interval is characterized by the probability that a random variable lies within a certain interval. A 90% confidence interval, for example, indicates that 90 out of 100 cases observed lie within this interval. A 90% confidence interval is limited by a 5% and 95% confidence limit.

The determination of confidence limits and the confidence interval is shown in the next analysis step.

Step 2: Determine failure probabilities $F(t_i)_{5\%}$ and $F(t_i)_{95\%}$ with Tables A.1 and A.3 in the appendix and enter the coordinates into the Weibull probability chart. Draw in the lines through all $F(t_i)_{5\%}$ and $F(t_i)_{95\%}$ coordinates respectively. The best fit lines represent the 5% and the 95% confidence limits. The region between the confidence limits is the 90% confidence interval.

The following values resulted out of the gear wheel test:

Table 6.1. Median values and confidence intervals

i	t_i	$F(t_i)_{5\%}$	$F(t_i)_{50\%}$ (median)	$F(t_i)_{95\%}$
1	7.9	0.5 %	6.7 %	25.9 %
2	10.0	3.7 %	16.3 %	39.4 %
3	12.2	8.7 %	25.9 %	50.7 %
4	13.5	15.0 %	35.6 %	60.8 %
5	14.3	22.2 %	45.2 %	69.7 %
6	15.1	30.4 %	54.8 %	77.8 %
7	17.3	39.3 %	64.4 %	85.0 %
8	18.2	49.3 %	74.1 %	91.3 %
9	24.6	60.6 %	83.7 %	96.3 %
10	30.5	74.1 %	93.3 %	99.5 %

The confidence interval can either be determined by the median distribution Weibull line or directly from the individual coordinates, so that the confidence interval is respectively either a curved-based or a coordinate-based interval.

In Figure 6.13, $F(t_i)_{5\%}$ and $F(t_i)_{95\%}$ are drawn in the Weibull probability chart as circles determined by the coordinates. By connecting all the circles for the different order statistics with an approximation curve, it is possible to acquire the limit curves of the confidence limits for the total failure time. The region between these confidence limits is the 90% confidence interval.

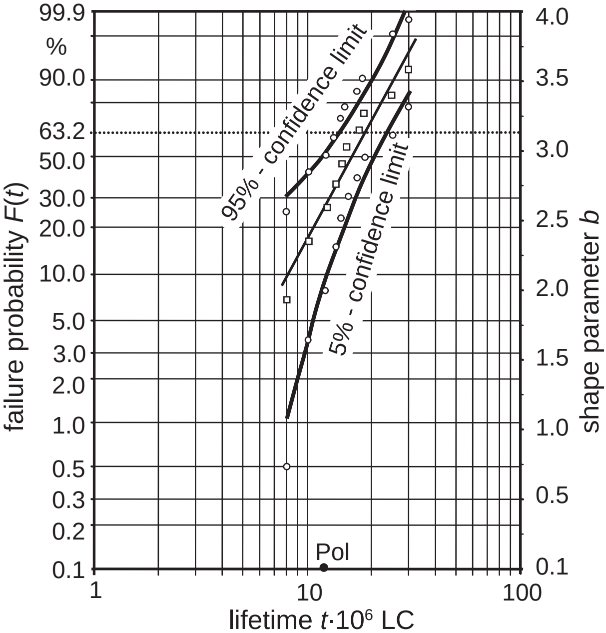


Figure 6.13. Weibull line and the 90% confidence interval

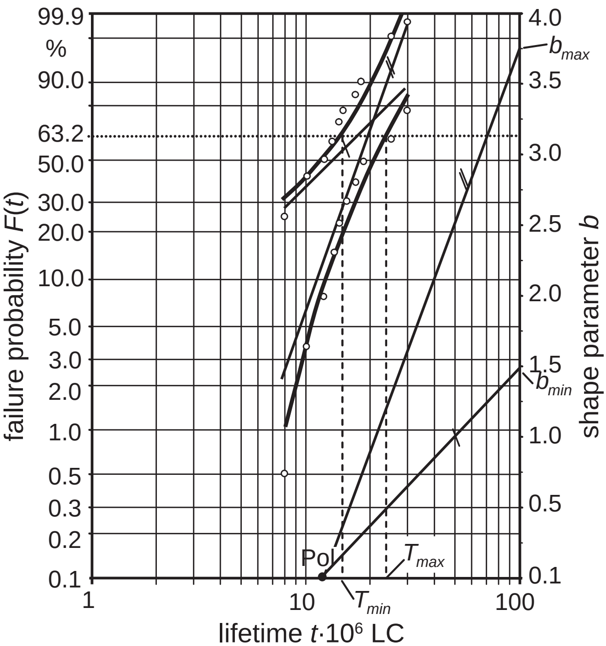


Figure 6.14. Confidence interval with minimal and maximal values for *T* and *b*

The Weibull line of the median values and the confidence interval can be interpreted as follows: When observed for several test specimens, the Weibull line drawn in Figure 6.13 in the middle is the mean result or estimation. 50% of the cases lie above this line and 50% lie below the Weibull line. However, it is possible that for an arbitrary test specimen, the position of the Weibull line lies within the confidence interval but not along the median values. The probability that the failure results lie outside of the confidence interval is only 10%. This means that in only one out of ten cases the confidence interval cannot be trusted. The minimal and maximal values for the parameters *T* and *b* for a 90% confidence interval are shown in Figure 6.14. A graphical analysis of the test results produces the following parameters for the two parametric Weibull distribution:

$T_{min} = 15 \cdot 10^6$ load cycles; $T_{median} = 18 \cdot 10^6$ load cycles; $T_{max} = 23 \cdot 10^6$ load cycles;
$b_{min} = 1.5$; $b_{median} = 2.7$; $b_{max} = 3.7$;
Confidence Interval: 90%

The range of dispersion for the characteristic lifetime and the shape parameter can also be calculated with simple approximation equations [6.10]. In this way, the second analysis step can be omitted.

The approximation equations for the characteristic lifetimes T_{min} and T_{max} are as follows:

$$T_{min} = T_{5\%} \approx T_{median} \cdot \left(1 - \frac{1}{9n} + 1.645\sqrt{\frac{1}{9n}}\right)^{-3/b_{median}}, \tag{6.14}$$

$$T_{max} = T_{95\%} \approx T_{median} \cdot \left(1 - \frac{1}{9n} - 1.645\sqrt{\frac{1}{9n}}\right)^{-3/b_{median}} \tag{6.15}$$

(T_{median}: corresponds to the characteristic lifetime *T* determined in Figure 6.11.)

The range of dispersion for the shape parameter can be approximately determined by the following equations:

$$b_{min} = b_{5\%} \approx \frac{b_{median}}{1 + \sqrt{\frac{1.4}{n}}}, \tag{6.16}$$

$$b_{max} = b_{95\%} \approx b_{median} \cdot \left(1 + \sqrt{\frac{1.4}{n}}\right) \tag{6.17}$$

(b_{median}: corresponds to the shape parameter *b* determined in Figure 6.11).

Observing the confidence interval is especially significant for small test specimen sizes, since then the confidence intervals can cover a very large range. For cases with only few test results, the confidence interval can be seen as a measurement for the desired parameters. With an increasing test specimen size *n*, the range covered by the confidence intervals becomes thinner and can only be completely neglected for $n > 50...100$.

6.3.3 Consideration of the Failure Free Time t_0 (three parametric Weibull Distribution)

If a failure free time t_0 exists, then the test yields coordinates that no longer lie along a straight line in the Weibull probability paper but rather along a bent convex curve, see Section 2.3.3.

Step 3.1: Check whether the coordinates in the Weibull probability paper can be better approximated linearly (best fit straight line) or nonlinearly (approximation curve). An approximation curve indicates a three parametric Weibull distribution with a failure free time t_0. The failure free time t_0 can either be determined with the graphical procedure described in the following section or more exactly with the analytical methods in Section 6.6.

Figure 6.15 shows the example that an approximation curve is a good estimation for a function describing the test result coordinates. Therefore, a three parametric Weibull distribution should be determined.

The occurrence of a failure free time t_0 can have several causes [6.1]. The most important causes are:

- Principally, no failure can occur before the time t_0. For example, before damage can occur on a brake disk, the brake lining must be worn down.
- A time shift occurs between production, delivery and operation of a product.
- The development and expansion of damage requires a certain amount of time, for example, the development of pittings or dimpling during a gear wheel test first occurs after cracks begin to form and spread.

An approximate determination of the failure free time t_0 can be acquired graphically. A simple estimation of t_0 is determined by extending the approximation curve to the x-axis as seen in Figure 6.15. The failure free time t_0 can be taken within a certain range in front of the point of intersection of the approximation curve with the abscissa, see Figure 6.15. The best approximation for the parameter t_0 has been won when the corrected failure times $t_i' = t_i - t_0$ display a straight line in the Weibull probability paper.

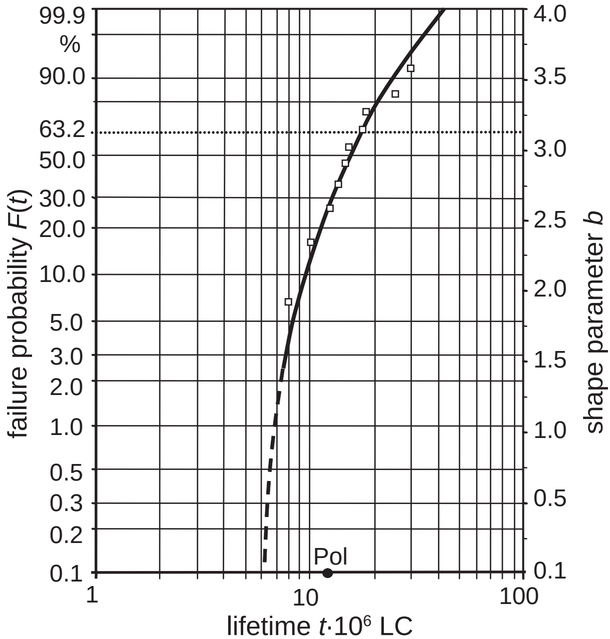


Figure 6.15. Approximation curve or a three parametric Weibull distribution through the test result coordinates (compare with Figure 6.10 and Figure 6.11)

Step 3.2: Transform the approximation curve into a Weibull line. For this it is necessary to transform the failure times: $t_i' = t_i - t_0$. The best approximation for t_0 is obtained if a straight Weibull line can be laid through the coordinates $\left(t_i', F(t_i')\right)$.

The most suitable failure free time t_0 can only be determined iteratively. It is necessary to try out various values for t_0. The best value in the gear wheel test comes out to t_0 = 6 million load cycles, see Figure 6.16.

The parameters of the Weibull line in Figure 6.16 can be determined with step 1.3. The characteristic lifetime comes to T = 18 million load cycles and the shape parameter to b = 1.6, see Figure 6.16. (The shape parameter b differs for a two parametric or a three parametric assessment, see Chapter 7).

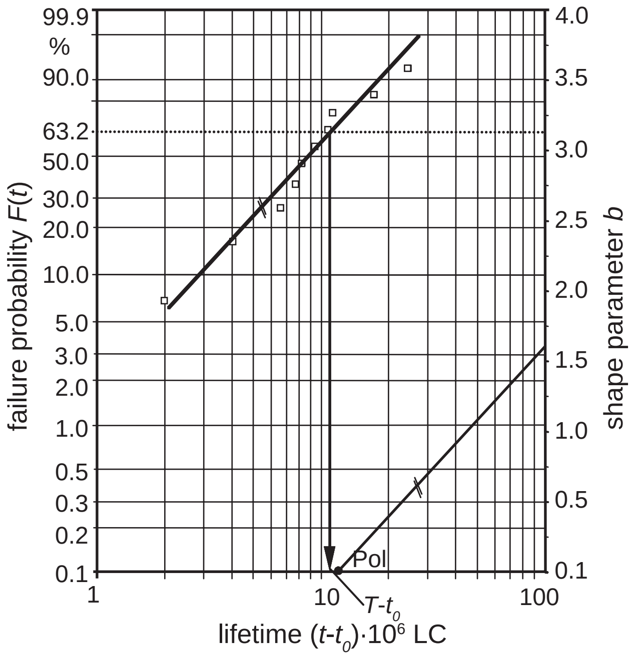


Figure 6.16. "Weibull line" for the failure times corrected with t_0

The failure behaviour of gear wheels can thus be described by the following three parametric Weibull distribution:

$$F(t) = 1 - e^{-\left(\frac{t - 6\cdot 10^6\ \text{LW}}{(18-6)\cdot 10^6\ \text{LW}}\right)^{1.6}} \quad . \tag{6.18}$$

The failure free time t_0 can also be approximately calculated with a procedure from Dubey [6.3]. This procedure is relatively simple and can be applied with little time. It can be described as follows:

- An approximation curve is drawn through the test results in the Weibull probability paper, see Figure 6.17.
- The ordinate (y-axis) is divided into two equal portions Δ and the corresponding lifetimes t_1, t_2 and t_3 are determined.
- The failure times t_1, t_2 and t_3, determined in Figure 6.17, determine the failure free time t_0 as follows:

$$t_0 = t_2 - \frac{(t_3 - t_2)\cdot(t_2 - t_1)}{(t_3 - t_2) - (t_2 - t_1)} \ . \tag{6.19}$$

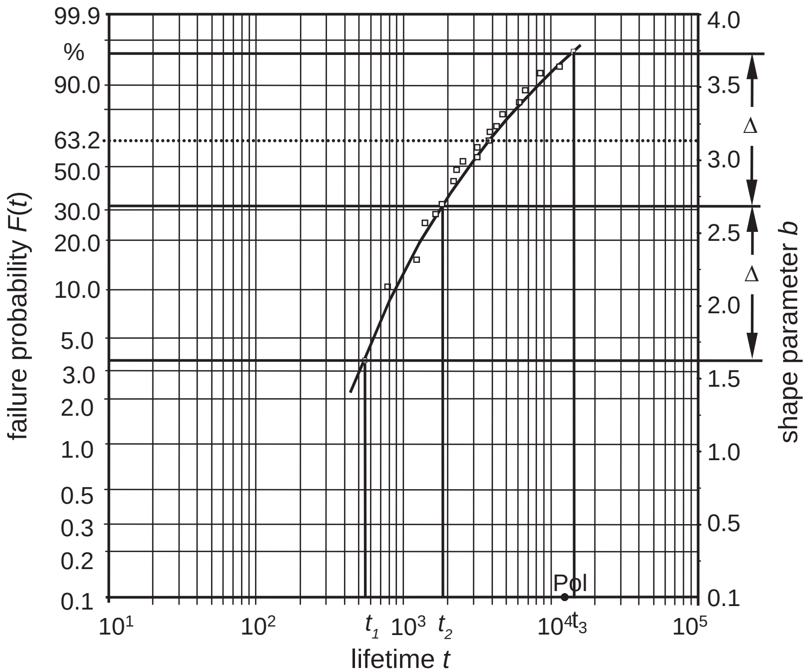


Figure 6.17. Determination of the failure free time according to Dubey [6.3]

Step 3.3: Determine the confidence intervals for the three parametric Weibull distribution for the corrected Weibull line as in step 2 (see Figure 6.16).

Figure 6.18 shows the 90% confidence interval for this example. The following values for the parameters result for the three parametric Weibull distribution:

$T_{min} = 13 \cdot 10^6$ load cycles; $T_{median} = 18 \cdot 10^6$ load cycles; $T_{max} = 25 \cdot 10^6$ load cycles;
$b_{min} = 0.8$; $b_{median} = 1.6$; $b_{max} = 2.5$;
$t_0 = 6$ million load cycles;
Confidence interval: 90%.

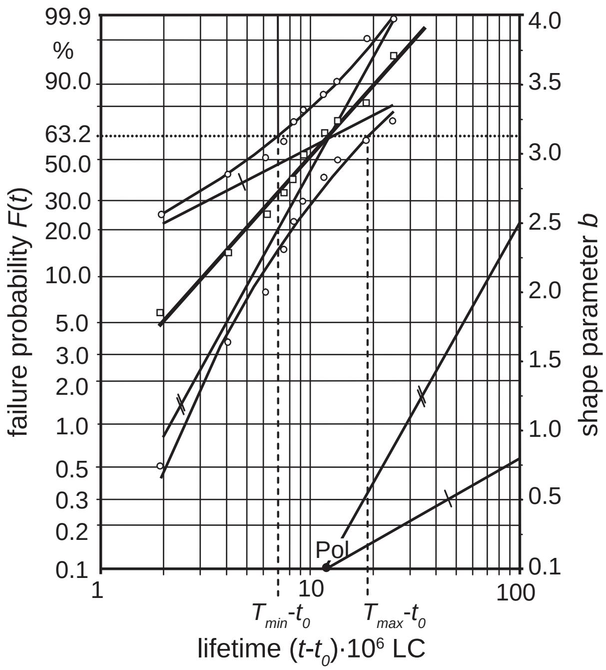


Figure 6.18. Confidence interval for the "Weibull line" and the distribution parameters

Due to the small test specimen size of the example test chosen, $n = 10$, no definite decision can be made between a two or a three parametric Weibull distribution. Both distribution types could present a possible solution after analysis is carried out. A three parametric Weibull distribution should only be used if it is known or assumed that a failure free time exists. For all other cases the failure behaviour description should be limited to a two parametric Weibull distribution since it offers a more conservative description.

6.4 Assessment of Incomplete (Censored) Data

As described in Section 6.1, the time and effort involved in testing can be significantly reduced by incomplete tests or by strategies for test run duration reduction. Several such often used procedures and methods will be introduced in this section. An overview of the procedures for the assessment of incomplete (censored) data can be found in Table 6.1.

Table 6.2. Overview of procedures for the assessment of incomplete (censored) data

Type of Data	Type of Censore	Description	Procedure	see Section
Complete Data $r = n$	No censoring	All units have failed	Median Procedure $F_i \approx \frac{i-0.3}{n+0.4} \forall i = 1(1)n$	6.3
Incomplete Data $r < n$	Censoring Type I or Type II	Lifetime characteristics (e.g. run times) of all intact units **are larger** than the lifetime characteristics of the unit r which failed last	Median Procedure $F_i \approx \frac{i-0.3}{n+0.4} \forall i = 1(1)r$	6.4.1
	Multiple Censoring	Lifetime characteristics of intact units **are unknown**	Sudden Death Test	6.4.3
		Lifetime characteristics of intact units **are known**	Procedure for the Consideration of Unoccured Events (assessment under variable conditions) – Johnson or VDA Procedure	6.4.3.2
		Information about intact units available in the form of an **"operational performance distribution"**	Procedure for the Consideration of Unoccured Events out of the Test Route	6.4.3.3

Variables:
r = number of failures
n = test specimen size

6.4.1 Censoring Type I and Type II

If, for example, a test stand trial is interrupted before all *n* test units have failed, then an "incomplete test specimen size" is produced. If the interruption (trial stop) occurs after a given time, then one is dealing with censoring of type I, see Figure 6.19 where "x" indicates a failure. Object numbers 4 and 5 endure until the end of the trial without failure. Thus, only the failure times for $r < n$ test units are known in this case. The only thing known concerning the remaining $n - r$ "survivors" is that they are still intact after interruption of the trial. The amount *r* of failures is a random variable, which is unknown before the trial begins.

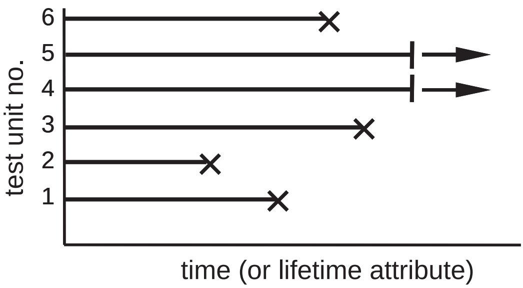


Figure 6.19. Schematic of type I censoring

If a trial is interrupted after a given amount of test units *r* has failed, then one is dealing with censoring of type II, see Figure 6.20. The trial is stopped after 4 failures. The object numbers 3 and 4 endure until the end of the trial without failure. In this case, the point in time at which the failure *r* occurs is a random variable, thus leaving the entire trial time length open until the end of the trial.

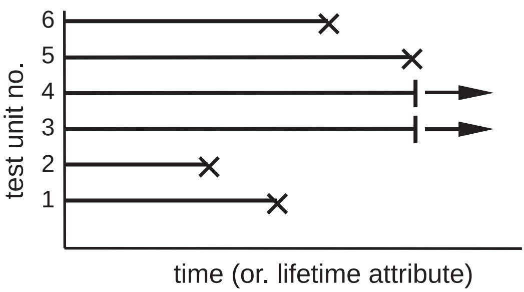


Figure 6.20. Schematic of type II censoring

In both of these cases (censoring of type I and II) it is impossible to carry through an evaluation as described in the previous Section 6.3.

There the cumulative frequencies were calculated according to an approximation equation in order to enter the failure behaviour in Weibull probability chart:

$$F(t_j) \approx \frac{i - 0.3}{n + 0.4} \qquad \text{for } i = 1, 2, \ldots, r.$$

The fact that $n - r$ test units have not failed is taken into account by substituting r for n in the denominator of the approximation equation.

For the evaluation of test runs with type I or II censoring it is often necessary to estimate the characteristic lifetime T in the Weibull chart by extrapolating the best fit line beyond the point in time of the last failure. This is generally problematic as long as further failure mechanisms cannot be neglected. In case of doubt, a statistical statement concerning the failure behaviour can only be made based on the shortest and longest observed lifetime (see the following proof).

Proof for Extrapolation in the Weibull Chart

The following proof can be generalized to the following statement: Complete as well as incomplete test specimens allow for the evaluation of information about the failure behaviour in Weibull chart, as long as the information is limited between the lowest and highest value of the lifetime characteristic.

As soon as no more information is available below the lowest value or above the highest value, it is generally problematic to extrapolate the entered coordinates (neither above nor below).

Example: Censoring of Type I or Type II

Population: $n = 6$

Number of failures: $r = 4$ $\quad r \rightarrow n_f(t)$

Failure Probability: $F_i = F(t_i) \approx \dfrac{i - 0.3}{n + 0.4} \, \forall i = 1(1)r$

Rank i	1	2	3	4	5	6
Order Statistic t_i	900	1300	1900	2300	?	?
Failure Prob. F_i	10.94%	26.26%	42.19%	57.81%		

Graphic (Weibull chart):

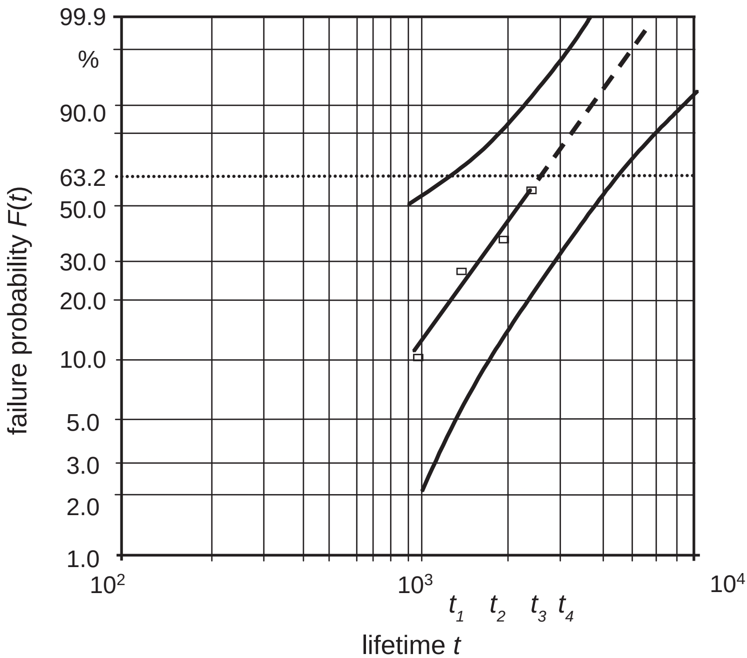


Figure 6.21. Example for extrapolating in Weibull chart

6.4.2 Multiple Censored Data

In lifetime tests it is often the case that the test objects must be removed from the trial before failure. Contrary to type I censoring, where the surviving test units are all removed from the trial at the same (predetermined) point in time, it is possible that the "removal" can occur at varying (random) points in time (multiple censored data), see Figure 6.22 where "x" indicates a failure. The arrow implies that the object remained intact according to the failure mechanism observed up until the point of its removal from the trial.

Such a case is especially common if products should be tested under the failure mode A (e.g. failure of an electronic component), but the failure occurs because of the failure mode B (e.g. due to a mechanical defect).

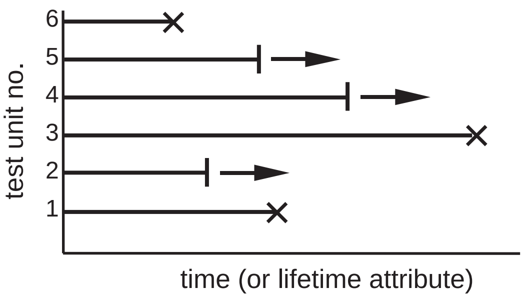


Figure 6.22. Schematic for the visualization of multiple censored data

6.4.3 Sudden Death Test

In a Sudden Death Test, a test specimen is divided into m inspection lots of the same size, Figure 6.23. For example, if a test specimen consists of n = 30 components it is possible to divide it into m = 6 inspection lots of the same size, each consisting of k = 5 components:

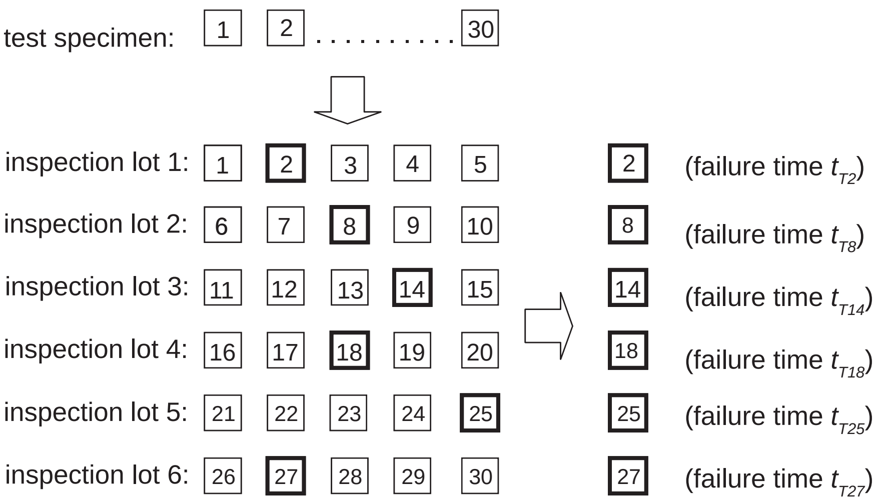


Figure 6.23. Breakdown of the test specimen into inspection lots in a Sudden Death Test

The components of each inspection lot are tested simultaneously until the first component fails. In order to achieve this, several trial stands are necessary (as in example 5) or the trials must be carried out in intervals (e.g. each component is tested for x hours). After one test component has failed, the remaining components of the inspection lot are not tested further. Thus, the result is the running time of the first failed component for each inspection lot, Figure 6.23, right.

Afterwards, the failure times are ordered in increasing value:

$$t_{T8} < t_{T27} < t_{T14} < t_{T2} < t_{T18} < t_{T25} \quad \text{or} \tag{6.20}$$

$$t_1 < t_2 < t_3 < t_4 < t_5 < t_6 \,. \tag{6.21}$$

The following assessment can be executed in two ways. In the first procedure, each determined failure time is assigned a hypothetical rank, which considers the undamaged components. The shortest failure time t_{T8} in Equation (6.20) also corresponds to the shortest failure time in the total test specimen. However, the second shortest failure time t_{T27} is not necessarily the second shortest failure time in the total test specimen. It is quite possible that in the inspection lot 2 that after the failure time $t_{T8,}$ the next component fails after a time that is shorter than t_{T27}. The determined hypothetical rank takes such a phenomenon into consideration. The corrected average rank can be calculated as follows:

The average rank $j(t_j)$ is equal to the previous rank $j(t_j\text{-}1)$ plus the incremental growth $N(t_j)$:

$$j(t_j) = j(t_{j-1}) + N(t_j)\,; \qquad j(0) = 0\,. \tag{6.22}$$

The incremental growth $N(t_j)$ is:

$$N(t_j) = \frac{n + 1 - j(t_{j-1})}{1 + (\text{number of remaining parts})}\,. \tag{6.23}$$

The number of remaining parts refers to the number of still remaining test units including the currently regarded failed unit. Thus, $N(t_j)$ can also be determined as:

$$N(t_j) \;=\; \frac{n + 1 - j(t_{j-1})}{1 + (n - \text{number of previous parts})}\,. \tag{6.24}$$

Seen on the basis of an example:

$j_1 = j_0 + N_1; \quad j_0 = 0; \qquad N_1 = \dfrac{30 + 1 - 0}{1 + 30 - 0} = \dfrac{31}{31} = 1.0$

$j_1 = 0 + 1.0 = 1.0$

$j_2 = j_1 + N_2; \quad j_1 = 1.0; \quad N_2 = \dfrac{30 + 1 - 1{,}0}{1 + 30 - 5} = \dfrac{30}{26} = 1.15$

$j_2 = 1.0 + 1.15 = 2.15$

$j_3 = j_2 + N_3$; $j_2 = 2.15$; $N_3 = \frac{30+1-2.15}{1+30-10} = \frac{28.85}{21} = 1.37$

$j_3 = 2.15 + 1.37 = 3.52$

etc.

The calculation of the failure probability is carried out with the knowledge from Equation (6.10) for the median of the order statistics:

$$F(t_j) \approx \frac{j(t_j)-0.3}{n+0.4}. \tag{6.25}$$

The further procedure for the assessment of the failure times reflects a normal assessment in Weibull probability chart.

A second assessment procedure can be carried out directly with the help from the Weibull chart. As in the previous procedure, the lifetime values are ordered according to increasing value, Equation (6.21), and are entered in Weibull probability paper. Each of the first failures is assigned the median of the order statistics, Figure 6.24, where *m* is the number of inspection lots.

$$F(t_i) \approx \frac{i-0.3}{m+0.4} \tag{6.26}$$

A straight line results in the probability paper for the failures. The theory tells us that the slope of a line, that is the shape parameter *b* in a Weibull function, is the same for a part of a test specimen as for the total test specimen. This implies that the slope of the first failures also corresponds to the shape parameter of the total distribution, and can thus be determined in this way. For an exact representation of the failure behaviour, the line must be shifted to the right. The dimension of this shift is won from the fact that the first failure of an inspection lot can be assigned a failure probability of $F_1* = 0.7 / (k + 0.4)$ and that a representative value for the first failure is taken to be the median (50% value) of the determined first failures. A vertical line is drawn from the intersection of the 50% line with the straight line of the first failures, whose intersection with the F_1* line of the first failure represents a point on the straight line for the total distribution. The straight line of the first failures must now only be shifted parallel through this point, Figure 6.24. In the example above, with $k = 5$, $F_1* = 12,9\%$.

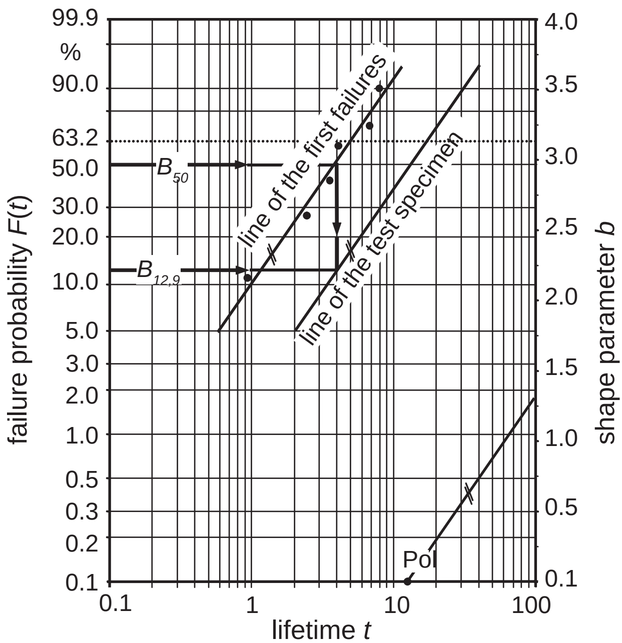


Figure 6.24. Graphical assessment of a Sudden Death Test

6.4.3.1 Application of "Sudden Death Testing" in field tests, when no information is given concerning intact parts

The Sudden Death Test assessment can also be applied in field tests, by dividing all delivered machines into equally large inspection lots [6.4, 6.12, 6.14]. The number of inspection lots is determined out of the sum of failures plus one. This ensures that the faultless units with run times shorter than the first failure are taken into account. The size of the inspection lot is determined as follows:

$$k = \frac{n - n_f}{n_f + 1} + 1. \tag{6.27}$$

Here, k is the size of the inspection lot, n is the number of all delivered machines at the observed point in time and n_f is the number of machines which have failed. The assessment can be carried out as given above.

In the following example, the Sudden Death Test assessment will be shown on behalf of an example.

Given:

n = 4800 parts in one production month that have been delivered to customers

n_f = 16 failed parts with their respective lengths of run time

which are:

t_{f1} = 1,500 km with (cumulative frequency: 4.2% from Table A.2 for n = 16)

t_{f2} = 2,300 km t_{f3} = 2,800 km t_{f4} = 3,400 km t_{f5} = 3,900 km

t_{f6} = 4,200 km t_{f7} = 4,800 km t_{f8} = 5,000 km t_{f9} = 5,300 km

t_{f10} = 5,500 km t_{f11} = 6,200 km t_{f12} = 7,000 km t_{f13} = 7,600 km

t_{f14} = 8,000 km t_{f15} = 9,000 km

t_{f16} = 11,000 km (cumulative frequency: 95.8% from Table A.2 for n = 16)

The component set k (the inspection lot size) is calculated as

$$k = \frac{n - n_f}{n_f + 1} + 1. \tag{6.28}$$

Note: The simpler equation $k = \frac{n}{n_f + 1}$ is also sufficient and leads approximately to the same result. For example above this yields:

$$k = \frac{4800 - 16}{16 + 1} + 1 = \frac{4784}{17} + 1 = 281.4 + 1 \;\Rightarrow\; k \approx 282\,.$$

Thus, there are 281 intact components between each failure, that is, before the first failure, between the first and second, second and third … and after the 16th failure. The number of undamaged components:

$$\sum n_s(t) = m \cdot (k - 1) \text{ with } m = 17 \text{ and } k = 282\,,$$

$$\Rightarrow \sum n_s(t) = 4777\,,$$

in comparison to the exact values: 4800 – 16 = 4784.

The total population is estimated at:

$$n = m \cdot (k - 1) + n_f = m \cdot k \text{ with } m = 17 \text{ and}$$
$$k = 282\,,$$

$$\Rightarrow \quad n \quad = \quad 4793$$

or exactly: 4800.

At the same time it is assumed that nothing is known about the 281 components until the first failure, so that undetermined cases must be assumed.

As done in the previous section, here each first failure is represented by the median rank $\frac{1-0.3}{k+0.4} \cdot 100\,\% = 0.25\,\%$ for $k = 282$ in the total population of n = 4800 parts.

The line for the total failure distribution for the 4800 components can be determined by drawing a vertical line from the intersection of the 50% line with the "straight line for the first failures". From the intersection of the 0.25% line with the vertical line a new line is drawn parallel to the "straight line of the first failures". The result is the straight line for the total failure distribution sought for, see Figure 6.25.

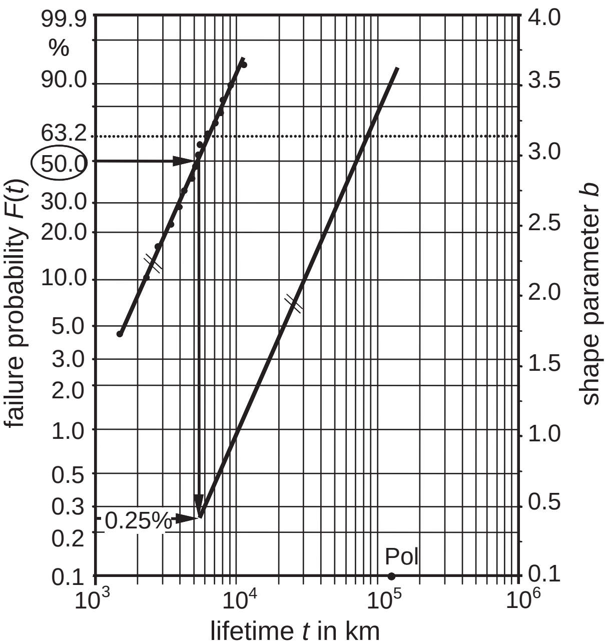


Figure 6.25. Total failure behaviour for the Sudden Death Testing

6.4.3.2 Individually Known Data of Damaged and Undamaged Components – Test Specimen

Principally, the procedure described in Section 6.4 can also be used for cases with variable conditions, if it is not possible to build equally large inspection lots which are tested under similar conditions [6.13]. For this assessment procedure the value of the lifetime characteristic must be known for the undamaged components.

The following example describes the calculation of the failure distribution for the case that the lifetime characteristic of the undamaged components is known.

The assessment below shows the calculation of the failure distribution for the case that the lifetime characteristics of damaged and undamaged components are known [6.13].

The following data is given:

$n = 50$ test specimen size with

$n_f = 12$ damaged and

$n_s = 38$ undamaged components.

The corresponding run times [km · 10^3] for the damaged components *f* and undamaged components *s* are sorted according to increasing value:

$t_{s_1} = 40$; $t_{s_2} = 51$; $t_{f_1} = 54$; $t_{f_2} = 55$; $t_{s_3} = 58$; $t_{s_4} = 59$; $t_{s_5} = 59$;
$t_{f_3} = 60$; $t_{s_6} = 60$; $t_{f_4} = 61$; $t_{s_7} = 62$; $t_{f_5} = 63$; $t_{f_6} = 65$; $t_{s_8} = 66$;
$t_{s_9} = 66$; $t_{f_7} = 67$; $t_{f_8} = 70$; $t_{s_{10}} = 70$; $t_{s_{11}} = 70$; $t_{s_{12}} = 70$; $t_{s_{13}} = 70$;
$t_{f_9} = 71$; $t_{s_{14}} = 72$; $t_{s_{15}} = 72$; $t_{s_{16}} = 72$; $t_{s_{17}} = 72$; $t_{s_{18}} = 72$; $t_{s_{19}} = 73$;
$t_{s_{20}} = 73$; $t_{s_{21}} = 73$; $t_{s_{22}} = 74$; $t_{f_{10}} = 75$; $t_{s_{23}} = 77$; $t_{s_{24}} = 78$; $t_{s_{25}} = 78$;
$t_{s_{26}} = 79$; $t_{s_{27}} = 80$; $t_{s_{28}} = 81$; $t_{s_{29}} = 81$; $t_{s_{30}} = 82$; $t_{s_{31}} = 82$; $t_{s_{32}} = 83$;
$t_{f_{11}} = 84$; $t_{s_{33}} = 85$; $t_{s_{34}} = 86$; $t_{s_{35}} = 86$; $t_{s_{36}} = 88$; $t_{f_{12}} = 91$; $t_{s_{37}} = 92$;
$t_{s_{38}} = 92$.

The pre-sorted run time values are sequentially assigned to one another, the undamaged component run times are assigned to the damaged run times, Table 6.3. The procedure is to note the value of an undamaged component in the line of the next highest value for a damaged component if its value is less. If the value of the undamaged component is the same as the damaged component then they are noted in the same line. The same also applies for several undamaged components.

After this sub step of allocation, a final overview is prepared in a table of the damaged and undamaged components, Table 6.3.

Table 6.3. Sub step for the allocation (grouping the run time values) [6.13]

Ordinal Number j	Lifetime characteristics in increasing order t_j [km · 10^3]	damaged	undamaged	Number of previous components
	40		X	
	51		X	
1	54	X		2
2	55	X		3
	58		X	
	59		X	
	59		X	
3	60	X		7
	60		X	
4	61	X		9
	62		X	
5	63	X		11
	.	.	.	
	.	.	.	
	.	.	.	
	etc.	etc.	etc.	

The undamaged component values t_{s37} and t_{s38} can not be assigned to damaged components because corresponding similar or larger values of damaged components do not exist. However, these values are taken into account by calculating with $n = 50$ and not (!) with $n = 48$.

The next steps encompass the calculation of the average ordinal number and thus that of the median rank. Here, the calculation procedure is only discussed for the first steps. The complete calculation can be carried out analogue to the steps in the previous sections.

Calculation of the Average Ordinal Number $j(t_j)$

The average ordinal number $j(t_j)$ is equal to the previous ordinal number $j(t_{j-1})$ plus the number of failures $n_f(t_j)$ times the growth $N(t_j)$

$$j(t_j) = j(t_{j-1}) + [n_f(t_j) \cdot N(t_j)], \tag{6.29}$$

$$N(t_j) = \frac{n + 1 - j(t_{j-1})}{1 + \text{number of remaining components}}. \tag{6.30}$$

The number of remaining parts is the difference between the test specimen size and the number of all previous components including the component currently regarded, see also Table 6.3:

$$N(t_j) = \frac{n + 1 - j(t_{j-1})}{1 + (n - \text{previous components})}. \tag{6.31}$$

For the example – where coincidentally $n_f(t_j)$ is always equal to 1 – one can find:

$$j_0 = 0$$

$$j_1 = j_0 + N_1 \text{ with } N_1 = \frac{50 + 1 - 0}{1 + (50 - 2)} = \frac{51}{49} = 1.04. \tag{6.32}$$

$$j_1 = 0 + 1.04 = 1.04$$

$$j_2 = j_1 + N_2 \text{ with } N_2 = \frac{50 + 1 - 1.04}{1 + (50 - 3)} = \frac{49.95}{48} = 1.04$$

$$j_2 = 1.04 + 1.04 = 2.08$$

$$j_3 = j_2 + N_3 \text{ with } N_3 = \frac{50 + 1 - 2.08}{1 + (50 - 7)} = \frac{48.92}{44} = 1.11$$

$$j_3 = 2.08 + 1.11 = 3.19$$

etc. $j_4 \ldots j_{12}$.

Calculation of the Median Ranks $F(t_j)$ [%]

For the calculation of the median ranks the approximation equation is used:

$$F_{median}(t_j) \approx \frac{j(t_j) - 0.3}{n + 0.4} \cdot 100\%. \tag{6.33}$$

The numerical values for this example are:

$$F_{median}(t_1) \approx \frac{j_1 - 0.3}{n + 0.4} \cdot 100\% = \frac{1.04 - 0.3}{50 + 0.4} \cdot 100\% = \underline{\underline{1.47\%}}$$

$$F_{median}(t_2) \approx \frac{j_2 - 0.3}{n + 0.4} \cdot 100\% = \frac{2.04 - 0.3}{50 + 0.4} \cdot 100\% = \underline{\underline{3.53\%}}$$

$$F_{median}(t_3) \approx \frac{j_3 - 0.3}{n + 0.4} \cdot 100\% = \frac{3.19 - 0.3}{50 + 0.4} \cdot 100\% = \underline{\underline{5.73\%}}$$

etc. $F_{median}\ (t_4)\ldots F_{median}\ (t_{12})$.

The following table includes several calculated values.

Table 6.4. Assessment results [6.13]

Ordinal No. j	Lifetime characteristics in increasing order [km ·10³] t_j	Number of damaged parts $n_f(t_j)$	Number of undamaged parts $n_s(t_j)$	Calculation: Number of previous parts	Growth $N(t_j)$	Mean ordinal number $j(t_j)$	Median rank [%] $F_{median}(t_j)$
1	54	1	2	2	1.04	1.04	1.47
2	55	1		3	1.04	2.08	3.53
3	60	1	3	7	1.11	3.19	5.73
4	61	1	1	9	1.14	4.33	7.99
5	63	1	1	11	1.16	5.49	10.31
6	65	1		12	1.17	6.66	12.62
7	67	1	2	15	1.23	7.89	15.07
8	70	1		16	1.23	9.12	17.51
9	71	1	4	21	1.40	10.52	20.28
10	75	1	9	31	2.02	12.54	24.30
11	84	1	10	42	4.28	16.82	32.77
12	91	1	4	47	8.54	25.36	49.73
	>91		2				
		$n_f(t) = 12$	$n_s(t) = 38$				
		$n = 50$					

The calculated median ranks $F_{median}(t_j)$ together with the lifetime values t_j build the coordinates of the points in the lifetime network according to Weibull. The Weibull best fit line represents the lifetime line, Figure 6.26.

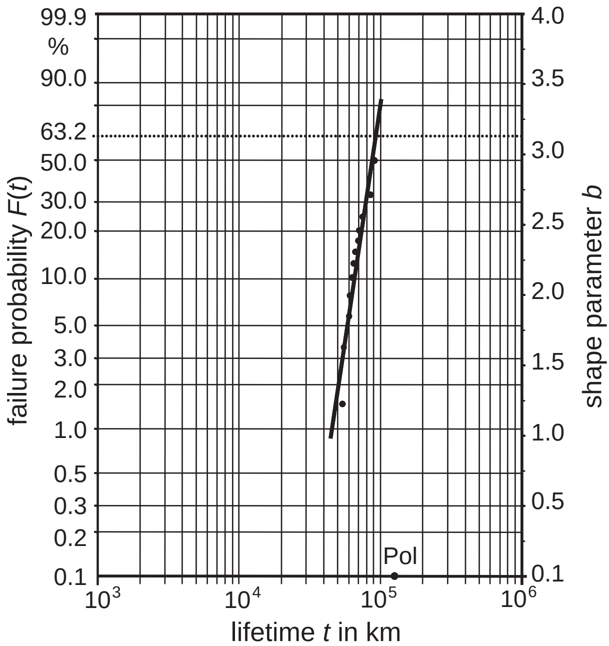


Figure 6.26. Weibull diagram with individually known data for damaged and undamaged components

The parameters for this example are:

- Shape parameter $b = 6.4$
- Characteristic lifetime $T = 92 \cdot 10^3$ km
- Average lifetime $MTTF = 91 \cdot 10^3$ km.

It has become clear that with a relatively small test specimen size ($n \leq 50$) of damaged and undamaged components, the same result can be calculated as with a large test specimen size ($n = 360$) of only damaged components. Thus, a remarkable shortening in testing time is achieved for collecting data, which in turn results to less time required for the calculation and assessment of this data.

For larger test specimens with $n \geq 50$ (damaged and undamaged components) it is reasonable to classify the collected data. The calculation of the ordinal numbers and median ranks is carried out as described above.

6.4.3.3 Calculation of the Failure Behaviour with Undamaged Parts from the Test Drive Distribution

The procedure for this calculation will be described in this section also on behalf of an example. Special aspects in the collection and processing of data are mentioned and possible limitations of this procedure are named [6.13].

In order to estimate the failure behaviour of components and aggregates for the time period after the guarantee period, it is necessary to work with representative test specimens with damaged and undamaged components. The determination and statistical assignment of individual values according to the lifetime characteristic, normally the test run in kilometres, can be executed without difficulty for undamaged components.

Especially within the warranty period information concerning damaged components is almost complete. The only problem is information gaps for undamaged components. If it is possible to close these gaps then it is possible to predict a trend for long period behaviour for components and aggregates relatively early. If a test run distribution exists for the vehicles in which the damaged components are assembled then the calculation of the undamaged components per test run interval is possible.

In the following example information about the damaged components is complete whereas only a total number of undamaged components is known. The calculation of the number of undamaged components per test run interval is based on the test run distribution of the total test specimen (damaged components plus undamaged components). The individual groups of undamaged components are won by subtracting the number of damaged components from the total number per test run.

To make things simpler it is convenient to base the test run distribution only on the population of undamaged components and then to calculate the missing values per test run. For this simplification, the number of damaged vehicles should be significantly smaller than the approved vehicles. This procedure will now be practically shown with data from the test field for a heavy duty vehicle component.

Example:

The damage cases from the warranty period are assigned to classes according to length of run orders for licensed $n_{auto} = 3780$ automobiles. (Table 6.5, first column)

Since the length of run class for 20,000…24,000 km does not show any damages, it can be neglected for the further calculation.

The portion of undamaged components (vehicles with no damaged components) is calculated into the individual length of run classes out of a length of run distribution in logarithmic probability paper (Figure 6.27). This length of run distribution is assumed to be known.

The upper limit of these length of run classes can be read from the length of run distribution. The upper limits are the corresponding ordinate percent values for each class. It is the sum of automobiles, which have reached an arbitrary length of run until the class upper limit. Here, 80% of the vehicles have reached a length of run of 40,000 km without any damaged components. This indicates that 20% of the vehicles achieve a length of run greater than 40,000 km.

The allocation base for the further calculation of the quantity of undamaged parts in each individual class is the number of vehicles produced within a certain manufacture period or the number of licensed vehicles on the market within a certain time period.

The length of the period of use must be about the same for all vehicles. If this is not the case then a corrective calculation must be carried out.

To ensure the case that the statistical spread of the vehicle length of runs at hand is lower than if an additional influence from variously long periods of use also comes into play. This implies that the probability for the occurrence of damage is the same for all vehicles with respect to the length of run.

In the example, $n_{auto} = 3780$ manufactured and licensed vehicles are given for the calculation. Out of the 3780 vehicles, $n_f(t) = 19$ prove to have a damaged component. Correspondingly, $n_s(t) = 3761$ vehicles do not have damaged components.

The first length of run class for the ranked damage cases has an upper limit of 4,000 km. Out of the length of run distribution in Figure Figure 6.27 a cumulative frequency of about 0.035% can be read off for 4,000 km.

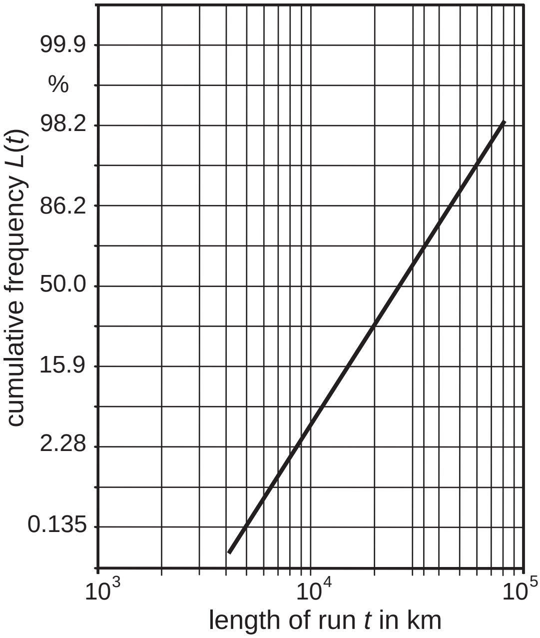


Figure 6.27. Length of run distribution

The portion of undamaged components in the class up to 4,000 km is equal to 0.035% of 3761 vehicles without damaged components. This is equal to about 1 vehicle ($n_s(t_1)$). By idealizing the length of run distribution, a certain inaccuracy occurs in the lower region of the distribution, which however has no effect on the final total outcome.

The next class upper limit is 8,000 km. The corresponding cumulative frequency value from the length of run distribution is 1.7% for undamaged components (vehicles). Since the length of run distribution depicts a cumulative frequency function, whereas in this procedure the percentage portions of the observed class is of interest, it is necessary to subtract the cumulative frequency of the previous class from each extracted value. Thus, for the class from 4,000 to 8,000 km a percentage portion of 1.7% - 0.035% = 1.665% ($n_s(t_2)$).

The remaining $n_s(t_j)$ values can be calculated in the same way, Table 6.5.

Table 6.5. Determination of $n_s(t_j)$ values [6.13]

Length of Run [km] t_j	Cumulative Frequency [%] $L(t_j)$	Single Frequency [%] $l(t_j) = L(t_j) - L(t_{j-1})$	Number of Undamaged Components $n_S(t_j)$
... 4,000	0.035	0.035	1
... 8,000	1.7	1.665	63
...12,000	8.6	6.9	260
...16,000	20.0	11.4	429
...20,000	33.5	13.5	508
...28,000	57.0	23.5	884
...32,000	67.0	10.0	376
...36,000	74.0	7.0	263
...40,000	80.0	6.0	226

The calculation of the failure behaviour according to the median range procedure is carried out using the $n_s(t_j)$ values calculated above, see Table 6.6.

Table 6.6. Calculation of the failure behaviour according to the median rank procedure [6.13]

Length of Run [km] t_j	Number of Damaged Components $n_f(t_j)$	Number of Undamaged Components $n_s(t_j)$	Calculations			
			Number of previous Components	Growth: Number of Damaged Components $N(t_j)$	Mean Ordinal No. $j(t_j)$	Median Rank [%] $F_{median}(t_j)$
... 4,000	5	1	1	5.00	5.00	0.12
... 8,000	2	63	69	2.03	7.03	0.17
...12,000	2	260	331	2.19	9.22	0.23
...16,000	2	429	762	2.50	11.72	0.30
...20,000	1	508	1,272	1.50	13.22	0.34
...28,000	1	884	2,157	2.32	15.54	0.40
...32,000	2	376	2,534	6.04	21.58	0.56
...36,000	3	263	2,799	11.48	33.06	0.86
...40,000	1	226	3,028	4.98	38.04	0.99
> 40,000		751				
	$n_f(t) = 19$	$n_s(t) = 3761$				
	$n_{Fzg}(t) = 3{,}780$					

The graphical assessment of Table 6.6 in Weibull probability paper shows that the values represent a mixed distribution, Figure 6.18.

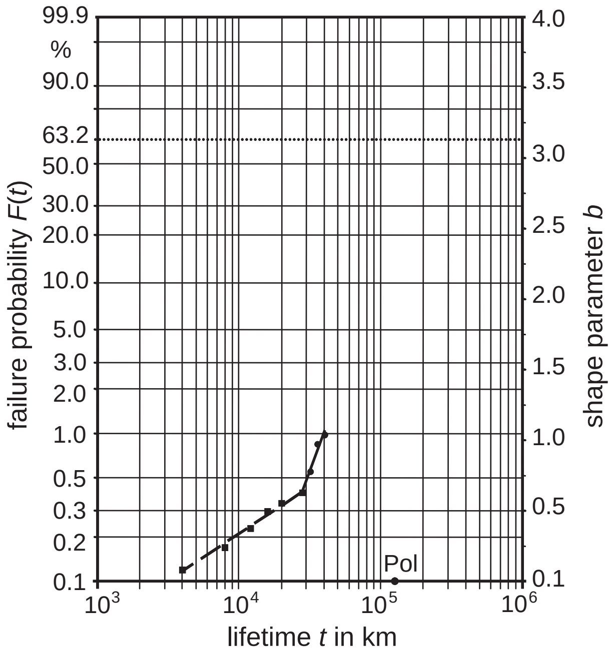


Figure 6.18. Weibull diagram for a mixed distribution [6.13]

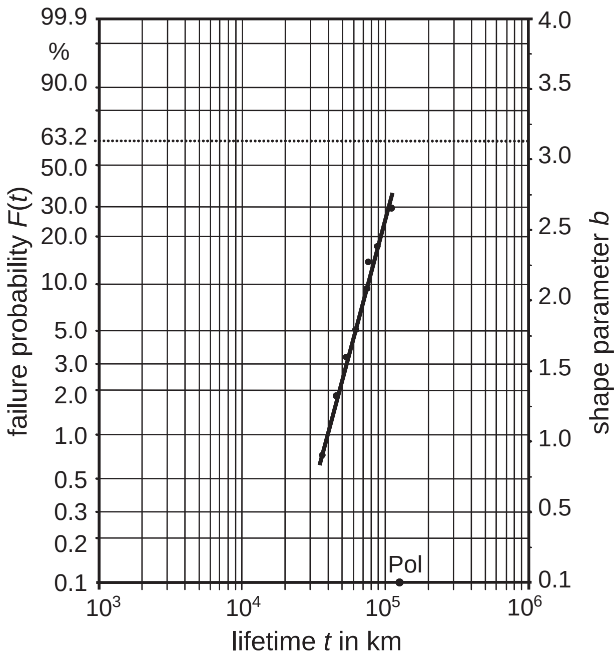


Figure 6.29. Weibull diagram with realistic long term data [6.13]

The expected long term behaviour caused by wearout is represented in the second section of the distribution by the straight line 2. This behaviour has been confirmed by field data at a later point in time with long term data, Figure 6.29.

By just observing the damaged components during the time of warranty, a completely different failure behaviour is depicted (Figure 6.18, dotted line 1), however it does not correctly reflect the field situation.

With this information, the following values result for a calculation with one damaged component per vehicle. A summary of the required field data and the calculated median ranks is shown in Table 6.7:

Number of vehicles	n_{veh}	=	140
Vehicles with one damaged component	$n_{veh\ f}(t)$	=	10
Vehicles without any damaged components	$n_{veh\ s}(t)$	=	130.

Table 6.7. Consideration of long term data in the median range procedure [6.13]

Length of Run [km] t_j	Number of Damaged Components $n_f(t_j)$	Number of Undamaged Components $n_s(t_j)$	Calculations: Number of Previous Components	Growth: Number of Damaged Components $N(t_j)$	Mean Ordinal Number $j(t_j)$	Median Rank [%] $F_{median}(t_j)$
36110	1	42	42	1.38	1.38	0.72
45311	1	19	62	1.68	3.06	1.83
53,000	1	22	85	2.24	5.30	3.32
61,125	1	9	95	2.60	7.90	5.05
72,700	2	11	107	6.51	14.41	9.38
75,098	2	2	111	6.83	21.24	13.92
87,000	1	14	127	5.4	26.64	17.51
110,000	1	16	144	17.77	44.41	29.33
>110,000		5				
	$n_s(t)=10$	$n_s(t)=140$				
	$n_s(t)=150$					

Comments regarding the application of the described procedure:

When preparing the warranty data, it must be assured, that the damaged components considered regard the first case of damage in each individual vehicle (first original parts). Only if this is the case, then the length of run and the respective values for the portion of damage have the same value.

If the damage frequency is already too large in the warranty period, so that more than one case of damage per vehicles occurs (the replacement part is also damaged), then it must be assured that only the first failures are taken into account. Furthermore, if possible, then all damaged components should have arrived.

6.5 Confidence Intervals for Low Summations

When considering short application periods and short lengths of run, for example 1 year or 15,000 km, or when dealing with electronic and electric components, then the summation values run very low, mostly lower than 10%.

For this low range, another procedure has proven to be more adequate for the determination of the confidence interval. This procedure is based on the determination of factors [6.13]. In this procedure, single t_q values are specified, which are dependent upon the inspection lot size n. Confidence interval factors V_q for $P_A = 90\%$ (double-sided) can be taken from the figures to be found in the appendix. These factors are dependent upon the test specimen size n and the Weibull shape parameter b. For interim values of b, V_q must be interpolated.

The lower lifetime limit for q percentage failure probability can be calculated as follows:

$$t_{qu} = t_q \cdot \frac{1}{V_q} \, . \tag{6.34}$$

The upper lifetime limit for q percentage failure probability can be calculated with the equation below:

$$t_{qo} = t_q \cdot V_q \, . \tag{6.35}$$

By connecting the individual coordinates for upper and lower limits, the entire confidence level can be determined.

Example:

For a test specimen size of $n = 100$, trials are executed up until the ordinal number $j = 10$. According to Table 6.8 t_j and F_j can be assigned as follows:

$$F_j \approx \frac{j - 0.3}{n + 0.4} \cdot 100\% \, . \tag{6.36}$$

Table 6.8. Assignment of t and F_j [6.13]

j	1	2	3	4	5	6	7	8	9	10
t_j [cycle]	62	190	288	332	426	560	615	780	842	1,000
F_j [%]	0.70	1.69	2.69	3.68	4.68	5.68	6.67	7.67	8.66	9.66

The individual coordinates are transferred into a Weibull chart. Subsequently, a linear smoothing function is laid through the coordinates. A Weibull distribution with $b = 1$ is yielded.

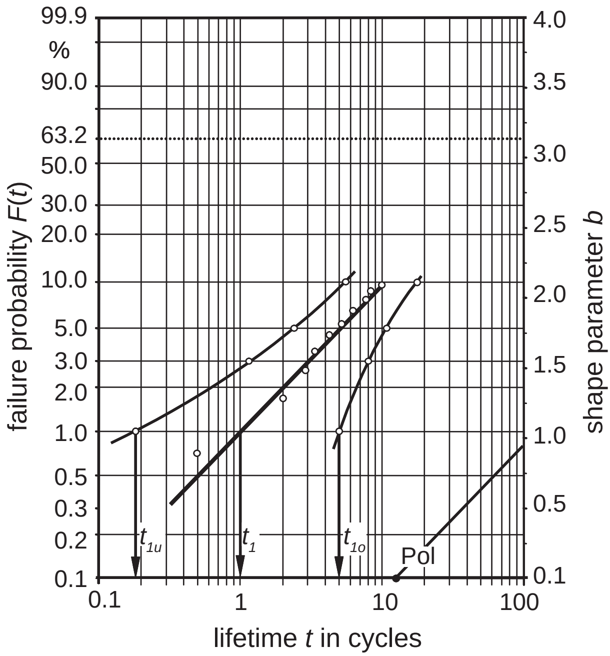


Figure 6.30. Confidence levels for low summations

The confidence interval factors V_q corresponding to $b = 1$ and $n = 100$ are taken from the figures to be found in the appendix for t_1, t_3, t_5 and t_{10}, see Table 6.9.

Table 6.9. V_q Factors

q [%]	t_q	V_q	$t_{qo} = t_q \cdot V_q$	$t_{qu} = t_q/V_q$
1	96	5.0	480	19.2
3	295	2.6	767	113.5
5	500	2.1	1050	238.1
10	1030	1.7	1751	606.0

6.6 Analytical Methods for the Assessment of Reliability Tests

The assessment of failure data can be carried out under various analytical methods. The most well known methods are:

- the Moment of Moments,
- the Regression Analysis (Method of the Lowest Failure Squares according to *Gauß*) and
- the Maximum Likelihood Method.

These methods will first be explained independent of a distribution. Their application can be depicted with an example for the Weibull distribution.

6.6.1 Method of Moments

In the Method of Moments the best distribution is determined by comparing the test specimen moments with the theoretical distribution moments. Moments are certain statistical values from a distribution. The most well know moments are:

- means,
- standard deviation or variance,
- skewness.

One statistical value alone offers very little information about a distribution. The mean only indicates where about the middle of the distribution lies. Several moments together offer a exact picture of the distribution, for which is sought.

With the Method of Moments only complete test specimens can be assessed. The parameter estimation is carried out, by comparing empirical test specimen moments with the theoretical distribution moments. Empirical inspection test specimen are the statistical values of a certain test specimen. If n values t_i , $i = 1(1)n$ are given, then the first empirical moment for the test specimen is the mathematical mean $\bar{t}$

$$\bar{t} = \frac{1}{n} \cdot \sum_{i=1}^{n} t_i \ . \tag{6.37}$$

The second empirical test specimen moment is either the standard deviation s or the variance s^2

$$s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (t_i - \bar{t})^2 = \frac{1}{n-1} \cdot \left[\sum_{i=1}^{n} t_i^2 - \frac{1}{n} \cdot \left(\sum_{i=1}^{n} t_i \right)^2 \right]. \tag{6.38}$$

Third empirical moment is referred to as skewness γ. The skewness is a value for the asymmetry of the distribution density:

$$\gamma = \frac{n}{(n-1)\cdot(n-2)} \cdot \frac{1}{s^3} \sum_{i=1}^{n} (t_i - \bar{t})^3 . \tag{6.39}$$

Theoretical distribution moments characterize probability distributions with continuous random values. The first theoretical distribution moment, mostly referred to as the expected value $E(t)$, can be determined by the improper integral of the density function $f(t)$ multiplied with the statistical variable t:

$$E(t) = m_1 = \int_{-\infty}^{\infty} t \cdot f(t) \cdot dt . \tag{6.40}$$

The general definition for a moment m_k to the order k with respect to the origin is:

$$m_k = \int_{-\infty}^{\infty} t^k \cdot f(t) \cdot dt \quad k = 1,2,\ldots \ . \tag{6.41}$$

Along with origin moments, central moments m_{kz} also exist, which can be defined by the following expression:

$$m_{kz} = \int_{-\infty}^{\infty} (t - m_1)^k \cdot f(t) \cdot dt \qquad k = 1,2,\ldots \ . \tag{6.42}$$

The second central moment is called variance $Var(t)$

$$Var(t) = m_{2z} = m_2 - m_1^2 . \tag{6.43}$$

The skewness $S_3(t)$ is the third theoretical distribution moment, which is defined by the expression:

$$S_3(t) = \frac{m_{3z}}{\sqrt{m_{2z}^2}} \tag{6.44}$$

By setting the empirical moments equal to the theoretical moments, a system of equations results, out of which it is possible to calculate the determining distribution parameters:

$$\begin{aligned} &E(t) = \bar{t}, \\ &Var(t) = s^2 \text{ and} \\ &S_3(t) = \gamma. \end{aligned} \tag{6.45}$$

With these three equations, the three unknown parameters can be calculated. For a one or two parametric distribution, only the first or the first two equations are necessary for determining the sought parameters.

Example: Three parametric Weibull distribution

The application of the Method of Moments for a Weibull distribution is complex. The theoretical moments can only be expressed with the help of the gamma function $\Gamma(x)$. For a three parametric Weibull distribution the following relationships are valid for the expected value $E(t)$, variance $Var(t)$ and skewness $S_3(t)$:

$$E(t) = (T - t_0) \cdot \Gamma\left(1 + \frac{1}{b}\right) + t_0 \,, \tag{6.46}$$

$$Var(t) = (T - t_0)^2 \cdot \left[\Gamma\left(1 + \frac{2}{b}\right) - \Gamma^2\left(1 + \frac{1}{b}\right)\right] \text{ and} \tag{6.47}$$

$$S_3(t) = \frac{\Gamma\left(1 + \frac{2}{b}\right) - \Gamma^2\left(1 + \frac{1}{b}\right)}{\sqrt[3]{\Gamma\left(1 + \frac{3}{b}\right) - 3 \cdot \Gamma\left(1 + \frac{2}{b}\right) \cdot \Gamma\left(1 + \frac{1}{b}\right) + 2 \cdot \Gamma^3\left(1 + \frac{1}{b}\right)}} \,. \tag{6.48}$$

For a two parametric distribution, $t_0 = 0$ in the equations above. According to Equation (6.39), the skewness is only a function of the shape parameter b. Since the empirical skewness γ is known out of Equation (6.38), b can be determined iteratively, for example with the *Newton Method*, with the assumption that $\gamma = S_3(t)$. If b is known, then t_0 can be determined with the Equations (6.46) and (6.47), in connection with the mathematical mean $\bar{t}$, Equation (6.37), and the standard deviation s, Equation (6.38) :

$$t_0 = \bar{t} - \frac{\Gamma\left(1 + \frac{1}{b}\right)}{\sqrt{\Gamma\left(1 + \frac{2}{b}\right) - \Gamma^2\left(1 + \frac{1}{b}\right)}} \cdot s \,. \tag{6.49}$$

The last parameter, the characteristic lifetime T, is calculated out of Equation (6.46):

$$T = \frac{\bar{t} - t_0}{\Gamma\left(1 + \frac{1}{b}\right)} + t_0 \,. \tag{6.50}$$

6.6.2 Regression Analysis

The Regression Analysis can also be referred to as the Method of the Lowest Failure Squares. The determination of the distribution is carried out with a linear smoothing function. The sum of the squares of the intervals between the coordinates (t_i, $F(t_i)$) and the linear smoothing function is minimized. The intervals are calculated to a general form of assumed straight line functions. These functions are summed together. Following, the known normal equations can be derived by differentiating.

In contrast to the Method of Moments, incomplete test specimens can be assessed with the Regression Analysis. If incomplete test specimen r trial values t_i, $i = 1(1)r$ are given for a test specimen size n. The trial values are ordered according to increasing value so that $t_1 \le t_2 \le ... \le t_i \le ... \le t_r$. The ordered values are now referred to as order statistics and the respective index i is referred to as the rank. Next, failure probabilities F_i are assigned to the order statistics. An estimation for the failure probability can be defined by various values related to the beta distribution with the help of the ranks:

Median: $$F_i \approx \frac{i-0.3}{n+0.4} \qquad i = 1(1)r\,, \tag{6.51}$$

Mean: $$F_i = \frac{i}{n+1} \qquad i = 1(1)r\,, \tag{6.52}$$

Mode: $$F_i = \frac{i-1}{n-1} \qquad i = 1(1)r\,. \tag{6.53}$$

These failure probabilities are then adapted to a line equation in the form

$$y(x) = m \cdot x + c \tag{6.54}$$

The probability distributions in a reliability analysis can be transformed into a straight line equation by respectively reforming the distributions. After such a transformation, the variable x becomes a function of the lifetime t:

$$x = x(t)\,. \tag{6.55}$$

The slope m and the ordinate intersection factor c become functions of the distribution factor k

$$\psi_\ell\,,\ \ell = 1(1)k\,, \tag{6.56}$$

which can be summarized together to form a parameter vector:

$$\vec{\psi} = (\psi_1, \ldots, \psi_\ell, \ldots, \psi_k)\,. \tag{6.57}$$

Since the straight line is uniquely determined by the slope and ordinate intersection, it is possible to define maximal two parameters with this adaptation. Equation (6.54) is transformed into the equation below:

$$y(x(t)) = m(\vec{\psi}) \cdot x(t) + c(\vec{\psi})\,, \tag{6.58}$$

for which the following applies when using order statistics:

$$x_i = x(t_i) \quad \text{and} \quad y(x(t_i)) = y(x_i)\,. \tag{6.59}$$

With this transformation it is necessary to transform the failure probabilities of the order statistics accordingly:

$$y_i = y(F_i)\,. \tag{6.60}$$

For the adaptation of the equation, the function value $y(x(t_i))$ is subtracted from the transformed failure probabilities y_i. The results of this subtraction are interpreted as the failure r_i, Figure 6.. A total of n equations results in the form

$$y(F_i) - y(x(t_i)) = y_i - m \cdot x_i - c = r_i\,. \tag{6.61}$$

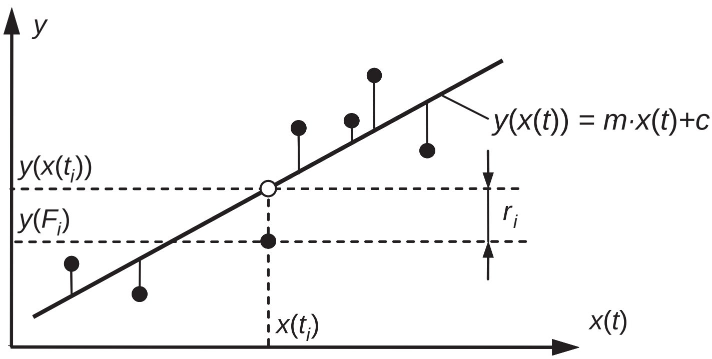


Figure 6.31. Line of regression

According to *Gauß*, a good estimate for the variables for the lines of regression sought for, m and c, can be achieved, if the failure squares sum ρ^2 is minimized:

$$\rho^2 = \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (y_i - m \cdot x_i - c)^2 \rightarrow min\,. \tag{6.62}$$

For the minimization, the first partial derivatives of ρ^2 with respect to *m* and *c* are set equal to zero:

$$\frac{\partial \rho^2}{\partial c} = -\sum_{i=1}^{n} 2 \cdot (y_i - m \cdot x_i - c) \Rightarrow \sum_{i=1}^{n} (y_i - m \cdot x_i - c) = 0\,,$$
$$\frac{\partial \rho^2}{\partial m} = -\sum_{i=1}^{n} 2 \cdot x_i \cdot (y_i - m \cdot x_i - c) \Rightarrow \sum_{i=1}^{n} x_i \cdot (y_i - m \cdot x_i - c) = 0\,. \tag{6.63}$$

The result is a linear system of equations (normal equations) for the two unknowns *m* und *c*:

$$n \cdot c + \left(\sum_{i=1}^{n} x_i \right) \cdot m = \sum_{i=1}^{n} y_i\,,$$
$$\left(\sum_{i=1}^{n} x_i \right) \cdot c + \left(\sum_{i=1}^{n} x_i^2 \right) \cdot m = \sum_{i=1}^{n} x_i \cdot y_i\,, \tag{6.64}$$

with which the following solutions can be found considering the mathematical Equation (6.37):

$$m = \frac{\sum_{i=1}^{n} (x_i - \overline{x}) \cdot (y_i - \overline{y})}{\sum_{i=1}^{n} (x_i - \overline{x})^2} \tag{6.65}$$

$$c = \overline{y} - m \cdot \overline{x}\,. \tag{6.66}$$

Since *m* and *c* are functions of the distribution parameters, a maximum of two parameters can be calculated out of a reverse transformation of the equations above. To judge the quality of the approximation of the line, the correlation coefficient *K* is determined:

$$K = \frac{\sum_{i=1}^{n} (x_i - \overline{x}) \cdot (y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \overline{x})^2 \cdot \sum_{i=1}^{n} (y_i - \overline{y})^2}}\,. \tag{6.67}$$

The correlation coefficient is a variable for the strength and direction of a relationship between coordinates. For a complete linear dependency the correlation coefficient is $K = -1$ or $K = 1$, depending upon whether the coordinates change in the same direction or in the opposite direction. Such a case is referred to as a functional dependency. If there is no dependency between the coordinates, then $K = 0$. A stochastic dependency is at hand if the absolute value of the correlation coefficient lies between 0 and 1 ($0 < |K| < 1$). Figure 6.32 shows the various dependency possibilities.

The quality of the approximation of a linearly transformed distribution function to the coordinates can also be judged by the correlation coefficient. The closer the absolute value of K is to 1.0, the better the approximation. For the approximation of a distribution for failure data, a stochastic dependency always results.

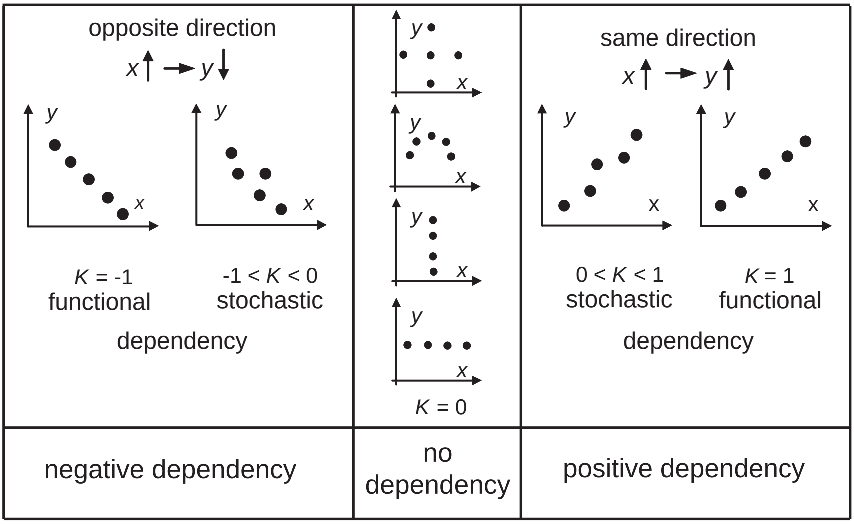


Figure 6.32. Forming the dependency between the coordinates (x, y) and the correlation coefficient K

Example: Weibull distribution

The origin of the Regression Analysis is the Weibull chart. Mathematically, this corresponds to the equation for the line

$$\underbrace{\ln\left(-\ln(1-F(t))\right)}_{y(x(t))} = \underbrace{b}_{m(b)} \cdot \underbrace{\ln(t)}_{x(t)} - \underbrace{b \cdot \ln(T)}_{c(b,T)}, \tag{6.68}$$

assuming the distribution to be a two parametric distribution. The transformed failure probabilities can be found by using the median value, for example,

$$y_i = ln\left(-ln\left(1 - \frac{i-0.3}{n+0.4}\right)\right). \tag{6.69}$$

By using the Regression Analysis, the following two Weibull parameters b and T can be found with the following equations:

$$b = \frac{\sum_{i=1}^{n}(x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\text{with} \quad y_i = ln(-ln(1-F_i)) \; ; \bar{y} = \frac{1}{n}\sum_{i=1}^{n} ln(-ln(1-F_i)) \tag{6.70}$$

$$\text{and} \quad x_i = ln(t_i) \; ; \; \bar{x} = \frac{1}{n}\sum_{i=1}^{n} ln(t_i)$$

$$\text{and } T = \exp\left(-\frac{\bar{y} - b \cdot \bar{x}}{b}\right). \tag{6.71}$$

The correlation coefficients can be taken from Equation (6.67). If the distribution contains a failure free time t_0 as a third parameter, then the calculation using the Regression Analysis is more complicated, since it must then be carried out iteratively. The calculation is carried out as for a two parametric distribution, but the measured values become $x_i = \ln(t_i - t_0)$. This is represented as a straight line in Weibull paper, if t_0 is to be sought. The quality of this approximation can again be judged by the value of the correlation coefficient, Equation (6.67). Thus, with a targeted variation of the parameter t_0 a maximum for the correlation coefficient can be determined. The iteration is done with the help the *Golden Section Search* algorithm [6.9].

6.6.3 Maximum Likelihood Method

One very good statistical method for the determination of unknown parameters of a distribution is the Maximum Likelihood Method from *R.A. Fisher*. The procedure assumes that the histogram of the failure frequency depicts the number of failures per interval, Figure 6.33.

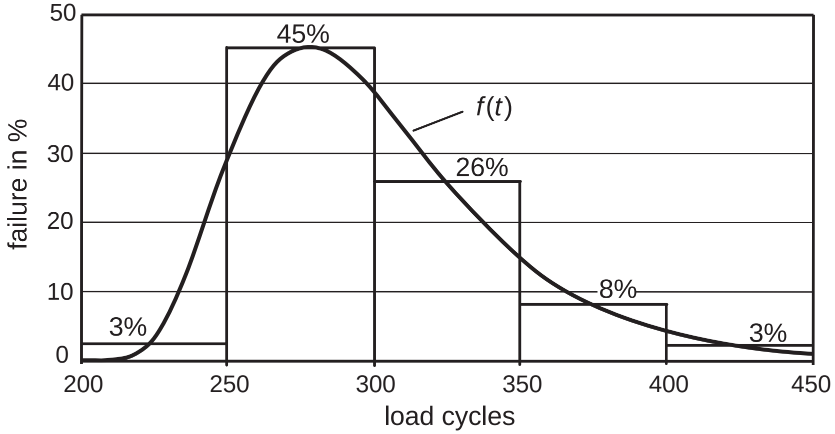


Figure 6.33. Histogram of failure frequency and density function

For larger test specimen sizes n it is possible to derive the density function out of the histogram and thus to exchange from the frequencies to the probabilities (law of large numbers). In this way it is possible to state, for example that during the first interval in Figure 6.33, probably 3% of all failures will occur. In the second interval it is most likely that 45% of the failures occur, etc. According to theory, the probability L, that exactly the test specimen is at hand as given in Figure 6.33, can be found by the product of the probabilities of the individual intervals:

$$L = f(t_1) \cdot f(t_2) \cdot \ldots \cdot f(t_n) \,. \tag{6.72}$$

This function is called the likelihood function. The idea of this procedure is to find a function f, for which the product L is maximized. Here, the function must possess high values of the density function f in the corresponding regions with several failure times t_i. At the same time in regions with few failures, only low values of f are found. Thus, the actual failure behaviour is accurately represented. The function f, if determined in this way, offers the best probability to describe the test specimen.

A test specimen with n observation values t_i, $i = 1(1)n$ is given, whose density function $f(t)$ possesses k unknown parameters ψ_ℓ, $\ell = 1(1)k$. These parameters are often summarized as $\vec{\psi} = (\psi_1, ..., \psi_\ell, ..., \psi_k)$. The likelihood function for such a case is as follows:

$$L(t_1, ..., t_i, ..., t_n; \psi_1, ..., \psi_\ell, ..., \psi_k) = \prod_{i=1}^{n} f(t_i; \psi_1, ..., \psi_\ell, ..., \psi_k) \,. \tag{6.73}$$

Commonly, the likelihood function is logarithmized. Thus the product equation becomes an addition equation, which greatly simplifies the differentiation. Since the natural log is a monotonic function, this step is mathematically logical. What remains is:

$$\ln(L) = \ln\left(L(t_1,...,t_i,...,t_n;\vec{\psi})\right) = \sum_{i=1}^{n} \ln\left(f(t_i;\vec{\psi})\right). \tag{6.74}$$

For the estimation of the k parameters ψ_ℓ according to the above considerations, those parameters are used for which the likelihood function reaches its maximum. These parameters can be found by setting the k partial differentiations of the likelihood function with respect to the k parameters equal to zero. The maximum of the logarithmized likelihood function and thus the statistically optimal parameters ψ_ℓ can be found out of the equations:

$$\frac{\partial \ln(L)}{\partial \psi_\ell} = \sum_{i=1}^{n} \frac{1}{f(t_i;\vec{\psi})} \cdot \frac{\partial f(t_i;\vec{\psi})}{\partial \psi_\ell} = 0 \ , \quad \ell = 1(1)k \ . \tag{6.75}$$

These equations can be nonlinear in the parameters; therefore it is often useful to apply appropriate numerical procedures. Figure 6.34 schematically shows these ratios for a two parametric Weibull distribution.

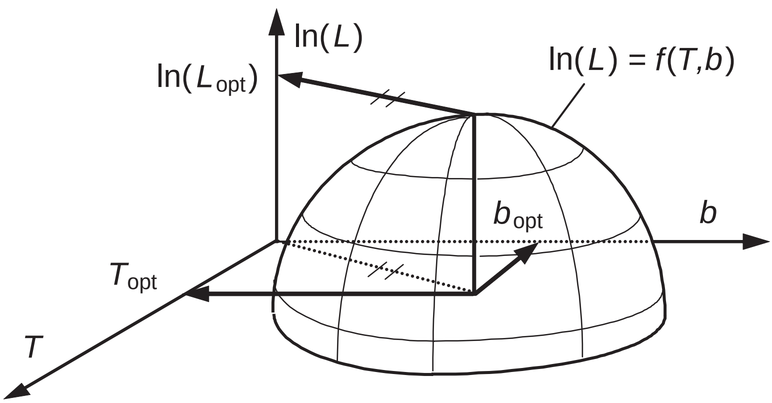


Figure 6.34. Schematic representation of the logarithmized likelihood function

Through the Likelihood function value the opportunity is given to estimate the quality of the adaptation of a distribution to the failure data, even if the calculation is not carried out by means of the Maximum Likelihood Method. The greater the likelihood function value is, the better the determined distribution function describes the actual failure behaviour. It often seems to be confusing, that negative values result for $\ln(L)$ through the logarithmized transformation. A better adaptation can be recognized by higher absolute values of $\ln(L)$.

Example: Weibull distribution

In order to carry out the Maximum Likelihood Method, the failure density will be used in a different form with the parameter $\eta = T - t_0$. This results to a density function as seen below:

$$f(t) = \frac{b}{\eta} \cdot \left(\frac{t - t_0}{\eta}\right)^{b-1} \cdot e^{-\left(\frac{t-t_0}{\eta}\right)^b} . \tag{6.76}$$

The logarithmized likelihood function results to:

$$\ln(L(t_1,\ldots,t_i,\ldots,t_n;b,\eta,t_0)) = n \cdot \ln\left(\frac{b}{\eta^b}\right) + \sum_{i=1}^{n}\left[(b-1)\cdot\ln(t_i - t_0) - \left(\frac{t_i - t_0}{\eta}\right)^b\right] . \tag{6.77}$$

The partial differentiation with respect to the parameters is as follows:

$$\frac{\partial \ln(L)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\left[\ln\left(\frac{t_i - t_0}{\eta}\right) \cdot \left\{1 - \left(\frac{t_i - t_0}{\eta}\right)^b\right\}\right] = 0 , \tag{6.78}$$

$$\frac{\partial \ln(L)}{\partial \eta} = -n + \frac{1}{\eta} \cdot \sum_{i=1}^{n}(t_i - t_0)^b = 0 , \tag{6.79}$$

$$\frac{\partial \ln(L)}{\partial t_0} = \sum_{i=1}^{n}\left[\frac{1-b}{t_i - t_0} + \frac{b}{\eta} \cdot (t_i - t_0)^{b-1}\right] = 0 . \tag{6.80}$$

This system of equations is nonlinear. Therefore it must be solved iteratively. First, however, several transformations must take place. The newly introduced parameter η out of Equation (6.79) is solved for:

$$\eta = \sqrt[b]{\frac{1}{n} \cdot \sum_{i=1}^{n}(t_i - t_0)^b} . \tag{6.81}$$

Substituting this equation into Equation (6.80) leads to:

$$\sum_{i=1}^{n}\left[\frac{1-b}{t_i-t_0}+n\cdot b\cdot\frac{(t_i-t_0)^{b-1}}{\sum_{j=1}^{n}(t_j-t_0)^b}\right]=0\,. \tag{6.82}$$

The following approach has proven to be successful:

1. Choose t_0 out of the range $0 < t_0 < t_1$.
2. Iteratively determine the shape parameter b at the point in time t_0 with Equation (6.82).
3. With these two values calculate η out of Equation (6.81).
4. With these values calculate the value for the likelihood function with Equation (6.77).
5. Vary t_0 and repeat from step 2 until the maximum is found.

In order to allow a quick determination of the maximum, the method according to *Brent* [6.9] is applied.

6.7 Exercises to Assessment of Lifetime Tests

Problem 6.1

In a pre-production test components were tested for their lifetime. All components failed. The corresponding failure times are recorded below:

69,000 km,	29,000 km,	24,000 km,	52,500 km,
128,000 km,	60,000 km,	12,800 km	98,000 km.

a) Calculate the mathematical mean, the standard deviation and the spread of the failure data.
b) Determine the order statistics and assign them a failure probability.
c) Determine the parameters for the Weibull distribution that describes the failure behaviour with help of the form sheet (Weibull chart).
d) Determine the B_{10} lifetime and the median.
e) With which probability can a component survive t_1 = 70,000 km?
f) Draw the 90% confidence intervals for the Weibull lines.
g) Calculate the 90% confidence intervals for the parameters b and T. Determine these confidence intervals graphically.

Problem 6.2

Complete failure data is given for a mechanical switch:

470; 550; 600; 800; 1,080; 1,150; 1,450; 1,800; 2,520; 3,030 operations.

a) Determine the parameters of the Weibull distribution graphically.
b) Determine the B_{10} lifetime and the median.
c) Draw the 90% confidence intervals of the Weibull function.

Problem 6.3

Lifetime values are given from torsion vibration trials of an uncensored test specimen of crenated drive shafts out of 41 Cr 4 with a stress amplitude of 200 MPa.

Lifetime values (in 10^3 load cycles):

264, 208, 222, 434, 382, 198, 380, 166, 435, 242.

Assess this lifetime data using a Weibull chart and determine the 90% confidence interval.

Problem 6.4

In a test 8 similar components were tested simultaneously on a test stand. The test was interrupted after the failure of the fifth component. Determine the failure behaviour of the components in a Weibull distribution, their 90% confidence interval and the confidence limits of the parameters.

Failure data (in h): 192, 135, 102, 214, 167

Problem 6.5

An incomplete test specimen is given, where the lifetime of planet carrier-head screws in farm tractors was recorded. A total of 1075 tractors were involved. After 10 failures due to broken head screws, an assessment should be carried out with the recorded data. The running time until breakage of the planet carrier-head screws are given:

99, 200, 260, 300, 340, 430, 499, 512, 654, 760.

The running times of the intact units at the time of the assessment are unknown.

a) Assess the test specimen graphically with the Sudden Death procedure (determine the straight line for the first failures and extrapolate the lifetime distribution for the total test specimen).
b) Calculate the test specimen using the Sudden Death procedure (determine the hypothetical ranks). Compare the results with part a).

Problem 6.6

A field study for a reliability analysis of a vehicle clutch was carried out. 20 clutches were available for the analysis. Up until the point of analysis $n_f = 8$ clutches failed, that means that $n_s = 12$ clutches were still functional. The following length of runs (in 10^3 km) of failed and intact clutches are given in the table below:

Failed components:	7, 24, 29, 53, 60, 69, 100, 148,
Intact components:	5, 6, 19, 32, 39, 40, 65, 70, 76, 85, 157, 160.

a) Determine the lifetime distributions under consideration of the intact clutches.
b) Determine the 90% confidence interval and the respective confidence limits of the parameters.

Problem 6.7

Warranty and amiability data of an omnibus transmission should be assessed after one year. A total of $n = 178$ transmissions were delivered and $r = 7$ of them have failed.

Failure data (in km):

18,290; 160,770; 51,450; 89,780; 130,580; 35,200; 51,450.

The length of run distribution for the omnibuses is described by a normal distribution with $\mu = 80{,}000$ km and $\sigma = 45{,}000$ km. Determine the Weibull distribution which describes the failure behaviour.

Problem 6.8

Information is given concerning the failure data of an uncensored test specimen: 42, 66, 87 and 99 h.

a) Use the Regression Analysis to calculate the parameters b and T for the two parametric Weibull distribution describing the failure behaviour.
b) What is the correlation coefficient?
c) Determine the logarithmized likelihood function value.

Problem 6.9

Give a generally valid relationship for the estimation of the failure rate λ and the failure free time t_0 of a two parametric form of the exponential distribution with the density

$$f(t) = \lambda \cdot \exp\left(-\lambda(t - t_0)\right)$$

with known failure data t_i, $i = 1(1)n$

a) using the Method of Moments,
b) using the Maximum Likelihood Method and
c) using the Regression Analysis.

References

[6.1] Bertsche B, (1989) Zur Berechnung der System Zuverlässigkeit von Maschinenbau-Produkten. Diss Universität Stuttgart, Institut für Maschinenelemente und Gestaltungslehre, Inst. Ber. Nr. 28

[6.2] Birolini A (2004), Reliability Engineering: theory and practice. Springer-Verlag Berlin, Heidelberg

[6.3] Dubey S D (1967) On Some Permissible Estimators of the Location Parameter of the Weibull and Certain Other Distributions. Technometrics, Vol 9, No. 2, May, p 293–307

[6.4] Eckel G (1976-9) Bestimmung des Anfangsverlaufs der Zuverlässigkeitsfunktion von Automobilteilen. Qualität und Zuverlässigkeit 22

[6.5] Forschungsvereinigung Antriebstechnik (1981) Einfluss moderner Schmierstoffe auf die Pittingbildung bei Wälz- und Gleitbeanspruchung. Arbeitsgruppe „Pitting-Ringversuch“, FVA-Forschungsreport, Wiesbaden

[6.6] Meeker W, Escobar L (1998) Statistical methods for reliability data. Wiley: New York

[6.7] Kapur K C, Lamberson L R (1977) Reliability in Engineering Design. John Wiley & Sons Inc., New York

[6.8] John P (1990) Statistical methods in engineering and quality assurance. Wiley: New York

[6.9] Press W H, Flannery B P, Teukolsky S A, Vetterling W T(1988) Numerical Recipes in C - The Art of Scientific Computing, Cambridge University Press

[6.10] Reichelt C (1978) Rechnerische Ermittlung der Kenngrößen der Weibull-Verteilung. Fortschr.-Ber. VDI-Z, Reihe 1 Nr. 56

[6.11] Tittes E (1973) Über die Auswertung von Versuchsergebnissen mit Hilfe der Weibullverteilung. Qualität und Zuverlässigkeit 18, Heft 5, S 108–113, Heft 7, S 163–165

[6.12] Uludag A I (2/1972) Aussagen über die zeitliche Entwicklung von Schadensfällen anhand weniger Informationen aus dem Felde – Anwendung von Computer und Weibull-Methode. Grundl. Landtechnik, S 47/48

[6.13] Verband der Automobilindustrie (2000) VDA 3.2 Zuverlässigkeitssicherung bei Automobilherstellern und Lieferanten. VDA, Frankfurt

[6.14] Weber H (8/1976) Statistische Auswertung von Lebensdauerversuchen nach Weibull bei Entwicklung von Bauelementen der Pneumatik. Hydraulik und Pneumatik, S 529–533

[6.15] Pham H (2003) Handbook of reliability engineering. Springer: London, Berlin, Heidelberg

[6.16] O’Connor P (2002) Practical reliability engineering. Wiley: Chichester

7 Weibull Parameters for Specifically Selected Machine Components

The failure behaviour of some components can be determined accurately by accordant extensive statistical analysis. The analysis can be carried out with results of tests, data for damage statistics or with data given in literature. Knowledge about the failure behaviour of components enables the forecasting of the expected failure behaviour of elements with comparable operating conditions. Also, the expected failure behaviour of systems can then be calculated with system theory. There is a dearth of published relevant contemporary information pertaining to the failure behaviour of mechanical components. As a consequence the Institute of Machine Components (IMA) has initiated a reliability base [7.1]. In the following text selected results from this data base will be shown for the machine components: gears, shafts and bearings, for which extensive information is available.

While beginning to compile this reliability date base it was discovered that in most cases only very few failure times were available ($n = 5,\ldots, 10,\ldots, 20$). As commonly known for all statistical methods, the significance of a statistical assessment increases remarkably with the amount of failed components. It is at least possible to estimate the dimension of the parameter to be determined with a high confidence intervals if numerous results are at hand.

A further problem in the set-up of the reliability data base is the fact that the statistical parameters b, T, t_0 are dependent upon various influential factors:

$$(b, T, t_0) = f(\text{shape, material, machining, stress}).$$

This means that it is possible, that each component requires special parameters depending on the case of operation. The analyzed tests and damage statistics have shown, however, that the shape parameter b and the factor $f_{tB} = t_0/B_{10}$ of a component under a certain stress level remain relatively constant for the occurred type of damage. The results available

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_7,

indicate that these parameters only depend on operational demands, the failure mechanism and partly on the stress. Consequently, it is sufficient to determine the shape parameter and the factor of the failure free time only once out of a very extensive test or to estimate these parameters from the results of many tests. Thus, in many cases the parameters introduced in the following sections can serve as a first orientation.

The statistical analyses of the reliability data base were executed with the calculation programs WEIBULL and SYSLEB. For the fatigue and wearout failures examined, a three parametric Weibull distribution was always used. This assumption is reinforced by new analyses in [7.2, 7.3, 7.5]. The method used for the analysis was the Regression Analysis. The differences to the Method of Moments and the Maximum Likelihood Method have proved to be minimal in several comparisons.

7.1 Shape Parameter b

A summary of the determined shape parameters is shown in Figure 7.1. The spread of the shape parameters represents the confidence intervals of the statistical analysis and a dependency on the stress level. For gears and shafts the shape parameter *b* must be chosen according to the stress. For these components a higher stress calls for a larger shape parameter.

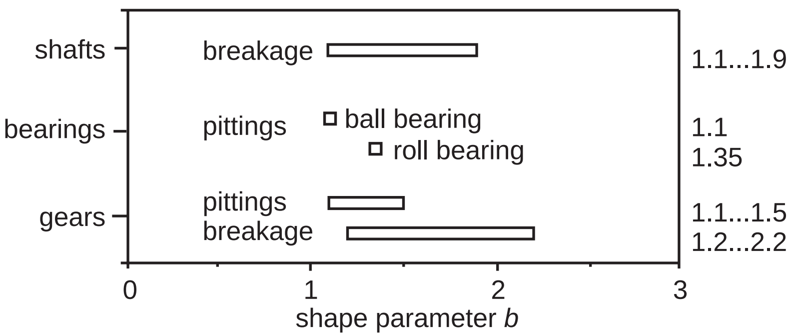


Figure 7.1. Determined shape parameters b for the three parametric Weibull distribution of select machine components (for gears and shafts: higher stress → larger shape parameter; lower stress → smaller shape parameter)

For the determination of the shape parameter *b* for shafts (type of damage: crack) two very interesting tests exist: one according to Maennig [7.6] and another according to Kitschke [7.5]. Maennig calls for tests carried out on many varying stress levels, where each test consists of $n = 20$ test values. Thus, the dependency between the shape parameter and the stress can be clearly demonstrated. Maennig began his tests in the fatigue strength region with loads close to the fatigue strength and then increased the stress step by step up to the static strength. The shape parameter thereby increased from b = 1.1 to b =1.9. Kitschke, on the other hand, carried out only a few tests with very many test values (n = 99 ... 112). Thus, the resulting statistical confidence is quite high. Furthermore, Kitschke was the only one who conducted an extensive statistical analysis, thus offering an excellent example for the determination of reliability parameters. The shape parameters he determined for medium stress lie between b = 1.5 and b = 1.9.

For roll bearings, extensive tests exist with up to n = 500 and provide a notably high statistical confidence. Furthermore, bearings are the only machine components for which the failure behaviour is documented in standards: DIN 622 and ISO DIN 281. Here, the shape parameter was determined for a two parametric Weibull distribution due to the fact, that the failure free time of bearings is relatively small (see Section 7.2). The analysis with a three parametric Weibull distribution also exhibits only small variances. According to Bergling [7.1], the shape parameter is independent of size, type and stress of the bearing. This simplifies the application considerably.

The determined and analyzed tests of gears were carried out for a relatively small amount of tests (n = 5 ... 20). The dependency of the shape parameter upon the stress can be seen here as well. With increasing stress the shape parameter b also increases. For the type of damage “crack” shape parameters are yielded similar to the parameters for shafts (type of damage: crack). For pittings, the range of dispersion of the gears is not as large as for cracks and the values for b are approximately in the same range as for pittings of roll bearing.

The determined shape parameters range from b ≈ 1 ... 2. With a closer look at Figure 7.2 one can see, that the failure behaviour of all elements has a left symmetric distribution. Such a left symmetric distribution seems to be typical for the failure behaviour of classic machine components

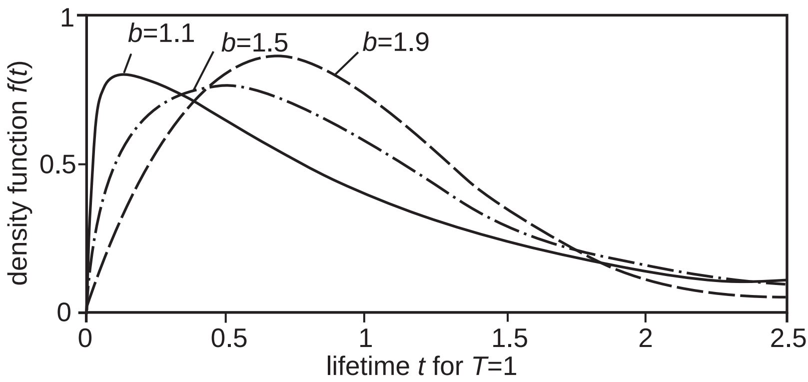


Figure 7.2. Density function of the Weibull distribution for b = 1.1 ... 1.9

The shape parameters b yielded for the three parametric Weibull distribution tend to be smaller than those yielded for the two parametric Weibull distribution. The cause for this discrepancy can be explained by Figure 7.3.

For most tests the histogram of the density function shows a left symmetrical shape and the lowest failure time is the time t_0. The two parametric Weibull distribution, according to its definition, must begin at $t = 0$ and attempts to describe the histogram with these conditions. Thus, this results in an almost symmetrical curve linearity ($b \approx 2$ ... 3).

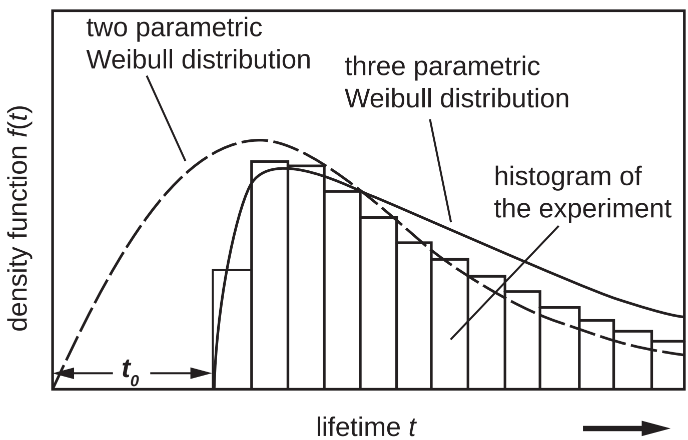


Figure 7.3. Histogram of a test and density distributions of the two and three parametric Weibull distributions

The three parametric Weibull distribution may begin with t_0 and thus offers a better approximation of the histogram. Afterwards, a left symmetric distribution ($b \approx 1$ ... 2) is yielded, whereas the shape parameter b is by definition smaller as for symmetric shape parameters.

7.2 Characteristic Lifetime T

The characteristic lifetime T is the scale parameter of the Weibull distribution and can therefore be regarded as the mean of the distribution. Increasing the characteristic lifetime T results in a shift of the complete failure behaviour to higher failure times.

Whereas the shape parameter b and the factor f_{tB} primarily depend upon the machine component and upon the type of damage, see Sections 7.1 and 7.3, the characteristic lifetime T can be regarded as a function of stress. For all components, lower stress leads to higher failure times and the characteristic lifetime T increases.

For the prognosis of the failure behaviour the characteristic lifetime T is generally determined by a lifetime calculation or an operational fatigue strength calculation. A certain calculation method leads to a lifetime which is combined with an expected failure probability F(t). For example, for the calculation of roll bearings one yields by defining the B10 lifetime (F(t) = 10%) and for the calculation of gears the B1 lifetime (F(t) = 1%). With these lifetimes one receives one point each on the probability net, see Figure 7.4. The complete statistical failure behaviour can be determined with the additional knowledge of the shape parameter b and by the potential failure free time t_0. For components, for which no secured lifetime calculation exists, one depends on experience (e.g. out of damage and warranty statistics), estimations or tests. Out of the B_1 and/or B_{10} lifetimes the corresponding characteristic lifetime T ($F(t) = 63{,}2\%$) can be calculated with the Equations (7.1) and (7.2)

The characteristic lifetime T for the determined B_1 lifetime:

$$T = \frac{B_1 - f_{tB} \cdot B_{10}}{\sqrt[b]{-\ln 0{,}99}} + f_{tB} \cdot B_{10}\,, \tag{7.1}$$

whereby
$$B_{10} = \frac{B_1}{(1 - f_{tB}) \cdot \sqrt[b]{\dfrac{\ln 0{,}99}{\ln 0{,}9}} + f_{tB}}\,.$$

The characteristic lifetime T for the determined B_{10} lifetime:

$$T = \frac{B_{10} - f_{tB} \cdot B_{10}}{\sqrt[b]{-\ln 0{,}9}} + f_{tB} \cdot B_{10}\,. \tag{7.2}$$

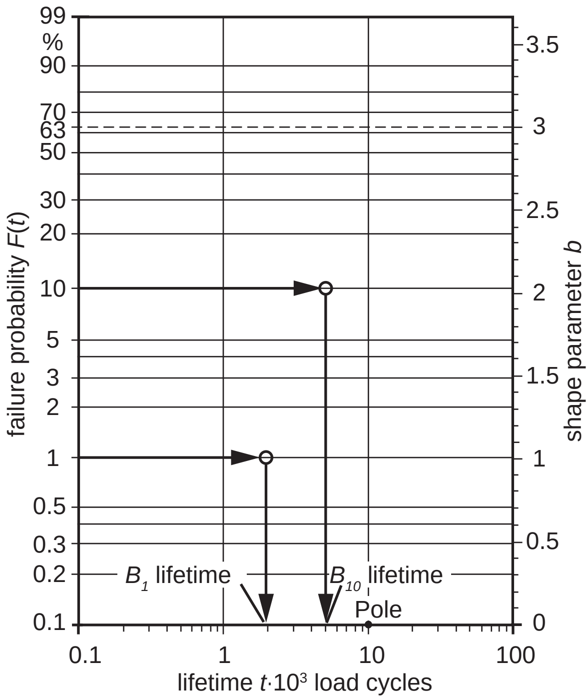


Figure 7.4. Calculated B_1 and B_{10} lifetime in Weibull probability paper (example)

For the case of a B_x lifetime, the corresponding characteristic lifetime T is:

$$T = \frac{B_x - f_{tB} \cdot B_{10}}{\sqrt[b]{- ln(1 - x)}} + f_{tB} \cdot B_{10} \tag{7.3}$$

$$\text{whereby} \quad B_{10} = \frac{B_x}{(1 - f_{tB}) \cdot \sqrt[b]{\dfrac{\ln(1 - x)}{\ln(0{,}9)}} + f_{tB}}$$

(The Equations (7.1) to (7.3) were derived from the general equations of the Weibull distribution).

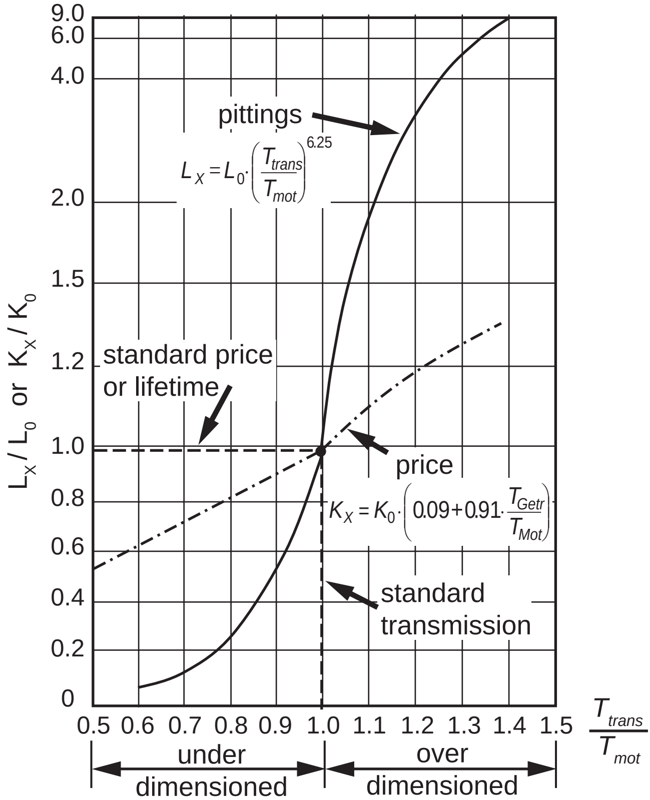


Figure 7.5. Dependency of pitting load-carrying capacity and transmission cost in relation to the dimension

The meaning and/or sensitivity of an exact determination of the characteristic lifetime *T* can be shown by the example of the pitting load-carrying capacity of a gear in Figure 7.5. The slightest incline of the Wöhler curve, expressed by the exponent $k = 6.25$, leads to an almost doubled lifetime for an over-dimensioning of only 10%. For the more critical case, an under-dimensioning of 10%, the gear already fails after half of the standard lifetime. The costs for the gear, however, only change insignificantly. In [7.2] it is also shown, that a change of the characteristic lifetime *T* has the most significant influence on the calculated system reliability. An appropriate accurate prognosis of the system lifetime can therefore only be achieved by a secured operational fatigue strength calculation.

7.3 Failure Free Time t_0 and Factor f_{tB}

As already mentioned, the failure behaviour of the components can only be described accurately with a three parametric Weibull distribution for most fatigue and wearout failures. Especially for the calculation of the system lifetime, the initial range of the failure behaviour must be collected very accurately. This consideration makes the three parametric Weibull distribution with its third additional parameter – the failure free time t_0 – absolutely necessary, see [7.2, 7.3, 7.4, 7.5].

The failure free time t_0 of fatigue and wearout failures implies that a certain time is required before damage appearance and propagation. Without this assumption, the failures caused by wearout, fatigue, aging etc. would have to appear already after short operation times. This, however, contradicts general knowledge and belief.

By executing the analysis of the data base it has proven to be useful not to specify the failure free time as an absolute value but rather in the form of the factor $f_{tB} = t_0/B_{10}$. With this factor, the values are much better for comparison. The lifetime B_{10} has been taken as a reference, because it shows no significant discrepancies between two and three parametric analysis, and not to mention, it is also in the initial range of the failure event. The summary of the determined factors f_{tB} is shown in Figure 7.6.

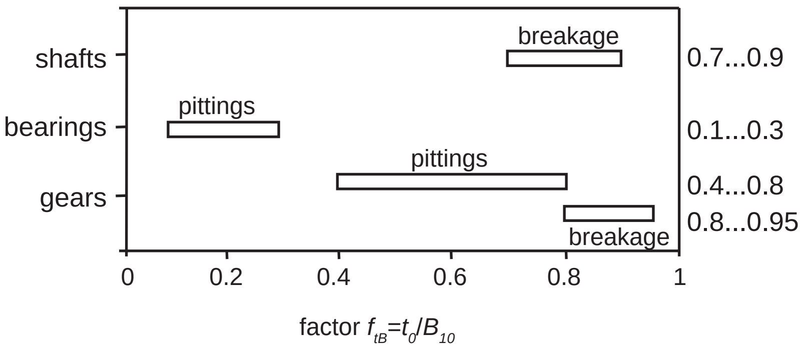


Figure 7.6. Factors $f_{tB} = t_0/B_{10}$ for select components

A dependency of the stress level on the factors could not been determined so far. Thus, for a conservative estimation a smaller factor should be chosen, whereas for an optimistic estimation a higher factor can be used.

The failure free time t_0 was determined in tests according to Mann, Scheuer and Fertig [7.7]. In this test, a significance level α was calculated for the period of time $0 < t_0 < t_1$. Because t_0 and the lifetime B_{10} are statistical variables, the factor f_{tB} was in each case determined with the median values of both parameters.

With the extensive tests from Kitschke [7.5] for shafts (type of damage: "crack") the nearly compulsory statistical proof could be executed, that there has to be a failure free time t_0. Interestingly enough, for roll bearings rather small values were yielded for the factor f_{tB}. For gears with pittings as the type of damage, only large ranges of failure free times could be determined. In analyzing further tests it should be possible to narrow this range down. For gears with "crack" as the type of damage, similar values were yielded as for shafts, with the same type of damage

References

[7.1] Bergling G (1976) Betriebszuverlässigkeit von Wälzlagern. SKF Kugellager / Zeitschrift 188, Jhrg 51

[7.2] Bertsche B (1989) Zur Berechnung der System-Zuverlässigkeit von Maschinenbau-Produkten. Dissertation Universität Stuttgart, Institut für Maschinenelemente und Gestaltungslehre, Inst. Ber. Nr 28

[7.3] Bertsche B, Lechner G (1986) Verbesserte Berechnung der Systemlebensdauer von Produkten des Maschinenbaus. Konstruktion 38, Heft 8, S 315-320

[7.4] Bertsche B, Lechner G (1987) Einfluß der Teileanzahl auf die System-Zuverlässigkeit. Antriebstechnik 26, Nr 7, S 40-43

[7.5] Kitschke E (1983) Wahrscheinlichkeitstheoretische Methoden zur Ermittlung der Zuverlässigkeitskenngrößen mechanischer Systeme auf der Grundlage der statistischen Beschreibung des Ausfallverhaltens von Komponenten. Dissertation Ruhr-Universität Bochum, Lehrstuhl für Maschinenelemente und Fördertechnik, Heft 83

[7.6] Maennig W W (1967) Untersuchungen zur Planung und Auswertung von Dauerschwingversuchen an Stahl in den Bereichen der Zeit- und der Dauerfestigkeit. VDI-Fortschrittberichte, Nr 5, August

[7.7] Mann N R, Fertig K W (1975) A Goodness of Fit Test for the Two Parameter vs. Three Parameter Weibull; Confidence Bounds for Threshold. Technometrics, vol 17, No 2, May

8 Methods for Reliability Test Planning

This chapter deals with the main principles and procedures of planning lifetime tests. Planning lifetime tests can be divided into statistical test planning and experimental-technical measurement planning, see Chapter 6. Common principles of correct test execution are valid for the latter of the two [8.3, 8.4, 8.8, 8.11, 8.12].

The size of the test specimen is the first aspect of statistical test planning and is closely related to the confidence intervals and the statistical spread of experimental values, see Section 6.3. The less components are tested, the greater the confidence interval is and the results of a statistical analysis become more uncertain. Thus, for a more precise result it is necessary that enough machine components are tested. This, however, can greatly increase the time and effort involved in testing.

How and which machine components should be tested, referred to as test specimen extraction, must also be determined for statistical test planning. The test specimen should represent a real random test specimen, which implies that the components tested are determined at random. Only then is the fundamental condition for a representative random test specimen fulfilled.

Furthermore, a suitable test strategy must be defined for statistical test planning. One differentiates between the various possibilities of:

- complete tests,
- incomplete (censored) tests and
- strategies for test time shortening.

The best statistical option is the complete test, where all machine components of a random test specimen are subjected to a lifetime test. The test is run until the last element has failed. The result is that the failure times of all elements are available for analysis.

In order to reduce the time and effort of testing, it may prove to be beneficial to carry out an incomplete test, which is also sometimes called a censored test. In this case, the test is run until a predetermined lifetime is reached or until a certain number of elements have failed. This type of test is not as meaningful and exact as a complete test but is associated with a considerably lower test effort.

Sudden Death Tests and tests with an increased load offer further possibilities for a significant shortening of the test time. Section 6.4 deals

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_8, © Springer-Verlag Berlin Heidelberg 2008

with the evaluation of incomplete tests and with strategies for shortening of test time in detail.

The fundamental task of test planning is to certify the achievement of the required reliability, which is given by the reliability demands of:

- the number of units to be tested (n = ?) and
- the required test duration (t_{test} = ?)

A minimal reliability at a certain lifetime is a common problem specifications in the practical field, for example, a required reliability of e.g. R(200,000 km) = 90%, which corresponds to a B_{10} lifetime of 200,000 km. Additionally, a confidence level is determined (e.g. 95%, 90% or 80%), with which the reliability requirement can be proven. Often, it is common that *no* failure is expected during a test run. This type of test is called a “success run”. Furthermore, cost and time conditions can be set.

This chapter will deal with life time tests:

- Statistical test planning,
- Measurement planning,
- In statistical planning greater confidence when larger number of tests considered,
- Discrimination in machine selection for test planning,
- Sudden death tests are considered.

8.1 Test Planning Based on the Weibull Distribution

An example is observed with a product requirement of $R(t)$ = 90% with a one-sided confidence level of P_A = 95%. The conditions are illustrated in a Weibull chart in Figure 8.1.

Example:

A reliability of R(200,000 km) = 90% is required with a confidence level of P_A = 95%. Out of the 95% confidence level table one searches for the column which offers a lower failure probability as the previously required failure probability of F(200,000 km) = 10% for i = 1. This is the case for n = 29. This situation is shown in a Weibull chart in Figure 8.2.

One can now make the following statement: if n = 29 test units reach a test time t = 200,000 km without failure, then $R(t)$ = 90% with a certainty /

probability of 95%. A universal procedure based on the binomial distribution will now be introduced.

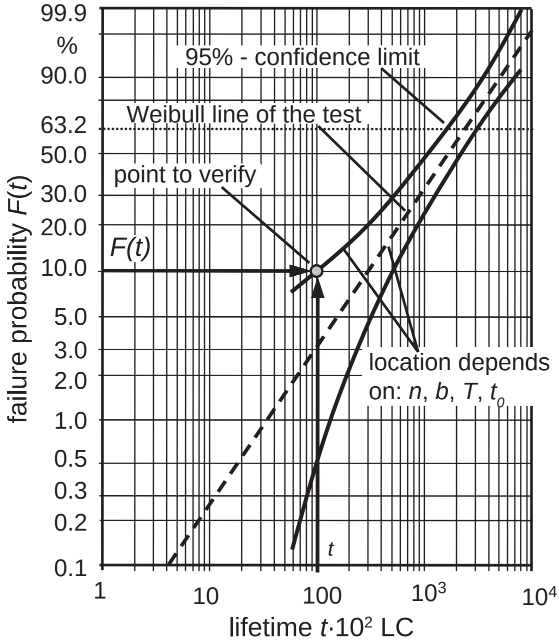


Figure 8.1. Test planning with the Weibull distribution

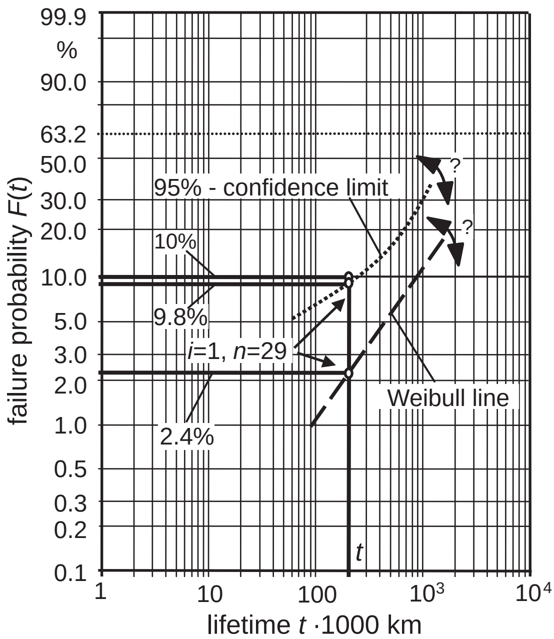


Figure 8.2. Example of test planning with the Weibull distribution

8.2 Test Planning Based on the Binomial Distribution

Here, we would like to begin with the observation of n test units. If the test units are identical, then they will all exhibit the same reliability $R(t)$, Figure 8.3.

At the time t, the reliabilities $R_1(t)$, $R_2(t)$, $R_3(t)$, …, $R_n(t)$ with $R_i(t) = R(t)$ are valid for each individual test units. For the probability that all n test units survive until time t, one uses the product law of probabilities $R(t)^n$.

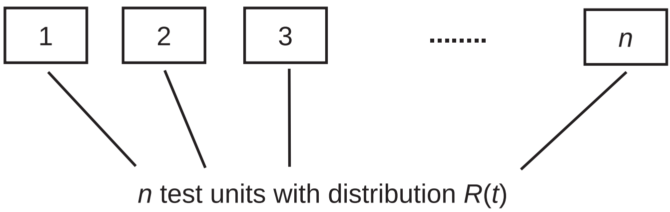


Figure 8.3. Starting point for test planning with the binomial distribution

If no failure can be observed while testing a random test specimen of the size n until the time t, which represents the required lifetime, and if $R(t)$ is the survival probability of the tested object, then the probability, that all n units tests will survive until the time t, is equal to $R(t)^n$. In other words, the probability, that at least one failure can be observed until the point in time t, is $P_A = 1–R(t)^n$.

The inversion of this consideration allows one to say, that if no failure has occurred during a test of a random test specimen of the size n until the time t, then the minimal reliability of a test unit is equal to $R(t)$ with a confidence level of P_A. This can be seen in the following equation:

$$P_A = 1 - R(t)^n \text{ or } R(t) = (1 - P_A)^{\frac{1}{n}} . \tag{8.1}$$

In literature and in the practical field, Equation (8.1) is often referred to as “success run”.

Example:

The following reliability requirement is given: R(200.000 km) = 90%. The verification should have a confidence level of P_A = 95%. The required random test specimen size can be attained by reforming the above equation:

$$R(t) = (1 - P_A)^{\frac{1}{n}} \Leftrightarrow n = \frac{ln(1 - P_A)}{ln(R(t))} \Rightarrow n = \frac{ln(0.05)}{ln(0.9)} = 28.4 .$$

Here, it is common to use diagrams. Figure 8.4 shows an example of a minimal reliability $R(t)$ as a function of the random test specimen size n for various confidence levels P_A, in the case that until the point in time t *no failures* have occurred (success run).

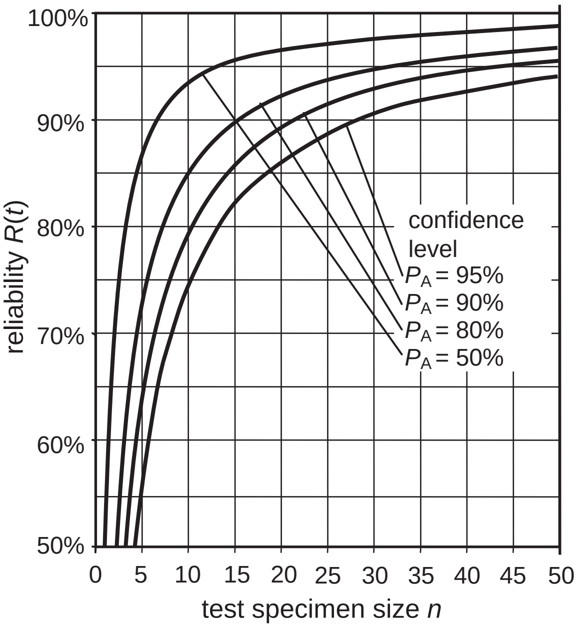


Figure 8.4. Minimal reliability $R(t)$ as a function of the test specimen size n and the confidence level P_A, if at the point in time t *no failure* has occurred (success run)

8.3 Lifetime Ratio

In this section, the effect of increasing or decreasing the test duration of the required test specimen size is observed. According to Weibull, $R(t) = \exp\left(-(t/T)^b\right)$. If a test runs until the time $t_{test} \neq t$, then $R(t_{test}) = exp\left(-(t_{test}/T)^b\right)$. Hence after simplification:

$$\frac{ln(R(t_{test}))}{ln(R(t))} = \left(\frac{t_{test}}{t}\right)^b = L_V{}^b, \tag{8.2}$$

which results to $R(t)^{L_V^b} = R(t_{test})$.

The ratio of the test duration t_{test} to the required lifetime t is signified as the lifetime ratio L_V.

$$L_V = \frac{t_{test}}{t} \tag{8.3}$$

If a failure free time is present, then

$$L_V = \frac{t_{test} - t_0}{t - t_0} \text{ and thus } t_{test} = L_V(t - t_0) + t_0 \,. \tag{8.4}$$

Substituting the lifetime ratio in Equation (8.1) results to:

$$R(t) = (1 - P_A)^{\frac{1}{L_V^b \cdot n}} \,. \tag{8.5}$$

Therefore, for a constant reliability $R(t)$ and confidence level P_A, an increase in the test duration t_{test} leads to a decrease in the required test specimen size n and vice versa, see Figure 8.5 and Figure 8.6.

Diagrams and Examples:

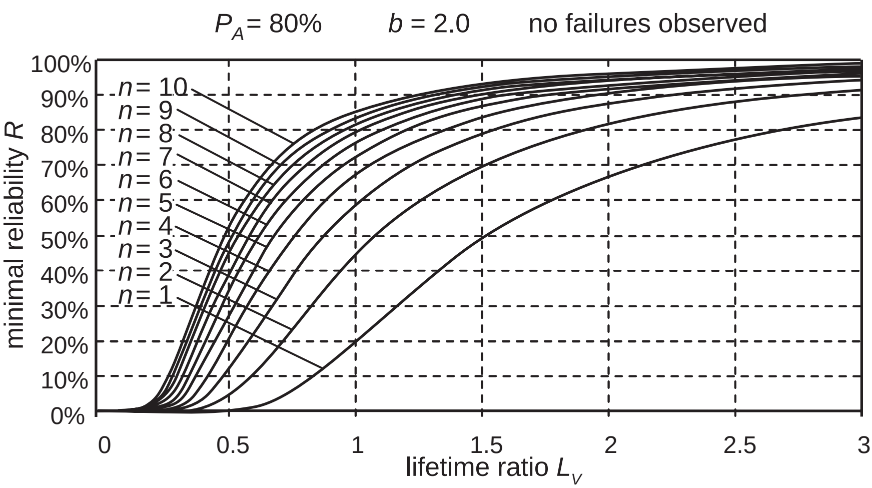


Figure 8.5. Reliability as a function of the lifetime ratio and test specimen size [8.2]

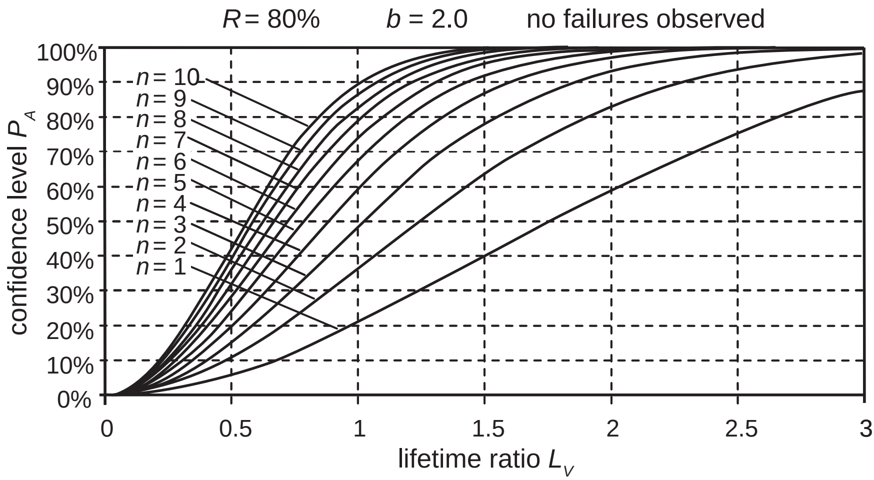


Figure 8.6. Confidence level as a function of the lifetime ratio and test specimen size [8.12]

Example 1: Verification of the reliability goal – determination of test duration and the test specimen size [8.2]

Given:

Budget for a lifetime test of 40,000 km without failure of a product; assumed shape parameter: $b = 2.0$.

Problem:

What is the number of test units required for the most effective and economical type of test execution in order to guarantee a reliability of $R = 80\%$ and a confidence level of $P_A = 80\%$?

Solution:

There are two ways to solve this problem: with the given information $b = 2.0$ and $R = 80\%$, the diagram in Figure 8.5 is used and with the information $P_A = 80\%$, the diagram in Figure 8.6 is applied. Both solutions yield the same result.

Determination of L_V:

Beginning on the y-axis at either R or $P_A = 80\%$, one moves towards the right until intersecting an n-curve. In both diagrams, the value perpendicular to the abscissa from the intersection point corresponds to the lifetime ratio L_V.

Result:

The most cost-efficient test is a test with one test unit (one unit, one trial, one person), thus $n = 1$. The perpendicular on the abscissa to the 80%

intersection point with the $n = 1$ curve is the corresponding lifetime ratio $L_V = 2.7$. For this test ($n = 1$) the test duration is $2.7 \cdot 40{,}000$ km = 108,000 km. The most cost-efficient test for the achievement of the required reliability goal (R and $P_A \geq 80$ %) is realized with one trial ($n = 1$) over a test duration of 108,000 km.

Example 2: Reliability test [8.2]

The decision for a certain cost-efficient reliability test needs to be founded.

Given:

- Three test specimens
- Budget for a test completion at 120,000 km
- Required minimal lifetime: 40,000 km
- Estimated shape parameter: $b = 2{,}0$
- Required confidence level: $P_A = 80\%$.

Problem:

What type of test execution is required:

a) testing one unit for a total of 120,000 km or
b) testing three units each for 40,000 km (a total of 120,000)?

Solution:

Diagram, Figure 8.5 (with $b = 2.0$ and $P_A = 80\%$)

a) Test with one unit ($n = 1$) over 120,000 km; lifetime ratio $L_V = 120{,}000 / 40{,}000 = 3$. According to Figure 8.5, the reliability is $R = 83.6\%$ for $L_V = 3$ and $n = 1$.
b) Test with three units over 40,000 km; $L_V = 1$; $n = 3$; $R = 58.5\%$.

Result:

Since the time and effort involved in total kilometres tested is the same for both tests, the procedure with the highest minimal reliability is given preference: execution of the test with one unit ($n = 1$) over 120,000 km. The achieved reliability is $R = 83.6\%$.

Note:

It is also possible to assume a constant value for the reliability and to determine the confidence level with Figure 8.6. In this case the higher confidence level would have been the deciding factor for achieving the set goal.

Example 3: Determination of the reliability [8.2]

How to determine the reliability, if one unit is taken out of the test before reaching the desired lifetime:

Given:

A test is run with the goal to verify a reliability of $R = 80\%$ and a confidence level of $P_A = 80\%$. This test requires that one unit is tested without failing until 2.7 times the desired lifetime. However, the unit was taken out of the test after 1.1 times the targeted lifetime. A Weibull shape parameter of $b = 2.0$ is assumed.

Problem:

How long does a second unit have to be tested without failure in order to verify the original reliability requirement of $R \geq 80\%$?

Solution:

Out of Figure 8.5 ($b = 2.0$ and $P_A = 80\%$) the number of required tests is found to be $n = 6$ in order to achieve a reliability of $R = 80\%$ with a lifetime ratio of $L_V = 1.1$. Since one unit has already been tested with $L_V = 1.1$, 5 further tests are required for which $L_V \geq 1.1$. Five tests with $L_V = 1.1$ correspond to the reliability statement of one test with $L_V = 2.45$.

The reliability requirement of $R \geq 80\%$ is fulfilled, if after the first unit is taken out of the test at $L_V = 1.1$, a further unit is successfully tested until 2.45 times of the required lifetime are reached.

8.4 Generalization for Failures during a Test

In general, the binomial law is valid for the confidence level:

$$P_A = 1 - \sum_{i=0}^{x} \binom{n}{i} \cdot (1 - R(t))^i \cdot R(t)^{n-i} \; . \tag{8.6}$$

Here, x is the number of failures during a time span t and n is the test specimen size. If one failure occurs during the test at the point in time t, then

$$P_A = 1 - R(t)^n - n \cdot (1 - R(t)) \cdot R(t)^{n-1} . \tag{8.7}$$

For this application, the use of diagrams has proved to be helpful. Figure 8.7 shows a Larson nomogram as an example (see for example [8.12]). A test specimen size of $n = 20$ elements is entered where $x = 2$ test units have failed during the test duration t. In order to determine the achieved reliability with a confidence level of $P_A = 90\%$, a line is drawn starting at

$P_A = 0.9$ going through the point ($n = 20$; $x = 2$) and the reliability can be read from the diagram at the intersection of this line with the R curves. The reliability $R(t)$ at the test duration is equal to 75% with a confidence level of 90%.

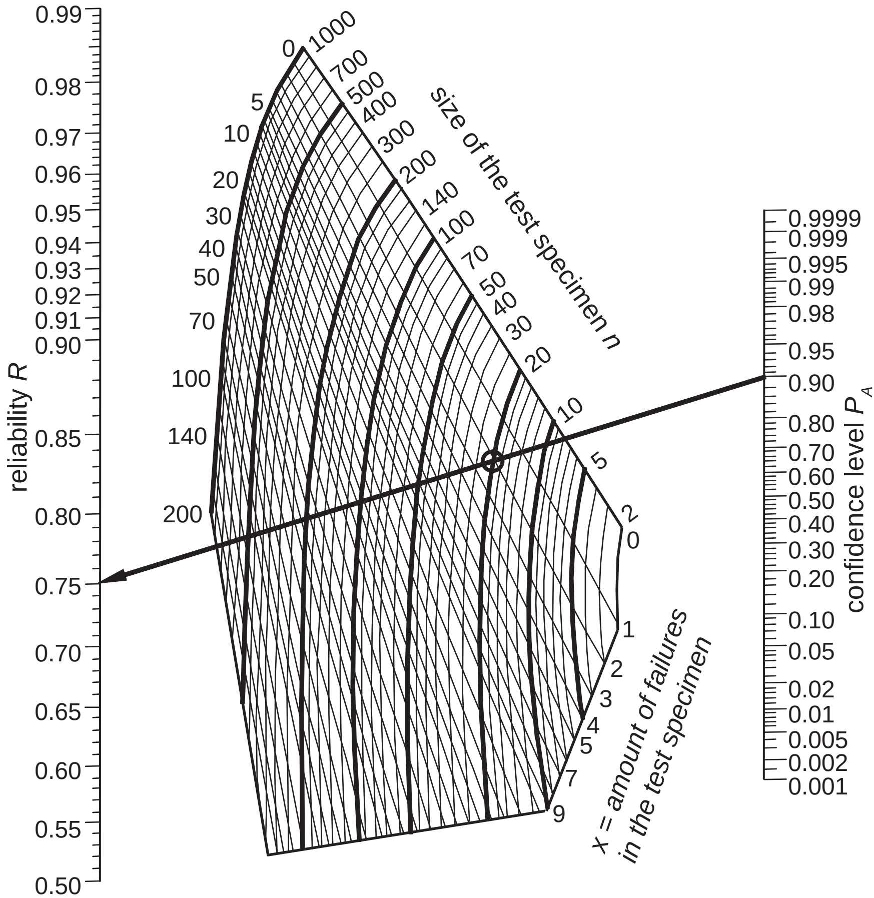


Figure 8.7. Larson nomogram

8.5 Consideration of Prior Information (Bayesians-Method)

The Bayesian law for the consideration of prior information can be used for the reduction of the required test specimen size n. Prior information is considered in the form of the a-priori distribution density with the density

$f(\vartheta)$. For a certain event A, the probability $P(A|\vartheta)$ is considered with the unknown parameter ϑ. If it is known that ϑ is distributed according to the density $f(\vartheta)$, then the a-posteriori distribution density is produced under consideration of prior information according to Bayesian law:

$$f(\vartheta \,|\, A) = \frac{P(A \,|\, \vartheta) \cdot f(\vartheta)}{\int_{-\infty}^{\infty} P(A \,|\, \vartheta) \cdot f(\vartheta) \cdot d\vartheta}\,. \tag{8.8}$$

With this density the confidence intervals can be calculated by integration:

$$P(a \le \vartheta \le b) = \int_a^b f(\vartheta \,|\, A) \cdot d\vartheta\,. \tag{8.9}$$

For a success run, if R is a probability value (by the rectangular distribution $0 \le R \le 1$) available as prior information, then the test specimen size can be reduced by one test unit ($n + 1$ instead of n in the exponent) when using the Bayesian method:

$$P_A = P\left(R_0 < R < 1\right) = \frac{\int_{R_0}^{1} R^n \cdot dR}{\int_0^1 R^n \cdot dR} = 1 - R^{n+1}\,. \tag{8.10}$$

Further references concerning this topic can be found in [8.6].

The difficulty for further application of the Bayesian method is the formation of the a-priori distribution.

8.5.1 Procedure from Beyer/Lauster

On practical approach to solve this problem stems from *Beyer/Lauster* [8.2]. Prior information concerning the reliability at the time t is regarded with a value R_0 which has confidence level of 63.2%. According to [8.2], under the consideration of prior information for Weibull distributed failure behaviours, one yields the following relationship for the confidence level:

$$P_A = 1 - R^{n \cdot L_V^b + 1/\ln(1/R_0)} \cdot \sum_{i=0}^{x} \binom{n + 1/(L_V^b \cdot \ln(1/R_0))}{i} \left(\frac{1 - R^{L_V^b}}{R^{L_V^b}} \right)^i . \qquad (8.11)$$

Here, b stands for the Weibull shape parameter and x stands for the number of failures until the time t. If no failures are allowed (success run), which implies $x = 0$, then

$$P_A = 1 - R^{n \cdot L_V^b + \frac{1}{\ln(1/R_0)}} . \qquad (8.12)$$

Solving this equation for the required test specimen size results to:

$$n = \frac{1}{L_v^b} \cdot \left[\frac{\ln(1 - P_A)}{\ln(R)} - \frac{1}{\ln(1/R_0)} \right] . \qquad (8.13)$$

This means that the required test specimen size can be reduced by considering the prior information R_0 by

$$n^* = \frac{1}{L_v^b \cdot \ln(1/R_0)} . \qquad (8.14)$$

Here again, the used of nomograms can prove to be helpful, Figure 8.8.

Example:

For the release of an aggregate, a lifetime test should be conducted. A lifetime of B_{10} = 20,000 h is required, which is equal to R(20,000 h) = 0.9.

The following knowledge has been gathered from previous comparable models:

- R_0 = 0.9 (with 63.2% confidence level) and
- shape parameter b = 2.

The verification should be proceeded with P_A = 85% and n = 5 test units. According to Figure 8.8, the lifetime ratio is L_v = 1.25 and thus the test duration is t_{test} = 25,000 h (line ❶).

The following statements can be made when analyzing the nomogram in Figure 8.8.

- If no prior information is considered, then n = 10 test units must be tested (with L_v = 1.25) (line ❷).
- If one aggregate fails, then the test specimen size increases to n = 14 (likewise with L_v = 1.25, line ❸).

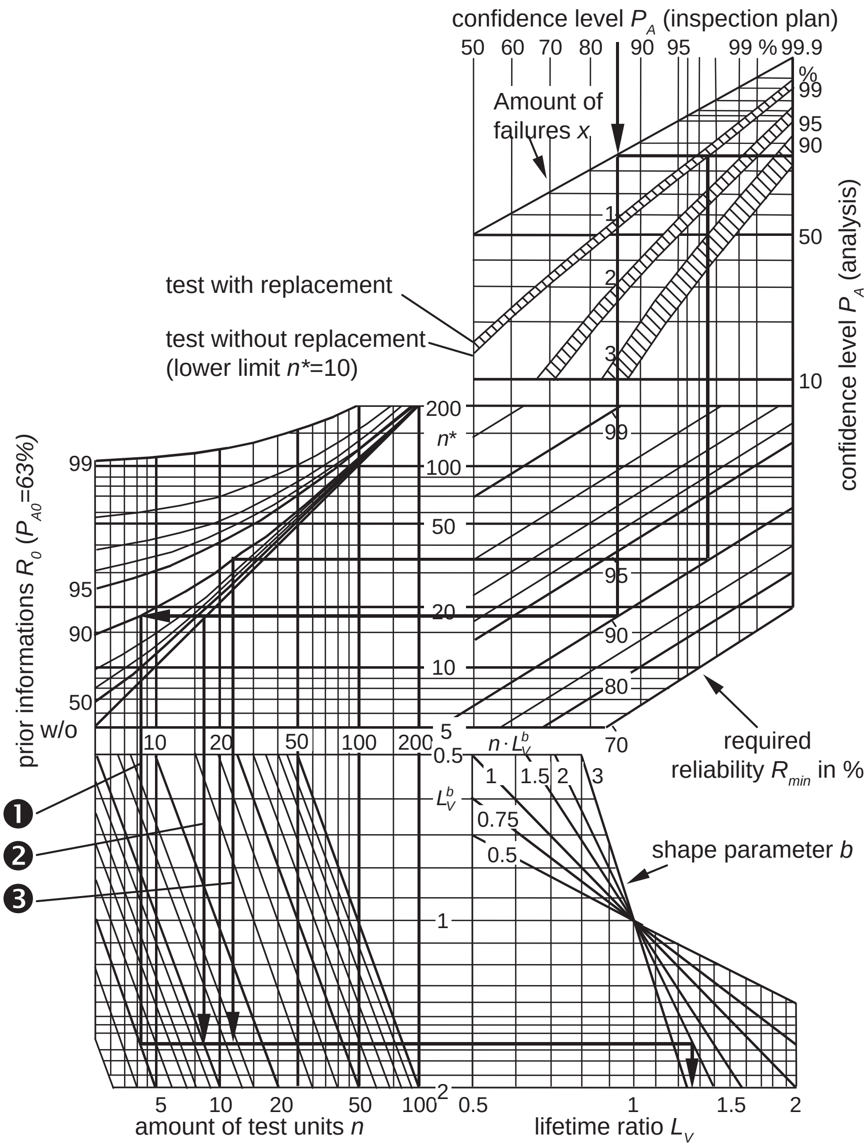


Figure 8.8. Nomogram from Beyer/Lauster [8.2]

8.5.2 Procedure from Kleyner et al.

A second procedure for the consideration of prior information was suggested by *Kleyner et al.* [8.5]. A mixture between a rectangular distribution and a beta distribution is used for the a-priori distribution. The proportion of each distribution is weighted with the “knowledge factor” ρ. If little is known regarding *R*, then many units must be tested in order to be able to make a reliable statement. In [8.5], field data is used from an older product in order to estimate a pre-distribution. This is the objective part of the procedure. The subjective part of the procedure comes into play with the estimation of the similarity between the new and old product. This is done by estimating a value for the “knowledge factor” ρ. No similarity, which implies no transferability of information from the old product onto the new product, is indicated by ρ = 0. The larger ρ becomes, the more similar the new and old products and the lower the necessary test specimen size will be. For ρ = 1, the a-priori distribution corresponds exactly to the beta distribution without any rectangular distribution influence. This indicates very good prior information of *R* and it is understandable that the necessary test specimen size then becomes relatively small. This subjective estimation of ρ is the main objective of the Kleyner method.

In [8.5], the execution of the calculation of this method is given. Under the assumption that no failures occur during the tests, the a-posteriori density can be calculated according to [8.5] with the application of the Bayesian law.

$$f(R) = \frac{(1-\rho)\cdot R^n + \rho \cdot \dfrac{R^{A+n-1}\cdot(1-R)^{B-1}}{\beta(A,B)}}{\dfrac{1-\rho}{n+1} + \rho\cdot\dfrac{\beta(A+n,B)}{\beta(A,B)}} \tag{8.15}$$

By integrating Equation (8.11), the confidence level is found:

$$P_A = \int_R^1 f(R)\cdot dR\,. \tag{8.16}$$

The estimated “knowledge factor” ρ must remain within the interval $0 \le \rho \le 1$. *A* and *B* are parameters for the beta distribution, which can be determined out of failure data for previous products.

The common density function for the beta distribution is as follows:

$$f(x) = \begin{cases} \dfrac{\Gamma(A+B)}{\Gamma(A)\cdot\Gamma(B)} x^{A-1}(1-x)^{B-1} & 0 < x < 1; A > 0; B > 0 \\ 0 & \text{otherwise} \end{cases}\,. \tag{8.17}$$

Here, $\Gamma(..)$ is the Euler gamma function. The a-priori distribution is gained with the “knowledge factor” out of the beta and rectangular distributions. Under the application of Bayesian law, the a-posteriori distribution is obtained according to Equation (8.15).

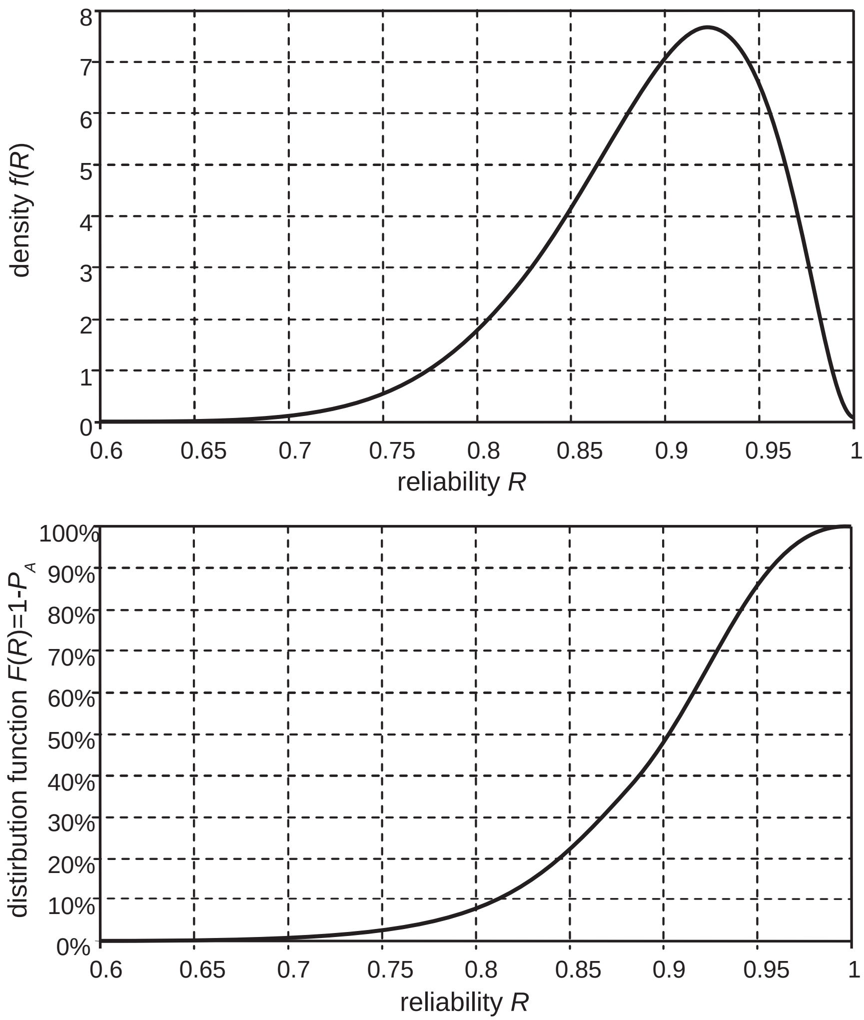


Figure 8.9. Beta density and distribution function of R with $A = 25$, $B = 3$

Normally, the reliability R and the confidence level P_A are already given. If A and B are known, then the only unknown variable is the test specimen size n. This can be numerically calculated by the integral or simply read from the diagram.

Figure 8.9 shows a beta density function with the parameters $A = 25$ and $B = 3$ and the corresponding beta distribution function of R. The mean (median) of this beta distribution is located at a reliability of R_{median} = 90.22%.

Example:

Figure 8.10 represents the confidence level P_A as a function of the necessary test specimen size n for various knowledge factors ($\rho = 0$; $\rho = 0{,}1$; $\rho = 0{,}2$; $\rho = 0{,}4$; $\rho = 0{,}6$; $\rho = 0{,}8$; $\rho = 1$) and for the success run according to Equation (8.1). The required reliability is $R(t_{test}) = 0.9$. The corresponding beta distribution for the a-priori density has the parameters $A = 25$ and $B = 3$, see Figure 8.9. The parameters were determined from a previous test. A reduction of n for an increasing ρ can be clearly seen in Figure 8.10.

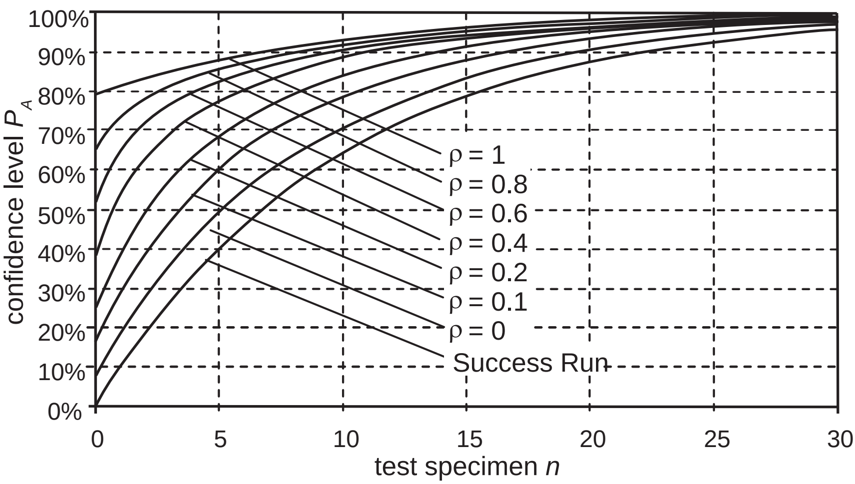


Figure 8.10. Confidence level P_A as a function of the necessary test specimen size for various knowledge factor values ρ with a reliability of R = 90%

Figure 8.11 shows the required number of test units only as a function of the knowledge factor. The confidence level as well as the reliability was set at 90%. This corresponds to values which are often prescribed in the practical field. If 22 test units are required for a success run, then the test specimen size can be reduced to seven units if the previous information is assumed to be correct ($\rho = 1$). For $\rho = 0$, the *Kleyner et al*. method is a pure rectangular distribution and the test specimen size can be reduced by one unit.

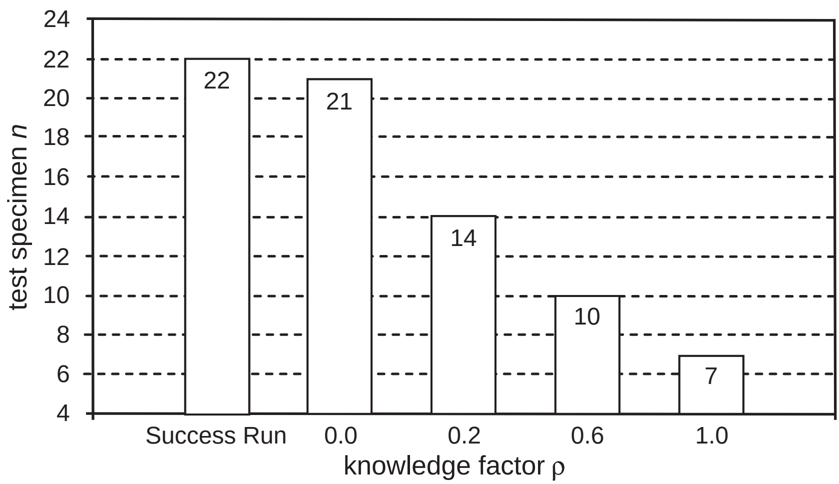


Figure 8.11. Necessary test specimen size *n* as a function of the knowledge factor ρ with a confidence level P_A of 90 % and a reliability *R* of 90%

A detailed description of the mathematics involved in this topic can be found in [8.7].

Another method is introduced in [8.6]. This method describes reliability information with the beta distribution and is simpler as the procedure from *Kleyner et al.* in regards to the difficulty of the calculation required. Prior information is transmitted with the so-called transformation factor. With the introduction of time-acceleration factors it is possible to use information acquired from an accelerated test to reduce the test specimen size. Furthermore, with the lifetime ratio, other tests with deviating test durations can be used to verify the lifetime.

8.6 Accelerated Lifetime Tests

This section deals with methods in which the lifetime under “normal” loads can be verified from the results from trials run under high load levels. This is achieved with physically founded models [8.12].

Logically, such a conclusion is only valid under the assumption that no changes in the failure mechanism are caused by a load increase. One special case of these tests is where the load is increased in predetermined increments (step stress test).

Trials for fatigue lifetime make up a second special case. Here, the endurance strength limit should be determined (Wöhler curve).

8.6.1 Time-Acceleration Factor

One practical procedure consists in recording load profiles during a test run under realistic operation conditions. Out of these profiles time or distance proportions are determined or operation or use frequencies (histograms) are developed. Finally, by extrapolating these histograms, the lifetime can be verified. Reproducing the load on test rigs with increased loads (using a physical model) results in a time-acceleration factor corresponding to the underlying model [8.12].

With the increased load, greater damage occurs during the trials than would occur under real operational conditions. This in turn leads to lower lifetimes.

The relationship between the lifetime under normal operation conditions and the lifetime in an accelerated test is described by the time-acceleration factor *AF:*

$$AF = \frac{t}{t_{acc}} . \qquad (8.18)$$

In Equation (8.18) it is assumed that the failure probability for both lifetimes is the same.

With accelerated tests it is possible to reduce the test duration by the time-acceleration factor. This is shown in the following example.

Example (“Understanding Accelerated Life-Testing Analysis” by Pantelis Vassiliou, from RAMS 2001 - Tutorial Notes):

In this example it is examined which failure behaviour results for a paper clip which is bent back and forth to a certain angle.

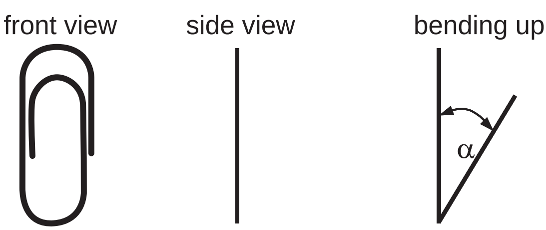


Figure 8.12. Paper clip and bending angle

In this experiment 6 paper clips are bent to different angles: 45°, 90° and 180°. The failure times given in load cycles are summarized in Table 8.1 (1 load cycle = paper clip is bent once to an angle α and brought back to its original position).

Table 8.1. Paper clip experiment results

No.	α = 45°	α = 90°	α = 180°
1	58	16	4
2	63	17	5
3	65	18	5
4	72	21	5.5
5	78	22	6
6	86	23	6.5

Figure 8.13 represents the Weibull lines for the different bending angles. Almost all Weibull distributions possess the same shape parameter, which means that the failure mechanism for paper clips does not change for these bending angles. Based on the characteristic lifetime, the time-acceleration factor for a bending angle of 180° is (in relation to a bend angle of 45°):

$$AF_{180^\circ} = \frac{t_{45^\circ}}{t_{180^\circ}} = \frac{74.85}{5.72} = 13$$

By testing paper clips with a bending angle of 180°, the test length can be reduced by a time-acceleration factor of 13 in comparison to a test with the bending angle 45°.

The time-acceleration factor for a bending angle of 90° (based on a bending angle of 45°) is:

$$AF_{90^\circ} = \frac{t_{45^\circ}}{t_{90^\circ}} = \frac{74.85}{20.78} = 3.6\,.$$

The paper clip test with a bending angle of 90° results to a test duration reduction by a time-acceleration factor 3.6.

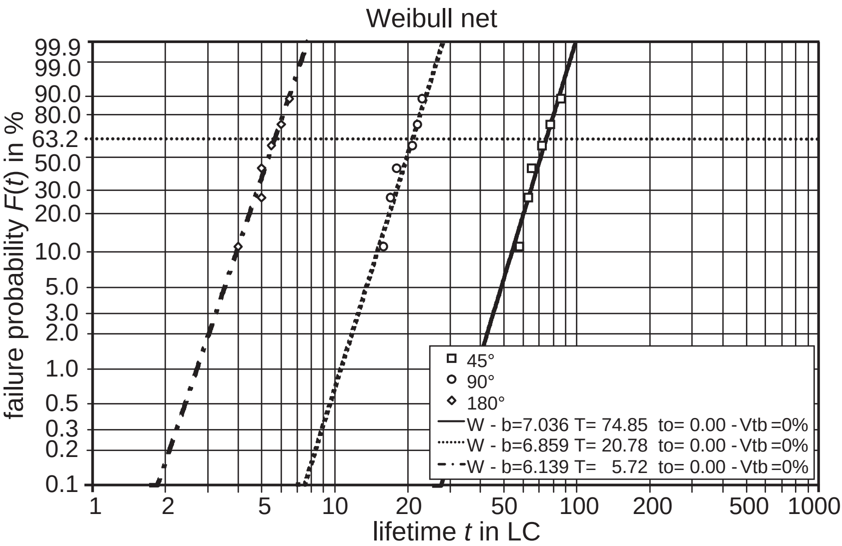


Figure 8.13. Failure distributions in a Weibull chart for various bending angles

8.6.2 Step Stress Method

A method for accelerated lifetime testing called the "step stress method" was introduced by *Nelson* [8.9] already in 1980. With incremental increase in the load after each failure, the test length is shortened. Here, it is important that the failure mechanism does not change. This implies that the slope of the Weibull lines remains the same. After evaluating the failure data gained by such a method, the original distribution is calculated. This method is somewhat debatable and is not experimentally verified for mechanical components. However, the application of this method is plausible as a version comparison and under a limited time. Furthermore, the relationship between load and lifetime must be known (e.g. Wöhler curve) for the proper application of this method. The principle of this method is shown in Figure 8.14.

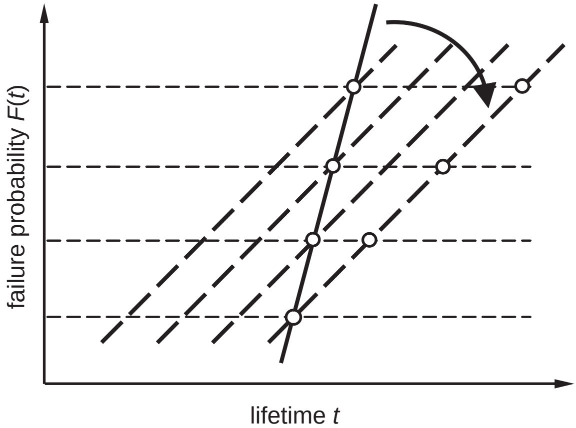


Figure 8.14. Principle of the step stress method shown in a Weibull network

8.6.3 HALT (Highly Accelerated Life Testing)

Since around 1993 a continuative method for accelerated testing under the acronym HALT has been discussed in literary resources [8.1]. HALT stands for *Highly Accelerated Life Testing* and refers to a method developed by *G. K. Hobbs* (see Hobbs Engineering Corporation, Westminster, Colorado, for reliability assurance of items manufactured in the design phase.

This method is mainly applied to electrical and electronic components, sometimes to electro-mechanical components, but less to mechanical assemblies.

Through tests with incremental load increase, relevant failure mechanisms are enhanced and this at minimal costs in the shortest time possible. HALT works with load levels which considerably exceed the load level applied to a product under normal conditions, or at least for which the product is specified. The HALT process begins with an analysis of possible loads, such as electrical (operation voltage, operation frequency, power), mechanical (vibration, shock) or thermal stress (extreme temperatures, fast temperature changes). This stress must be established for each individual product. For this determination, no predetermined load limits exist. As many failure mechanisms should be provoked as possible. The goal is the failure and not the survival of the test. The component must be monitored during the test.

This phase represents the beginning of an iterative process made up of the following steps:

- test with incremental load increase;
- analysis of test results (search for "root causes"); during this step each failure cause must be considered, even if the failures occur outside of the specification limits;
- implementation of corrective actions (e.g. design alterations, material, supplier, assembly);
- renewed test.

The operation limits (functional limitations) and the destruction limits (failure limits) are determined within the realms of trials with load increases. When exceeding (under-running) the highest (lowest) operation limit, the product reacts defectively, whereas under-running (exceeding) leads again to normal operation. In order to determine the destruction limits (upper and lower destruction limits), it is necessary to reach back step by step to the fundamental limits of technology (FLT). Beyond the destruction limits, the apparatus is continually damaged and fails irreversibly. A combination of several loads often yields to lower limits. HALT begins with the different modular units and then continues in increments to more complex levels.

Results from HALT should be reintroduced in the

- constructive dimensioning/design for successors,
- production processes and
- determination of stress profiles.

HALT is the most effective method because it can

- recognize design and production flaws, shortcomings or defects,
- determine and expand design limits,
- increase product reliability,
- shorten development time and
- estimate the effect of modifications.

Disadvantage: it is not possible within the realms of HALT to forecast reliability values on a statistical basis.

8.6.4 Degradation Test

It is possible that within the available test duration no failure of components is observed. In this case, it is not possible to make statements concerning the failure behaviour of machine components with the reliability tests discussed up to this point. Many failure causes, however, can be traced back to wearout processes which take place within a machine component. Material wearout leads to a weak point which in end effect can cause a component to fail. If the wearout can be measured, then important information concerning the behaviour of wear over time can be gained.

In the “degradation test” the occurred wear on a machine component is of interest. The determination of the failure behaviour of a machine component is possible without the necessity that the component has failed. The failure behaviour is determined over the measure of wear. If the relationship between load time and the measure of wear is known, then the lifetime end can be determined for each tested component based on its wear. Here, a certain wear limit is set as a type of failure. The lifetimes determined based on the actual wear can then be statistically analyzed and shown in a Weibull network.

In many cases of application it is possible to measure directly occurred wear during the test phase. The result is a function of the wear over time obtained during a set of trials (e.g. wear of a tire profile over mileage). In other applications, however, it is not possible to measure wear or at least not to measure it non-destructively. Other values can help to make the occurred wear measurable, for example the timely reduction of performance of functionality of a machine component. Depending on the situation, the wear measurement can take place continuously or after determined intervals of time. After many wear measurements have been taken, the function of wear over time can be determined for the respective machine component.

Various wear behaviours can be determined depending on the component. Some materials possess a run-in period. This indicates that the wear at the beginning is higher than after a certain period of time. There are also some machine components whose wear is less at the beginning of operation and afterwards increases. Figure 8.15 shows examples of various wear behaviours as a function of the operation time.

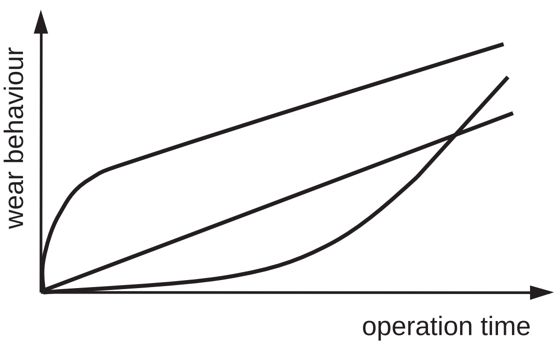


Figure 8.15. Various wear behaviours

The degradation test is based on normal operational conditions, just as the component is subjected to under normal use. The accelerated degradation test is a combination of an accelerated test, which is executed under an increase in load level and the traditional degradation test. Here, wear appearances of the components become more frequent. The relationship between the time-acceleration factor and wear must be known for an analysis of the failure behaviour under normal operational conditions.

8.7 Exercise Problems to Reliability Test Planning

Problem 8.1

In the specifications for an automobile transmission a lifetime of $B_{10} = 250{,}000$ km is required with a confidence level of $P_A = 95\%$. The shape parameter is $b = 1.5$. Determine the required number of transmissions which must be tested without any failure

a) based on the Weibull distribution and
b) based on the binomial distribution (success run).

The test should take place without failure under the following restrictions:

c) Due to time, each transmission can only be tested for a maximal of 150,000 test kilometers. How many transmissions must now be tested?
d) Due to costs, only $n = 15$ transmissions are available for testing. How long must these transmissions be tested without failure in order to ensure the required reliability?

The lifetime tests were carried out with a test specimen size of $n = 30$. However, three transmissions failed before the 250,000 km operational performance, i.e. $x = 3$. The other $n - x$ transmissions survived the required B_{10} operational performance without failure.

e) Which reliability can be issued to the transmission with an unchanged confidence level P_A?
f) With which confidence level is it possible to certify the B_{10} lifetime?
g) How many transmissions n^* must be additionally tested without failure due to the occurred failures until the B_{10} lifetime in order to certify the required reliability with the required confidence level?

Now, prior information will be considered from a previous model, for which the reliability $R_0 = 90\%$ is known. To solve the following questions use the procedure from *Beyer/Lauster* (the nomogram as well as the analytical relationships).

h) How many transmissions are to be tested without failure for $L_v = 1$?
i) Which test duration t_{test} must be tested without failure if only 12 transmissions are available for testing?

Problem 8.2

While testing an apparatus $n = 2$ experimental vehicles are used. The test is carried out until the point in time t, where $x = 1$ unit has failed before the time t. Which reliability $R(t)$ can be confirmed as a function of the confidence level P_A? Show this relationship qualitatively in a graph.

Problem 8.3

During a test, the determined characteristic lifetime $T = 1.2 \cdot 10^6$ load cycles of a gearwheel should be verified by operational strength calculations. The shape parameter is known to be $b = 1.4$ and the failure free time is set at $t_0 = 2 \cdot 10^5$ load cycles. A total of $n = 8$ gearwheels are available for testing. For which test duration t_{test} must the n gearwheels be tested in order to confirm the characteristic lifetime with a confidence level of $P_A = 90\%$?

Problem 8.4

According to the specifications, a vehicle assembly should reach a lifetime of $B_{10} = 250{,}000$ with a confidence level of $P_A = 95\%$. A two-parametric Weibull distribution is taken, whose shape parameter is $b = 1.5$. $n = 23$ transmissions are available for testing.

a) Determine the required test duration t_{test} for testing without failures.
b) For which test duration t_{test} must the n transmissions be tested if the calculated characteristic lifetime $T = 1.5 \cdot 10^6$ is considered as prior information? Use the Bayesian method from *Beyer/Lauster*.

References

[8.1] AT&T Strategic Technology Group (1997) HALT, HASS and HASA as applied at AT&T. AT&T Wireless Services, Strategic Technology Group
[8.2] Beyer R, Lauster E (1990) Statistische Lebensdauerprüfpläne bei Berücksichtigung von Vorkenntnissen. QZ 35, Heft 2, S 93-98
[8.3] Kececioglu D (1993) Reliability and life testing handbook. vol 1. Prentice Hall, cop. Engelwood Cliffs, N.J.
[8.4] Kececioglu D (1993) Reliability and life testing handbook. vol 2. Prentice Hall, cop. Engelwood Cliffs, N.J.

[8.5] Kleyner, Bhagath, Gasparini, Robinson, Bender (1997) Bayesian techniques to reduce sample size in automotive electronics attribute testing. Microelectronic Reliability, vol 37, No 6, pp 879-883.
[8.6] Krolo A, Bertsche B (2003) An Aproach for the Advanced Planning of a Reliability Demonstration Test based on a Bayes Procedure. Proc. Ann. Reliability & Maintainability Symp., 2003, S 288-294
[8.7] Martz H F, Waller R A (1982) Bayesian reliability analysis. John Wiley & Sons, New York,
[8.8] Meyna A, Pauli B (2003) Taschenbuch der Zuverlässigkeit. Hanser, München, Wien
[8.9] Nelson W (1980) Accelerated Life Testing - Step-Stress Models and Data Analysis. Trans. of Reliability, vol. R-29, No 2, June
[8.10] O'Connor P D T (1990) Zuverlässigkeitstechnik. VDH Verlag, Weinheim
[8.11] Tobias P A, Trindade D C (1994) Applied reliability. 2nd ed. Chapman & Hall
[8.12] Verband der Automobilindustrie (2000) VDA 3.2 Zuverlässigkeitssicherung bei Automobilherstellern und Lieferanten. VDA, Frankfurt

9 Lifetime Calculations for Machine Components

Lifetime calculations for machine components represent an important foundation for quantitative reliability methods. For this, determined fatigue strength and lifetime values are the input values for calculations. In this context, the reliability procedures are a form of extended strength calculation. Due to the large scale of this subject matter, only an overview of the procedures and aspects of lifetime calculations of mechanical components can be given in the following chapter. A detailed description and further explanation of these coherences can be found in technical literature [9.7, 9.14, 9.15, 9.33].

The goal of product development with respect to reliability is to develop products with a high and defined lifetime [9.5, 9.18]. For a prediction of lifetime all failure causes must be known. These can be divided into three categories:

1. *fatigue failures, aging failures, wearout failures* and *failures* caused by *environmental influences*, for example corrosion, etc., caused by changes in the used materials, dependent upon time, e.g. highly loaded components in automobile technology.
2. *Tolerance failures* lead to unreliable deviations, which forbid an efficient function, for example machine tools, which no longer achieve desired production precision, or seals, which show an unreliably high leakage.
3. *Failures*, caused by *faults*, which occur during *production*, *assembly* or during the *operation* of machines.

While the failure behaviours in categories 2 and 3 can only be described statistically up to now, procedures exist for a calculational lifetime prediction of material fatigue, see Figure 9.1. For an optimal design the operational loads occurring on critical locations of a component, caused by external forces, have to comply with the tolerable loads, caused by material, design, production and environmental influences [9.15].

Depending on the load case, the component is designed either statically or dynamically. Dynamic design targets are fatigue strength, endurance strength or operational fatigue strength. Well proven strength calculation

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_9, © Springer-Verlag Berlin Heidelberg 2008

procedures exist for static, fatigue strength and endurance strength design. A largely increasing number of publications describe the measurement of operationally endurable components [9.12].

Due to uncertainties in the prediction of operational loads and the often inexact linear damage accumulation hypotheses, lifetime calculations are often subjected to large deviation ranges. Despite these uncertainties, detailed procedures are used today for pre-dimensioning in connection with trials for optimization and lifetime proofs. Before series products can be released, today, trials are compulsory.

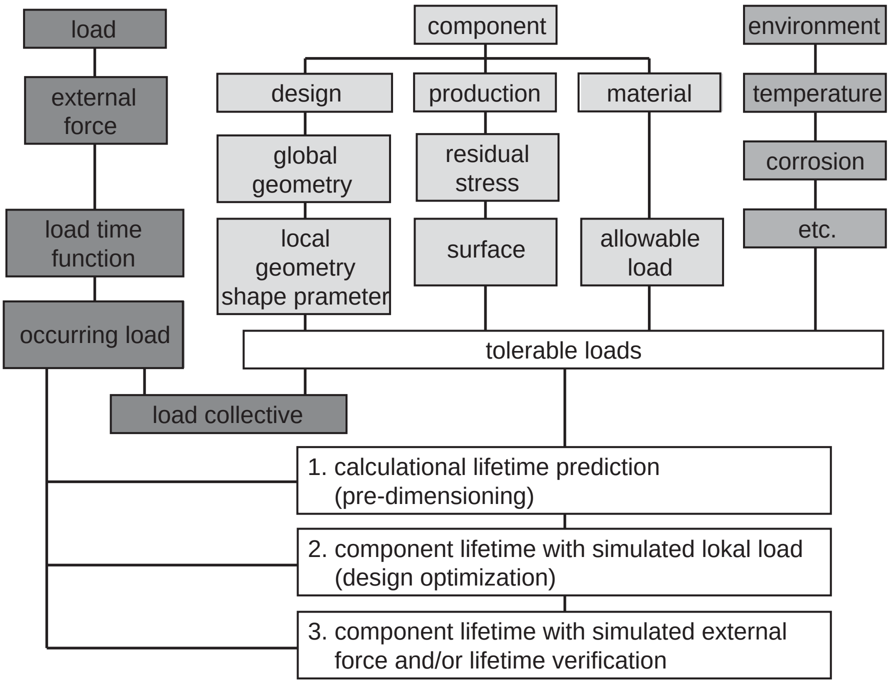


Figure 9.1. Calculational lifetime determination with damage accumulation hypotheses

9.1 External Loads, Tolerable Loads and Reliability

For static and fatigue endurable design of mechanical components, the designer uses methods from strength of materials involving nominal values or local load peaks from occurred loads as well as corresponding values for tolerable load capacities together with a safety factor, see Figure 9.2. The safety factor is chosen in such a way, so that preferably all uncertainties in the calculation procedure and load assumptions, such as statistical spreads in material variables, are taken into consideration [9.7]. Concerning load assumption, it is possible that the operational conditions are collected through an operation factor, e.g. the dynamic factor K_A for the design of gear wheels according to DIN 3990. This procedure has proven to be successful.

- **static design**
  - safety factor
- **dynamic endurance strength design**
  - safety factor
- **dynamic fatigue stength or operational fatigue strength**
  - Wöhler curve
  - lifetime curve

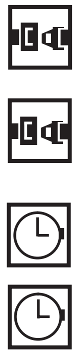


Figure 9.2. Failure modes of mechanically calculable failure causes

9.1.1 Static and Endurance Strength Design

For most products, however, the load and load capacity are random variables. Thus they are distributed statistically, see Figure 9.3. An example of this is the load on a vehicle transmission. The load is caused by a torsion moment on the transmission input shaft as a function of time. It is dependent upon the applied engine concept, the engine characteristic map, the mass of the vehicle including payload, the drive concept, the transmission ratios, the road profiles and especially upon the driver [9.1]. The load capacity of a component is not only dependent upon the material itself, but also upon the quality of production. If the load σ_B (density f_B) and load capacity σ_W distribution (density f_W) are known along with their overlapping, then statements can be made concerning the failure probability and reliability of machines and their elements on a statistical basis, see Figure 9.3. This coherence between load, load capacity and failure probability is known as the stress strength interference. It is unimportant which type of distribution is at hand.

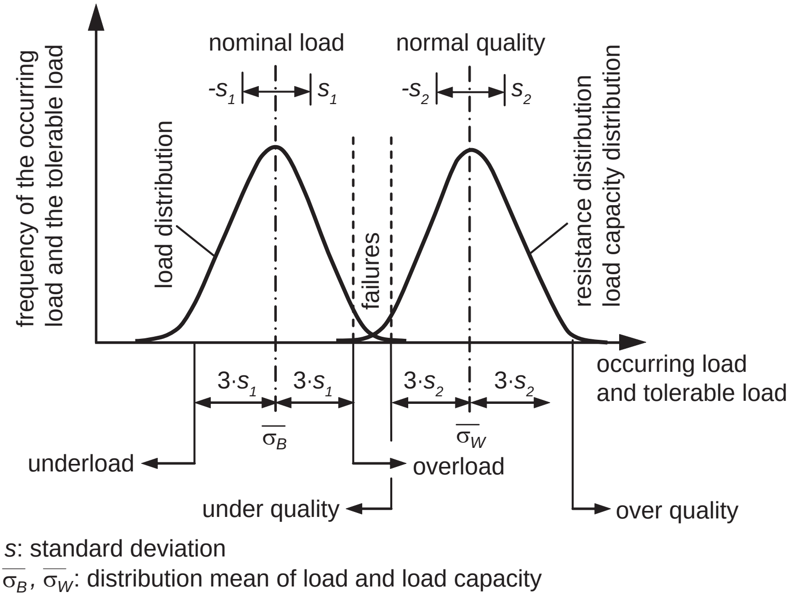


Figure 9.3. Coherence between occurred and tolerable stress or strain

The reliability R is the probability that the actual load does not surpass the tolerable load:

$$R = P(\sigma_W > \sigma_B)\,. \tag{9.1}$$

For $\sigma_W > \sigma_X$, all components will not fail for the load σ_X. According to Figure 9.4, the number of reliable components or their probability can be described with the density integral:

$$\int_{\sigma_X}^{\infty} f_W(\sigma_W) \cdot d\sigma_W\,. \tag{9.2}$$

However, the load $\sigma_B = \sigma_X$ only occurs with the relative probability

$$f_B(\sigma_B) \cdot d\sigma_B\,. \tag{9.3}$$

The reliability of a component, that is the probability that the actual load does not surpass the tolerable load, can be obtained for the actual load σ_X according to the multiplication theorem for independent probabilities, Figure 9.4:

$$f_B(\sigma_B) \cdot d\sigma_B \cdot \int_{\sigma_x}^{\infty} f_W(\sigma_W) \cdot d\sigma_W \;. \tag{9.4}$$

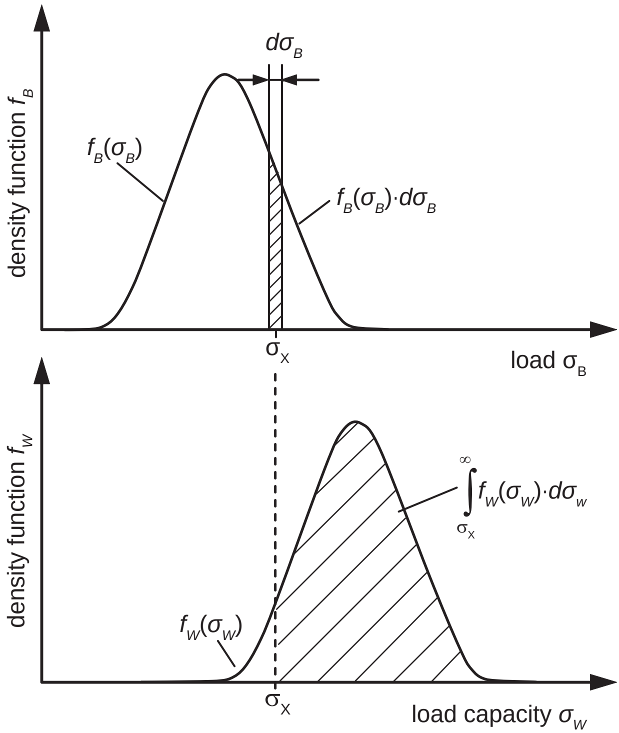


Figure 9.4. Determination of the reliability out of external loads and load capacity

If all possible actual loads are considered, then the reliability can be obtained for all loads with Equation (9.4):

$$R = \int_{-\infty}^{\infty} f_B(\sigma) \cdot \left[\int_{\sigma_x}^{\infty} f_W(\sigma) \cdot d\sigma \right] \cdot d\sigma \;. \tag{9.5}$$

The variables used are

- σ_W: tolerable load,
- σ_B: actual load and
- $_{B,\,W}$: the indexes for the load and load capacity.

Equation (9.5) says, that the reliability R of a component can be calculated, if the density functions for the tolerable load $f_W(\sigma_W)$ and for the actual load $f_B(\sigma_B)$ are known. This is shown in Figure 9.5.

The random variable Y is a measurement for the distance between the actual load and the tolerable load [9.22]:

$$Y = \sigma_W - \sigma_B \text{ with } \overline{Y} = \overline{\sigma}_W - \overline{\sigma}_B \,. \tag{9.6}$$

- $P_R = P(Y \geq 0)$ is the probability that $Y \geq 0$: the reliability,
- $P_F = P(Y < 0)$ is the probability that $Y < 0$: the failure probability.

If the random variables load σ_B and tolerable load σ_W are assumed to be normally distributed due to their numerable random influences, then along with the parameters mean and statistical spread, $(\overline{\sigma}_B, s_B), (\overline{\sigma}_W, s_W)$, the density function of a normal distributed load can be determined as follows:

$$f_B(\sigma_B) = \frac{1}{s_B \cdot \sqrt{2\pi}} \cdot e^{-\left(\frac{(\sigma_B - \overline{\sigma}_B)^2}{2 \cdot s_B^{\,2}}\right)} . \tag{9.7}$$

The density function of the load capacity can be determined in the same way. The random variable Y is likewise normally distributed. One transforms

$$Z = \frac{Y - \overline{Y}}{s_Y} \text{ with } s_Y = \sqrt{s_W^{\,2} + s_B^{\,2}} \,. \tag{9.8}$$

With Equations (9.5), (9.7) and (9.8) the normally distributed reliability R is calculated tp:

$$R(z) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-z_0}^{\infty} e^{-\frac{z^2}{2}} \cdot dz \text{ with } z_0 = \frac{\overline{y}}{s_y} \,. \tag{9.9}$$

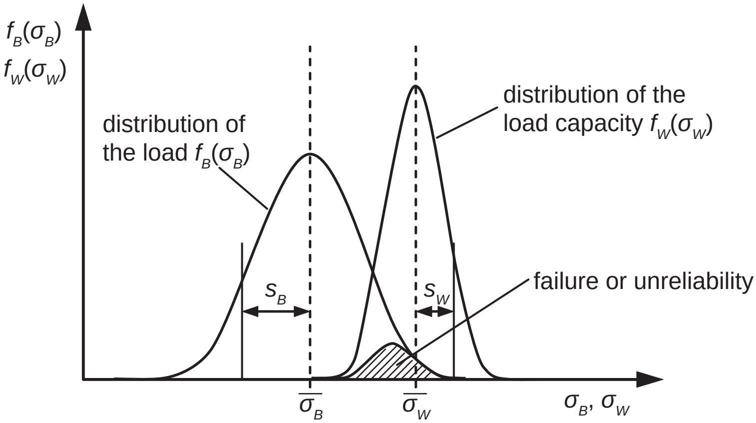


Figure 9.5. Failure probability or unreliability of the load and load capacity

With the safety distance *SM*:

$$SM = \frac{(\bar{\sigma}_W - \bar{\sigma}_B)}{\sqrt{s_W{}^2 + s_B{}^2}} \tag{9.10}$$

the reliability can be simply calculated

$$R = \phi\left(\frac{(\bar{\sigma}_W - \bar{\sigma}_B)}{\sqrt{s_W{}^2 + s_B{}^2}}\right), \tag{9.11}$$

where ϕ is the normal distribution function [9.6, 9.22, 9.30, 9.31]. For this calculation, the standardized normal distribution (i.e. mean 0 and standard deviation 1) with the *Error-Function* erfc(x) can be calculated with either table calculation programs or be read from tables.

In contrast, the common strength calculation of the safety factor S_F is given as the quotient of the mean values:

$$S_F = \frac{\bar{\sigma}_W}{\bar{\sigma}_B}. \tag{9.12}$$

Figure 9.6 shows the decisive influence of the statistical spread of the load and tolerable load on the failure probability.

Instead of the normal distribution, other distribution functions can also be used, for example the logarithmic normal distribution or the Weibull distribution. These distributions estimate the extreme values of the distribution, which are of special interest in a better way [9.15].

Example:

A component series has a load capacity that is normally distributed with a mean of 5,000 N and a standard deviation of 400 N. The load is likewise normally distribution with a mean of 3,500 N and a standard deviation of 400 N. What is the reliability of the component?

The safety factor for this component is:

$$S_F = \frac{\bar{\sigma}_W}{\bar{\sigma}_B} = \frac{5000}{3500} = 1.4 . \tag{9.13}$$

The reliability of this component can be calculated to:

$$R = \phi\left(\frac{5{,}000 - 3{,}500)}{\sqrt{400^2 + 400^2}}\right) = \phi(2.65) = 0.996 . \tag{9.14}$$

9.1.2 Fatigue Strength and Operational Fatigue Strength

The former observations are only suitable for statically loaded or dynamically loaded mechanical elements, as long as they lie within the region of endurance strength. For the fatigue strength region it can be assumed, that both scatter bands for the tolerable and acting load and stress amplitudes approach one another when stressed under operational load conditions. The failure probability increases with increasing lifetime, Figure 9.6, which causes increasing component damage in the fatigue strength region. If the component's Wöhler curve is known with its lifetime exponent *k*, the following can be derived out of the line equations in the double logarithmic diagram in Figure 9.6.

$$\frac{\sigma}{\sigma_D} = \left(\frac{N}{N_D}\right)^{-\frac{1}{k}} . \tag{9.15}$$

By substituting Equation (9.15) into Equation (9.11), the reliability is calculated piecewise for the individual load values in the fatigue strength region [9.29]

$$R = \phi\left(\frac{\left[\left(\sigma_D \cdot N_D^{1/k}\right)\cdot N^{-1/k}\right] - \bar{\sigma}_B}{\sqrt{s_w^{\,2} + s_B^{\,2}}}\right). \tag{9.16}$$

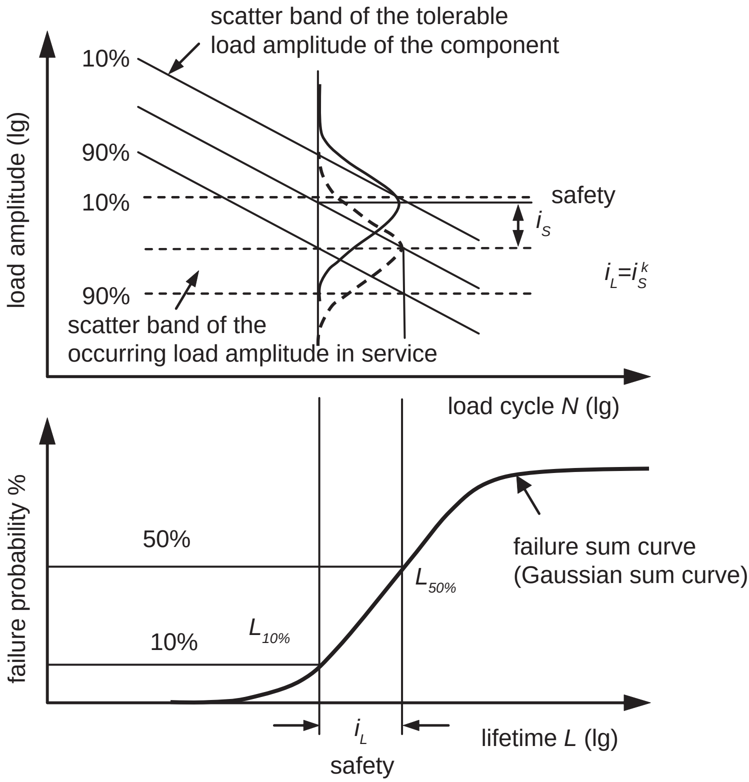


Figure 9.6. Increase in failure probability with increasing lifetime [9.15]

For components operated at temperatures exceeding the crystal recovery temperature of the used material, a similar behaviour can be observed: creeping. However, in creep strength calculations the shifting of the components resistance strength is both a function of time *t* and temperature *T*, Figure 9.7. In technical literature, approaches such as the exponential model are known and used in order to describe the time and temperature dependent behaviour of high-temperature materials [9.23]. In the case of operational fatigue strength, Figure 9.8, the actual loads can lie in the fatigue strength region as well as in the endurance strength region.

The frequency and amplitude of the actual load are also random variables. The tolerable load is likewise scattered and statements concerning the reliability of operationally loaded machines can be made with the help of the damage accumulation hypothesis. Operationally endurable design of a component is conducted with the goal to prevent component failure during a predetermined period of operation with a certain necessary confidence. For this purpose the component load must first be described over the predetermined period of operation. Thus, under consideration of the dynamical system behaviour, it is possible to obtain load curves, which can be summarized by suitable classification of the loads. The load capacity is determined by the material and the geometric parameters of the component, such as form, size and surface finish. The comparison between the load spectrum and the Wöhler curve can then be conducted with the help of a damage hypothesis, generally with the linear damage accumulation hypothesis from Palmgren Miner.

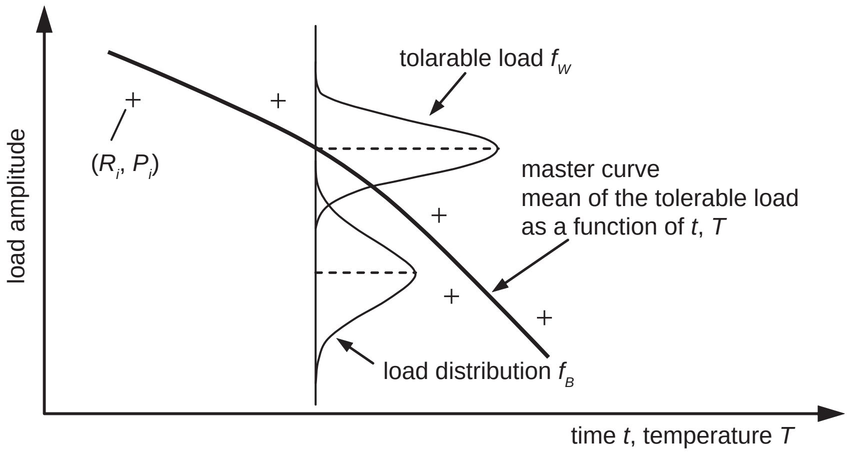


Figure 9.7. Creep strength

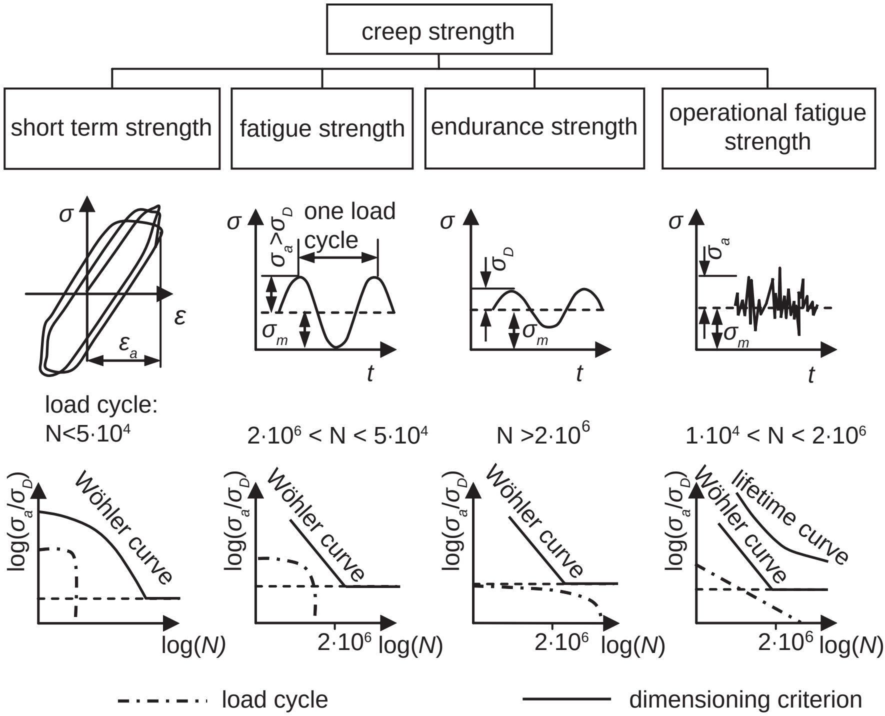


Figure 9.8. Oscillation strength with lifetime measurement curves [9.7]

In order to make statements on reliability according to Equation (9.11), instead of the Wöhler curve of an actual, existing, representative load spectrum with a certain probability, the lifetime curve, Figure 9.8, or the load spectrum must be transformed into an equivalent single stage replacement load spectrum. This new load spectrum must represent the same damage result as the Wöhler curve. The determination of the load spectrum, however, is difficult and time-consuming. Thus, the distribution function f_B is normally not known. The load is then determined under various, unsuitably high stresses. The individual load portions are pieced together regarding their expected frequency and the spectrum is extrapolated for the entire time of operation [9.1]. For this unsuitable load the input probability is normally estimated. Then the reliability can be simply calculated from the distribution of the tolerable load, Figure 9.9.

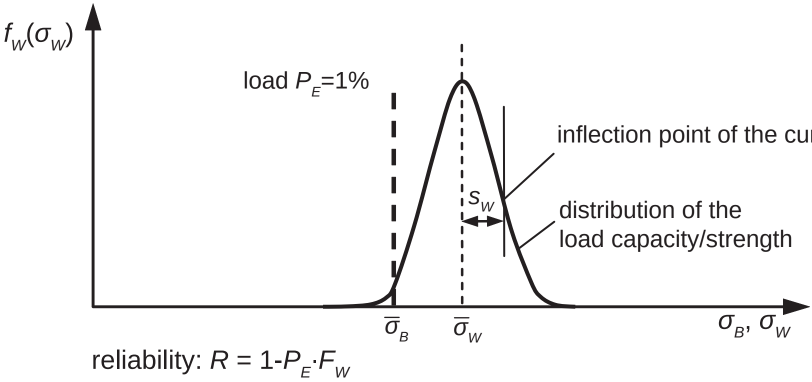


Figure 9.9. Simplified coherence between load and load capacity

9.2 Load

When observing operational stress and strain on most components, it can be established that constant load amplitudes are quite seldom in technology, Figure 9.10. The loads follow a more or less random curve.

For example, passenger cars possess completely random stochastic load curves due to the streets' roughness and the driver. The same applies for ships and drilling rigs because of sea disturbance, Figure 9.10.

Often, pure stochastic processes are superimposed with deterministic processes. For example, a mean load change occurs on the wing of a transportation airplane when it taxis, takes off or lands. This is a deterministic, precisely foreseeable process, which to a greater or lesser extent is superimposed with random processes due to the gust loads in the air or the rolling movement on the ground. Processes taking place in a reverse rolling mill are similar. On the other hand, the load of the blade in a gas turbine in a transportation airplane is to a large extent deterministic, but the load sequence is still variable. The cause for this is that the revolution speed is almost completely deterministically preset for the pilot during a certain flight and the load of the disc is mainly dependent upon the square of the revolution speed. In order to use these load-time-writs for a lifetime prediction, it must be assessed with a statistical procedure.

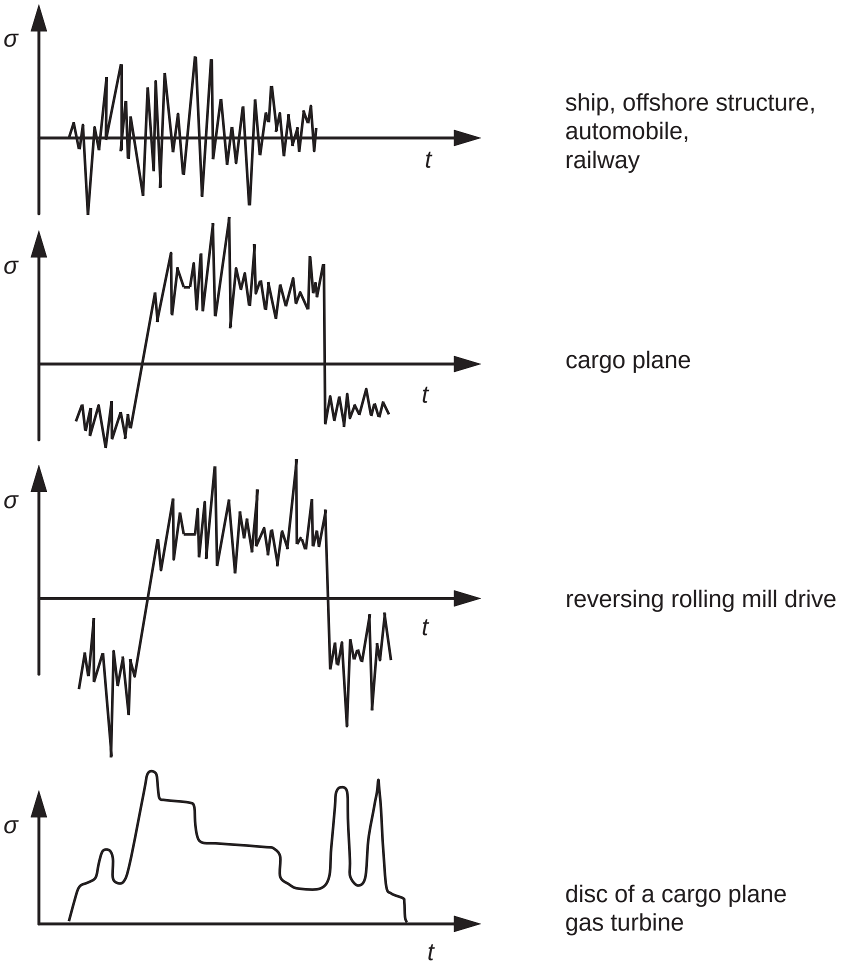


Figure 9.10. Stochastic and deterministic load curves [9.15]

9.2.1 Determination of Operational Load

For the determination of the operational load a load-time-writ or load-path-writ is required for the load. These load-time-writs can be determined with various possibilities, measurements or simulations, Figure 9.11.

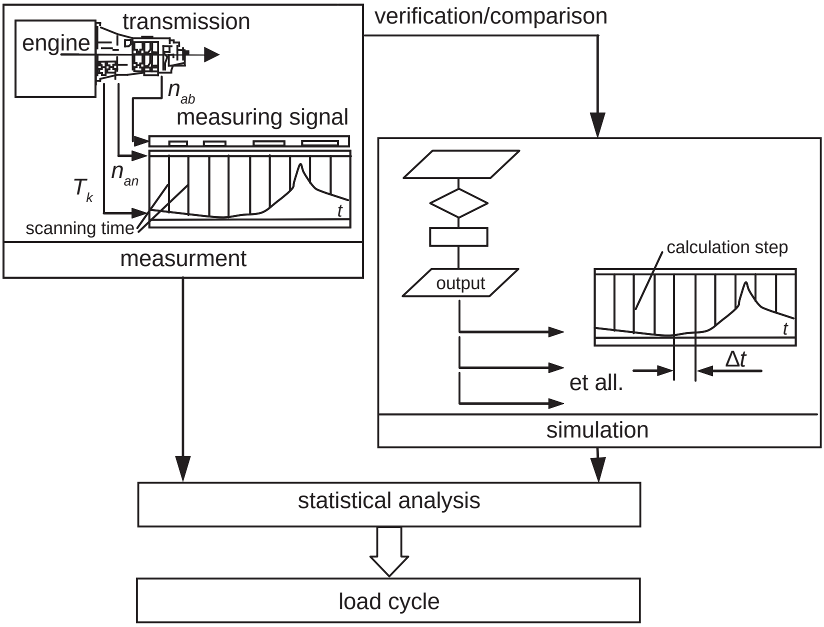


Figure 9.11. Determination of the load with simulations and measurements on a vehicle transmission

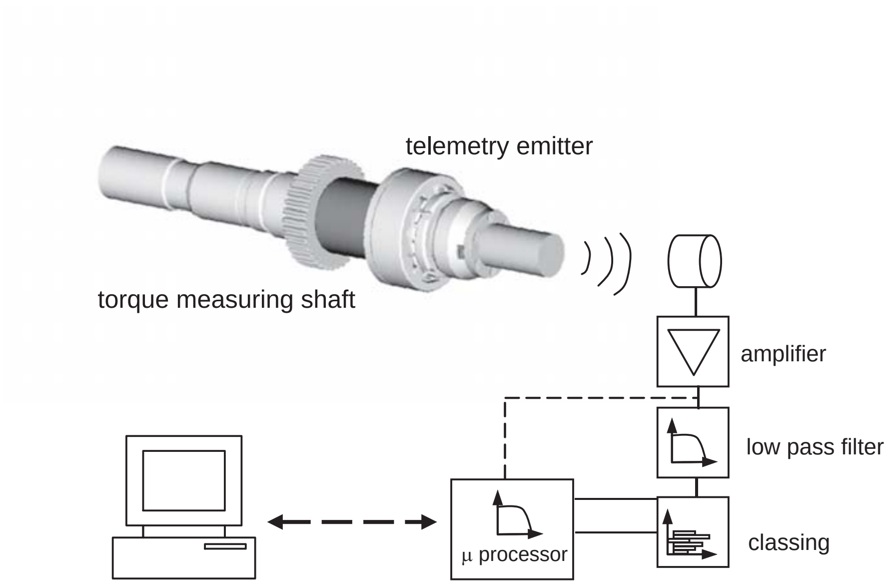


Figure 9.12. Determination of the torque on transmission shafts through measurement

Firstly, stress and strain curves for a component can be directly measured during operation. However, a measurement during operation is quite time-consuming and on certain locations often even impossible, for example gear tooth tension in a transmission. A block diagram for the measurement of torque curves in vehicle transmissions is shown in Figure 9.12.

For the mobile use of vehicles the torque curve is classified online with the help of a micro processor, or it is only recorded and classified with a delay. High sampling rates and long measurement cycles require online processing, since the number of measurement values to be saved would be too high. Today, with simple algorithms and fast processors an online classification is even possible for high frequency load-time-functions [9.10, 9.31].

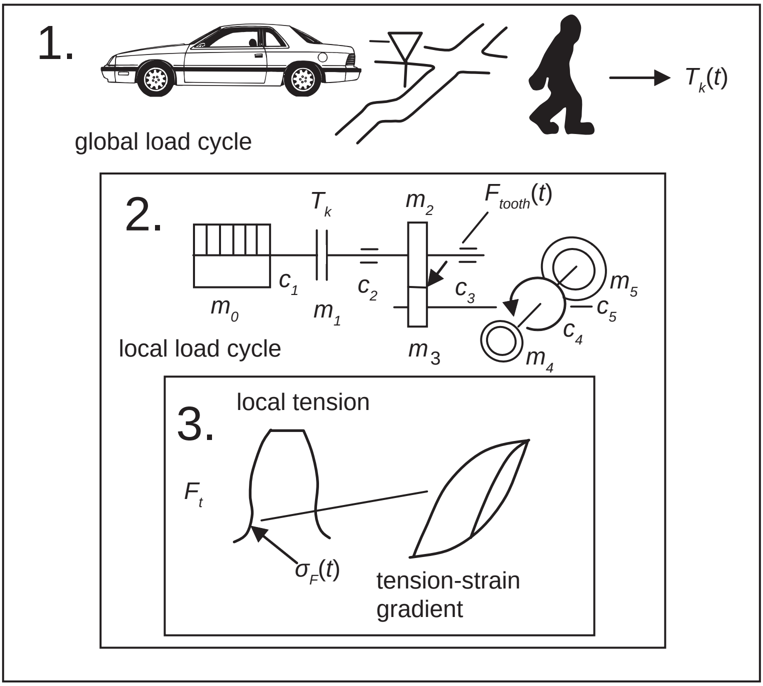


Figure 9.13. Determination of local operational loads out of a load nominal function [9.16]

The time and effort involved in measurements done directly on a component can often be reduced, if for one location in the force or torque flow, the load nominal function is determined and transferred onto the remaining components through calculation. For a vehicle, for example, this would involve measuring the clutch torque. Out of the measured torque, the torque on the gear wheels can be determined and out of that, the resulting local tensions can be acquired. Since, however, the connection between the clutch and the gear wheels and the gear wheels themselves are not rigid, but rather they possess mass, rigidity and dampers, the measured values taken from the clutch can only be transferred to the other individual components by considering the total dynamic behaviour of the system, see Figure 9.13.

For further analysis suitable programs exist for the simulation of rigid or elastic multi-body systems (MBS), finite elements (FEM) or boundary-element programs (BEM), see Figure 9.13.

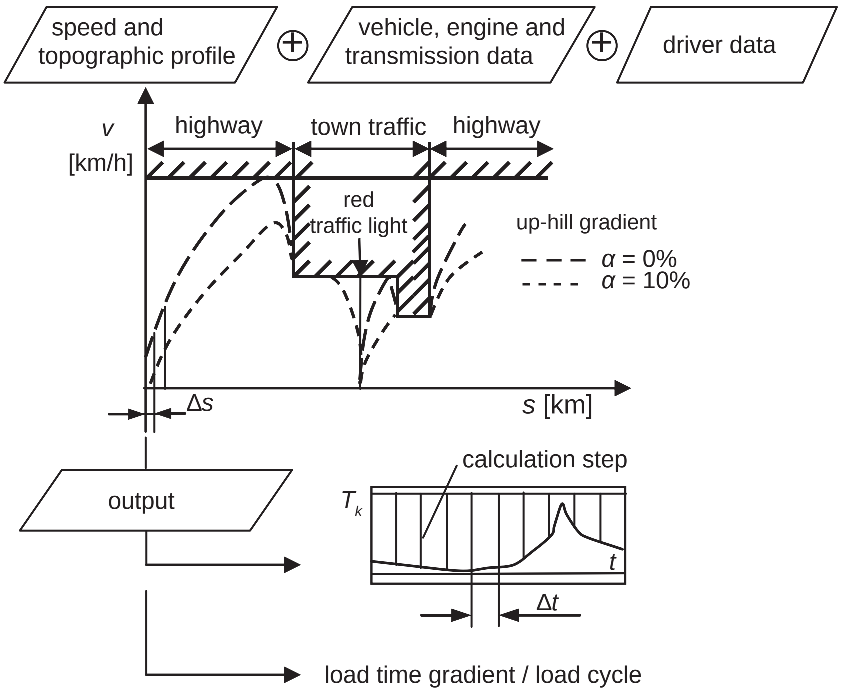


Figure 9.14. Simulated drive tour of a vehicle

Simulation offers a second method to determine stress and strain curves. This method is displayed in Figure 9.14 for vehicle power trains [9.25].

Simulations also require measured data such as path, vehicle and driver data for the example of a vehicle. In addition, an algorithm is necessary, which, dependent upon the stationary input values, allows the determination of the dynamic curve of the load as a time or path dependent variable. Then a simulation is able to produce the same results for the stress and strain curve as the measurements taken during operation, as long as the marginal conditions and algorithm are representative of the reality. The decision whether the nominal load or the local load should be determined is dependent upon the detail of modulation of the used model. In certain cases, local loads can be derived out of the nominal loads.

A third method completely avoids the time-consuming identification of the load-time-function [9.19]. The load assumptions are mostly made as a type of load spectrum. The form of the spectrum can be assumed to be normally distributed, for example, since the observed process is a random process [9.7, 9.15]. It can also be already given for rule-type processes, for example for the design of cranes, or it may be known from several long-time measurements, see Figure 9.15.

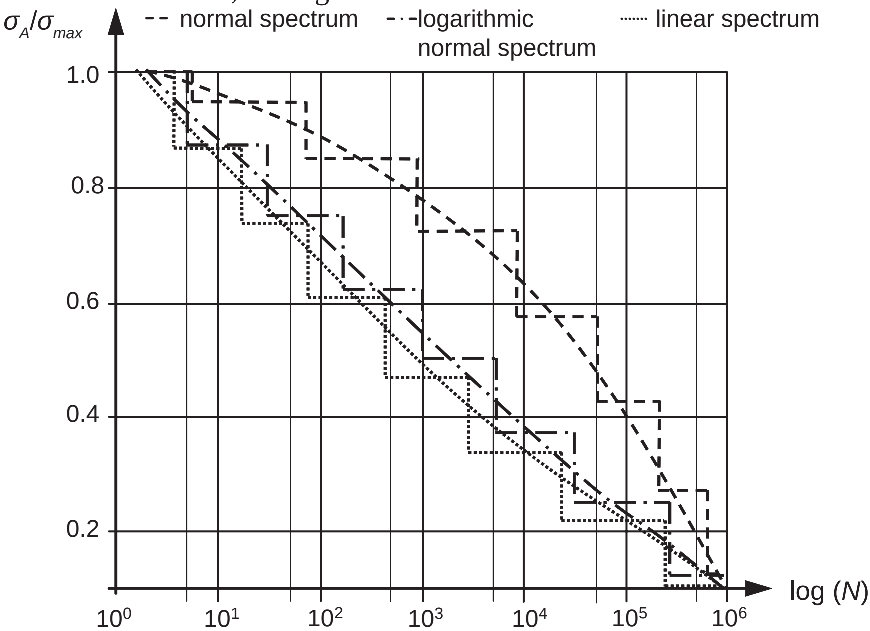


Figure 9.15. Various standardized standard spectrums with 8 stages

As a further simplification the load spectrum is summarized as an operation factor. Here, a single stage equivalent load spectrum is determined with the same damage result. The ratio of the equivalent load to the nominal load $\sigma_{equ}/\sigma_{nenn}$ is added to the nominal load as an operation factor.

9.2.2 Load Spectrums

For the lifetime calculations the measured or simulated load-time-writs must be evaluated with statistical counting methods [9.32]. This procedure is known as classification. In DIN 45667 the single parametric classification methods are introduced in detail [9.9]. Along with the single parametric classification methods, two parametric methods also exist, which have proven themselves useful for the classification of load-time-functions. For lifetime estimations the size of the stress or strain and its frequency is of major interest. The frequency of the load-time function and the sequence of the occurrence of the results are thus not taken into consideration, unless one is dealing with high temperatures, corrosion, or if the load-time-functions must be redesigned out of the load spectrum for trials to be conducted, see Figure 9.16. These assumptions are generally allowed, however, they should be closely checked for each individual case [9.32].

In classification the load curve is divided into individual oscillation cycles as well as possible, in order to establish a correlation to the Wöhler trials, which are carried out with a sine type of load in the single stage procedure.

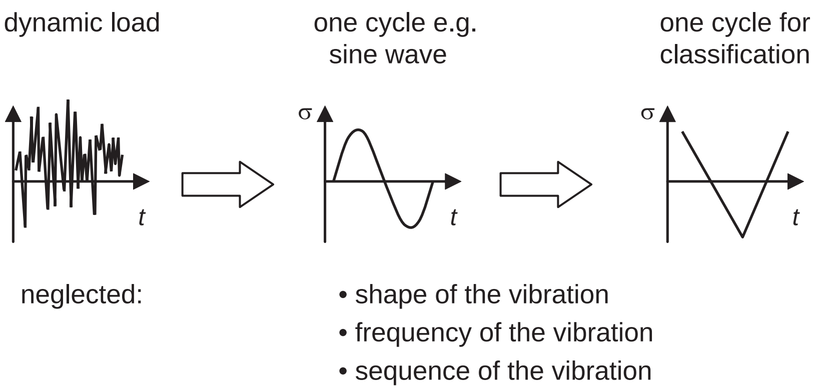


Figure 9.16. Oscillations for lifetime predictions and simplifications in the classification

The transformation of the load-time-curve is carried out by arranging the individual oscillation cycles as a number value in a class grid. The precision of the recording of the load amplitude is determined by the fineness of the class grid. 16 to 24 classes offer a sufficient classification. Creating the spectrum gives the class occupation and the sum occupation. Class occupation indicates how many recorded oscillation cycles lie within the boundaries of a certain class. The sum occupation indicates how many oscillation cycles are lower than or the same as the upper boundary of the observed class. The form of the spectrum has a decisive influence on the component's lifetime.

In a half logarithmic representation, a load spectrum is described by the amplitude *H* (cumulative frequency) and the maximal and mean values of the stress and/or strain (σ_o, σ_m), see Figure 9.17.

In the following section, the applicable single and two parametric counting methods will be introduced for lifetime determination. Single parametric methods only count the amplitude or the class boundary. Two parametric methods count the amplitude and mean or maximum and minimum. Additionally, these methods can be further distinguished between methods which count the oscillation cycles and thus collect the stress-strain characteristic for material mechanics and methods, which have a time, revolution or angle dependent sampling of the signal, e.g. in drive engineering, out of which the load on the individual components can be determined.

9.2.2.1 Single Parametric Counting Methods

Level Crossing Counting

The principle method for classification should be shown on an example of level crossing counting, see Figure 9.17. In level crossing counting a count is provoked at the crossing of one class boundary. The class widths are determined by the number of classes and the statistical spread of the measured values. For positive classes all the class transitions located above the boundary are counted, while for negative classes all the class transitions located below the class boundary are counted. Passes through the reference line (neutral axis) should be counted to the first positive class. This classification method is shown in Figure 9.17 for a stochastic function. It is shown as a histogram with the class number j over the absolute class occupation number n_j and vice versa for n_j over j. By adding the absolute occupation numbers, the cumulative frequency H_j is yielded. Furthermore, a histogram of the absolute cumulative frequency of the sums is given. Level crossing counting describes the actual load elevations. Since, however, the amplitudes of the individual load cycles have been lost, the amplitude load spectrum must be reproduced out of the level crossing counting spectrum for the execution of a damage calculation. For this it is required that for each point in the spectrum, the crossing number is equal to the cumulative frequency of the load at the amplitude and corresponds to the upper and lower limit of the spectrum load. Strictly speaking, this is only the case if all oscillation cycles cross one class.

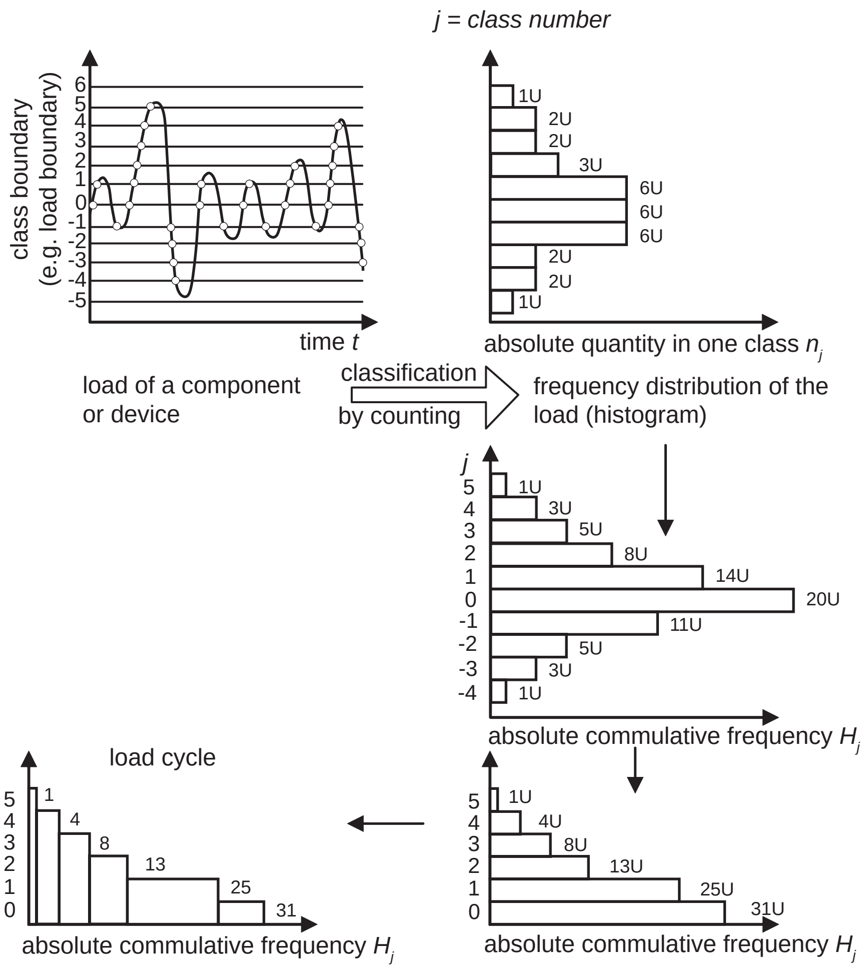


Figure 9.17. Classification with level crossing counting

In order to create a step-like amplitude load spectrum, it is necessary to insert blocks into the level crossing counting spectrum. The height of each block corresponds to the average difference between the upper and lower spectrum load and represents the respective amplitude load. The breadth of each block describes the respective load cycle number. This procedure only provides the amplitude loads required for damage accumulation calculations after a transformation.

In the past, level crossing counting was often used for lifetime estimations. For variable mean loads separate spectrums must be set up.

Range Counting and Range Pair Counting

Range counting and range pair counting are standardized in DIN 45667, see Figure 9.18.

For these methods the maximal and minimal values must be known in order for the load curve to be assessed. The difference between two successive extreme values is referred to as the range and is recorded as half an oscillation cycle in range counting. Either increasing or decreasing ranges are counted.

This method reacts sensitively to small in-between oscillations, which are of no importance to the damage, however they deconstruct large ranges. Thus, due to the exponential law of damage, the calculated total damage is strongly decreased. If possible, small in-between oscillations should be filtered out either during or before the classification process. This method is not suitable for lifetime estimation.

Range pair counting determines the cumulative frequency of range pairs, which are composed of equally large increasing and decreasing ranges. The ranges can be composed of several range sections which occur timely delayed to one another and do not necessarily need to lie at the same elevation as its equal counterpart. Thus, a characterization of the mean load is not possible. The absolute values for the maxima and minima are lost. Overlapped in-between oscillations, however, are recorded in addition to the main load cycle, without deconstructing it.

Since the result of range pair counting is a cumulative frequency distribution, out of which the frequencies of the individual classes can only be determined after the first completion of the counting, online assessment cannot be applied. Range pair counting is often used for lifetime estimations. However, caution should be taken during the assessment that only regions with the same mean load are added together.

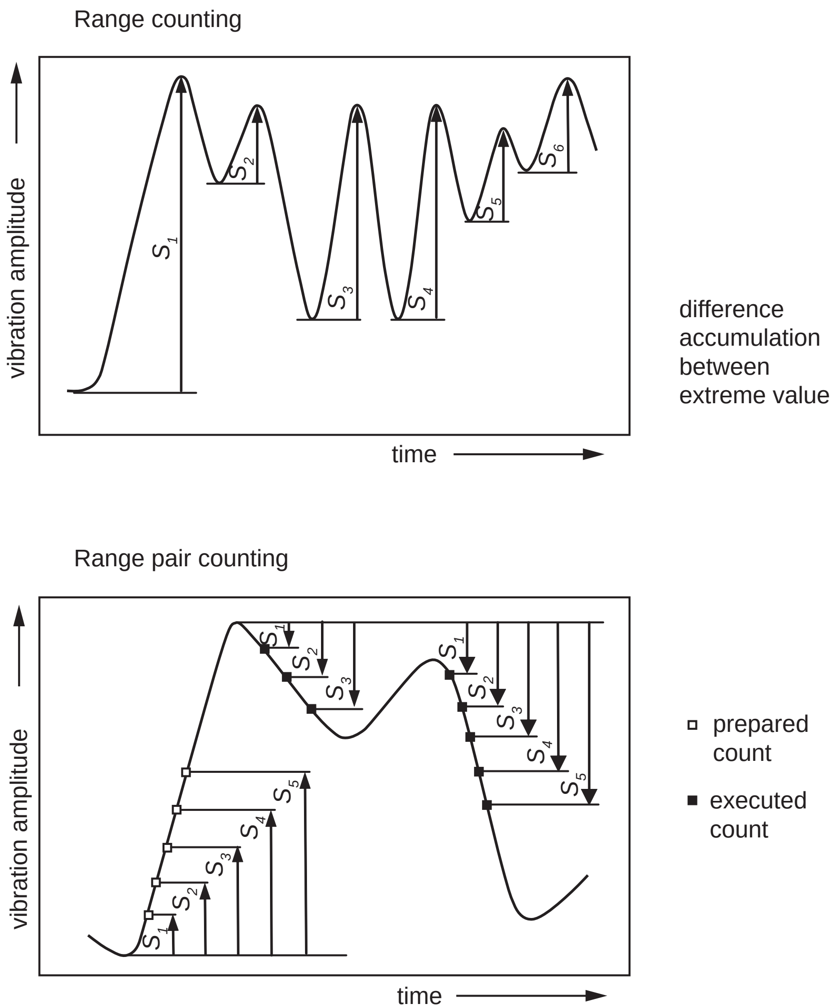


Figure 9.18. Range counting and range pair counting [9.15]

Time at Level Counting and Level Distribution Counting

Time at level counting and level distribution counting shown in Figure 9.19 and Figure 9.20 belong to the revolution speed dependent sampling methods for a vehicle transmission. These counting methods, especially level distribution counting, are the standard procedures for gear and bearing lifetime calculations today, since the individual stresses of the gear and the bearing stresses can be determined out of the torque-time-function. For time at level counting the sum of times is determined, that the signal remained within the individual class boundaries.

For level distribution counting the signal is read after equal time intervals and counted in the respective class. The frequency of counting per class is a measurement for the time spent in that class.

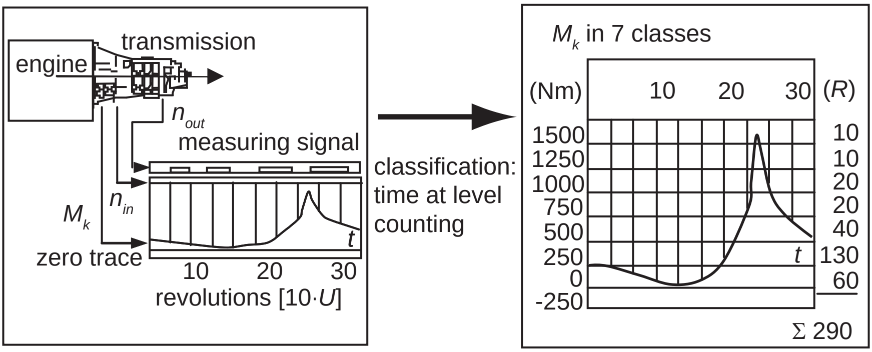


Figure 9.19. Recording a transmission load spectrum (torque) with time at level counting

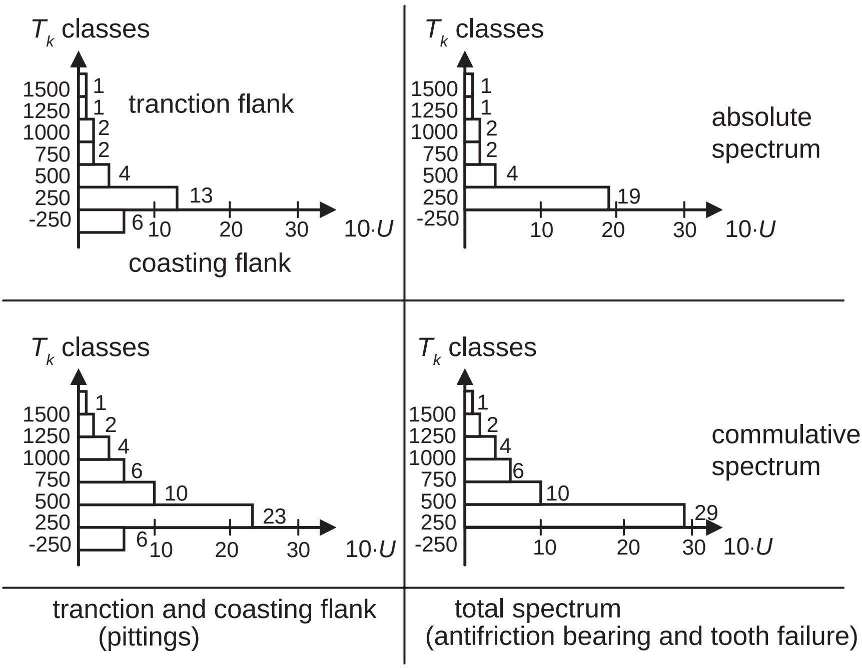


Figure 9.20. Creating a transmission load spectrum

For small sampling intervals the counting result corresponds to time at level counting. If, for the example of stress on gears, the lifetime is to be evaluated with respect to pittings (front and back flank seen separately), it is necessary to distinguish between the stress on the driving and the driven flank. For the determination of the roll bearings' lifetime and the evaluation of tooth failure the driven flank and driving flank spectra are merged into one total spectrum, due to the fact that during driven operation and driving operation, the same gear tooth is stressed.

9.2.2.2 Two Parametric Counting

For single parametric counting only the amplitude or the class boundaries are counted. For two parametric counting the maximum-minimum or the amplitude-mean is counted, see

Range Mean Counting

Range mean counting, see a, is an expansion of the single parametric range pair counting. The counting result is a frequency matrix for ranges and mean values. This counting method is not widely used, since the transition matrix of the following method is more efficient.

From-To-Counting in a Transition Matrix

The positive and negative flanks of a load-time-function are entered into a matrix one after the other, see Figure 9.21b. This matrix can be referred to as the from-to-matrix, transition matrix, correlation matrix or Markov matrix. The increasing flanks are found in the upper triangular matrix and the decreasing flanks are found in the lower one. The diagonal remains empty. The transition matrix shows the contents of the load-time-function clearly (extreme values, etc.). The results from single parametric counting functions can easily be derived, see Figure 9.25. An online classification is possible with arbitrarily long measurement cycles.

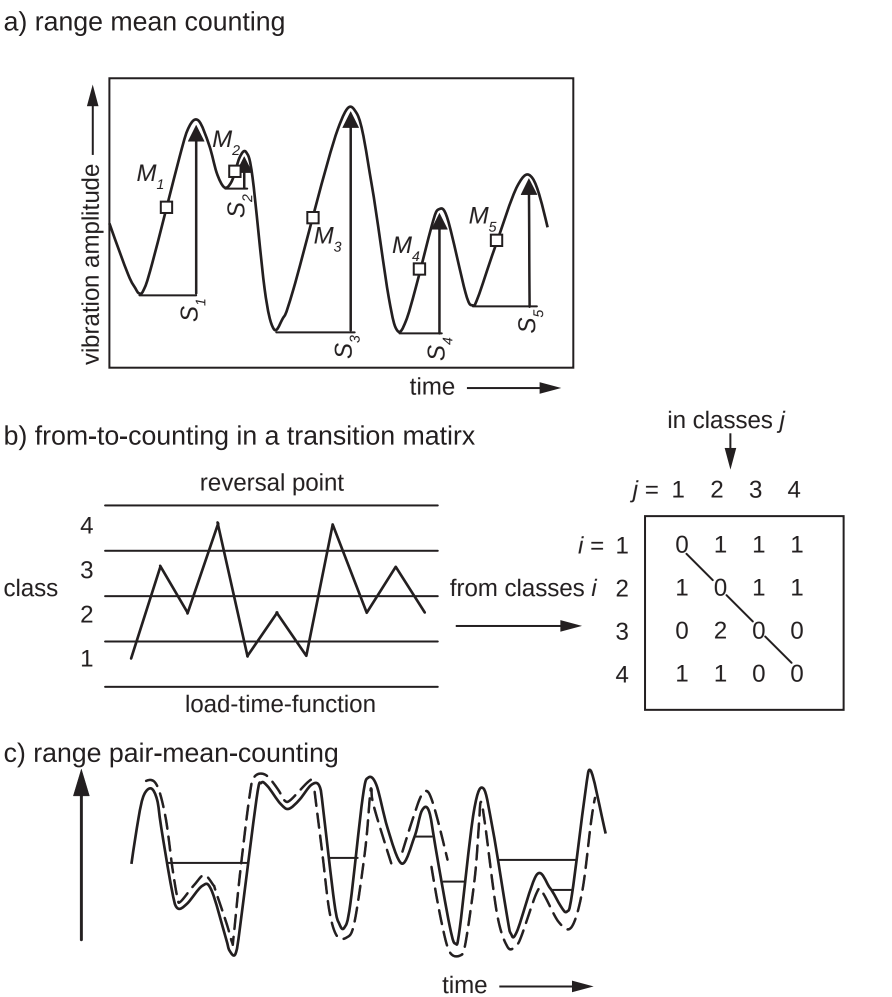


Figure 9.21. Two parametric counting

Range Pair-Mean-Counting

This counting procedure, see Figure 9.21c, corresponds to range pair counting, except that in this procedure the mean is also registered and the result is entered into a matrix. The result is identical to the result gained in rain flow counting, except for the residuum, which can only be determined with rain flow counting.

Two Parametric Level Distribution Classification

Two parametric level distribution classification connects the level distribution classification of two signals. The values are entered into a matrix.

This method is standard for torque and revolution speed classifications for the determination of bearing stress and gear tooth spectra. With the value of the revolution speed, the number of revolutions can be determined [9.24].

Rain flow Counting

Rain flow counting is a concept developed in Japan by Matsuishi and Endo [9.11] and in the USA for the segmentation of any arbitrary stress curve into complete oscillation cycles. Rain flow counting counts closed hysteresis loops in a load-time-function, which are decisive for the damage of metal materials. Open hysteresis curves are stored as a residual, see Figure 9.22.

The problem, which led to the development of rain flow counting, was to develop a procedure to classify the stress-strain characteristic of a non-single stage elastic-plastic loaded material in such a way, that characteristic values of the material behaviour, which, after recent observation, are associated with fatigue damage (disruption), can be determined and a storage for these values can be made accessible. Such characteristics are typical variables for single stage trials, which have an automatically closed stress-strain hysteresis loop, and characteristic variables for the case of non single-stage load processes of hysteresis loops, which completely close during the load-time functions, see Figure 9.22.

The total strain oscillation amplitude (ε_{tot}) and the plastic strain oscillation amplitude (ε_{pl}) count as such characteristic variables. They are characteristic variables for which alone the strain-time function must be known in order to determine them. Naturally, all stress variables can be classified with the rain flow method. Normally, the variables for the outer loads are classified.

The following assumptions are valid for rain flow counting [9.8, 9.21]:

- Cyclic stable material behaviour, that means that the cyclic stress-strain curve remains constant, thus no hardening or softening of the material takes place.
- Validity of the Masing hypothesis, which means that the form of the hysteresis loop branches correspond to the double of the initial load curve.
- Memory behaviour of the material, compare with Figure 9.22, which means that after a closed hysteresis loop, a previously not yet completely closed hysteresis loop follows the same σ, ε path.

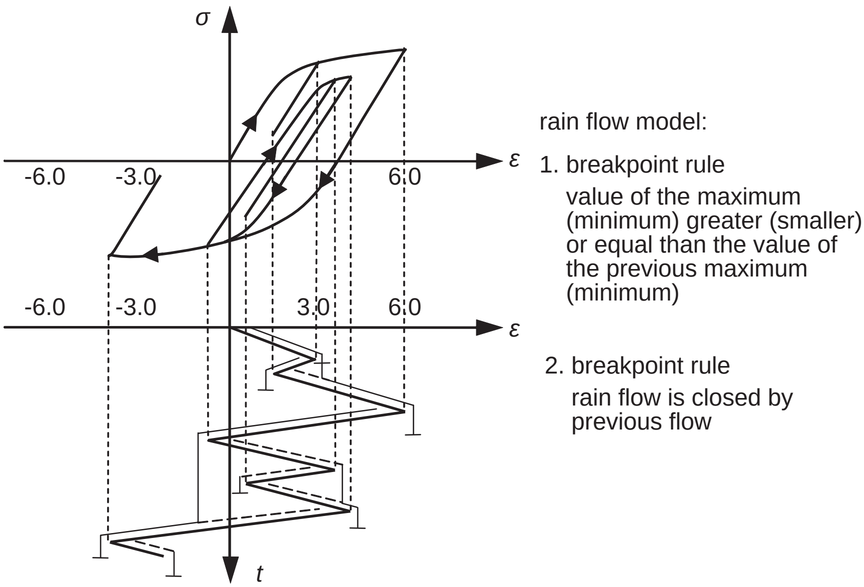


Figure 9.22. Acquisition of the load-time behaviour with the rain flow counting method

Several algorithms exist for the automation of this assessment, which only slightly differ from one another. The two most common algorithms are the push-down-list [9.32] and HCM (Hysteresis Counting Method) [9.8]. The latter of the two is more suitable for a computerized supported assessment.

9.2.2.3 Comparison of the Different Counting Methods

Figure 9.23 and Figure 9.24 show classification results for constant and variable mean loads in order to compare the different counting methods. For constant mean stress, see Figure 9.23, the oscillation cycles are completely recorded. For variable mean loads, see Figure 9.24, the rain flow and the range pair counting methods do now deviate from one another. Level crossing counting indicates a higher portion of larger oscillation cycles (more damage intensive) for smaller spectrum sizes. For range counting it is exactly the opposite.

If, for example in drive engineering, the exact stress-strain curve is not taken directly from the most critically loaded position, but rather the mean value of the load function in the power flow either before or after this position is taken, then the time at level counting or level distribution counting methods should be applied.

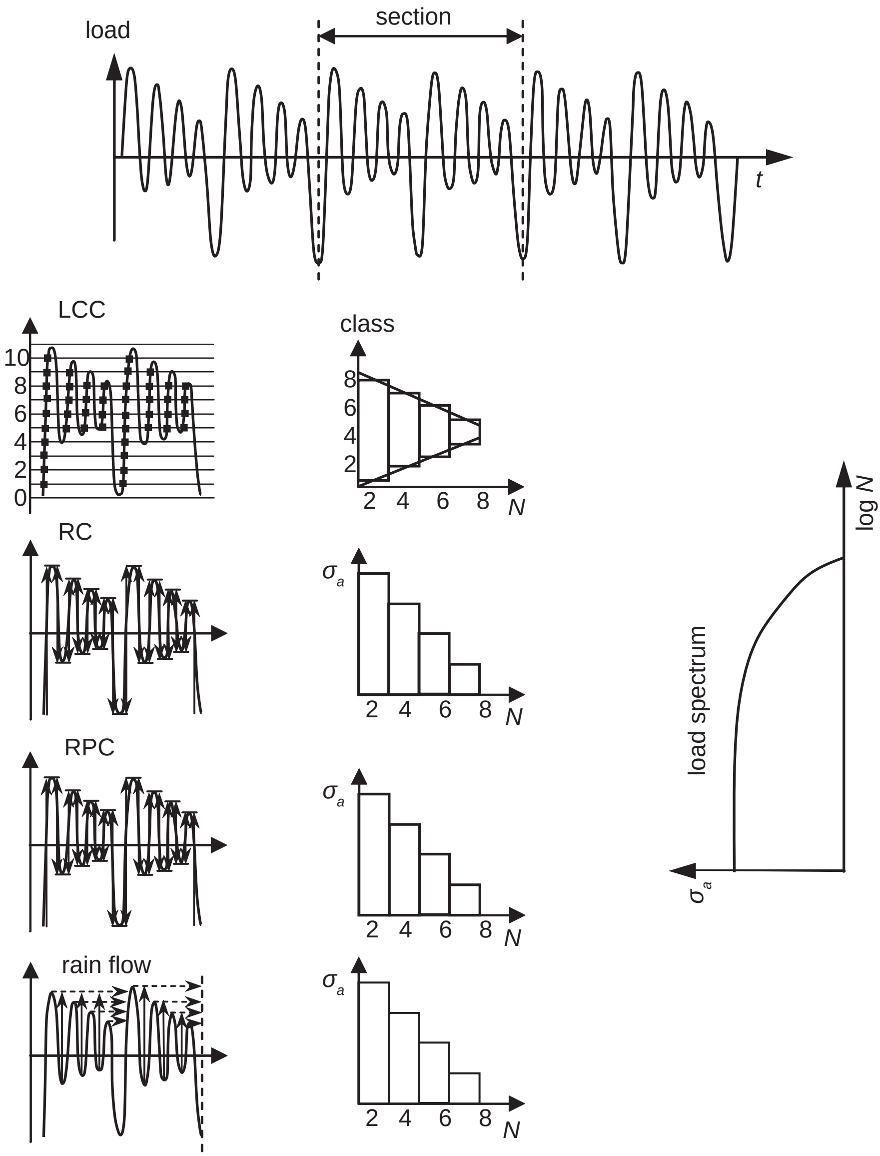


Figure 9.23. Comparison of classification methods for loads with a constant mean stress

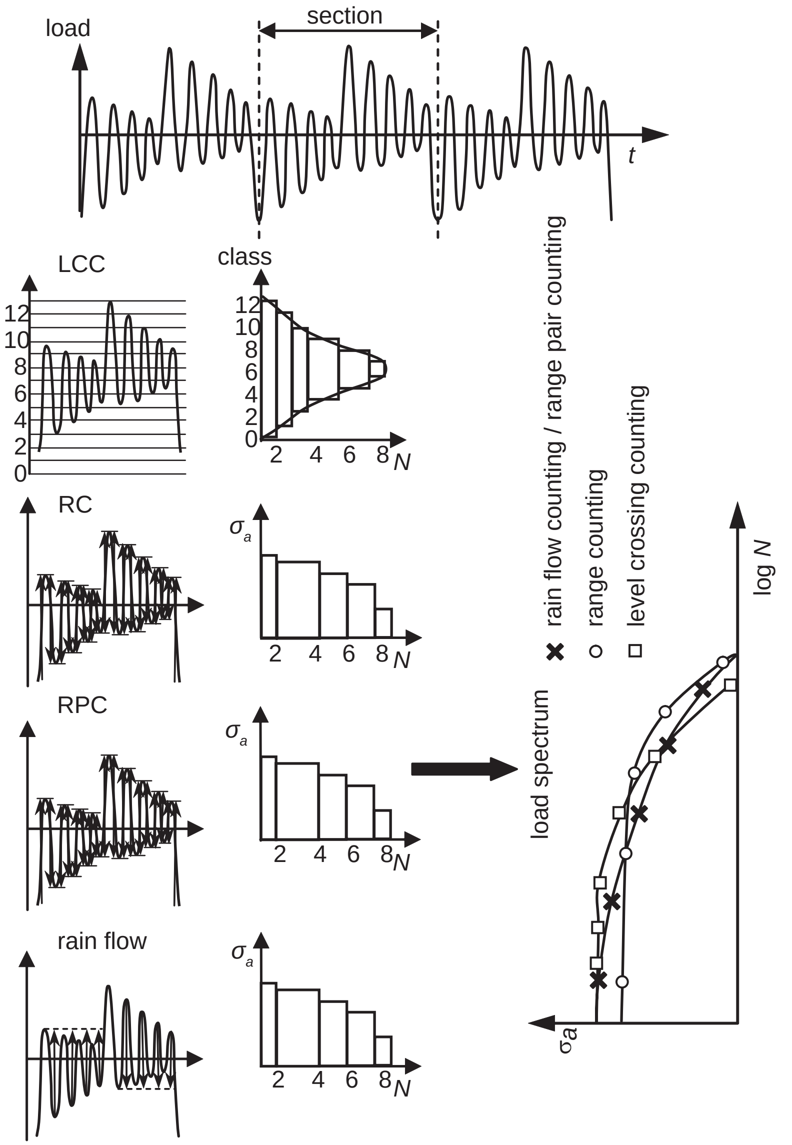


Figure 9.24. Comparison of classification methods with variable mean stresses

Since calculated lifetime estimations are afflicted with large uncertainties, it is desired to reconstruct the stochastic load-time functions out of the load spectrums, in order to carry out experimental lifetime proofs with servo-hydraulic facilities. However, the reconstruction of a representative load-time function is not possible with the load spectra alone. Figure 9.25

gives an overview of single parametric counting results which can be derived from two parametric counting results.

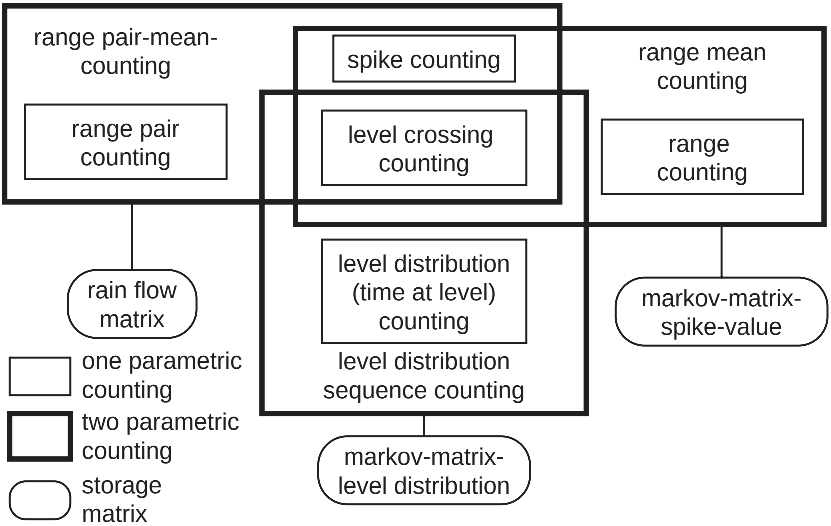


Figure 9.25. Interrelationship between single and two parametric counting methods

In conclusion one can say: the counting method chosen influences the result of the lifetime estimation. Contemporary knowledge suggests that, the two parametric rain flow counting method is the most suitable method for the acquisition of the local stress-strain hysteresis curves. However, the most well known and applied methods are level crossing counting and region pair counting.

In drive engineering level crossing counting as well as the single and two parametric time at level counting and level distribution counting are used.

9.3 Tolerable Load, Wöhler Curves, SN-Curve

The Wöhler curve, often also referred to as the SN-Curve, is required as a description of the material behaviour for the calculation of fatigue strength and operational fatigue strength. Two types of Wöhler curves exist: stress controlled and strain controlled.

9.3.1 Stress and Strain Controlled Wöhler Curves

Stress controlled Wöhler curves describe material behaviour as a correlation between the tolerable load cycles to failure *N* for a certain stress amplitude, Figure 9.26.

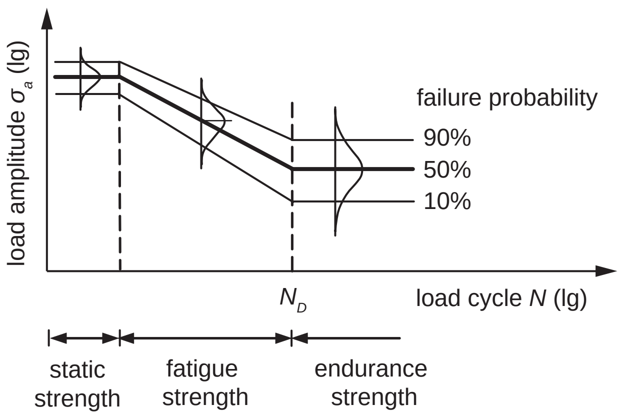


Figure 9.26. Stress controlled Wöhler curve

There are three zones to distinguish between in the double logarithmic representation of Wöhler curves:

1. *Quasi-static fatigue*, up to ca. $N = 10^1 - 10^3$ oscillation cycles,
2. *Fatigue strength*, the zone of the sloped lines, until the corner load cycles to failure $N_D = 10^6 - 10^7$,
3. *Endurance strength*, zone of the horizontal lines starting from $N > N_D$. However, several materials such as austenitic steels do not possess a distinct endurance strength

In the fatigue strength zone, the Wöhler curve can be described by the following equation if represented in double logarithmic form:

$$N = N_D \cdot \left(\frac{\sigma_a}{\sigma_D} \right)^{-k} . \tag{9.17}$$

In contrast to the stress controlled Wöhler curves, strain controlled Wöhler curves describe material behaviour for constant strain, see Figure 9.27.

With strain controlled Wöhler curves material damage can be better described, since for oscillating loads, the remaining strain which occurs in every load cycle is virtually the same as the total strain for the case of large strains, and thus can be seen as damaging.

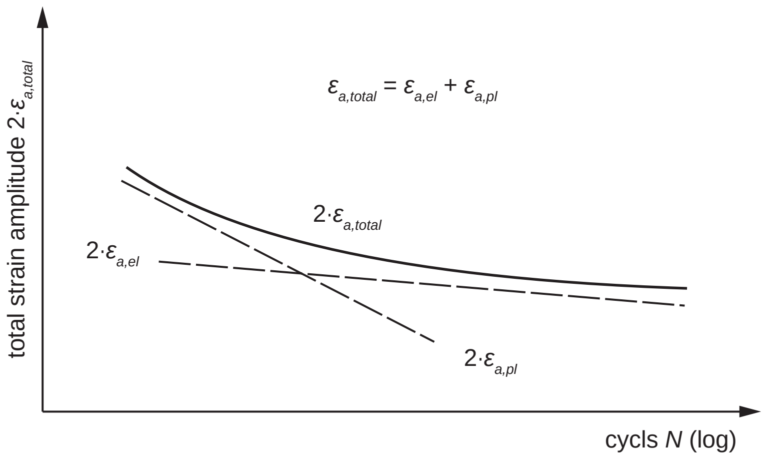


Figure 9.27. Strain Wöhler curves with constant stress

For double logarithmic representation, the lines of the elastic and plastic strain amplitudes can be formed for example with the Manson-Coffin equations:

$$\varepsilon_{a,el} = (k_{el} / E) \cdot N_A{}^b \,, \tag{9.18}$$

$$\varepsilon_{a,pl} = k_{pl} \cdot N_A{}^c \,. \tag{9.19}$$

k_{el}, k_{pl} are the respective stress and strain coefficients of the material. b, c are the stress and strain exponents. The strain Wöhler curve is normally given as the crack Wöhler curve. This indicates that the cause of damage is cracking in the material.

9.3.2 Determination of the Wöhler Curves

If possible, the determination of Wöhler curves for operational fatigue strength calculations should be carried out on a real component. Often, however, due to cost and time limitations, the calculations are only carried out on special test specimens.

The resulting load cycles to failure are random variables, which means that they lie scattered around the mean value. Today, the transformation of results won from a tension/compression trial onto a real component is difficult [9.26]. Thus, the exact determination of a notch over the entire load cycle zone is still not possible today. Therefore, one is forced to rely on tests and trials.

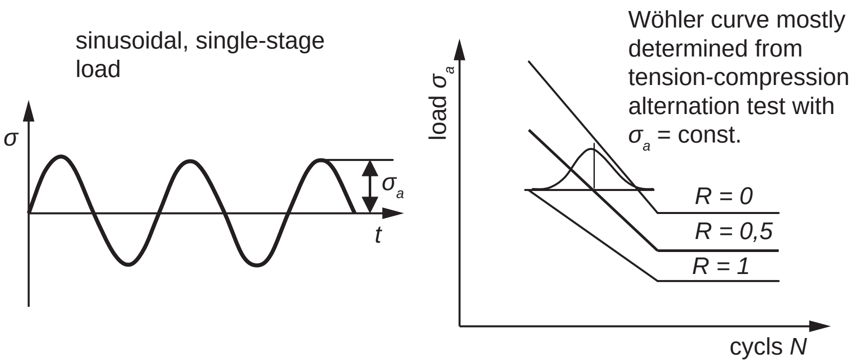


Figure 9.28. Material properties, Wöhler curve

For the calculation the shape variable α_k and the notch variable β_k are used. These variables show how much greater the local stress is than the nominal stress in the notch. Additionally, mean stress influences the lifetime, whereas a tension mean stress shortens the lifetime and a compression mean stress increases the lifetime. The effect of a tension mean stress is dependent upon the material. High-strength materials are very sensitive to tension mean stress, whereas low-strength materials are not so sensitive. Casting materials are more sensitive to tension mean stress than malleable materials. In general, welding connections behave like casting materials.

Likewise, residual stress can strongly influence a material's lifetime. This type of stress acts like a mean stress of the same amplitude and the same algebraic sign, as long as the residual stress is not decreased again during operation due to high temperatures for example. The influence of residual stress on the lifetime is difficult to determine quantitatively. Furthermore, residual stress can disappear during the lifetime because of oscillating loads. In general, the technological characteristic influence, for example through poor deformation while forging, leads to unfavourable material properties and material flaws. The geometric characteristic influence covers the uneven stress distribution within a component. The statistical characteristic influence covers the number of possible failures with respect to the volume of a component. The type of load is another influencing parameter. Bending stress, for example, causes the support effect. This influence is considered by a support digit. The surface finish also plays an influential role. Components with a smooth surface achieve longer lifetimes than components with rough surfaces. Further influencing factors come from the surrounding environment, such as corrosion or temperature.

If all these parameters are considered with the various types of positive and negative influences on the lifetime, it is also necessary, aside from all other difficulties, to consider interdependencies of the influences among one another. Therefore, until now it has not yet been successfully achieved to develop a scientific irreproachable method for lifetime estimation for even the simple case of constant stress amplitudes, which for example is based on the metallic characteristics of a material. Thus, the only option for lifetime estimation is the Wöhler trial, at best carried out on the original component itself.

For cases where the necessary Wöhler curve is not available, the "calculational fatigue strength proof" guide [9.12] can be used as an aid. Another approach can be used according to Hück [9.17]. This approach deals with a statistically ascertained formula, developed out of several Wöhler curves, which takes the influential parameters like type of material, shape variable, type of load, level stress ratio, surface finish and production procedure into consideration.

In all of these estimations the danger exists that significant influential factors are not taken into consideration. The example in Figure 9.29 shows a comparison of a component Wöhler curve of a straight toothed spur gear with the Wöhler curve according to DIN 3990 for gear tooth stresses. The component Wöhler curve of the gear was determined in the transmission on an electrical torque test rig [9.4].

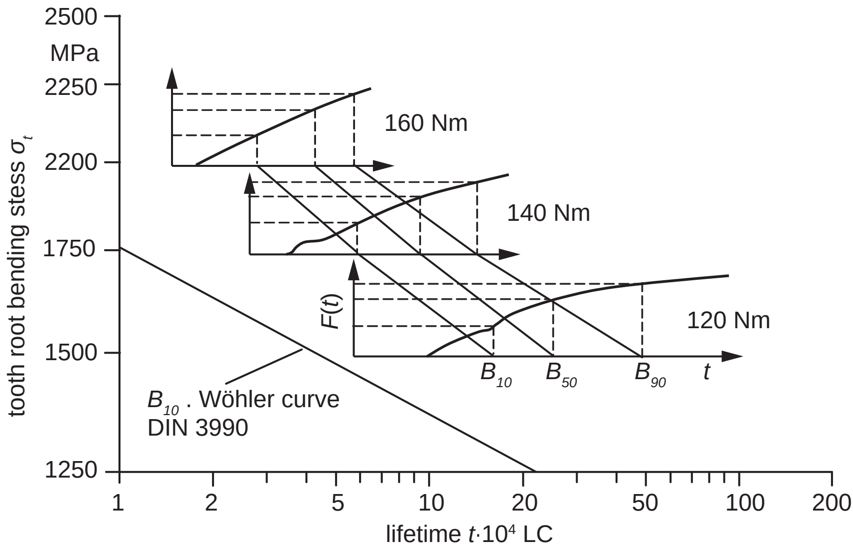


Figure 9.29. Comparison of a DIN Wöhler curve and a component Wöhler curve for gear tooth stress

9.4 Lifetime Calculations

In lifetime calculations, occurring loads (load spectrum) are compared with the tolerable load. In principle, three different calculation concepts exist:

- nominal stress concept,
- local concept or notch base concept and
- fracture mechanics concepts.

The fracture mechanics concept assumes that the component has already begun to crack and calculates the remaining lifetime of the crack's progress until the final fracture occurs [9.15]. Because this concept has little relevance for mechanical components, it will not be observed here in further detail.

In Section 9.4.1 and 9.4.2 the general procedure shown on an example using the nominal stress concept is shown. Section 9.4.3 discusses the differences between the nominal stress concept and the local concept.

9.4.1 Damage Accumulation

Oscillating loads cause an effect in materials, which is often referred to as "damage" as soon as this load surpasses a certain limit. It is assumed that this damage accumulates from the individual load cycles and leads to a material disruption (material fatigue). For an exact calculation this damage must be collected and recorded quantitatively. This, however, has not yet been achieved with success.

Despite this fact, in order to gather information concerning the lifetime L out of the results of Wöhler trials with irregular load cycle effects, around the year 1920, *Palmgren* developed the fundamental idea of linear accumulation, specific for roll bearing calculations. In 1945, *Miner* published the same idea in a general form.

Miner assumes that a component absorbs work during the fatigue process. The ratio of already absorbed work to the maximal work which can be absorbed is a measurement for the current damage. Thus, the ratio of the load cycle number n to the load cycles to failure N, which is determined in the single-stage zone with the corresponding amplitude, is equal to the ratio of absorbed work w to absorbable work W. This is denoted as the damage portion:

$$\frac{w}{W} = \frac{n}{N}. \tag{9.20}$$

The prerequisite that the absorbable fracture work W is the same for all occurring load sizes, allows the addition of the individual damage portions for load cycles of different sizes:

$$\frac{n_1}{N_1}+\frac{n_2}{N_2}+...+\frac{n_m}{N_m}=\frac{w_1}{W}+\frac{w_2}{W}+...+\frac{w_m}{W}. \tag{9.21}$$

The limiting condition of strength comes into play when the absorbed work and absorbable work are the same:

$$\frac{w_1+w_2+...+w_m}{W}=1. \tag{9.22}$$

By substituting this equation into Equation (9.21), the non quantifiable work sizes disappear and a condition evolves which can be used for dimensioning tasks:

$$\frac{n_1}{N_1}+\frac{n_2}{N_2}+...+\frac{n_m}{N_m}\leq 1. \tag{9.23}$$

Using the fundamental equation of the damage accumulation hypothesis requires knowledge concerning the load cycles to failure N_i for the corresponding load absolute values σ_i. These can be taken, for example, out of a Wöhler curve in a double logarithmic coordinate system defined by the endurance strength pivot point (σ_D, N_D) and the slope k. Out of the equation for a straight line for this Wöhler curve, the Equation (9.17) evolves for the tolerable load cycles to failure N:

$$N=N_D\cdot\left(\frac{\sigma_a}{\sigma_D}\right)^{-k}. \tag{9.24}$$

After substituting Equation (9.24) in (9.23), Equation (9.25) describes the damage with the damage sum S of a discontinuous spectrum with m load stages σ_i:

$$S=\sum_{i=1}^{m}\frac{n_i}{N_D}\cdot\left(\frac{\sigma_i}{\sigma_D}\right)^{k};\qquad \sigma_D\leq\sigma_i\leq\sigma_{\max}. \tag{9.25}$$

Miner confined the applicability of this equation by the following conditions:

- sinus formed load curve;
- no hardening or softening appearances in the material;

- the begin of a crack is considered as an incipient damage;
- some loads lie above the endurance strength.

By not considering the conditions above, especially the last condition, the results of the calculations will be on the unsafe side in many cases. The Palmgren-Miner hypothesis is shown in Figure 9.30.

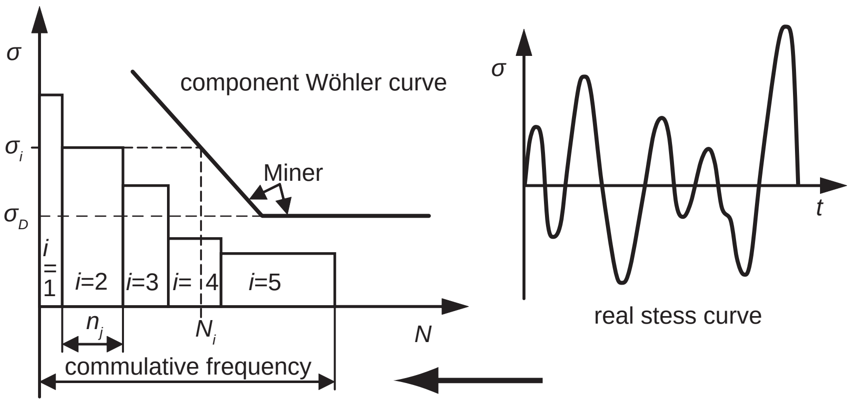


Figure 9.30. Linear damage accumulation hypothesis according to Palmgren Miner

Numerous different researchers have occupied themselves with the damage accumulation hypothesis, so that currently several variations exist. In general, the variations only distinguish themselves by the fundamental Wöhler curve used: either fictitiously extrapolated or the real curve itself, Figure 9.31. The hypotheses from *Haibach*, *Corten-Dolan* [9.15] and *Zenner-Liu* [9.20, 9.34] also assume damage for loads, which occur in the endurance strength zone.

The fundamental Miner procedure from Corten and Dolan is an application of the law of Palmgren-Miner on a Wöhler curve, which is elongated straightly until $\sigma = 0$ without taking the existence of an endurance strength into consideration. Thus, damage portions of stress changes less than the endurance strength are considered:

$$S = \sum_{i=1}^{m} \frac{n_i}{N_D} \cdot \left(\frac{\sigma_i}{\sigma_D} \right)^k ; \qquad 0 \le \sigma_i \le \sigma_{max} . \tag{9.26}$$

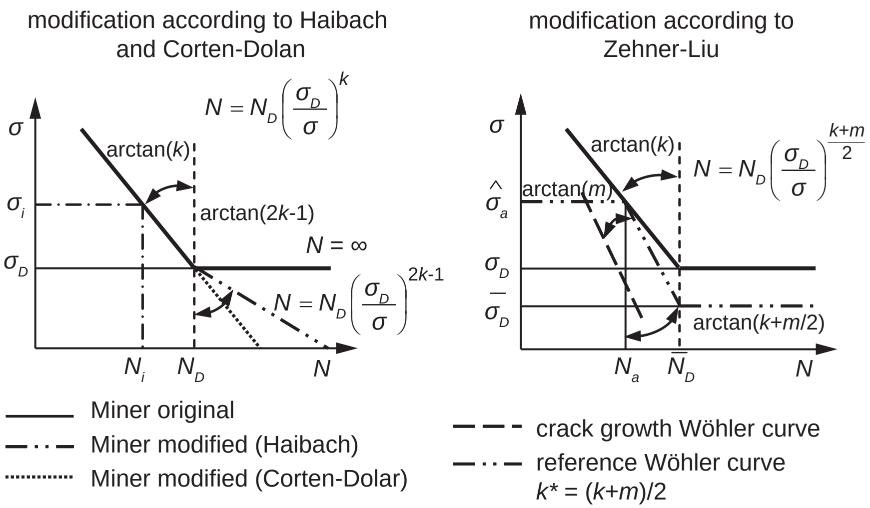


Figure 9.31. Most important modifications of the law of Miner

This assumption that no endurance strength exists yields results which lie on the safe side, especially if a large portion of the loads lie below the endurance strength. For a decreasing portion of load cycles lower than the endurance strength, the discrepancy to the result is decreased when using the law of Palmgren-Miner.

The Miner procedure modified by Haibach is oriented on a thesis supported by experimental results, which implies that the endurance strength decreases with increasing damage. The iterative calculation of damage growth under consideration of the straight existing degree of damage (consequential Miner modification), which can only be done with much time and effort, is altogether avoided in Haibach's approach through the definition of a fictitious extended fatigue strength below the endurance strength. The calculation of a spectrum's damage now takes place with the Wöhler curve slope k for loads, which are greater than the endurance strength, and with the slope $(2k - 1)$ of the fictitious fatigue strength line for loads, which are lower than the endurance strength:

$$S = \sum_{i=1}^{m} \frac{n_i}{N_D} \cdot \left(\frac{\sigma_i}{\sigma_D} \right)^k + \sum_{j=1}^{l} \frac{n_j}{N_D} \cdot \left(\frac{\sigma_i}{\sigma_D} \right)^{2k-1} \quad , \tag{9.27}$$

$$\sigma_1 \geq \sigma_i \geq \sigma_D ; \quad \sigma_D \geq \sigma_j \geq 0 . \tag{9.28}$$

The consequential Miner modification differs from the modified Miner procedure by the fact that the lifetime line merges to the endurance strength like an asymptote.

Another improved approach was suggested by *Zenner* and *Liu* [9.20, 9.34]. This approach claims that the component Wöhler curve is not an adequate reference for lifetime calculations. Since most of the time, damage is caused by two different phases: crack formation and crack progression, the crack progression line is assumed to have the slope $m = 3.6$ independent of the type of material. The fictitious reference Wöhler curve is then formed out of the component Wöhler curve and the crack progression line. The pivot point in the reference Wöhler curve is at the spectrum's highest value and has the slope:

$$k^* = \frac{k + m}{2} . \tag{9.29}$$

The endurance strength of the reference Wöhler curves is half of the endurance strength of the component Wöhler curve:

$$\bar{\sigma}_D = \frac{\sigma_D}{2} . \tag{9.30}$$

Thus, the damage of a component can be calculated analogue to Equation (9.25):

$$S = \sum_{i=1}^{l} \frac{n_i}{N_D} \cdot \left(\frac{\sigma_i}{\sigma_D} \right)^{\frac{k+m}{2}} , \tag{9.31}$$

$$\hat{\sigma}_a \geq \sigma_i \geq \frac{\sigma_D}{2} . \tag{9.32}$$

This procedure is evaluated differently in different literature sources. *Melzer* [9.21] and *Zenner* [9.20] claim an improvement in the informational value, while other literature sources [9.13, 9.28] claim a shift of the results to the unsafe side.

If stress or strain spectrums and a Wöhler curve are available, then with the help of the damage accumulation hypothesis the lifetime of a component can be calculated. In practice, however, it has often been proved that for a failure the damage sum $S = 1$ often does not comply. Thus, the calculations are carried out with another damage sum $S = \text{constant}$, which is gained from experimental operation fatigue strength trials [9.2]. This procedure is referred to as the relative Miner procedure and is found quite often in practice [9.13]. The survival probability results from the initial probabilities.

9.4.2 Two Parametric Damage Calculations

The calculation equations for damage accumulation shown in the previous sections only consider the amplitude stress as the most important influential factor for the assessment of the individual load cycles. Considering other parameters, such as the mean stress or frequency for example, is basically possible, if a respective identification of the load and the necessary material property values are at hand.

Since after the amplitude stress, the mean stress as a second parameter has the largest influence on the lifetime, an additional consideration of the mean stress is conducted in a two parametric damage calculation. In some cases the limit stress ratio $R = \sigma_u / \sigma_o$ is the preferred designation of the mean stress.

Both variables are made available through a classification done with rain flow counting for example. The execution of the damage calculation requires the deflection stress spectrum and the component Wöhler curve for each observed mean stress, Figure 9.32.

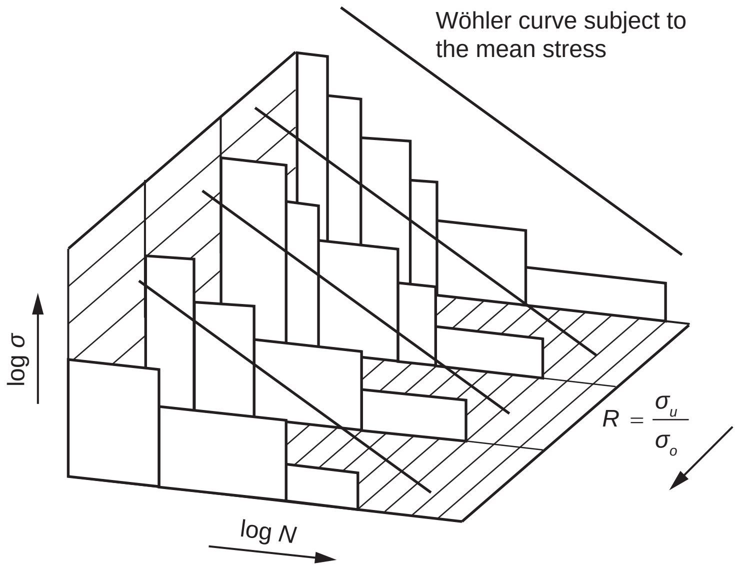


Figure 9.32. Two parametric damage calculation under consideration of the mean stress

The equation for the calculation of the relative damage includes the compilation of the individual deflection stress classes in the inner summation, which, depending on the outer summation, is carried out for each of the observed mean stress classes (limit stress ratio classes), for example for the fundamental Miner modification:

$$S = \sum_{j=1}^{q} \left(\sum_{i=1}^{p} \frac{n_{ij}}{N_{Dj}} \cdot \left(\frac{\sigma_{ij}}{\sigma_{Dj}} \right)^{k_j} \right). \tag{9.33}$$

An alternative approach is the possibility of including the mean stress with the use of the modified Haigh graph. This graph describes the relationship (Gerber parabola or Goodman line) between mean stress and amplitude stress for a constant limit oscillation cycle number. Here, with rain flow counting, for example, through a transformed amplitude with the mean stress 0

$$\sigma_{a,trans} = f(\sigma_a, M, \sigma_m) \tag{9.34}$$

each matrix element is replaced with the mean stress sensibility (compare with Figure 9.33):

$$M = \frac{\sigma_a(R=-1) - \sigma_a(R=0)}{\sigma_m(R=0)} = -1. \tag{9.35}$$

The result is an amplitude stress spectrum [9.17].

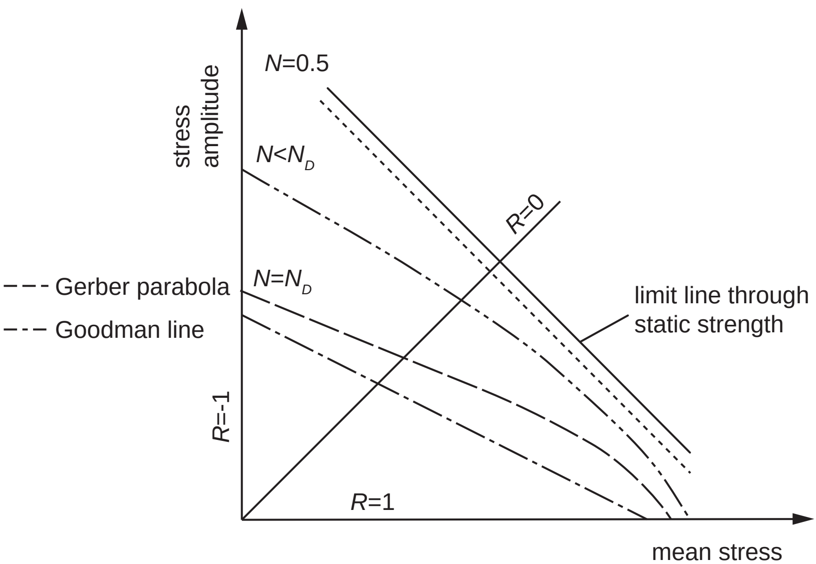


Figure 9.33. Haigh graph with the Goodman line and the Gerber parabola

Further trials have shown that the sequence in which the stress amplitudes working on the component occur has a significant influence on the lifetime. A longer lifetime can be expected, if for the same load spectrum the oscillation load cycles with lower amplitudes occur on the component before the larger amplitudes. This leads to the requirement, that the trial loads, upon which the lifetime estimations are based, should show the same intermixture of amplitudes as the loads in operation. Otherwise, large deviations in the results will occur. In general, in calculational lifetime predictions uncertainties can arise. Along with the operational load, the strength values are inflicted with uncertainties and a calculational linear accumulation of component damage based on knowledge of fracture mechanics is only conditionally correct. Thus, operational fatigue strength measurements of a component must be supported by trials.

9.4.3 Nominal Stress Concept and Local Concept

Normally, the lifetime of components is estimated based on nominal stress (nominal stress concept). This procedure, see Figure 9.34, will be shown in the last section of this chapter.

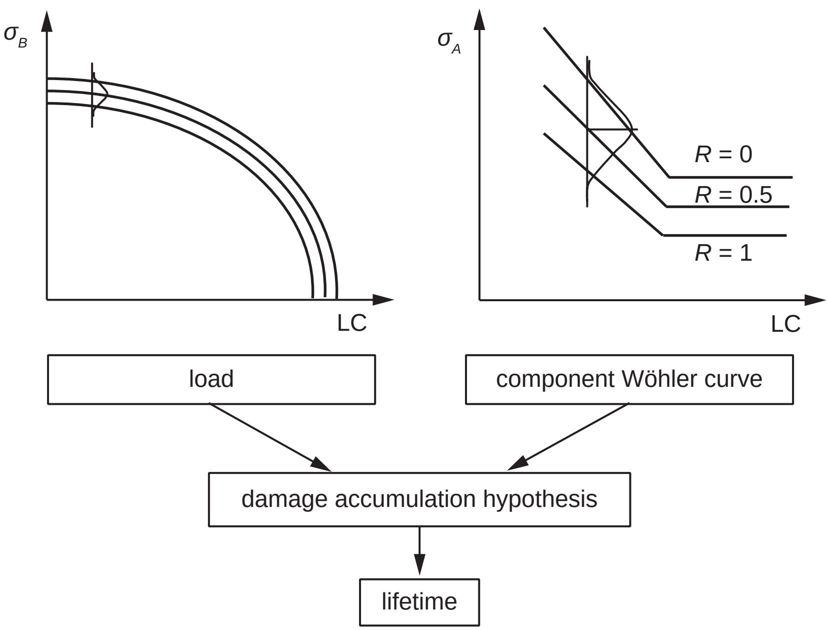


Figure 9.34. Lifetime calculation with the nominal stress concept

Beginning with a load spectrum determined with a counting method, most often with the rain flow method, possibly under consideration of the mean stress influences, the lifetime is determined by the relative linear damage accumulation according to Palmgren Miner together with the component Wöhler curve.

This method has proven itself to be quite successful, however it is somewhat deficient in several aspects. Therefore, it is often better to determine the local load-time function (local concept, notch base concept) [9.2, 9.3, 9.13, 9.27, 9.28, 9.34]. In other words, the local stress-strain curve is determined for the highest loaded locations on a component due to outer loads. Furthermore, only one single material Wöhler curve is necessary.

In the local concept the local load process is classified with the rain flow method, see Figure 9.35. The relationship between the external stress and the local strain, which due to the alternating plastification is not always liner, is determined with stress analyses, for example with finite element analyses.

The cyclic stress-strain curve, which can be derived from the stress-strain hystereses, compare with Figure 9.22, represents the coherence between the actual stresses and strains. Hardening and softening of the material is not considered. Residual stress is considered if applicable.

Based on these parameters, a suitable damage parameter is chosen, typically according to *Smith*, *Watson* and *Topper* [9.27] for example:

$$P_{SWT} = \sqrt{\sigma_{\max} \cdot E \cdot \varepsilon_{a,ges}} \; . \tag{9.36}$$

Other damage parameters, which under certain circumstances can incorporate mean stress influences more adequately, are described in [9.3, 9.15]. Alternatively, a component specific Haigh diagram can be used to estimate the mean stress influence.

With the help of a damage accumulation hypothesis the damage portion is calculated on a standard test specimen with the strain Wöhler curve of a material.

The advantage of this concept is that local stresses can be directly compared to material property values. However, the local concept involves several uncertainties due to the numerous influential parameters. In practice, a combination of the nominal stress concept and local concept is found, so that, for example, the component Wöhler curve is used, since here the production and surface finish influences are accounted for, together with experimental proofs [9.13].

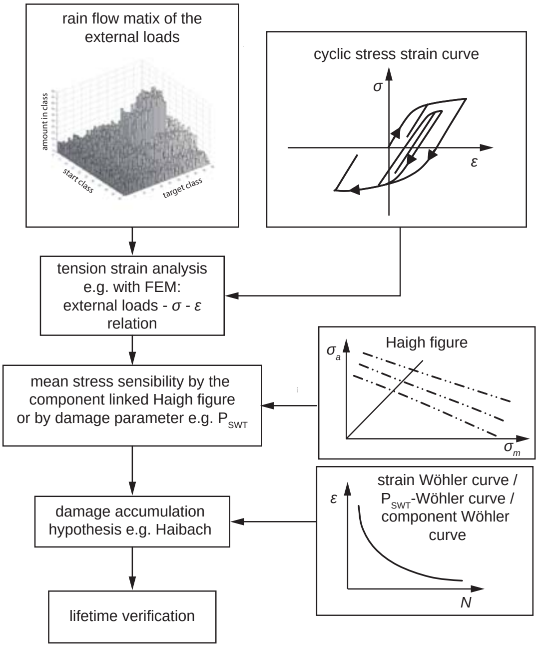


Figure 9.35. Lifetime calculation according to the local concept or the notch base concept

9.5 Conclusion

With knowledge of the load spectrum and the tolerable material load in the form of a Wöhler diagram, a lifetime prediction can be made for a mechanical element with the help of a damage accumulation hypothesis. Here it should be considered, that this prediction can only be made with a certain probability, since amongst other things the load spectrum as well as the load capacity expressed in the form of a Wöhler curve are random variables. Likewise, the damage accumulation hypotheses known today have only been proven empirically in materials science. Therefore, a practical lifetime prediction requires a balance between field tests, test stand trials, calculations and a careful assessment and evaluation of the data, if the prediction should be able to serve as an effective tool for the designer, Figure 9.36.

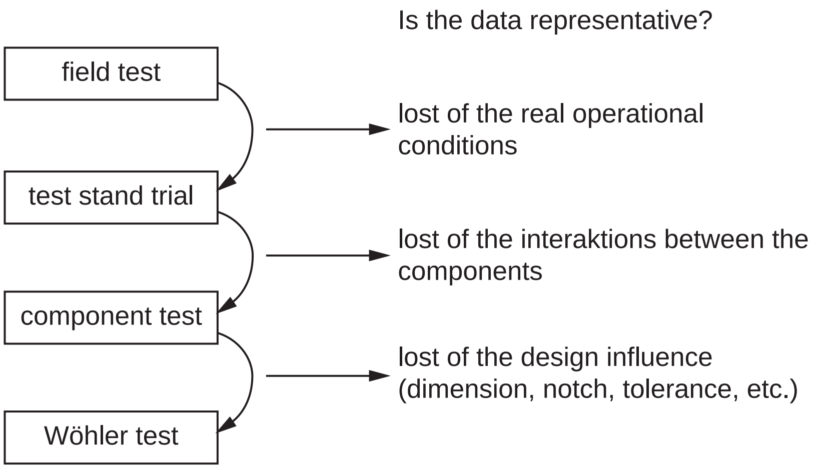


Figure 9.36. Trial variations

References

[9.1] Best R, Klätschke H (1996) Belastungsanalyse manuell geschalteter Fahrzeuggetriebe - Voraussetzung zur beanspruchungsgerechten Dimensionierung. VDI-Berichte Nr 1230: Getriebe in Fahrzeugen, S 257-272

[9.2] Borenius G (1990) Zur rechnerischen Schädigungsakkumulation in der Erprobung von Kraftfahrzeugteilen bei stochastischer Belastung mit variabler Mittellast. Dissertation Universität Stuttgart

[9.3] Britten A (1994) Lebensdauerberechnung randschichtgehärteter Bauteile basierend auf der FE-Methode: Anwendung des örtlichen Konzepts auf ein Zweischichtmodell. VDI-Berichte Nr 1153, S 61-72

[9.4] Brodbeck P, Dörr C, Lechner G (1995) Lebensdauerversuche an Zahnrädern - Einfluss unterschiedlicher Betriebsbedingungen. DVM-Berichte 121: Maschinenelemente und Lebensdauer, S 137-147

[9.5] Brunner F J,(1985) Lebensdauervorhersage bei Materialermüdung. Automobilindustrie H 2, S 157-162

[9.6] Brunner F (1987) Angewandte Zuverlässigkeitstechnik bei der Fahrzeugentwicklung. ATZ, Nr 89, S 291-296, 399-404

[9.7] Buxbaum O (1992) Betriebsfestigkeit - Sichere und wirtschaftliche Bemessung schwingbruchgefährdeter Bauteile. 2. Aufl. Verlag Stahleisen

[9.8] Clormann U H, Seeger T (1986) Rainflow - HCM Ein Zählverfahren für Betriebsfestigkeitsnachweise auf werkstoffmechanischer Grundlage. Stahlbau 3, S 65-71

[9.9] Deutsches Institut für Normung (1985) DIN 45667 Klassiervorgänge für regellose Schwingungen. Beuth , Berlin

[9.10] Dörr C, Hirschmann K H, Lechner G (1996) Verbesserung der Beurteilung der Wiederverwendbarkeit hochwertiger, gebrauchter Teile mit einem Beschleunigungsmessverfahren. Arbeitsbericht zum DFG Forschungsvorhaben
[9.11] Endo T (1974) Damage Evaluation of Metals for Random or Varying Loading. Proc. of the 1974 Symposium on Mechanical. Behaviour of Materials, Society of Mat. Scienc Japan
[9.12] Forschungskuratorium Maschinenbau (1994) FKM-Richtlinie Rechnerischer Festigkeitsnachweis für Maschinenbauteile. FKM-Forschungshefte 183-1 und 183-2
[9.13] Foth J, Jauch F (1995) Betriebsfestigkeit von torsionsbelasteten Wellen für Automatgetriebe. DVM-Berichte 121, S 161-173
[9.14] Gudehus H, Zenner H (1999) Leitfaden für eine Betriebsfestigkeitsrechnung, 4. Aufl. Verl Stahleisen, Düsseldorf
[9.15] Haibach E (2002) Betriebsfestigkeit - Verfahren und Daten zur Bauteilberechnung. 2. Aufl. Springer, Berlin
[9.16] Hanschmann D, Schelke E, Zamow J (1994) Rechnerisches mehraxiales Betriebsfestigkeitsvorhersage-Konzept für die Dimensionierung von KFZ-Komponenten in der frühen Konstruktionsphase. VDI-Berichte Nr. 1153, S 89-112
[9.17] Hück M, Thrainer L, Schütz W (1981) Berechnung der Wöhlerlinien für Bauteile aus Stahl, Stahlguss und Grauguss. Bericht der ABF Nr ll, Mai
[9.18] Kapur K C, Lamberson L R (1977) Reliability in Engineering Design. John Wiley & Sons, New York
[9.19] Liu J (1995) Lastannahmen und Festigkeitsberechnung. VDI-Berichte 1227: Festigkeitsberechnung metallischer Bauteile, S 179-198
[9.20] Liu J, Zenner H (1995) Berechnung von Beuteilwöhlerlinien unter Berücksichtigung der statistischen und spannungsmechanischen Stützziffer. Mat.-wiss. u. Werkstofftech. 26, S 14-21
[9.21] Melzer F (1995) Symbolisch-numerische Modellierung elastischer Mehrkörpersysteme mit Anwendung auf rechnerische Lebensdauervorhersagen. VDI-Fortschritts-Berichte Reihe 20 Nr 139
[9.22] O'Connor P D T (2001) Practical Reliability Engineering. John Wiley & Sons
[9.23] Peralta-Duran, Wirsching (1985) Creep-Rapture Reliability Analysis. Transactions Wirsching of the ASME, Journal of Vibration; Acoustics, Stress and Reliability in Design. Juli, 107, S 339-346.
[9.24] Pinnekamp W (1987) Lastkollektiv und Betriebsfestigkeit von Zahnrädern. VDI-Bericht Nr 626, S 131-145
[9.25] Schiberna P, Spörl T, Lechner G (1995) Triebstrangsimulation - FASIMA II, ein modulares Triebstrangsimulationsprogramm. VDI Berichte Nr. 1175
[9.26] Schütz W (1982) Zur Lebensdauer in der Rissentstehungs- und Rissfortschrittsphase. Der Maschinenschaden, 55, H 5, S 237-256
[9.27] Smith K N, Watson P, Topper T H (1970) A Stress-Strain Function for the Fatigue of Metals. Journal of Materials, JMLSA, vol 5, No 4, pp 767-778

[9.28] Sonsino C M, Kaufmann H, Grubisic V (1995) Übertragbarkeit von Werkstoffkennwerten am Beispiel eines betriebsfest auszulegenden geschmiedeten Nutzfahrzeug- Achsschenkels. Konstruktion 47, S 222-232

[9.29] Thum H (1995) Zur Bewertung der Zuverlässigkeit und Lebensdauer mechanischer Strukturen. VDI-Berichte Nr 1239, S 135-146

[9.30] Thum H (1996) Lebensdauer, Zuverlässigkeit und Sicherheit von Zahnradpassungen. VDI-Berichte Nr 1230, S 603-614

[9.31] Westerholz A (1985) Die Erfassung der Bauteilschädigung bestriebsfester Systeme, ein Mikrorechner geführtes On-Line-Verfahren. Diss Ruhr-Universität Bochum, Institut für Konstruktionstechnik. Heft 85.2

[9.32] Westermann-Friedrich A, Zenner H (1988) Zählverfahren zur Bildung von Kollektiven aus Zeitfunktionen - Vergleich der verschiedenen Verfahren und Beispiele. Merkblatt des AK Lastkollektive der FVA Nr 0/14 Forschungsvereinigung Antriebstechnik, Frankfurt

[9.33] Zammert W U (1985) Betriebsfestigkeitsberechnung – Grundlagen, Verfahren und technische Anwendungen. Vieweg, Braunschweig

[9.34] Zenner H (1994) Lebensdauervorhersage im Automobilbau. VDI-Berichte Nr 1153, S 29-42

10 Maintenance and Reliability

Maintenance, reliability and costs are dealt with in this chapter. Each of these aspects are interdependent. The aim is to design of the system so that availability and costs optimised. Thus, a one-sided view often does not suffice in achieving the goal of product design. In general, this goal is the optimal design of a system, in order to achieve the best possible equilibrium between availability and costs.

The term "life cycle costs" has continued to grow in importance during the last few years. The costs which occur during the entire planned operational lifetime of a technical system can very much influence any necessary investment decisions, and thus make up the focus of the first part of this chapter. Information concerning the reliability and planned maintenance methods during the operational lifetime are necessary for a prognosis of life cycle costs.

Various calculation models have already been developed for the analysis of the reliability and availability in connection with maintenance processes. These models vary considerably in complexity, to the effect that some models have certain restrictions as to which maintenance procedure can be recaptured by the model. Thus, in the second part of this chapter an overview of possible calculation models is given along with the parameters which can be determined.

10.1 Fundamentals of Maintenance

Along with the failure behaviour, *maintenance* considerably influences the availability of a technical system in mechanical engineering [10.8]. Maintenance can be defined as follows [10.14, 10.46]:

> Maintenance signifies methods for the determination and evaluation of the current status as well as for the preservation and reestablishment of the nominal status of facilities, machines and components.

Maintenance methods can be divided into preventive and non-preventive (corrective) methods. Maintenance methods include service, inspection, overhauling and repair. In terms of maintenance strategies, inspection

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_10, © Springer-Verlag Berlin Heidelberg 2008

intervals, extent of service, repair priorities and repair capacities in the form of replacement parts and repair labor are determined. In providing security for the maintenance, it is necessary to have the required replacement parts in the required amount and quality at hand analog to the requirements of material logistics [10.26]. This includes logistic aspects such as transportation and effective storage. Similar to the reliability and availability, maintainability can be described as a probability.

One assumes that the goal of general maintenance work is that the required availability is reached or maintained at its current status.

10.1.1 Maintenance Methods

Maintenance methods can be distinguished as methods for preventive maintenance, for corrective maintenance and for condition-based maintenance. These methods will be described in detail in the following sections.

10.1.1.1 Preventive Maintenance

Preventive maintenance deals with maintenance methods which are carried out preventively, that is, at a predetermined time or periodically after a certain amount of operational hours. Preventive maintenance methods allow for the determination and evaluation of the current status as well as for the preservation of the nominal status of facilities, machines and components [10.37].

Preventive maintenance methods include:

- *service:* methods for the preservation of the nominal status, e.g. cleaning, refilling of lubricants and cooling mediums, adjusting, calibrating.
- *inspection:* methods for the determination and judgment of the current status, e.g. inspection of wearout, corrosion, leaks, loosened connections, periodical or continual measuring and analysis.
- *overhauling:* disassembling until certain components, assemblies or elements can be reached and if needed, changing of components, assemblies and components.

Preventive maintenance methods are most often carried out without consideration of the current status of the machine. Preventive work is work done on a machine even though there is no current technical disturbance. The purpose of preventive maintenance is to avoid failures and breakdowns caused by wearout, aging, corrosion and contamination as well as to prevent any failure effects which could arise from these circumstances. Thus, preventive maintenance can be seen as precautionary maintenance.

10.1.1.2 Condition-Based Maintenance

Condition-based maintenance avoids exact inspection and overhauling intervals and thus avoids the periodical renewal of fully functional components and assemblies [10.37]. Furthermore, a reduction in the availability due to preventive maintenance carried out too often can be avoided. With continuous or periodical measurements and observations of certain values on components, assemblies and components and how these values have altered, one is able to determine wearout during operation. These measurements, observations and evaluations are known as condition monitoring. With condition monitoring it is possible to reduce the expenses of repairs and maintenance without compromising the reliability and security of a product.

In [10.39], the term condition-based maintenance is defined. The goal of condition-based maintenance is to plan and carry out maintenance methods optimally regarding time, quality and costs. With this maintenance strategy an intensive inspection of critical components and equipment during operation is carried out (e.g. with automatic measuring devices). Thus, it is possible to predict when a failure could occur. With this prediction the required maintenance methods can be applied, for example, the renewal of parts, before a failure occurs.

The application of condition-based maintenance is suitable for systems and components for which the operational conditions can be measured and inspected over time. Inspection techniques for condition-based maintenance include among others:

- thermographical inspection,
- nondestructive material tests,
- oil analysis and
- vibration analysis.

Indicators on vehicle break systems measure the extent of the brake lining wearout, thus allowing for a prediction of the remaining lifetime before a renewal of the brake lining is necessary. Figure 10.1 shows the relative frequency of procedures for machine status monitoring which are applied today. Bearing diagnoses and the examination of machine oscillations are among the most commonly used procedures.

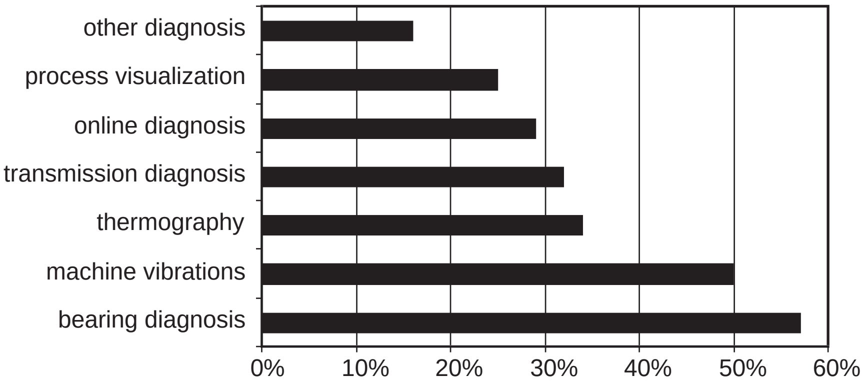


Figure 10.1. Procedures for machine condition monitoring [10.9]

On the one hand, these procedures require much work due to the required data gathering and evaluation, and on the other hand, they lower the total maintenance costs considerably without interfering in any way with the safety or reliability of the device. For this reason, many American airlines have applied this strategy with success [10.8].

10.1.1.3 Corrective Maintenance

Corrective maintenance methods are required for partial and total failures of facilities, devices and components. Such methods serve to the reestablishment of the nominal condition [10.37] and are described by the term *repair* [10.7]. It should be noted that preventive maintenance methods such as inspections may incorporate corrective maintenance.

If only one maintenance level is present, corrective maintenance methods can be divided into the following individual methods:

- Determination of the interference or failure (failure recognition)*,
- Notification of the responsible maintenance personnel,
- Maintenance personnel come to the site of interference,
- Preparation of tools and test control units,
- Localization of the interference on the level of the device or component (failure localization)*,
- dismantle of the defective device (component)*,
- Preparation of the required replacement parts,
- Replacement of the defective device (failure elimination)*,
- Adjustment, calibration and testing of the repaired device (component)*,
- Assembly of the repaired device (component) in the facility*,
- Functionality test of the complete facility*.

Similar to the preventive maintenance methods, this method requires a certain amount of time, personnel and material. The sum of the time required for each individual method gives the total down time. The actual repair time is made up of the time required for those individual methods denoted with a *.

10.1.2 Maintenance Levels

If several maintenance levels are present, the defective device (component) is replaced by a new, fully functional device [10.37]. The defective device enters a maintenance cycle. Thus, the down time and repair time are decreased for the facility; however, there is no reduction in the total maintenance work.

Decisive criteria for the introduction of repair levels are, for example, the effects on the availability and the on-site repairability [10.8].

10.1.3 Repair Priorities

The assignment of repair priorities is suggested if one system component is more important than another one. The significance of a component is defined by the system operator. Economically speaking, the most significant component is the component whose failure leads to the highest costs. This can be clarified very well by examining a conveyor belt facility. A conveyor belt consists of a motor as an actuation unit (component 1) and three conveyor belts working parallel to one another (components 2-4), which are driven by the actuation unit. The corresponding reliability block diagram is shown in Figure 10.2.

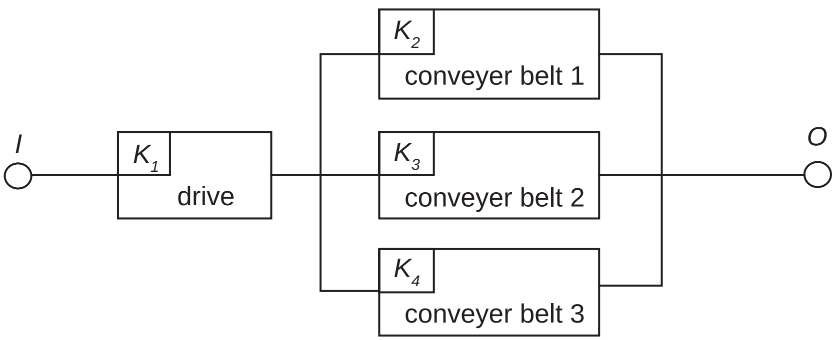


Figure 10.2. Reliability block diagram of a conveyor belt facility

If the motor fails, then the complete facility stops. If, however, only one conveyor belt fails, the remaining two belts continue to rotate and the operator of the facility continues to win financial revenues. In this example it is clear that the motor is the component with the highest repair priority, since it is the most significant component for the profit of the facility.

10.1.4 Maintenance Capacities

In stochastic processes commonly used for the calculation of the attainable availability (renewal process, Markov process, …), it is always assumed that all required maintenance methods are carried out without any delay [10.5, 10.30, 10.43]. However, in reality this is seldom the case, since an economical compromise must always be made for the assessment of the maintenance capacities between the complication involved in the preparation of maintenance capacity (infrastructure, personnel, tools and devices, replacement parts) and waiting times caused by the momentary lack of required maintenance capacities.

Thus, limited maintenance resources must be considered in order to create a model of a technical system, which is close to reality. This can be done with the implementation of repair teams and/or stocking up on replacement parts and further maintenance resources.

10.1.4.1 Repair Teams

The type and length of preventive and corrective maintenance work can be estimated by a maintainability analysis [10.8]. From the analysis it is possible to derive the required number of personnel and their qualifications. Repair teams are organized for the maintenance work within the framework of the maintenance strategy. These teams are generally limited in size.

10.1.4.2 Fundamentals of Replacement Part Stock

According to *Pfohl*, “stock is a buffer between input and output flows of material”. Within the realm of maintenance, material, here, is replacement parts required for the maintenance methods. Economically, an unnecessarily large storage is negative, due to the high costs caused. Thus, the goal is to conduct an optimal stock management for the needs of the maintenance. Several basic principles for stockkeeping will be discussed in the following sections.

Use of Storage

Stockpiling replacement parts in storage provides the following advantages:

- Prevention of waiting times: waiting times avoided by the immediate availability of the replacement parts for unexpected failures.
- Size digression effects: storage allows the opportunity to take advantage of so-called size degression effects, for example, quantity rebates.
- Storage used as a protection against prognosis uncertainty.
- Storage used for long-term assurance of replacement part availability.

The Storage Function

Now, the basic terms related to storage will be further explained. Figure 10.3 shows storage as a function of time $S(t)$.

Storage $S(t)$ contains $S_{nominal}$ replacement parts at the point in time $t = 0$. The removal of replacement parts as needed by the repair teams results in a continuous decrease in stock. If the storage falls below a certain storage limit S_{order}, then a certain quantity of replacement parts should be reordered. The time t_{order}, at which the reordering takes place, is called the order point. The amount of replacement parts ordered is the order quantity N_{order}. Until the order quantity arrives, a demand for replacement parts during the time between "order placed" and "order arrived" is estimated. The order point should be chosen so that the security storage S_S is not reached before the reordered replacement parts have arrived. Due to imponderability, each storage is provided with security storage.

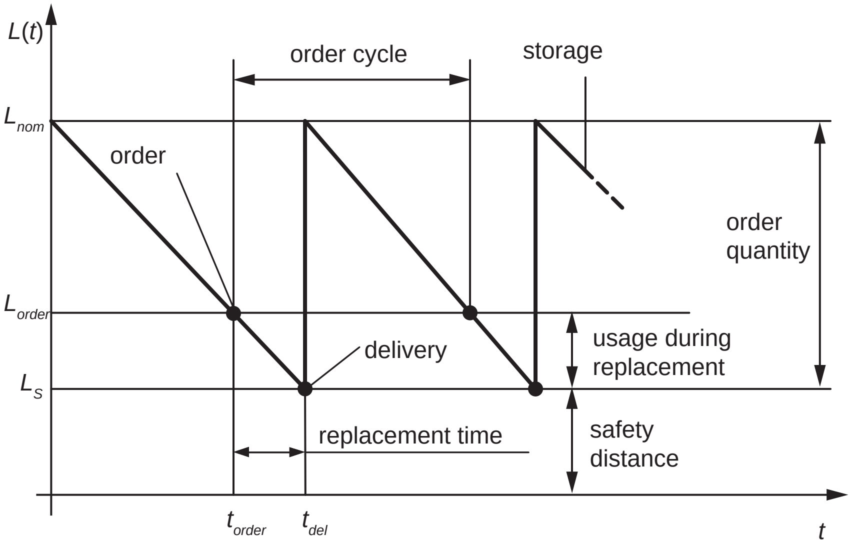


Figure 10.3. Storage function

The order quantity N_{order} is determined as the difference between the nominal storage $S_{nominal}$ and the security storage S_S: $N_{order} = S_{nominal} - S_S$. If the order quantity is estimated correctly, then the nominal storage $S_{nominal}$ is reached at the point in time $t_{delivery}$ when the replacement parts are delivered.

10.1.5 Maintenance Strategies

According to DIN 31051 [10.14], maintenance methods include the coordination of maintenance goals to the company's goals and to the determination of the corresponding maintenance strategy. The optimal maintenance strategy is the result of a conflict of goals between achieved availability of a facility and the required maintenance costs.

The maintenance strategy determines the following parameters for the maintenance of a system and its components:

- type and frequency of maintenance measures taken (i.e. inspection intervals and repair complexity),
- strategy of replacement parts storage,
- quantity and qualification of the repair teams,
- repair priorities,
- maintenance levels.

The maintenance strategy makes up the fundamental of maintenance, as shown in Figure 10.4.

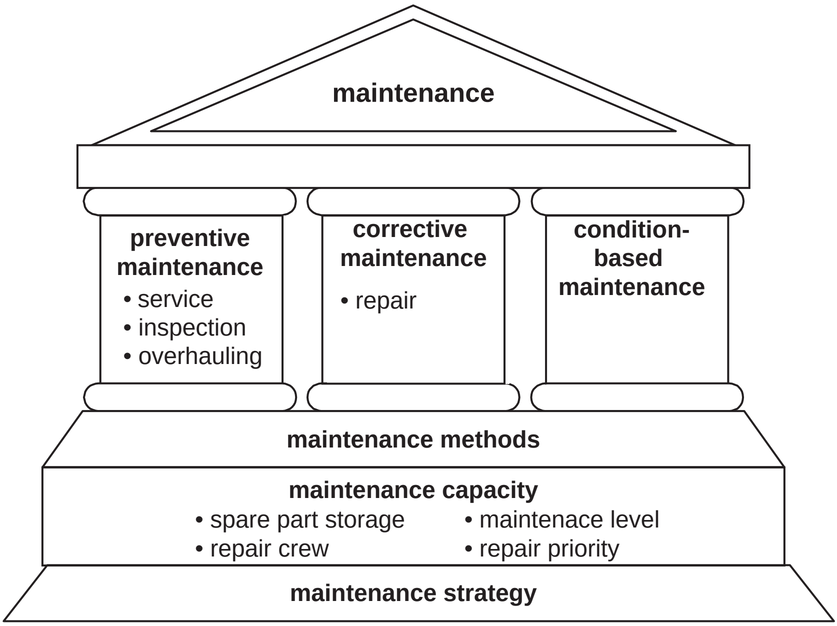


Figure 10.4. The three columns of maintenance

Maintenance methods can be applied according to the following strategies:

- exclusive corrective maintenance methods,
- maintenance methods with preventive monitoring,
- exclusive preventive maintenance methods,
- combination of preventive and corrective maintenance methods,
- condition-based maintenance.

10.2 Life Cycle Costs

Reliability, maintainability and availability have a large influence on the costs which occur during the product use. In order to asses the use and profit of reliability methods, the consideration of cost aspects is required. Therefore, the concept of life cycle costs will now be more closely observed. The portions of life cycle costs, which can be directly influenced by reliability engineering, are the reliability costs and the maintenance costs.

The time span from the first idea or contract during the development, production and use up to the disposal of a product is characterized as the life cycle of a product or as the product lifetime. During this time, costs continually accrue, which the user must bear directly (i.e. in the form of operational costs) or indirectly (i.e. production costs above the acquisition price). The sum of these costs is known as life cycle costs, which include all costs incurred by the product user due to purchase costs as well as costs incurred during the use of the product (facility, machine, or device) in the course of the product lifetime.

The main purchase criterion for a customer is often only the purchase price. However, with this attitude some operators have suffered “shipwreck”. The difficulty in the incurred life cycle costs is exemplified by the representation of a cost iceberg in, see Figure 10.5. The life cycle costs *LCC* consist of purchase costs, non-recurring costs, operational costs, maintenance costs and miscellaneous costs.

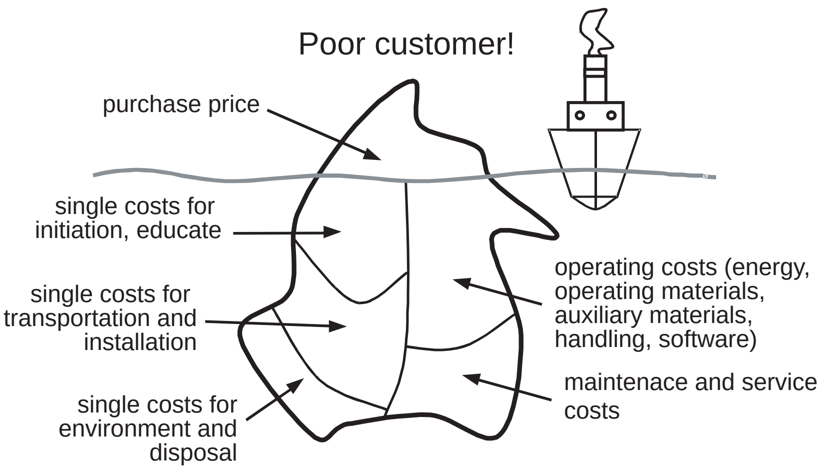


Figure 10.5. The iceberg of life cycle costs from the view of the user

The accrual of costs is more simply represented in Figure 10.6, where the summed costs are shown over the length of the product lifetime.

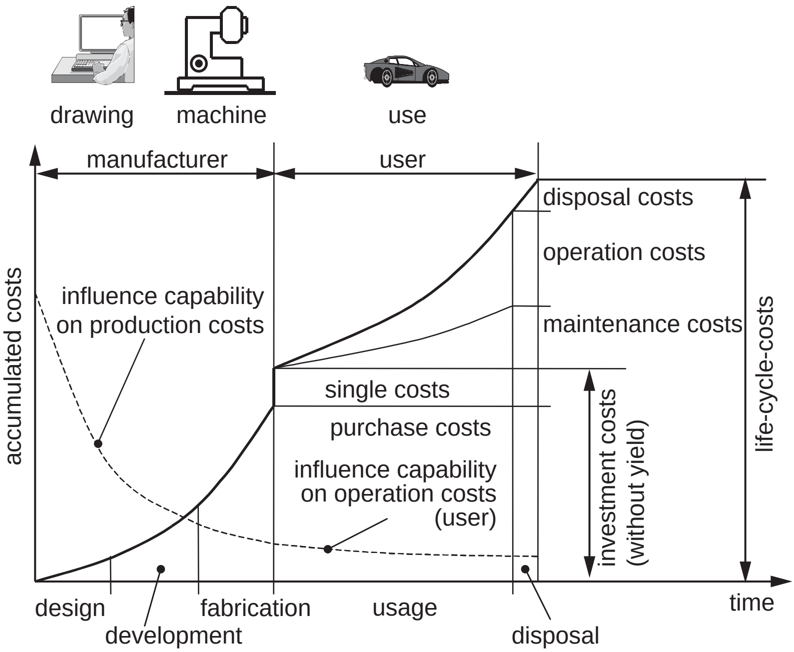


Figure 10.6. Life cycle costs during the product lifetime (without miscellaneous costs)

Here, despite relatively low design and design costs, rapidly increasing costs can be determined for the product during its later use. The purchase price can be seen as the investment costs of the user, here given as a defined amount without interest. During the utilization period of the machine, operation costs and maintenance costs also incur, which continually increase until the utilization period comes to an end, thus playing a fairly significant role in the investment costs. The goal of cost optimal development (value management) is the minimization of the life cycle costs incurred during the use of a product. Often, the user is not aware where the main portions of life cycle costs lie. The life cycle costs are strongly dependent upon the variables reliability and maintenance.

Failure costs also include costs caused by production failures during unavailability of a facility. For facilities relevant to security, compensation costs can also incur in the case of a damage event.

Each product type possesses its own specific life cycle cost structure, as represented in Figure 10.7.

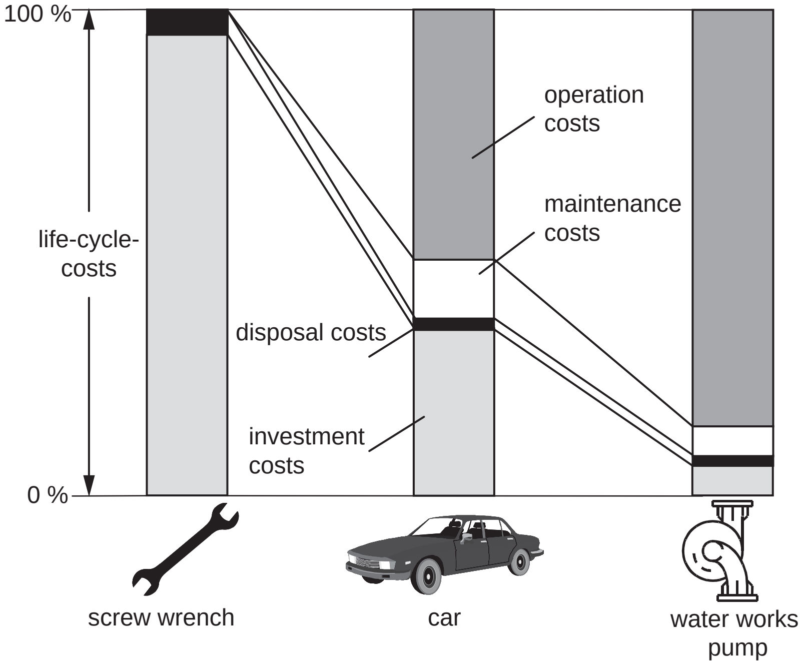


Figure 10.7. Structure of life cycle cost portions for various products

In this figure several portions of life cycle costs are shown. For simple devices, for example a screw wrench, only investment costs and disposal costs occur, e.g. neither operational nor maintenance costs incur. For vehicles, however, all three cost types are relevant. The dominating costs for a water works pump are the energy costs (approx. 96%).

The unavailability of a product can influence its *LCC* significantly. Therefore, the availability of a product must be optimized in order to reach the lowest resulting *LCC*. Figure 10.8 shows the relationship between availability and the *LCC* in a simplified form. High reliability and quick maintainability lead to increased purchase prices. Likewise, the better the maintenance organization is formed, the higher the maintenance costs are. High investments in these two cost elements result in an increasing availability. At the same time, the costs caused by down time decrease with increasing availability.

The sum of purchase and maintenance costs along with down time costs minimizes at a certain availability A_{opt}. At this point the lowest life cycle costs are reached with the optimal availability.

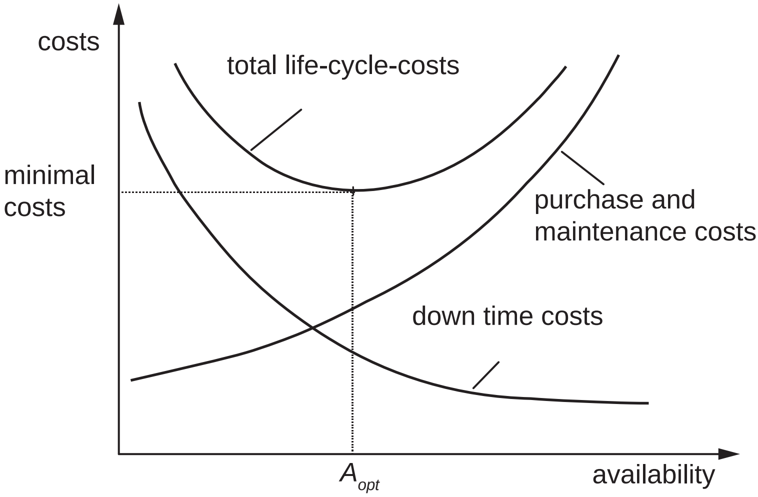


Figure 10.8. Simplified relationship between availability and life cycle costs

10.3 Reliability Parameters

Normally, a facility is not in operation at all times. Down time is caused by failures or preventive maintenance. Delays result from waiting for maintenance personnel or for missing replacement parts. Individual time periods can be assigned to certain conditions.

10.3.1 The Condition Function

For any system, a flow of events is yielded over time due to the failure behaviour of components and pending maintenance methods. Such a flow of events is shown as an example in Figure 10.9.

The various conditions can be represented by a condition indicator $c(t)$, which can assume various values over time. By assigning the condition indicator the following values:

$$c(t) = \begin{cases} 1, & \text{for active operation} \\ 0, & \text{in repair} \\ -1, & \text{in maintenance} \\ -2, & \text{maintenance delay time} \\ -3 & \text{supply delay time,} \end{cases}$$

then Figure 10.9 can describe a possible condition function of a system over time: at the time $t = 0$ the system begins operation. The system is active until the failure of one of its components. A renewal of the defect component is required because of the failure, which means that the component is replaced by a new component. Since the required replacement part is not available, it is necessary to wait upon its arrival. Because the repair personnel are needed more urgently elsewhere, it is also necessary to wait for their arrival. Only after both have arrived is it possible to proceed with the actual repair procedure. After all repairs are concluded, the assembly returns to operation. Due to a planned preventive shutdown of the facility within the realm of the maintenance strategy, the facility again enters a period of down time in order to carry out maintenance included in the maintenance plan. Since this inspection work has already been carefully planned, no delay due to missing parts is caused. After the inspection work has been completed, the facility finally enters full operation.

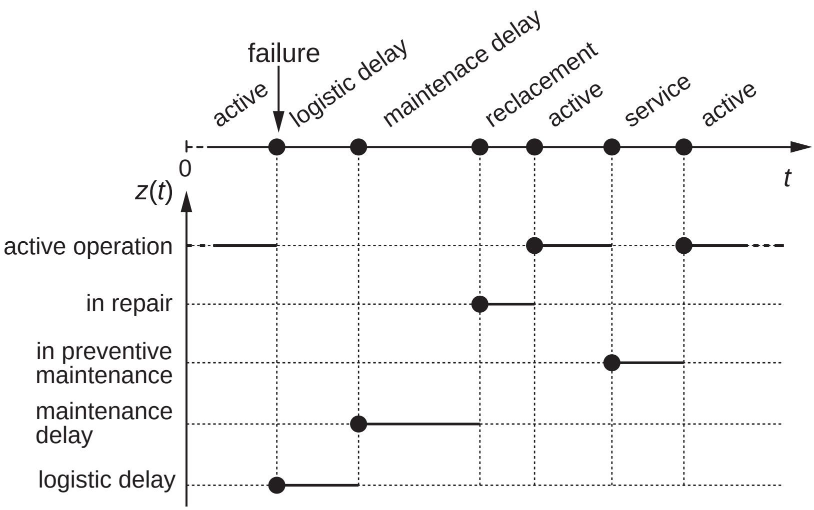


Figure 10.9. Example of a flow of events over time with respective conditions

The condition function over time can be divided into certain activities and delays:

- *Supply Delay Time, SDT*: includes waiting on production and/or delivery of replacement parts, administrative (management) cycle delays, production delays, purchasing delays, and transportation delays. For the most part, these times are influenced by the spectrum and quantity of replacement parts in storage, which are available for maintenance. The logistical delay time disappears if the replacement part is directly available.

- *Maintenance Delay Time, MDT*: the waiting time for maintenance capacities or maintenance provisions. This includes time needed to inform the necessary persons as well as travel time. Maintenance capacities are personnel, testing and measuring devices, tools, manuals or other technical data. Provisions are repair workshops, test stands, airplane hangars, etc. The maintenance time is influenced by the amount of available repair channels. A repair channel is defined as collectivity of all maintenance capacities and provisions required for the successful execution of a repair. If a repair channel is directly available when a failure occurs, the maintenance delay time disappears.

Since supply and maintenance delay time are influenced by external parameters, they do not belong to the system characteristics, which means that they cannot be influenced by means of design.

10.3.2 Maintenance Parameters

The time required for all activities and delays involved in maintenance methods are not uniquely defined, but rather can vary.Thus, they are characterized as random variables, which are again characterized by the maintenance parameters.

Similar to reliability, maintainability can also be understood as a probability. Maintainability can be defined as follows [10.7]:

> The maintainability describes the probability that if the maintenance is carried out under defined material and personnel conditions, then the required time period for a repair or for an inspection is shorter than a given time interval.

The random variable τ_M is the duration of the maintenance methods, as shown in Figure 10.10. The index M stands for maintenance.

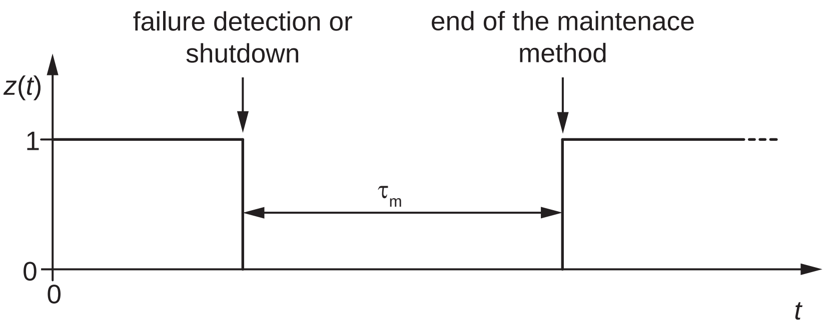


Figure 10.10. Duration of the maintenance methods as a random variable

The maintainability seen as a random variable encompasses not only the actual maintenance work, but also the entire timeframe between failure recognition (shutdown) of the observed unit and its reconnection (including delay times for the provision of replacement parts or measuring devices, breaks, administrative work, etc.).

The maintenance parameters are defined analogous to the failure process. The distribution function of the *maintenance duration* τ_M

$$G(t) = P(\tau_M \leq t) \tag{10.1}$$

is known as the *maintainability* $G(t)$. The respective density function is the *maintenance density* $g(t)$. The *maintenance rate* $\mu(t)$ has the meaning

$$\mu(t) = P(\text{M ends in } [t, t + \mathrm{d}t] \mid \text{M while } [0, t]).$$

M: maintenance

The *expected value* $E(\tau_M)$ of the maintenance duration τ_M is defined as

$$MTTM = E(\tau_M) = \int_0^\infty t g(t)\mathrm{d}t = \int_0^\infty (1 - G(t))dt \,. \tag{10.2}$$

The abbreviation *MTTM* stands for Mean Time To Maintenance and points to the average maintenance duration τ_M.

The lognormal distribution is often used for the description of the maintainability. Figure 10.11 represents an example of the density function of the maintainability as a lognormal distribution and as an exponential distribution for the same *MTTM*.

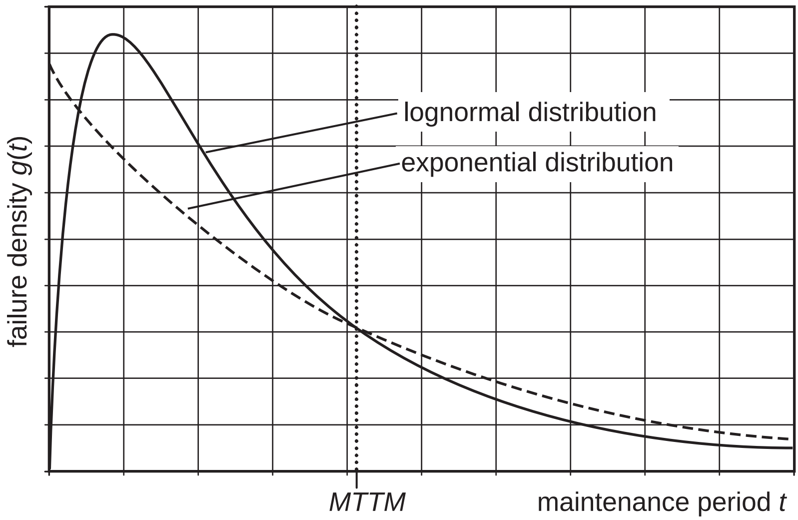


Figure 10.11. Maintainability $G(t)$ as a lognormal and exponential distribution for the same *MTTM*

Maintenance methods can be divided into preventive and corrective methods. Variables for preventive maintenance receive the index *PM* (*P*reventive *M*aintenance) while variables for corrective maintenance receive the index *R* (*R*epair). Depending on the type of maintenance methods, the respective maintenance durations, τ_{PM} for preventive maintenance and τ_R for corrective maintenance, are used. In most German literature, preventive methods are brought under the term "inspection". Consequently, the maintenance is divided into *serviceability* $G_{PM}(t)$ and *maintainability* $G_R(t)$ (repairability) [10.7].

The following terms are common in characterizing the service and maintenance duration analog to Equation (10.2) [10.7]:

- *MTTPM* (Mean Time To Preventive Maintenance) for the average service duration and
- *MTTR* (Mean Time To Repair) for the average repair duration

In Table 10.1, the described survival or failure behaviour parameters are summarized along with the maintenance parameters.

Table 10.1. Summary of the survival parameters and the maintenance parameters

parameter	Random Variables			
	lifetime	maintenance duration	service duration	repair duration
symbol for random variable	τ_L	τ_M	τ_{PM}	τ_R
distribution function	$F(t)$	$G(t)$	$G_{PM}(t)$	$G_R(t)$
survival probability	$R(t)$	-	-	-
density function	$f(t)$	$g(t)$	$g_{PM}(t)$	$g_R(t)$
exit risk	$\lambda(t)$	$\mu(t)$	$\mu_{PM}(t)$	$\mu_R(t)$
expected value	*MTTF*	*MTTM*	*MTTPM*	*MTTR*

The maintainability serves qualitatively as a measurement for the simplicity with which the maintenance work on a system or its components can be carried out. Due to the direct influence of the availability of a machine and the rash increase in maintenance costs, the maintainability is of great significance. Maintainability is already "designed into" a system during the development phase. The maintainability achieved during operation is equally dependent upon the installation of the machine or facility and upon the organization of the maintenance.Design measurements which directly influence the maintainability of a component are [10.25]:

- the integration of function tests (BIT's),
- modular design,
- technical design of a component (i.e. electrical vs. mechanical),
- ergonomic factors,
- labeling and coding,
- displays and indicators,
- standardization,
- interchangeability/compatibility.

The time involved in the discovery and removal of an interference can be remarkably reduced by already observing these aspects during the design phase.

10.3.3 Availability Parameters

The duration of application for a technical system is normally not ended by the first failure of an element. Rather, a system is brought back to its operational condition with the help of the maintenance methods. The quality of the reliability and maintainability greatly influence the availability of a system.

The general definition of availability can be found in the references [10.7 and 10.27]:

> The *availability* is the probability that a system is in a functional condition at the time *t* or during a defined time span, under the condition that it is operated and maintained correctly.

The active operational condition is defined as $c = 1$ in the condition diagram. The *availability* $A(t)$, or to be more exact, the point availability [10.7] is defined under the following condition for the expected value of the condition indicator $c(t)$:

$$A(t) = \left(c(t) = 1 \,\middle|\, \text{as good as new at the point in time } t = 0 \right) = E\left(c(t)\right) \quad (10.3)$$

The following relationship applies for the *average availability* $A_{Av}(t)$:

$$A_{Av}(t) = \frac{1}{t} \int_0^t A(x)\,\mathrm{d}x\,. \quad (10.4)$$

The representation of the average availability can be simplified to the interval availability [10.25]

$$A_{Int}(t) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A(x)\,\mathrm{d}x\,. \quad (10.5)$$

The interval availability describes the average availability during the interval $[t_1, t_2]$. For longer times t, the function of the (point) availability and the average availability converge to a constant value which is independent of the initial conditions at the point in time $t = 0$.

In general, the *steady state availability* A_D can be defined as

$$A_D = \lim_{t \to \infty} A(t) = \frac{MTTF}{MTTF + \overline{M}} = \frac{1}{1 + \dfrac{\overline{M}}{MTTF}} \quad (10.6)$$

with the average down time $\overline{M}$.

Figure 10.12 shows an example of the point availability and the steady state availability of a component. The failure behaviour of the component is described by a Weibull distribution with $b = 3.5$ and $T = 1{,}000$ h. The distribution function of the repair duration is a Weibull distribution with $b = 3.5$ and $T = 10$ h. The parameters have been specifically chosen so that the transient effect can be clearly seen in the initial status followed by a transition to a constant value for the steady state availability.

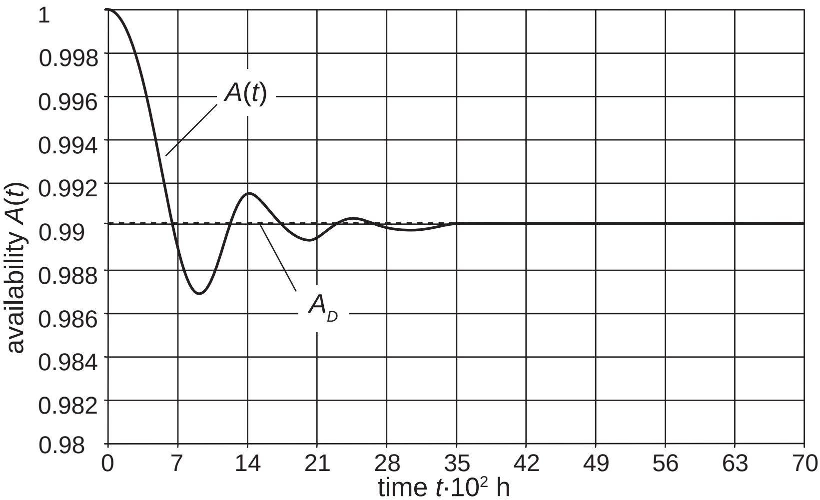


Figure 10.12. Point availability and steady state availability

Depending upon which time intervals are considered for the average down time, the following types of steady state availability are defined.

The *inherent steady state availability* $A_D^{(i)}$ is defined as [10.25, 10.47],

$$A_D^{(i)} = \frac{MTTF}{MTTF + MTTR} \quad \text{with } \overline{M} = MTTR. \tag{10.7}$$

The inherent steady state availability regards the failure behaviour of a system in connection with corrective maintenance. It is based on the distribution functions for the failure probability $F(t)$ and for the maintainability $G_R(t)$. Thus, this availability can be used as an assessment criterion for the quality of a product.

The *technical steady state availability* $A_D^{(t)}$ is defined as [10.38]

$$A_D^{(t)} = \frac{MTTF}{MTTF + MTTPM + MTTR} \tag{10.8}$$

with $\overline{M} = MTTM = MTTPM + MTTR.$

It considers the failure behaviour of a system, preventive maintenance methods as well as repairs.

The *operational steady state availability* $A_D^{(o)}$ is defined as [10.38]

$$A_D^{(o)} = \frac{MTTF}{MTTF + MTTPM + MTTR + SDT + MDT} \tag{10.9}$$

with $\overline{M} = MTTPM + MTTR + SDT + MDT.$

The supply delay time *SDT* essentially includes waiting on the production or delivery of replacement parts. Maintenance delay *MDT* includes waiting on maintenance capacities or provisions. The operational availability is thus a useful assessment criterion to estimate the quantity of replacement parts and the number of repair channels. Along with design parameters (reliability and maintainability), the operational availability considers the quality of the maintenance organization.

The *total steady state availability* $A_D^{(p)}$ is the most general way to describe the steady state availability. It considers both the failure behaviour of a system, all maintenance methods and administrative down times as well as any logistic delays. In addition, external causes for unavailability are considered which cannot be controlled by the system operator.

Table 10.2 summarizes the various definitions of steady state availability. The parameters are marked as they apply to each respective steady state availability. Furthermore, the measured value and/or expected value is given for each parameter.

Table 10.2. Overview of the steady state availabilities

	design related		preventive maintenance	availability of replacement parts	repair teams	maintenance provisions	external influences
	reliability	maintainability					
$A_D^{(i)}$	●	●	-	-	-	-	-
$A_D^{(t)}$	●	●	●	-	-	-	-
$A_D^{(o)}$	●	●	●	●	●	●	-
$A_D^{(p)}$	●	●	●	●	●	●	●
parameter	*MTTF*	*MTTR*	*MTTPM*	*SDT*	*MDT*		-

10.4 Models for the Calculation of Repairable Systems

The first failure of a machine does not normally result to a shutdown in its operation. Rather, its functionality is sustained over a longer time span with the help of maintenance methods such as inspection and repair. A system is characterized as repairable if it is integrated into a maintenance process. Various calculation models have been developed for the analysis of the reliability and availability of repairable systems. These models vary significantly in their complexity; to the effect that many models limit which maintenance events can be represented. Thus, the second part of this chapter will introduce an overview of possible calculation models, with which parameters of a system can be determined.

The models to be discussed have been taken for the most part out of the references [10.30, 10.43, and 10.50].

The reliability of a component can be improved by maintenance methods which are carried out at certain predetermined intervals.

Repairable systems can be dealt with according to the Markov method. With this method it is possible to determine the availability of a system or component. However, a fundamental pre-condition for this method is that the failure and repair behaviours must be able to be described by exponential distributions.

If a system consists of repairable system elements independent from one another, then the Boole-Markov model can be used for the calculation of the steady state availability of this system.

The common renewal process provides an approximation for the calculation of required replacement parts over time in order to uphold the maintenance process for a component or system. In this case, maintenance signifies the replacement of a defective component with a brand new component. However, the duration of the renewal is neglected in the common renewal process.

If the duration of the renewal or repair of a failed component is not neglected, then an alternating renewal process comes into play. With this process it is easier to simulate reality, since normally the discovery of a defective component, as well as its repair or renewal, requires a certain amount of time. Thus, the availability can be calculated.

For systems which can be described by renewal processes, it is hardly necessary to limit the calculation model to certain distributions. However, it is only possible to describe simply structured systems with these processes. The number of statuses is limited to two – “in operation” and “in repair”. With the semi-Markov process more than two statuses can be represented, for example, a third condition: “in inspection”.

System transport theory yields the most general description possibility for technical systems. It allows for the modeling of complex systems with arbitrary structures, arbitrary distribution functions for the description of the failure and repair behaviour of components and arbitrary interactions of components within a system. A multitude of maintenance strategies can be represented and the replacement part logistics can be taken into consideration as well.

10.4.1 Periodical Maintenance Model

Often, the reliability of complex systems or components can be improved with the help of preventive maintenance, where the negative influences of aging and wearout can be avoided. Furthermore, the time of use and application can be considerably increased if a unit is well maintained.

10.4.1.1 Fundamentals

The models described in the following sections assume that the maintained unit is brought back to a “good as new” condition after each maintenance. This means that the maintenance methods include renewal and/or overhauling. $R(t)$ is the reliability of a unit with preventive maintenance. The maintenance methods are carried out according to predetermined maintenance intervals T_{PM} (PM = Preventive Maintenance) independent of the status of the unit.

In order to determine the reliability of an element with preventive maintenance $R_{PM}(t)$, it is necessary to consider a maintenance plan over time as shown in Figure 10.13.

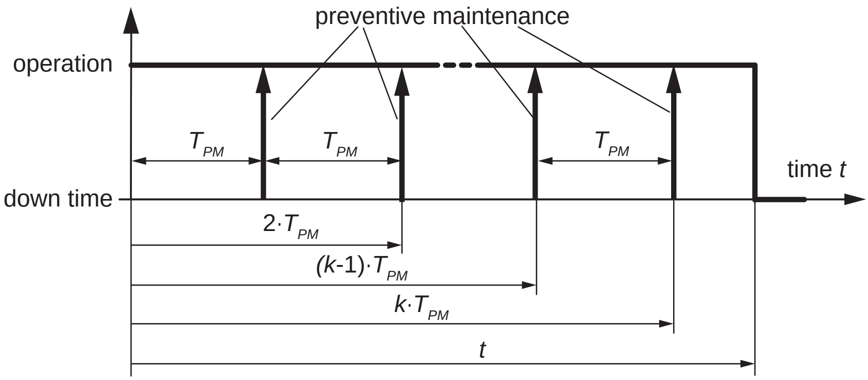


Figure 10.13. Maintenance plan over time

The following assumptions form the basis for the maintained model:

- The down time required for the renewal is negligible.
- After every renewal, the observed unit is brought back to its original status.
- The renewal is carried out periodically in constant intervals T_{PM}.
- The failure behaviour before and after inspection is stochastically independent.
- Even if a failure occurs after the k^{th} renewal during the ensuing operation period, the next renewal will first take place at the next maintenance interval point $(k+1)\cdot T_{PM}$.

Since the flow of events is seen as stochastically independent, the resulting reliability function for an inspected unit is [10.28]

$$R_{PM}(t) = R(T_{PM})^k \cdot R(t - k \cdot T_{PM})$$
$$\text{for } k \cdot T_{PM} \le t \le (k+1)T_{PM} \text{ and } k = 0(1)\infty. \tag{10.10}$$

The term $R(T_{PM})^k$ describes the probability that k renewal periods have succeeded without any failures having occurred. $R(t - k\cdot T_{PM})$ is the survival probability in the operation period after the last (k^{th}) completed maintenance.

The expected value of a component after each periodical renewal $MTTF_{PM}$ is [10.25]

$$MTTF_{PM} = \int_0^\infty R_{PM}(t)\mathrm{d}t = \frac{\int_0^{T_{PM}} R(t)\mathrm{d}t}{1 - R(T_{PM})} . \tag{10.11}$$

10.4.1.2 Periodical Renewal of Components with Constant Failure Rates

If the failure behaviour of an element can be described by an exponential distribution, then the periodical renewal does not influence the failure behaviour of that element, since

$$R_{PM}(t) = e^{-k\cdot\lambda\cdot T_{PM}} \cdot e^{-\lambda(t-k\cdot T_{PM})} = e^{-\lambda\cdot t} = R(t) \tag{10.12}$$

This means that the failure behaviour is the same with or without periodical renewal. For constant failure rates it is not possible to detect any aging appearances, thus it is easy to understand why the failure behaviour is not affected in this case.

10.4.1.3 Periodical Renewal of Components with Time Dependent Failure Rates

If the failure rate is time dependent, then the failure behaviour of the component is influenced by the length of the renewal interval. If the failure behaviour of the component is described by a three parametric Weibull distribution, then the reliability $R_{PM}(t)$ is

$$R_{PM}(t) = \exp\left[-\left(k \cdot \left(\frac{T_{PM} - t_0}{T - t_0} \right)^b + \left(\frac{t - k \cdot T_{PM} - t_0}{T - t_0} \right)^b \right) \right] \tag{10.13}$$

for $k\cdot T_{PM} \leq t \leq (k+1)T_{PM}$.

The reliability function for a component with and without periodical renewal is shown in Figure 10.14 (shape parameter $b = 2.0$, characteristic lifetime $T = 2{,}000$ h and failure free time $t_0 = 500$ h). Here, the maintenance interval is given as $T_{PM} = 1{,}000$ h.

In Figure 10.14, a reliability function is represented according to Equation (10.13). After each renewal, a break occurs in the survival probability function, which in turn leads to discontinuous failure densities and failure rates. The renewal is executed at times T_{PM}, $2\cdot T_{PM}$, $3\cdot T_{PM}$,.... The function

$R_{PM}(t)$ clearly lies above the function $R(t)$, which means that the survival probability with periodical renewal is much greater as the probability where no measures are taken.

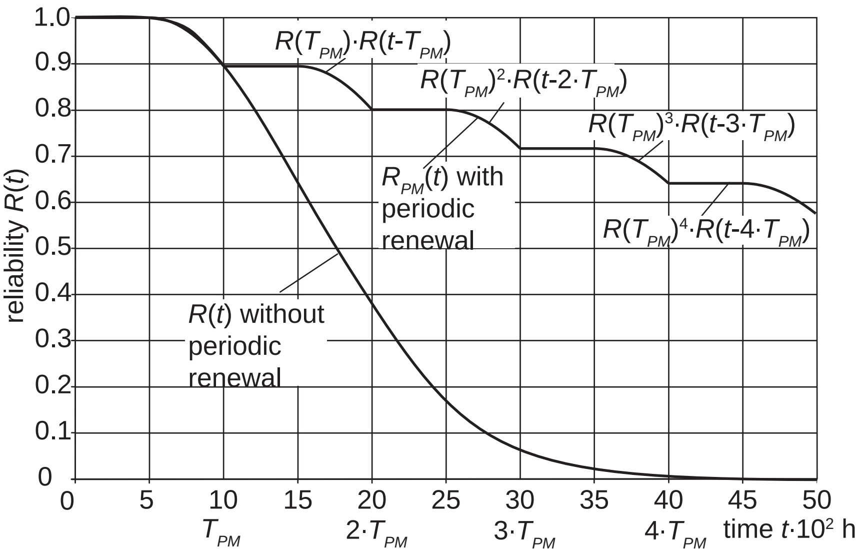


Figure 10.14. Reliability with and without periodical renewal

Figure 10.15 shows the expected value $MTTF_{PM}$ (standardized to the characteristic lifetime T) of a component as a function of the shape parameter b and the maintenance interval T_{PM} as a parameter. Here, a failure free time $t_0 = 0$ for the component was assumed.

It is shown that for a shape parameter $b > 1$, the expected value for preventive maintenance $MTTF_{PM}$ is greater as the $MTTF$ value, which is determined without maintenance ($T_{PM} = \infty$). For any given shape parameter b, $MTTF_{PM}$ increases strongly dependent upon the maintenance interval T_{PM}. In general, the greater b is, and thus the greater the influence of aging and wearout on the failure cause, the greater the positive effect is, with which periodical renewals can be carried out. For shape parameters $b < 1$, this effect is reversed, which means that the average lifetime is decreased by renewals. For the shape parameter $b = 1$, as already shown in Section 10.4.1.2, maintenance methods have no influence on the reliability and thus have no influence on the average lifetime.

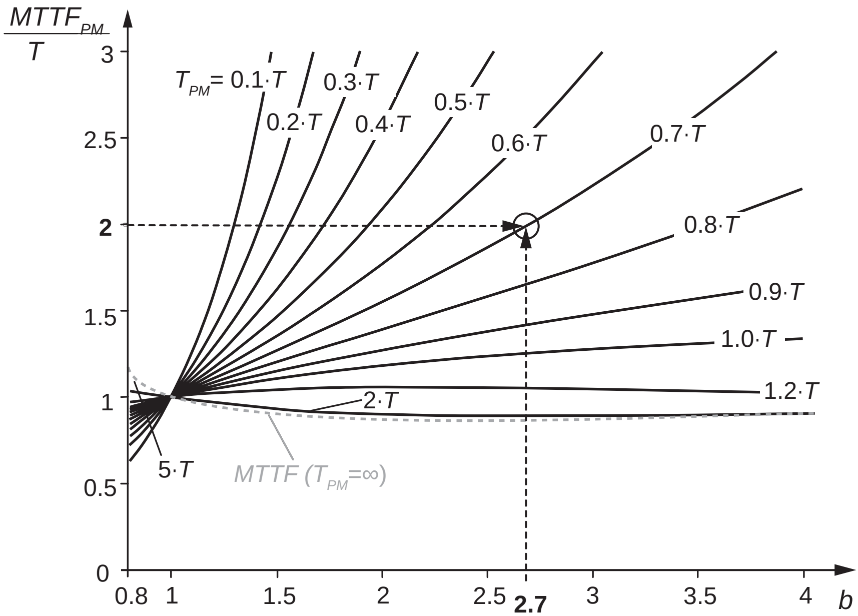


Figure 10.15. $MTTF_{PM}$ of a component as a function of the shape parameter *b* and the maintenance interval T_{PM}

Example:

For a certain component, a failure behaviour in the form of a Weibull distribution with the parameters $b = 2.7$ and $T = 1{,}000$ h was determined through lifetime tests. The average lifetime *MTTF* is 889.3 h, however, an average lifetime of 2,000 h is required. This required average lifetime can be reached with the help of periodical renewals within the realm of maintenance methods. What maintenance interval is required in order to reach the desired $MTTF_{PM}$? By observing the intersection of given shape parameters *b* and desired $MTTF_{PM}$ values in Figure 10.15, we see that $T_{PM} = 0.7 \cdot T = 700$ h.

In Figure 10.16 the steady state availability A_D of a component according to Equation (10.7) is shown as a function of the shape parameter *b* with the maintenance interval T_{PM} as a parameter. Here, it is assumed that the preventive renewals do not require any additional down time, but rather they are carried out during shift breaks. If an (unexpected) failure occurs, then the repair of the failed component begins immediately. The average repair time for the component is estimated at $MTTR = 0.1 \cdot T$.

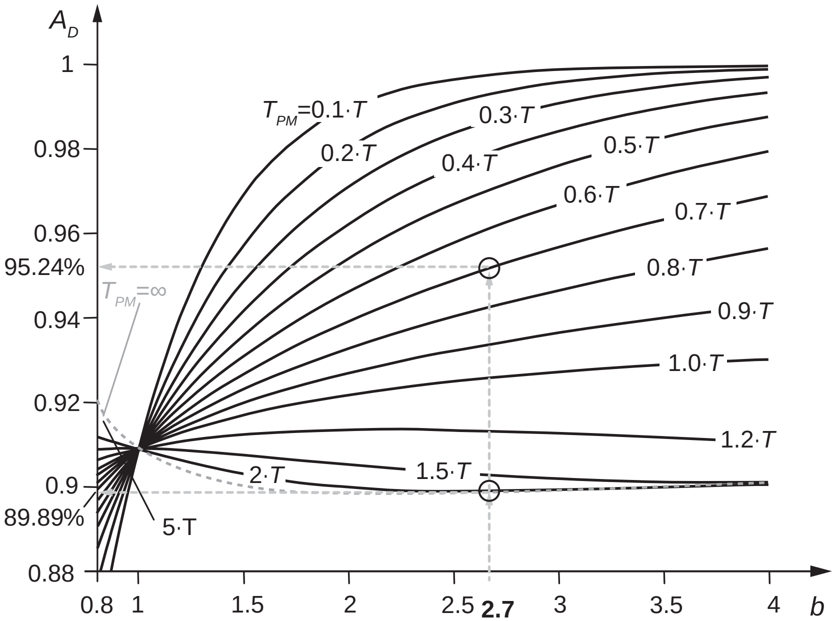


Figure 10.16. Steady state availability of a component with periodical maintenance with an average repair time of $MTTR = 0.1 \cdot T$

Example:

The component from the above example with $MTTR = 0.1 \cdot T = 100$ h yielded a steady state availability of $A_D = 89.89$ % according to Equation (10.7). After the introduction of a maintenance program, a steady state availability of $A_D = 95.24$ % can be achieved. This results in an increase in steady state availability of 5.95 %.

10.4.2 Markov Model

With the Markov model [10.7, 10.31, 10.35] it is possible to deal with repairable systems. The goal of this model is to determine the availability of a system or component. The following requirements have been established for the simplification of the model and its calculations:

- The unit to be observed switches continually between a state of operation and repair.
- After each maintenance action the repaired unit is as good as new.
- The time required for operation and repair for each unit observed is continuous and stochastically independent.

- The influence of any switch device is not taken into consideration.

The Markov method is based on the Markov process, which is a stochastic process $X(t)$ with a limited number of statuses (or states of condition) C_0, C_1, ..., C_n, for which the further development of the process is only dependent upon the present condition and the time t for each arbitrary time t. This means that only those systems can be dealt with using the Markov model, whose elements possess constant failure and repair rates. Further, the method is based on an equilibrium between the possible alterations between statuses in the form of equilibrium equations. The result is a system of consititutive differential equations, out of which the availability of the observed unit can be determined as a function of time.

10.4.2.1 Availability of an Individual Element

To begin with, the Markov method procedure will be shown step by step on an individual element.

a) Defining the Status

Each element can only assume one of two statuses: either "functional" or "failed".

- status (or condition) C_0: the element is functional and in operation.
- status (or condition) C_1: the element has failed and is currently in a state of repair.

The corresponding status probabilities are signified by $P_0(t)$ and $P_1(t)$.

b) Creating Status Graphs

The status graph represents the change in status for an element. An element crosses from one status to the next with a certain transition probability. The sum of the transition probabilities possessed by all arrows pointing away from a node (status) is always equal to 1. To simplify the Markov graph, the transition rates, failure rate λ and repair rate μ, are given. Such a Markov graph is shown in Figure 10.17 for an individual element.

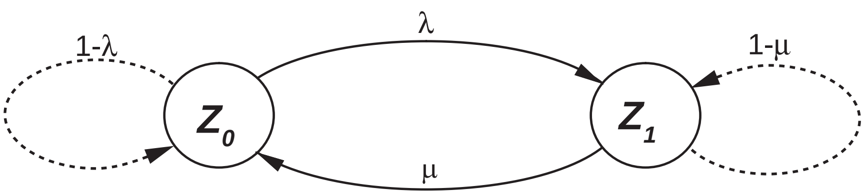


Figure 10.17. Markov graph for an individual element

c) Deriving the Constitutive Differential Equations

To derive the status differential equations it is first necessary to balance the probabilities for all possible changes in status. A change in status probability can be calculated by the addition of all transition probabilities. These transition probabilities can be obtained from the multiplication of the status probabilities with the corresponding transition rates. All arrows pointing away from a status are negative and all arrows pointing to a status are positive. This results in the following two differential equations in the case of an individual element:

$$\frac{\mathrm{d}\,P_0(t)}{\mathrm{d}\,t} = -\lambda \cdot P_0(t) + \mu \cdot P_1(t) \;\text{ and} \tag{10.14}$$

$$\frac{\mathrm{d}\,P_1(t)}{\mathrm{d}\,t} = -\mu \cdot P_1(t) + \lambda \cdot P_0(t)\,. \tag{10.15}$$

d) Standardization and Initial Conditions

Since the element must be located in one of the statuses at all times, the sum of all status probabilities at any point in time is always 1. Thus, the standardization condition is

$$P_0(t) + P_1(t) = 1\,. \tag{10.16}$$

The initial condition tells which status the element is in at the time $t = 0$. In the beginning, the observed element is normally functional and as good as new. Thus, the initial conditions are

$$P_0(t=0) = 1 \;\text{ and }\; P_1(t=0) = 0\,. \tag{10.17}$$

e) Solving for the Status Probability

The following equation for $P_0(t)$ is obtained from the differential Equations (10.14) and (10.15), the standardization condition (10.16) and the initial conditions (10.17)

$$P_0(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} \cdot e^{-(\lambda+\mu)\cdot t}\,. \tag{10.18}$$

The status probability is won with help of the standardization condition

$$P_1(t) = 1 - P_0(t)\,. \tag{10.19}$$

f) Determining the Availability

The availability $A(t)$, which an element possesses at the point in time t in a functional operational state, is equal to the status probability $P_0(t)$:

$$A(t) = P_0(t)\,. \tag{10.20}$$

The unavailability $U(t)$ is the complement of the availability

$$U(t) = 1 - A(t) = P_1(t)\,. \tag{10.21}$$

Stationary Solution

The availability converges to the limit of the stationary solution for $t \to \infty$. This steady state availability A_D is normally expressed by the

- expected value of the operation duration $MTTF = 1/\lambda$ (Mean Time To Failure) and the
- expected value of the repair time $MTTR = 1/\mu$ (Mean Time To Repair)

and is of great practical importance in maintenance:

$$A_D = \lim_{t\to\infty} A(t) = \frac{\mu}{\lambda+\mu} = \frac{MTTF}{MTTF + MTTR} = \frac{1}{1+\dfrac{MTTR}{MTTF}}\,. \tag{10.22}$$

The steady state availability is only dependent upon the quotient *MTTR*/*MTTF*, as graphically shown in Figure 10.18. As this quotient becomes larger, the steady state availability becomes smaller.

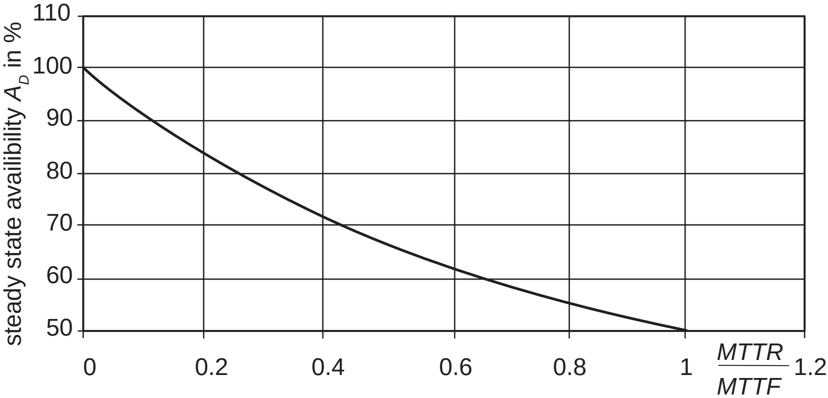


Figure 10.18. Steady state availability A_D as a function of the quotient *MTTR/MTTF*

Example:

Here, in order to exhibit the Markov method, an element is dealt with whose failure and repair behaviours are exponentially distributed. The availability is shown for various repair rates with a constant failure rate λ. Table 10.3 shows the various parameters. The determined availabilities $A(t)$ are shown in Figure 10.19.

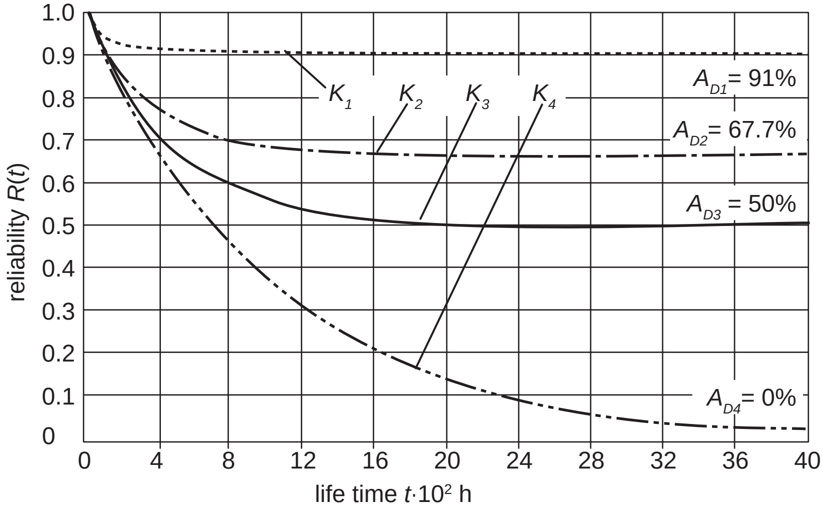


Figure 10.19. Availability $A(t)$ for various repair rates μ_i with a constant failure rate λ

Here, it may be observed that the convergence to the steady state availability for long periods of time t. For $\mu = 0$, which corresponds to an infinitely long repair duration ($MTTR \to \infty$), the reliability and availability coincide. The steady state availability decreases with increasing repair duration.

Table 10.3. Parameters for the Markov example

No.	λ [1/h]	*MTTF* [h]	μ [1/h]	*MTTR* [h]	A_D [%]
C_1	0.001	1,000	0.01	100	91
C_2	0.001	1,000	0.002	500	66.7
C_3	0.001	1,000	0.001	1,000	50
C_4	0.001	1,000	0	∞	0

10.4.2.2 Markov Model for Several Elements

When analyzing a system consisting of several elements, it is necessary to consider the interaction of all elements to one another. If n elements interact with one another, then the Markov model can assume 2^n statuses for all thinkable combinations of failures and transitions. The simplest scenario is a system made up of two elements K_1 and K_2, for which four statuses are possible, Table 10.4.

Table 10.4. Description of the statuses for a system with two elements

Condition	Description	Probability
C_0	both elements K_1 and K_2 are intact	$P_0(t)$
C_1	K_1 defect and K_2 intact	$P_1(t)$
C_2	K_1 intact and K_2 defect	$P_2(t)$
C_3	both elements K_1 and K_2 are defect	$P_3(t)$

Figure 10.20 shows the corresponding Markov status graph with all thinkable change overs. The rates λ_1 and μ_1 describe the transition behaviour of the element K_1, while λ_2 and μ_2 describe that of K_2. The transition from C_0 to C_3 and from C_1 to C_2 are not taken into account because such a change in the status would mean that both elements would change their condition simultaneously.

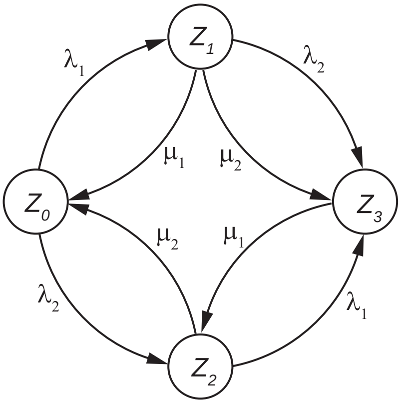


Figure 10.20. Markov status graph for two elements

The differential system of equations for the condition probabilities again results in an equilibrium of the status transitions in the Markov graph.

Thus:

$$\left.\begin{aligned}
\frac{\mathrm{d}\,P_0(t)}{\mathrm{d}\,t} &= -(\lambda_1+\lambda_2)\cdot P_0(t)+\mu_1\cdot P_1(t)+\mu_2\cdot P_2(t)\,,\\
\frac{\mathrm{d}\,P_1(t)}{\mathrm{d}\,t} &= \lambda_1\cdot P_0(t)-(\lambda_2+\mu_1)\cdot P_1(t)+\mu_2\cdot P_3(t)\,,\\
\frac{\mathrm{d}\,P_2(t)}{\mathrm{d}\,t} &= \lambda_2\cdot P_0(t)-(\lambda_1+\mu_2)\cdot P_2(t)+\mu_1\cdot P_3(t) \text{ and}\\
\frac{\mathrm{d}\,P_3(t)}{\mathrm{d}\,t} &= \lambda_2\cdot P_1(t)+\lambda_1\cdot P_2(t)-(\mu_1+\mu_2)\cdot P_3(t)\,.
\end{aligned}\right\} \quad (10.23)$$

As before, the standardization condition can be determined as the sum of the status probabilities

$$P_0(t)+P_1(t)+P_2(t)+P_3(t)=1\,. \quad (10.24)$$

Here, the initial conditions are

$$P_0(t=0)=1 \text{ and } P_i(t=0)=0\,\forall i=1(1)3\,. \quad (10.25)$$

Under consideration of the standardization and initial conditions, the differential system of equations can be solved, for example, with help of the Laplace transformation. However, this solution is very complex and time-consuming.

After extensive calculations have been carried out, the various possible status probabilities result to:

$$P_0(t) = \frac{\lambda_1 \cdot \lambda_2 \cdot e^{-(\lambda_1+\mu_1+\lambda_2+\mu_2)\cdot t} + \mu_1 \cdot \lambda_2 \cdot e^{-(\lambda_2+\mu_2)\cdot t} + \mu_2 \cdot \lambda_1 \cdot e^{-(\lambda_1+\mu_1)\cdot t} + \mu_2 \cdot \mu_1}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)},$$

(10.26)

$$P_1(t) = -\frac{\lambda_1 \cdot \lambda_2 \cdot e^{-(\lambda_1+\mu_1+\lambda_2+\mu_2)\cdot t} - \lambda_2 \cdot e^{-(\lambda_2+\mu_2)\cdot t} + \mu_2 \cdot e^{-(\lambda_1+\mu_1)\cdot t} - \mu_2}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)},$$

$$P_2(t) = \frac{\lambda_2 \cdot -\lambda_1 \cdot e^{-(\lambda_1+\mu_1+\lambda_2+\mu_2)\cdot t} - \mu_1 \cdot e^{-(\lambda_2+\mu_2)\cdot t} + \lambda_1 \cdot e^{-(\lambda_1+\mu_1)\cdot t} + \mu_1}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)},$$

and $$P_3(t) = \frac{\lambda_1 \cdot \lambda_2 \cdot \left(e^{-(\lambda_1+\mu_1+\lambda_2+\mu_2)\cdot t} - e^{-(\lambda_1+\mu_1)\cdot t} - e^{-(\lambda_2+\mu_2)\cdot t} + 1\right)}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)}.$$

In order to determine the availability, it is now necessary to consider the system structure. Since we are dealing with two elements, it is possible that these elements are connected to one another either serial or parallel. The respective availabilities are yielded:

for a serial connection $A(t) = P_0(t)$ and (10.30)

for a parallel connection $A(t) = P_0(t) + P_1(t) + P_2(t) = 1 - P_3(t)$. (10.31)

The status probabilities remain constant for stationary cases, thus, the changes in status reach zero:

$$\lim_{t\to\infty} P_i(t) = p_i = \not\subset \text{ and thus, } \lim_{t\to\infty} \frac{\mathrm{d}\,P_i(t)}{\mathrm{d}\,t} = 0 \quad \forall i = 0(1)3. \tag{10.32}$$

Therefore, for stationary cases, the differential system of equations becomes a linear algebraic system of equations. Out of Equations (10.27) to (10.30), the following stationary solutions are yielded:

$$\left.\begin{aligned} p_0 &= \frac{\mu_1 \cdot \mu_2}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)}, \quad p_1 = \frac{\lambda_1 \cdot \mu_2}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)}, \\ p_2 &= \frac{\lambda_2 \cdot \mu_1}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)} \text{ and } p_3 = \frac{\lambda_1 \cdot \lambda_2}{(\lambda_1 + \mu_1)\cdot(\lambda_2 + \mu_2)}. \end{aligned}\right\} \tag{10.33}$$

This results in relationships between the steady state availabilities for serial and parallel connections for both elements.

for a serial connections

$$A_D = \frac{1}{\left(1+\frac{\lambda_1}{\mu_1}\right)\cdot\left(1+\frac{\lambda_2}{\mu_2}\right)} \quad (10.34)$$

for a parallel connections

$$A_D = 1-\frac{1}{\left(1+\frac{\mu_1}{\lambda_1}\right)\cdot\left(1+\frac{\mu_2}{\lambda_2}\right)}. \quad (10.35)$$

Example:

The failure and repair behaviour of both system components is exponentially distributed. The parameters are given in Table 10.5.

Table 10.5. Parameters for the failure and repair distributions

No.	Failure Behaviour		Repair Behaviour		Steady state Availability
	λ [1/h]	*MTTF* [h]	μ [1/h]	*MTTR* [h]	A_{Di} [%]
C_1	0.001	1,000	0.01	100	90.9
C_2	0.002	500	0.02	50	90.9

The availability Equations (10.30) and (10.31) for serial and parallel connections for both components are illustrated in Figure 10.21.

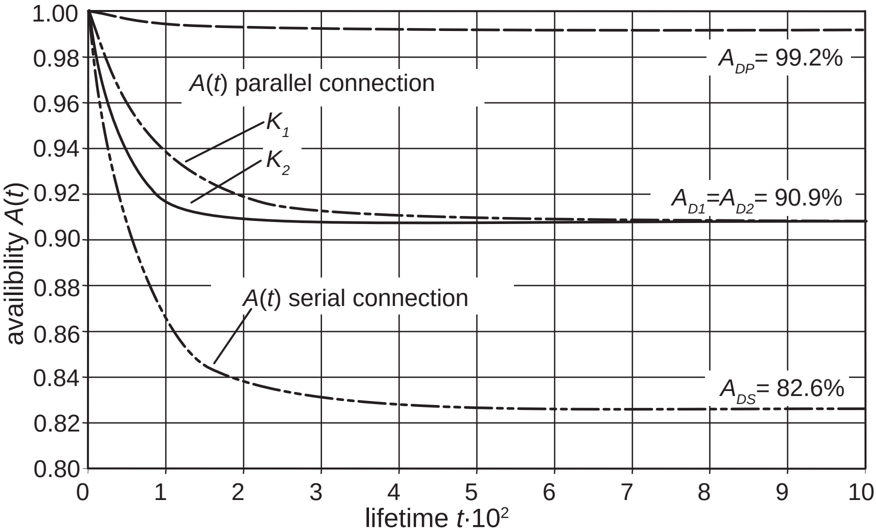


Figure 10.21. Availability for serial and parallel connections of two components

The steady state availabilities calculated with the Equations (10.34) and (10.35) are also given. Here, it may be shown how quickly the stationary status is reached.

10.4.3 Boole-Markov Model

If a system is made up of repairable system elements independent of one another, then the so-called Boole-Markov model can be used, Figure 10.22. Thus, the repairable system is observed as a system with repairable elements. As already seen, the Markov model determines the steady state availability of individual system elements. The connections between these elements are achieved with the Boolean model.

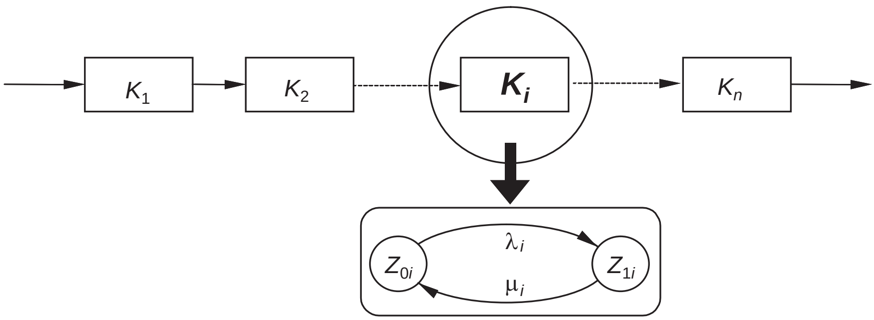


Figure 10.22. Boole-Markov model

Contemporary acceptance of the Markov model for repairable systems was limited by the fact that the corresponding elements must possess constant failure and repair rates. For time dependent failure or repair rates, it is impossible to solve the status probabilities as functions of time. Thus, in the Boole-Markov model only the stationary status, that is the steady state availability, is observed. The following time-dependent transition rates have been determined for the steady state availability A_{Di} for each individual element:

$$A_{Di} = \lim_{t \to \infty} A_i(t) = \frac{\mu_i}{\lambda_i + \mu_i} = \frac{MTTF_i}{MTTF_i + MTTR_i}. \qquad (10.36)$$

Here, it is necessary to calculate the expected value of the operating hours $MTTF_i$ and the expected value of the repair time $MTTR_i$ as the expected value $E(t)$ of the failure or repair distribution. The system steady state availability can now be estimated with the help of the Boolean model:

serial system: $$A_{DS} = \prod_{i=1}^{n} A_{Di} = \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i + \mu_i} = \prod_{i=1}^{n} \frac{MTTF_i}{MTTF_i + MTTR_i}, \quad (10.37)$$

parallel system: $$A_{DS} = 1 - \prod_{i=1}^{n} (1 - A_{Di}) = 1 - \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \mu_i} = 1 - \prod_{i=1}^{n} \frac{MTTR_i}{MTTF_i + MTTR_i}. \quad (10.38)$$

Example:

A system made up of three components in serial connection with Weibull distributions desicribing the failure behaviour is considered as an example for the calculation of system availability. The repair behaviour of each component is described by an exponential distribution with a repair mean of $MTTR = 100$ h. Table 10.6 summarizes the parameters for the distributions and the calculated steady state availabilities for the individual components as well as the system steady state availability.

Table 10.6. Parameters for the system components – calculation of the system steady state availability

Nr.	Failure Behaviour				Repair Behaviour	Steady state Availability
	b	T [h]	t_0 [h]	$MTTF$ [h]	$MTTR$ [h]	A_{Di}
K_1	2.0	3,000	0	2,658	100	0.9637
K_2	1.8	3,200	500	2,901	100	0.9667
K_1	1.5	2,500	1,000	2,354	100	0.9593

System Steady state Availability: $$A_{DS} = \prod_{i=1}^{n} \frac{MTTF_i}{MTTF_i + MTTR_i} = \prod_{i=1}^{n} A_{Di} = 0.8937$$

10.4.4 Common Renewal Processes

Renewal theory originates from studies of “renewing populstions”. Over time, however, renewal theory leaned more towards studies of general incidents over the sum of independent and positive random variables in probability theory [10.13, 10.14]. Early research in the field of renewal theory can be found compiled in the literature by *Lotka* [10.33].

Common renewal processes [10.1, 10.2, 10.3, 10.7, 10.13, 10.14, 10.45] are also part of the class of stochastic point processes and describe the fundamental principle of an individual component in continuous operation. It is assumed that at the end of its lifetime, a failed component is immediately replaced (renewed) by a new, statistically identical component. For common renewal processes, this means that the duration of repair is neglected in comparison to the duration of operation, thus it is necessary that *MTTF* >> *MTTR*. This simplification results, however, in the fact that the availability cannot be calculated with a common renewal process. Despite this limitation, we would like to take a closer look at the common renewal process.

10.4.4.1 Time until the n^{th} Renewal

A common renewal process is symbolically illustrated in Figure 10.23.

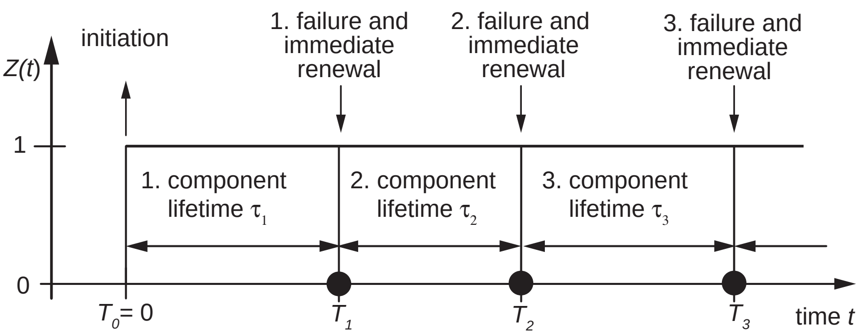


Figure 10.23. Procedure of a common renewal process

The points T_1, T_2, ... are signified as renewal points or regeneration points. The value T_n describes the distance from the origin to the n^{th} renewal and thus the time until the n^{th} renewal. Renewal processes produce sequences of points, whose renewal times are independent of one another – thus this process is often characterized as a point process. The lifetimes τ_n are positive, independent random variables, which all possess the same distribution function $F(t)$. The following equation is valid for the common renewal process:

$$T_n = \sum_{i=1}^{n} \tau_i \ , \ n = 1\ (1)\ \infty \ . \tag{10.39}$$

The point of origin does not count as a point of renewal. It is an exception for which $T_0 = 0$. The distribution of the n^{th} renewal, and thus at the point in time T_n, is given by the n^{th} convolution power of $F(t)$

$$F_n(t) = F^{*(n)}(t) \tag{10.40}$$

and corresponds to the distribution of the sum of n lifetimes. The n^{th} convolution power of $F(t)$ can be calculated recursively with

$$F^{*(n)}(t) = \int_0^t F^{*(i-1)}(t-t') f(t') \mathrm{d}t' \;\; \forall \; i = 2(1)n \tag{10.41}$$

where $F^{*(1)}(t) \equiv F(t)$.

10.4.4.2 Number of Renewals

The number of renewal points $N(t)$ during the time span $[0, t]$ is a discrete random variable for which

$$N(t) = \begin{cases} 0 \;\text{ for } & t < T_1 \\ n \;\text{ for } & T_n \le t < T_{n+1} \end{cases}. \tag{10.42}$$

for the probability that exactly n renewal points lie between 0 and t, it is necessary that $W_n(t) = P(N(t) = n)$. This results to the following expression:

$$W_n(t) = F^{*(n)}(t) - F^{*(n+1)}(t)\,. \tag{10.43}$$

The probability that no renewals occur between the beginning of operation of the first component and the point in time t is equal to the reliability

$$W_0(t) = 1 - F(t) = R(t)\,. \tag{10.44}$$

10.4.4.3 Renewal Function and Renewal Density

The renewal function $H(t)$ is defined as the expected value for the number of renewals in the time span $[0, t]$. From Equation (10.43) one concludes:

$$H(t) = E\big(N(t)\big) = \sum_{n=1}^{\infty} n W_n(t) = \sum_{n=1}^{\infty} n\left[F^{*(n)}(t) - F^{*(n+1)}(t)\right] = \sum_{n=1}^{\infty} F^{*(n)}(t) \quad \text{for } t \ge 0\,. \tag{10.45}$$

The renewal function serves as a foundation for the determination of the replacement part demand, since it predicts with a probability of 50% how many renewals will be conducted up to the point in time t. That is to say that if the renewal process should be maintained with a probability of 50%, then it is necessary to have a total of $H(t)$ replacement components at hand at the time t.

Deriving the renewal function results in the renewal density

$$h(t) = \frac{\mathrm{d}H(t)}{\mathrm{d}t} = \sum_{n=1}^{\infty} f^{*(n)}(t) \tag{10.46}$$

as an infinite sum of the convolution power of the failure densities. This can be recursively calculated with the equation

$$f_S(t) = f^{*(i)}(t) = \int_0^t f^{*(i-1)}(t-t') f(t') \mathrm{d}t' \;\forall\; i = 2(1)n \text{ where}$$
$$f^{*(1)}(t) \equiv f(t) \,. \tag{10.47}$$

The expression $h(t)\mathrm{d}t$ is the average probability for the number of failures during the interval $[t, t+\mathrm{d}t]$. Thus, the renewal density describes the average number of failures per unit of time. Figure 10.24 illustrates an example of the relationship for a normally distributed failure density with $\mu = 36$ h and $\sigma = 6$ h.

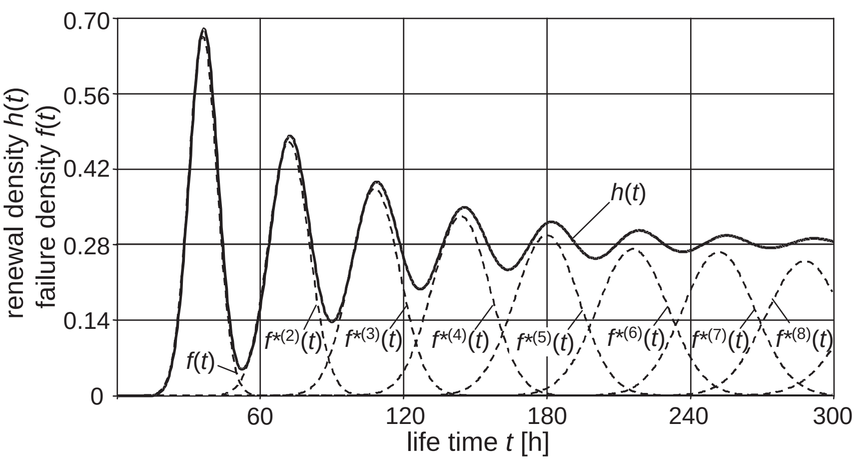


Figure 10.24. Renewal density as an infinite sum of the convolution power of the failure density [10.28]

10.4.4.4 Renewal Equations

The Laplace transformation of the renewal can be represented as a geometric series [10.6]. By applying the convolution theorem of the Laplace transformation to Equation (10.46), the following is yielded:

$$\begin{aligned} L\{h(t)\} &= \tilde{h}(s) = \sum_{n=1}^{\infty} \tilde{f}^{\,n}(s) = \tilde{f}(s)(1+\tilde{f}(s)+\tilde{f}^{\,2}(s)+\tilde{f}^{\,3}(s)+\ldots) \\ &= \frac{\tilde{f}(s)}{s(1-\tilde{f}(s))} \end{aligned} \quad . \qquad (10.48)$$

The renewal function results to the following equation under consideration of the integration theorem of the Laplace transformation

$$L\{H(t)\} = \tilde{H}(s) = \sum_{n=1}^{\infty} \tilde{F}^{\,n}(s) = \frac{1}{s}\sum_{n=1}^{\infty} \tilde{f}^{\,n}(s) = \frac{\tilde{f}(s)}{s(1-\tilde{f}(s))} . \qquad (10.49)$$

Another interpretation can be seen by rewriting Equation (10.48) and Equation (10.49) to

$$\tilde{h}(s) = \tilde{f}(s) + \tilde{h}(s)\tilde{f}(s) \text{ and} \qquad (10.50)$$

$$\tilde{H}(s) = \tilde{F}(s) + \tilde{H}(s)\tilde{f}(s) . \qquad (10.51)$$

By carrying out the reverse transformation of Equation (10.50) and Equation (10.51) while at the same time observing the convolution law of the Laplace transformation, the following equations result

$$h(t) = f(t) + h * f(t) = f(t) + \int_0^t h(t-t') f(t') \mathrm{d}t' \text{ and} \qquad (10.52)$$

$$H(t) = f(t) + H * f(t) = F(t) + \int_0^t H(t-t') f(t') \mathrm{d}t' . \qquad (10.53)$$

These equations are called the integral equations of renewal theory or simply renewal equations. They often form the point of origin for further studies.

10.4.4.5 Estimation of Replacement Part Demand

Asymptotes for $H(t)$ have already been discovered for the common renewal process according to [10.3]. For large values of time t, these asymptotes yield approximations for $H(t)$ as well as for the distribution $N(t)$. Here, it is continually necessary that $E(\tau) = MTTF < \infty$ and $\mathrm{Var}(\tau) < \infty$.

The fundamental law of renewal theory allows for further asymptotical statements concerning the renewal process. One such important implication is that the straight line described in the equation below represents the asymptote of the curve $H(t)$

$$\hat{H}(t) = \frac{t}{MTTF} + \frac{Var(\tau) + MTTF^2}{2 \cdot MTTF^2} - 1 = \frac{t}{MTTF} + \frac{Var(\tau) - MTTF^2}{2 \cdot MTTF^2} . \tag{10.54}$$

$\hat{H}(t)$ yields an approximation solution for the calculation of the required replacement parts over time in order to maintain a renewal process for a component or system.

10.4.4.6 Comment to Availability

Since it is assumed in the common renewal process that after a failure the component is replaced by a new component without delay, the following availability can be assumed for the common renewal process:

$$A(t) = 1 \quad \forall t \geq 0 . \tag{10.55}$$

10.4.4.7 Analysis of the Common Renewal Process

The renewal Equations (10.52) and (10.53) are linear Volterra integral equations of the 2^{nd} type. The solution of such equations with the application of numerical integration procedures is shown in [10.30] and [10.27].

10.4.5 Alternating Renewal Processes

If the repair duration or the time for renewal of a failed component is not neglected, then one deals with an alternating renewal process [10.1, 10.2, 10.3, 10.7, 10.13, 10.14, 10.40, 10.51]. With this process, it is easier to model reality, since normally the discovery of a defect component as well as its repair or renewal takes a certain amount of time. Thus, it is possible to calculate the availability.

In 1959, *Cane* [10.10] and *Page* [10.34] published the first applications of alternating renewal processes for problems in animal ethology as well as in the inspection of electronic computers [10.13].

10.4.5.1 Alternating Renewal Process Procedure

The situation illustrated in Figure 10.25 is modeled on behalf of an alternating renewal process.

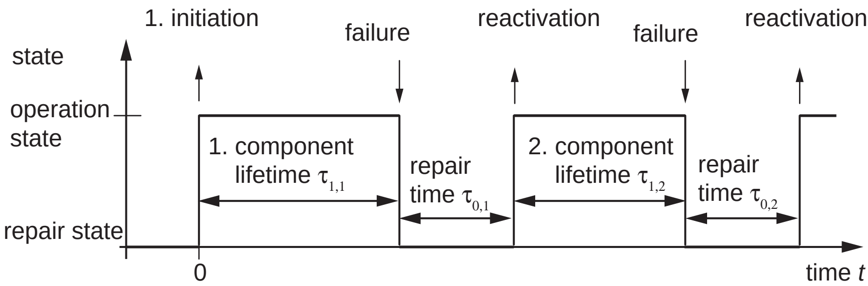


Figure 10.25. Alternating renewal process procedure

The first component begins operation at the time $t = 0$. This component is in an active state of operation during its first lifetime $\tau_{1,1}$. At the end of its lifetime, the component fails and is then in a failed status or a repair status. During the repair duration $\tau_{0,1}$, the defect component is either repaired or replaced by a replacement component. After the repair or renewal has been completed, the component immediately returns to a status. After the random lifetime $\tau_{1,2}$ has expired, it is again repaired or renewed for the random time $\tau_{0,2}$. The lifetimes and the repair durations take place alternating to one another. The time $T_{1,n}$ in which each lifetime ends as well as the points in time $T_{0,n}$ in which a repair duration ends are shown along the time axis in Figure 10.26.

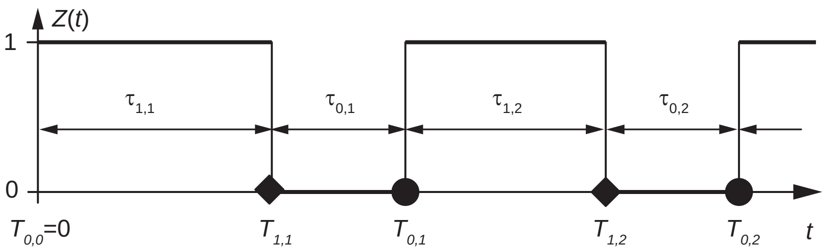


Figure 10.26. Alternating renewal process over time

The lifetimes $\tau_{1,n}$ are given as

$$\tau_{1,n} = T_{1,n} - T_{0,n-1}\,,\ \ n = 1(1)\infty, \tag{10.56}$$

which does not include the time at the beginning of operation $T_{0,0} = 0$ as a renewal point.

The repair durations can be calculated by:

$$\tau_{0,n} = T_{0,n} - T_{1,n}\,,\ \ n = 1(1)\infty. \tag{10.57}$$

A sequence of time $T_{1,1}$, $T_{0,1}$, $T_{1,2}$, $T_{0,2}$, $T_{1,3}$, ... signifies an alternating renewal process if the given differences in Equations (10.56) and (10.57) are independent, positive, random variables. Furthermore, it is necessary that all lifetimes $\tau_{1,n}$ and all repair durations $\tau_{0,n}$ possess the same distribution. Since it is assumed that at the time $t = 0$ a new component enters operation, the distribution of the first lifetime $\tau_{1,1}$ is the same as the distribution of the following lifetimes $\tau_{1,n}$. In order to distinguish those alternating renewal processes which do not fulfil this requirement, such processes are given the name common alternating renewal processes [10.7, 10.45].

The behaviour of the lifetimes $\tau_{1,n}$ is characterized by $F(t)$, $f(t)$ and *MTTF* and the behaviour of the repair durations $\tau_{0,n}$ is given by $G(t)$, $g(t)$ and *MTTR*. The lifetime $\tau_{1,n}$ comes to an end when it reaches the point of renewal $T_{1,n}$, which is the reason why this point is also called the point of failure. The point of renewal $T_{0,n}$, which ends the repair duration $\tau_{0,n}$, is also characterized as the point in time for the restart of operation.

10.4.5.2 Renewal Equations

Similar to the common renewal process, the renewal equations for embedded processes are found to be integral equations after Laplace transformations, geometrical series development, rewriting in the Laplace field and Laplace reverse transformation. The renewal equation for the renewal density of an embedded 1-renewal process composed of points of failure is given in the equation below:

$$h_1(t) = f(t) + h_1 * (f * g(t)) = f(t) + \int_0^t h_1(t-t')\big(f * g(t')\big)\,\mathrm{d}t' \,. \tag{10.58}$$

The renewal equation for the renewal function is as follows:

$$H_1(t) = F(t) + H_1 * \big(f * g(t)\big) = F(t) + \int_0^t H_1(t-t')\big(f * g(t')\big)\,\mathrm{d}t' . \tag{10.59}$$

In order to approximate the replacement part demand, it is practical to use the renewal function of failures $H_1(t)$, so that the required replacement components are available during the state of repair, which begins after a failure occurs.

The renewal equation for the renewal density of an imbedded 0-renewal process is as follows:

$$h_0(t) = f * g(t) + h_0 * \big(f * g(t)\big) = f * g(t) + \int_0^t h_0(t - t')\big(f * g(t')\big)\mathrm{d}t' \qquad (10.60)$$

The renewal equation for the renewal function is

$$H_0(t) = F * g(t) + H_0 * \big(f * g(t)\big) = F * g(t) + \int_0^t H_0(t - t')\big(f * g(t')\big)\mathrm{d}t' \cdot \qquad (10.61)$$

10.4.5.3 Estimation of Replacement Part Demand

For both embedded renewal processes it is possible to specify renewal laws. These laws yield approximations for $H_1(t)$ and $H_0(t)$ for large values of time t. It is required that $MTTF < \infty$, $MTTR < \infty$, $Var(\tau_1) < \infty$ and $Var(\tau_0) < \infty$.

It can be determined that the straight line below describes the asymptote of the curve $H_1(t)$ for an imbedded 1-renewal process [10.27]

$$\hat{H}_1 = \frac{t}{MTTF + MTTR} + \frac{Var(\tau_1) + Var(\tau_0) + MTTR^2 - MTTF^2}{2(MTTF + MTTR)^2} \, . \qquad (10.62)$$

The following straight line corresponds to the asymptote of the curve $H_0(t)$ for an imbedded 0-renewal process [10.27]

$$\hat{H}_0 = \frac{t}{MTTF + MTTR} + \frac{Var(\tau_1) + Var(\tau_0) + (MTTF + MTTR)^2}{2(MTTF + MTTR)^2} - 1 \, . \qquad (10.63)$$

Thus, Equations (10.62) and (10.63) offer a closer approximation of the renewal functions $H_1(t)$ and $H_0(t)$ for large values of t as does the elementary renewal approach. At the same time, these equations offer the possibility to estimate the replacement part demand for large values of t. Here, the approximation of the renewal function $H_1(t)$ should be used, since at the time of failure, the replacement part should already be ready at hand.

10.4.5.4 Point Availability

Point availability receives an ever increasing interest in the practical field as a performance feature of a technical system. According to Equation (10.3), point availability describes the probability that a component is in a state of operation at the point in time *t*. Point availability can be determined in various different ways based on alternating renewal processes. Three of these methods will be introduced and discussed in the following text.

Method I:

Point availability is seen as a special case of interval reliability [10.30] for $x = 0$:

$$A(t) = R(t) + R * h_0(t) = R(t) + \int_0^t R(t - t')h_0(t')\,\mathrm{d}t' \,. \qquad (10.64)$$

For the calculation of the point availability in Equation (10.64), it is required that the renewal density $h_0(t)$ is known.

Method II:

A second way to calculate the point availability $A(t)$ of a component, without having to explicitly know the renewal density $h_0(t)$, is described in [10.7] and [10.36]. It is assumed that a component starts in an operational status (1-state) at $t = 0$. Only the time $T_{0,n,}$ where the renewed component is brought back to a state of operation, are taken into consideration. The first renewal after the end of the first repair duration is at the time t', Figure 10.27.

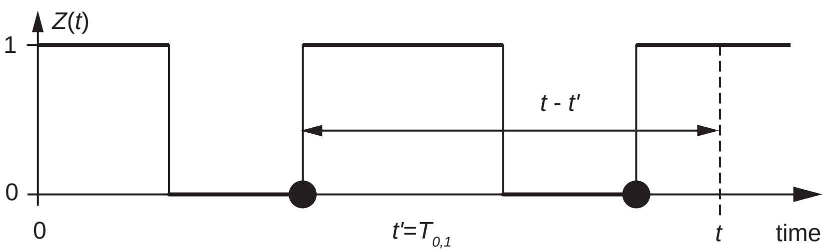


Figure 10.27. Progression of status for an alternating renewal process

The distributed density of the time of the first reconnections $T_{0,1}$ is equal to $f * g(t')$. Under the requirement that the first reconnection takes place at t' for $t' \leq t$, the probability of the 1-state at t is equal to $A(t - t')$.

The integration over all possible t''s yields:

$$P(Z(t)=1 \mid T_{0,1}=t' \le t)=\int_0^t A(t'-t)\big(f * g(t')\big)\mathrm{d}t' \,. \qquad (10.65)$$

Furthermore, it is necessary to consider the case that the first reconnection $T_{0,1}$ takes place after the time t. For this case, the probability of the 1-state at the time t is

$$P(Z(t)=1 \mid T_{0,1}=t' > t)=1-F(t)=R(t)\,. \qquad (10.66)$$

The recursion formula for point availability can be directly won out of the Equations (10.65) and (10.66), each with disjunctive conditions for the point in time $T_{0,1}$, along with the law for total probability:

$$A(t)=R(t)+A*\big(f * g(t)\big)=R(t)+\int_0^t A(t-t')\big(f * g(t')\big)\mathrm{d}t' \,. \qquad (10.67)$$

Method III:

According to Equation (10.3), point availability is defined as the expected value of the condition indicator $C(t)$ at the point in time t. The calculation of the condition indicator is carried out with the help of the number functions $N_1(t)$ and $N_0(t)$.

$N_1(t)$ signifies the number of occurred failures during the interval $[0, t]$ and $N_0(t)$ signifies the number of finished repairs during the interval $[0, t]$. As can be seen in Figure 10.28, at the time t, the condition indicator $C(t)$ is

$$C(t)=1+N_0(t)-N_1(t)\,, \qquad (10.68)$$

Through expected value formation, a further form of point availability is gained from the description of the condition indicator. Under consideration of the rules for sums of random variables along with the fact that the expected value of a constant value is constant, the following point availability can be achieved [10.1]:

$$A(t)=E\big(Z(t)\big)=1+E\big(N_0(t)\big)-E\big(N_1(t)\big)=1+H_0(t)-H_1(t)\,. \qquad (10.69)$$

With a Laplace transformation it is possible to show that Equations (10.64), (10.67) and (10.69) represent equivalent expressions for the calculation of the point availability $A(t)$ [10.30].

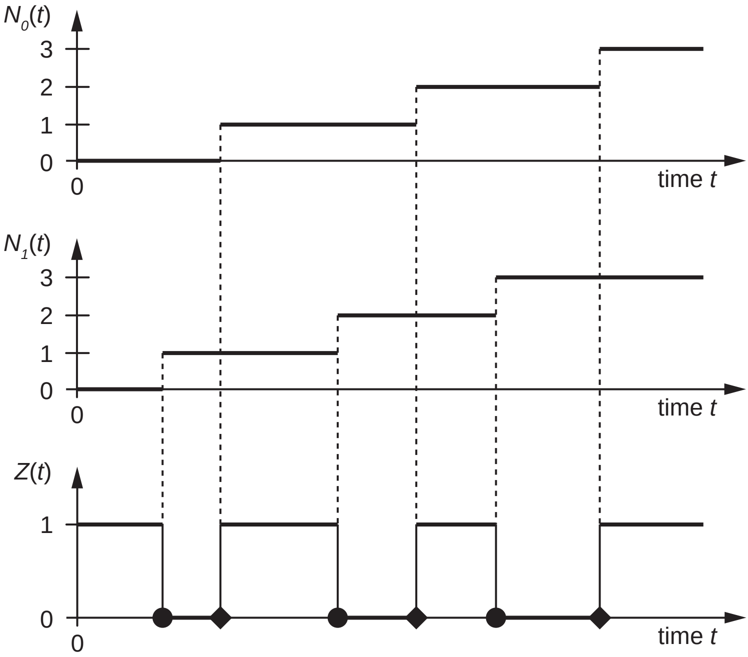


Figure 10.28. Relationship between the number function and condition function

10.4.5.5 Asymptotic Behaviour

The availability $A(t)$ converges for long periods of time to a constant value independent of the initial condition at the time $t = 0$. The steady state availability can be determined with the help of the fundamental theorem of renewal theory.

$$A_D = \lim_{t \to \infty} A(t) = \frac{MTTF}{MTTF + MTTR}. \tag{10.70}$$

By characterizing the time span between two neighboring renewal points as the renewal cycle, the steady state availability becomes equal to the expected value of the work time relative to the expected value of the cycle length.

10.4.5.6 Analysis of the Alternating Renewal Process

The renewal Equations (10.58) to (10.61) are linear Volterra integral equations of the 2^{nd} type. The solution achieved by the application of numerical integration procedures is shown in [10.27] and [10.30]. The Equations (10.64), (10.67) and (10.69) for the calculation of point availability $A(t)$ can also be calculated numerically.

10.4.5.7 Example

Figure 10.29 represents examples of renewal densities, renewal functions, failure and repair densities as well as the availability for various Weibull failure distributions for identical Weibull repair distributions. The repair distribution is similar to a normal distribution with $b = 3.5$. The various failure distributions have been chosen in such a way that the *MTTF* value remains constant. The shape parameter of the failure distribution is varied 5 times, which also leads to various characteristic lifetimes T for a constant value *MTTF*. The parameters for the distributions used have been compiled in Table 10.7.

Table 10.7. Parameters for failure and repair distributions

Nr.	Failure Distribution $F(t)$				Repair Distribution $G(t)$			
	b	*MTTF* [h]	T [h]	$\sqrt{Var(\tau)}$ [h]	b	*MTTR* [h]	T [h]	$\sqrt{Var(\tau)}$ [h]
1	1.0	1,000	1,000	1,000	3.5	600	666.85	189.87
2	1.5	1,000	1,107.73	678.97	3.5	600	666.85	189.87
3	2.0	1,000	1,128.38	522.72	3.5	600	666.85	189.87
4	3.0	1,000	1,119.85	363.44	3.5	600	666.85	189.87
5	4.0	1,000	1,103.26	280.54	3.5	600	666.85	189.87

Figure 10.29 shows the convergence towards the stationary value for the renewal densities $h_\infty = 1/(MTTF + MTTR) = 1/1600\ \text{h}^{-1} = 6{,}25\ 10^{-4}\ \text{h}^{-1}$. The renewal densities assume different forms dependent upon the shape parameter. The larger the shape parameter b is, the more strongly the renewal density oscillates around the stationary value h_∞.

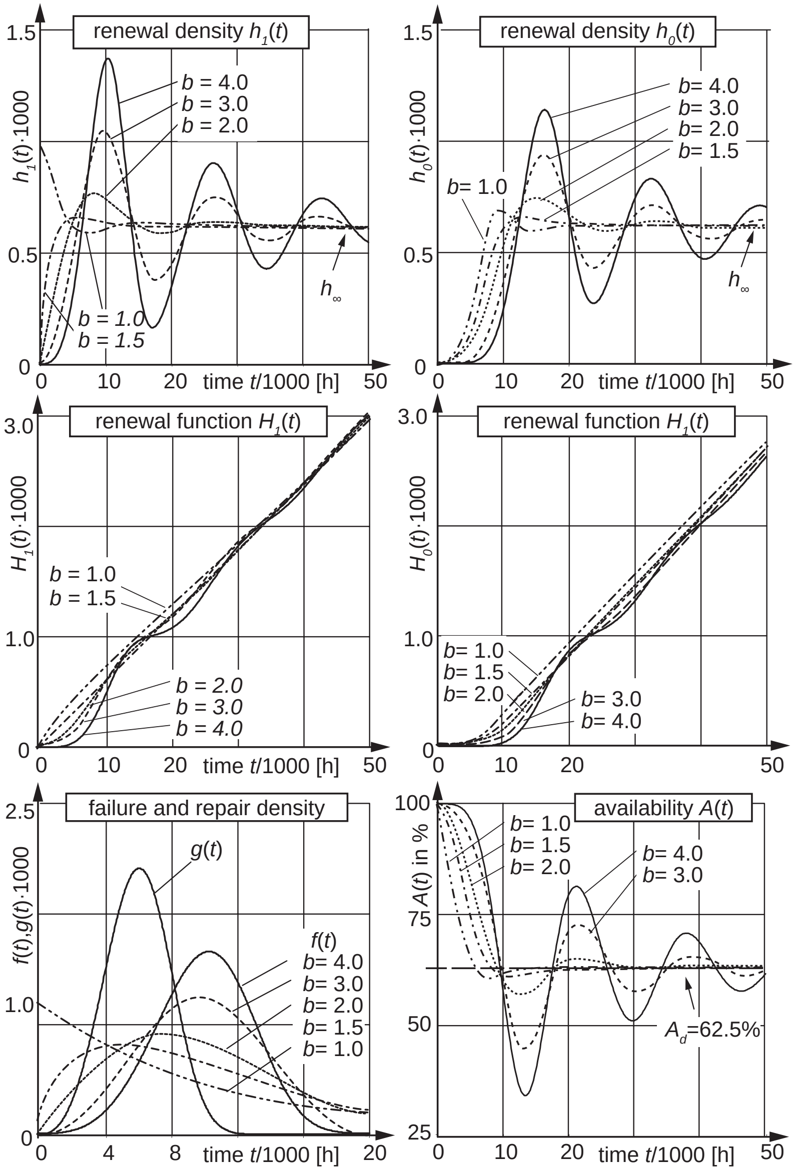


Figure 10.29. Renewal densities, renewal functions, failure density, repair density and availability for Weibull distributed failure and repair behaviours

The lower the variance is for failure distribution, the faster the renewal density swings to the convergent value.

The renewal functions show a corresponding behaviour. Here, it may be shown the convergence towards a linear asymptote according to Equations

(10.62) and (10.63), whereas the slope of the renewal functions converts towards $1/h_\infty$. The fact that the renewal functions are shifted in the horizontal direction can be accounted for by the different variances.

The availability also assumes different forms in dependency upon the shape parameter. The larger the shape parameter b is, the more the availability oscillates around the steady state availability $A_D = MTTF\ /\ (MTTF + MTTR) = 10\ /\ 16 = 62{,}5\%$.

10.4.6 Semi-Markov Processes (SMP)

For systems described by a renewal process, it is not necessary to limit the requirements to a certain distribution. However, with these processes it is only possible to realize simply structured systems. With the common Markov process it is possible to describe complicated systems. It is required however, that the systems are described by exponential distributions. The Semi-Markov process (SMP) to a certain extent combines the positive characteristics of the renewal and the Markov processes. *Lévy* [10.31] and *Smith* [10.35] were the first to formulate this process in 1954 (*Bernet* [10.4]). A summary of applications of the SMP as well as a deeper look into SMP theory with derivations and proofs can be found, for example, in *Cocozza-Thivent et al.*[10.11, 10.12] *Fahrmeir et al.* and *Gaede*.

10.4.6.1 Semi-Markov Process Procedure

SMP is a stochastic process with $m + 1$ statuses ($C_0, ..., C_m$), with the following characteristics: if the status C_i is occupied at a certain point in time t, then the next status is determined by the Semi-Markov transition probability (SMT) $Q_j(t)$. This allows for the last start time t' to be incorporated into the calculations. An example of the condition indicator function $c(t)$ under the Semi-Markov process is shown in Figure 10.30.

The distribution function of the unconditional time spent in the various statuses C_i is won by summation:

$$Q_i(t) = \sum_{j=0}^{m} Q_{ij}(t)\ . \tag{10.71}$$

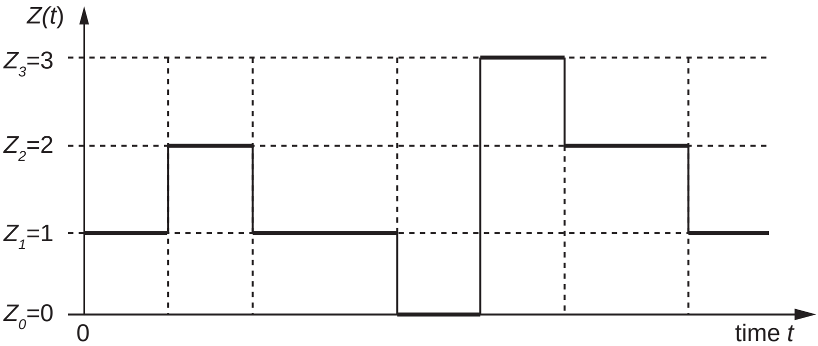


Figure 10.30. Example of the condition indicator function under the Semi-Markov process

The time spent in each status C_i including the transition to status C_j is a positive random variable with the distribution function $F_{ij}(t)$. The process, also called the Markov renewal process, is fully determined by $F_{ij}(t)$ and the given initial conditions.

In contrast to the renewal processes, the semi-Markov process allows for the modeling of more than two statuses. Together with the maintenance of repairable systems it is not only possible to recreate the state of operation and the state or failure or repair, but also further statuses can be modeled, for example, “down time due to preventive maintenance” or “waiting for the arrival of replacements parts”.

10.4.6.2 Probability Distribution and Availability

In reliability theory, the status probability is mostly of interest

$$P_{i,j}(t) = P(Z(t) = Z_j \mid Z(0) = Z_i)\,, \tag{10.72}$$

which corresponds to the probability distribution at the point in time t in the status j, if the process in the status i is started at the point in time $t = 0$. This probability function is defined by a system of integral equations, which in many references are often referred to as *Kolmogorov* equations of a SMP:

$$P_{i,j}(t) = \delta_{ij}(1 - Q_i(t)) + \sum_{k=0}^{m} \int_0^t q_{ik}(t') P_{k,j}(t - t')\mathrm{d}t'\,, \tag{10.73}$$

with the Kronecker delta $\delta_{ij} = 0$ for $j \neq i$, $\delta_{ii} = 1$ and the SMT density

$$q_{ij}(t) = \frac{dQ_{ij}(t)}{dt} . \tag{10.74}$$

For the determination of the availability it is useful to set up two complementary subsets out of the statuses for the process: Γ_S is the subset for all statuses in which the observed unit is functional and Γ_F is the subset for all statuses in which the observed unit has failed. Thus, the point availability can be calculated as follows:

$$A(t) = \sum_{j \in \Gamma_S} P_{i,j}(t) . \tag{10.75}$$

10.4.7 System Transport Theory

Dubi developed the system transport theory during his attempts to find an all-inclusive theory of availability analysis [10.16 - 10.26]. This theory is based on an analogy to particle transport theory in matter. The following sections introduce this analogy and the fundamental idea of this theory for the description of system behaviour.

10.4.7.1 Analogy to Physical Particle Transport Theory

Dubi found a closely related mathematical analogy between the physical transportation of particles in a medium and the failure and repair behaviour of a system over time [10.16]. The analogy concludes that when a particle floats in three dimensional space, it collides with other particles and undergoes a change in its status. This analogy has also been researched and published by *Devooght* [10.16], *Labeau* [10.27, 10.36] and *Lewins* [10.32, 10.49].

Within a medium in space, a particle (neutron) moves along a straight line until it collides with an atomic nucleus. The location of the collision is described by the vector r. The vector Ω describes the direction in which the neutron enters the collision. Before the collision, the particle has the energy E. The particle enters the collision with the state space vector $P = (r, \Omega, E)$. A nuclear reaction takes place when the neutron collides with the atomic nucleus, in which the particle is absorbed or repelled. In the latter case, the particle leaves the collision at the same location r, however, with a new direction Ω' and a new energy E'. Thus, the particle exits the location of collision with the state space vector $P' = (r, \Omega', E')$. After this event, the particle moves again along a straight line from r to r' where the next collision occurs. Thus, the particle enters the next collision at the point in state space $P' = (r', \Omega', E')$. This process continues until the particle is absorbed or until the particle leaves the limits of the matter. The process which controls the transportation from the event P to the next event P' is described by the neutron transport theory, which is shown schematically in Figure 10.31.

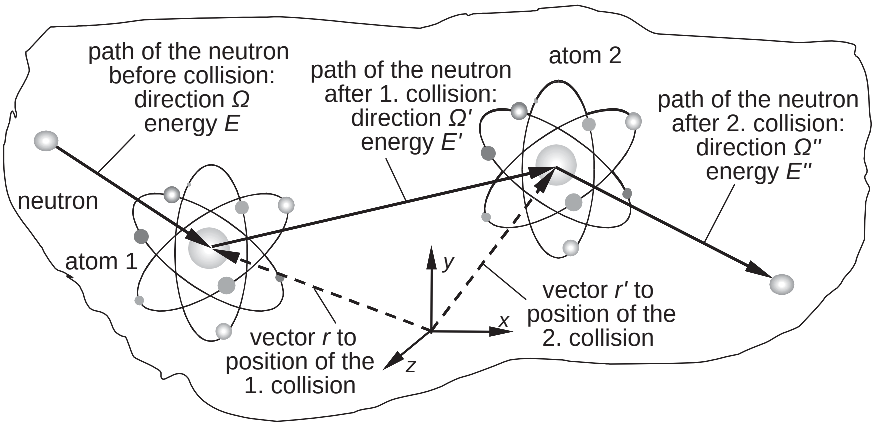


Figure 10.31. Neutron transportation process in matter

This process can be divided into two parts. The first part is the collision itself and the second part is the free flight towards the next collision. The collision itself is described by the relationship between energy and direction before and after the collision. For this purpose, the collision kernel $C(r;\Omega,E \rightarrow \Omega',E')$ is defined as the probability that a particle, which enters a collision at the point r with the direction Ω and the energy E, leaves the collision with the direction Ω' and the energy E'. The kernel $T(\Omega',E';r \rightarrow r')$ describes the free flight of the particle as a probability density that the particle, after having left a collision at r with the direction Ω' and energy E', will enter the next collision at point r'. The product of the collision kernel and the free flight kernel results in the transportation kernel:

$$K(P \rightarrow P') = C(r;\Omega;\Omega \rightarrow \Omega',E')\,T(\Omega',E';r \rightarrow r'). \tag{10.76}$$

This is the probability density that a particle, which underwent a collision at the point P, will undergo its next collision at the point P'. Based on this, the collision density or event density $\psi(P)$ is introduced, which stands for the number of collisions at the point P. The definition of the collision density is given by the Boltzmann transport equation [10.16, 10.22], which is

$$\psi(P) = S(P) + \int_P \psi(P') K(r,\Omega,\Omega \to r',\Omega',E') dP' , \tag{10.77}$$

Here, $S(P)$ is the so-called source term, which describes the first collision in the matter. The Boltzmann transport equation describes the relationships of the successive collisions and is the fundamental and only equation to be solved for the analysis of the behaviour of a particle in a medium. Analytical solutions for this equation only exist for very few and simple cases. Numerical solutions exist for one and two dimensional approximations. A complete solution, in all dimensions, is only possible with the help of the Monte-Carlo method.

10.4.7.2 General Form of the System Transport Equation

A situation in reliability theory is comparable to the situation described above in physical particle transport theory. Thus, it is possible to transpose methods from particle transport theory to reliability theory [10.16, 10.22].

A system is observed with n components. Each component is assigned a condition indicator b_i and the respective entry time τ_i, $i = 1(1)n$. The condition indicator can assume as many different values as the component can assume statuses. For example, $b_i = \{0,1,2\}$ could stand for the statuses: failed, functional and in reserve. The value τ_i stands for the point in time in which the i^{th} component enters status b_i.

All status indicators n are brought together to a system status vector B and system entry time vector τ, represented as:

$$B = (b_1, b_2, ..., b_i, ..., b_n) \text{ and} \tag{10.78}$$

$$\tau = (\tau_1, \tau_2, ..., \tau_i, ..., \tau_n) . \tag{10.79}$$

Both vectors are now combined with the continuous successive system time t to the status space vector P:

$$P = (B, \tau, t) . \tag{10.80}$$

This vector expresses that at the time t, a system is in the status B, which was achieved at the times τ_i. All possible vectors P together result to the set $\{P\}$, which is characterised as the state space for the system. The system now switches from one condition vector to the next, in an n dimensional discrete space for statuses, in dependency upon the continuous system time t.

This procedure can be illustrated as follows: At the point in time $t = t_0 = 0$, the system is in its initial status $B = B_0$. At a certain point in time t_1, a change in the system’s status occurs. This event causes a change in the condition indicator b_i of a system component. The change can then further provoke the immediate change of status in other system components. The linking of these changes must be defined by logical connections in the system model. A unique condition vector B_1 is assigned to the time t_1. The system remains in status B_1 until a new status change occurs at the time t_2. This transportation process continues until the end of the observed period of time T_{max}. Figure 10.32 shows an illustration of this transportation process. In *Dubis* terminology, this constant switching between changes in the status and free flight phases is referred to as collision and free flight. Until the time T_{max} is reached, p changes in status and p+2 free flight phases take place.

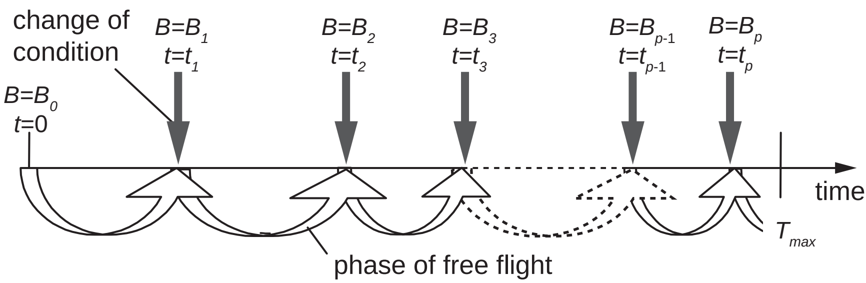


Figure 10.32. Transportation of a system in state space

This sequence of successive switching between changes in status and free flight phases can be summarized as a history of the system C_k. This history describes the random effects of the changes in status with the corresponding points in time of changes in status:

$$C_k = \left(c_1, c_2, \ldots, c_{p-1}, c_p\right) = \left((B_1, t_1), (B_1, t_1), \ldots, (B_{p-1}, t_{p-1}), (B_p, t_p)\right). \tag{10.81}$$

Since the time of the change in status as well as the change in status itself are stochastic values, the effect C_k is not an exact solution of the system transport problem, but rather just a possible sequence.

Dubi proposed the following equation for the collision density, the fundamental value for the calculation of the availability, in an equivalent form to the Boltzmann transport equation (Equation (10.77)) [10.20]:

$$\psi(B, \tau, t) = P(B_0)\delta(t) + \sum_{B'} \iint_{\tau\, t'} \psi(B', \tau', t') K(B', \tau', t' \rightarrow B, \tau, t)\mathrm{d}\tau' \mathrm{d}t' \,. \tag{10.82}$$

This transport equation represents the foundation for the calculation of the availability of systems relative to time. The availability $A(t)$ can be found in [10.20] as

$$A(t) = \sum_{B \in \Gamma_S} \int_0^t \psi(B, \tau, t') R_S(B, \tau, t') \mathrm{d}t' \,. \tag{10.83}$$

Here, the expression $R_S(B, \tau, t)$ is defined as the system state reliability and as the fictitious serial connection of the individual system components as a function of B and τ. A universal analytical assessment of this relationship has not yet been found and can only be evaluated for exceptions.

10.4.7.3 Application and Analysis of the System Transport Theory

The system transport theory allows for the modeling of complex systems possessing an arbitrary structure, arbitrary distribution function for the description of the failure and repair behaviours of the components and arbitrary interaction of the components within the system [10.18, 10.20, 10.24]. Many numerous maintenance strategies can be recreated and replacement part logistic can be taken into consideration.

The only applicable method for the solution of the system transport equations is Monte-Carlo simulation [10.17, 10.21, 10.22, 10.30]. With this system, a “game” is played with the modeled system. In other words, a large number of various flows of events are generated in many numerous simulation runs. A flow of events describes the function of the status for a system and its components over time. Based on the flow of events it is possible to determine variables of the system, for example, availability or required replacement parts.

10.4.8 Comparison of the Calculation Models

In Table 10.8, the calculation models are summarized.

Table 10.8. Comparison of the models

model	individual components	complex structure	preventive maintenance	repairs	maintenance strategy	statuses of component	complexities, dependencies	failure behaviour	repair behaviour	type of description	solution options	reliability $R(t)$	availability $A(t)$	steady state availability $A_D(t)$	replacement part demand
periodical maintenance model	●	-	●	-	-	2	-	random	-	algebraic	analytical	●	-	●	●
Markov	●	●	-	●	-	n	-	exponential	exponential	diff. eq. system	analytical	-	●	●	-
Boole-Markov	●	-	-	●	-	2	-	random	random	algebraic	analytical	-		●	-
common renewal process	●	-	-	●	-	2	-	random	-	integral system	numerical	-	-	-	●
alternating renewal process	●	-	-	●	-	2	-	random	random	integral system	numerical	-	●	●	●
Semi-Markov process	●	●	-	●	-	n	-	random	random	integral eq. system	numerical/MC simulation	-	●	●	●
system transportation theory	●	●	●	●	●	n	●	random	random	syst. trans. equations	MC Simulation	●	●	●	●

For each model, the aspects which can be considered by the system description are noted. Furthermore, the types of distribution functions which can be used for the description of failure and repair behaviour are shown. Each model possesses a specific type of describing equations. The respective solution options for the analysis of each model are also given. Lastly, the system variables or component variables which can be calculated with each model are listed.

10.5 Exercise Problems to Repairable Systems

Here, our readers can find comprehension questions for the individual chapter sections. These questions serve as a measure for the testing and understanding of the material dealt with in this chapter. These comprehension questions are followed by several calculation problems. The focus of the calculation problems is Section 10.5. Additionally, examples in Sections 10.5.1.3, 10.5.2.1, 10.5.2.2, 10.5.3 and 10.5.5.7 can be found for the individual calculation models.

10.5.1 Comprehension Questions

Section 10.1

1. How is maintenance defined?
2. What is the goal of maintenance work?
3. What are the 3 categories into which maintenance methods can be divided?
4. What is the general idea of preventive maintenance?
5. Which methods can be carried out within the scope of preventive maintenance?
6. Describe the term condition-based maintenance.
7. Which procedures can be used for status monitoring?
8. What is the general idea of corrective maintenance?
9. How can the methods for corrective maintenance be characterized?
10. What are the advantages of replacement part storage?
11. Describe the storage function for one order cycle.
12. How is the order limit chosen in replacement parts storage?
13. What can be determined by a maintenance strategy?

Section 10.2

1. What components make up life cycle costs?
2. Which lifetime phase has the greatest influence on the life cycle costs?
3. Why does a minimum occur in life cycle costs for a certain availability?

Section 10.3

1. What is supply delay time?
2. What is maintenance delay time?

3. When does the maintenance delay time reach zero?
4. Why can't supply delay time and maintenance delay time be influenced by design methods?
5. How is maintainability defined?
6. Which distribution functions are often used for the description of maintainability?
7. What can be qualitatively described by maintainability?
8. With the help of which design actions can the maintainability be improved?
9. What is the general definition of availability?
10. What is the steady state availability A_D of a system as a function of the average lifetime MTTF and the average down time $\overline{M}$?
11. What types of steady state availabilities are there?
12. Which type of steady state availability can be used as an assessment criterion for the design quality of a product? Explain your answer!
13. Which periods of time make up the average down time $\overline{M}$ when calculating the operational availability $A_D^{(o)}$? Which of these periods of time can be influenced by the manufacturer and which can be influenced by the operator?

Section 10.4:

1. The failure behaviour of an element is described by an exponential distribution. Show why periodical renewal is not able to improve the reliability of this element.
2. The failure distribution of an element is described by a Weibull distribution. For which shape parameters b is it possible to increase the average lifetime $MTTF_{PM}$ with periodical renewals?
3. The failure behaviour of an element is described by a Weibull distribution. The repair duration is estimated with an exponential distribution. Can the Markov process be used for the calculation of the availability A(t)?
4. Why is the availability always A(t) = 100 % for the common renewal process of a component?
5. Which time span is described by the alternating renewal process?
6. Why is the approximation of the renewal function $H_1(t)$ preferably used for the determination of the replacement part demand?
7. What is the fundamental variable of system transport theory for the calculation of the availability?
8. Name the only applicable method for solving system transport equations.

10.5.2 Calculation Problems

Problem 10.1:

A component has a *MTTF* value of 5,000 h. What is the maximal allowed *MTTR* value for the component, so that an availability of $A_D = 99\ \%$ is achieved?

Problem 10.2:

A serial system consists of three identical components. The *MTTF* value of one component is 1,500 hours. The system has a steady state availability of 90 %. What is the *MTTR* value of the component?

Problem 10.3:

A system consists of three identical components in parallel connection. The system has a steady state availability A_{DS} of 99.9 %. What is the steady state availability A_{Di} of one component?

Problem 10.4:

A system consists of three identical components in parallel connection. The *MTTF* value of one component is 1,500 hours. The system has a steady state availability of 99 %. What is the *MTTR* value of the component?

Problem 10.5:

A system consists of 3 components connected as shown in the reliability block diagram below. The system should reach a steady state availability A_{DS} of 95 %. The steady state availabilities A_{D2} and A_{D3} of the components 2 and 3 are both 90 %. Component 1 has an average lifetime *MTTF* of 1,000 hours.

Figure 10.33. Diagram for Problem 10.5

a) Calculate the required steady state availability A_{D1} for component 1 so that the system steady state availability is achieved.
b) Which *MTTR* value must be reached for component 1 in order to reach the required steady state availability A_{D1} from a)?

Problem 10.6:

The size of the replacement part storage should be determined for a component. The lifetime τ_1 of the component can be described by an exponential distribution with the failure rate $\lambda = 0.002$ 1/h. The repair duration τ_0 is likewise described by an exponential distribution with the repair rate $\mu = 0.1$ 1/h. Note: $\mathrm{Var}(\tau_1) = 1 / \lambda^2$ and $\mathrm{Var}(\tau_0) = 1 / \mu^2$. The storage size is characterized with I (initial stock).

a) Using the approximation equation $\hat{H}_1(t)$ from the alternating renewal process, give the general equation for the stock $S(t)$.
b) Determine the storage size I, so that the replacement part supply is guaranteed for an operation period of 8,760 hours, in other words, so that the stock $S(t)$ does not reach zero before the operational period is over.

Problem 10.7:

The failure rate $\lambda = 0.03$ 1/h and the repair rate $\mu = 0.2$ 1/h are known for a single component. The single component enters operation for the first time at the point in time $t = 0$ h.

a) Calculate the steady state availability A_D for the single component.
b) What availability $A(t)$ does the single component have at the point in time $t = 2.1$ h?

Problem 10.8:

The failure rate $\lambda = 0.01$ 1/h and the repair rate $\mu = 0.1$ 1/h are known for a single component.

a) Calculate the steady state availability A_D for the single component.
b) At what point in time t^* has the single component reached an availability of $A(t^*) = 95\ \%$?

References

[10.1] Aven T, Jensen U (1999) Stochastic Models in Reliability. Springer
[10.2] Beichelt F (1995) Stochastik für Ingenieure. Teubner, Stuttgart
[10.3] John P (1990) Statistical methods in engineering and quality assurance. Wiley, New York
[10.4] Osaki S (2002) Stochastic Moduls in Reliability and Maintenance. Springer, Berlin; Heidelberg; New York
[10.5] Bertsche B (1989) Zur Berechnung der System-Zuverlässigkeit von Maschinenbau-Produkten. Dissertation Universität Stuttgart, Institut für Maschinenelemente. Inst. Ber. 28
[10.6] Birolini A (1985) On the Use of Stochastic Processes in Modeling Reliability Problems. Habilitationsschrift, ETH Zürich, Springer, Berlin
[10.7] Birolini A (2004) Reliability Engineering. Springer, Berlin; Heidelberg

[10.8] Bitter P et al (1986) Technische Zuverlässigkeit. Herausgegeben von der Messerschmitt-Bölkow-Blohm GmbH. München, Springer
[10.9] Brumby, Lennart (2000) Marktstudie Fremdinstandhaltung 2000. Ergebnisse einer Expertenstudie des Forschungsinstituts für Rationalisierung (FIR) an der RWTH Aachen, Sonderdruck 5/2000, 1.Auflage
[10.10] Cane V R (1959) Behaviour Sequences as Semi-Markov Chains, Royal Statistic Journal, no 21, pp 36-58
[10.11] Cocozza-Thivent C (2000) Some Models and Mathematical Results for Reliability of Systems of Components, MMR 2000 (International Conference on Mathematical Methods in Reliability), Juli 2000, Bordeaux, Frankreich, in Nukulin M & Limnios N (eds): Recent Advances in Reliability Theory. Birkhäuser, pp 55-68
[10.12] Cocozza-Thivent C, Roussignol M (1997) Semi-Markov Processes for Reliability Studies. ESAIM: Probability and Statistics, May, vol 1, pp 207-223, http://www.emath.fr/ps
[10.13] Cox D R (1962) Renewal Theory. John Wiley & Sons Inc., New York
[10.14] Deutsches Institut für Normung (1985) DIN 31 051 Instandhaltung – Begriffe und Maßnahmen. Beuth, Berlin
[10.15] Devooght J (1997) Dynamic Reliability, Advances in Nuclear Science and Technology. vol 25, pp 215-279
[10.16] Dubi A (1986) Monte-Carlo Calculations for Nuclear Reactor, in Ronen, Y.(Ed): Handbook of Nuclear Reactor Calculations, CRC Press
[10.17] Dubi A (1990) Stochastic modeling of realistic systems with the Monte-Carlo Method. Tutorial notes for the annual R&M Symposium, Malchi Science corp. contract
[10.18] Dubi A (1994) Reliability & Maintainability - An Approach to System Engineering, Notes, Nucl. Eng. Department, Ben Gurion University of the Negev, Israel
[10.19] Dubi A (1997) Analytic Approach & Monte-Carlo Method for Realistic Systems. IMACS Seminar on Monte-Carlo Methods, Bruxelles, April
[10.20] Dubi A (1999) Monte-Carlo Applications in System Engineering, John Wiley & Sons Ltd., New York
[10.21] Dubi A, Gurvitz N (1995) A note on the analysis of systems with time dependent transition rates. Ann. Nucl. Energy, vol 22, no 3/4, pp 215-248
[10.22] Dubi A, Gurvitz N (1996) Aging, Availability and Maintenance Models in the System Transport Equations. Department of Nuclear Engineering, Ben Gurion University of the Negev, Beer-Sheva
[10.23] Dubi A, Gandini A, Goldfeld A, Righini R, Simonot H (1991) Analysis of non-markov systems by a Monte-Carlo Method. Ann. nucl. Energy, vol 18, no 3, pp 125-130
[10.24] Dubi A, Gurvitz N, Claasen S J (1993) The Concept of Age in System Analysis. South Africa Journal of Industrial Engineering, no 7, pp 12-23
[10.25] Ebeling C E, (1997) An Introduction to Reliability and Maintainability Engineering. McGraw-Hill

[10.26] Stroh, MB (2001) A Pracitcal Guide to Transportation and Logistics. Logistics Netroser Grochla E (1992) Grundlagen der Materialwirtschaft. Gabler

[10.27] Hendrickx I, Labeau P-E (2000) Partially unbiased estimators for unavailability calculations. Proc. ESREL 2000 Conference, 15.-17. Mai, Edinburgh, Schottland, Balkema Publishers, Rotterdam, pp 1619-1624

[10.28] Nachas, J (2005) Reliability Engineering. Taylor & Francis

[10.29] Labeau P-E (1999) The transport framework for Monte-Carlo based Reliability and Availability estimations, Workshop on Variance Reduction Methods for Weight-controlled Monte-Carlo Simulation of complex dynamical Systems. ESREL ´99-Conference, München, 13.-17. September

[10.30] Lechner G, Naunheimer H (1999) Automotive Transmissions. Springer

[10.31] Lévy P (1954) Processus semi-Markoviens. Proc. Int. Congr. Math., Amsterdam

[10.32] Lewins J D (1999) Classical Perturbation theory for Monte-Carlo Studies of System Reliability, Workshop on Variance Reduction Methods for Weight-controlled Monte-Carlo Simulation of complex dynamical Systems. ESREL ´99-Conference, 13.-17. September, München

[10.33] Lotka A J (1939) A Contribution to the Theory of Self-Renewing Aggregats, with Special Reference to Industrial Replacement. Ann. Math. Statistics, no 18, pp 1-35

[10.34] Page E S (1959) Theoretical Considerations of Routine Maintenance. The Computer Journal, vol 2, pp 199-204

[10.35] Smith W L (1954) Regenerative stochastic processes. Proc. Int. Congr. Math., Amsterdam

[10.36] Verein deutscher Ingenieure (1984) VDI-Richtlinie 4008 Blatt 8 Erneuerungsprozesse. Beuth, Berlin

[10.37] Verein deutscher Ingenieure (1986) VDI-Richtlinie 4004 Blatt 3 Kenngrößen der Instandhaltbarkeit.

[10.38] Verein deutscher Ingenieure (1986) VDI Richtlinie 4004 Blatt 4 Verfügbarkeitskenngrößen.

[10.39] Verein deutscher Ingenieure (1999) VDI-Richtlinie 2888 Zustandsorientierte Instandhaltung.

[10.40] Wu Y-F, Lewins J D (1991) System Reliability Perturbation Studies by a Monte-Carlo Method. Ann. nucl. Energy, vol 18, no 3, pp 141-146

[10.41] Zhao M (1994) Availability for Repairable Components and Serial Systems. IEEE, Transactions on Reliability, vol 43, no 2, June

[10.43] Nakagawa T (2005) Maintenance Theory of Reliability. Springer, London

[10.44] Gross, JM (2002) Fundamentals of Preventive Maintenance. Amacom, New York

11 Reliability Assurance Program

11.1 Introduction

The kernel of the excellence of engineering products in present day competitive engineering manufacture is their reliability, which has been expostulated throughout this book. The present chapter deals with a comprehensive study encapsulating the importance of optimizing reliability through analysis and design. It provides a comprehensive reliability assurance programme which should prove efficacious to both designer and product developer. It is argued that reliability assurance will in the future be the touchstone of a product's veracity and a reliability audit will be a requisite for a product's sale transaction.

The design of reliable products is conducted under marginal conditions which continually intensify (see Figure 11.3). Especially the large complexity of products and short development times require a more frequent and expanded use of reliability measures taken by the product developer. Well-engineered design methods and procedures alone are thus no longer sufficient to achieve high product reliability. The increased requirements can only be met under the application of special analytical reliability methods (see Figure 11.6). Such actions should encompass the entire product life cycle in order to optimize comprehensively. The result is a comprehensive reliability assurance program [11.2].

Controlling reliability consists of process steps made up of a succession of events which can be applied in each individual phase of a product life cycle. An example of such a procedure is illustrated in Figure 11.1.

B. Bertsche, *Reliability in Automotive and Mechanical Engineering.* VDI-Buch,
doi: 10.1007/978-3-540-34282-3_11, © Springer-Verlag Berlin Heidelberg 2008

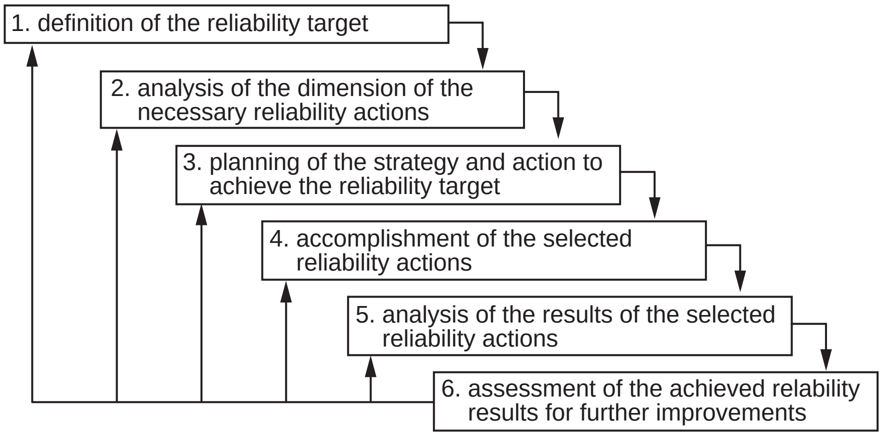


Figure 11.1. Systematical process steps for the control of reliability according to DIN EN 60300-1 [11.3]

The integration of feedback loops in the various process steps allows for a continuous improvement of the product at critical points.

A further example taken from the practical field should help to clarify in detail how the application and use of a reliability assurance program can be set up. Here, reference should be made to the procedure conditions in addition to the description of the process steps, compare with Figure 11.2.

These examples illustrate the already applied implementation of reliability methods in the development process. In the future, the necessity for reliability assurance will continue to increase and most likely be valued as a prerequisite for a successful product.

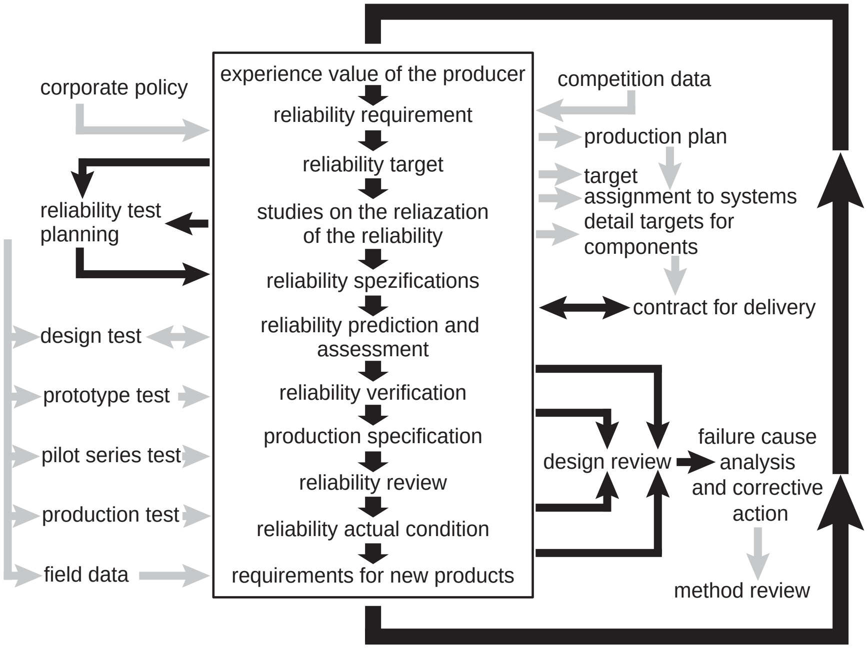


Figure 11.2. Elements of a reliability program [11.1]

In the following, the fundamentals of a reliability assurance program will be represented.

11.2 Fundamentals of the Reliability Assurance Program

11.2.1 Product Definition

Each new development begins with a planning phase where the procedure and clarification of a problem are established [11.5]. As the saying "You'll never reach a goal you don't have" implies, the first part of reliability work consists in determining the planned reliability, see Figure 11.3. Logically, this goal value set is taken from customer expectations or oriented on competition positions. Thus, for example, it is possible to determine a measurement of either percentage increase of the previously achieved reliability or of lower failure quota in relation to the competitors. In some cases statutory requirements must be adopted.

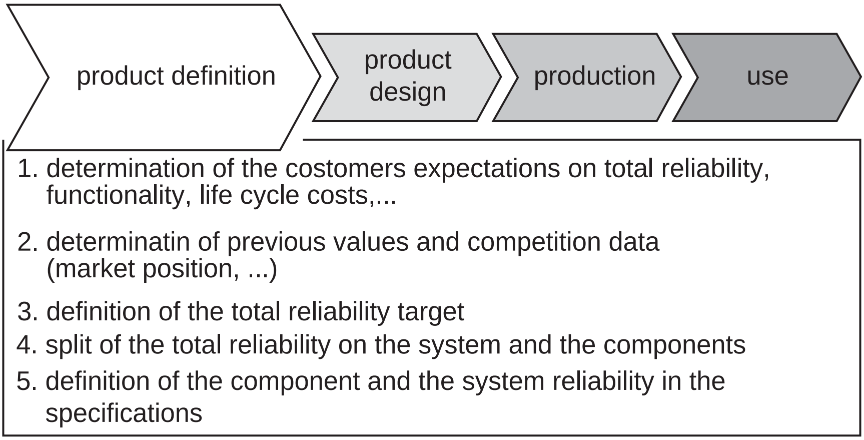


Figure 11.3. Reliability measurements during product definition

In the next step, this reliability goal must be distributed among the systems and machine components of the product. Normally, this can be easily done with the help of Boolean theory. For the most common case of a serial reliability structure and a low desired total failure quota, it is possible to estimate the total failure quota as the sum of all failure quotas for the systems and machine components.

The acquired reliability characteristics should be entered into requirements or product specifications. Here, the completeness of all entries is important. To achieve the completeness of reliability data, the definition of the term reliability is given. Thus, all decisive function and surrounding conditions must first be described, see Figure 11.4. Since this content is often a part of other specification lists, a mere reference is sufficient.

SPECIFICATIONS for [component/system] **chapter reliability**
1. functional and environmetal conditions (possibly q.v. chapter ...) 2. definition of the failures failure mode and resultant effects for the total system 3. reliability requirements maximum tolerable failure rate and/or B_{10}-value etc. 4. reliability verification test bench run, verification procedure, test conditions and test period

Figure 11.4. Requirement catalogue chapter: Reliability

The definition of failures is of importance since reliability characteristics are assigned to each failure. Common values used in reality are the survival probability $R(t)$, the failure rate $\lambda(t)$, a B_x lifetime (B_{10} lifetime), a *MTBF* value (mean time between failures) or simply a failure quota (mostly a ppm value), see also Chapter 2. Sometimes it is possible that one reliability characteristic covers several failure types of a component. The exact choice of a value is based on the desired exactness and on branch-specific circumstances. The specification catalogue is concluded by information concerning reliability proofs to be carried out. This proof should document the achieved product reliability at certain points. This can normally be done with experimental testing and trials.

11.2.2 Product Design

Product design is the most important phase for a product developer. In this phase the product is planned, is designed and all details are worked out. Many various reliability measurements are taken; mostly dealing with special methods for the analysis and optimization of reliabilities, see Figure 11.5 and the previous chapter of this book. Reliability methods can be divided into quantitative and qualitative methods.

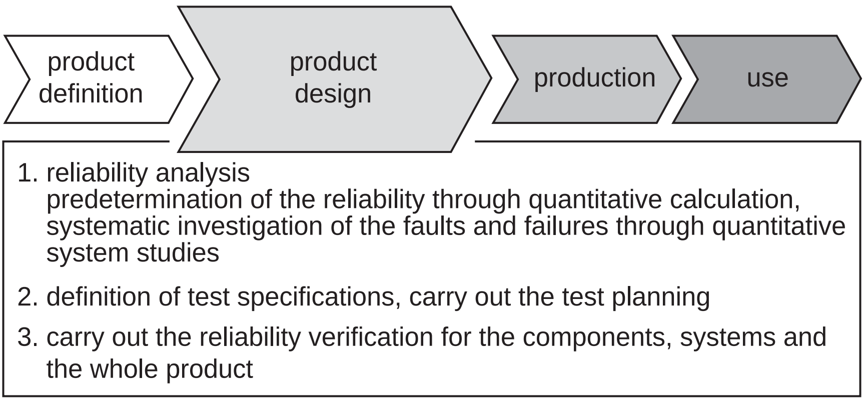


Figure 11.5. Reliability measurements taken during the product design phase

Qualitative methods determine all possible faults and failures along with their consequences and effects with the help of planned and systematic procedures.

In most cases, a qualitative ranking of weak points is gained. Figure 11.6 offers an overview of the most common methods used.

The most well known qualitative method of today is FMEA (Failure Mode and Effects Analysis), which has obtained a wide application in the practical field.

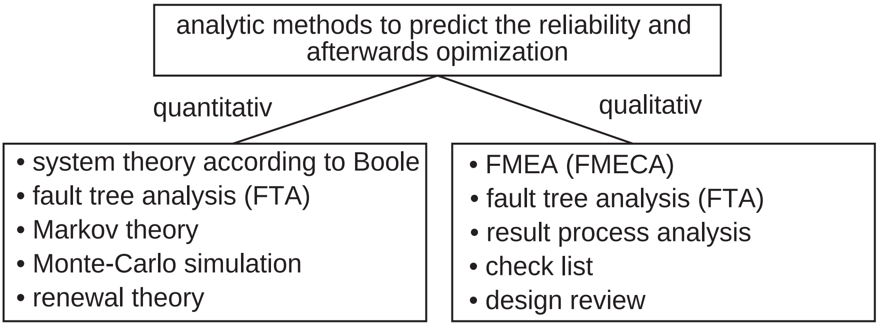


Figure 11.6. Overview of qualitative and quantitative reliability

Quantitative methods directly produce probability values for the expected reliability with the help of calculation models. These methods are based on terms and procedures from statistics and probability theory. For quantitative methods it is first necessary to know the failure behaviour of system elements and their connections to one another, in order to be able to determine exact reliability characteristics. Here, it is important that the respective system theory is suitable for the corresponding situation, see Figure 11.7.

$$R_{System} = f\,(\,R_{Systemelement\ 1},\ R_{Systemelement\ 2},\ \ldots\,)$$

system modeling / system theory	failure behaviour system element/ component distribution
Boolean model: $R_S(t)=\sum_{j=1}^{m}\varphi_S(j)\left(x(j)\right)\cdot\prod_{j=1}^{n}\left(R_i(t)\right)^{x_i(j)}\cdot\left(1-R_i\right)^{1-x_i(j)}$	Weibull distribution: $R(t)=e^{-\left(\frac{t-t_0}{T-t_0}\right)^{b}}$
Markov process: $\frac{dP_i(t)}{dt}=-\sum_{j=1}^{n}\alpha_{ij}\cdot P_i(t)+\sum_{j=1}^{n}\alpha_{ij}\cdot P_j(t)$	Exponential distribution: $R(t)=e^{-\lambda t}$
Monte Carlo simulation: $A(t)=\sum_{B\in\Gamma_S}\int_{0}^{1}\psi(B,\tau)\cdot R_S(B,t-\tau)\cdot d\tau$	Gausian distribution: $R(t)=\frac{1}{\sigma\cdot\sqrt{2\cdot\pi}}\cdot\int_{t}^{\infty}e^{-\frac{(\tau-\mu)^2}{2\cdot\sigma^2}}\cdot d\tau$
renewal theory: $h(t)=f(t)+\int_{0}^{1}h(\tau)\cdot f(t-\tau)\cdot d\tau$	Lognormal distribution: $R(t)=\frac{1}{\sigma\cdot\sqrt{2\cdot\pi}}\cdot\int_{t}^{\infty}\frac{1}{(\tau-t_0)}\cdot e^{-\frac{1}{2}\left(\frac{\ln(\tau-t_0)-\mu}{\sigma}\right)^2}\cdot d\tau$
....	....

Figure 11.7. Determination of quantitative system reliability

Diverse mathematical descriptions exist for element distributions as well as for system models. These mathematical models have a wide range of application and are highly sophisticated, thanks to the careful research done in the area of mathematical-theoretical description. They make up the foundation of all system reliability calculations. However, certain enhancements and especially practical application should continue to be investigated. In mechanical engineering, the Weibull distribution and the Boolean model are used the most often [11.4].

These analyses describe the expected reliability behaviour of machine components as well as entire systems. Figure 11.8 shows an example for a bus transmission, see also Figure 11.9. The analysis results can be used to improve weak points in the reliability as well as to optimize costs for components whose role in the reliability are uncritical.

In the product design phase it is necessary, not only to carry out theoretical trials, but also to prove the reliability in the product specifications. For this, exact testing information must be obtained and corresponding tests must be conducted. A reliability proof should include at least critical systems and components.

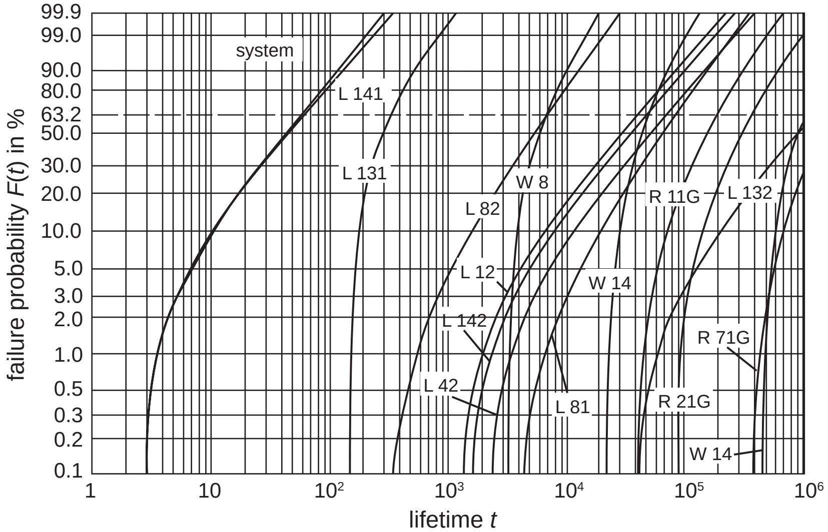


Figure 11.8. Failure behaviour of a transmission (Figure 11.9) and the system elements in the Weibull chart [11.2]

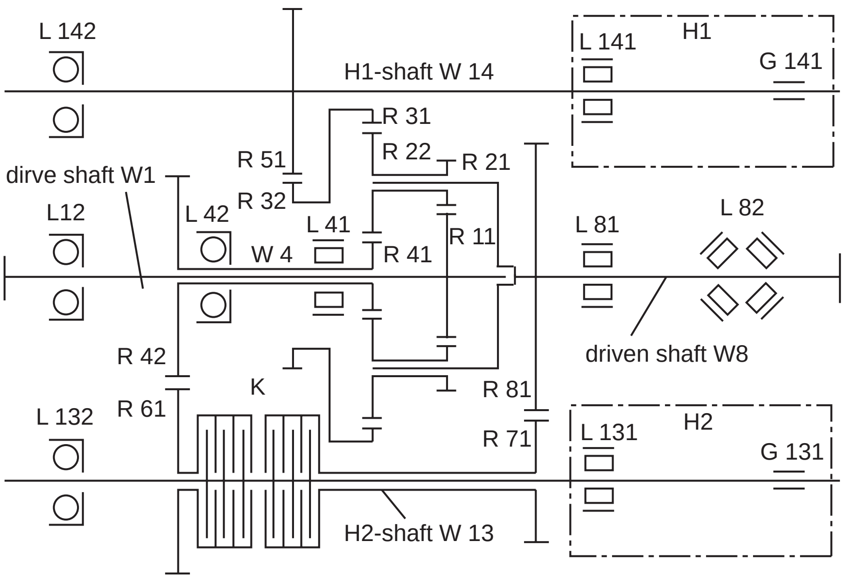


Figure 11.9. Scheme of a transmission for a mechanical power splitter transmission with hydraulic coupled hydro units (H1, H2) for continuously variable ratio. Drive torque T_{max} = 900 Nm, ratio i_{gmax} = 14 [11.2]

11.2.3 Production and Operation

The product life phases "production" and "operation" cannot be directly, but rather indirectly traced back to the designer. Figure 11.10 shows a summary of reliability actions in these phases. During production, the reliability of the processes must be ensured and tested for assemblies and the final product in appropriate audits. Failures which occur during production and which are directly related to the design of the product are of special interest to the product developer, as they must lead to a revision of all the previously completed reliability work.

Failures which occur during the operation phase, the field data, are equally important. These failures show the actual failure behaviour of a product. By analyzing these failures it is possible to compare them with the prognosis of achievable reliability which was done during the design stage. Thus, reliability calculations can be improved and reliability information can be derived for future products.

In the operation phase, the final reliability proof takes place, since here the most failures can be observed. The ideal case is the achievement of the given target reliability.

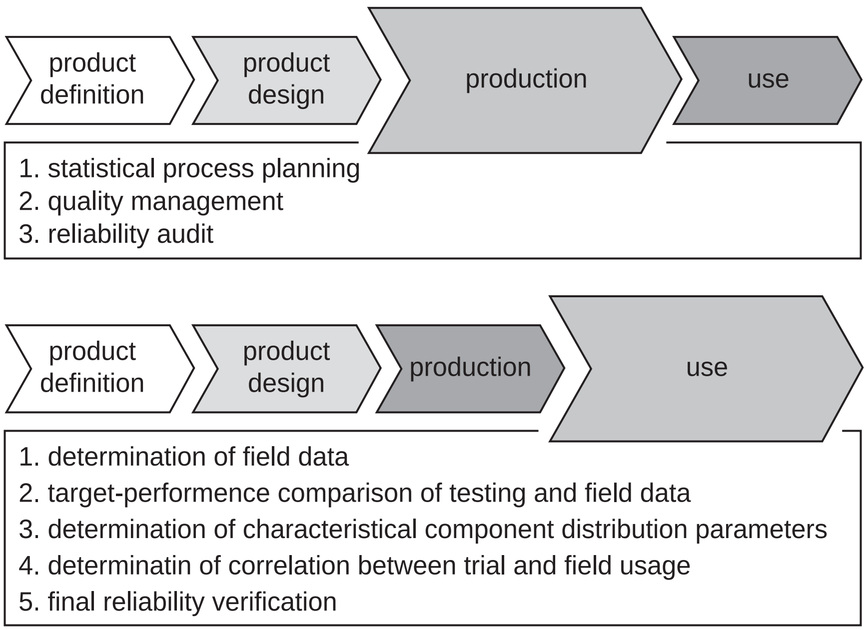


Figure 11.10. Reliability measurements during production and operation

11.2.4 Further Actions in the Product Design Cycle

Further supporting measurements can accompany those measurements focused on the product life cycle. The most important of these are:

- Setup of a comprehensive reliability data system as a foundation for prognosis calculations and feedback systems;
- Further education of employees on topics regarding reliability;
- Information system for the management and employees about reliability work (newsletter, reports, summaries, …);
- Further research of reliability methods and counselling during their application;
- Use of computers, which includes the introduction and use of analysis programs, CAD/CAE and product lifetime systems.

For the most efficient results, these further methods should already be implemented in the product design cycle. Their proper use and establishment in product design assures optimal reliability during the entire development process.

11.3 Conclusion

Reliability belongs to the most important characteristics of the quality of a product or process. Reliability actions can already prove to be profitable and effective in stages of product innovation and design. Here, high performance methods are available which can be directly applied in practical situations. A complete process observation covering all phases in the product life cycle is required in order to fulfil the high and always increasing reliability demands. Thus, it is necessary to develop a reliability assurance program whose essential elements have been described. Here, for the product developer, the determination of the reliability targets, the exact definition of specifications for reliability variables and actions as well as the introduction of existing reliability analyses during the product design should be included. The introduction and improvement of quantitative methods should especially be supported.

References

[11.1] Allen A T (1985) Die Straße der Zuverlässigkeit: eine Übersicht zur Zuverlässigkeitstechnik im Zusammenhang mit Kraftfahrzeugen. Joint Research Comitte, Zuverlässigkeitsgruppe

[11.2] Bertsche B, Marwitz H, Ihle H, Frank R (1998) Entwicklung zuverlässiger Produkte. Konstruktion 50

[11.3] Deutsches Institut für Normung (2004) DIN EN 60300 Teil 1 Zuverlässigkeitsmanagement. Deutsche Fassung EN 60300-1:2003. Beuth, Berlin

[11.4] Lechner G (1994) Zuverlässigkeit und Lebensdauer von Systemen, Jahresband der Universität Stuttgart

[11.5] Pahl G, Beitz W (2003) Konstruktionslehre: Grundlagen erfolgreicher Produktentwicklung; Methoden und Anwendung. Springer, Heidelberg Berlin

[11.6] Verband der Automobilindustrie (2000) VDA 3.2 Zuverlässigkeitssicherung bei Automobilherstellern und Lieferanten. VDA, Frankfurt

Solutions

Solution 2.1

a) Classification

First, it is advisable for evaluation to sort the failure times according to their value (LC = load cycle):

t_1= 59,000 LC ,	t_2= 66,000 LC ,	t_3= 69,000 LC,	t_4= 80,000 LC,
t_5= 87,000 LC,	t_6= 90,000 LC,	t_7= 97,000 LC,	t_8= 98,000 LC,
t_9= 99,000 LC,	t_{10}= 100,000 LC,	t_{11}= 107,000 LC,	t_{12}= 109,000 LC,
t_{13}= 117,000 LC,	t_{14}= 118,000 LC,	t_{15}= 125,000 LC,	t_{16}= 126,000 LC,
t_{17}= 132,000 LC,	t_{18}= 158,000 LC,	t_{19}= 177,000 LC,	t_{20}= 186,000 LC.

Number of classes for the class division according to the approximation equation (2.3): $n_C \approx \sqrt{n} = \sqrt{20} = 4.5$.

Chosen: 5 classes for a better visualization of the distribution.

Calculation of the class size:

$$\Delta_C = \frac{t_{20} - t_1}{n_C} = \frac{186{,}000\,\text{load cycles} - 59{,}000\,\text{load cycles}}{5} = 26{,}000\,\text{load cycles}.$$

Starting with the shortest failure time, the following classes can be set up:

Class 1:	59,000 load cycles	...	85,000 load cycles,
Class 2:	85,000 load cycles	...	111,000 load cycles,
Class 3:	111,000 load cycles	...	137,000 load cycles,
Class 4:	137,000 load cycles	...	163,000 load cycles,
Class 5:	163,000 load cycles	...	189,000 load cycles.

b) Density function

Number of failures and the relative frequency according to Equation (2.2) in the individual classes:

Class 1:	4 failures;	$h_{\text{rel},1}$	=	4/20	=	20%,
Class 2:	8 failures;	$h_{\text{rel},2}$	=	8/20	=	40%,
Class 3:	5 failures;	$h_{\text{rel},3}$	=	5/20	=	25%,
Class 4:	1 failure;	$h_{\text{rel},4}$	=	1/20	=	5%,
Class 5:	2 failures;	$h_{\text{rel},5}$	=	2/20	=	10%.

For the histogram of the failure frequencies and the empirical density function $f^*(t)$ see figure below.

c) Failure Probability

The histogram of the cumulative frequency and the empirical failure probability $F^*(t)$ are calculated by the addition of the failure frequencies according to Equation (2.8):

Class 1:	cumulative frequency	H_1	=	$h_{rel,1}$	=	20%	=	20%,
Class 2:	cumulative frequency	H_2	=	$H_1 + h_{rel,2}$	=	20% + 40%	=	60%,
Class 3:	cumulative frequency	H_3	=	$H_2 + h_{rel,3}$	=	60% + 25%	=	85%,
Class 4:	cumulative frequency	H_4	=	$H_3 + h_{rel,4}$	=	85% + 5%	=	90%,
Class 5:	cumulative frequency	H_5	=	$H_4 + h_{rel,5}$	=	90% + 10%	=	100%.

For the histogram of the cumulative frequency and the empirical failure probability $F^*(t)$ see figure below.

d) Survival probability

The simplest way to calculate the survival probability is with Equation (2.11) where the survival probability is the complement to the failure probability.

Class 1:	survival probability	$R_1^* = 100\% - H_1 = 100\% - 20\%$	$= 80\%$,
Class 2:	survival probability	$R_2^* = 100\% - H_2 = 100\% - 60\%$	$= 40\%$,
Class 3:	survival probability	$R_3^* = 100\% - H_3 = 100\% - 85\%$	$= 15\%$,
Class 4:	survival probability	$R_4^* = 100\% - H_4 = 100\% - 90\%$	$= 10\%$,
Class 5:	survival probability	$R_5^* = 100\% - H_5 = 100\% - 100\%$	$= 0\%$.

For the histogram of the survival probability and the empirical survival probability $R^*(t)$ see figure below.

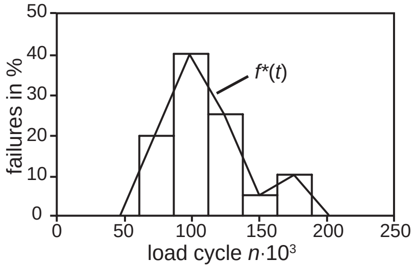


Figure Solution to Problem 2.1b

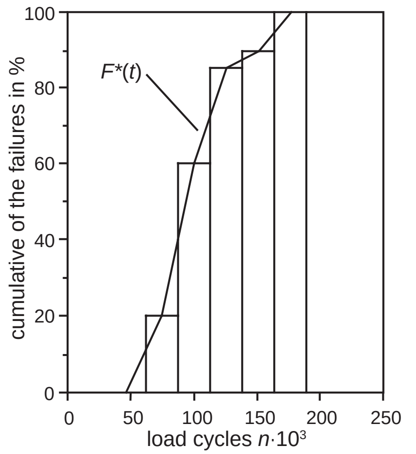


Figure Solution to Problem 2.1c

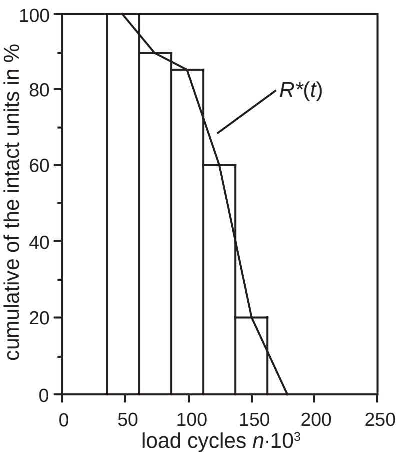


Figure Solution to Problem 2.1d

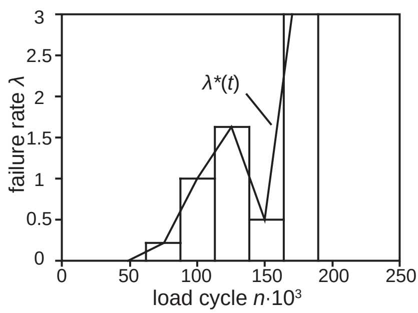


Figure Solution to Problem 2.1e

e) Failure rate
To determine the failure rate, the already calculated relative failure frequencies and the survival probability can be used. The failure rate is the quotient of these two values according to Equation (2.12):

Class 1:	failure rate	λ_1	=	$h_{rel,1}/R^*_1$	=	20% / 80%	=	0.25,
Class 2:	failure rate	λ_2	=	$h_{rel,2}/R^*_2$	=	40% / 40%	=	1.00,
Class 3:	failure rate	λ_3	=	$h_{rel,3}/R^*_3$	=	25% / 15%	=	1.67,
Class 4:	failure rate	λ_4	=	$h_{rel,4}/R^*_4$	=	5% / 10%	=	0.50,
Class 5:	failure rate	λ_5	=	$h_{rel,5}/R^*_5$	=	10% / 0%	=	∞.

For the histogram of the survival probability and the empirical survival probability $\lambda^*(t)$ see figure above.

Solution 2.2

a) Mean, median and mode (measures of central tendency)
The empirical arithmetic mean according to Equation (2.14) is:

$$t_m = \frac{t_1 + t_2 + ... + t_n}{n} = \frac{59 + 66 + ... + 186}{20} \cdot 10^3 \text{ load cycles} = 110.000 \text{ load cycles}.$$

The median can be most easily calculated with the empirical failure probability $F^*(t)$ from the solution to Problem 2.1c as the intersection with the 50% line of the cumulate frequency. Thus, the median for the trial shafts is

$$t_{median} \approx 95.000 \text{ load cycles}.$$

The mode t_{mode} is the failure time corresponding to the maximum of the density function and can thus be determined out of the solution to Problem 2.1a. The mode for the trial shafts is t_{mode} = 98.000 load cycles.

b) Variance and standard deviation (statistical variables)
The empirical variance of the trial set is calculated with the Equation (2.15) to:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(t_i - t_m)^2$$

$$= \frac{1}{19}\left[(59-110)^2 + (66-110)^2 + ... + (186-110)^2\right] \cdot 10^6 \text{ load cycles}^2 \text{ Th}$$

$$= 1.170.400.000 \text{ load cycles}^2.$$

e empirical standard deviation is the square root of the variance:

$$s = \sqrt{s^2} = 34.200 \text{ load cycles}.$$

Solution 2.3

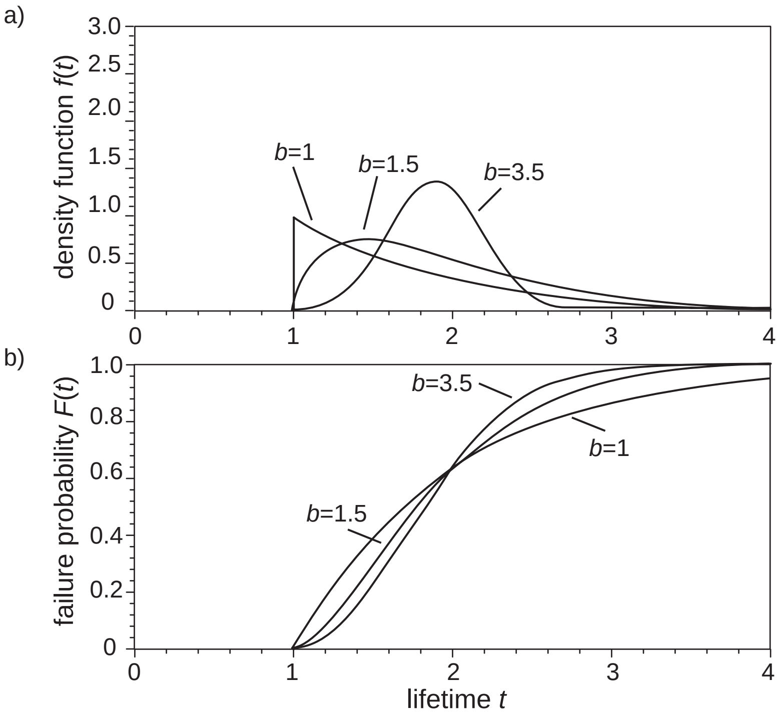


Figure Solution to Problem 2.3

Solution 2.4

Calculation using the conversion table:

$$F(t) = \int_0^t f(\tau)\cdot d\tau \overset{a\le t\le b}{=} \int_a^t \frac{1}{b-a}\cdot d\tau = \left.\frac{\tau}{b-a}\right|_a^t = \frac{t}{b-a} - \frac{a}{b-a} = \frac{t-a}{b-a}$$

$$F(t) = \begin{cases} 0 & \text{for} \quad t < a \\ \dfrac{t-a}{b-a} & \text{for} \quad a \le t \le b \\ 1 & \text{for} \quad t > b \end{cases}$$

$$R(t) = 1 - F(t) \overset{a \le t \le b}{=} 1 - \frac{t-a}{b-a} = \frac{b-a+a-t}{b-a} = \begin{cases} 1 & \text{for } t < a \\ \dfrac{b-t}{b-a} & \text{for } a \le t \le b \\ 0 & \text{for } t > b \end{cases}$$

$$\lambda(t) = \frac{f(t)}{R(t)} = \begin{cases} \dfrac{1}{b-t} & \text{for } a \le t \le b \\ 0 & \text{otherwise} \end{cases} \qquad (\hat{=}\ \text{hyperbola})$$

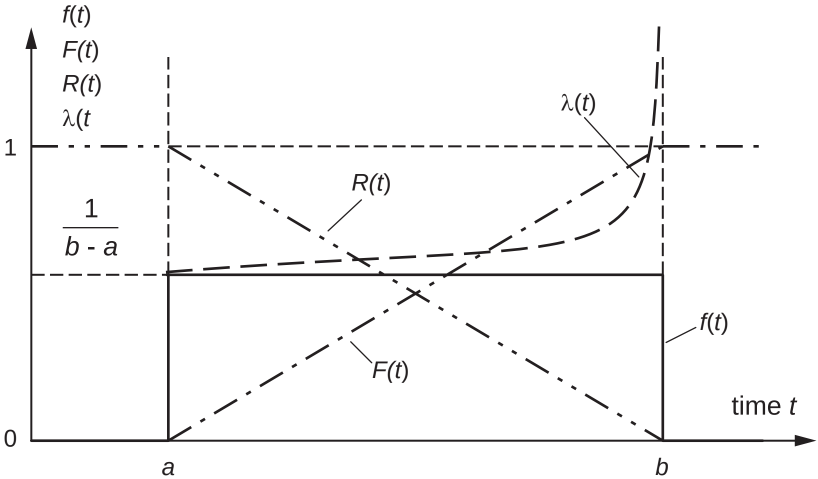


Figure Solution to Problem 2.4

Solution 2.5

The Rayleight distribution corresponds to a two parametric ($t_0 = 0$) Weibull distribution with the shape parameter $b = 2.0$ and a characteristic lifetime of $T = \dfrac{1}{\lambda}$.

The calculation of the reliability variables is executed with the help of the conversion table:

$$F(t) = 1 - R(t) = 1 - \exp\left(-(\lambda \cdot t)^2\right) \qquad t \ge 0$$

$$f(t) = \frac{dF(t)}{dt} \overset{\text{chain rule}}{=} \frac{d\left(-(\lambda \cdot t)^2\right)}{dt} \cdot \frac{dF(t)}{d(-(\lambda \cdot t)^2)} =$$

$$= -2 \cdot (\lambda \cdot t) \cdot \lambda \cdot \left(-exp(-(\lambda \cdot t)^2\right) = 2 \cdot \lambda^2 \cdot t \cdot \left(exp(-(\lambda \cdot t)^2\right)$$

$$\lambda(t) = \frac{f(t)}{R(t)} = 2 \cdot \lambda^2 \cdot t \qquad \text{(linear increasing failure rate)}$$

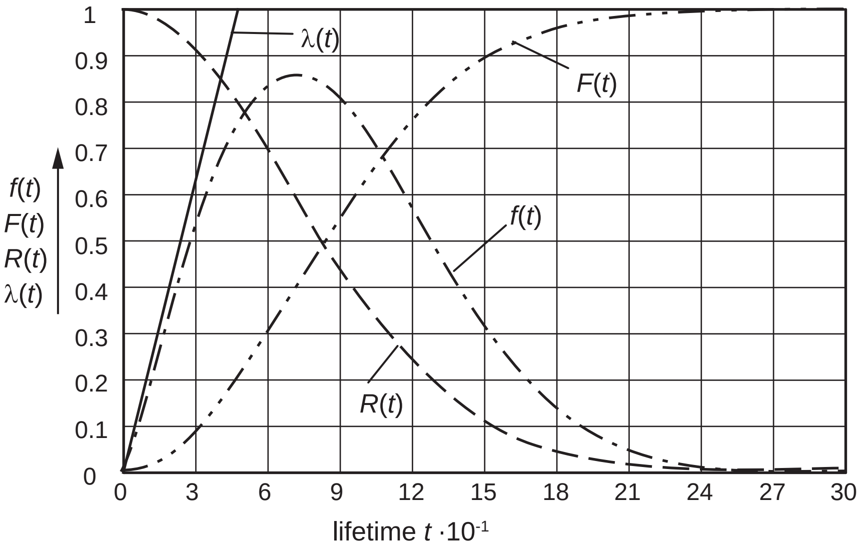


Figure Solution to Problem 2.5

Solution 2.6

a) See graphic: normal distribution network

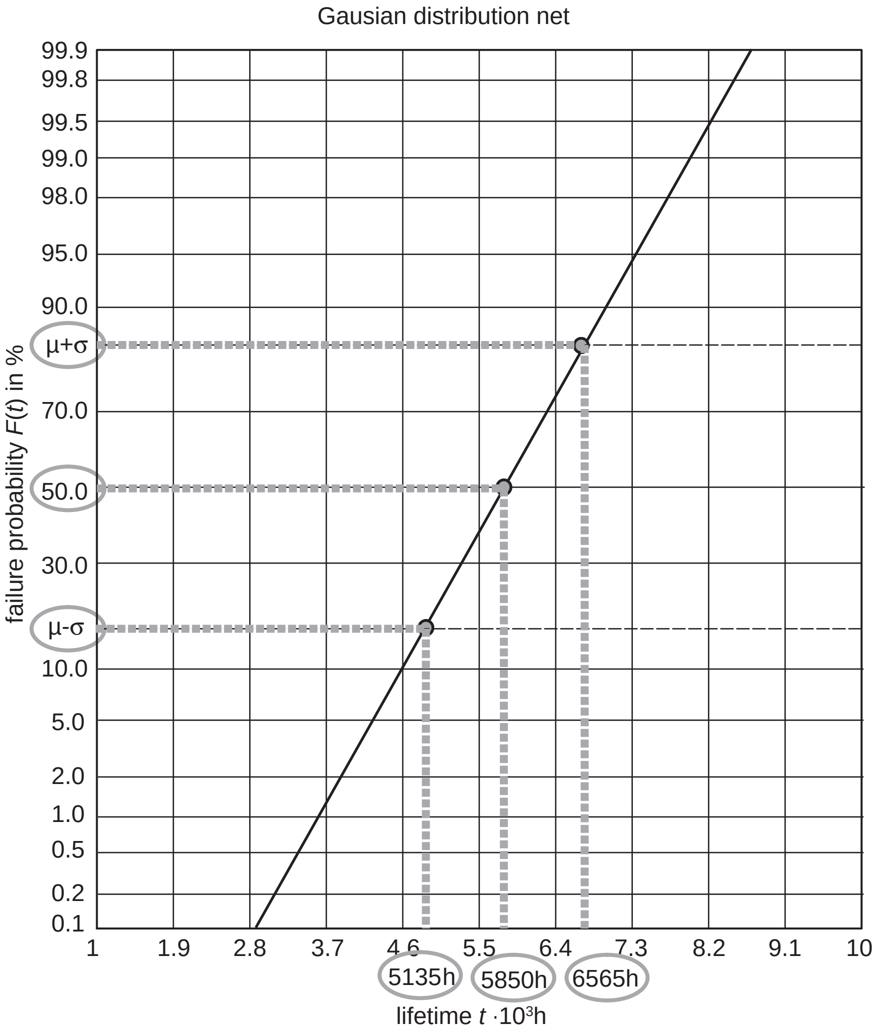


Figure Solution to Problem 2.6a

Procedure for using the normal distribution network:
1) Draw in μ: t = 5,850h; F =50%
2) Draw in μ + σ: t = 5,850h + 715h = 6,565h; F = 84%
3) Draw in μ - σ: t = 5,850h - 715h = 5,135h; F = 16%
4) Connect the three points to make a line

b) Searched for: $P(t > t_1) = 1 - P(t \le t_1) = 1 - F(t_1) = R(t_1)$

Transformation: $x_1 = \dfrac{t_1 - \mu}{\sigma} = \dfrac{4500\text{h} - 5850\text{h}}{715\text{h}} = -1.8882$

Value for $F(t_1)$ taken from the table:

$$F(t_1) = \phi(-1.8882) = 1 - \phi(1.8882) = 1 - 0.9699 = 0.0301 \approx 3\%$$
$$\underline{\underline{R(t_1)}} = 1 - F(t_1) = 0.9699 \mathrel{\hat{=}} \underline{\underline{96.99\%}}$$

c) Searched for: $P(t \le t_2) = F(t_2)$

Transformation: $x_2 = \dfrac{t_2 - \mu}{\sigma} = \dfrac{6{,}200 - 5{,}850}{715} = 0.4895$

$$\underline{\underline{F(t_2)}} = \phi(0.4895) = 0.6879 \mathrel{\hat{\approx}} \underline{\underline{68.8\%}}$$

d) $\mu + \sigma = 6{,}565\text{h} = t_u$ $\qquad$ $\mu - \sigma = 5{,}135\text{h} = t_0$

Searched for:
$$P(t_u \le t \le t_0) = F(t_0) - F(t_u) = P(5{,}135 \le t \le 6{,}565) = F(6{,}565) - F(5{,}135)$$
Transformations:
$$x_u = \frac{t_u - \mu}{\sigma} = \frac{5{,}135 - 5{,}850}{715} = -1 \quad \text{and} \quad x_0 = \frac{t_0 - \mu}{\sigma} = \frac{6{,}565 - 5{,}850}{715} = 1$$
$$\underline{\underline{P(t_u \le t \le t_0)}} = \phi(x_0) - \phi(x_u) = \phi(x_0) - (1 - \phi(-x_u)) = \phi(1) - 1 + \phi(1)$$
$$= 2 \cdot \phi(1) - 1 = 2 \cdot 0.8413 - 1 = 0.6826 \mathrel{\hat{=}} \underline{\underline{68.26}}$$

e) Required condition: $P(t_3 < t) = 1 - F(t_3) \overset{!}{=} 0.9$; thus x_3 is needed, then t_3 results from inverse transformation!

Out of the table: $\phi(x_3) = 0.1$? $\qquad$ doesn't exist in the table!!

With the help of this equation:
$$\phi(x_3) = 1 - \phi(-x_3) = 0.1 \Rightarrow \phi(-x_3) = 0.9 \Rightarrow -x_3 = 1.28 \Rightarrow x_3 = -1.28$$
Inverse transformation: $t_3 = x_3 \cdot \sigma + \mu = -1.28 \cdot 715\text{h} + 5{,}850\text{h} = 4{,}934.8\text{h}$

Solution 2.7

a) Note that for the LNV it is necessary that:

$t_{0.5} = exp(\mu)$ and $t_{\mu \pm \sigma} = exp(\mu \pm \sigma)$.

Thus:
$$t_{10.5} = exp(\mu) = exp(10.1) = 24{,}343\text{h}; \quad F = 50\%$$
$$t_{\mu+\sigma} = exp(\mu + \sigma) = exp(10.1 + 0.8) = 54176.4; \quad F = 84\%$$
$$t_{\mu-\sigma} = exp(\mu - \sigma) = exp(10.1 - 0.8) = 10938; \quad F = 16\%$$
Draw in the straight line (see graphic lognormal network)

b) Searched for: $P(t_1 < t) = 1 - P(t_1 \ge t) = 1 - F(t_1) = R(t_1)$

Transformation: $x_1 = \dfrac{ln(t_1) - \mu}{\sigma} = \dfrac{ln(10{,}000\text{h}) - 10.1}{0{,}8} = -1.112$

$\phi(x_1) = 1 - \phi(-x_1) = 1 - \phi(1.112) = 1 - 0.8665 = 0.1335 \hat{=} 13.35\%$,

thus $\underline{\underline{R(t_1)}} = 1 - F(t_1) = \underline{\underline{86.55\%}}$

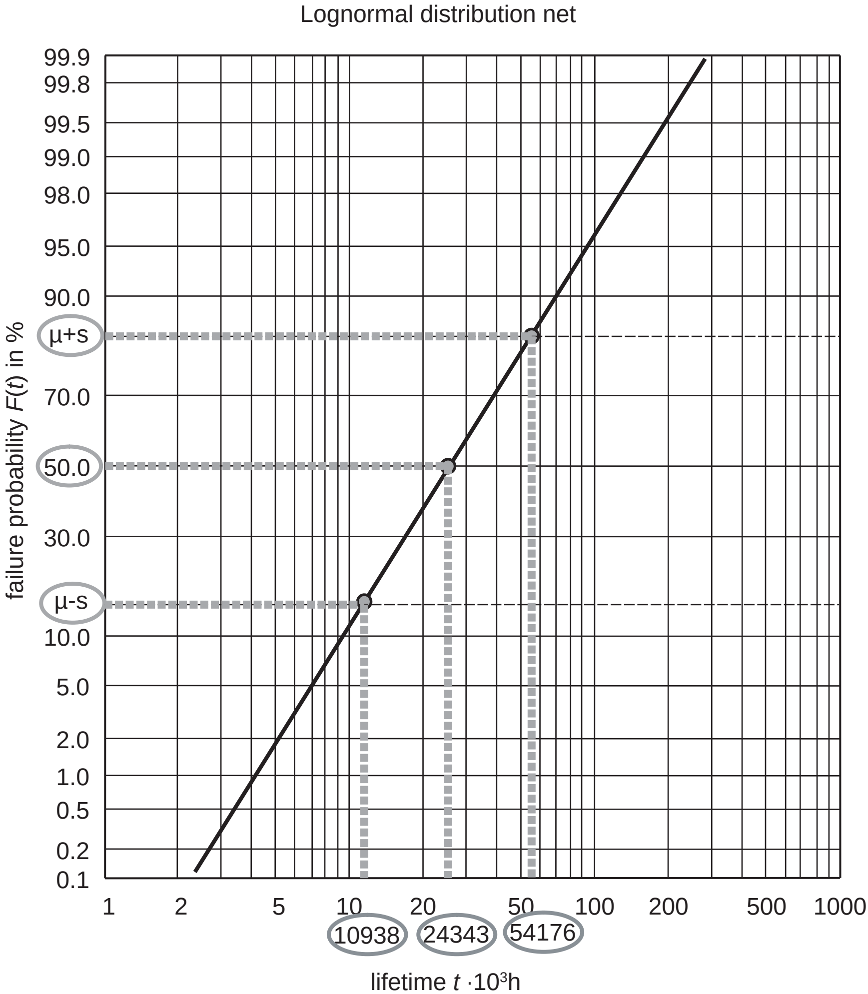


Figure Solution to Problem 2.7a

c) Searched for: $P(t_2 \geq t) = F(t_2)$

Transformation: $x_2 = \dfrac{ln(t_2) - \mu}{\sigma} = \dfrac{ln(35000) - 10.1}{0.8} = 0.4538$

$\phi(x_2) = \phi(0.4538) = 0.6736 \hat{=} 67.36\%$, thus $\underline{\underline{F(t_2) = 67.36\%}}$

d) Searched for:

$$\underline{\underline{P(t_1 \le t \le t_2)}} = F(t_2) - F(t_1) = 0.6736 - 0.1335 = 0.5401 \mathrel{\hat{=}} \underline{\underline{54.01\%}}$$

e) Required condition: $P(t_3) = 1 - F(t_3) \overset{!}{=} 0{,}9 \Rightarrow F(t_3) = 0{,}1$.

Thus, is x_3 needed, then t_3 results from inverse transformation.
Out of the table $\phi(x_3) = 0.1$?

Does not exist in the table but it is known that $\phi(x_3) = 1 - \phi(-x_3) = 0.1$

and thus $\phi(-x_3) \overset{!}{=} 0.9 \Rightarrow -x_3 = 1.28 \Rightarrow x_3 = -1.28$

Inverse transformation:

$$\underline{\underline{t_3}} = exp(\mu + x_3 \cdot \sigma) = exp(10.1 - 1.28 \cdot 0.8) = \underline{\underline{8{,}742.92\text{h}}}$$

Solution 2.8

a) Searched for:

$$\underline{\underline{P(t_1 \le t)}} = 1 - F(t_1) = R(t_1) = exp(-\lambda \cdot t_1) = exp(-\frac{200\text{h}}{500\text{h}}) = 0.6703 \mathrel{\hat{=}} \underline{\underline{67.03\%}}$$

b) Searched for:

$$\underline{\underline{P(t_2 \ge t)}} = F(t_2) = 1 - exp(-\lambda \cdot t_2) = 1 - exp(-\frac{100}{500}) = 0.1813 \mathrel{\hat{=}} \underline{\underline{18.13\%}}$$

c) Searched for:

$$\underline{\underline{P(t_3 \le t \le t_4)}} = F(t_4) - F(t_3) = 1 - exp(-\lambda \cdot t_4) - 1 + exp(-\lambda \cdot t_3)$$

$$= -exp(-\frac{300}{500}) + exp(-\frac{200}{500}) = -0.5488 + 0.6703 = 0.1215$$

$$\mathrel{\hat{=}} \underline{\underline{12.15\%}}$$

d) Required condition:

$$P(t_5 < t) = 1 - P(t_5 \ge t) = 1 - P(t_5 \ge t) = 1 - F(t_5)$$

$$= R(t_5) = exp(-\lambda \cdot t_5) \overset{!}{=} 0.9$$

$$\Rightarrow \underline{\underline{t_5}} = -\frac{ln(0.9)}{\lambda} = -ln(0.9) \cdot 500\text{h} = \underline{\underline{52.68\text{h}}}$$

With at least 90%: all times $t \le t_5$

e) Required condition: $P(50 \le t) = R(50) = exp(-\lambda \cdot 50) \overset{!}{=} 0.9$

$$\Rightarrow \underline{\underline{\lambda}} = -\frac{ln(0.9)}{50\text{h}} = +\underline{\underline{0.0021072\%}}$$

Solution 2.9

Hint: Conversion for the expectancy: $\int t \cdot f \to \int R$

$$E(t) = \int_0^\infty t \cdot f(t) \cdot dt = \int_0^\infty t \cdot \frac{dF(t)}{dt} \cdot dt \quad \text{mit} \quad \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$

$$\Rightarrow E(t) = \int_0^\infty t \cdot f(t) \cdot dt = -\int_0^\infty t \cdot \frac{dR(t)}{dt} \cdot dt$$

Use integration by parts: $\int_a^b u' \cdot v \cdot dx = u \cdot v\Big|_a^b - \int_a^b u \cdot v' \cdot dx$

This results to: $\underline{\underline{E(t)}} = \underbrace{\left[-t \cdot R(t)\right]_0^\infty}_{\to 0} + \int_0^\infty R(t) \cdot dt = \underline{\underline{\int_0^\infty R(t) \cdot dt}}$ q.e.d

---

(Derivation just for information – not necessary for the solution of this problem)

Expectancy (mean): $E(t) = \int_0^\infty t \cdot f(t) \cdot dt = \int_0^\infty R(t) \cdot dt$

Three parametric Weibull distribution:

$$f(t) = \frac{b}{T - t_0} \cdot \left(\frac{t - t_0}{T - t_0}\right)^{b-1} \cdot \exp\left[-\left(\frac{t - t_0}{T - t_0}\right)^b\right]$$

Inserted results to: $E(t) = \int_0^\infty \frac{t \cdot b}{T - t_0} \cdot \left(\frac{t - t_0}{T - t_0}\right)^{b-1} \cdot \exp\left[-\left(\frac{t - t_0}{T - t_0}\right)^b\right] \cdot dt$

Substitution: $t' = \frac{t - t_0}{T - t_0}$ and $\frac{dt'}{dt} = \frac{1}{T - t_0}$

$$\Rightarrow t = t' \cdot (T - t_0) + t_0 \quad \text{and} \quad dt = dt' \cdot (T - t_0)$$

Insert: $E(t) = \int_0^\infty \frac{t' \cdot (T - t_0) \cdot b + t_0 \cdot b}{T - t_0} \cdot (t')^{b-1} \cdot \exp(-(t')^b) \cdot (T - t_0) \cdot dt'$

Substitution again: $x = (t')^b$ and $\frac{dx}{dt'} = b \cdot (t')^{b-1}$

$$\Rightarrow t' = x^{1/b} \quad \text{and} \quad dt' = \frac{dx}{b \cdot (t')^{b-1}}$$

Thus: $E(t) = \int_0^{\infty} \left(x^{1/b} \cdot (T - t_0) \cdot b + t_0 \cdot b \right) \cdot (t')^{b-1} \cdot \exp(-x) \cdot \frac{dx}{b(t')^{b-1}}$

After simplification:

$$E(t) = \int_0^{\infty} x^{1/b} \cdot (T - t_0) \cdot \exp(-x) \cdot dx + \underbrace{\int_0^{\infty} t_0 \cdot \exp(-x) \cdot dx}_{t_0}$$

Compared with the gamma function $\Gamma(z) = \int_0^{\infty} \exp(-y) \cdot y^{z-1} \cdot dz$ (tabulated)

Here it is necessary: $x = y$ and $\frac{1}{b} = z\text{-}1 \Rightarrow \mathrm{z} = \frac{1}{\mathrm{b}} + 1$

Thus, the expectancy is: $\underline{\underline{E(t) = (T - t_0) \cdot \Gamma\left(1 + \frac{1}{b}\right) + t_0}}$

Similar: $VAR(t) = (T - t_0)^2 \cdot \left[\Gamma\left(1 + \frac{2}{b}\right) - \Gamma^2\left(1 + \frac{1}{b}\right) \right]$ is called variance

a) $b = 1.0$; $T = 1{,}000\mathrm{h}$; $t_0 = 0$

$$\underline{\underline{MTBF = E(t)}} = 1{,}000\mathrm{h} \cdot \Gamma\left(\underbrace{1 + \frac{1}{1}}_{2} \right) = 1{,}000\mathrm{h} \cdot 1 = \underline{\underline{1{,}000\mathrm{h}}}$$

(Compared with the exponential distribution: $E(t) = \frac{1}{\lambda} = T$)

b) $b = 0.8$; $T = 1{,}000\mathrm{h}$; $t_0 = 0$

$$\underline{\underline{E(t)}} = 1{,}000\mathrm{h} \cdot \Gamma\left(1 + \frac{1}{0{,}8}\right) = 1{,}000\mathrm{h} \cdot \Gamma(2{,}25) = 1{,}000\mathrm{h} \cdot 1.25 \cdot \Gamma(1.25) =$$

$$= 1{,}000\mathrm{h} \cdot 1.25 \cdot 0.906402477 = \underline{\underline{1{,}133.00\mathrm{h}}}$$

c) $b = 4.2$; $T = 1{,}000\mathrm{h}$; $t_0 = 100\mathrm{h}$

$$\underline{\underline{MTBF}} = E(t) = (1{,}000\mathrm{h} - 100\mathrm{h}) \cdot \Gamma\left(1 + \frac{1}{4.2}\right) + 100$$

$$= 900\mathrm{h} \cdot \Gamma\left(\underbrace{1.238}_{1.24} \right) + 100 = 900\mathrm{h} \cdot 0.908521 + 100 = \underline{\underline{917.67\mathrm{h}}}$$

d) $b = 0.75$; $T = 1{,}000\text{h}$; $t_0 = 200\text{h}$

$$\underline{\underline{MTBF}} = E(t) = (1{,}000\text{h} - 200\text{h}) \cdot \Gamma\left(1 + \frac{1}{0.75}\right) + 200$$

$$= 800\text{h} \cdot \Gamma\left(2.3\overline{3}\right) + 200\text{h} = 800\text{h} \cdot 1.3\overline{3} \cdot \Gamma\left(1.3\overline{3}\right) + 200\text{h}$$

$$= 800\text{h} \cdot 1.3\overline{3} \cdot 0.89337 + 200\text{h} = \underline{\underline{1150.54\text{h}}}$$

Solution 2.10

a)

$$B_{10} = \frac{B_y}{(1 - f_{tB}) \cdot \sqrt[b]{\frac{ln(1-y)}{ln(1-0.1)}} + f_{tB}} \overset{y=50\%}{=} \frac{6\,000\,000}{(1 - 0.25) \cdot \sqrt[1.11]{\frac{ln(1-0.5)}{ln(1-0.1)}} + 0.25}$$

$$= \underline{\underline{1{,}381{,}265.5\ LW}}$$

b) $\underline{\underline{t_0}} = B_{10} \cdot f_{tB} = 1{,}381{,}265.5 \cdot 0.25 = \underline{\underline{345{,}316.4\ LW}}$

$Aus\ F(B_{50}) = 0.5$

$$\Rightarrow \underline{\underline{T}} = t_0 + \frac{B_{50} - t_0}{\sqrt[b]{-ln(1-0.5)}} = 345\,316.4 + \frac{6{,}000{,}000 - 345{,}316.4}{\sqrt[1.11]{-ln 0.5}}$$

$$= \underline{\underline{8{,}212{,}310\ \text{LW}}}$$

c) $P(t_1 \le t \le t_2) = F(t_2) - F(t_1)$

$$= 1 - exp\left(-\left(\frac{9{,}000{,}000 - 345{,}316.4}{8{,}212{,}310 - 345{,}316.4}\right)^{1.11}\right) - 1 + exp\left(-\left(\frac{2{,}000{,}000 - 345{,}316.4}{8{,}212{,}310 - 345{,}316.4}\right)^{1.11}\right)$$

$$= -0.3289 + 0.8376 = 0.508 \mathrel{\hat{=}} 50.8\%$$

d) $P(t_3 > t) = R(t_3) \overset{!}{=} 0.99 = exp\left(-\left(\frac{t - t_0}{T - t_0}\right)^b\right)$

$$\Rightarrow ln(0.99) = -\left(\frac{t_3 - t_0}{T - t_0}\right)^b \Rightarrow \sqrt[b]{-ln(0.99)} = \frac{t_3 - t_0}{T - t_0}$$

$$\Rightarrow t_3 = (T - t_0) \cdot \sqrt[b]{-\ln(0.99)} + t_0$$
$$= (8{,}212{,}310 - 345{,}316.4) \cdot \sqrt[1.11]{-\ln(0.999)} + 345{,}316.4$$
$$= \underline{\underline{470{,}046.6 \text{ LW}}}$$

e) $P(5{,}000{,}000 > t) = R(5{,}000{,}000) \overset{!}{=} 0{,}5$

$$R(t) = exp\left(-\left(\frac{t - t_0}{T - t_0}\right)^b\right)$$

$$\Rightarrow ln(R(t)) = -\left(\frac{t - t_0}{T - t_0}\right)^b \Rightarrow ln(-ln(R(t)) = b \cdot ln\left(\frac{t - t_0}{T - t_0}\right)$$

$$\Rightarrow \underline{\underline{b}} = \frac{ln(-ln(R(t)))}{ln\left(\frac{t - t_0}{T - t_0}\right)} = \frac{ln(-ln(0.5))}{ln\left(\frac{5{,}000{,}000 - 345{,}316.4}{8{,}212{,}310 - 345{,}316.4}\right)} = \underline{\underline{0.698}}$$

Solution 2.11

Required condition for the mode: $\tilde{t}\ :\ \frac{df(\tilde{t})}{dt} \overset{!}{=} 0$

$$\frac{df(t)}{dt} = \frac{d}{dt}\left(\frac{b}{T - t_0} \cdot \left(\frac{t - t_0}{T - t_0}\right)^{b-1} \cdot \exp\left(-\left(\frac{t - t_0}{T - t_0}\right)^b\right)\right)$$

$$\overset{(a \cdot b)' = a' \cdot b + a \cdot b'}{=} \frac{b}{T - t_0} \cdot \left(\frac{d}{dt}\left(\frac{t - t_0}{T - t_0}\right)^{b-1} \cdot R(t) - \left(\frac{t - t_0}{T - t_0}\right)^{b-1} \cdot f(t)\right)$$

$$= \frac{b}{T - t_0}\left(\left(\frac{b - 1}{T - t_0}\right) \cdot \left(\frac{t - t_0}{T - t_0}\right)^{b-2} \cdot R(t) - \left(\frac{t - t_0}{T - t_0}\right)^{b-1} \cdot f(t)\right)$$

$$= \frac{b \cdot (b - 1)}{(T - t_0)^2} \cdot \left(\frac{t - t_0}{T - t_0}\right)^{b-2} \cdot \exp\left(-\left(\frac{t - t_0}{T - t_0}\right)^b\right) - \frac{b^2}{(T - t_0)^2} \cdot \left(\frac{t - t_0}{T - t_0}\right)^{2b-2} \cdot \exp\left(-\left(\frac{t - t_0}{T - t_0}\right)\right)$$

$$\text{now}: \quad \frac{df(\tilde{t}\,)}{dt} \overset{!}{=} 0 \qquad \text{substution}: \tilde{x} = \frac{\tilde{t} - t_0}{T - t_0}$$

$$0 \overset{!}{=} e^{-x} \cdot \frac{b}{(T - t_0)^2} \cdot \left((b-1) \cdot \tilde{x}^{b-2} - b \cdot \tilde{x}^{2b-2}\right)$$

$$\Rightarrow (b-1) \cdot \tilde{x}^{b-2} = b \cdot \tilde{x}^{2b-2}$$

$$\text{required condition}: \frac{b-1}{b} > 0 \Rightarrow b > 1 \,(\text{only then does the mode exist!!!})$$

$$ln\left(\frac{b-1}{b}\right) + (b-2) \cdot ln\,\tilde{x} = (2b-2) \cdot ln\,\tilde{x}$$

$$ln\left(\frac{b-1}{b}\right) = (2b - 2 - b + 2) \cdot ln\,\tilde{x}$$

$$\ln \tilde{x} = \frac{1}{b} \cdot \ln\left(\frac{b-1}{b}\right)$$

$$\tilde{x} = \left(\frac{b-1}{b}\right)^{1/b}$$

$$\underline{\underline{\tilde{t} = (T - t_0) \cdot \left(\frac{b-1}{b}\right)^{1/b} + t_0}} \qquad (b > 1\ !)$$

Example calculation for a Weibull distribution with:
$b = 1.8;\quad T = 1000\text{h};\ t_0 = 500\text{h}$

$$\underline{\underline{\tilde{t}}} = (1000 - 500) \cdot \left(\frac{0.8}{1.8}\right)^{1/1.8} + 500 = \underline{\underline{818.64\text{h}}}$$

Control (with calculation):

A possible condition: $f(800) < f(\tilde{t}\,) \wedge f(850) < f(\tilde{t}\,)$

$$f(800) = \frac{1.8}{500} \cdot \left(\frac{300}{500}\right)^{0.8} \cdot exp\left(-\left(\frac{300}{500}\right)^{1.8}\right) = 0.0016057$$

$$f(850) = \frac{1.8}{500} \cdot \left(\frac{350}{500}\right)^{0.8} \cdot exp\left(-\left(\frac{350}{500}\right)^{1.8}\right) = 0.001599$$

$$f(818.64) = \frac{1.8}{500} \cdot \left(\frac{318.64}{500}\right)^{0.8} \cdot exp\left(-\left(\frac{318.64}{500}\right)^{1.8}\right) = 0.0016097$$

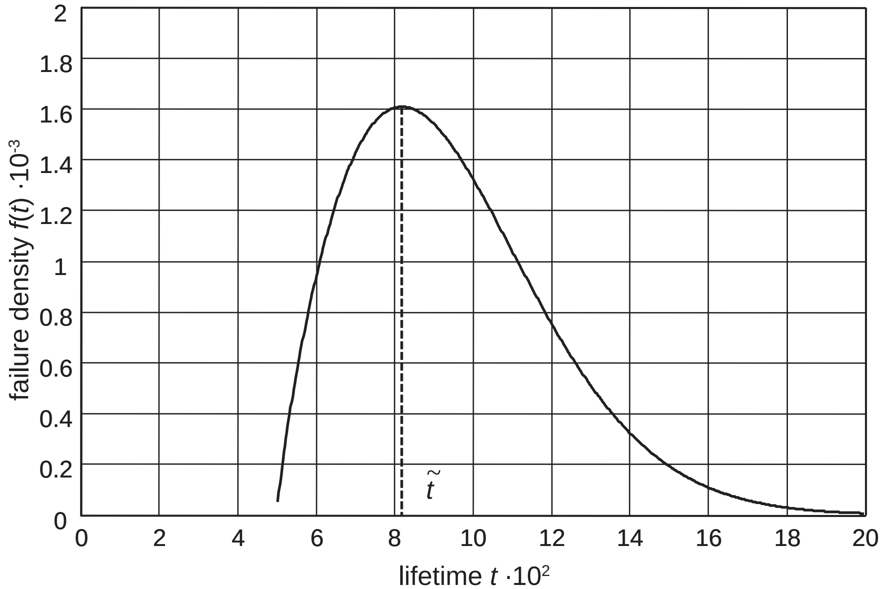


Figure Solution to Problem 2.11

Solution 2.12

Given is: t_1, x_1, t_2, x_2

Required conditions: $x_1 = 1 - \exp\left(-\left(\frac{t_1}{T}\right)^b\right) \wedge x_2 = 1 - \exp\left(-\left(\frac{t_2}{T}\right)^b\right)$

Reform:

$$\ln(1 - x_i) = -\left(\frac{t_i}{T}\right)^b \quad \Rightarrow \quad \ln(-\ln(1 - x_i)) = b \cdot \ln\left(\frac{t_i}{T}\right)$$

$$\text{Solve for } b \Rightarrow b = \frac{\ln(-\ln(1 - x_1))}{\ln(t_1) - \ln(T)} \overset{!}{=} \frac{\ln(-\ln(1 - x_2))}{\ln(t_2) - \ln(T)} \qquad (*)$$

$$\text{Substitution}: \ \Lambda_i = \ln(-\ln(1 - x_i))$$

$$\frac{\ln(t_1)-\ln(T)}{\Lambda_1}=\frac{\ln(t_2)-\ln(T)}{\Lambda_2}$$

$$\ln(T)\cdot\left(\frac{1}{\Lambda_2}-\frac{1}{\Lambda_2}\right)=\frac{\ln(t_2)}{\Lambda_2}-\frac{\ln(t_1)}{\Lambda_1}$$

$$\Rightarrow T=\exp\left(\frac{\left(\frac{\ln(t_2)}{\Lambda_2}-\frac{\ln(t_1)}{\Lambda_1}\right)\cdot\Lambda_2\cdot\Lambda_1}{\Lambda_1-\Lambda_2}\right)=\exp\left(\frac{\frac{(\Lambda_1\cdot\ln(t_2)-\Lambda_2\cdot\ln(t_1))\cdot\Lambda_1\cdot\Lambda_2}{\Lambda_1\cdot\Lambda_2}}{\Lambda_1-\Lambda_2}\right)$$

$$\Rightarrow T=\exp\left(\frac{\ln(-\ln(1-x_1))\cdot\ln(t_2)-\ln(-(\ln(1-x_2)))\cdot\ln(t_1)}{\ln(-\ln(1-x_1))-\ln(-(1-x_2))}\right)$$

ln(*T*) and in (*)

$$b=\frac{\ln(-\ln(1-x_1))}{\ln(t_1)-\frac{\ln(-\ln(1-x_1))\cdot\ln(t_2)-\ln(-\ln(1-x_2))\cdot\ln(t_1)}{\ln(-\ln(1-x_1))-\ln(-\ln(1-x_2))}}$$

Solution 2.13

a) $R_S=R_3\cdot R_E$

$R_E=1-(1-R_1)\cdot(1-R_2)$

$\Rightarrow R_S=R_3\cdot(1-(1-R_1)\cdot(1-R_2))$

b) $R_{p1}=1-(1-R_1)\cdot(1-R_2)$

$R_{p2}=1-(1-R_3)\cdot(1-R_4)$

$\Rightarrow R_S=R_{p1}\cdot R_{p2}$

$\Rightarrow R_S=(1-(1-R_1)\cdot(1-R_2))\cdot(1-(1-R_3)\cdot(1-R_4))$

c) $R_S=1-(1-R_E)\cdot(1-R_3)$

$R_E=R_1\cdot R_2$

$\Rightarrow R_S=1-(1-R_1\cdot R_2)\cdot(1-R_3)$

d) $R_E=1-(1-R_2)\cdot(1-R_3)\cdot(1-R_4)$

$R_S=R_1\cdot R_E\cdot R_5=R_1\cdot R_5\cdot(1-(1-R_2)\cdot(1-R_3)\cdot(1-R_4))$

e) $R_S=1-(1-R_{E1})\cdot(1-R_{E3})$

$$R_{E1} = R_1 \cdot R_2$$
$$R_{E3} = R_{E2} \cdot R_5$$
$$R_{E2} = 1 - (1 - R_3) \cdot (1 - R_4)$$
Substitution :
$$\underline{\underline{R_S = 1 - (1 - R_1 \cdot R_2) \cdot (1 - R_5 \cdot (1 - (1 - R_3) \cdot (1 - R_4)))}}$$

Solution 2.14

The following equations are valid for a serial connection (n = number of components)

$$R_S(t) = \prod_{i=1}^{n} R_i(t) \qquad F_i(t) = 1 - R_i(t) \qquad \text{and} \qquad \mathrm{F_S(t) = 1 - R_S(t)}$$

$$\Rightarrow 1 - F_S(t) = \prod_{i=1}^{n} (1 - F_i(t)) \;\Rightarrow\; \underline{\underline{F_S(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t))}}$$

Density $f_S(t) = ?$

$$f_S(t) = \frac{dF_S(t)}{dt} = \frac{d(1 - R_S(t))}{dt} = -\frac{dR_S(t)}{dt} = \frac{d}{dt}\left(\prod_{i=1}^{n} R_i(t)\right) \rightarrow \text{Difficult to}$$

differentiate $\rightarrow$ logarithmize, then a product becomes a sum :

$$log(a \cdot b) = log\, a + log\, b \Rightarrow ln(R_S(t)) = \sum_{i=1}^{n} ln(R_i(t))$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{dt}}(ln(R_S(t))) = \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{dt}}(ln(R_i(t)))$$

Using logarithmic differentiation:

$$\text{In general } (f = \text{function}) : \frac{d}{dt}(ln(f(t))) = \frac{df(t)}{dt} \cdot \frac{1}{f(t)}$$

$$\Rightarrow \frac{dR_S(t)}{dt} \cdot \frac{1}{R_S(t)} = \sum_{i=1}^{n} \frac{dR_i(t)}{dt} \cdot \frac{1}{R_i(t)}$$

$$\Rightarrow -f_S(t) = R_S(t) \cdot \sum_{i=1}^{n} - \underbrace{f_i(t) \cdot \frac{1}{R_i(t)}}_{=\lambda_i(t)}$$

System failure density for a series connection:

$$\boxed{f_S(t) = R_S(t) \cdot \sum_{i=1}^{n} \lambda_i(t)}$$

System failure rate for a series connection:

$$\boxed{\lambda_S(t) = \frac{f_S(t)}{R_S(t)} = \sum_{i=1}^{n} \lambda_i(t)}$$

(= sum of the failure rates for the system components)

Solution 2.15

a) Successively summarize the system structure:

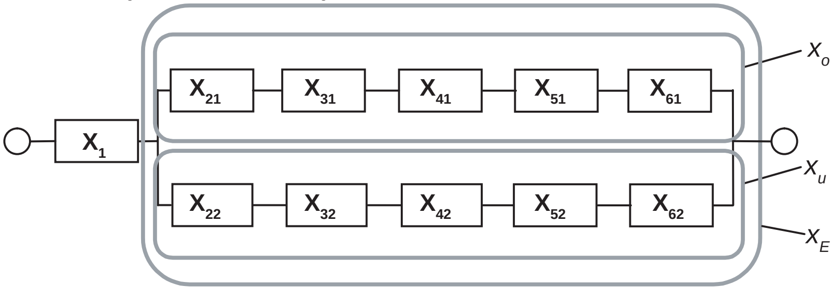


Figure Solution 2.15a

$$R_o = R_{21} \cdot R_{31} \cdot R_{41} \cdot R_{51} \cdot R_{61} = \prod_{i=2}^{6} R_{i1};$$

$$R_u = R_{22} \cdot R_{32} \cdot R_{42} \cdot R_{52} \cdot R_{62} = \prod_{i=2}^{6} R_{i2}$$

$$R_E = 1-(1-R_o)\cdot(1-R_u) = 1-\left(1-\prod_{i=2}^{6} R_{i1}\right)\cdot\left(1-\prod_{i=2}^{6} R_{i2}\right)$$

$$\underline{\underline{R_S}} = R_1 \cdot R_E$$

$$= \underline{\underline{R_1 - R_1 \cdot (1-R_{21}\cdot R_{31}\cdot R_{41}\cdot R_{51}\cdot R_{61})\cdot(1-R_{22}\cdot R_{32}\cdot R_{42}\cdot R_{52}\cdot R_{62})}}$$

b) All components with an exponential distribution, use the power rule:

$$R_i = \exp-(\lambda_i \cdot t) \qquad \prod R_i = \prod \exp-(\lambda_i \cdot t) = \exp-\left(\sum \lambda_i \cdot t\right)$$

$$R_S = \exp(-\lambda_1 \cdot t) - \exp(-\lambda_1 \cdot t)\cdot\left(1-\exp\left(-\underbrace{(\lambda_{21}+\lambda_{31}+\lambda_{41}+\lambda_{51}+\lambda_{61})}_{\lambda^*}\cdot t\right)\right)\cdot$$

$$\left(1-\exp\left(-\underbrace{(\lambda_{22}+\lambda_{32}+\lambda_{42}+\lambda_{52}+\lambda_{62})}_{\lambda^*}\cdot t\right)\right)$$

$$\lambda^* = (7+5+0.2+1.5+0.3)\cdot 10^{-3}\cdot\frac{1}{a} = 14\cdot 10^{-3}\frac{1}{a}$$

$$\underline{\underline{R_S}} = \exp(-\lambda_1\cdot t) - \exp(-\lambda_1\cdot t)\cdot\left(1-\exp(-\lambda^*\cdot t)\right)\cdot\left(1-\exp(-\lambda^*\cdot t)\right)$$

$$= \underline{\underline{\exp(-\lambda_1\cdot t)\cdot\left(1-\left(1-\exp(-\lambda^*\cdot t)\right)^2\right)}}$$

$$\underline{\underline{R_S(10\text{a})}} = \exp\left(-4\cdot 10^{-3}\cdot 10\text{a}\right)\cdot\left(1-\left(1-\exp\left(-14\cdot 10^{-3}\cdot 10\text{a}\right)\right)^2\right)$$

$$= 0.944 \mathrel{\hat{=}} \underline{\underline{94.4\%}}$$

$$F_S(10\,a) = 1 - R(10\,a) = 0.0556 \mathrel{\hat{=}} 5.56\%$$

5 ABS systems have failed out of 100.

c) $$R_S = \exp(-\lambda_1\cdot t) - \exp(-\lambda_1\cdot t)\cdot\left(1-\exp(-\lambda^*\cdot t)\right)^2$$

$$= \exp(-\lambda_1\cdot t) - \exp(-\lambda_1\cdot t)\cdot\left(1-2\cdot\exp(-\lambda^*\cdot t)+\exp(-\lambda^*\cdot 2\cdot t)\right)$$

$$= \exp(-\lambda_1\cdot t) - \exp(-\lambda_1\cdot t) + 2\cdot\exp\left(-\left(\lambda^*+\lambda_1\right)\cdot t\right) - \exp\left(-\left(2\cdot\lambda^*+\lambda_1\right)\cdot t\right)$$

$$\underline{\underline{MTBF}} = \int_0^\infty R_S(t)\cdot dt = \int_0^\infty \left(2\cdot exp\left(-\left(\lambda^*+\lambda_1\right)\cdot t - exp\left(-\left(2\cdot\lambda^*+\lambda_1\right)\cdot t\right)\right)\right)dt$$

$$= -\left[\frac{-2}{\lambda_1+\lambda^*} + \frac{1}{2\cdot\lambda^*+\lambda_1}\right] = \frac{2}{\lambda_1+\lambda^*} - \frac{1}{2\cdot\lambda^*+\lambda_1}$$

$$= \frac{2\cdot a}{18\cdot 10^{-3}} - \frac{1\cdot a}{(28+4)\cdot 10^{-3}} = \underline{\underline{79.86\,a}}$$

d) Iterative calculation → Newton procedure:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad \text{here}: \; x \mathrel{\hat{=}} B_{10}$$

$$\text{Condition}: F_S(B_{10}) = 0.1 = 1 - 2\cdot\exp\left(-\left(\lambda^*+\lambda_1\right)\cdot B_{10}\right) + \exp\left(-\left(2\cdot\lambda^*+\lambda_1\right)\cdot B_{10}\right)$$

$$\Rightarrow f(B_{10}) \overset{!}{=} 0 = 0.9 - 2\cdot\exp\left(-\left(\lambda^*+\lambda_1\right)\cdot B_{10}\right) + \exp\left(-\left(2\cdot\lambda^*+\lambda_1\right)\cdot B_{10}\right)$$

$$f'(B_{10}) = 2\cdot\left(\lambda^*+\lambda_1\right)\cdot\exp\left(-\left(\lambda^*+\lambda_1\right)\cdot B_{10}\right) - \left(2\cdot\lambda^*+\lambda_1\right)\cdot\exp\left(-\left(2\cdot\lambda^*+\lambda_1\right)\cdot B_{10}\right)$$

$$\text{thus}: B_{10}^{i+1} = \underline{\underline{B_{10}^i - \frac{0.9 - 2\cdot\exp\left(-\left(\lambda^*+\lambda_1\right)\cdot B_{10}^i\right) + \exp\left(-\left(2\cdot\lambda^*+\lambda_1\right)\cdot B_{10}^i\right)}{2\cdot\left(\lambda^*+\lambda_1\right)\cdot\exp\left(-\left(\lambda^*+\lambda_1\right)\cdot B_{10}^i\right) - \left(2\cdot\lambda^*+\lambda_1\right)\cdot\exp\left(-\left(2\cdot\lambda^*+\lambda_1\right)\cdot B_{10}^i\right)}}}$$

Start value: $R(10\text{a}) = 94.4\%$ → $F(10\text{a}) = 5.56\% < 10\%$

→ Choose $B_{10}^0 = 12\,\text{a}$

e) The following statement is valid in general for the survival probability that a system remains intact until the point in time *t* under consideration that the system has already survived the point in time t_1 (previous knowledge) (conditional probability):

$$P(t > 10\,\text{a} \mid t > t_1) = \frac{P(t > 10\,\text{a})}{P(t > t_1)} = \frac{R_S(10\,\text{a})}{R_S(t_1)} = R_S(10\,\text{a} \mid R_S(t_1)),$$

For components as well as a system, here

$$R_S(10\,a) = 0.944$$

$$R_S(t_1 = 5\,a) = 2 \cdot exp(-18 \cdot 10^{-3} \cdot 5) - exp(-32 \cdot 10^{-3} \cdot 5) = 0.97572$$

$$\underline{\underline{R_S(10\,a \mid R_S(5\,a))}} = \frac{0.944}{0.975} = 0.9682 \mathrel{\hat{=}} \underline{\underline{96.82\%}}$$

Solution 2.16

a) First the system equation:

$$R_{E1} = 1-(1-R_2)\cdot(1-R_3) = 1-(1-R_3-R_2+R_3\cdot R_2) = R_2+R_3-R_3\cdot R_2$$

$$R_{E2} = R_{E1}\cdot R_4 = R_2\cdot R_4 + R_3\cdot R_4 - R_2\cdot R_3\cdot R_4$$

$$\begin{aligned} R_S &= 1-(1-R_1)\cdot(1-R_{E2}) = R_1 + R_{E2} - R_1\cdot R_{E2} \\ &= R_1 + R_2\cdot R_4 + R_3\cdot R_4 - R_2\cdot R_3\cdot R_4 - R_1\cdot R_2\cdot R_4 \\ &\quad - R_1\cdot R_3\cdot R_4 + R_1\cdot R_2\cdot R_3\cdot R_4 \end{aligned}$$

apply distributions, use the power law :

$$\begin{aligned} R_S &= exp(-\lambda_1\cdot t) + exp(-(\lambda_2+\lambda_4)\cdot t) + exp(-(\lambda_3+\lambda_4)\cdot t) \\ &\quad - exp(-(\lambda_2+\lambda_3+\lambda_4)\cdot t) - exp(-(\lambda_1+\lambda_2+\lambda_4)\cdot t) \\ &\quad - exp(-(\lambda_1+\lambda_3+\lambda_4)\cdot t) + exp(-(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\cdot t) \end{aligned}$$

Summarize, use $\lambda_2 = \lambda_3$ and replacement part failure rates :

$$\begin{aligned} R_S &= exp(-\lambda_1\cdot t) + 2\cdot exp\left(-\underbrace{(\lambda_2+\lambda_4)}_{\lambda_b}\cdot t\right) - exp\left(-\underbrace{(\lambda_2+\lambda_3+\lambda_4)}_{\lambda_c}\cdot t\right) \\ &\quad - 2\cdot exp\left(-\underbrace{(\lambda_1+\lambda_2+\lambda_4)}_{\lambda_d}\cdot t\right) + exp\left(-\underbrace{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}_{\lambda_e}\cdot t\right) \end{aligned}$$

$$\lambda_a = \lambda_1 = 2.2 \cdot 10^{-3} \mathrm{h}^{-1},$$

$$\lambda_b = (4+3.6) \cdot 10^{-3} \mathrm{h}^{-1} = 7.6 \cdot 10^{-3} \mathrm{h}^{-1},$$

$$\lambda_c = (4+4+3.6) \cdot 10^{-3} \mathrm{h}^{-1} = 11.6 \cdot 10^{-3} \mathrm{h}^{-1},$$

$$\lambda_d = (2.2+4+3.6) \cdot 10^{-3} \mathrm{h}^{-1} = 9.8 \cdot 10^{-3} \mathrm{h}^{-1},$$

$$\lambda_e = (2.2+8+3.6) \cdot 10^{-3} \mathrm{h}^{-1} = 13.8 \cdot 10^{-3} \mathrm{h}^{-1}.$$

Thus

$$R_S(t) = \exp(-\lambda_a \cdot t) - 2 \cdot \exp(-\lambda_b \cdot t) - \exp(-\lambda_c \cdot t) - 2 \cdot \exp(-\lambda_d \cdot t) + \exp(-\lambda_e \cdot t)$$

Desired:

$$\underline{\underline{P(t > 100\,h)}} = R_S(t = 100\,h)$$

$$= exp(-0.22 \cdot 1) + 2 \cdot exp(-0.76 \cdot 1) - exp(-1.16) - exp(-0.98) + exp(-1.38)$$

$$= 0.9253 \,\hat{=}\, \underline{\underline{92.53\%}}$$

b) $F_S(100\,h) = 1 - R_S(100\,h) = 0.0746;$

$$n_f = N \cdot F_s(t) = 250 \cdot 0.0746 = 18.66 \Rightarrow 18 \text{ Systems}$$

c) $\mathrm{MTBF_S} = \int_0^\infty R_S(t)dt$

$$= \int_0^\infty \left(\exp(-\lambda_a \cdot t) + 2 \cdot \exp(-\lambda_b \cdot t) - \exp(-\lambda_c \cdot t) - 2 \cdot \exp(-\lambda_d \cdot t) + \exp(-\lambda_d \cdot t)\right) dt$$

$$= \Bigg[-\frac{1}{\lambda_a} \cdot \exp(-\lambda_a \cdot t) - \frac{2}{\lambda_b} \cdot \exp(-\lambda_b \cdot t) + \frac{1}{\lambda_c} \cdot \exp(-\lambda_c \cdot t) + \frac{2}{\lambda_d} \cdot \exp(-\lambda_d \cdot t) - \frac{1}{\lambda_e} \exp(-\lambda_d \cdot t) \Bigg]_0^\infty$$

$$= 0 - \left[-\frac{1}{\lambda_a} - \frac{2}{\lambda_b} + \frac{1}{\lambda_c} + \frac{2}{\lambda_d} - \frac{1}{\lambda_e} \right] = \underline{\underline{\frac{1}{\lambda_a} + \frac{2}{\lambda_b} - \frac{1}{\lambda_c} - \frac{2}{\lambda_d} + \frac{1}{\lambda_e}}}$$

$$\underline{\underline{\mathrm{MTBF}}} = 10^3 \cdot \left[\frac{1}{2{,}2} + \frac{2}{7{,}6} - \frac{1}{11{,}6} - \frac{2}{9{,}8} + \frac{1}{13{,}8} \right] h = 499.8\,h \approx \underline{\underline{500\mathrm{h}}}$$

d) Condition: $F_S(B_{10}) \overset{!}{=} 0.1 \Rightarrow f(B_{10}) = F_S(B_{10}) - 0.1$

$$F_S(t) = 1 - R_S(t)$$
$$= 1 - \exp(-\lambda_a \cdot t) - 2\cdot\exp(-\lambda_b \cdot t) + \exp(-\lambda_c \cdot t) + 2\cdot\exp(-\lambda_d \cdot t) - \exp(-\lambda_e \cdot t)$$

$$f_S(t) = \frac{dF_S(t)}{dt} \mathrel{\hat{=}} f'(B_{10})$$
$$= 0 + \lambda_a \cdot \exp(-\lambda_a \cdot t) + 2\cdot\lambda_b \cdot \exp(-\lambda_b \cdot t) - \lambda_c \cdot \exp(-\lambda_c \cdot t)$$
$$- 2\cdot\lambda_d \cdot \exp(-\lambda_d \cdot t) + \lambda_e \cdot \exp(-\lambda_e \cdot t)$$

Iterative→ Newton Procedure:

$$B_{10}^{i+1} = B_{10}^{i} - \frac{f\left(B_{10}^{i}\right)}{f'\left(B_{10}^{i}\right)} =$$

$$= B_{10}^{i} - \frac{0.9 - e^{\left(-\lambda_a \cdot B_{10}^{i}\right)} - 2\cdot e^{\left(-\lambda_b \cdot B_{10}^{i}\right)} + e^{\left(-\lambda_c \cdot B_{10}^{i}\right)} + 2\cdot e^{\left(-\lambda_d \cdot B_{10}^{i}\right)} - e^{\left(-\lambda_e \cdot B_{10}^{i}\right)}}{\lambda_a \cdot e^{\left(-\lambda_a \cdot B_{10}^{i}\right)} + 2\cdot\lambda_b \cdot e^{\left(-\lambda_b \cdot B_{10}^{i}\right)} - \lambda_c \cdot e^{\left(-\lambda_c \cdot B_{10}^{i}\right)} - 2\cdot\lambda_d \cdot e^{\left(-\lambda_d \cdot B_{10}^{i}\right)} + \lambda_e \cdot e^{\left(-\lambda_e \cdot B_{10}^{i}\right)}}$$

Start value: $R(100\,h) = 0.92 \Rightarrow F(100\,\text{h}) = 8\% \Rightarrow$ start value $B_{10}^{0} = 105\text{h}$

Solution 2.17

$$\text{Series:}\quad R_S(t) = \prod_{i=1}^{n} R_i(t) \overset{!}{=} [R_i(t)]^n \qquad (*)$$

a i's are the same

Insert and solve for T.

$$R_S(B_{10S}) = 1 - F(B_{10S}) = 1 - 0.1 = 0.9$$

System:

$$0.9 = \left[exp\left(-\left(\frac{B_{10S} - t_0}{T - t_0} \right)^b \right) \right]^n (= R_S(t))$$

$$\sqrt[n]{0.9} = exp\left(-\left(\frac{B_{10S} - t_0}{T - t_0} \right)^b \right)$$

$$-\ln\left(\sqrt[n]{0.9}\right) = +\left(\frac{B_{10S}-t_0}{T-t_0}\right)^b$$

$$\sqrt[b]{-\ln\left(\sqrt[n]{0.9}\right)} = \frac{(B_{10S}-t_0)/B_{10}}{(T-t_0)/B_{10}} \qquad \text{now with } f_{tB} = \frac{t_0}{B_{10}} = 0.85$$

$$\sqrt[b]{-\ln\left(\sqrt[n]{0.9}\right)} = \frac{\frac{B_{10S}}{B_{10}} - f_{tB}}{\frac{T}{B_{10}} - f_{tB}}$$

$$\frac{T}{B_{10}} - f_{tB} = \frac{\frac{B_{10S}}{B_{10}} - f_{tB}}{\sqrt[b]{-\ln\left(\sqrt[n]{0.9}\right)}}$$

$$T = B_{10}\cdot\left(\frac{\frac{B_{10S}}{B_{10}} - f_{tB}}{\sqrt[b]{-\ln\left(\sqrt[n]{0.9}\right)}} + f_{tB}\right) \qquad (**)$$

Now $B_{10} = ?$ (gear wheel)

Out of (*): $R_S(B_{10S}) = R_i(B_{10S})^n$

$\Rightarrow R_i(B_{10S}) = \sqrt[n]{R_S(B_{10S})} = \sqrt[n]{0.9} = 0.98836$

$\Rightarrow x = 1 - R_i(B_{10S}) = 0.0116385) = F_i(B_{10s})$

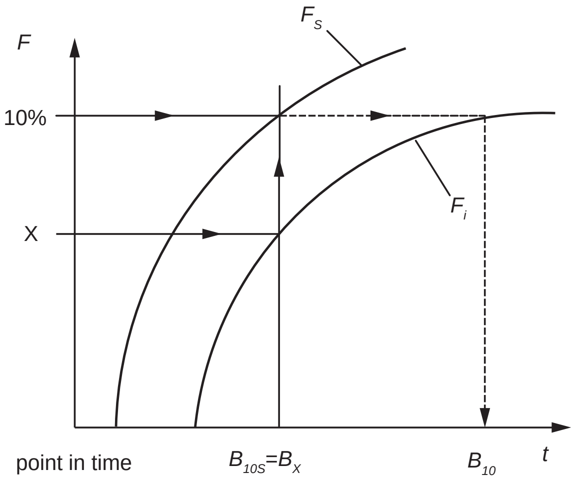


Figure Solution 2.17

Gear wheel:

$$B_{10} = \frac{B_x(\overset{!}{=} B_{10S}!\,)}{(1-f_{tB})\cdot\sqrt[b]{\frac{ln(1-x)}{ln(0.9)}}+f_{tB}} = \frac{100\,000}{0.15\cdot\sqrt[1,8]{\frac{ln(0.98836)}{ln(0.9)}}+0.85} = 111824.6\ \mathrm{LW}$$

in *(**)*

$$T = 111824.6\cdot\left(\frac{\frac{100\,000}{111824.6}-0.85}{\sqrt[1,8]{-ln(\sqrt[9]{0.9})}}+0.85\right)$$

$$\Rightarrow \underline{\underline{T = 153\,613.09\ LW}}$$

$$\underline{\underline{t_0 = f_{tB}\cdot B_{10} = 95\,050.91\ LW}}$$

Solution 5.1

$$y = (x_1 \wedge x_2) \vee (x_3 \wedge \overline{x_4})$$

$$\rightarrow R_S = 1-(1-R_1R_2)\cdot(1-R_3\underbrace{(1-R_4)}_{F_4})$$

Negate and apply the law from de Morgan:

$$\overline{y} = \overline{(x_1 \wedge x_2) \vee (x_3 \wedge \overline{x_4})}$$

$$\overline{y} = \overline{(x_1 \wedge x_2)} \wedge \overline{(x_3 \wedge \overline{x_4})}$$

$$\overline{y} = (\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_3} \vee \overline{\overline{x_4}})$$

Fault tree:

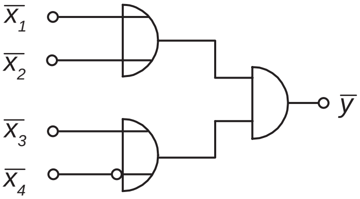


Figure Solution 5.1

Solution 5.2

a)

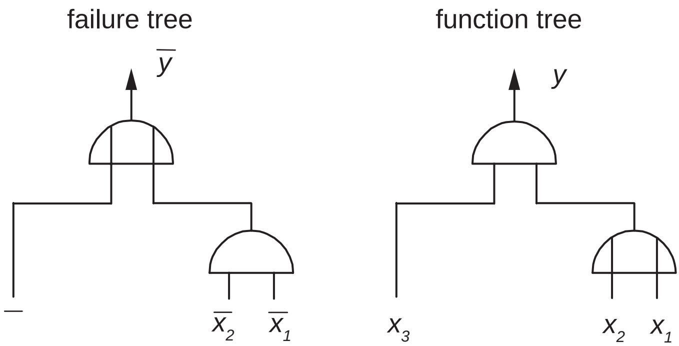


Figure Solution 5.2a

$$\overline{y} = \overline{x} \vee (\overline{x} \wedge \overline{x}$$

$$F_S = 1 - (1 - F_3) \cdot (1 - F_1 \cdot F_2)$$

$$F_S = 1 - R_3 \cdot (1 - (1 - R_1) \cdot (1 - R_2))$$

b)

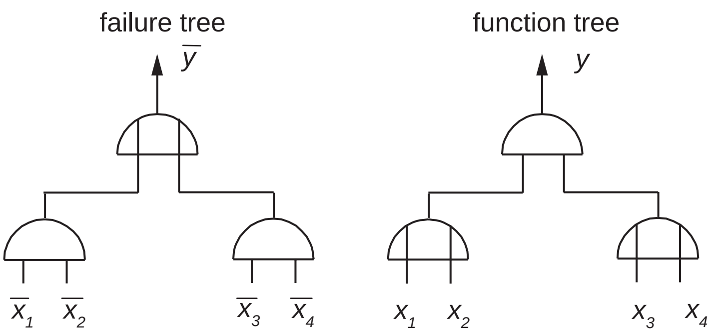


Figure Solution 5.2b

$$\overline{y} = (\overline{x} \wedge \overline{x} \vee (\overline{x} \wedge \overline{x}$$

$$F_S = 1 - (1 - F_1 \cdot F_2) \cdot (1 - F_3 \cdot F_4)$$

$$F_S = 1 - (1 - (1 - R_1) \cdot (1 - R_2)) \cdot (1 - (1 - R_3) \cdot (1 - R_4))$$

c)

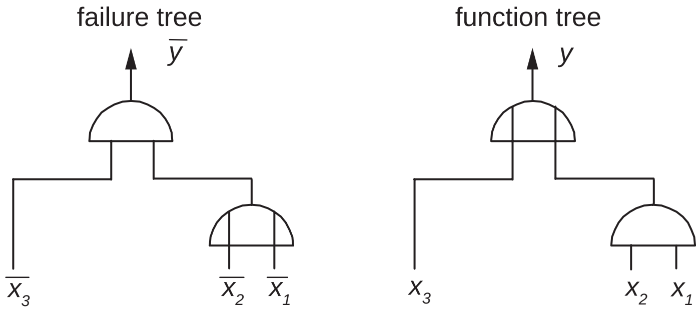


Figure Solution 5.2c

$$\bar{y} = \bar{x}_3 \wedge (\bar{x}_1 \vee \bar{x}_2)$$

$$F_S = F_3 \cdot (1-(1-F_1)\cdot(1-F_2))$$

$$F_S = (1-R_3)\cdot(1-R_1 \cdot R_2)$$

d)

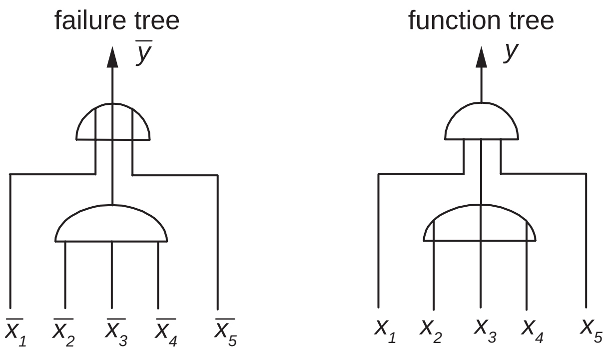


Figure Solution 5.2d

$$\bar{y} = \bar{x}_1 \vee (\bar{x}_2 \wedge \bar{x}_3 \wedge \bar{x}_4) \vee \bar{x}_5$$

$$F_S = 1-(1-F_1)\cdot(1-F_2 \cdot F_3 \cdot F_4)\cdot(1-F_5)$$

$$F_S = 1-R_1 \cdot (1-(1-R_2)\cdot(1-R_3)\cdot(1-R_4))\cdot R_5$$

e)

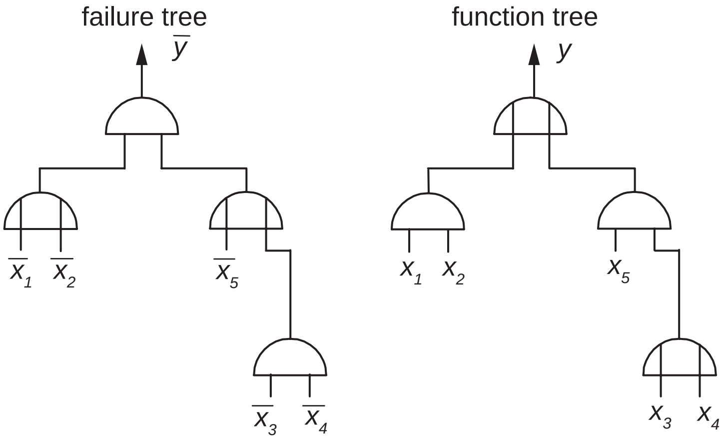


Figure Solution 5.2e

$$\bar{y} = (\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_5} \vee (\overline{x_3} \wedge \overline{x_4}))$$

$$F_S = F_a \cdot F_b \text{ and } F_a = 1 - (1 - F_1) \cdot (1 - F_2)$$

$$F_b = 1 - (1 - F_5) \cdot (1 - F_3 \cdot F_4)$$

$$F_S = (1 - (1 - F_1) \cdot (1 - F_2)) \cdot (1 - (1 - F_5) \cdot (1 - F_3 \cdot F_4))$$

$$F_S = (1 - R_1 \cdot R_2) \cdot (1 - R_5 \cdot (1 - (1 - R_3) \cdot (1 - R_4)))$$

Solution 5.3

a) Fault tree derived from the system description and function sketch:

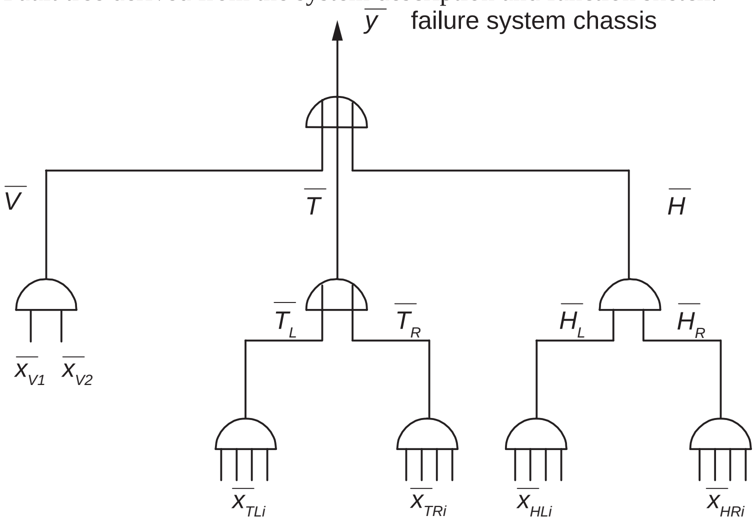


Figure Solution 5.3a

b) Determine the Boolean system function for the failure of the system "undercarriage".

out of a) $\bar{y} = \bar{V} \vee \bar{T} \vee \bar{H}$

with $\bar{V} = \bar{x}_{V1} \wedge \bar{x}_{V2}$

and $\bar{T} = \bar{T}_L \vee \bar{T}_R$

with $\bar{T}_L = \bigwedge_{i=1}^{4} \bar{x}_{TLi}$, $\bar{T}_R = \bigwedge_{i=1}^{4} \bar{x}_{TRi}$ and $\bar{H} = \bar{H}_L \wedge \bar{H}_R$

with $\bar{H}_L = \bigwedge_{i=1}^{4} \bar{x}_{HLi}$ and $\bar{H}_R = \bigwedge_{i=1}^{4} \bar{x}_{HRi}$.

Inserting into the above equation results to:

$$\bar{y} = \left(\bar{x}_{V1} \wedge \bar{x}_{V2}\right) \vee \left(\left(\bigwedge_{i=1}^{4} \bar{x}_{TLi}\right) \vee \left(\bigwedge_{i=1}^{4} \bar{x}_{TRi}\right)\right) \vee \left(\left(\bigwedge_{i=1}^{4} \bar{x}_{HLi}\right) \wedge \left(\bigwedge_{i=1}^{4} \bar{x}_{HRi}\right)\right).$$

c) Determine the system equation for the failure probability F_S.

Out of the additional notes:

$$F_S = 1 - (1 - F_V) \cdot (1 - F_T) \cdot (1 - F_H)$$

with $F_V = F_{V1} \cdot F_{V2}$, $F_T = 1 - \left(1 - \prod_{i=1}^{4} F_{TLi}\right) \cdot \left(1 - \prod_{i=1}^{4} F_{TRi}\right)$ and

$$F_H = \prod_{i=1}^{4} F_{HLi} \cdot \prod_{i=1}^{4} F_{HRi}.$$

d) Determine the Boolean system function for the reliability of the undercarriage.

Negate the left and right side of the system function out of b)

$$\bar{\bar{y}} = \overline{\bar{V} \vee \bar{T} \vee \bar{H}}$$

Apply de Morgan: $y = V \wedge T \wedge H$ with $V = x_{V1} \vee x_{V2}$

and $T = T_L \wedge T_R$, $T_L = \bigvee_{i=1}^{4} x_{TLi}$ and $T_R = \bigvee_{i=1}^{4} x_{TRi}$

and $H = H_L \vee H_R$, $H_L = \bigvee_{i=1}^{4} x_{HLi}$ and $H_R = \bigvee_{i=1}^{4} x_{HRi}$

$$\Rightarrow y = \left(x_{V1} \vee x_{V2}\right) \wedge \left(\left(\bigvee_{i=1}^{4} x_{TLi}\right) \wedge \left(\bigvee_{i=1}^{4} x_{TRi}\right)\right) \wedge \left(\left(\bigvee_{i=1}^{4} x_{HLi}\right) \vee \left(\bigvee_{i=1}^{4} x_{HRi}\right)\right).$$

e) i. Determine the system equation for the reliability R_S.

$R_S = R_V \cdot R_T \cdot R_H$ (out of the handout or $R = 1–F$)

with $R_V = 1 - (1 - R_{V1}) \cdot (1 - R_{V2})$,

$$R_T = \left(1 - \prod_{i=1}^{4}\left(1 - R_{TLi}\right)\right) \cdot \left(1 - \prod_{i=1}^{4}\left(1 - R_{TRi}\right)\right) \text{ and}$$

$$R_H = 1 - \left(1 - \left(1 - \prod_{i=1}^{4}\left(1 - R_{HLi}\right)\right) \cdot \left(1 - \left(1 - \prod_{i=1}^{4}\left(1 - R_{HRi}\right)\right)\right)\right).$$

ii. Create the corresponding block diagram
(out of the Boolean system function for the operability of the system „undercarriage“).

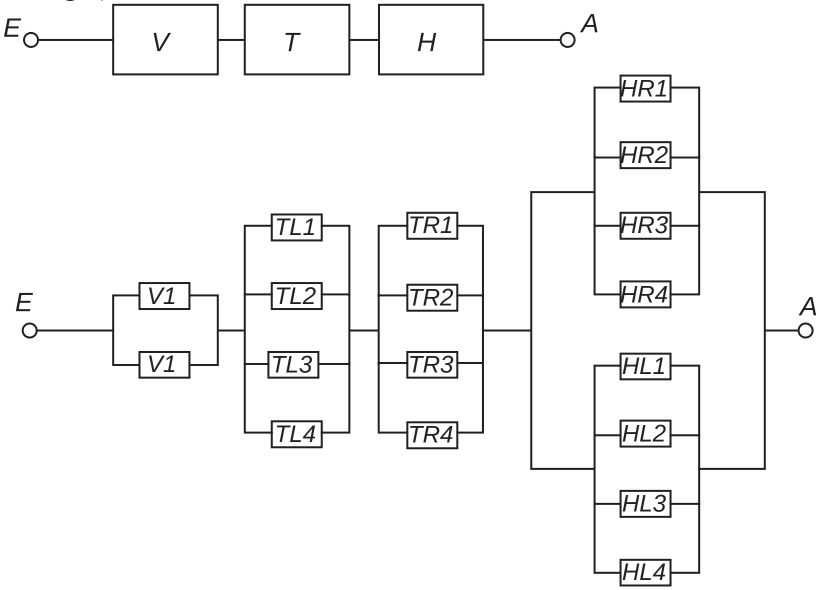


Figure Solution 5.3f

Solution 5.4

x_2 continuously operational — $R_I(t)$ x₂ continuously failed — $R_{II}(t)$

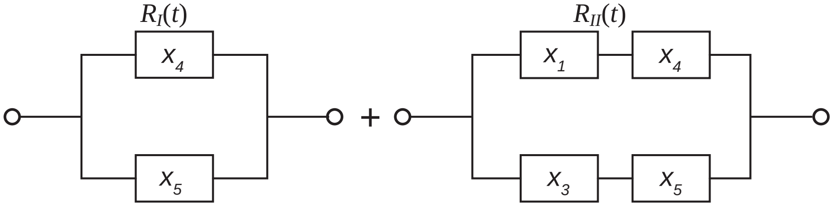


Figure Solution 5.4

$$R = R_2 \cdot R_I + \left(1 - R_2\right) \cdot R_{II}$$

with $R_I = 1 - \left(1 - R_4\right) \cdot \left(1 - R_5\right)$

$R_{II} = 1 - \left(1 - R_1 \cdot R_4\right) \cdot \left(1 - R_3 \cdot R_5\right)$

$$R = R_2\left(1-\left(1-R_4\right)\cdot\left(1-R_5\right)\right)+\left(1-R_2\right)\cdot\left(1-\left(1-R_1\cdot R_4\right)\cdot\left(1-R_3\cdot R_5\right)\right).$$

Solution 5.5

a) Determine the system function fort he failure of the control unit.

$$\overline{y} = \overline{x_1} \vee \overline{x_2} \vee \overline{x_{34}} \vee \overline{x_5} \vee \overline{x_{69}} \vee \overline{x_{10}}$$
$$\overline{x_{34}} = \overline{x_3} \wedge \overline{x_4}$$
$$\overline{x_{69}} = \overline{x_{68}} \wedge \overline{x_9}$$
$$\overline{x_{68}} = \overline{x_6} \vee \overline{x_7} \vee \overline{x_8}$$
$$\overline{y} = \overline{x_1} \vee \overline{x_2} \vee \left(\overline{x_3} \wedge \overline{x_4}\right) \vee \overline{x_5} \vee \left(\left(\overline{x_6} \vee \overline{x_7} \vee \overline{x_8}\right) \wedge \overline{x_9}\right) \vee \overline{x_{10}}$$

b) Calculate the failure probability of the system.

$$F_S = 1-\left(1-F_1\right)\cdot\left(1-F_2\right)\cdot\left(1-F_{34}\right)\cdot\left(1-F_5\right)\cdot\left(1-F_{69}\right)\cdot\left(1-F_{10}\right)$$
$$F_{34} = F_3 \cdot F_4$$
$$F_{69} = F_{68} \cdot F_9$$
$$F_{68} = 1-\left(1-F_6\right)\cdot\left(1-F_7\right)\cdot\left(1-F_8\right)$$
$$F_S = 1-\left(1-F_1\right)\cdot\left(1-F_2\right)\cdot\left(1-F_3F_4\right)\cdot\left(1-F_5\right)$$
$$\cdot\left(1-\left[1-\left(1-F_6\right)\cdot\left(1-F_7\right)\cdot\left(1-F_8\right)\right]\cdot F_9\right)\cdot\left(1-F_{10}\right)$$

c) Determine the system function for the operability of the control unit. Negation followed by application of the de Morgan law

$$\overline{\overline{y}} = \overline{\overline{x_1} \vee \overline{x_2} \vee \overline{x_{34}} \vee \overline{x_5} \vee \overline{x_{69}} \vee \overline{x_{10}}}$$
$$y = x_1 \wedge x_2 \wedge x_{34} \wedge x_5 \wedge x_{69} \wedge x_{10}$$
$$x_{34} = x_3 \vee x_4$$
$$x_{69} = x_{68} \vee x_9$$
$$x_{68} = x_6 \wedge x_7 \wedge x_8$$
$$y = x_1 \wedge x_2 \wedge \left(x_3 \vee x_4\right) \wedge x_5 \wedge \left(\left(x_6 \wedge x_7 \wedge x_8\right) \vee x_9\right) \wedge x_{10}$$

d) Create the block diagram.
Out of the system function for the operability $\rightarrow$ block diagram

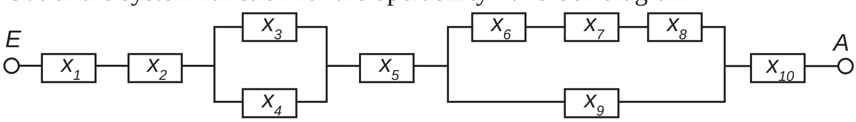


Figure Solution 5.5d

Solution 6.1

a) Average: $\bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i$ here: $n = 8$

$$\underline{\underline{\bar{t}}} = \frac{1}{8}\cdot\left(69+29+24+52{,}5+128+60+12{,}8+98\right)\cdot 10^3 = \underline{\underline{59{,}162.5\ \text{km}}}$$

Standard deviation: $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(t_i - \bar{t}\right)^2}$

$$s = \sqrt{\frac{1}{8-1}\cdot\left[(69-59.162)^2 + (29-59.162)^2 + \ldots + (98-59.162)^2\right]\cdot 10^6}$$

$s = 39068.65\ \text{km}$

Range: $r = t_{max} - t_{min}$

$r = (128 - 12{,}8)\cdot 10^3 = 115{,}200\ \text{km}$

b) Ranking order means: $t_i \le t_{i+1}$ $\quad i = 1(1)n-1$

thus: sort the values in ascending order

Table Analysis

Rank i	Ranking Order t_i [km]	Failure Probability $F_i = \frac{i-0.3}{n+0.4}$	Failure Probability (according to the table)
1	12,800	0.083 = 8.3%	8,3%
2	24,000	0.202 = 20.2%	20.1%
3	29,000	0.321 = 32.1%	32.0%
4	52,000	0.440 = 44.0%	44.0%
5	60,000	0.559 = 55.9%	55.9%
6	69,000	0.678 = 67.8%	67.9%
7	98,000	0.797 = 79.7%	79.9%
8	128,000	0.916 = 91.6%	91.7%

Failure probabilities:

- calculate $F_i = \frac{i-0.3}{n+0.4}$ (median)
- take from table: $F_i = f(i, n)$

c) Draw in the Weibull network chart (see chart paper)

Draw a straight line → two parametric

$b = 1.43$; $T = 66{,}000\ \text{km}$

d) Read from the graph:

$B_{10} = F^{-1}(0{,}1) = 14{,}000\ \text{km}$;

$t_{50} = F^{-1}(0{,}5) = 52{,}000\ \text{km}$.

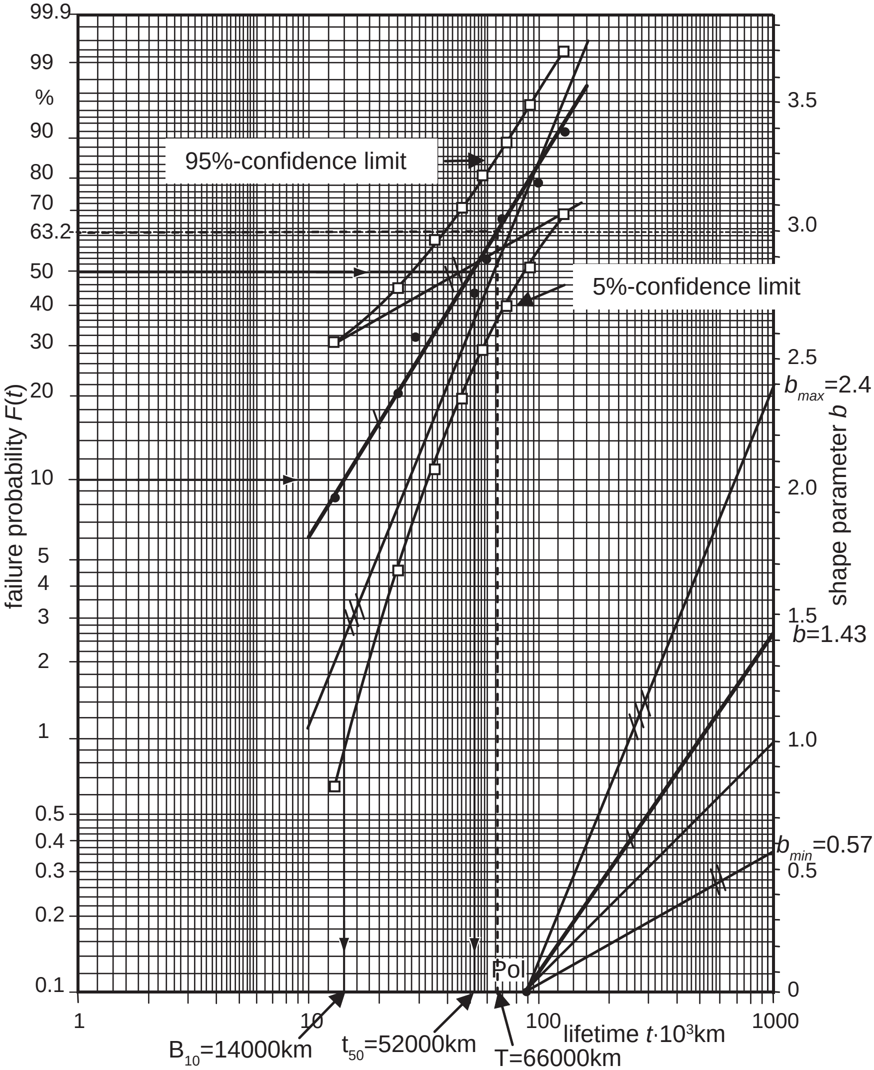


Figure Solution 6.1c

e) Searched for: $R(t_1 = 70{,}000) = 1 - F(t_1)$

$F(t_1 = 70{,}000) \approx 66\% \quad \rightarrow \quad \underline{\underline{R(t_1) = 34\%}}$

f) Take values for the 5% and 95% confidence levels out of the table and enter into the graph going out from the straight line, thus line-based confidence levels are determined.

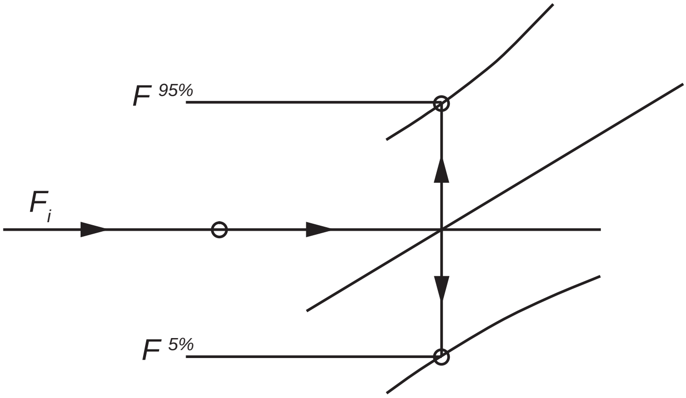


Figure Solution 6.1f

g) Shape parameter b:

calculational: (Equation 6.16)

$$\underline{\underline{b_{5\%}}} = \frac{b_{median}}{1+\sqrt{\frac{1.4}{n}}} = \frac{1.43}{1+\sqrt{\frac{1.4}{8}}} = \underline{\underline{1.008}}$$

$$\underline{\underline{b_{95\%}}} = b_{median} \cdot \left(1+\sqrt{\frac{1.4}{n}}\right) = 1.43 \cdot \left(1+\sqrt{\frac{1.4}{8}}\right) = \underline{\underline{2.028}}$$

graphical:

$b_{min} \approx 0.57$; $b_{max} \approx 2.4$

Characteristic lifetime T:

calculational: (Equation 6.14)

$$\underline{\underline{T_{5\%}}} = T \cdot \left(1-\frac{1}{9n}+1.645\cdot\sqrt{\frac{1}{9n}}\right)^{-\frac{3}{b}}$$

$$= 66{,}000 \cdot \left(1-\frac{1}{9\cdot 8}+1.645\cdot\sqrt{\frac{1}{9\cdot 8}}\right)^{-\frac{3}{1.43}} = \underline{\underline{46{,}640.3\text{ km}}}$$

$$\underline{\underline{T_{95\%}}} = T \cdot \left(1-\frac{1}{9n}-1.645\cdot\sqrt{\frac{1}{9n}}\right)^{-\frac{3}{b}}$$

$$= 66{,}000 \cdot \left(1-\frac{1}{72}-1.645\cdot\sqrt{\frac{1}{72}}\right)^{-\frac{3}{1.43}} = \underline{\underline{107{,}578.5\text{ km}}}$$

graphical:

$T_{5\%} \approx 37{,}000$ km ; $T_{95\%} \approx 110{,}000$ km

Solution 6.2

a) $n = 10$: Data is already sorted according to ascending value
$\rightarrow$ Rank order F_i taken from Table A.2, median

Table Analysis

Rank i	Ranking Order t_i	F_i (median) in %
1	470	6.7
2	550	16.2
3	600	25.9
4	800	35.5
5	1080	45.2
6	1150	54.8
7	1450	64.5
8	1800	74.1
9	2520	83.8
10	3030	93.3

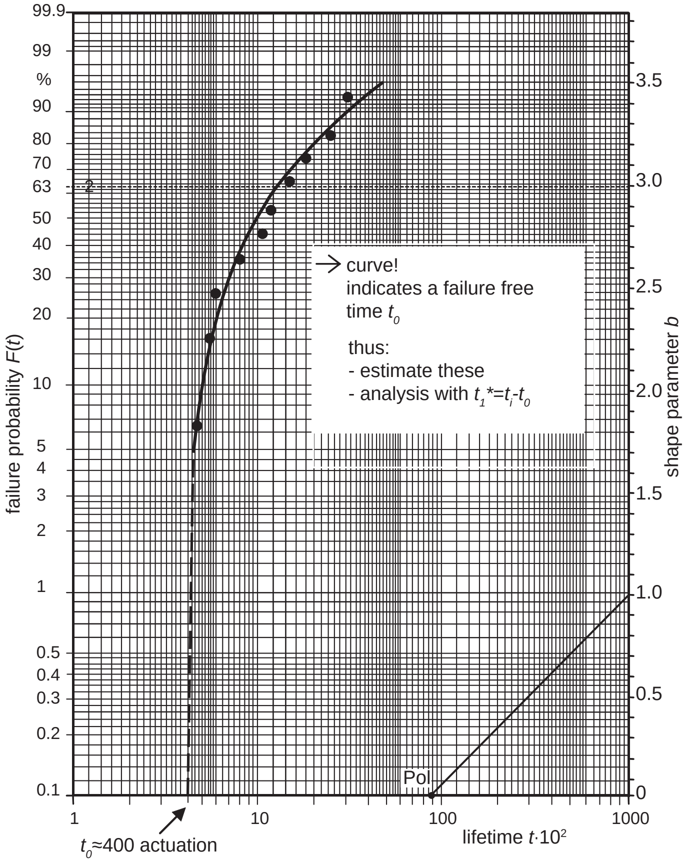


Figure Solution 6.2a

Draw in the chart → Curve formed by the points on the graph indicate a failure free time

→ estimate t_0: $t_0 \approx 400$ operations

Repeat analysis with $t_i = t_i - t_0$

Table Analysis with t_0

I	$t_i - t_0$	F_i in %
1	70	6.7
2	150	16.2
3	200	25.9
4	400	35.5
5	680	45.2
6	750	54.8
7	1,050	64.5
8	1,400	74.1
9	2,120	83.8
10	2,630	93.3

Enter values in a new Weibull chart → Linear approximation is now better
→ Confirmation that a failure free time exists
Read parameters:

$\underline{\underline{b = 0.95}}$, $\underline{\underline{t_0 = 400}}$

$T - t_0 = 930 \quad \rightarrow \quad \underline{\underline{T}} = 930 + 400 = \underline{\underline{1{,}330 \text{ operations}}}$

b) Read from the graph:

$B_{10} - t_0 = 90 \quad \rightarrow \quad \underline{\underline{B_{10}}} = 90 + t_0 = \underline{\underline{490 \text{ operations}}}$;

$t_{50} - t_0 = 640 \quad \rightarrow \quad \underline{\underline{t_{50}}} = 640 + 400 = \underline{\underline{1{,}040 \text{ operations}}}$

c) Draw in the graph: see Weibull chart

Table Analysis

Rank i	$t_i - t_0$	$F_i^{5\%}$	$F_i(median)$	$F_i^{95\%}$
1	70	0.5116	6.7	25.8866
2	150	3.6771	16.2	39.4163
3	200	8.7264	25.9	50.6901
4	400	15.0028	35.5	60.6624
5	680	22.2441	45.2	69.9493
6	750	30.3537	54.8	77.7559
7	1,050	39.3376	64.5	84.9972
8	1,400	49.3099	74.1	91.2736
9	2,120	60.5836	83.8	96.3229
10	2,630	74.1134	93.3	99.4884

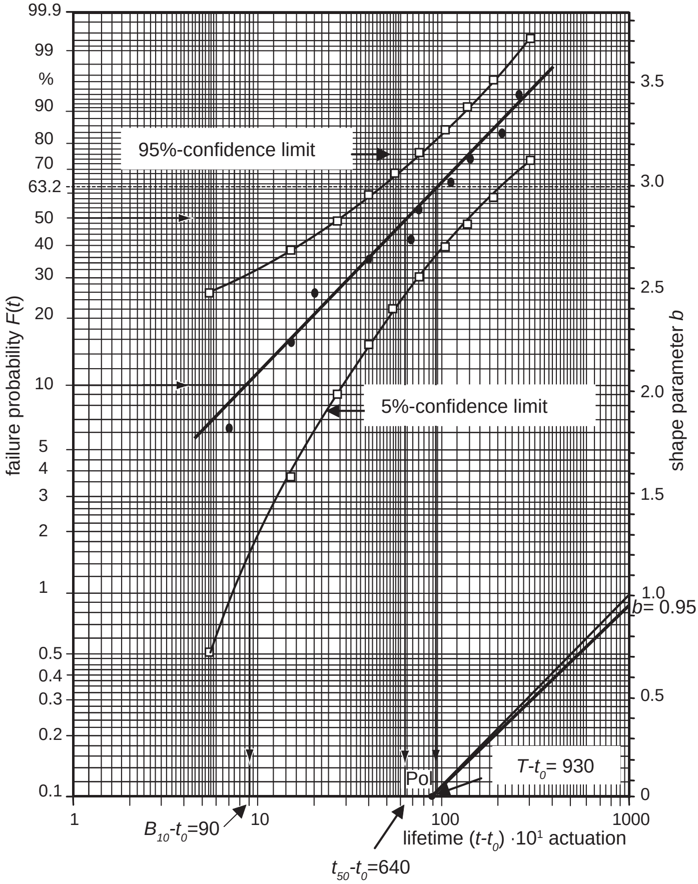


Figure Solution 6.2c

Solution 6.3

Sort values and assign Failure probabilities.

Table Analysis

i	$t_i \left[\cdot 10^3 \text{ LW}\right]$	$F_i = \dfrac{i-0.3}{n+0.4}$
1	166	6.7%
2	198	16.3%
3	208	26.0%
4	222	35.6%
5	242	45.2%
6	264	54.8%
7	380	64.4%
8	382	74.0%
9	434	83.7%
10	435	93.3%

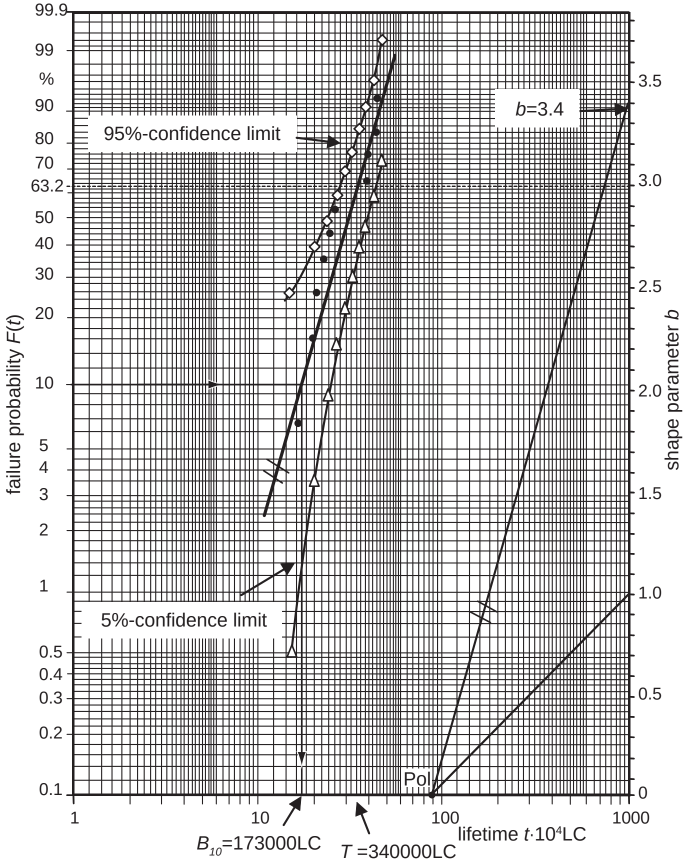


Figure Solution 6.3

Draw: The entered data points do indicate neither a straight line nor a typical curve for a failure free time; it looks more like a mixture distribution.

<u>Despite this:</u> Analysis points to a two parametric Weibull distribution $\rightarrow t_0 = 0$

Read parameters from the chart: $b = 3.4$; $T = 340{,}000$ load cycles

Draw in confidence levels.

Solution 6.4

Sample size: $n = 8$
Number of failures: $r = 5$

$\Big\}$ $n \neq r \rightarrow$ incomplete or censured

Tested timely parallel and stopped after the 5^{th} failure $\rightarrow$ Censor type II

Table Analysis

Rank i	Ranking Order $t_i[h]$	Median F_i	5% $F_i^{5\%}$	95% $F_i^{95\%}$
1	102	8.3%	0.6%	31.2%
2	135	22.1%	4.6%	47.0%
3	167	32.0%	11.1%	60.0%
4	192	44.0%	19.2%	71.1%
5	214	56.0%	29.0%	80.7%

Taken from the table with $n = 8$ until $i = 5$!

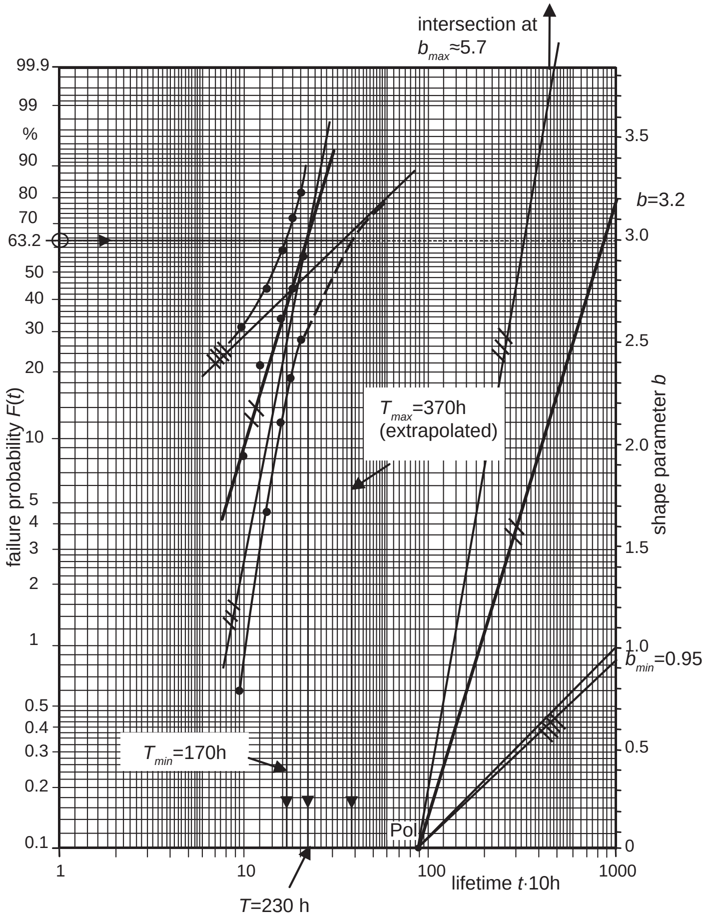


Figure Solution 6.4

Draw in graph, read b and T (for T extrapolated!) $\rightarrow$ $b = 3.2$; $T = 230\,\text{h}$

Draw confidence levels, extrapolate confidence levels

$\rightarrow$ read from graph: $b_{min} = 0.95$; $b_{max} = 5.7$; $T_{min} = 170$ h ; $T_{max} = 370\,\text{h}$

Solution 6.5

a) Data is already ordered → ranking order

$n = 1{,}075 \quad \hat{=} \quad$ sample size

$n_f = r = 10 \quad \hat{=} \quad$ number of failures

Inspection lot size:

$$k = \frac{n-r}{r+1} + 1 = \frac{1{,}075 - 10}{10+1} + 1 = 97.8 \approx 98$$

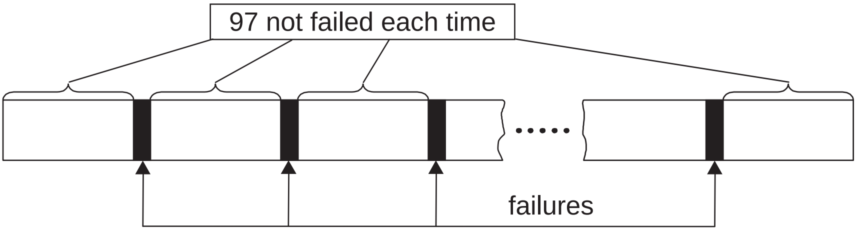


Figure Solution 6.5a

About 97 tractors which have not failed lie between the individual failures:

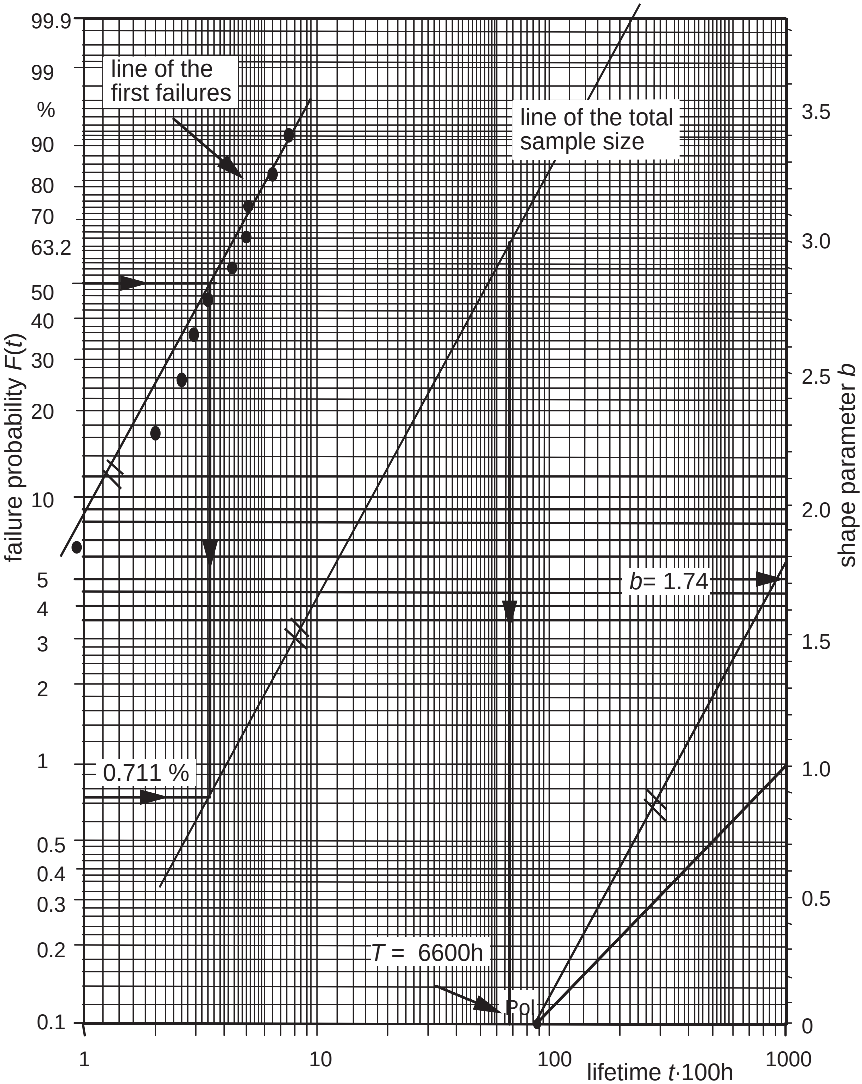


Figure Solution 6.5a

Control: $97 \cdot 11 + 10 = 1{,}077 \approx 1075$ ✓

Draw in the straight line for the first failures; F_i out of the table with $n = 10$.

Shift line: 50% value for the straight line for the first failures is assigned a new failure probability $F(t_{50}) = \dfrac{1-0.3}{k+0.4} = \dfrac{0.7}{98.4} = 0.711\%$. The slope b remains unchanged; thus, shift the line → straight line for the entire sample.

Read parameters: $b = 1.74$; $T = 6{,}600\,\text{h}$

b) Hypothetical ranks:

$j_i = j_{i-1} + N_i \qquad j_0 = 0$

$$N_i = \frac{n+1-j_{i-1}}{1+n-previos} \qquad F(t_i) = \frac{j_i - 0.3}{n+0.4}$$

For "Sudden Death" the "previous" value can be calculated, since the "in-between" value remains constant:

here $$N_i = \frac{n+1-j_{i-1}}{1+n-\left(\underbrace{i\cdot k}_{\text{in-between}}+\underbrace{(i-1)}_{\text{previously failed}}\right)} \qquad \forall i = 1(1)r$$

Note: for experimental analysis $N_i = \dfrac{n+1-j_{i-1}}{1+n-(i-1)\cdot(k+1)}$

Table Hypothetical ranks

i	t_i	previous	N_i	j_i	F_i [%]
1	99	97	1.099	1.099	0.075
2	200	195	1.22	2.32	0.189
3	260	293	1.37	3.69	0.315
4	300	391	1.56	5.25	0.461
5	340	489	1.82	7.07	0.630
6	430	587	2.18	9.25	0.833
7	499	685	2.72	11.97	1.086
8	512	783	3.62	15.59	1.423
9	654	881	5.41	21.00	1.926
10	760	979	10.76	31.76	2.930

$r = 10 \quad ; \quad n = 1{,}075$

Draw in the graph: $b = 1.77 \quad ; \quad T = 6{,}800h$

<u>Comparison with the graphical method:</u>

- Both methods agree with one another !
- Differences <u>only</u> due do drawing inaccuracies !

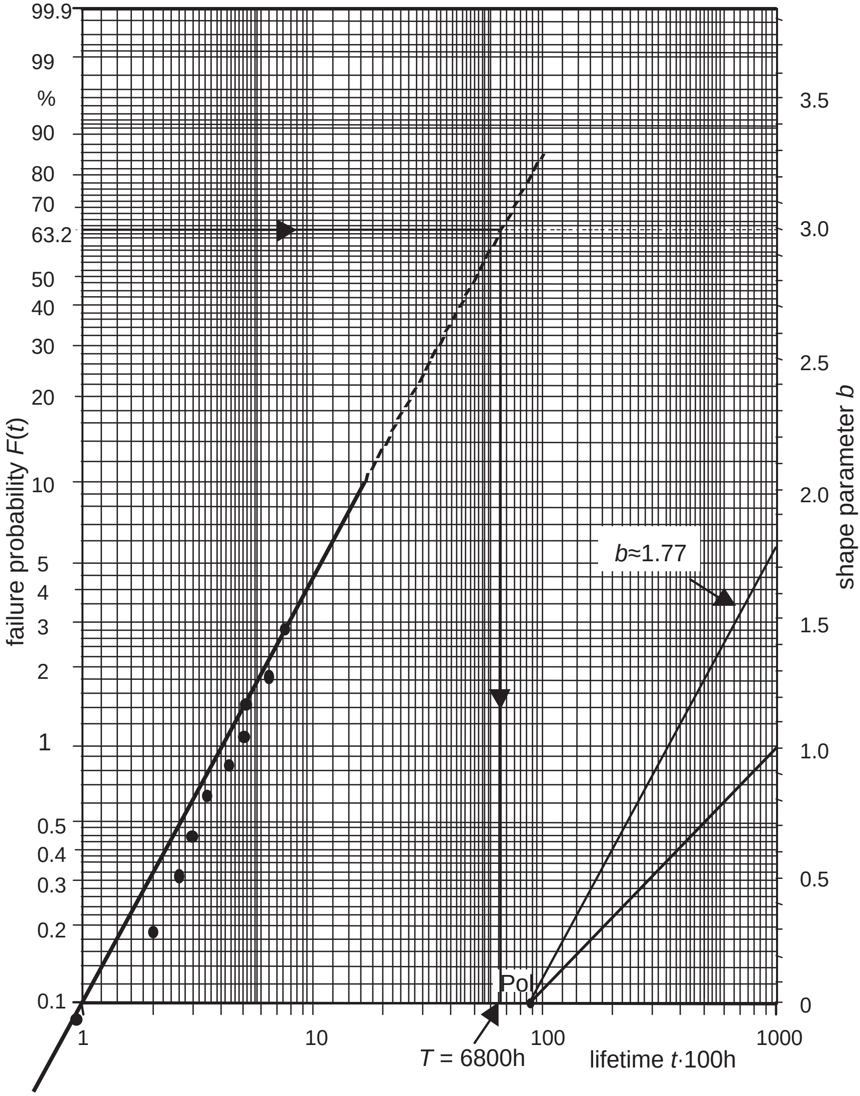


Figure Solution 6.5b

Solution 6.6

a) $n_f = 8$ (f stands for "failure")

$n_s = 12$ (s stands for "survivor")

$n = 20 \;\hat{=}\;$ sample size

Analysis under consideration of the parts without failure
$\Rightarrow$ hypothetical ranks !

$$j_0 = 0 \qquad j_i = j_{i-1} + N_i \quad \forall i = 1(1)n_f$$

$$N_i = \frac{n+1-j_{i-1}}{1+(n-previous)} \qquad F_i = \frac{j_i - 0{,}3}{n+0{,}4}$$

$\rightarrow$ Table

Table Hypothetical ranks

i	Time in 10^3 km	Without failure	failed	previous	N_i	j_i	F_i [%]
	5	X					
	6	X					
1	7		X	2	1.10	1.10	3.92
	19	X					
2	24		X	4	1.17	2.28	9.68
3	29		X	5	1.17	3.45	15.42
	32	X					
	39	X					
	40	X					
4	53		X	9	1.46	4.91	22.59
5	60		X	10	1.46	6.37	29.76
	65	X					
6	69		X	12	1.62	8.00	37.73
	70	X					
	76	X					
	85	X					
7	100		X	16	2.60	10.60	50.48
8	148		X	17	2.60	13.20	63.23
	157	X					
	160	X					
		$n_s = 12$	$n_f = 8$				
		$n = n_s + n_f = 20$					

$\rightarrow$ Draw in the graph, read the parameters: $b = 1.15$; $T = 150 \cdot 10^3 km$

b) Confidence level $\rightarrow$ 2 possibilities

↙ interpolation in table

↘ V_q procedure (Chapter 6.5)

1) Interpolation with the tables for 5% and 95% with n = 20.

Procedure:

- Form whole number rank numbers m_i, so that $m_i < j_i < m_{i+1}$
- Calculate the increment $\Delta j_i = j_i - m_i$
- Read out of the table: $F^{5\%}(m_i)$; $F^{5\%}(m_{i+1})$; $F^{95\%}(m_i)$; $F^{95\%}(m_{i+1})$;
- Interpolation
  $$F^{5\%}(j_i) = \left(F^{5\%}(m_{i+1}) - F^{5\%}(m_i)\right) \cdot \Delta j_i + F^{5\%}(m_i)$$
  $$F^{95\%}(j_i) = \left(F^{95\%}(m_{i+1}) - F^{95\%}(m_i)\right) \cdot \Delta j_i + F^{95\%}(m_i)$$
- Draw in the graph

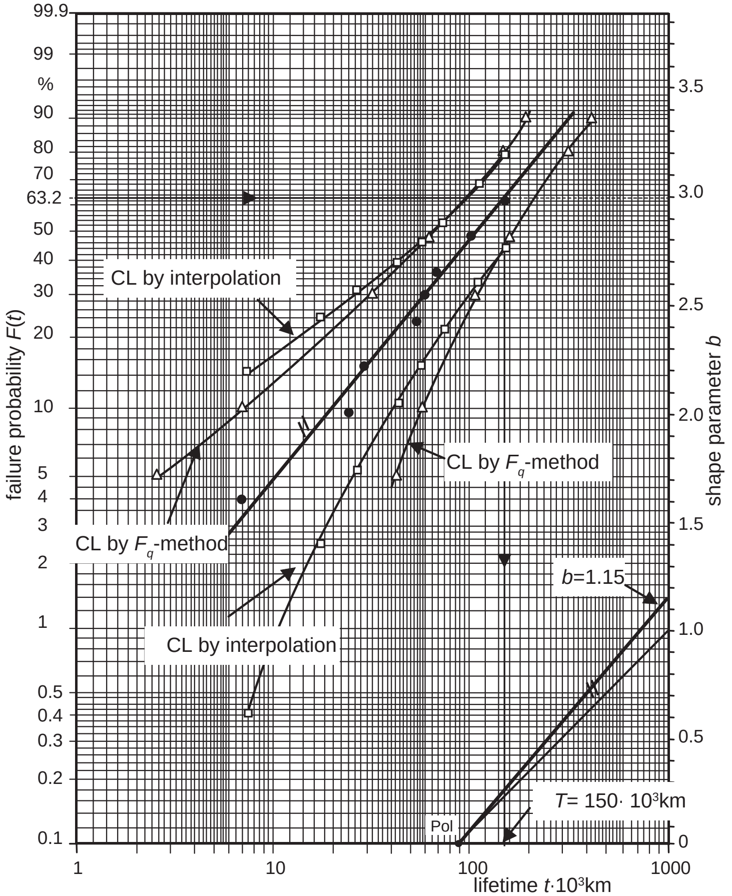


Figure Solution 6.6

Table Interpolation

i	j_i	m_i	Δj_i	$F^{5\%}(m_i)$	$F^{5\%}(m_{i+1})$	$F^{5\%}(j_i)$	$F^{95\%}(m_i)$	$F^{95\%}(m_{i+1})$	$F^{95\%}(j_i)$
1	1.10	1	0.10	0.256	1.807	0.41	13.911	21.611	14.67
2	2.28	2	0.28	1.807	4.217	2.48	21.611	28.262	23.47
3	3.45	3	0.45	4.217	7.135	5.53	28.262	34.366	31.00
4	4.91	4	0.91	7.135	10.408	10.11	34.366	40.103	39.59
5	6.37	6	0.37	13.956	17.731	15.35	45.558	50.781	47.49
6	8.00	8	1.00	21.707	--	21.707	55.804	--	55.804
7	10.60	10	0.60	30.196	34.692	32.89	65.308	69.804	68.01
8	13.20	13	0.20	44.196	49.219	45.20	78.293	82.269	79.06

(time-consuming calculation)

Enter these columns into the graph

2) V_q procedure

$n = 20 \rightarrow$ can be drawn starting at t_5 ($b = 1.15$)

Table V_q procedure

q	$t_q \cdot 10^3$ km	V_q	$t_{qo} = t_q \cdot V_q$	$t_{qu} = t_q / V_q$
5	10.8	4	43.2	2.7
10	20.5	2.9	59.5	7.1
30	59	1.8	106.2	32.8
50	106	1.6	169.6	66.3
80*	215	1.5	322.5	143.3
90*	290	1.49	432.1	194.6

*extrapolated

Enter these columns into the graph

(better)

Parameter confidence levels: draw, read from graph

Solution 6.7

Sample size: $n = 178$

Number of failures: $r = 7$

→ number of transmission which have not failed $n_s = n - r = 171$

Division of the transmissions which have not failed is achieved through an operational performance distribution.

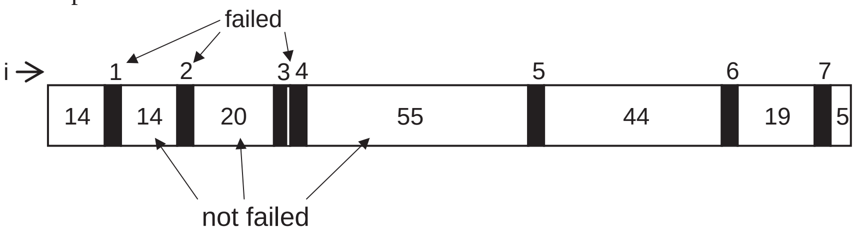


Figure Solution 6.7

Table Failure probability over the operational performance distribution

i	t_i	Occurrence Probability $L(t_i)$	Individual Frequency $\Delta L_i = L(t_i) - L(t_{i-1})$	Number of transmissions "in-between" $n_s(t_i) = \Delta L \cdot n_s$
1	18,290	8%	8%	14
2	35,200	16%	8%	14
3	51,450	28%	12%	20
4	51,450	28%	0%	0
5	89,780	60%	32%	55
6	130,580	86%	26%	44
7	160,770	97%	11%	19
	>160,770		3%	5
				$\sum 171$

Round off so that the value fits

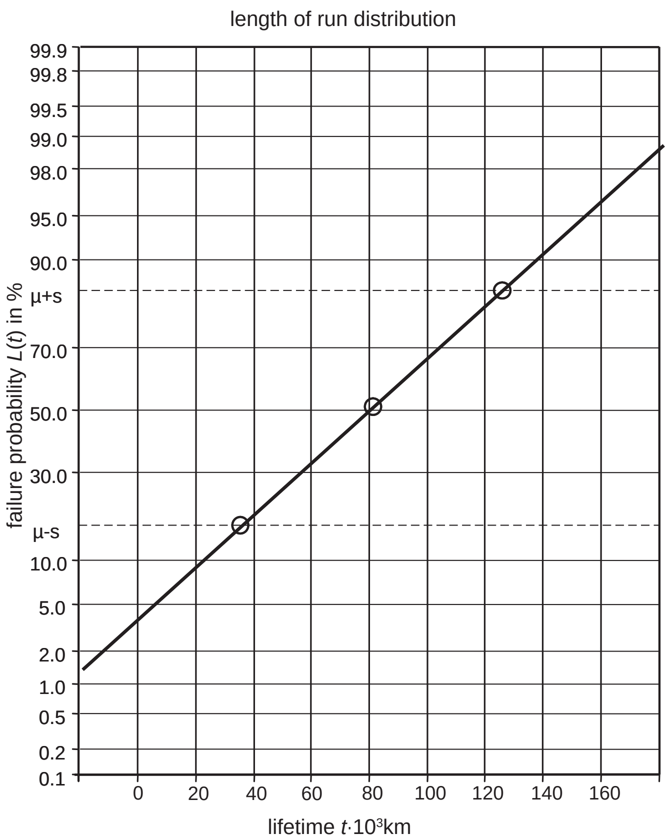


Figure Solution 6.7

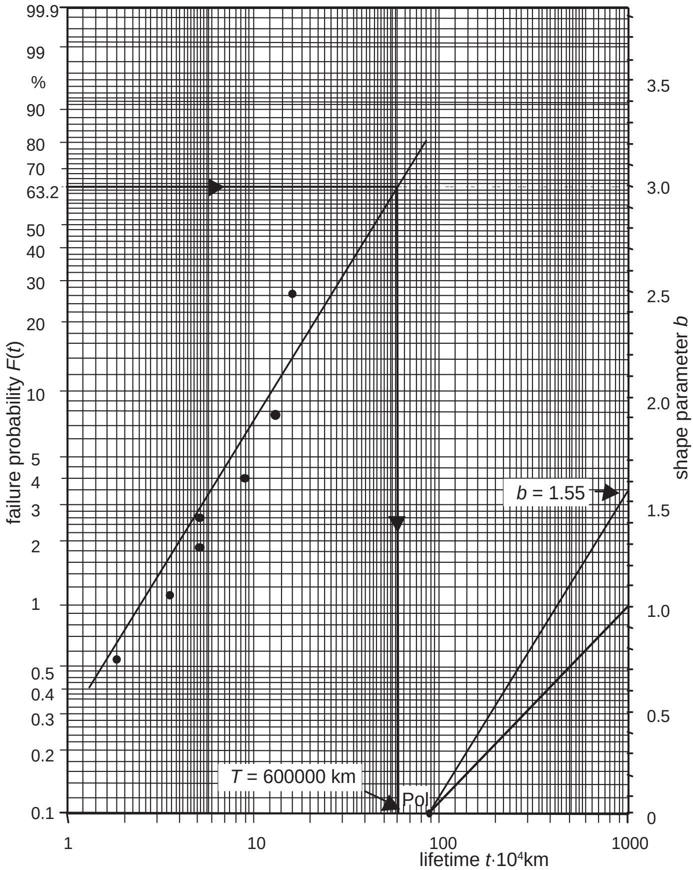


Figure Solution 6.7

Table Failure probability over hypothetical ranks

i	t_i	$n_s(t_i)$	Previous	N_i	j_i	F_i [%]
1	18,290	14	14	1.08	1.08	0.6
2	35,200	14	29	1.20	2.28	1.11
3	51,450	20	50	1.37	3.65	1.87
4	51,450	0	51	1.37	5.02	2.65
5	89,780	55	107	2.41	7.44	4.00
6	130,580	44	152	6.86	14.30	7.85
7	160,770	19	172	32.93	47.23	26.3

Enter these values in the Weibull chart

Read from the chart: $b = 1.55;\ T = 600{,}000 \text{km}$

Solution 6.8

Uncensored = complete → $n = r = 4$

a) Regression + Weibull → Equation (6.70) and Equation (6.71)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} \ln(t_i)$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} \ln(-\ln(1-F_i))$$

Result: $b = 2.63$; $T = 83.84$ h

b) $K_{Wei} = 0.98958$

c) $ln(L) = -18.380$

Solution 6.9

a) 2 parameters → first two moments are sufficient
empirical sample moments:

$$\bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i \quad (1) \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(t_i - \bar{t}\right)^2 \quad (2)$$

theoretical moments:
In comparison with the Weibull distribution for

$$b = 1 \text{ and } \lambda = \frac{1}{T - t_0} \quad \rightarrow \quad T = \frac{1}{\lambda} + t_0$$

$$E(t) = \underbrace{(T - t_0)}_{\hat{=}\frac{1}{\lambda}} \cdot \underbrace{\Gamma\left(\underbrace{1 + \frac{1}{b}}_{=2}\right)}_{\hat{=}\,1} + t_0 \qquad \text{Note: } \Gamma(n) = (n-1)!$$

$$\underline{\underline{E(t) = \frac{1}{\lambda} + t_0}}$$

$$\underline{\underline{Var(t)}} = \underbrace{(T - t_0)^2}_{\hat{=}\frac{1}{\lambda^2}} \cdot \left[\underbrace{\Gamma\left(\underbrace{1 + \frac{2}{b}}_{=3}\right)}_{=2} - \underbrace{\Gamma^2\left(1 + \frac{1}{b}\right)}_{=1}\right] = \underline{\underline{\frac{1}{\lambda^2}}}$$

Moment method:

$$\bar{t} = E(t) = \frac{1}{\lambda} + t_0; \qquad s^2 = Var(t) = \frac{1}{\lambda^2} \quad \Rightarrow \quad \underline{\underline{\lambda = \frac{1}{s}}}$$

$$\bar{t} = \frac{1}{\lambda} + t_0 = s + t_0 \quad \rightarrow \quad \underline{\underline{t_0 = \bar{t} - s}}$$

b) Maximum Likelihood

$$L(t_i, \lambda, t_0) = \prod_{i=1}^{n} f(t_i, \lambda, t_0) = \prod_{i=1}^{n} \left(\lambda \cdot e^{-\lambda(t_i - t_0)}\right)$$

logarithmize: $\ln(L) = \sum_{i=1}^{n} \ln\left(\lambda \cdot e^{-\lambda(t_i - t_0)}\right)$

Derive: $\dfrac{\partial \ln(L)}{\partial \lambda} = 0 \qquad \dfrac{\partial \ln(L)}{\partial t_0} = 0$

In general through logarithmic differentiation:

$$\frac{\partial\, ln(L)}{\partial \Psi_i} = \sum \frac{1}{f\left(t_i, \vec{\Psi}\right)} \cdot \frac{\partial f\left(t_i, \vec{\Psi}\right)}{\partial \Psi_i}$$

$$\frac{\partial f}{\partial \lambda} = e^{-\lambda(t_i - t_0)} + \lambda \cdot \left(-(t_i - t_0)\right) \cdot e^{-\lambda(t_i - t_0)} = \left(1 - \lambda \cdot (t_i - t_0)\right) \cdot e^{-\lambda(t_i - t_0)}$$

$$\frac{\partial\, ln(L)}{\partial \lambda} \overset{!}{=} 0 = \sum_{i=1}^{n} \frac{1}{\lambda \cdot e^{-\lambda(t_i - t_0)}} \cdot \left(1 - \lambda \cdot (t_i - t_0)\right) \cdot e^{-\lambda(t_i - t_0)}$$

$$\rightarrow \quad 0 = \sum_{i=1}^{n} \frac{\left(1 - \lambda \cdot (t_i - t_0)\right)}{\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} (t_i - t_0)$$

$$\rightarrow \qquad = \frac{n}{\lambda} - \underbrace{\sum_{i=1}^{n} t_i}_{= n \cdot \bar{t}} + \underbrace{\sum_{i=1}^{n} t_0}_{= n \cdot t_0}$$

$$\rightarrow \quad 0 = \frac{n}{\lambda} - n \cdot \bar{t} + n \cdot t_0$$

$$\lambda \cdot \left(n \cdot \bar{t} - n \cdot t_0\right) = n$$

$$\rightarrow \quad \lambda = \frac{n}{n \cdot \bar{t} - n \cdot t_0} = \frac{1}{\bar{t} - t_0}$$

$$\frac{\partial f(t_i, b, t_0)}{\partial t_0} = \lambda^2 \cdot e^{-\lambda(t_i - t_0)}$$

$$\frac{\partial\, ln(L)}{\partial t_0} = \sum_{i=1}^{n} \frac{1}{\lambda \cdot e^{-\lambda(t_i - t_0)}} \cdot \lambda^2 \cdot e^{-\lambda(t_i - t_0)} = 0$$

$$0 = \sum_{i=1}^{n} \lambda = n \cdot \lambda \quad \text{contradiction} \Rightarrow \text{estimate } t_0 = t_1$$

c) Regression:

$$f(t) = \lambda \cdot \exp(-\lambda \cdot (t - t_0))$$

$$\Rightarrow \quad F(t) = 1 - e^{-\lambda(t - t_0)} \quad \Rightarrow \quad 1 - F(t) = e^{-\lambda(t - t_0)}$$

Transformation:

$$\underbrace{\ln(1 - F(t))}_{y(x(t))} = -\lambda \cdot (t - t_0) = -\lambda \cdot t + \underbrace{\lambda \cdot t_0}_{c(\lambda, t_0)}$$

$-\lambda \rightarrow m(\lambda)$, $t \rightarrow x(t) = t$

Transformed failure probabilities:

$$y_i = \ln\left(1 - \frac{i - 0{,}3}{n + 0{,}4}\right)$$

Out of the appendix:

$$m = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sum_{i=1}^{n} x_i^2 - n \cdot \bar{x}}$$

$$c = \bar{y} - m \cdot \bar{x}$$

here: $x_i = t_i \quad \bar{x} = \bar{t}$

$$y_i = \ln\left(1 - \frac{i - 0{,}3}{n + 0{,}4}\right) \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\rightarrow \quad \underline{\underline{\lambda = -m}} \qquad \underline{\underline{t_0 = \frac{c}{\lambda}}}$$

Solution 8.1

a) $\underbrace{B_{10}} = 250000$ km

$\quad \longrightarrow$ failure probability $F(t = B_{10}) = 10\%$

$\rightarrow$ required reliability $R(t = B_{10}) = 90\%$

confidence level $P_A = 95\%$

Look in the table for the 95% confidence level for the column whose value for $i = 1$ is just less than 10% $\Rightarrow \underline{\underline{n}}$

here (extract):

Table Extract from the 95% confidence level

	$n = 27$	$n = 28$	$n = 29$	$n = 30$
$i = 1$	10.502	10.147	9.814	9.503
$i = 2$	16.397	15.851	15.340	14.859

Thus: $\underline{\underline{n = 29}}$

b) $R(t) = (1 - P_A)^{\frac{1}{n}}$

$$ln(R(t)) = \frac{1}{n} ln(1 - P_A)$$

$$\underline{\underline{n}} = \frac{ln(1 - P_A)}{ln(R(t))} = \frac{ln(1 - 0.95)}{ln(0.)} = 28.43 = \underline{\underline{29}}$$

(Result matches the result from a) very well!)

c) $t_{test\ max} = 150000$ km and $t_{soll} = 250000$ km

$$\left(\frac{t_{test}}{t}\right)^b = L_v^b = \left(\frac{150000}{250000}\right)^{1.5} = 0.46$$

now: $R(t) = (1 - P_A)^{\frac{1}{L_v^b \cdot n}}$

$$ln(R(t)) = \frac{1}{L_v^b \cdot n} \cdot ln(1 - P_A)$$

$$n = \frac{1}{L_v^b} \cdot \frac{ln(1 - P_A)}{ln(R(t))} = \frac{1}{0.46} \cdot \underbrace{\frac{ln(1 - P_A)}{ln(R(t))}} = \frac{28.43}{0.46} = 61.17$$

$$\underline{\underline{n_{erf} = 62 \text{ Getriebe !}}} \qquad \hat{=}\; n \text{ aus b)}$$

d) $n = 15 \quad ; \quad t_{test} = ? \quad t_{test} = L_v \cdot t_{soll}$

$$R(t) = (1 - P_A)^{\frac{1}{L_v^b \cdot n}}$$

$$\rightarrow \quad L_v^b = \frac{1}{n} \cdot \frac{\ln(1 - P_A)}{\ln(R(t))} = \left(\frac{t_p}{t_{soll}}\right)^b$$

$$\rightarrow \quad t_{test} = t_{soll} \cdot \sqrt[b]{\frac{1}{n} \cdot \frac{ln(1 - P_A)}{ln(R(t))}} = 250000 \cdot \sqrt[1.5]{\frac{1}{15} \cdot \frac{ln(1 - 0.95)}{ln(0.9)}}$$

$$\underline{\underline{t_{test} = 382909.3 \text{ km}}}$$

e) Larson nomogram: $P_A = 0.95\,;\, n = 30\,;\, x = 3 \rightarrow \underline{\underline{R = 76\%}}$

f) Larson nomogram: $R = 0.9\,;\, n = 30\,;\, x = 3 \rightarrow \underline{\underline{P_A = 31\%\ !}}$

g) Larson nomogram:

$$P_A = 0.95\,;\, R = 0.9\,;\, x = 3 \rightarrow \underline{\underline{n_{notw} = 80}} \rightarrow \underline{\underline{n^*}} = n_{notw} - n = 80 - 30 = \underline{\underline{50}}$$

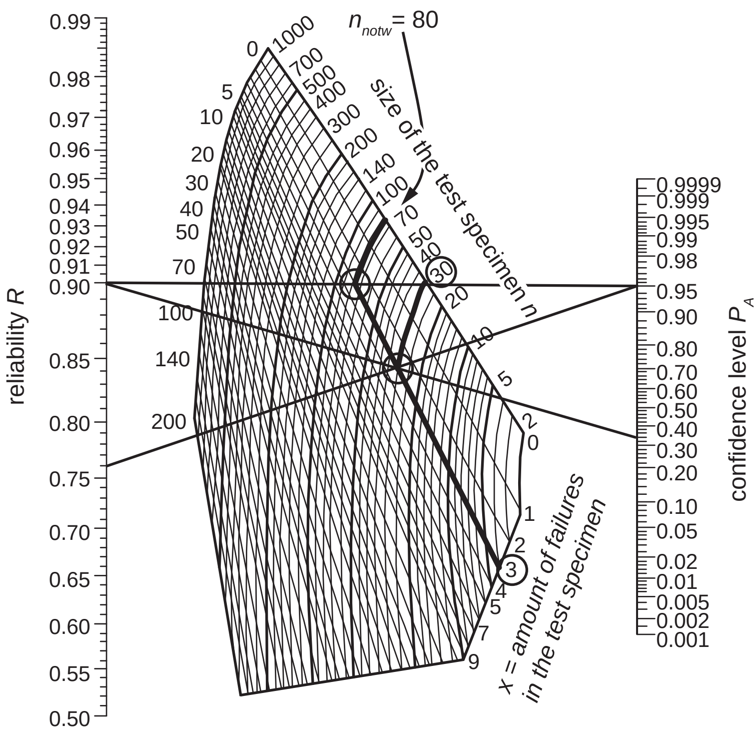


Figure Solution 8.1e-g

h) According to Beyer/Lauster, with the previous knowledge $R_0 = 0{,}9$ (with a confidence level of 63,2%):

$$\underline{\underline{n}} = \frac{1}{L_v^b} \cdot \left[\frac{ln(1-P_A)}{ln(R)} - \frac{1}{ln\left(\frac{1}{R_0}\right)} \right] \quad (*)$$

$$= \frac{1}{1^{1.5}} \left[\frac{ln(1-0.95)}{ln(0.9)} - \frac{1}{ln\left(\frac{1}{0.9}\right)} \right] = 28.43 - 9.49 = 18.93 \approx \underline{\underline{19}}$$

i) Equation (*) out of h) solved for $t_{test} = L_v \cdot t$:

$$\underline{\underline{t_{test}}} = t \cdot \left(\frac{1}{n} \cdot \left[\frac{ln(1-P_A)}{ln(R)} - \frac{1}{ln\left(\frac{1}{R_0}\right)} \right] \right)^{\frac{1}{b}}$$

$$= 250000 \text{ km} \cdot \left(\frac{1}{12} \cdot \left[\frac{ln(1-0.95)}{ln(0.9)} \right] - \frac{1}{ln\left(\frac{1}{0.9}\right)} \right)^{\frac{1}{1.5}}$$

$$= 338{,}781.43\text{km} \approx \underline{\underline{340{,}000 \text{ km}}}$$

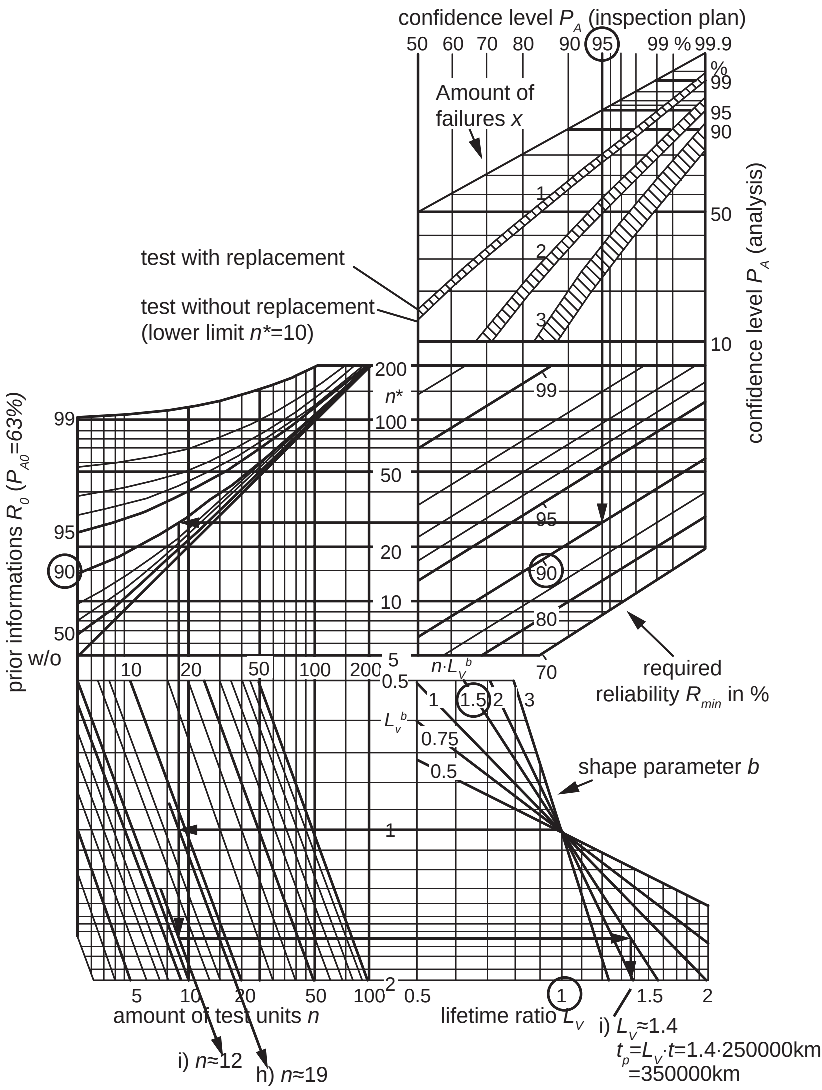


Figure Solution 8.1h-i

Solution 8.2

$n = 2 \quad ; \quad x = 1$

Failures during the test → generalized binomial approach

$P_A = 1 - R^n - n \cdot (1 - R) \cdot R^{n-1}$ for $x = 1$

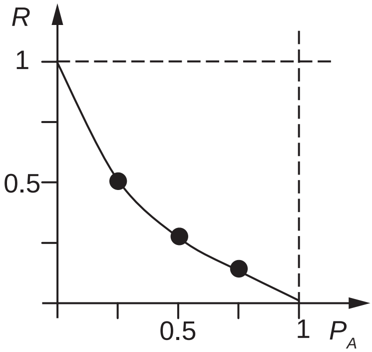


Figure Solution 8.2

$$P_A = 1 - R^2 - 2 \cdot (1 - R) \cdot R$$
$$= 1 - R^2 - 2 \cdot R + 2 \cdot R^2$$
$$= 1 - 2 \cdot R + R^2$$
$$= (1 - R)^2$$
$$\sqrt{P_A} = \pm(1 - R)$$
$$\Rightarrow R_1 = \sqrt{P_A} + 1$$

Contradiction $\notin [0,1]$

$$\Rightarrow R_2 = \underline{\underline{R = 1 - \sqrt{P_A}}}$$

Solution 8.3

$$F(T) = 63.2\% \quad \rightarrow \quad R(T) = 36.8\%$$

$$L_V = \sqrt[b]{\frac{1}{n} \cdot \frac{ln(1 - P_A)}{ln(R(T))}} = \frac{t_{test} - t_0}{T - t_0} \quad \leftarrow \text{due to } t_0$$

$$\rightarrow t_{test} = (T - t_0) \sqrt[b]{\frac{1}{n} \cdot \frac{ln(1 - P_A)}{ln(R(T))}} + t_0 = (12 - 2) \cdot 10^5 \cdot \sqrt[1.4]{\frac{1}{8} \cdot \frac{ln(0.1)}{ln(0.368)}} + 2 \cdot 10^5$$

$$= \underline{\underline{610{,}925 \text{ load cycles}}}$$

Solution 8.4

Given: $B_{10} = 250{,}000$ km $\rightarrow$ $R(B_{10}) = 90\%$

a) $$t_{test} = B_{10} \cdot \sqrt[b]{\frac{1}{n} \cdot \frac{ln(1 - P_A)}{ln(R(B_{10}))}} = 250000 \cdot \sqrt[1.5]{\frac{1}{23} \cdot \frac{ln(0.05)}{ln(0.9)}}$$

$$\rightarrow \underline{\underline{t_{test} = 287{,}964 \text{ km}}}$$

b) Previous knowledge: $T = 1.5 \cdot 10^6$ km

Beyer/Lauster: previous knowledge $R_0(B_{10})$ required !

$$\underline{\underline{R_0(B_{10})}} = e^{-\left(\frac{B_{10}}{T}\right)^b} = e^{-\left(\frac{250}{1500}\right)^{1.5}} = 0.9342 \mathrel{\hat{=}} \underline{\underline{93.4\%}}$$

$$\underline{\underline{t_{test}}} = B_{10} \cdot \sqrt[b]{\frac{1}{n} \cdot \left[\frac{ln(1-P_A)}{ln(R(B_{10}))} - \frac{1}{ln\left(\frac{1}{R_0}\right)} \right]}$$

$$= 250000 \cdot \sqrt[1.5]{\frac{1}{23} \cdot \left[\frac{ln(0.05)}{ln(0.9)} - \frac{1}{ln\left(\frac{1}{0.9342}\right)} \right]} = \underline{\underline{177{,}339.66 \text{ km}}}$$

Solution 10.1

$$A_{Di} = \frac{MTTF}{MTTF + MTTR} = \frac{1}{1 + \frac{MTTR}{MTTF}}$$

$$\Rightarrow \quad MTTR = \left(\frac{1}{A_D} - 1\right) \cdot MTTF = \left(\frac{1}{0.99} - 1\right) \cdot 5{,}000\,\text{h} = \underline{\underline{50.51\,\text{h}}}$$

Solution 10.2

Duration availability of an individual component:

$$A_{Di} = \frac{MTTF}{MTTF + MTTR} = \frac{1}{1 + \frac{MTTR}{MTTF}}$$

The following equation is valid for three **identical** components:

$$A_{DS} = A_{Di}^{\;3} = \left(\frac{1}{1 + \frac{MTTR}{MTTF}}\right)^3$$

$$\Rightarrow \quad MTTR = \left(\frac{1}{\sqrt[3]{A_{DS}}} - 1\right) \cdot MTTF = \left(\frac{1}{\sqrt[3]{0{,}9}} - 1\right) \cdot 1{,}500\,\text{h} = \underline{\underline{53.62\ \text{h}}}$$

Solution 10.3

$$A_{DS} = 1 - (1 - A_{Di})^3$$

$$\Rightarrow \quad A_{Di} = 1 - \sqrt[3]{1 - A_{DS}} = 1 - \sqrt[3]{1 - 0.999} = \underline{\underline{90\,\%}}$$

Solution 10.4

$$A_{DS} = 1 - (1 - A_{Di})^3$$

The following equation is valid for one individual component:

$$1 - A_{Di} = 1 - \frac{MTTF}{MTTF + MTTR} = \frac{MTTR}{MTTF + MTTR} = \frac{1}{\frac{MTTF}{MTTR} + 1}$$

$\Rightarrow$ For three **identical** components the following relationship is valid:

$$A_{DS} = 1 - \left(\frac{1}{\frac{MTTF}{MTTR} + 1} \right)^3 \Rightarrow \quad MTTR = \frac{MTTF}{\frac{1}{\sqrt[3]{1 - A_{DS}}} - 1} = \frac{1500\,\text{h}}{\frac{1}{\sqrt[3]{1 - 0.99}} - 1} = \underline{\underline{411.91\,\text{h}}}$$

Solution 10.5

a) $A_{DS} = A_{D1} \cdot (1 - (1 - A_{D2}) \cdot (1 - A_{D3}));$ $\qquad A_{D2} = A_{D3}$

$$\Rightarrow \quad A_{D1} = \frac{A_{DS}}{1 - (1 - A_{D2})^2} = \underline{\underline{95.96\,\%}}$$

b) $MTTR = \left(\frac{1}{A_{D1}} - 1 \right) \cdot MTTF = \underline{\underline{42.1\,\text{h}}}$

Solution 10.6

a) $S(t)$ = stock at the point in time t
I = initial stock

$$S(t) = I - \hat{H}_1(t)$$

$$S(t) = I - \left(\frac{t}{MTTF + MTTR} + \frac{Var(\tau_1) + Var(\tau_0) + MTTR^2 - MTTF^2}{2 \cdot (MTTF + MTTR)^2} \right)$$

b) Solve for I:

$$I = S(t) + \left(\frac{t}{MTTF + MTTR} + \frac{Var(\tau_1) + Var(\tau_0) + MTTR^2 - MTTF^2}{2 \cdot (MTTF + MTTR)^2} \right)$$

with

$$Var(\tau_1) = \frac{1}{\left(0.002 \frac{1}{\text{h}}\right)^2} = 250\,000\,\text{h}^2\,; \qquad MTTF = \frac{1}{\lambda} = 500\,\text{h}$$

$$Var(\tau_0) = \frac{1}{\left(0.1\,\frac{1}{h}\right)^2} = 100\,h^2\,; \qquad MTTR = \frac{1}{\mu} = 10\,h$$

$$S(8760\,\text{h}) = 0$$

$$I = 17.18\,\text{Teile} \quad \Rightarrow \quad \underline{\underline{18\,\text{Teile}}}$$

Solution 10.7

a) $MTTF = \dfrac{1}{\lambda} = 33.3\,h$

$$MTTR = \frac{1}{\mu} = 5\,\text{h}$$

$$\Rightarrow \quad A_D = \frac{MTTF}{MTTF + MTTR} = \underline{\underline{86.96\,\%}}$$

b) $A(t) = \dfrac{\mu}{\mu+\lambda} + \dfrac{\lambda}{\mu+\lambda} \cdot e^{-(\lambda+\mu)\cdot t}$

$$A(2.1\,\text{h}) = \underline{\underline{95\,\%}}$$

Solution 10.8

a) $MTTF = \dfrac{1}{\lambda} = 100\,\text{h}\,; \; MTTR = \dfrac{1}{\mu} = 100\,\text{h}$

$$A_D = \frac{MTTF}{MTTF + MTTR} = \underline{\underline{90.91\%}}$$

b) $A(t) = \dfrac{\mu}{\mu+\lambda} + \dfrac{\lambda}{\mu+\lambda} \cdot e^{-(\mu+\lambda)\cdot t}$

$$\Rightarrow \; t = \frac{\ln\left(\frac{(\mu+\lambda)\cdot A(t)}{\lambda} - \frac{\mu}{\lambda}\right)}{-(\mu+\lambda)}$$

$$A(t^*) = 95\,\% \; \Rightarrow \; t^* = \underline{\underline{7.26\,\text{h}}}$$

Appendix

Table A.1. 5 %-confidence limit

Table A.1.1. Failure probability in % for the 5 %-confidence limit for a sample size of *n* ($1 \leq n \leq 10$) and the rank *i*

	n = 1	2	3	4	5	6	7	8	9	10
i =1	5,0000	2,5321	1,6952	1,2742	1,0206	0,8512	0,7301	0,6391	0,5683	0,5116
2		22,3607	13,5350	9,7611	7,6441	6,2850	5,3376	4,6389	4,1023	3,6771
3			36,8403	24,8604	18,9256	15,3161	12,8757	11,1113	9,7747	8,7264
4				47,2871	34,2592	27,1338	22,5321	19,2903	16,8750	15,0028
5					54,9281	41,8197	34,1261	28,9241	25,1367	22,2441
6						60,6962	47,9298	40,0311	34,4941	30,3537
7							65,1836	52,9321	45,0358	39,3376
8								68,7656	57,0864	49,3099
9									71,6871	60,5836
10										74,1134

Table A.1.2. Failure probability in % for the 5 %-confidence limit for a sample size of *n* ($11 \leq n \leq 20$) and the rank *i*

	n = 11	12	13	14	15	16	17	18	19	20
i =1	0,4652	0,4265	0,3938	0,3657	0,3414	0,3201	0,3013	0,2846	0,2696	0,2561
2	3,3319	3,0460	2,8053	2,5999	2,4226	2,2679	2,1318	2,0111	1,9033	1,8065
3	7,8820	7,1870	6,6050	6,1103	5,6847	5,3146	4,9898	4,7025	4,4465	4,2169
4	13,5075	12,2851	11,2666	10,4047	9,6658	9,0252	8,4645	7,9695	7,5294	7,1354
5	19,9576	18,1025	16,5659	15,2718	14,1664	13,2111	12,3771	11,6426	10,9906	10,4081
6	27,1250	24,5300	22,3955	20,6073	19,0865	17,7766	16,6363	15,6344	14,7469	13,9554
7	34,9811	31,5238	28,7049	26,3585	24,3727	22,6692	21,1908	19,8953	18,7504	17,7311
8	43,5626	39,0862	35,4799	32,5028	29,9986	27,8602	26,0114	24,3961	22,9721	21,7069
9	52,9913	47,2674	42,7381	39,0415	35,9566	33,3374	31,0829	29,1201	27,3946	25,8651
10	63,5641	56,1894	50,5350	45,9995	42,2556	39,1011	36,4009	34,0598	32,0087	30,1954
11	76,1596	66,1320	58,9902	53,4343	48,9248	45,1653	41,9705	39,2155	36,8115	34,6931
12		77,9078	68,3660	61,4610	56,0216	51,5604	47,8083	44,5955	41,8064	39,3585
13			79,4184	70,3266	63,6558	58,3428	53,9451	50,2172	47,0033	44,1966
14				80,7364	72,0604	65,6175	60,4358	56,1118	52,4203	49,2182
15					81,8964	73,6042	67,3807	62,3321	58,0880	54,4417
16						82,9251	74,9876	68,9738	64,0574	59,8972
17							83,8434	76,2339	70,4198	65,6336
18								84,6683	77,3626	71,7382
19									85,4131	78,3894
20										86,0891

Table A.1.3. Failure probability in % for the 5 %-confidence limit for a sample size of n ($21 \leq n \leq 30$) and the rank i

	n = 21	22	23	24	25	26	27	28	29	30
i=1	0,2440	0,2329	0,2228	0,2135	0,2050	0,1971	0,1898	0,1830	0,1767	0,1708
2	1,7191	1,6397	1,5674	1,5012	1,4403	1,3842	1,3323	1,2841	1,2394	1,1976
3	4,0100	3,8223	3,6515	3,4953	3,3520	3,2199	3,0978	2,9847	2,8796	2,7816
4	6,7806	6,4596	6,1676	5,9008	5,6563	5,4312	5,2233	5,0308	4,8520	4,6855
5	9,8843	9,4109	8,9809	8,5885	8,2291	7,8986	7,5936	7,3114	7,0494	6,8055
6	13,2448	12,6034	12,0215	11,4911	11,0056	10,5597	10,1485	9,7682	9,4155	9,0874
7	16,8176	15,9941	15,2480	14,5686	13,9475	13,3774	12,8522	12,3669	11,9169	11,4987
8	20,5750	19,5562	18,6344	17,7961	17,0304	16,3282	15,6819	15,0851	14,5322	14,0185
9	24,4994	23,2724	22,1636	21,1566	20,2378	19,3960	18,6220	17,9077	17,2465	16,6326
10	28,5801	27,1313	25,8243	24,6389	23,5586	22,5700	21,6617	20,8243	20,0496	19,3308
11	32,8109	31,1264	29,6093	28,2356	26,9853	25,8424	24,7934	23,8271	22,9340	22,1059
12	37,1901	35,2544	33,5148	31,9421	30,5130	29,2082	28,0120	26,9111	25,8944	24,9526
13	41,7199	39,5156	37,5394	35,7564	34,1389	32,6642	31,3139	30,0725	28,9271	27,8669
14	46,4064	43,9132	41,6845	39,6785	37,8622	36,2089	34,6972	33,3090	32,0296	30,8464
15	51,2611	48,4544	45,9544	43,7107	41,6838	39,8424	38,1613	36,6197	35,2005	33,8893
16	56,3024	53,1506	50,3565	47,8577	45,6067	43,5663	41,7069	40,0044	38,4392	36,9948
17	61,5592	58,0200	54,9025	52,1272	49,6359	47,3838	45,3360	43,4645	41,7464	40,1629
18	67,0789	63,0909	59,6101	56,5309	53,7791	51,3002	49,0522	47,0021	45,1235	43,3945
19	72,9448	68,4087	64,5067	61,0861	58,0480	55,3234	52,8608	50,6211	48,5730	46,6914
20	79,3275	74,0533	69,6362	65,8192	62,4595	59,4646	56,7698	54,3269	52,0988	50,0561
21	86,7054	80,1878	75,0751	70,7727	67,0392	63,7405	60,7902	58,1272	55,7064	53,4927
22		87,2695	80,9796	76,0199	71,8277	68,1758	64,9380	62,0330	59,4034	57,0066
23			87,7876	81,7108	76,8960	72,8098	69,2374	66,0598	63,2004	60,6053
24				88,2654	82,3879	77,7107	73,7261	70,2309	67,1127	64,2991
25					88,7072	83,0169	78,4700	74,5830	71,1628	68,1029
26						89,1170	83,6026	79,1795	75,3861	72,0385
27							89,4981	84,1493	79,8439	76,1402
28								89,8534	84,6608	80,4674
29									90,1855	85,1404
30										90,4966

Table A.2. Median values

Tabelle A.2.1. Median values in % for a sample size of n ($1 \le n \le 10$) and the rank i

	n = 1	2	3	4	5	6	7	8	9	10
i =1	50,0000	29,2893	20,6299	15,9104	12,9449	10,9101	9,4276	8,2996	7,4125	6,6967
2		70,7107	50,0000	38,5728	31,3810	26,4450	22,8490	20,1131	17,9620	16,2263
3			79,3700	61,4272	50,0000	42,1407	36,4116	32,0519	28,6237	25,8575
4				84,0896	68,6190	57,8593	50,0000	44,0155	39,3085	35,5100
5					87,0550	73,5550	63,5884	55,9845	50,0000	45,1694
6						89,0899	77,1510	67,9481	60,6915	54,8306
7							90,5724	79,8869	71,3763	64,4900
8								91,7004	82,0380	74,1425
9									92,5875	83,7737
10										93,3033

Tabe A.2.2. Median values in % for a sample size of n ($11 \le n \le 20$) and the rank i

	n = 11	12	13	14	15	16	17	18	19	20
i =1	6,1069	5,6126	5,1922	4,8305	4,5158	4,2397	3,9953	3,7776	3,5824	3,4064
2	14,7963	13,5979	12,5791	11,7022	10,9396	10,2703	9,6782	9,1506	8,6775	8,2510
3	23,5785	21,6686	20,0449	18,6474	17,4321	16,3654	15,4218	14,5810	13,8271	13,1474
4	32,3804	29,7576	27,5276	25,6084	23,9393	22,4745	21,1785	20,0238	18,9885	18,0550
5	41,1890	37,8529	35,0163	32,5751	30,4520	28,5886	26,9400	25,4712	24,1543	22,9668
6	50,0000	45,9507	42,5077	39,5443	36,9671	34,7050	32,7038	30,9207	29,3220	27,8805
7	58,8110	54,0493	50,0000	46,5147	43,4833	40,8227	38,4687	36,3714	34,4909	32,7952
8	67,6195	62,1471	57,4923	53,4853	50,0000	46,9408	44,2342	41,8226	39,6603	37,7105
9	76,4215	70,2424	64,9837	60,4557	56,5167	53,0592	50,0000	47,2742	44,8301	42,6262
10	85,2037	78,3314	72,4724	67,4249	63,0330	59,1774	55,7658	52,7258	50,0000	47,5421
11	93,8931	86,4021	79,9551	74,3916	69,5480	65,2950	61,5313	58,1774	55,1699	52,4580
12		94,3874	87,4209	81,3526	76,0607	71,4114	67,2962	63,6286	60,3397	57,3738
13			94,8078	88,2978	82,5679	77,5255	73,0600	69,0793	65,5091	62,2895
14				95,1695	89,0604	83,6346	78,8215	74,5288	70,6780	67,2048
15					95,4842	89,7297	84,5782	79,9762	75,8457	72,1195
16						95,7603	90,3218	85,4190	81,0115	77,0332
17							96,0047	90,8494	86,1729	81,9450
18								96,2224	91,3225	86,8526
19									96,4176	91,7490
20										96,5936

Table A.2.3. Median values in % for a sample size of n ($21 \leq n \leq 30$) and the rank i

	n = 21	22	23	24	25	26	27	28	29	30
i =1	3,2468	3,1016	2,9687	2,8468	2,7345	2,6307	2,5345	2,4451	2,3618	2,2840
2	7,8644	7,5124	7,1906	6,8952	6,6231	6,3717	6,1386	5,9221	5,7202	5,5317
3	12,5313	11,9704	11,4576	10,9868	10,5533	10,1526	9,7813	9,4361	9,1145	8,8141
4	17,2090	16,4386	15,7343	15,0879	14,4925	13,9422	13,4323	12,9583	12,5166	12,1041
5	21,8905	20,9107	20,0147	19,1924	18,4350	17,7351	17,0864	16,4834	15,9216	15,3968
6	26,5740	25,3844	24,2968	23,2986	22,3791	21,5294	20,7419	20,0100	19,3279	18,6909
7	31,2584	29,8592	28,5798	27,4056	26,3241	25,3246	24,3983	23,5373	22,7350	21,9857
8	35,9434	34,3345	32,8634	31,5132	30,2695	29,1203	28,0551	27,0651	26,1426	25,2809
9	40,6288	38,8102	37,1473	35,6211	34,2153	32,9163	31,7123	30,5932	29,5504	28,5764
10	45,3144	43,2860	41,4315	39,7292	38,1613	36,7125	35,3696	34,1215	32,9585	31,8721
11	50,0000	47,7620	45,7157	43,8375	42,1075	40,5089	39,0271	37,6500	36,3667	35,1679
12	54,6856	52,2380	50,0000	47,9458	46,0537	44,3053	42,6847	41,1785	39,7749	38,4639
13	59,3712	56,7140	54,2843	52,0542	50,0000	48,1018	46,3423	44,7071	43,1833	41,7599
14	64,0566	61,1898	58,5685	56,1625	53,9463	51,8982	50,0000	48,2357	46,5916	45,0559
15	68,7416	65,6655	62,8527	60,2708	57,8925	55,6947	53,6577	51,7643	50,0000	48,3520
16	73,4260	70,1408	67,1366	64,3789	61,8386	59,4911	57,3153	55,2929	53,4084	51,6480
17	78,1095	74,6156	71,4202	68,4868	65,7847	63,2875	60,9729	58,8215	56,8167	54,9441
18	82,7911	79,0894	75,7032	72,5944	69,7305	67,0837	64,6304	62,3500	60,2251	58,2401
19	87,4687	83,5614	79,9853	76,7014	73,6759	70,8797	68,2877	65,8785	63,6333	61,5361
20	92,1356	88,0296	84,2657	80,8076	77,6209	74,6754	71,9449	69,4068	67,0415	64,8320
21	96,7532	92,4876	88,5425	84,9121	81,5650	78,4706	75,6017	72,9349	70,4496	68,1279
22		96,8984	92,8094	89,0132	85,5075	82,2649	79,2581	76,4627	73,8574	71,4236
23			97,0313	93,1048	89,4467	86,0578	82,9136	79,9900	77,2650	74,7191
24				97,1532	93,3769	89,8474	86,5677	83,5166	80,6721	78,0143
25					97,2655	93,6283	90,2187	87,0417	84,0784	81,3091
26						97,3693	93,8614	90,5639	87,4834	84,6032
27							97,4655	94,0779	90,8855	87,8959
28								97,5549	94,2798	91,1859
29									97,6382	94,4683
30										97,7160

Table A.3. 95 %-confidence limit

Table A.3.1. Failure probability in % for the 95 %-confidence limit for a sample size of n ($1 \leq n \leq 10$) and the rank i

	n = 1	2	3	4	5	6	7	8	9	10
i =1	95,0000	77,6393	63,1597	52,7129	45,0720	39,3038	34,8164	31,2344	28,3129	25,8866
2		97,4679	86,4650	75,1395	65,7408	58,1803	52,0703	47,0679	42,9136	39,4163
3			98,3047	90,2389	81,0744	72,8662	65,8738	59,9689	54,9642	50,6901
4				98,7259	92,3560	84,6839	77,4679	71,0760	65,5058	60,6624
5					98,9794	93,7150	87,1244	80,7097	74,8633	69,6463
6						99,1488	94,6624	88,8887	83,1250	77,7559
7							99,2699	95,3611	90,2253	84,9972
8								99,3609	95,8977	91,2736
9									99,4317	96,3229
10										99,4884

Table A.3.2. Failure probability in % for the 95 %-confidence limit for a sample size of n ($11 \leq n \leq 20$) and the rank i

	n = 11	12	13	14	15	16	17	18	19	20
i =1	23,8404	22,0922	20,5817	19,2636	18,1036	17,0750	16,1566	15,3318	14,5868	13,9108
2	36,4359	33,8681	31,6339	29,6734	27,9396	26,3957	25,0125	23,7661	22,6375	21,6106
3	47,0087	43,8105	41,0099	38,5389	36,3442	34,3825	32,6193	31,0263	29,5802	28,2619
4	56,4374	52,7326	49,4650	46,5656	43,9785	41,6572	39,5641	37,6679	35,9425	34,3664
5	65,0188	60,9137	57,2620	54,0005	51,0752	48,4397	46,0550	43,8883	41,9120	40,1028
6	72,8750	68,4763	64,5201	60,9585	57,7444	54,8347	52,1918	49,7828	47,5797	45,5582
7	80,0424	75,4700	71,2951	67,4972	64,0435	60,8989	58,0295	55,4046	52,9967	50,7818
8	86,4925	81,8975	77,6045	73,6415	70,0013	66,6626	63,5991	60,7845	58,1935	55,8034
9	92,1180	87,7149	83,4341	79,3926	75,6273	72,1397	68,9171	65,9402	63,1885	60,6415
10	96,6681	92,8130	88,7334	84,7282	80,9135	77,3308	73,9886	70,8799	67,9913	65,3069
11	99,5348	96,9540	93,3950	89,5953	85,8336	82,2234	78,8092	75,6039	72,6054	69,8046
12		99,5735	97,1947	93,8897	90,3342	86,7889	83,3638	80,1047	77,0279	74,1349
13			99,6062	97,4001	94,3153	90,9748	87,6229	84,3656	81,2496	78,2931
14				99,6343	97,5774	94,6854	91,5355	88,3574	85,2530	82,2689
15					99,6586	97,7321	95,0102	92,0305	89,0093	86,0446
16						99,6799	97,8682	95,2975	92,4706	89,5919
17							99,6987	97,9889	95,5535	92,8646
18								99,7154	98,0967	95,7831
19									99,7304	98,1935
20										99,7439

Table A.3.3. Failure probability in % for the 95 %-confidence limit for a sample size of n ($21 \leq n \leq 30$) and the rank i

	n = 21	22	23	24	25	26	27	28	29	30
i =1	13,2946	12,7306	12,2123	11,7346	11,2928	10,8830	10,5019	10,1466	9,8145	9,5034
2	20,6725	19,8122	19,0204	18,2893	17,6121	16,9831	16,3975	15,8507	15,3392	14,8596
3	27,0552	25,9467	24,9249	23,9801	23,1040	22,2893	21,5300	20,8205	20,1561	19,5326
4	32,9211	31,5913	30,3637	29,2273	28,1723	27,1902	26,2739	25,4170	24,6139	23,8598
5	38,4408	36,9091	35,4932	34,1807	32,9608	31,8242	30,7627	29,7691	28,8372	27,9615
6	43,6976	41,9800	40,3899	38,9139	37,5405	36,2595	35,0620	33,9402	32,8873	31,8971
7	48,7389	46,8494	45,0975	43,4692	41,9520	40,5354	39,2098	37,9670	36,7995	35,7009
8	53,5936	51,5456	49,6435	47,8728	46,2209	44,6767	43,2302	41,8728	40,5966	39,3947
9	58,2801	56,0868	54,0456	52,1423	50,3642	48,6998	47,1391	45,6731	44,2936	42,9934
10	62,8099	60,4844	58,3155	56,2893	54,3933	52,6162	50,9478	49,3789	47,9012	46,5073
11	67,1891	64,7456	62,4607	60,3215	58,3162	56,4337	54,6640	52,9979	51,4270	49,9439
12	71,4200	68,8737	66,4853	64,2436	62,1378	60,1576	58,2931	56,5355	54,8765	53,3086
13	75,5005	72,8687	70,3906	68,0579	65,8611	63,7911	61,8387	59,9956	58,2536	56,6055
14	79,4250	76,7276	74,1757	71,7645	69,4871	67,3358	65,3028	63,3803	61,5608	59,8371
15	83,1824	80,4437	77,8364	75,3611	73,0147	70,7918	68,6861	66,6909	64,7996	63,0052
16	86,7552	84,0059	81,3656	78,8434	76,4414	74,1576	71,9880	69,9275	67,9704	66,1108
17	90,1156	87,3966	84,7520	82,2040	79,7622	77,4300	75,2066	73,0889	71,0728	69,1536
18	93,2193	90,5891	87,9785	85,4313	82,9696	80,6039	78,3383	76,1728	74,1056	72,1331
19	95,9901	93,5404	91,0191	88,5089	86,0525	83,6718	81,3780	79,1757	77,0660	75,0474
20	98,2809	96,1776	93,8324	91,4115	88,9944	86,6226	84,3181	82,0923	79,9504	77,8941
21	99,7560	98,3603	96,3485	94,0992	91,7709	89,4404	87,1478	84,9149	82,7535	80,6691
22		99,7671	98,4326	96,5047	94,3437	92,1014	89,8515	87,6331	85,4678	83,3674
23			99,7772	98,4988	96,6480	94,5688	92,4064	90,2318	88,0831	85,9815
24				99,7865	98,5597	96,7801	94,7767	92,6886	90,5845	88,5013
25					99,7950	98,6158	96,9022	94,9692	92,9506	90,9126
26						99,8029	98,6677	97,0153	95,1480	93,1944
27							99,8102	98,7159	97,1204	95,3145
28								99,8170	98,7606	97,2184
29									99,8233	98,8024
30										99,8292

Table A.4. Standard Normal Distribution

The table contains values of the Standard Normal Distribution $\phi(x) = NV(\mu = 0, \sigma = 1)$ for $x \geq 0$. For $x < 0$ one considers $\phi(-x) = 1 - \phi(x)$.

Transformation of a Normal Distribution: $x = \dfrac{t-\mu}{\sigma}$.

Transformation of a LogNormal Distribution: $x = \dfrac{\ln(t-t_0)-\mu}{\sigma}$.

x	+0,00	+0,01	+0,02	+0,03	+0,04	+0,05	+0,06	+0,07	+0,08	+0,09
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990

Table A.5. Gamma Function

The Gamma function was defined by Euler as improper parameter integral (second Euler integral): For real number $x > 0$ is $\Gamma(x) = \int_0^\infty e^{-t} \cdot t^{x-1} \cdot dt$.

The following functional equations are valid:

$$\Gamma(x=1) = 1 , \; \Gamma(x+1) = x \cdot \Gamma(x) , \; \Gamma(x) = \frac{\Gamma(x+1)}{x} , \; \Gamma(x) = (x-1) \cdot \Gamma(x-1) .$$

x	$\Gamma(x)$	x	$\Gamma(x)$	x	$\Gamma(x)$	x	$\Gamma(x)$
1,00	1	1,25	0,906402477	1,50	0,886226925	1,75	0,919062527
1,01	0,994325851	1,26	0,904397118	1,51	0,886591685	1,76	0,921374885
1,02	0,988844203	1,27	0,902503064	1,52	0,887038783	1,77	0,923763128
1,03	0,983549951	1,28	0,900718476	1,53	0,887567628	1,78	0,926227306
1,04	0,978438201	1,29	0,899041586	1,54	0,888177659	1,79	0,92876749
1,05	0,973504266	1,30	0,897470696	1,55	0,888868348	1,80	0,931383771
1,06	0,968743649	1,31	0,896004177	1,56	0,889639199	1,81	0,934076258
1,07	0,964152042	1,32	0,894640463	1,57	0,890489746	1,82	0,936845083
1,08	0,959725311	1,33	0,893378053	1,58	0,891419554	1,83	0,939690395
1,09	0,955459488	1,34	0,892215507	1,59	0,892428214	1,84	0,942612363
1,10	0,95135077	1,35	0,891151442	1,60	0,893515349	1,85	0,945611176
1,11	0,947395504	1,36	0,890184532	1,61	0,894680608	1,86	0,948687042
1,12	0,943590186	1,37	0,889313507	1,62	0,895923668	1,87	0,951840185
1,13	0,93993145	1,38	0,888537149	1,63	0,897244233	1,88	0,955070853
1,14	0,936416066	1,39	0,887854292	1,64	0,89864203	1,89	0,958379308
1,15	0,933040931	1,40	0,887263817	1,65	0,900116816	1,90	0,961765832
1,16	0,929803067	1,41	0,886764658	1,66	0,901668371	1,91	0,965230726
1,17	0,926699611	1,42	0,88635579	1,67	0,903296499	1,92	0,968774309
1,18	0,923727814	1,43	0,886036236	1,68	0,90500103	1,93	0,972396918
1,19	0,920885037	1,44	0,885805063	1,69	0,906781816	1,94	0,976098907
1,20	0,918168742	1,45	0,88566138	1,70	0,908638733	1,95	0,979880651
1,21	0,915576493	1,46	0,885604336	1,71	0,91057168	1,96	0,98374254
1,22	0,913105947	1,47	0,885633122	1,72	0,912580578	1,97	0,987684984
1,23	0,910754856	1,48	0,885746965	1,73	0,914665371	1,98	0,991708409
1,24	0,908521058	1,49	0,885945132	1,74	0,916826025	1,99	0,99581326
						2,00	1

Examples:

a) $\Gamma(1{,}35) = 0{,}891151442$

b) $\Gamma(0{,}8) = \dfrac{\Gamma(1{,}8)}{0{,}8} = \dfrac{0{,}931383771}{0{,}8} = 1{,}16497971375$

c) $\Gamma(3{,}2) = 2{,}2 \cdot \Gamma(2{,}2) = 2{,}2 \cdot 1{,}2 \cdot \Gamma(1{,}2) = 2{,}2 \cdot 1{,}2 \cdot 0{,}918168742 = 2{,}42397$

Graphics for the determination of the confidence interval according to the V_q-procedure:

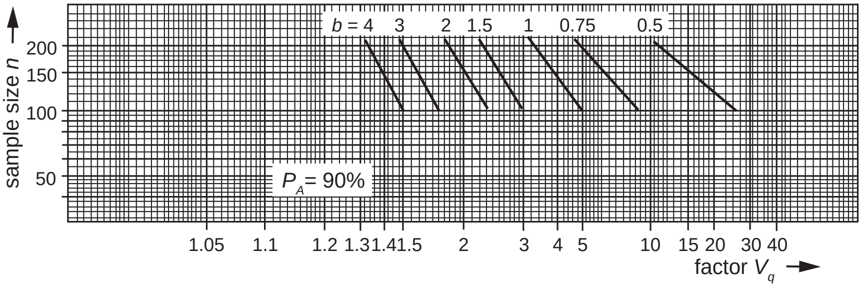


Fig. A1 Confidence interval of t_1-lifetime values ($q = 1$ %) for different b-values according to the V_q-procedure [VDA 4.2]

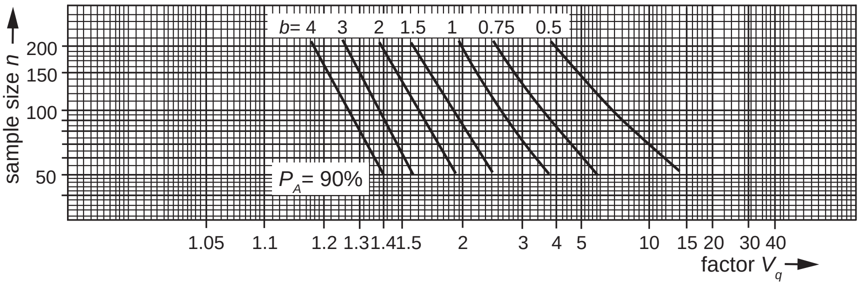


Fig. A2 Confidence interval of t_3-lifetime values ($q = 3$ %) for different b-values according to the V_q-procedure [VDA 4.2]

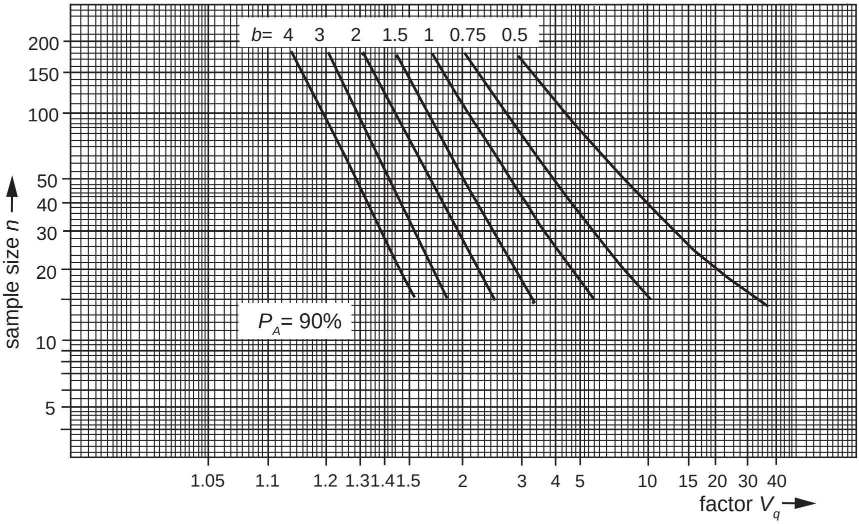


Fig. A3 Confidence interval of t_5-lifetime values ($q = 5$ %) for different *b*-values according to the V_q-procedure [VDA 4.2]

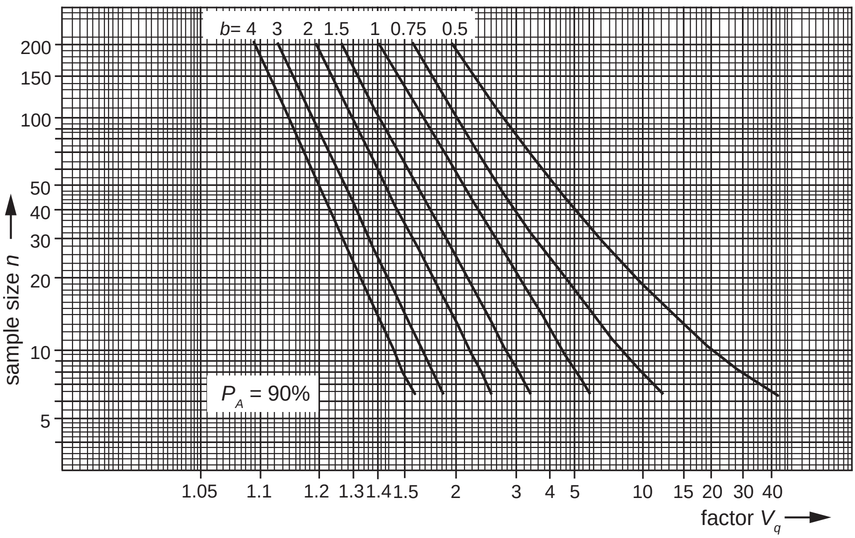


Fig. A4 Confidence interval of t_{10}-lifetime values ($q = 10$ %) for different *b*-values according to the V_q-procedure [VDA 4.2]

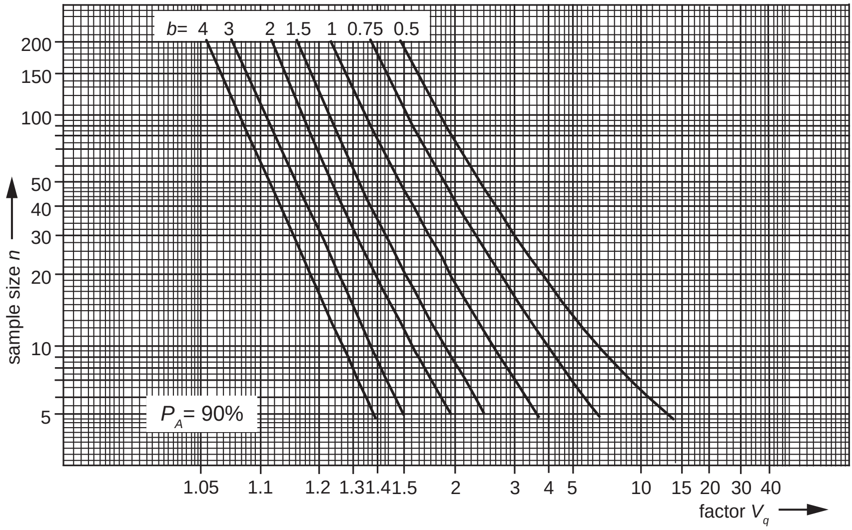


Fig. A5 Confidence interval of t_{30}-lifetime values (q = 30 %) for different b-values according to the V_q-procedure [VDA 4.2]

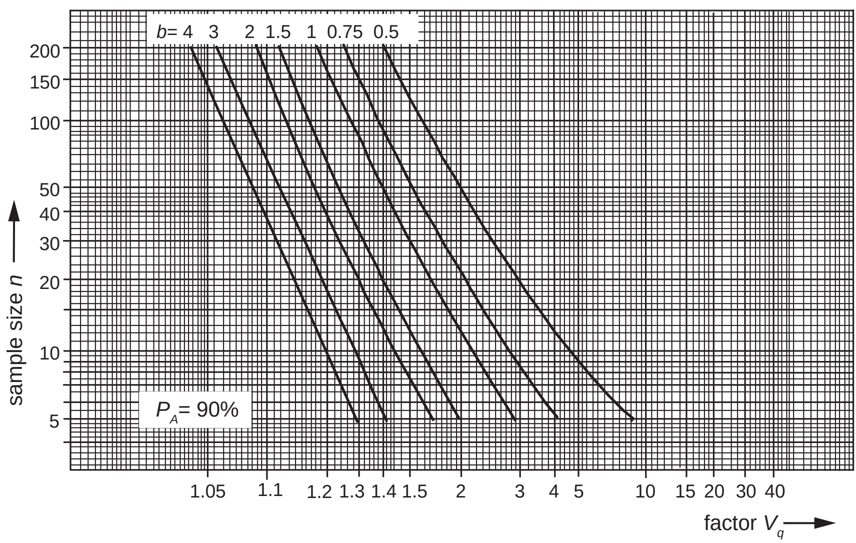


Fig. A6 Confidence interval of t_{50}-lifetime values (q = 50 %) for different b-values according to the V_q-procedure [VDA 4.2]

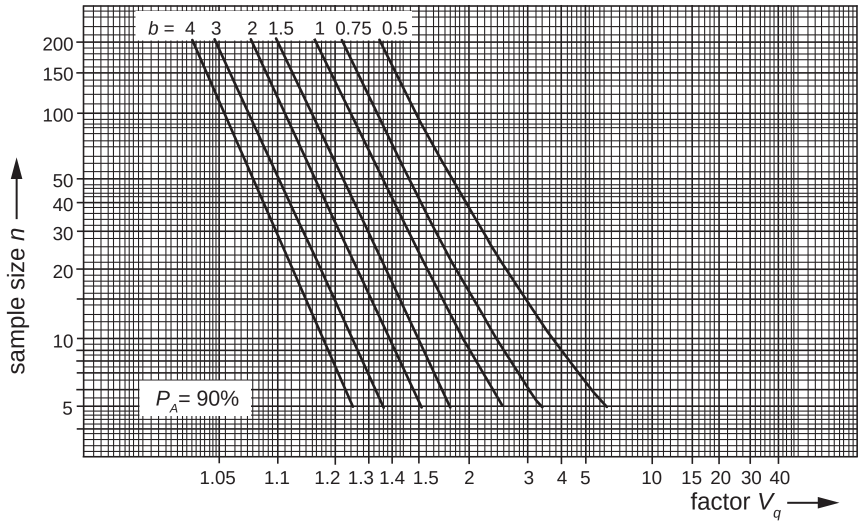


Fig. A7 Confidence interval of t_{80}-lifetime values (q = 80 %) for different b-values according to the V_q-procedure [VDA 4.2]

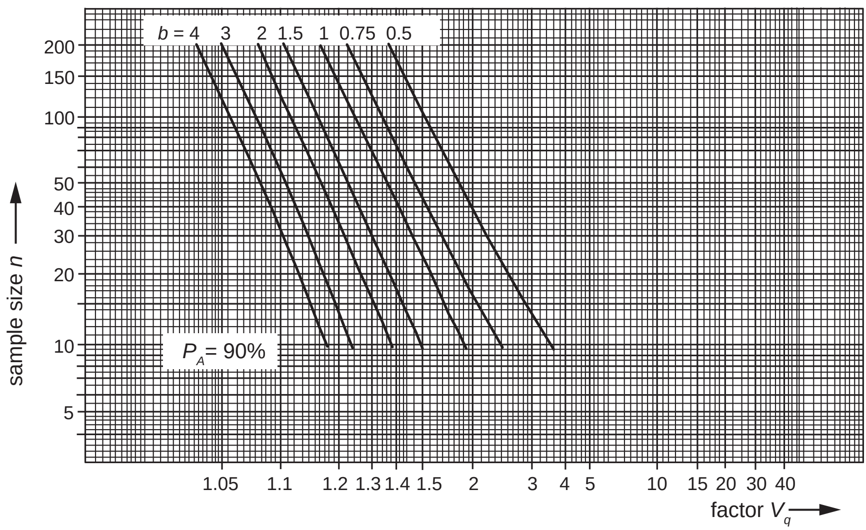


Fig. A8 Confidence interval of t_{90}-lifetime values (q = 90 %) for different b-values according to the V_q-procedure [VDA 4.2]

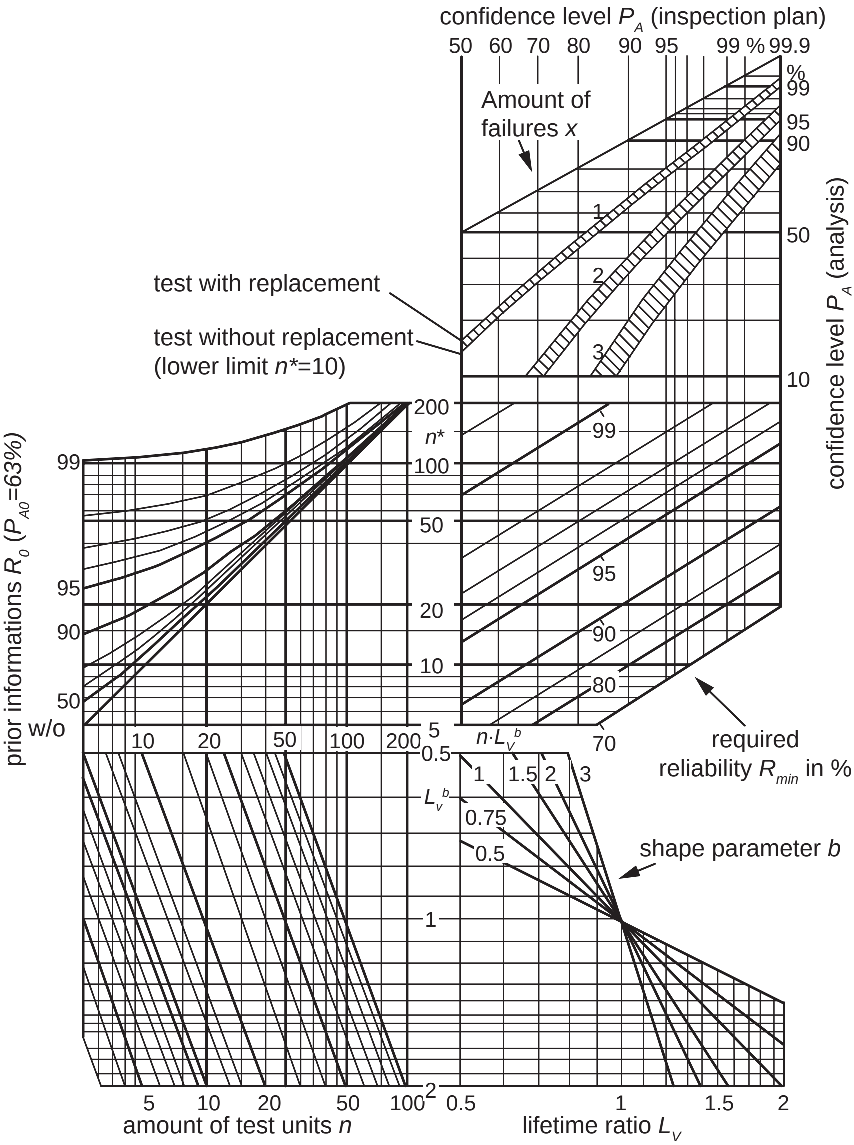


Fig. A9 Beyer-Lauster Nomogramm

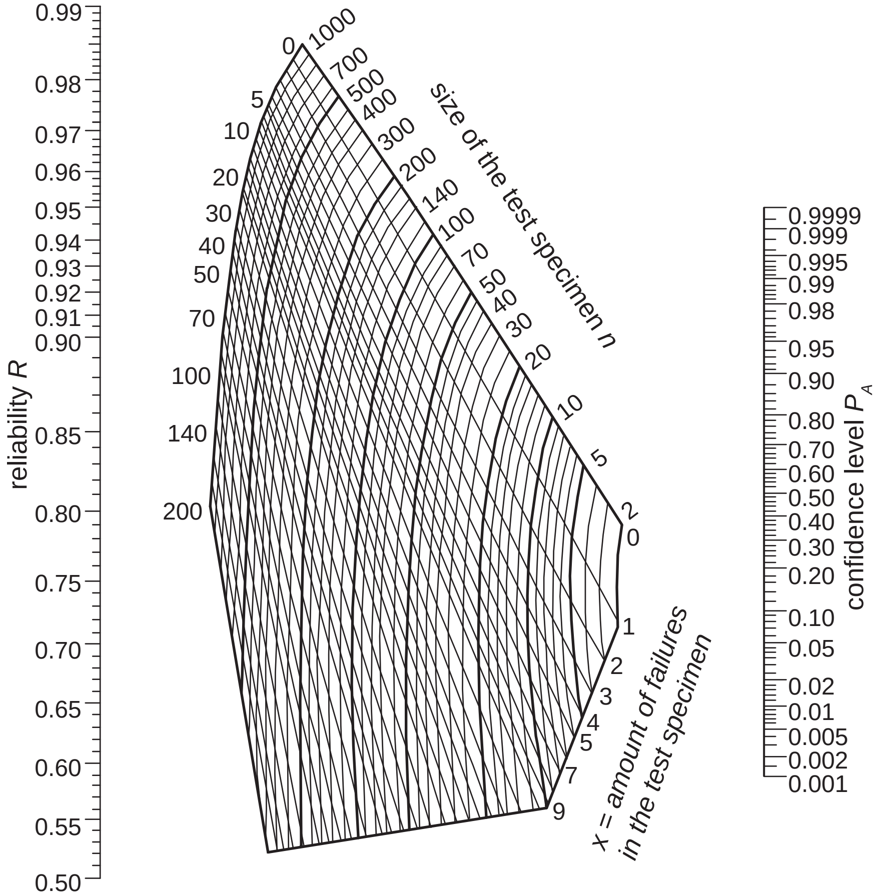


Fig. A10 Larson-Nomogramm

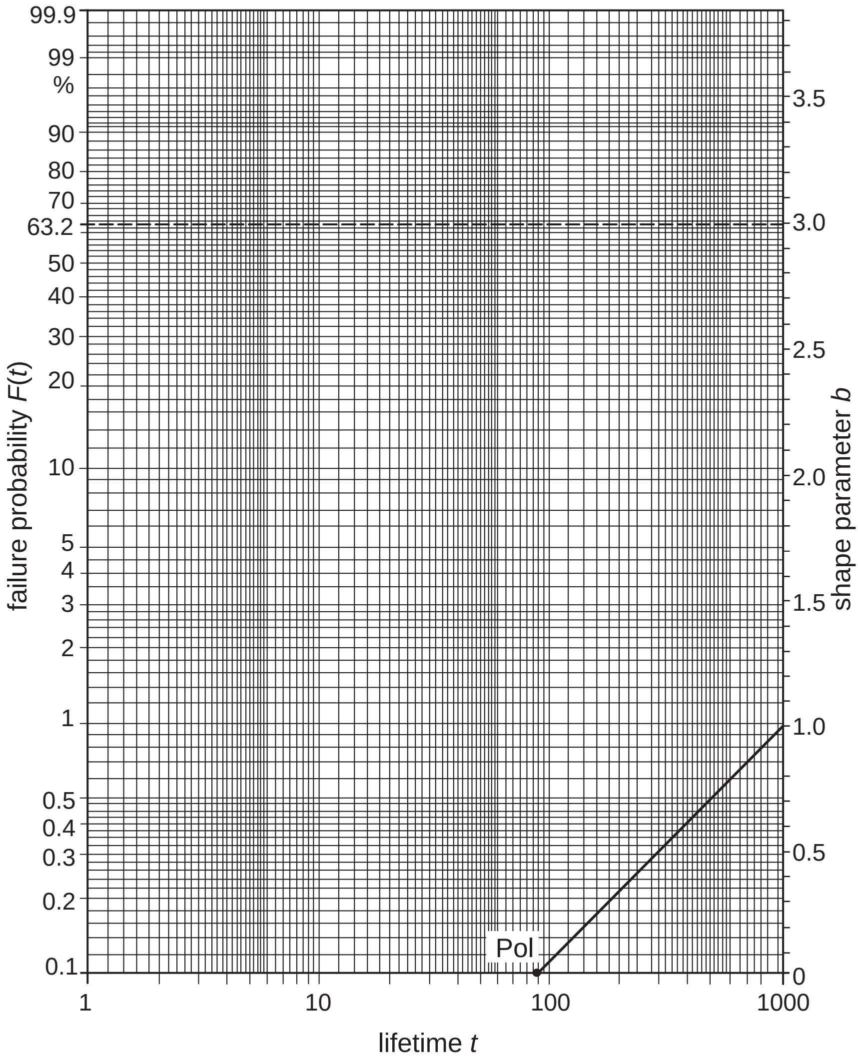


Fig. A11 Weibull net

Index