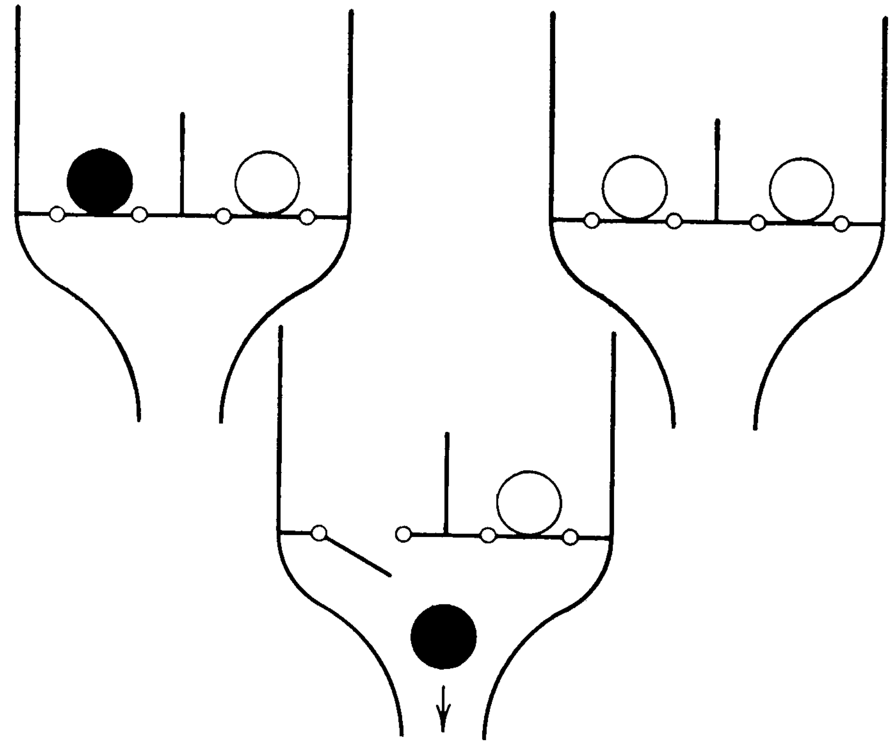
Robert B. Ash



INFORMATION THEORY

PREFACE

Statistical communication theory is generally regarded as having been founded by Shannon (1948) and Wiener (1949), who conceived of the communication situation as one in which a signal chosen from a specified class is to be transmitted through a channel, but the output of the channel is not determined by the input. Instead, the channel is described statistically by giving a probability distribution over the set of all possible outputs for each permissible input. At the output of the channel, a received signal is observed, and then a decision is made, the objective of the decision being to identify as closely as possible some property of the input signal.

The Shannon formulation differs from the Wiener approach in the nature of the transmitted signal and in the type of decision made at the receiver. In the Shannon model, a randomly generated message produced by a source of information is "encoded," that is, each possible message that the source can produce is associated with a signal belonging to a specified set. It is the encoded message which is actually transmitted. When the output is received, a "decoding" operation is performed, that is, a decision is made as to the identity of the particular signal transmitted. The objectives are to increase the size of the vocabulary, that is, to make the class of inputs as large as possible, and at the same time to make the probability of correctly identifying the input signal as large as possible. How well one can do these things depends essentially on the properties of the channel, and a fundamental concern is the analysis of different channel models. Another basic problem is the selection of a particular input vocabulary that can be used with a low probability of error.

In the Wiener model, on the other hand, a random signal is to be communicated directly through the channel; the encoding step is absent. Furthermore, the channel model is essentially fixed. The channel is generally taken to be a device that adds to the input signal a randomly generated "noise." The "decoder" in this case operates on the received signal to produce an estimate of some property of the input. For example, in the prediction problem the decoder estimates the value of the input at some future time. In general, the basic objective is to design a decoder, subject to a constraint of physical realizability, which makes the best estimate, where the closeness of the estimate is measured by an appropriate criterion. The problem of realizing and implementing an optimum decoder is central to the Wiener theory.

I do not want to give the impression that every problem in communication theory may be unalterably classified as belonging to the domain of either Shannon or Wiener, but not both. For example, the radar reception problem contains some features of both approaches. Here one tries to determine whether a signal was actually transmitted, and if so to identify which signal of a specified class was sent, and possibly to estimate some of the signal parameters. However, I think it is fair to say that this book is concerned entirely with the Shannon formulation, that is, the body of mathematical knowledge which has its origins in Shannon's fundamental paper of 1948. This is what "information theory" will mean for us here.

The book treats three major areas: first (Chapters 3, 7, and 8), an analysis of channel models and the proof of coding theorems (theorems whose physical interpretation is that it is possible to transmit information reliably through a noisy channel at any rate below channel capacity, but not at a rate above capacity); second, the study of specific coding systems (Chapters 2, 4, and 5); finally, the study of the statistical properties of information sources (Chapter 6). All three areas were introduced in Shannon's original paper, and in each case Shannon established an area of research where none had existed before.

The book has developed from lectures and seminars given during the last five years at Columbia University; the University of California, Berkeley; and the University of Illinois, Urbana. I have attempted to write in a style suitable for first-year graduate students in mathematics and the physical sciences, and I have tried to keep the prerequisites modest. A course in basic probability theory is essential, but measure theory is not required for the first seven chapters. All random variables appearing in these chapters are discrete and take on only a finite number of possible values. For most of Chapter 8, the random variables, although continuous, have probability density functions, and therefore a knowledge of basic probability should suffice. Some measure and Hilbert space theory is helpful for the last two sections of Chapter 8, which treat time-continuous channels. An appendix summarizes the Hilbert space background and the results from the theory of stochastic processes that are necessary for these sections. The appendix is not self-contained, but I hope it will serve to pinpoint some of the specific equipment needed for the analysis of time-continuous channels.

Chapters 1 and 3 are basic, and the concepts developed there appear throughout the book. Any of Chapters 4 through 8 may be read immediately after Chapters 1 and 3, although the reader should browse through the first five sections of Chapter 4 before looking at Chapter 5. Chapter 2 depends only on Chapter 1.

In Chapter 4, the exposition is restricted to binary codes, and the generalization to codes over an arbitrary finite field is sketched at the end of the chapter. The analysis of cyclic codes in Chapter 5 is carried out by a matrix development rather than by the standard approach, which uses abstract algebra. The matrix method seems to be natural and intuitive, and will probably be more palatable to students, since a student is more likely to be familiar with matrix manipulations than he is with extension fields.

I hope that the inclusion of some sixty problems, with fairly detailed solutions, will make the book more profitable for independent study.

The historical notes at the end of each chapter are not meant to be exhaustive, but I have tried to indicate the origins of some of the results.

I have had the benefit of many discussions with Professor Aram Thomasian on information theory and related areas in mathematics. Dr. Aaron Wyner read the entire manuscript and supplied helpful comments and criticism. I also received encouragement and advice from Dr. David Slepian and Professors R. T. Chien, M. E. Van Valkenburg, and L. A. Zadeh.

Finally, my thanks are due to Professor Warren Hirsch, whose lectures in 1959 introduced me to the subject, to Professor Lipman Bers for his invitation to publish in this series, and to the staff of Interscience Publishers, a division of John Wiley and Sons, Inc., for their courtesy and cooperation.

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Robert B. Ash

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CHAPTER ONE

A Measure of Information

1.1. Introduction

Information theory is concerned with the analysis of an entity called a "communication system," which has traditionally been represented by the block diagram shown in Fig. 1.1.1. The source of messages is the person or machine that produces the information to be communicated. The encoder associates with each message an "object" which is suitable for transmission over the channel. The "object" could be a sequence of binary digits, as in digital computer applications, or a continuous waveform, as in radio communication. The channel is the medium over which the coded message is transmitted. The decoder operates on the output of the channel and attempts to extract the original message for delivery to the destination. In general, this cannot be done with complete reliability because of the effect of "noise," which is a general term for anything which tends to produce errors in transmission.

Information theory is an attempt to construct a mathematical model for each of the blocks of Fig. 1.1.1. We shall not arrive at design formulas for a communication system; nevertheless, we shall go into considerable detail concerning the theory of the encoding and decoding operations.

It is possible to make a case for the statement that information theory is essentially the study of one theorem, the so-called "fundamental theorem of information theory," which states that "it is possible to transmit information through a noisy channel at any rate less than channel capacity with an arbitrarily small probability of error." The meaning of the various terms "information," "channel," "noisy," "rate," and "capacity" will be clarified in later chapters. At this point, we shall only try to give an intuitive idea of the content of the fundamental theorem.

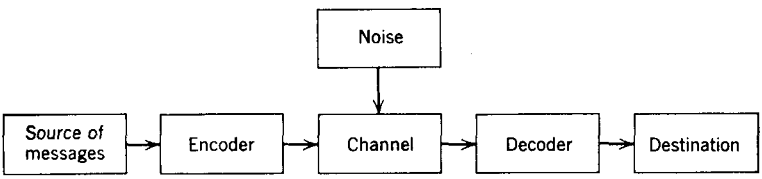


Fig. 1.1.1. Communication system.

Imagine a "source of information" that produces a sequence of binary digits (zeros or ones) at the rate of 1 digit per second. Suppose that the digits 0 and 1 are equally likely to occur and that the digits are produced independently, so that the distribution of a given digit is unaffected by all previous digits. Suppose that the digits are to be communicated directly over a "channel." The nature of the channel is unimportant at this moment, except that we specify that the probability that a particular digit

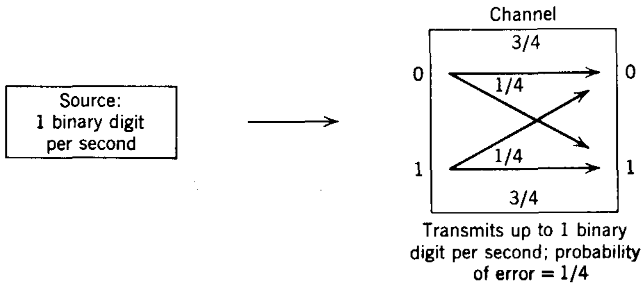


Fig. 1.1.2. Example.

is received in error is (say) 1/4, and that the channel acts on successive inputs independently. We also assume that digits can be transmitted through the channel at a rate not to exceed 1 digit per second. The pertinent information is summarized in Fig. 1.1.2.

Now a probability of error of 1/4 may be far too high in a given application, and we would naturally look for ways of improving reliability. One way that might come to mind involves sending the source digit through the channel more than once. For example, if the source produces a zero at a given time, we might send a sequence of 3 zeros through the channel; if the source produces a one, we would send 3 ones. At the receiving end of the channel, we will have a sequence of 3 digits for each one produced by the source. We will have the problem of *decoding* each sequence, that is, making a decision, for each sequence received, as to the identity of the source digit. A "reasonable" way to decide is by means of a "majority selector," that is, a rule which specifies that if more ones than zeros are received, we are to decode the received sequence as a "1"; if more zeros than ones appear, decode as a "0." Thus, for example, if the source produces a one, a sequence of 3 ones would be sent through the channel. The first and third digits might be received incorrectly; the received sequence would then be 0 1 0; the decoder would therefore declare (incorrectly) that a zero was in fact transmitted.

We may calculate the probability that a given source digit is received in error; it is the probability that at least 2 of a sequence of 3 digits will be received incorrectly, where the probability of a given digit's being incorrect is 1/4 and the digits are transmitted independently. Using the standard formula for the distribution of successes and failures in a sequence of Bernoulli trials, we obtain

$$\binom{3}{2}\left(\frac{1}{4}\right)^2\frac{3}{4}+\left(\frac{1}{4}\right)^3=\frac{10}{64}<\frac{1}{4}.$$

Thus we have lowered the probability of error; however, we have paid a price for this reduction. If we send 1 digit per second through the channel, it now takes 3 seconds to communicate 1 digit produced by the source, or three times as long as it did originally. Equivalently, if we want to synchronize the source with the channel, we must slow down the rate of the source to $\frac{1}{3}$ digit per second while keeping the channel rate fixed at 1 digit per second. Then during the time (3 seconds) it takes for the source to produce a single digit, we will be able to transmit the associated sequence of 3 digits through the channel.

Now let us generalize this procedure. Suppose that the probability of error for a given digit is $\beta < 1/2$, and that each source digit is represented by a sequence of length $2n + 1$; a majority selector is used at the receiver. The effective transmission rate of the source is reduced to $1/(2n + 1)$ binary digits per second while the probability of incorrect decoding is

$$p(e) = P\{n + 1 \text{ or more digits in error}\} = \sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\beta^k(1-\beta)^{2n+1-k}.$$

Since the expected number of digits in error is $(2n + 1)\beta < n + 1$, the weak law of large numbers implies that $p(e) \to 0$ as $n \to \infty$. (If S_{2n+1} is the number of digits in error, then the sequence $S_{2n+1}/(2n + 1)$ converges in probability to β, so that

$$\begin{aligned}p(e) &= P\{S_{2n+1} \geq n + 1\} = P\left\{\frac{S_{2n+1}}{2n+1} \geq \frac{n+1}{2n+1}\right\}.\\ &= P\left\{\frac{S_{2n+1}}{2n+1} \geq \beta + \varepsilon\right\} \to 0 \quad \text{as} \quad n \to \infty.)\end{aligned}$$

Thus we are able to reduce the probability of error to an arbitrarily small figure, at the expense of decreasing the effective transmission rate toward zero.

The essence of the fundamental theorem of information theory is that in order to achieve arbitrarily high reliability, *it is not necessary to reduce the transmission rate to zero*, but only to a number called the *channel capacity.*

The means by which these results are obtained is called *coding*. The process of coding involves the insertion of a device called an "encoder" between the source and the channel; the encoder assigns to each of a specified group of source messages a sequence of symbols called a *code word* suitable for transmission through the channel. In the above example, we have just seen a primitive form of coding; we have assigned to the source digit 0 a sequence of zeros, and to the source digit 1 a sequence of ones. The received sequence is fed to a decoder which attempts to determine the identity of the original message. In general, to achieve reliability without sacrificing speed of transmission, code words are not assigned to single digits but instead to long blocks of digits. In other words, the encoder waits for the source to produce a block of digits of a specified length, and then assigns a code word to the entire block. The decoder examines the received sequence and makes a decision as to the identity of the transmitted block. In general, encoding and decoding procedures are considerably more elaborate than in the example just considered.

The discussion is necessarily vague at this point; hopefully, the concepts introduced will eventually be clarified. Our first step in the clarification will be the construction of a mathematical measure of the information conveyed by a message. As a preliminary example, suppose that a random variable X takes on the values 1, 2, 3, 4, 5 with equal probability. We ask how much information is conveyed about the value of X by the statement that $1 \leq X \leq 2$. Originally, if we try to guess the value of X, we have probability 1/5 of being correct. After we know that X is either 1 or 2, we have a higher probability of guessing the right answer. In other words, there is less *uncertainty* about the second situation. Telling us that $1 \leq X \leq 2$ has reduced the uncertainty about the actual value of X. It thus appears that if we could pin down the notion of uncertainty, we would be able to measure precisely the transfer of information. Our approach will be to set up certain requirements which an uncertainty function should "reasonably" satisfy; we shall then prove that there is only one function which meets all the requirements. We must emphasize that it is not important *how* we arrive at the measure of uncertainty. The axioms of uncertainty we choose will probably seem reasonable to most readers, but we definitely will not base the case for the measure of uncertainty on intuitive grounds. The usefulness of the uncertainty measure proposed by Shannon lies in its operational significance in the construction of codes. Using an appropriate notion of uncertainty we shall be able to define the information transmitted through a channel and establish the existence of coding systems which make it possible to transmit at any rate less than channel capacity with an arbitrarily small probability of error.

1.2. Axioms for the uncertainty measure

Suppose that a probabilistic experiment involves the observation of a discrete random variable X. Let X take on a finite number of possible values $x_1, x_2, \ldots, x_M$, with probabilities $p_1, p_2, \ldots, p_M$, respectively. We assume that all p_i are strictly greater than zero. Of course $\sum_{i=1}^{M} p_i = 1$. We now attempt to arrive at a number that will measure the uncertainty associated with X. We shall try to construct two functions h and H. The function h will be defined on the interval $(0, 1]$; $h(p)$ will be interpreted as the uncertainty associated with an event with probability p. Thus if the event $\{X = x_i\}$ has probability p_i, we shall say that $h(p_i)$ is the uncertainty associated with the event $\{X = x_i\}$, or the uncertainty removed (or information conveyed) by revealing that X has taken on the value x_i in a given performance of the experiment. For each M we shall define a function H_M of the M variables $p_1, \ldots, p_M$ (we restrict the domain of H_M by requiring all p_i to be > 0, and $\sum_{i=1}^{M} p_i = 1$). The function $H_M(p_1, \ldots, p_M)$ is to be interpreted as the average uncertainty associated with the events $\{X = x_i\}$; specifically, we require that $H_M(p_1, \ldots, p_M) = \sum_{i=1}^{M} p_i h(p_i)$. [For simplicity we write $H_M(p_1, \ldots, p_M)$ as $H(p_1, \ldots, p_M)$ or as $H(X)$.] Thus $H(p_1, \ldots, p_M)$ is the average uncertainty removed by revealing the value of X. The function h is introduced merely as an aid to the intuition; it will appear only in this section. In trying to justify for himself the requirements which we shall impose on $H(X)$, the reader may find it helpful to think of $H(p_1, \ldots, p_M)$ as a weighted average of the numbers $h(p_1), \ldots, h(p_M)$.

Now we proceed to impose requirements on the functions H. In the sequel, $H(X)$ will be referred to as the "uncertainty of X"; the word "average" will be understood but will, except in this section, generally not be appended. First suppose that all values of X are equally probable. We denote by $f(M)$ the average uncertainty associated with M equally likely outcomes, that is, $f(M) = H(1/M, 1/M, \ldots, 1/M)$. For example, $f(2)$ would be the uncertainty associated with the toss of an unbiased coin, while $f(8 \times 10^6)$ would be the uncertainty associated with picking a person at random in New York City. We would expect the uncertainty of the latter situation to be greater than that of the former. In fact, our first requirement on the uncertainty function is that

$f(M) = H(1/M, \ldots, 1/M)$ should be a monotonically increasing function of M; that is, $M < M'$ implies $f(M) < f(M')$ $(M, M' = 1, 2, 3, \ldots)$.

Now consider an experiment involving two independent random variables X and Y. Let X take on the values $x_1, x_2, \ldots, x_M$ with equal probability, and let Y take on the values $y_1, y_2, \ldots, y_L$, also with equal probability.

Thus the joint experiment involving X and Y has ML equally likely outcomes, and therefore the average uncertainty of the joint experiment is $f(ML)$. If the value of X is revealed, the average uncertainty about Y should not be affected because of the assumed independence of X and Y. Hence we expect that the average uncertainty associated with X and Y together, minus the average uncertainty removed by revealing the value of X, should yield the average uncertainty associated with Y. Revealing the value of X removes, on the average, an amount of uncertainty equal to $f(M)$, and thus the second requirement on the uncertainty measure is that

$$f(ML) = f(M) + f(L) \qquad (M, L, = 1, 2, \ldots).$$

At this point we remove the restriction of equally likely outcomes and turn to the general case. We divide the values of a random variable X into two groups, A and B, where A consists of $x_1, x_2, \ldots, x_r$ and B consists of $x_{r+1}, x_{r+2}, \ldots, x_M$. We construct a compound experiment as follows. First we select one of the two groups, choosing group A with probability $p_1 + p_2 + \cdots + p_r$ and group B with probability $p_{r+1} + p_{r+2} + \cdots + p_M$. Thus the probability of each group is the sum of the probabilities of the values in the group. If group A is chosen, then we select x_i with probability $p_i/(p_1 + \cdots + p_r)$ $(i = 1, \ldots, r)$, which is the

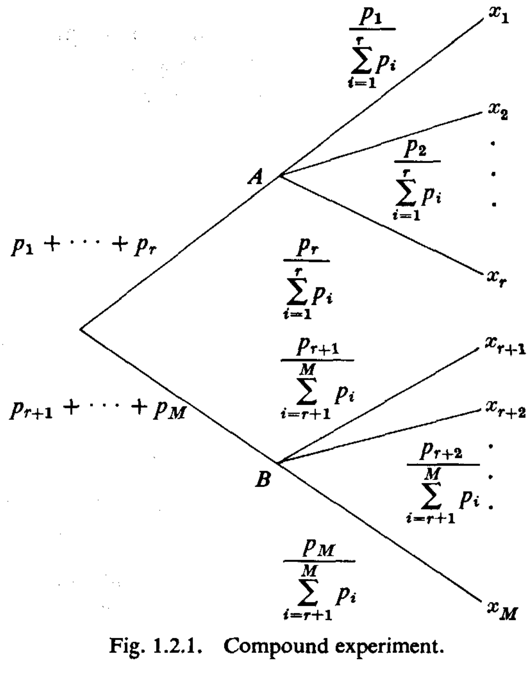


Fig. 1.2.1. Compound experiment.

conditional probability of x_i given that the value of X lies in group A. Similarly, if group B is chosen, then x_i is selected with probability $p_i/(p_{r+1} + \cdots + p_M)$ $(i = r + 1, \ldots, M)$. The compound experiment is diagrammed in Fig. 1.2.1. It is equivalent to the original experiment associated with X. For if Y is the result of the compound experiment, the probability that $Y = x_1$ is

$$\begin{aligned} P\{Y = x_1\} &= P\{A \text{ is chosen and } x_1 \text{ is selected}\} \\ &= P\{A \text{ is chosen}\}P\{x_1 \text{ is selected} \mid A \text{ is chosen}\} \\ &= \left(\sum_{i=1}^{r} p_i\right)\frac{p_1}{\sum_{i=1}^{r} p_i} = p_1. \end{aligned}$$

Similarly, $P\{Y = x_i\} = p_i (i = 1, 2, \ldots, M)$ so that Y and X have the same distribution. Before the compound experiment is performed, the average uncertainty associated with the outcome is $H(p_1, \ldots, p_M)$. If we reveal which of the two groups A and B is selected, we remove on the average an amount of uncertainty $H(p_1 + \cdots + p_r, p_{r+1} + \cdots + p_M)$. With probability $p_1 + \cdots + p_r$ group A is chosen and the remaining uncertainty is

$$H\left(\frac{p_1}{\sum_{i=1}^{r} p_i}, \frac{p_2}{\sum_{i=1}^{r} p_i}, \ldots, \frac{p_r}{\sum_{i=1}^{r} p_i}\right);$$

with probability $p_{r+1} + \cdots + p_M$, group B is chosen, and the remaining uncertainty is

$$H\left(\frac{p_{r+1}}{\sum_{i=r+1}^{M} p_i}, \frac{p_{r+2}}{\sum_{i=r+1}^{M} p_i}, \ldots, \frac{p_M}{\sum_{i=r+1}^{M} p_i}\right).$$

Thus *on the average* the uncertainty remaining after the group is specified is

$$(p_1 + \cdots + p_r)H\left(\frac{p_1}{\sum_{i=1}^{r} p_i}, \ldots, \frac{p_r}{\sum_{i=1}^{r} p_i}\right) + (p_{r+1} + \cdots + p_M)H\left(\frac{p_{r+1}}{\sum_{i=r+1}^{M} p_i}, \ldots, \frac{p_M}{\sum_{i=r+1}^{M} p_i}\right).$$

We expect that the average uncertainty about the compound experiment minus the average uncertainty removed by specifying the group equals the average uncertainty remaining after the group is specified. Hence, the third requirement we impose on the uncertainty function is that

$$H(p_1, \ldots, p_M) = H(p_1 + \cdots + p_r, p_{r+1} + \cdots + p_M) + (p_1 + \cdots + p_r)H\left(\frac{p_1}{\sum_{i=1}^{r} p_i}, \ldots, \frac{p_r}{\sum_{i=1}^{r} p_i}\right) + (p_{r+1} + \cdots + p_M)H\left(\frac{p_{r+1}}{\sum_{i=r+1}^{M} p_i}, \ldots, \frac{p_M}{\sum_{i=r+1}^{M} p_i}\right).$$

As a numerical example of the above requirement, we may write

$$H(\underbrace{\tfrac{1}{2}, \tfrac{1}{4}}_{A}, \underbrace{\tfrac{1}{8}, \tfrac{1}{8}}_{B}) = H(\tfrac{3}{4}, \tfrac{1}{4}) + \tfrac{3}{4}H(\tfrac{2}{3}, \tfrac{1}{3}) + \tfrac{1}{4}H(\tfrac{1}{2}, \tfrac{1}{2}).$$

Finally, we require for mathematical convenience that $H(p, 1 - p)$ be a continuous function of p. (Intuitively we should expect that a small change in the probabilities of the values of X will correspond to a small change in the uncertainty of X.)

To recapitulate, we assume the following four conditions as axioms:

1. $H(1/M, 1/M, \ldots, 1/M) = f(M)$ is a monotonically increasing function of M $(M = 1, 2, \ldots)$.

2. $f(ML) = f(M) + f(L) \qquad (M, L = 1, 2, \ldots)$.

3. $$H(p_1, \ldots, p_M) = H(p_1 + \cdots + p_r, p_{r+1} + \cdots + p_M) + (p_1 + \cdots + p_r)H\left(\frac{p_1}{\sum_{i=1}^{r} p_i}, \ldots, \frac{p_r}{\sum_{i=1}^{r} p_i}\right) + (p_{r+1} + \cdots + p_M)H\left(\frac{p_{r+1}}{\sum_{i=r+1}^{M} p_i}, \ldots, \frac{p_M}{\sum_{i=r+1}^{M} p_i}\right)$$
$$(r = 1, 2, \ldots, M - 1).$$

(Axiom 3 is called the *grouping axiom.*)

4. $H(p, 1 - p)$ is a continuous function of p.

The four axioms essentially determine the uncertainty measure. More precisely, we prove the following theorem.

Theorem 1.2.1. The only function satisfying the four given axioms is

$$H(p_1, \ldots, p_M) = -C \sum_{i=1}^{M} p_i \log p_i, \tag{1.2.1}$$

where C is an arbitrary positive number and the logarithm base is any number greater than 1.

Proof. It is not difficult to verify that the function (1.2.1) satisfies the four axioms. We shall prove that any function satisfying Axioms 1 through 4 must be of the form (1.2.1). For convenience we break the proof into several parts.

[We assume that all logarithms appearing in the proof are expressed to some fixed base greater than 1. No generality is lost by doing this; since $\log_a x = \log_a b \log_b x$, changing the logarithm base is equivalent to changing the constant C in (1.2.1).]

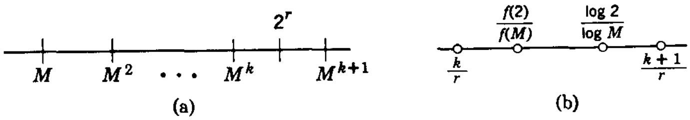


Fig. 1.2.2. Proof of Theorem 1.2.1.

a. $f(M^k) = kf(M)$ for all positive integers M and k. This is readily established by induction, using Axiom 2. If M is an arbitrary but fixed positive integer, then (a) is immediately true for $k = 1$. Since $f(M^k) = f(M \cdot M^{k-1}) = f(M) + f(M^{k-1})$ by Axiom 2, the induction hypothesis that (a) is true for all integers up to and including $k - 1$ yields $f(M^k) = f(M) + (k - 1)f(M) = kf(M)$, which completes the argument.

b. $f(M) = C \log M$ $(M = 1, 2, \ldots)$ where C is a positive number. First let $M = 1$. We have $f(1) = f(1 \cdot 1) = f(1) + f(1)$ by Axiom 2, and hence $f(1) = 0$ as stated in (b). This agrees with our intuitive feeling that there should be no uncertainty associated with an experiment with only one possible outcome. Now let M be a fixed positive integer greater than 1. If r is an arbitrary positive integer, then the number 2^r lies somewhere between two powers of M, that is, there is a nonnegative integer k such that $M^k \le 2^r < M^{k+1}$. (See Fig. 1.2.2a.) It follows from Axiom 1 that $f(M^k) \le f(2^r) < f(M^{k+1})$, and thus we have from (a) that $kf(M) \le rf(2) < (k + 1)f(M)$, or $k/r \le f(2)/f(M) < (k + 1)/r$. The logarithm is a montone increasing function (as long as the base is greater than 1) and hence $\log M^k \le \log 2^r < \log M^{k+1}$, from which we obtain $k \log M \le r \log 2 < (k + 1) \log M$, or $k/r \le (\log 2)/(\log M) < (k + 1)/r$. Since $f(2)/f(M)$ and $(\log 2)/(\log M)$ are both between k/r and $(k + 1)/r$, it follows that

$$\left| \frac{\log 2}{\log M} - \frac{f(2)}{f(M)} \right| < \frac{1}{r}. \qquad \text{(See Fig. 1.2.2b.)}$$

Since M is fixed and r is arbitrary, we may let $r \to \infty$ to obtain

$$(\log 2)/(\log M) = f(2)/f(M)$$

or $f(M) = C \log M$ where $C = f(2)/\log 2$. Note that C must be positive since $f(1) = 0$ and $f(M)$ increases with M.

c. $H(p, 1 - p) = -C[p \log p + (1 - p) \log (1 - p)]$ if p is a rational number. Let $p = r/s$ where r and s are positive integers. We consider

$$f(s) = H\Big(\underbrace{\frac{1}{s}, \dots, \frac{1}{s}}_{r}, \underbrace{\frac{1}{s}, \dots, \frac{1}{s}}_{s-r}\Big)$$

$$= H\left(\frac{r}{s}, \frac{s-r}{s}\right) + \frac{r}{s} f(r) + \frac{s-r}{s} f(s - r)$$

(by the grouping axiom 3).

Using (b) we have

$$C \log s = H(p, 1 - p) + Cp \log r + C(1 - p) \log (s - r).$$

Thus

$$\begin{aligned} H(p, 1 - p) &= -C[p \log r - \log s + (1 - p) \log (s - r)] \\ &= -C[p \log r - p \log s + p \log s \\ &\qquad - \log s + (1 - p) \log (s - r)] \\ &= -C\left[p \log \frac{r}{s} + (1 - p) \log \frac{s - r}{s}\right] \\ &= -C[p \log p + (1 - p) \log (1 - p)]. \end{aligned}$$

d. $H(p, 1 - p) = -C[p \log p + (1 - p) \log (1 - p)]$ for all p. This is an immediate consequence of (c) and the continuity axiom 4. For if p is any number between 0 and 1, it follows by continuity that

$$H(p, 1 - p) = \lim_{p' \to p} H(p', 1 - p').$$

In particular, we may allow p' to approach p through a sequence of rational numbers. We then have

$$\begin{aligned} \lim_{p' \to p} H(p', 1 - p') &= \lim_{p' \to p} [-C(p' \log p' + (1 - p') \log (1 - p'))] \\ &= -C[p \log p + (1 - p) \log (1 - p)]. \end{aligned}$$

e. $H(p_1, \dots, p_M) = -C \sum_{i=1}^{M} p_i \log p_i (M = 1, 2, \dots)$. We again proceed by induction. We have already established the validity of the above formula for $M = 1$ and 2. If $M > 2$, we use Axiom 3, which yields

$$\begin{aligned} H(p_1, \dots, p_M) &= H(p_1 + \cdots + p_{M-1}, p_M) \\ &\quad + (p_1 + \cdots + p_{M-1}) H\left(\frac{p_1}{\sum_{i=1}^{M-1} p_i}, \dots, \frac{p_{M-1}}{\sum_{i=1}^{M-1} p_i}\right) + p_M H(1). \end{aligned}$$

Assuming the formula valid for positive integers up to $M - 1$, we obtain:

$$H(p_1, \ldots, p_M) = -C[(p_1 + \cdots + p_{M-1}) \log (p_1 + \cdots + p_{M-1}) + p_M \log p_M]$$

$$- C(p_1 + \cdots + p_{M-1}) \left[\frac{p_1}{\sum_{i=1}^{M-1} p_i} \log \left(\frac{p_1}{\sum_{i=1}^{M-1} p_i} \right) + \cdots + \frac{p_{M-1}}{\sum_{i=1}^{M-1} p_i} \log \left(\frac{p_{M-1}}{\sum_{i=1}^{M-1} p_i} \right) \right] + p_M(0)$$

$$= -C\left[\left(\sum_{i=1}^{M-1} p_i \right) \log \left(\sum_{i=1}^{M-1} p_i \right) + p_M \log p_M \right]$$

$$- C\left[\sum_{i=1}^{M-1} p_i \log p_i - \left(\sum_{i=1}^{M-1} p_i \right) \log \sum_{i=1}^{M-1} p_i \right]$$

$$= -C \sum_{i=1}^{M} p_i \log p_i.$$

The proof is complete.

Unless otherwise specified, we shall assume $C = 1$ and take logarithms to the base 2. The units of H are sometimes called *bits* (a contraction of *binary digits*). Thus the units are chosen so that there is one bit of uncertainty associated with the toss of an unbiased coin. Biasing the coin tends to decrease the uncertainty, as seen in the sketch of $H(p, 1 - p)$ (Fig. 1.2.3).

We remark in passing that the average uncertainty of a random variable X does not depend on the values the random variable assumes, or on anything else except the probabilities associated with those values. The average uncertainty associated with the toss of an unbiased coin is not changed by adding the condition that the experimenter will be shot if the coin comes up tails.

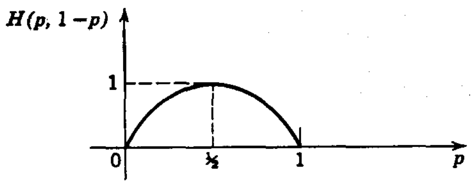


Fig. 1.2.3. $H(p, 1 - p) = -p \log p - (1 - p) \log (1 - p)$.

We also remark that the notations

$$\sum_{i=1}^{M} p_i \log p_i \quad \text{and} \quad -\sum_{i=1}^{M} p(x_i) \log p(x_i)$$

will be used interchangeably.

Finally, we note that Theorem 1.2.1 implies that the function h must be of the form $h(p) = -C \log p$, provided we impose a requirement that h be continuous (see Problem 1.11).

An alternate approach to the construction of an uncertainty measure involves a set of axioms for the function h. If A and B are independent events, the occurrence of A should not affect the odds that B will occur. If $P(A) = p_1, P(B) = p_2$, then $P(AB) = p_1 p_2$ so that the uncertainty associated with AB is $h(p_1 p_2)$. The statement that A has occurred removes an amount of uncertainty $h(p_1)$, leaving an uncertainty $h(p_2)$ because of the independence of A and B. Thus we might require that

$$h(p_1 p_2) = h(p_1) + h(p_2), \qquad 0 < p_1 \leq 1, \qquad 0 < p_2 \leq 1.$$

Two other reasonable requirements on h are that $h(p)$ be a decreasing function of p, and that h be continuous.

The only function satisfying the three above requirements is $h(p) = -C \log p$ (see Problem 1.12).

1.3. Three interpretations of the uncertainty function

As inspection of the form of the uncertainty function reveals, $H(p_1, \ldots, p_M)$ is a weighted average of the numbers $-\log p_1, \ldots, -\log p_M$, where the weights are the probabilities of the various values of X. This suggests that $H(p_1, \ldots, p_M)$ may be interpreted as the expectation of a random variable $W = W(X)$ which assumes the value $-\log p_i$ with probability p_i. If X takes the value x_i, then $W(X) = -\log P\{X = x_i\}$. Thus the values of $W(X)$ are the negatives of the logarithms of the probabilities associated with the values assumed by X. The expected value of $W(X)$ is $-\sum_{i=1}^{M} P\{X = x_i\} \log P\{X = x_i\} = H(X)$. An example may help to clarify the situation. If $p(x_1) = 2/3, p(x_2) = 1/6, p(x_3) = p(x_4) = 1/12$, then the random variable $W(X)$ has the following distribution:

$$P\left\{W = \log \frac{3}{2}\right\} = \frac{2}{3}; \qquad P\left\{W = \log 6\right\} = \frac{1}{6};$$

$$P\left\{W = \log 12\right\} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}.$$

The expectation of W is

$$\frac{2}{3} \log \frac{3}{2} + \frac{1}{6} \log 6 + \frac{1}{6} \log 12,$$

which is
$$H\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}\right).$$

There is another interpretation of $H(X)$ which is closely related to the construction of codes. Suppose X takes on five values x_1, x_2, x_3, x_4, x_5 with probabilities .3, .2, .2, .15, and .15, respectively. Suppose that the value of X is to be revealed to us by someone who cannot communicate except by means of the words "yes" and "no." We therefore try to

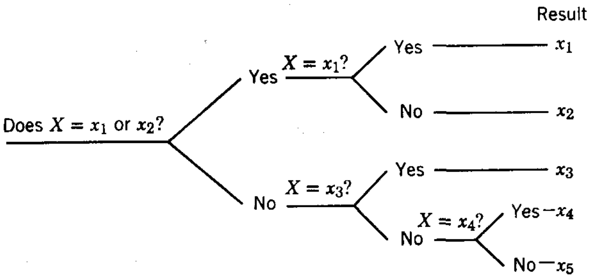


Fig. 1.3.1. A scheme of "yes or no" questions for isolating the value of a random variable.

arrive at the correct value by a sequence of "yes" and "no" questions, as shown in Fig. 1.3.1. If in fact X equals x_4, we receive a "no" answer to the question "Does $X = x_1$ or x_2?" We then ask "Does $X = x_3$?" and receive a "no" answer. The answer to "Does $X = x_4$?" is then "yes" and we are finished. Notice that the number of questions required to specify the value of X is a random variable that equals 2 whenever $X = x_1$, x_2, or x_3, and equals 3 when $X = x_4$ or x_5. The average number of questions required is
$$(0.3 + 0.2 + 0.2)2 + (0.15 + 0.15)3 = 2.3.$$

The essential content of the "noiseless coding theorem," to be proved in Chapter 2, is that the average number of "yes or no" questions needed to specify the value of X can never be less than the uncertainty of X. In this example, $H(X) = -0.3 \log 0.3 - 0.4 \log 0.2 - 0.3 \log 0.15 = 2.27$. We shall see in Chapter 2 that no scheme can be devised which on the average uses fewer questions than the one described above. We can, however, improve the efficiency of our guesswork if we assume that the experiment associated with X is performed independently n times and that we are allowed to wait until all n observations are recorded and then guess all n results simultaneously. For example, if X takes on two values, x_1 and x_2, with probabilities .7 and .3 respectively, one question is needed to specify the value of X. If, however, we are allowed to make guesses about the outcome of a joint experiment involving two independent observations of X, we may use the scheme shown in Fig. 1.3.2, which uses $0.49 + 2(0.21) + 3(0.21 + 0.09) = 1.81$ questions on the average, or 0.905 questions *per value of X*. We show later that by making guesses about longer and longer blocks, the average number of questions per value of X may be made to approach $H(X)$ as closely as desired. In no case can the

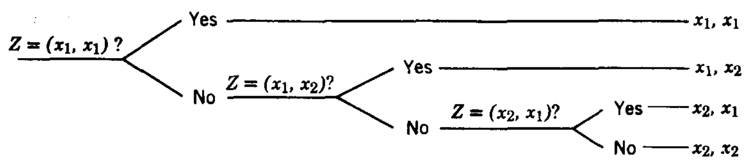


Fig. 1.3.2. "Yes or no" questions associated with two independent observations of X.

average number of questions be less than $H(X)$. Thus we may state the second interpretation of the uncertainty measure:

$H(X)$ = the minimum average number of "yes or no" questions required to determine the result of one observation of X.

There is a third interpretation of uncertainty which is related to the asymptotic behavior of a sequence of independent, identically distributed random variables. Let X be a random variable taking on the values $x_1, \ldots, x_M$ with probabilities $p_1, \ldots, p_M$ respectively. Suppose that the experiment associated with X is performed independently n times. In other words, we consider a sequence $X_1, \ldots, X_n$ of independent, identically distributed random variables, each having the same distribution as X. Let $f_i = f_i(X_1, \ldots, X_n)$ be the number of times that the symbol x_i occurs in the sequence $X_1, \ldots, X_n$; then f_i has a binomial distribution with parameters n and p_i. Given $\varepsilon > 0$ choose any positive number k such that $1/k^2 < \varepsilon/M$; fix ε and k for the remainder of the discussion. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a sequence of symbols, each α_i being one of the elements $x_1, \ldots, x_M$. We say that the sequence α is *typical* if

$$\left| \frac{f_i(\alpha) - np_i}{\sqrt{np_i(1 - p_i)}} \right| < k \quad \text{for all} \quad i = 1, 2, \ldots, M.$$

(The definition of a typical sequence depends on k, and hence on ε; however, the results we are seeking will be valid for any ε and appropriate k.) Thus in a typical sequence, each symbol x_i occurs approximately with its expected frequency np_i, the difference between the actual and expected frequency being of the order of $\sqrt{n}$ and hence small in comparison with n. We now show that, for large n, there are approximately 2^{Hn} typical sequences of length n, each with a probability of approximately 2^{-Hn}, where $H = H(X)$ is the uncertainty of X. More precisely, we prove the following result.

Theorem 1.3.1. 1. The set of nontypical sequences of length n has a total probability $<\varepsilon$.

2. There is a positive number A such that for each typical sequence α of length n,

$$2^{-nH-A\sqrt{n}} < P\{(X_1, \ldots, X_n) = \alpha\} < 2^{-nH+A\sqrt{n}}.$$

3. The number of typical sequences of length n is $2^{n(H+\delta_n)}$ where $\lim_{n\to\infty} \delta_n = 0$.

Proof. The probability that $(X_1, \ldots, X_n)$ is not a typical sequence is

$$P\left\{\left|\frac{f_i - np_i}{\sqrt{np_i(1-p_i)}}\right| \geq k \text{ for at least one } i\right\} \leq \sum_{i=1}^{M} P\left\{\left|\frac{f_i - np_i}{\sqrt{np_i(1-p_i)}}\right| \geq k\right\}.$$

But by Chebyshev's inequality,

$$P\left\{\left|\frac{f_i - np_i}{\sqrt{np_i(1-p_i)}}\right| \geq k\right\} = P\{|f_i - np_i| \geq k\sqrt{np_i(1-p_i)}\}$$

$$\leq \frac{E[|f_i - np_i|^2]}{k^2 np_i(1-p_i)} = \frac{1}{k^2} < \frac{\varepsilon}{M}$$

since f_i has a binomial distribution with parameters n and p_i, and hence has variance $np_i(1-p_i)$. Thus

$$P\{(X_1, \ldots, X_n) \text{ is not typical}\} < \varepsilon,$$

proving the first part of the theorem. Now let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a typical sequence. We then have

$$np_i - k\sqrt{np_i(1-p_i)} < f_i(\alpha) < np_i + k\sqrt{np_i(1-p_i)}, \qquad i = 1, \ldots, M. \tag{1.3.1}$$

Also $P\{(X_1, \ldots, X_n) = \alpha\} = p_1^{f_1(\alpha)} p_2^{f_2(\alpha)} \cdots p_M^{f_M(\alpha)}$ because of the independence of the X_i. Writing $p(\alpha)$ for $P\{(X_1, \ldots, X_n) = \alpha\}$ we have $-\log p(\alpha) = -\sum_{i=1}^{M} f_i(\alpha) \log p_i$ or, using (1.3.1),

$$-\sum_{i=1}^{M}(np_i \log p_i - k\sqrt{np_i(1-p_i)} \log p_i) < -\log p(\alpha)$$

$$< -\sum_{i=1}^{M}(np_i \log p_i + k\sqrt{np_i(1-p_i)} \log p_i). \tag{1.3.2}$$

If we let $A = -k \sum_{i=1}^{M} \sqrt{p_i(1-p_i)} \log p_i > 0$, then (1.3.2) yields

$$nH - A\sqrt{n} < -\log p(\alpha) < nH + A\sqrt{n}$$

or

$$2^{-nH-A\sqrt{n}} < p(\alpha) < 2^{-nH+A\sqrt{n}},$$

proving part 2. Finally, let S be the set of typical sequences. We have proved (part 1) that $1 - \varepsilon < P\{(X_1, \ldots, X_n) \in S\} \leq 1$ and (part 2) that for each $\alpha \in S$,

$$2^{-nH-A\sqrt{n}} < P\{(X_1, \ldots, X_n) = \alpha\} < 2^{-nH+A\sqrt{n}}.$$

Now if S is a subset of a finite sample space with a probability that is known to be between (say) $\frac{3}{4}$ and 1, and each point of S has a probability of between (say) $\frac{1}{16}$ and $\frac{3}{16}$, then the number of points of S is at least $\frac{3}{4}/\frac{3}{16} = 4$ and at most $1/\frac{1}{16} = 16$; for example, the fewest number of points would be obtained if S had the minimum probability $\frac{3}{4}$, and each point of S has the maximum probability $\frac{3}{16}$. By similar reasoning, the number of typical sequences is at least

$$(1-\varepsilon)2^{nH-A\sqrt{n}} = 2^{nH-A\sqrt{n}+\log(1-\varepsilon)} = 2^{n[H-An^{-1/2}+n^{-1}\log(1-\varepsilon)]}$$

and at most $2^{nH+A\sqrt{n}} = 2^{n(H+An^{-1/2})}$; part 3 follows.

1.4. Properties of the uncertainty function; joint and conditional uncertainty

In this section we derive several properties of the uncertainty measure $H(X)$ and introduce the notions of *joint uncertainty* and *conditional uncertainty*.

We first note that $H(X)$ is always nonnegative since $-p_i \log p_i \geq 0$ for all i. We then establish a very useful lemma.

Lemma 1.4.1. Let $p_1, p_2, \ldots, p_M$ and $q_1, q_2, \ldots, q_M$ be arbitrary positive numbers with $\sum_{i=1}^{M} p_i = \sum_{i=1}^{M} q_i = 1$.

Then $-\sum_{i=1}^{M} p_i \log p_i \leq -\sum_{i=1}^{M} p_i \log q_i$, with equality if and only if $p_i = q_i$ for all i. For example, let

$$p_1 = \tfrac{1}{2}, \quad p_2 = p_3 = \tfrac{1}{4}, \quad q_1 = \tfrac{1}{3}, \quad q_2 = \tfrac{4}{9}, \quad q_3 = \tfrac{2}{9}.$$

Then

$$-\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{4}\log\tfrac{1}{4} - \tfrac{1}{4}\log\tfrac{1}{4} = 1.5$$

and

$$-\tfrac{1}{2}\log\tfrac{1}{3} - \tfrac{1}{4}\log\tfrac{4}{9} - \tfrac{1}{4}\log\tfrac{2}{9} = 1.63.$$

Proof. For convenience we use natural logarithms instead of logarithms to the base 2. Since $\log_2 x = \log_2 e \log_e x$, the statement of the lemma is unaffected by this change. The logarithm is a convex function; in other words, $\ln x$ always lies below its tangent. By considering the tangent at $x = 1$, we obtain $\ln x \leq x - 1$ with equality if and only if $x = 1$ (see

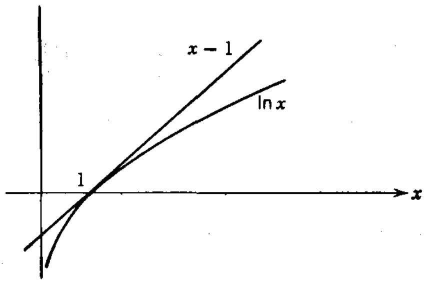


Fig. 1.4.1. Proof of Lemma 1.4.1.

Fig. 1.4.1). Thus $\ln (q_i/p_i) \leq q_i/p_i - 1$ with equality if and only if $p_i = q_i$. Multiplying the inequality by p_i and summing over i we obtain

$$\sum_{i=1}^{M} p_i \ln \frac{q_i}{p_i} \leq \sum_{i=1}^{M} (q_i - p_i) = 1 - 1 = 0$$

with equality if and only if $p_i = q_i$ for all i. Thus

$$\sum_{i=1}^{M} p_i \ln q_i - \sum_{i=1}^{M} p_i \ln p_i \leq 0$$

which proves the lemma.

We would expect that a situation involving a number of alternatives would be most uncertain when all possibilities are equally likely. The uncertainty function $H(X)$ does indeed have this property, as shown by the following theorem.

Theorem 1.4.2. $H(p_1, p_2, \ldots, p_M) \leq \log M$, with equality if and only if all $p_i = 1/M$.

Proof. The application of Lemma 1.4.1 with all $q_i = 1/M$ yields

$$-\sum_{i=1}^{M} p_i \log p_i \leq -\sum_{i=1}^{M} p_i \log \frac{1}{M} = \log M \sum_{i=1}^{M} p_i = \log M$$

with equality if and only if $p_i = q_i = 1/M$ for all i.

We turn now to the problem of characterizing the uncertainty associated with more than one random variable. Suppose X and Y are two discrete random variables associated with the same experiment. Let X and Y have a joint probability function

$$p(x_i, y_j) = P\{X = x_i \text{ and } Y = y_j\} = p_{ij} \qquad (i = 1, \ldots, M;\ j = 1, \ldots, L).$$

We therefore have an experiment with ML possible outcomes; the outcome $\{X = x_i, Y = y_j\}$ has probability $p(x_i, y_j)$. It is natural to define the *joint uncertainty* of X and Y as

$$H(X, Y) = -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j) \log p(x_i, y_j).$$

Similarly, we define the joint uncertainty of n random variables $X_1, X_2, \ldots, X_n$ as

$$H(X_1, X_2, \ldots, X_n) = -\sum_{x_1, x_2, \ldots, x_n} p(x_1, x_2, \ldots, x_n) \log p(x_1, \ldots, x_n),$$

where $p(x_1, x_2, \ldots, x_n) = P\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\}$ is the joint probability function of $X_1, X_2, \ldots, X_n$. A connection between joint uncertainty and individual uncertainty is established by the following theorem.

Theorem 1.4.3. $H(X, Y) \le H(X) + H(Y)$, with equality if and only if X and Y are independent.

Proof. Since $p(x_i) = \sum_{j=1}^{L} p(x_i, y_j)$ and $p(y_j) = \sum_{i=1}^{M} p(x_i, y_j)$, we may write:

$$H(X) = -\sum_{i=1}^{M} p(x_i) \log p(x_i) = -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j) \log p(x_i)$$

and

$$H(Y) = -\sum_{j=1}^{L} p(y_j) \log p(y_j) = -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j) \log p(y_j).$$

Thus

$$\begin{aligned} H(X) + H(Y) &= -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j)[\log p(x_i) + \log p(y_j)] \\ &= -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j) \log p(x_i)p(y_j) \\ &= -\sum_{i=1}^{M}\sum_{j=1}^{L} p_{ij} \log q_{ij} \end{aligned}$$

where $q_{ij} = p(x_i)p(y_j)$.

We have $H(X, Y) = -\sum_{i=1}^{M}\sum_{j=1}^{L} p_{ij} \log p_{ij}$. We may now apply Lemma 1.4.1 to obtain:

$$-\sum_{i=1}^{M}\sum_{j=1}^{L} p_{ij} \log p_{ij} \le -\sum_{i=1}^{M}\sum_{j=1}^{L} p_{ij} \log q_{ij}$$

with equality if and only if $p_{ij} = q_{ij}$ for all i, j. The fact that we have a double instead of a single summation is of no consequence since the double sum can be transformed into a single sum by reindexing. We need only check that the hypothesis of the lemma is satisfied, that is,

$$\sum_{i=1}^{M} \sum_{j=1}^{L} q_{ij} = \sum_{i=1}^{M} p(x_i) \sum_{j=1}^{L} p(y_j) = 1 \cdot 1 = 1.$$

Thus, $H(X, Y) \leq H(X) + H(Y)$ with equality if and only if $p(x_i, y_j) = p(x_i)p(y_j)$ for all i, j; that is, if and only if X and Y are independent.

An argument identical to that used above may be used to establish the following results.

COROLLARY 1.4.3.1. $H(X_1, \ldots, X_n) \leq H(X_1) + \cdots + H(X_n)$, with equality if and only if $X_1, \ldots, X_n$ are independent.

COROLLARY 1.4.3.2. $H(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \leq H(X_1, \ldots, X_n) + H(Y_1, \ldots, Y_m)$, with equality if and only if the "random vectors" $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ are independent, that is, if and only if

$$P\{X_1 = \alpha_1, \ldots, X_n = \alpha_n, Y_1 = \beta_1, \ldots, Y_m = \beta_m\}$$
$$= P\{X_1 = \alpha_1, \ldots, X_n = \alpha_n\} P\{Y_1 = \beta_1, \ldots, Y_m = \beta_m\}$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$.

We turn now to the idea of conditional uncertainty. Let two random variables X and Y be given. If we are given that $X = x_i$, then the distribution of Y is characterized by the set of conditional probabilities $p(y_j \mid x_i)$ $(j = 1, 2, \ldots, L)$. We therefore define the *conditional uncertainty of Y given that $X = x_i$* as

$$H(Y \mid X = x_i) = -\sum_{j=1}^{L} p(y_j \mid x_i) \log p(y_j \mid x_i).$$

We define the *conditional uncertainty of Y given X* as a weighted average of the uncertainties $H(Y \mid X = x_i)$, that is,

$$H(Y \mid X) = p(x_1)H(Y \mid X = x_1) + \cdots + p(x_M)H(Y \mid X = x_M)$$

$$= -\sum_{i=1}^{M} p(x_i) \sum_{j=1}^{L} p(y_j \mid x_i) \log p(y_j \mid x_i).$$

Using the fact that $p(x_i, y_j) = p(x_i)p(y_j \mid x_i)$ we have

$$H(Y \mid X) = -\sum_{i=1}^{M} \sum_{j=1}^{L} p(x_i, y_j) \log p(y_j \mid x_i).$$

We may define conditional uncertainties involving more than two random variables in a similar manner. For example,

$$H(Y, Z \mid X) = -\sum_{i,j,k} p(x_i, y_j, z_k) \log p(y_j, z_k \mid x_i)$$

$$= \text{the uncertainty about } Y \text{ and } Z \text{ given } X.$$

$$H(Z \mid X, Y) = -\sum_{i,j,k} p(x_i, y_j, z_k) \log p(z_k \mid x_i, y_j)$$

$$= \text{the uncertainty about } Z \text{ given } X \text{ and } Y.$$

$$H(Y_1, \dots, Y_m \mid X_1, \dots, X_n) =$$

$$-\sum_{x_1,\dots,x_n,y_1,\dots,y_m} p(x_1, \dots, x_n, y_1, \dots, y_m) \log p(y_1, \dots, y_m \mid x_1, \dots, x_n)$$

$$= \text{the uncertainty about } Y_1, \dots, Y_m \text{ given } X_1, \dots, X_n.$$

We now establish some intuitively reasonable properties of conditional uncertainty. If two random variables X and Y are observed, but only the value of X is revealed, we hope that the remaining uncertainty about Y is $H(Y \mid X)$. This is justified by the following result.

Theorem 1.4.4. $H(X, Y) = H(X) + H(Y \mid X) = H(Y) + H(X \mid Y)$.

Proof. The theorem follows directly from the definition of the uncertainty function. We write

$$\begin{aligned}
H(X, Y) &= -\sum_{i=1}^{M} \sum_{j=1}^{L} p(x_i, y_j) \log p(x_i, y_j) \\
&= -\sum_{i=1}^{M} \sum_{j=1}^{L} p(x_i, y_j) \log p(x_i) p(y_j \mid x_i) \\
&= -\sum_{i=1}^{M} \sum_{j=1}^{L} p(x_i, y_j) \log p(x_i) - \sum_{i=1}^{M} \sum_{j=1}^{L} p(x_i, y_j) \log p(y_j \mid x_i) \\
&= -\sum_{i=1}^{M} p(x_i) \log p(x_i) + H(Y \mid X) \\
&= H(X) + H(Y \mid X).
\end{aligned}$$

Similarly, we prove that $H(X, Y) = H(Y) + H(X \mid Y)$.

A corresponding argument may be used to establish various identities involving more than two random variables. For example,

$$\begin{aligned}
H(X, Y, Z) &= H(X) + H(Y \mid X) + H(Z \mid X, Y) \\
&= H(X, Y) + H(Z \mid X, Y) \\
&= H(X) + H(Y, Z \mid X);
\end{aligned}$$

$$H(X_1, \dots, X_n, Y_1, \dots, Y_m) = H(X_1, \dots, X_n) + H(Y_1, \dots, Y_m \mid X_1, \dots, X_n).$$

It is also reasonable to hope that the revelation of the value of X should not increase the uncertainty about Y. This fact is expressed by Theorem 1.4.5.

Theorem 1.4.5. $H(Y \mid X) \leq H(Y)$, with equality if and only if X and Y are independent.

Proof. By Theorem 1.4.4, $H(X, Y) = H(X) + H(Y \mid X)$.

By Theorem 1.4.3, $H(X, Y) \leq H(X) + H(Y)$, with equality if and only if X and Y are independent. Theorem 1.4.5 then follows.

Similarly, it is possible to prove that

$$H(Y_1, \ldots, Y_m \mid X_1, \ldots, X_n) \leq H(Y_1, \ldots, Y_m)$$

with equality if and only if the random vectors

$$(X_1, \ldots, X_n) \quad \text{and} \quad (Y_1, \ldots, Y_m)$$

are independent.

A joint or conditional uncertainty may be interpreted as the expectation of a random variable, just as in the case of an individual uncertainty. For example,

$$H(X, Y) = -\sum_{i,j} p(x_i, y_j) \log p(x_i, y_j) = E[W(X, Y)]$$

where

$$W(X, Y) = -\log p(x_i, y_j) \quad \text{whenever } X = x_i \text{ and } Y = y_j$$

$$H(Y \mid X) = -\sum_{i,j} p(x_i, y_j) \log p(y_j \mid x_i) = E[W(Y \mid X)]$$

where

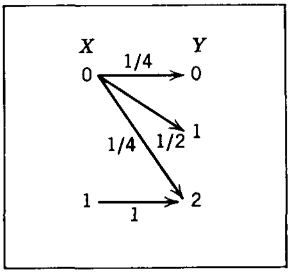
$$W(Y \mid X) = -\log p(y_j \mid x_i) \quad \text{whenever } X = x_i \text{ and } Y = y_j.$$

1.5. The measure of information

Consider the following experiment. Two coins are available, one unbiased and the other two-headed. A coin is selected at random and tossed twice, and the number of heads is recorded. We ask how much information is conveyed about the identity of the coin by the number of heads obtained. It is clear that the number of heads does tell us something about the nature of the coin. If less than 2 heads are obtained, the unbiased coin must have been used; if both throws resulted in heads, the evidence favors the two-headed coin. In accordance with the discussion at the beginning of this chapter, we decide to measure information as a *reduction in uncertainty*. To be specific, let X be a random variable that has the value 0 or 1 according as the unbiased or the two-headed coin is chosen. Let Y be the number of heads obtained in two tosses of the chosen coin. A diagram representing the experiment, together with various associated probability distributions, is shown in Fig. 1.5.1. The initial uncertainty about the identity of the coin is $H(X)$. After the number of heads is revealed, the uncertainty is $H(X \mid Y)$. We therefore define the *information conveyed about X by Y* as

$$I(X \mid Y) = H(X) - H(X \mid Y). \tag{1.5.1}$$

$P\{X = 0\} = 1/2$
$P\{X = 1\} = 1/2$



$P\{Y = 0\} = 1/8$ $P\{X = 0 \mid Y = 0\} = 1$
$P\{Y = 1\} = 1/4$ $P\{X = 0 \mid Y = 1\} = 1$
$P\{Y = 2\} = 5/8$ $P\{X = 0 \mid Y = 2\} = 1/5$

Fig. 1.5.1. A coin-tossing experiment.

The formula (1.5.1) is our general definition of the information conveyed about one random variable by another. In this case the numerical results are:

$$\begin{aligned} H(X) &= \log 2 = 1, \\ H(X \mid Y) &= P\{Y = 0\}H(X \mid Y = 0) + P\{Y = 1\}H(X \mid Y = 1) \\ &\qquad + P\{Y = 2\}H(X \mid Y = 2) \\ &= \tfrac{1}{8}(0) + \tfrac{1}{4}(0) - \tfrac{5}{8}(\tfrac{1}{5}\log\tfrac{1}{5} + \tfrac{4}{5}\log\tfrac{4}{5}) \\ &= 0.45; \\ I(X \mid Y) &= 0.55. \end{aligned}$$

The information measure may be interpreted as the expectation of a random variable, as was the uncertainty measure. We may write

$$\begin{aligned} I(X \mid Y) &= H(X) - H(X \mid Y) \\ &= -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j)\log p(x_i) + \sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j)\log p(x_i \mid y_j) \\ &= -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i, y_j)\log\frac{p(x_i)}{p(x_i \mid y_j)}. \end{aligned}$$

Thus $I(X \mid Y) = E[U(X \mid Y)]$ where $X = x_i$, $Y = y_j$ implies $U(X \mid Y) = -\log\,[p(x_i)/p(x_i \mid y_j)]$; we may write

$$U(X \mid Y) = W(X) - W(X \mid Y)$$

where $W(X)$ and $W(X \mid Y)$ are as defined in Sections 1.3 and 1.4.

A suggestive notation that is sometimes used is

$$H(X) = E[-\log p(X)], \qquad H(X \mid Y) = E[-\log p(X \mid Y)],$$

$$I(X \mid Y) = E\left[-\log \frac{p(X)}{p(X \mid Y)}\right];$$

the expression $-\log p(X)$ is the random variable $W(X)$ defined previously, and similarly for the other terms.

The information conveyed about X by Y may also be interpreted as the difference between the minimum average number of "yes or no" questions required to determine the result of one observation of X before Y is observed and the minimum average number of such questions required after Y is observed. However, the fundamental significance of the information measure is its application to the reliable transmission of messages through noisy communication channels. We shall discuss this subject in great detail beginning with Chapter 3. At this point we shall be content to derive a few properties of $I(X \mid Y)$. By Theorem 1.4.5, $H(X \mid Y) \leq H(X)$ with equality if and only if X and Y are independent. Hence $I(X \mid Y) \geq 0$ with equality if and only if X and Y are independent. By Theorem 1.4.4, $H(X \mid Y) = H(X, Y) - H(Y)$; thus $I(X \mid Y) = H(X) + H(Y) - H(X, Y)$. But $H(X, Y)$ is the same as $H(Y, X)$, and therefore

$$I(X \mid Y) = I(Y \mid X).$$

The information measure thus has a surprising symmetry; the information conveyed about X by Y is the same as the information conveyed about Y by X. For the example of Fig. 1.5.1, we compute

$$H(Y) = -\tfrac{1}{8} \log \tfrac{1}{8} - \tfrac{1}{4} \log \tfrac{1}{4} - \tfrac{5}{8} \log \tfrac{5}{8} = 1.3;$$

$$\begin{aligned} H(Y \mid X) &= P\{X = 0\} H(Y \mid X = 0) + P\{X = 1\} H(Y \mid X = 1) \\ &= \tfrac{1}{2} H(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}) + \tfrac{1}{2} H(1) = 0.75. \end{aligned}$$

$$I(Y \mid X) = H(Y) - H(Y \mid X) = 0.55, \qquad \text{as before.}$$

When the conditional probabilities $p(y_j \mid x_i)$ are specified at the beginning of a problem, as they are here, it is usually easier to compute the information using $H(Y) - H(Y \mid X)$ rather than $H(X) - H(X \mid Y)$.

More generally, we may define the information conveyed about a set of random variables by another set of random variables. If $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are random variables, the *information conveyed about* $X_1, \ldots, X_n$ *by* $Y_1, \ldots, Y_m$ is defined as

$$I(X_1, \ldots, X_n \mid Y_1, \ldots, Y_m) = H(X_1, \ldots, X_n) - H(X_1, \ldots, X_n \mid Y_1, \ldots, Y_m).$$

Proceeding as before, we obtain

$$\begin{aligned} I(X_1, \ldots, X_n \mid Y_1, \ldots, Y_m) &= H(X_1, \ldots, X_n) \\ &\quad + H(Y_1, \ldots, Y_m) - H(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \\ &= I(Y_1, \ldots, Y_m \mid X_1, \ldots, X_n). \end{aligned}$$

1.6. Notes and remarks

The axioms for the uncertainty measure given in the text are essentially those of Shannon (1948). A weaker set of axioms which determine the same uncertainty function has been given by Fadiev (1956); Fadiev's axioms are described in Feinstein (1958). The weakest set of axioms known at this time may be found in Lee (1964). The properties of the uncertainty and information measures developed in Sections 1.4 and 1.5 are due to Shannon (1948); a similar discussion may be found in Feinstein (1958). The interpretation of $H(X)$ in terms of typical sequences was discovered by Shannon (1948). The notion of typical sequence is used by Wolfowitz (1961) as the starting point for his proofs of coding theorems; Wolfowitz's term "π-sequence" corresponds essentially to our "typical sequence."

We have thus far required that the arguments $p_1, \ldots, p_M$ of H be strictly positive. It is convenient, however, to extend the domain so that zero values are allowed; we may do this by writing

$$H(p_1, \ldots, p_M) = -\sum_{i=1}^{M} p_i \log p_i \qquad (\text{all } p_i \geq 0, \ \sum_{i=1}^{M} p_i = 1)$$

with the proviso that an expression which appears formally as $0 \log 0$ is defined to be 0. This convention preserves the continuity of H. In Lemma 1.4.1, we may allow some of the p_i or q_i to be 0 if we interpret $0 \log 0$ as 0 and $-a \log 0$ as $+\infty$ if $a > 0$.

The quantity $H(X)$, which we have referred to as the "uncertainty of X," has also been called the "entropy" or "communication entropy" of X.

PROBLEMS

1.1 The inhabitants of a certain village are divided into two groups A and B. Half the people in group A always tell the truth, three-tenths always lie, and two-tenths always refuse to answer. In group B, three-tenths of the people are truthful, half are liars, and two-tenths always refuse to answer. Let p be the probability that a person selected at random will belong to group A. Let $I = I(p)$ be the information conveyed about a person's truth-telling status by specifying his group membership. Find the maximum possible value of I and the percentage of people in group A for which the maximum occurs.

1.2 A single unbiased die is tossed once. If the face of the die is 1, 2, 3, or 4, an unbiased coin is tossed once. If the face of the die is 5 or 6, the coin is tossed twice. Find the information conveyed about the face of the die by the number of heads obtained.

1.3 Suppose that in a certain city, $\frac{3}{4}$ of the high-school students pass and $\frac{1}{4}$ fail. Of those who pass, 10 percent own cars, while 50 percent of the failing students own cars. All of the car-owning students belong to fraternities, while 40 percent of those who do not own cars but pass, as well as 40 percent of those who do not own cars but fail, belong to fraternities.

a. How much information is conveyed about a student's academic standing by specifying whether or not he owns a car?

b. How much information is conveyed about a student's academic standing by specifying whether or not he belongs to a fraternity?

c. If a student's academic standing, car-owning status, and fraternity status are transmitted by three successive binary digits, how much information is conveyed by each digit?

1.4 Establish the following:

a. $$H(Y, Z \mid X) \leq H(Y \mid X) + H(Z \mid X)$$

with equality if and only if $p(y_j, z_k \mid x_i) = p(y_j \mid x_i)p(z_k \mid x_i)$ for all i, j, k.

b. $$H(Y, Z \mid X) = H(Y \mid X) + H(Z \mid X, Y).$$

c. $$H(Z \mid X, Y) \leq H(Z \mid X)$$

with equality if and only if $p(y_j, z_k \mid x_i) = p(y_j \mid x_i)p(z_k \mid x_i)$ for all i, j, k.

Note that these results hold if the random variables are replaced by random vectors, that is,

$$X = (X_1, \ldots, X_n), \qquad Y = (Y_1, \ldots, Y_m), \qquad Z = (Z_1, \ldots, Z_r).$$

The condition $p(y_j, z_k \mid x_i) = p(y_j \mid x_i)p(z_k \mid x_i)$ becomes

$$p(y_1, \ldots, y_m, z_1, \ldots, z_r \mid x_1, \ldots, x_n) = p(y_1, \ldots, y_m \mid x_1, \ldots, x_n)p(z_1, \ldots, z_r \mid x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r$. This condition is sometimes expressed by saying that *Y and Z are conditionally independent given X*.

1.5 Use Lemma 1.4.1 to prove the *inequality of the arithmetic and geometric means*:

Let $x_1, \ldots, x_n$ be arbitrary positive numbers; let $a_1, \ldots, a_n$ be positive numbers whose sum is unity. Then

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \leq \sum_{i=1}^{n} a_i x_i$$

with equality if and only if all x_i are equal.

Note that the inequality still holds if some of the a_i are allowed to be zero (keeping $\sum_{i=1}^{n} a_i = 1$) since if $a = 0$ we may multiply the left-hand side of the inequality by x^a and add ax to the right-hand side without affecting the result. However, the condition for equality becomes:

All x_i corresponding to positive a_i are equal.

1.6 A change that tends in the following sense to equalize a set of probabilities $p_1, \ldots, p_M$ always results in an increase in uncertainty:

Suppose $p_1 > p_2$. Define

$$\begin{aligned} p_1' &= p_1 - \Delta p \\ p_2' &= p_2 + \Delta p \\ p_i' &= p_i, \; i = 3, \ldots, M \end{aligned}$$

where $\Delta p > 0$ and $p_1 - \Delta p \geq p_2 + \Delta p$. Show that $H(p_1', \ldots, p_M') > H(p_1, \ldots, p_M)$.

1.7 (Feinstein 1958). Let $A = [a_{ij}]$ be a doubly stochastic matrix, that is, $a_{ij} \geq 0$ for all i, j; $\sum_{j=1}^M a_{ij} = 1$, $i = 1, \ldots, M$; $\sum_{i=1}^M a_{ij} = 1, j = 1, \ldots, M$. Given a set of probabilities $p_1, \ldots, p_M$, define a new set of probabilities $p_1', \ldots, p_M'$ by

$$p_i' = \sum_{j=1}^{M} a_{ij} p_j, \qquad i = 1, 2, \ldots, M.$$

Show that $H(p_1', \ldots, p_M') \geq H(p_1, \ldots, p_M)$ with equality if and only if $(p_1', \ldots, p_M')$ is a rearrangement of $(p_1, \ldots, p_M)$. Show also that Problem 1.6 is a special case of this result.

1.8 Given a discrete random variable X with values $x_1, \ldots, x_M$, define a random variable Y by $Y = g(X)$ where g is an arbitrary function. Show that $H(Y) \leq H(X)$. Under what conditions on the function g will there be equality?

1.9 Let X and Y be random variables with numerical values $x_1, \ldots, x_M$; $y_1, \ldots, y_L$ respectively. Let $Z = X + Y$.

a. Show that $H(Z \mid X) = H(Y \mid X)$: hence if X and Y are independent, $H(Z \mid X) = H(Y)$ so that $H(Y) \leq H(Z)$, and similarly $H(X) \leq H(Z)$.

b. Give an example in which $H(X) > H(Z)$, $H(Y) > H(Z)$.

1.10 Prove the *generalized grouping axiom.*

$$\begin{aligned} &H(p_1, \ldots, p_{r_1}; p_{r_1+1}, \ldots, p_{r_2}; \cdots ; p_{r_{k-1}+1}, \ldots, p_{r_k}) \\ &\quad = H(p_1 + \cdots + p_{r_1}, p_{r_1+1} + \cdots + p_{r_2}, \ldots, p_{r_{k-1}+1} + \cdots + p_{r_k}) \\ &\qquad + \sum_{i=1}^{k} (p_{r_{i-1}+1} + \cdots + p_{r_i}) H\left(\frac{p_{r_{i-1}+1}}{\sum_{j=r_{i-1}+1}^{r_i} p_j}, \ldots, \frac{p_{r_i}}{\sum_{j=r_{i-1}+1}^{r_i} p_j} \right) \end{aligned}$$

1.11 Show that if $h(p)$, $0 < p \leq 1$, is a continuous function such that $\sum_{i=1}^M p_i h(p_i) = -C \sum_{i=1}^M p_i \log p_i$ for all M and all $p_1, \ldots, p_M$ such that $p_i > 0$, $\sum_{i=1}^M p_i = 1$, then $h(p) = -C \log p$.

1.12 Given a function $h(p)$, $0 < p \leq 1$, satisfying

a. $h(p_1 p_2) = h(p_1) + h(p_2), \qquad 0 < p_1 \leq 1, \quad 0 < p_2 \leq 1.$

b. $h(p)$ is a monotonically decreasing and continuous function of p, $0 < p \leq 1$.

Show that the only function satisfying the given conditions is $h(p) = -C \log_b p$ where $C > 0$, $b > 1$.

CHAPTER TWO

Noiseless Coding

2.1. *Introduction*

Our first application of the notion of uncertainty introduced in Chapter 1 will be to the problem of efficient coding of messages to be sent over a "noiseless" channel, that is, a channel allowing perfect transmission from input to output. Thus we do not consider the problem of error correction; our only concern is to maximize the number of messages that can be sent over the channel in a given time. To be specific, assume that the messages to be transmitted are generated by a random variable X whose values are $x_1, \ldots, x_M$. A noiseless channel may be thought of intuitively as a device that accepts an input from a specified set of "code characters" $a_1, \ldots, a_D$ and reproduces the input symbol at the output with no possibility of error. (The formal definition of an information channel will be deferred until Chapter 3; it will not be needed here.) If the symbols x_i are to be communicated properly, each x_i must be represented by a sequence of symbols chosen from the set $\{a_1, \ldots, a_D\}$. Thus, we assign a sequence of code characters to each x_i; such a sequence is called a "code word." Since the problem of error correction does not arise, efficient communication would involve transmitting a given message in the shortest possible time. If the rate at which the symbols a_j can be sent through the channel is fixed, the requirement of efficiency suggests that we make the code words as short as possible. In calculating the long-run efficiency of communication, the *average* length of a code word is of interest; it is this quantity which we choose to minimize.

To summarize, the ingredients of the noiseless coding problem are:

1. A *random variable* X, taking on the values $x_1, \ldots, x_M$ with probabilities $p_1, \ldots, p_M$ respectively. X is to be observed independently over and over again, thus generating a sequence whose components belong to the set $\{x_1, \ldots, x_M\}$; such a sequence is called a *message*.

2. A set $\{a_1, \ldots, a_D\}$ called the set of *code characters* or the *code alphabet;* each symbol x_i is to be assigned a finite sequence of code characters called the *code word* associated with x_i (for example, x_1 might correspond to a_1a_2, and x_2 to $a_3a_7a_3a_8$). The collection of all code words is called a *code*. The code words are assumed to be distinct.

3. The *objective* of noiseless coding is to minimize the average code-word length. If the code word associated with x_i is of length n_i, $i = 1, 2, \ldots, M$, we will try to find codes that minimize $\sum_{i=1}^{M} p_i n_i$.

2.2. *The problem of unique decipherability*

It becomes clear very quickly that some restriction must be placed on the assignment of code words. For example, consider the following binary code:

x_1	0
x_2	010
x_3	01
x_4	10

The binary sequence 010 could correspond to any one of the three messages x_2, x_3x_1, or x_1x_4. Thus the sequence 010 cannot be decoded accurately. We would like to rule out ambiguities of this type; hence the following definition.

A code is *uniquely decipherable* if every finite sequence of code characters corresponds to at most one message.

One way to insure unique decipherability is to require that no code word be a "prefix" of another code word. If A_1 and A_2 are finite (nonempty) sequences of code characters then the *juxtaposition* of A_1 and A_2, written A_1A_2, is the sequence formed by writing A_1 followed by A_2. We say that the sequence A is a *prefix* of the sequence B if B may be written as AC for some sequence C.

A code having the property that no code word is a prefix of another code word is said to be *instantaneous*. The code below is an example of an instantaneous code.

x_1	0
x_2	100
x_3	101
x_4	11

Notice that the sequence 11111 does not correspond to any message; such a sequence will never appear and thus can be disregarded. Before turning to the problem of characterizing uniquely decipherable codes, we note that *every instantaneous code is uniquely decipherable, but not conversely*. For given a finite sequence of code characters of an instantaneous code, proceed from the left until a code word W is formed. (If there is no such word, the unique decipherability condition is vacuously satisfied.) Since W is not the prefix of another code word, W must correspond to the first symbol of the message. Now continue until another code word is formed. The process may be repeated until the end of the message.

For example, in the instantaneous code {0, 100, 101, 11} above, the sequence 101110100101 is decoded as $x_3x_4x_1x_2x_3$.

Now consider the code

x_1	0
x_2	01

This code is not instantaneous since 0 is a prefix of 01. The code is uniquely decipherable, however, since any sequence of code characters may be decoded by noting the position of the ones in the sequence. For example, the sequence 0010000101001 is decoded as $x_1x_2\, x_1x_1x_1\, x_2x_2x_1x_2$. The word "instantaneous" refers to the fact that a sequence of code characters may be decoded step by step. If, proceeding from the left, W is the first word formed, we known immediately that W is the first word of the message. In a uniquely decipherable code which is not instantaneous, we may have to wait a long time before we know the identity of the first word. For example, if in the code

$$\begin{array}{ll} x_1 & 0 \\ x_2 & \underbrace{00\cdots001}_{n} \end{array}$$

we received the sequence $\underbrace{00\cdots001}_{n+1}$ we would have to wait until the end of the sequence to find out that the corresponding message starts with x_1.

We now present a testing procedure that can always be used to determine whether or not a code is uniquely decipherable. To see how the procedure works, consider the code of Fig. 2.2.1, which is not instantaneous but could conceivably be uniquely decipherable. We construct a sequence of sets $S_0, S_1, S_2, \ldots$, as follows. Let S_0 be the original set of code words. To form S_1, we look at all pairs of code words in S_0. If a code word W_i is a prefix of another code word W_j, that is, $W_j = W_iA$, we place the suffix A in S_1. In the above code, the word a is a prefix of the word abb, so that bb is one of the members of S_1. In general, to form S_n, $n > 1$, we compare S_0 and S_{n-1}. If a code word $W \epsilon S_0$ is a prefix of a sequence $A = WB \epsilon S_{n-1}$, the suffix B is placed in S_n, and if a sequence $A' \epsilon S_{n-1}$ is a prefix of a code word $W' = A'B' \epsilon S_0$, we place the suffix $B' \epsilon S_n$. The sets S_n, $n = 0, 1, \ldots$, for the code of Fig. 2.2.1 are shown in Fig. 2.2.2.

x_1	a
x_2	c
x_3	ad
x_4	abb
x_5	bad
x_6	deb
x_7	$bbcde$

Fig. 2.2.1. A code.

We shall prove

Theorem 2.2.1. A code is uniquely deciperable if and only if none of the sets $S_1, S_2, S_3, \ldots$ contains a code word, that is, a member of S_0.

In Fig. 2.2.1, the code word *ad* belongs to the set S_5; hence, according to Theorem 2.2.1, the code is not uniquely decipherable. In fact, the sequence *abbcdebad* is ambiguous, having the two possible interpretations *a*, *bbcde*, *bad* and *abb*, *c*, *deb*, *ad*. A systematic method of constructing ambiguous sequences will be given as part of the proof of Theorem 2.2.1.

S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7
a	*d*	*eb*	*de*	*b*	*ad*	*d*	*eb*
c	*bb*	*cde*			*bcde*		
ad							
abb							
bad				S_n empty $(n > 7)$			
deb							
bbcde							

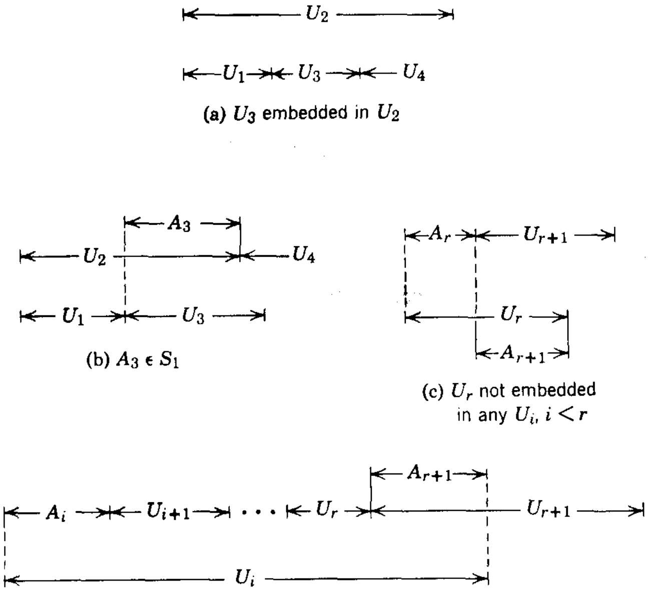
Fig. 2.2.2. Test for unique decipherability.

Proof. First suppose that the code is not uniquely decipherable, so that there is a sequence of code characters which is ambiguous, that is, corresponds to more than one possible message. Pick an ambiguous sequence G with the smallest possible number of symbols. Then G may be written in at least two distinct ways:

$$G = W_1 W_2 \cdots W_n = W_1' W_2' \cdots W_m'$$

where the W_i and W_j' are code words (assume $n \geq 2$, $m \geq 2$; otherwise the conclusion is immediate).

Now define the *index* of the word W_i (respectively W_j') in G as the number of letters in $W_1 W_2 \cdots W_{i-1}$ (respectively $W_1' \cdots W_{j-1}'$), $i = 2, \ldots, n$, $j = 2, \ldots, m$. The minimality of the number of letters of G implies that the indices of $W_2, \ldots, W_n, W_2', \ldots, W_m'$ are distinct. If W_1 has fewer letters than W_1', define the index of W_1 to be -1 and that of W_1' to be 0; reverse this procedure if W_1' has fewer letters than W_1. (Note that W_1' cannot equal W_1 for if so, $W_2' \cdots W_m' = W_2 \cdots W_n$, contradicting the minimality of the number of letters of G.) Let $U_1, U_2, \ldots, U_{n+m}$ be the words of G, arranged in order of increasing index. If $j < i$ and index $U_i >$ index U_j, but index $U_{i+1} <$ index U_j + the number of letters in U_j, we say that U_i is *embedded* in U_j. We claim that for each $i = 3, \ldots, n + m$, either U_i is embedded in some U_j, $j < i$, or the subsequence A_i of G which begins with the first letter of U_i and ends with the letter immediately preceding the first letter of U_{i+1}, is in one of the sets S_n, $n \geq 1$. (The sequence A_{n+m} is defined to be U_{n+m} itself.) The claim is true for $i = 3$ by inspection. The various possibilities are indicated in Fig. 2.2.3a and b. If the claim has been verified for $i \leq r$, consider U_{r+1}. If U_{r+1} is not embedded in some U_j, $j < r + 1$, we have:



(a) U_3 embedded in U_2

(b) $A_3 \in S_1$

(c) U_r not embedded in any U_i, $i < r$

(d) U_r embedded in U_i, $i < r$

Fig. 2.2.3. Proof of Theorem 2.2.1.

CASE 1. U_r is not embedded in any $U_i, i < r$ (see Fig. 2.2.3c). By induction hypothesis, $A_r \in$ some S_n, $n \geq 1$. But then $A_{r+1} \in S_{n+1}$.

CASE 2. U_r is embedded in some $U_i, i < r$ (see Fig. 2.2.3d). Then $U_i = A_i U_{i+1} \cdots U_r A_{r+1}$. By induction hypothesis, $A_i \in$ some S_n, $n \geq 1$. By definition of the sets S_n, $U_{i+1} \cdots U_r A_{r+1} \in S_{n+1}$, $U_{i+2} \cdots U_r A_{r+1} \in S_{n+2}, \ldots, U_r A_{r+1} \in S_{n+r-i}$, $A_{r+1} \in S_{n+r-i+1}$.

Now U_{n+m} cannot be embedded in any U_i; hence $A_{n+m} = U_{n+m} \in$ some S_n, $n \geq 1$, and the first half of the theorem is proved.

Conversely, suppose that one of the sets S_k, $k \geq 1$, contains a code word. Let n be the smallest integer ≥ 1 such that S_n contains a code word W. If we retrace the steps by which W arrived in S_n, we obtain a sequence

$$A_0, W_0, A_1, W_1, \ldots, A_n, W_n$$

such that $A_0, W_0, W_1, \ldots, W_n$ are code words, $A_1, \ldots, A_n$ are sequences of code characters such that $A_i \in S_i$, $i = 1, \ldots, n$, $A_n = W_n$, $W_0 = A_0 A_1$, and for each $i = 1, 2, \ldots, n-1$ either $A_i = W_i A_{i+1}$ or $W_i = A_i A_{i+1}$. For example, for the code of Fig. 2.2.2, we obtain

$A_0 = a$	$A_1 = bb \in S_1$	$A_2 = cde \in S_2$	$A_3 = de \in S_3$	$A_4 = b \in S_4$	$A_5 = ad \in S_5$
$W_0 = abb$	$W_1 = bbcde$	$W_2 = c$	$W_3 = deb$	$W_4 = bad$	$W_5 = ad$

We now give a systematic way of constructing an ambiguous sequence. We construct two sequences, one starting with A_0W_1 and the other with W_0. The sequences are formed in accordance with the following rules. Having placed W_i at the end of one of the sequences:

CASE 1. $A_i = W_iA_{i+1}$. Add W_{i+1} at the end of the sequence containing W_i.

CASE 2. $W_i = A_iA_{i+1}$. Add W_{i+1} at the end of the sequence not containing W_i. Continue until W_n is reached.

We shall illustrate this procedure for the code of Fig. 2.2.2.

$$A_0W_1 = abbcde, \quad W_0 = abb$$

$$W_1 = A_1A_2 \text{ so form } W_0W_2 = abbc.$$

(Notice that the sequence A_0W_1 is longer than the sequence W_0, hence W_2 is added to the shorter sequence.)

$$A_2 = W_2A_3 \text{ so form } W_0W_2W_3 = abbcdeb.$$

(After the addition of W_2, the sequence beginning with W_0 is still the shorter, so W_3 is added to that sequence.)

$$W_3 = A_3A_4 \text{ so form } A_0W_1W_4 = abbcdebad.$$

(After the addition of W_3, the sequence beginning with W_0 exceeds the sequence beginning with A_0 in length, and thus W_4 is added to the latter sequence.)

$$W_4 = A_4A_5 \text{ so form } W_0W_2W_3W_5 = abbcdebad.$$

The sequence $abbcdebad = A_0W_1W_4 = W_0W_2W_3W_5$ is ambiguous.

We now show that the procedure outlined above always yields an ambiguous sequence. We may establish by induction that after the word W_i is assigned ($i = 1, \ldots, n-1$), one of the sequences is a prefix of the other. By inspection, this is true after W_1 is assigned. The inductive step is accomplished by a tedious case-by-case analysis. For example, suppose that W_i is assigned to the first sequence and that after W_i is written down, sequence 2 is a prefix of sequence 1. In addition, suppose that W_{i-1} was assigned to sequence 2 (see Fig. 2.2.4). Then necessarily $W_{i-1} = A_{i-1}A_i$ and $W_i = A_iA_{i+1}$, therefore W_{i+1} is assigned to sequence 2. If $W_{i+1} = A_{i+1}A_{i+2}$ then A_{i+1} is a prefix of W_{i+1}. If $A_{i+1} = W_{i+1}A_{i+2}$ then W_{i+1} is a prefix of A_{i+1}. In either case, one of the sequences is still a prefix of the other. The other cases may be handled in a similar fashion. Now after W_{n-1} is assigned we have either $W_{n-1} = A_{n-1}W_n$ or $A_{n-1} = W_{n-1}W_n$. Another case-by-case analysis shows that after W_n is assigned, the two sequences are identical. Since $W_0 \neq A_0$, we have produced an ambiguous sequence.

We remark that since the sequences in the sets S_i, $i \geq 0$, cannot be longer than the longest code word, only finitely many of the S_i can be distinct. Thus eventually the S_n must exhibit a periodicity; that is, there must be integers N and k such that $S_i = S_{i+k}$ for $i \geq N$ (it may happen that S_i is empty for $i \geq N$; in fact a code is instantaneous if and only if S_i is empty for $i \geq 1$). Thus the testing procedure to determine unique decipherability must terminate in a finite number of steps, and an upper bound on the number of steps required may be readily calculated for a given code.

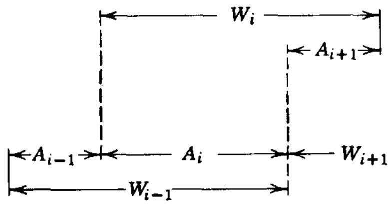


Fig. 2.2.4. Proof of Theorem 2.2.1.

2.3. *Necessary and sufficient conditions for the existence of instantaneous codes*

As we have seen in the previous section, an instantaneous code is quite easy to decode compared to the general uniquely decipherable code. Moreover, we shall prove in Section 2.6 that for the purpose of solving the noiseless coding problem, we may without loss of generality restrict our attention to instantaneous codes. Thus it would be desirable to examine the properties of such codes. We start by posing the following problem. Given a set of symbols $x_1, x_2, \ldots, x_M$, a code alphabet $a_1, a_2, \ldots, a_D$, and a set of positive integers $n_1, n_2, \ldots, n_M$, is it possible to construct an instantaneous code such that n_k is the length of the code word corresponding to x_k? For example, if $M = 3$, $D = 2$, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, a possible code is $\{0, 10, 110\}$. If $M = 3$, $D = 2$, $n_1 = n_2 = n_3 = 1$, no uniquely decipherable code, instantaneous or otherwise, will meet the specifications. The complete solution to the problem is provided by the following theorem.

Theorem 2.3.1. An instantaneous code with word lengths $n_1, n_2, \ldots, n_M$ exists if and only if $\sum_{i=1}^{M} D^{-n_i} \leq 1$ (D = size of the code alphabet).

Proof. To prove the theorem we make use of the idea of a *tree of order D and size k*, which is simply a system of points and lines such that

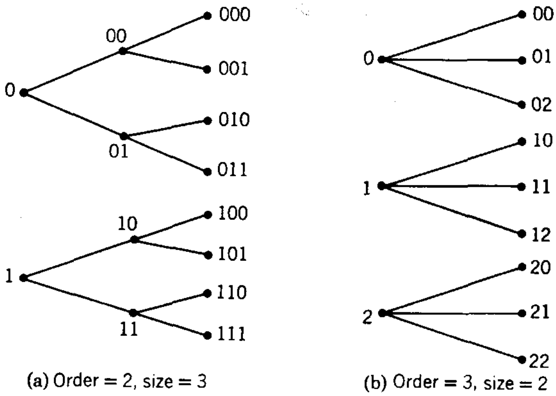


Fig. 2.3.1. Trees.

each sequence s of length $\leq k$ formed from the symbols $\{0, 1, \ldots, D-1\}$ is represented by a distinct point V_s, and such that if the sequence s' is formed by adding a single digit to s, then V_s and V_s' are connected by a line. Typical trees are shown in Fig. 2.3.1.

Now suppose we are given a base D instantaneous code with word lengths $n_1, n_2, \ldots, n_M$. We may assume without loss of generality that the code alphabet is $\{0, 1, \ldots, D-1\}$. Assume for simplicity that $n_1 \leq n_2 \leq \cdots \leq n_M$. Each code word may be identified with a point on the tree of order D and size n_M. For example, the binary code $\{0, 10, 111\}$ may be represented as shown in Fig. 2.3.2. Since no code word can be a prefix of another code word, once a point P on the tree is reserved for a particular code word, no other code word can correspond to any point on the branches emanating from P. In Fig. 2.3.2, the dashed lines indicate the forbidden parts of the tree. By construction of the tree, a code word of

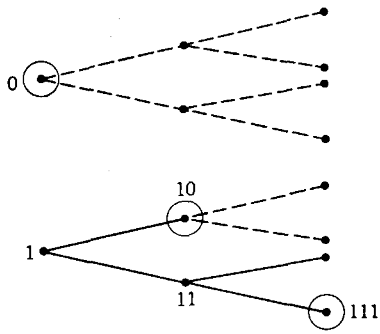


Fig. 2.3.2. Tree representation of a binary code.

length n_k excludes $D^{n_M-n_k}$ "terminal points," that is, points corresponding to sequences of length n_M. Thus the total number of terminal points excluded by the given code is $\sum_{i=1}^M D^{n_M-n_i}$. Since a tree of order D and size n_M has D^{n_M} terminal points, the total number of terminal points excluded cannot exceed D^{n_M}. Thus, $\sum_{i=1}^M D^{n_M-n_i} \leq D^{n_M}$, or $\sum_{i=1}^M D^{-n_i} \leq 1$.

Conversely, suppose we are given positive integers $n_1, n_2, \ldots, n_M$ satisfying $\sum_{i=1}^M D^{-n_i} \leq 1$. Again assume $n_1 \leq n_2 \leq \cdots \leq n_M$. To construct the required instantaneous code, select any point on the tree of order D and size n_M corresponding to a sequence of length n_1. This excludes $D^{n_M-n_1}$ terminal points. Since $\sum_{i=1}^M D^{-n_i} \leq 1$, or equivalently $\sum_{i=1}^M D^{n_M-n_i} \leq D^{n_M}$, we have $D^{n_M-n_1} < D^{n_M}$, and at least one terminal point remains. Hence a point corresponding to a sequence of length n_2 remains, and we may choose this point to correspond to the code word of length n_2. The total number of points excluded is now $D^{n_M-n_1} + D^{n_M-n_2}$, and again it follows from the hypothesis that this number is less than D^{n_M}, and hence we may choose a point corresponding to a sequence of length n_3. The process may be continued until all code words are assigned.

We remark that the condition $\sum_{i=1}^M D^{-n_i} \leq 1$ is equivalent to

$$\sum_{j=1}^r \omega_j D^{-j} \leq 1,$$

where ω_j is the number of code words of length j, and r is the maximum code-word length. This follows since the sum $\sum_{i=1}^M D^{-n_i}$ may be written as

$$\underbrace{D^{-1} + D^{-1} + \cdots + D^{-1}}_{\omega_1 \text{ times}} + \underbrace{D^{-2} + \cdots + D^{-2}}_{\omega_2} + \cdots + \underbrace{D^{-r} + \cdots + D^{-r}}_{\omega_r}.$$

2.4. *Extension of the condition $\sum_{i=1}^M D^{-n_i} \leq 1$ to uniquely decipherable codes*

We have shown that the condition $\sum_{i=1}^M D^{-n_i} \leq 1$ is necessary and sufficient for the existence of a base D instantaneous code with word lengths $n_1, n_2, \ldots, n_M$. In this section, we prove a stronger result, namely, that the same condition is necessary and sufficient for the existence of a *uniquely decipherable* code. The sufficiency part is immediate, since every instantaneous code is uniquely decipherable. Thus it remains to prove necessity.

Theorem 2.4.1. If a uniquely decipherable code has word lengths $n_1, n_2, \ldots, n_M$, then $\sum_{i=1}^M D^{-n_i} \leq 1$.

Proof. For convenience we use $\sum_{j=1}^{r} \omega_j D^{-j}$ instead of $\sum_{i=1}^{M} D^{-n_i}$. Now

$$\left(\sum_{j=1}^{r} \omega_j D^{-j}\right)^n = (\omega_1 D^{-1} + \cdots + \omega_r D^{-r})^n. \tag{2.4.1}$$

Each term in the expansion of (2.4.1) is of the form $\omega_{i_1} D^{-i_1} \omega_{i_2} D^{-i_2} \cdots \omega_{i_n} D^{-i_n}$ where $1 \le i_k \le r$ for each k, and hence $n \le i_1 + \cdots + i_n \le nr$. Thus

$$\left(\sum_{j=1}^{r} \omega_j D^{-j}\right)^n = \sum_{k=n}^{nr} N_k D^{-k} \tag{2.4.2}$$

where

$$N_k = \sum_{i_1 + \ldots + i_n = k} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_n}. \tag{2.4.3}$$

We claim that N_k is the total number of messages using n (not necessarily distinct) symbols x_i whose coded representation is of length k. To see this, we note that the number of sequences of code characters that can be formed using code words of length $i_1, \ldots, i_n$, *in that order*, where $i_1 + \ldots + i_n = k$ (that is, the number of sequences that start with a code word of length i_1, follow with a code word of length $i_2, \ldots$, and end with a code word of length i_n) is $\omega_{i_1} \ldots \omega_{i_n}$. By unique decipherability, every sequence of code characters corresponds to at most one message. Thus N_k cannot exceed the total number of coded sequences of length k, that is, $N_k \le D^k$. By (2.4.2),

$$\left(\sum_{j=1}^{r} \omega_j D^{-j}\right)^n \le \sum_{k=n}^{nr} 1 = nr - n + 1 \le nr.$$

Taking nth roots, we have

$$\sum_{j=1}^{r} \omega_j D^{-j} \le n^{1/n} r^{1/n}. \tag{2.4.4}$$

Since (2.4.4) holds for any n, we may let $n \to \infty$ to obtain $\sum_{j=1}^{r} \omega_j D^{-j} \le 1$.

2.5. *The noiseless coding theorem*

Let us return to the noiseless coding problem. A random variable X takes on values $x_1, \ldots, x_M$ with probabilities $p_1, \ldots, p_M$ respectively. Code words $W_1, \ldots, W_M$, of length $n_1, \ldots, n_M$ respectively, are assigned to the symbols $x_1, \ldots, x_M$. The code words are combinations of characters taken from a code alphabet $a_1, \ldots, a_D$. We wish to construct a uniquely decipherable code which minimizes the average code-word length $\bar{n} = \sum_{i=1}^{M} p_i n_i$. We shall approach the problem in three steps. First we establish an absolute lower bound on $\bar{n}$; then we find out how close we can come to the lower bound; then we attempt to synthesize the "best" code. The lower bound is set by the following theorem.

Theorem 2.5.1. (Noiseless Coding Theorem). If $\bar{n} = \sum_{i=1}^{M} p_i n_i$ is the average code-word length of a uniquely decipherable code for the random variable X, then $\bar{n} \geq H(X)/\log D$ with equality if and only if $p_i = D^{-n_i}$ for $i = 1, 2, \ldots, M$. Note that $H(X)/\log D$ is the uncertainty of X computed using logs to the base D, that is,

$$\frac{H(X)}{\log D} = -\sum_{i=1}^{M} p_i \frac{\log_2 p_i}{\log_2 D} = -\sum_{i=1}^{M} p_i \log_D p_i.$$

Proof. The condition $\bar{n} \geq H(X)/\log D$ is equivalent to

$$\log D \sum_{i=1}^{M} p_i n_i \geq -\sum_{i=1}^{M} p_i \log p_i.$$

Since $p_i n_i \log D = p_i \log D^{n_i} = -p_i \log D^{-n_i}$, the above condition may be written as

$$-\sum_{i=1}^{M} p_i \log D^{-n_i} \geq -\sum_{i=1}^{M} p_i \log p_i.$$

This looks suspiciously like Lemma 1.4.1, except that the terms D^{-n_i} do not necessarily add to unity. This difficulty may be circumvented by defining $q_i = D^{-n_i}/\sum_{j=1}^{M} D^{-n_j}$. Then the q_i's add to unity, and Lemma 1.4.1 yields

$$-\sum_{i=1}^{M} p_i \log p_i \leq -\sum_{i=1}^{M} p_i \log \frac{D^{-n_i}}{\sum_{j=1}^{M} D^{-n_j}} \tag{2.5.1}$$

with equality if and only if $p_i = D^{-n_i}/\sum_{j=1}^{M} D^{-n_j}$ for all i. Hence by (2.5.1),

$$H(X) \leq -\sum_{i=1}^{M} p_i \log D^{-n_i} + \left(\sum_{i=1}^{M} p_i\right) \log \left(\sum_{j=1}^{M} D^{-n_j}\right),$$

or

$$H(X) \leq \bar{n} \log D + \log \left(\sum_{j=1}^{M} D^{-n_j}\right) \tag{2.5.2}$$

with equality if and only if $p_i = \dfrac{D^{-n_i}}{\sum_{j=1}^{M} D^{-n_j}}$ for all i.

But by unique decipherability (Theorem 2.4.1) $\sum_{j=1}^{M} D^{-n_j} \leq 1$; hence

$$\log \left(\sum_{j=1}^{M} D^{-n_j}\right) \leq 0,$$

and $H(X) \leq \bar{n} \log D$.

Now if $p_i = D^{-n_i}$ for all i, then

$$H(X) = -\sum_{i=1}^{M} p_i \log p_i = \sum_{i=1}^{M} p_i n_i \log D = \bar{n} \log D.$$

It therefore remains to prove that if $H(X) = \bar{n} \log D$, then $p_i = D^{-n_i}$ for all i. By (2.5.2), $H(X) = \bar{n} \log D$ implies that

$$\log \left(\sum_{j=1}^{M} D^{-n_j} \right) \geq 0;$$

by unique decipherability

$$\log \left(\sum_{j=1}^{M} D^{-n_j} \right) \leq 0,$$

and thus

$$\log \left(\sum_{j=1}^{M} D^{-n_j} \right) = 0$$

or $\sum_{j=1}^{M} D^{-n_j} = 1$. It now follows from (2.5.2) that $p_i = D^{-n_i}$ for all i. This completes the proof.

We shall call a code that achieves the lower bound set by the noiseless coding theorem *absolutely optimal.* An example of an absolutely optimal binary code is the following.

X	Probabilities	Code Words	
x_1	1/2	0	
x_2	1/4	10	
x_3	1/8	110	$H(X) = \bar{n} = \frac{7}{4}$.
x_4	1/8	111	

In general we cannot hope to construct an absolutely optimal code for a given set of probabilities $p_1, p_2, \ldots, p_M$, since if we choose n_i to satisfy $p_i = D^{-n_i}$, then $n_i = (-\log p_i)/(\log D)$ may not be an integer. However, we can do the next best thing and select the integer n_i such that

$$\frac{-\log p_i}{\log D} \leq n_i < \frac{-\log p_i}{\log D} + 1 \qquad (i = 1, 2, \ldots, M). \tag{2.5.3}$$

We claim that an instantaneous code can be constructed with word lengths $n_1, n_2, \ldots, n_M$. To prove this, we must show that $\sum_{i=1}^{M} D^{-n_i} \leq 1$. From the left-hand inequality of (2.5.3), it follows that $\log p_i \geq -n_i \log D$, or $p_i \geq D^{-n_i}$. Thus $\sum_{i=1}^{M} D^{-n_i} \leq \sum_{i=1}^{M} p_i = 1$. To estimate the average code-word length, we multiply (2.5.3) by p_i and sum over i, to obtain

$$-\sum_{i=1}^{M} p_i \frac{\log p_i}{\log D} \leq \sum_{i=1}^{M} p_i n_i < -\sum_{i=1}^{M} p_i \frac{\log p_i}{\log D} + \sum_{i=1}^{M} p_i .$$

Thus we have proved the following theorem.

Theorem 2.5.2. Given a random variable X with uncertainty $H(X)$, there exists a base D instantaneous code for X whose average code-word length $\bar{n}$ satisfies

$$\frac{H(X)}{\log D} \le \bar{n} < \frac{H(X)}{\log D} + 1.$$

Consequently the average code-word length may be brought within one digit of the lower bound set by the noiseless coding theorem. In fact, we can always approach the lower bound as closely as desired if we are allowed to use "block coding." Suppose that instead of assigning a code word to each symbol x_i, we take a series of s independent observations of X and assign a code word to the resulting group of s symbols. In other words, we construct a code for the random vector $Y = (X_1, X_2, \ldots, X_s)$, where the X_i are independent and each X_i has the same distribution as X. If X (and hence each X_i) assumes M values, then Y assumes M^s values. Block coding will in general decrease the average code-word length *per value of* X as illustrated by the following example.

X	p	Code Word	$Y = (X_1, X_2)$	p	Code Word
x_1	3/4	0	x_1x_1	9/16	0
x_2	1/4	1	x_1x_2	3/16	10
			x_2x_1	3/16	110
			x_2x_2	1/16	111

$\bar{n} = 1$

$$\begin{aligned}\bar{n} &= \tfrac{9}{16} + \tfrac{3}{16}(2) + \tfrac{1}{4}(3)\\ &= \tfrac{27}{16} \text{ code characters/2 values of } X\\ &= \tfrac{27}{32} \text{ code characters/value of } X\end{aligned}$$

By Theorem 2.5.2, we may construct an instantaneous code for Y whose average code-word length $\bar{n}_s$ satisfies

$$\frac{H(Y)}{\log D} \le \bar{n}_s < \frac{H(Y)}{\log D} + 1 \text{ code characters/value of } Y.$$

Since $X_1, \ldots, X_s$ are independent and identically distributed, $H(Y) = H(X_1, \ldots, X_s) = H(X_1) + \cdots + H(X_s) = sH(X)$. Thus

$$\frac{sH(X)}{\log D} \le \bar{n}_s < \frac{sH(X)}{\log D} + 1,$$

or

$$\frac{H(X)}{\log D} \le \frac{\bar{n}_s}{s} < \frac{H(X)}{\log D} + \frac{1}{s}. \tag{2.5.4}$$

But $\bar{n}_s/s$ is the average number of code characters per value of X; it follows from (2.5.4) that $\bar{n}_s/s$ may be made arbitrarily close to $H(X)/\log D$ by choosing s large enough. Thus $H(X)/\log D$ may be interpreted as the minimum average number of base D digits required to encode one observation of X. If we consider the case $D = 2$, we make the observation that any instantaneous binary code corresonds to a sequence of "yes or no" questions, as illustrated in Fig. 2.5.1. In the example, an answer "yes" corresponds to a zero in a code word; an answer "no" to a one. Thus,

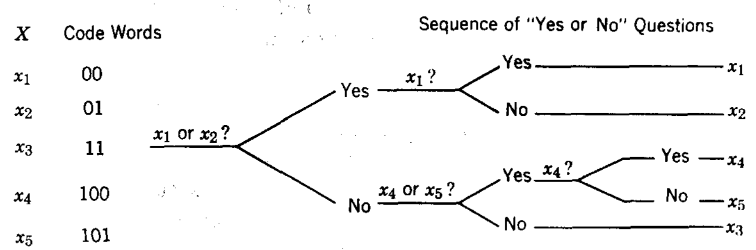


Fig. 2.5.1. A binary instantaneous code and its corresponding sequence of "yes or no" questions.

as we indicated in Section 1.3, the uncertainty $H(X)$ may be interpreted as the minimum average number of "yes or no" questions required to determine the result of one observation of X.

2.6. *Construction of optimal codes*

The only part of the noiseless coding problem that remains is the construction of the code which minimizes the average code-word length for a given set of probabilities $p_1, p_2, \ldots, p_M$. We first show that in the search for optimal codes, we may restrict our attention to instantaneous codes. More precisely, we prove:

Lemma 2.6.1. Suppose a code C is optimal within the class of instantaneous codes for the given probabilities $p_1, p_2, \ldots, p_M$; in other words, no other instantaneous code for $p_1, \ldots, p_M$ has a smaller average code-word length than C. Then C is optimal within the entire class of uniquely decipherable codes.

Proof. Suppose a uniquely decipherable code C' has a smaller average code-word length than C. Let $n_1', n_2', \ldots, n_M'$ be the code-word lengths of C'. By Theorem 2.4.1, $\sum_{i=1}^{M} D^{-n_i'} \leq 1$. But by Theorem 2.3.1 there exists an instantaneous code C'' with word lengths $n_1', \ldots, n_M'$. Hence the average word length of C'' is the same as the average word length of C', contradicting the fact that C is the best instantaneous code.

Before turning to the synthesis of optimal instantaneous codes, we imagine that we have constructed an optimal code and determine what properties such a code must have. In other words, we shall derive *necessary* conditions on an optimal instantaneous code. To simplify the presentation we consider only the binary case. For an extension to nonbinary code alphabets see Problem 2.5. The results are summarized as follows.

Lemma 2.6.2. Given a binary instantaneous code C with word lengths $n_1, n_2, \dots, n_M$ associated with a set of symbols with probabilities $p_1, p_2, \dots, p_M$. For convenience, assume that the symbols are arranged in order of decreasing probability ($p_1 \geq p_2 \geq \cdots \geq p_M$) and that a group of symbols with the same probability is arranged in order of increasing code-word length. (If $p_i = p_{i+1} = \cdots = p_{i+r}$, then $n_i \leq n_{i+1} \leq \cdots \leq n_{i+r}$.) Then if C is optimal within the class of instantaneous codes, C must have the following properties:

a. Higher probability symbols have shorter code words, that is, $p_j > p_k$ implies $n_j \leq n_k$.

b. The two least probable symbols have code words of equal length, that is, $n_{M-1} = n_M$.

c. Among the code words of length n_M there must be at least two words that agree in all digits except the last. For example, the following code cannot be optimal since code words 4 and 5 do not agree in the first three places.

x_1	0
x_2	100
x_3	101
x_4	1101
x_5	1110

Proof. To prove (a), we note that if $p_j > p_k$ and $n_j > n_k$, it is possible to construct a better code C' by interchanging code words j and k. The difference between the average code-word length of C' and of C is

$$(\bar{n})' - \bar{n} = p_j n_k + p_k n_j - (p_j n_j + p_k n_k) = (p_j - p_k)(n_k - n_j) < 0;$$

hence C' is better than C.

To prove (b) we note first that $n_{M-1} \leq n_M$, for if $p_{M-1} > p_M$ then $n_{M-1} \leq n_M$ by (a); if $p_{M-1} = p_M$, then $n_{M-1} \leq n_M$ by our assumption about arrangement of code words associated with equally probable symbols. Now, if $n_M > n_{M-1}$, we may drop the last digit of the M^{th} code word to obtain a code that is still instantaneous and better than the original code.

Finally, (c) is proved by observing that if no two code words of maximal length agree in all places but the last, then we may drop the last digit of all such code words to obtain a better code.

We now describe a method, due to Huffman, for the construction of optimal (instantaneous) codes. If we have an array of symbols $x_1, \ldots, x_M$ with probabilities $p_1, \ldots, p_M$ (again assume $p_1 \geq p_2 \geq \cdots \geq p_M$), then we combine the last two symbols x_{M-1} and x_M into an equivalent symbol $x_{M,M-1}$ with probability $p_M + p_{M-1}$. Now suppose that somehow we could construct an optimal code C_2 for the new set of symbols (see Table 2.6.1). We now construct a code C_1 for the original set of symbols $x_1, \ldots, x_M$. The code words associated with $x_1, x_2, \ldots, x_{M-2}$ are exactly the same as the corresponding code words of C_2. The code words associated with x_{M-1} and x_M are formed by adding a zero and a one, respectively, to the code word $W_{M,M-1}$ associated with the symbol $x_{M,M-1}$ in C_2. We now claim that C_1 *is an optimal code for the set of probabilities* $p_1, p_2, \ldots, p_M$.

To establish the claim, suppose that C_1 is not optimal. Let C_1' be an optimal instantaneous code for $x_1, x_2, \ldots, x_M$. Then C_1' will have code words $W_1', W_2', \ldots, W_M'$ with word lengths $n_1', n_2', \ldots, n_M'$ respectively. By Lemma 2.6.2b, $n'_{M-1} = n_M'$. By Lemma 2.6.2c, at least two words of length n_M' agree in all digits but the last. We may assume without loss of generality that W'_{M-1} and W_M' are two of these words. (If necessary we may interchange words of the same length without affecting the average word length.) At this point we again combine the symbols x_{M-1} and x_M and construct a code C_2' by taking as the code word for $x_{M,M-1}$ the word W_M' (or W'_{M-1}) with the last digit removed (see Table 2.6.2). We now establish that C_2' has a smaller average word length than C_2, contradicting the optimality of C_2. The average word length of C_2' is

$$\begin{aligned}(\bar{n}_2)' &= p_1 n_1' + \cdots + p_{M-2}n'_{M-2} + (p_{M-1} + p_M)(n'_{M-1} - 1) \\ &= p_1 n_1' + \cdots + p_{M-2}n'_{M-2} + p_{M-1}n'_{M-1} \\ &\qquad\qquad + p_M n_M' - (p_{M-1} + p_M).\end{aligned}$$

Since C_1' has a smaller average word length than C_1,

$$\begin{aligned}p_1 n_1' + \cdots + p_{M-2}n'_{M-2} + p_{M-1}n'_{M-1} + p_M n'_M &< p_1 n_1 + \cdots \\ &\quad + p_{M-2}n_{M-2} + p_{M-1}n_{M-1} + p_M n_M.\end{aligned}$$

Using the fact that $n_{M-1} = n_M$ (by construction of C_1) we obtain

$$(\bar{n}_2)' < p_1 n_1 + \cdots + p_{M-2}n_{M-2} + (p_{M-1} + p_M)n_{M-1} - (p_{M-1} + p_M)$$

or

$$(\bar{n}_2)' < p_1 n_1 + \cdots + p_{M-2}n_{M-2} + (p_{M-1} + p_M)(n_{M-1} - 1).$$

Table 2.6.1. Construction of the Huffman code

X	Probabilities	Code words of C_1	Word lengths	X	Probabilities	Code words of C_2	Word lengths
x_1	p_1	W_1	n_1	x_1	p_1	W_1	n_1
x_2	p_2	W_2	n_2	x_2	p_2	W_2	n_2
.				.	.	.	
.				.	.	.	
.				.	.	.	
				$x_{M,M-1}$	p_M+p_{M-1}	$W_{M,M-1}$	$n_{M,M-1}$
				.	.	.	
				.	.	.	
x_{M-2}	p_{M-2}	W_{M-2}	n_{M-2}	.	.	.	
x_{M-1}	p_{M-1}	$[W_{M,M-1}\ 0]$	n_{M-1}	x_{M-2}	p_{M-2}	W_{M-2}	n_{M-2}
x_M	p_M	$[W_{M,M-1}\ 1]$	n_M				

Table 2.6.2. Proof of optimality of the Huffman procedure

X	Probabilities	Code words of C_1'	Word lengths	X	Probabilities	Code words of C_2'	Word lengths
x_1	p_1	W_1'	n_1'	x_1	p_1	W_1'	n_1'
x_2	p_2	W_2'	n_2'	x_2	p_2	W_2'	n_2'
.				.			
.				.			
.				.			
				$x_{M,M-1}$	p_M+p_{M-1}	U'	n'_M-1 $=n'_{M-1}-1$
				.			
				.			
x_{M-2}	p_{M-2}	W'_{M-2}	n'_{M-2}	.			
x_{M-1}	p_{M-1}	W'_{M-1}	n'_{M-1}	x_{M-2}	p_{M-2}	W'_{M-2}	n'_{M-2}
x_M	p_M	W_M'	$n_M'=n'_{M-1}$				

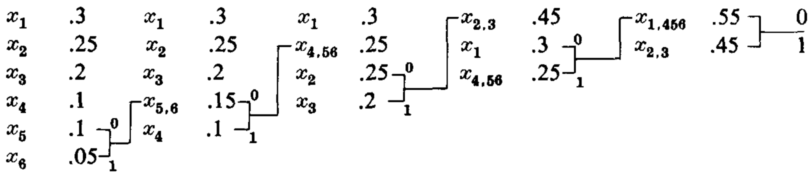
$U' = W_M'$ with last digit removed.

Since $n_{M-1} - 1 = n_{M,M-1}$, the expression on the right is the average word length of C_2, and thus the proof is complete.

The above result tells us exactly how to construct optimal codes. An example is carried through in Fig. 2.6.1. The idea is simply to combine the two least probable symbols until only two symbols remain. An optimal binary code for two symbols must consist of the two code words 0 and 1; we can then work backwards, constructing optimal codes for each array until we arrive at the original set of symbols.

2.7. *Notes and remarks*

The material of this chapter has become fairly standard, and is treated in Feinstein (1958), Abramson (1963), and Fano (1961). Theorem 2.2.1 is due to Sardinas and Patterson (1950, 1953), whose proof exploited the fact that the collection of finite sequences of code characters is a semigroup under juxtaposition. The somewhat cumbersome direct argument in the present text is the author's. The phrase "instantaneous code" was



Optimal Codes

$x_{1,456}$	0	$x_{2,3}$	1	x_1	00	x_1	00	x_1	00
$x_{2,3}$	1	x_1	00	$x_{4,56}$	01	x_2	10	x_2	10
		$x_{4,56}$	01	x_2	10	x_3	11	x_3	11
				x_3	11	$x_{5,6}$	010	x_4	011
						x_4	011	x_5	0100
								x_6	0101

Fig. 2.6.1. Huffman code.

coined by Abramson (1963). Theorem 2.3.1 is due to Kraft (1949); Theorem 2.4.1 was proved by McMillan (1956); the proof in the text is due to Karush (1961). Theorem 2.5.1 is due to Shannon (1948); the proof follows Feinstein (1958). The construction of Section 2.6 is due to Huffman (1952).

PROBLEMS

2.1 Determine whether or not each of the following codes is uniquely decipherable. If a code is not uniquely decipherable, construct an ambiguous sequence.

a.			b.		
	x_1	010		x_1	*abc*
	x_2	0001		x_2	*abcd*
	x_3	0110		x_3	*e*
	x_4	1100		x_4	*dba*
	x_5	00011		x_5	*bace*
	x_6	00110		x_6	*ceac*
	x_7	11110		x_7	*ceab*
	x_8	101011		x_8	*eabd*

2.2 a. For the binary code below, let $N(k)$ be the number of messages that can be formed using exactly k code characters. For example, $N(1) = 1$ (that is, x_1), $N(2) = 3$ (x_1x_1, x_2, x_3), $N(3) = 5$ $(x_1x_1x_1, x_1x_2, x_1x_3, x_2x_1, x_3x_1)$.
Find a general expression for $N(k)$ $(k = 1, 2, \ldots)$.

x_1	0
x_2	10
x_3	11

b. Repeat part (a) for the code below.

x_1	0
x_2	10
x_3	110
x_4	111

2.3 Construct a Huffman code for the symbols below. Compare the average code-word length with the uncertainty $H(X)$.

Symbols	Probabilities	Symbols	Probabilities
x_1	.2	x_7	.059
x_2	.18	x_8	.04
x_3	.1	x_9	.04
x_4	.1	x_{10}	.04
x_5	.1	x_{11}	.04
x_6	.061	x_{12}	.03
		x_{13}	.01

2.4 Show that the average word length of a Huffman binary code satisfies

$$\bar{n} < H(X) + 1$$

2.5 Explain how to construct a base D instantaneous code that minimizes the average code-word length and apply the results to construct an optimal ternary code for the set of symbols below.

Symbols	Probabilities	Symbols	Probabilities
x_1	.3	x_5	.1
x_2	.2	x_6	.08
x_3	.15	x_7	.05
x_4	.1	x_8	.02

CHAPTER THREE

The Discrete Memoryless Channel

3.1. Models for communication channels

In this chapter we shall begin the analysis of communication channels whose inputs are subject to random disturbances in transmission. In the usual communication situation, an object of some kind (for example, a letter of the alphabet, a pulse or other waveform) is selected from a specified class of inputs. The channel is a device that acts on the input to produce an output belonging to another specified class. The random nature of the channel may in many cases be described by giving a probability distribution over the set of possible outputs. The distribution will in general depend on the particular input chosen for transmission and in addition may depend on the internal structure of the channel at the time the input is applied. We shall try to arrive at a definition of an information channel that reflects these considerations.

We are going to specialize to the so-called "discrete case," that is, the situation in which the information to be transmitted consists of a sequence of symbols, each symbol belonging to a finite alphabet. More general models will be discussed in Chapter 8. If we apply a sequence $\alpha_1, \ldots, \alpha_n$ at the input of a channel, then at the output, perhaps after an appropriate delay, we will receive a sequence $\beta_1, \ldots, \beta_n$. It is reasonable to describe the action of the channel by giving a probability distribution over the output sequences $\beta_1, \ldots, \beta_n$ for each input sequence $\alpha_1, \ldots, \alpha_n$; the family of distributions should also reflect the fact that the "internal state" of the channel at the time the input is applied will affect the transmission of information. Physically, we expect that many channels have "memory"; that is, the distribution of the output symbol β_n may depend on previous inputs and outputs. We do not, however, expect "anticipatory" behavior; in our model the distribution of β_n should not depend on future inputs or outputs. Thus in giving the distribution of $\beta_1, \ldots, \beta_n$, we need not consider inputs beyond $\alpha_1, \ldots, \alpha_n$. We are led to the following definition.

Definition. Given finite sets Γ and Γ', to be called respectively the *input alphabet* and *output alphabet*, and an arbitrary set S called the *set of states*, a *discrete channel* is a system of probability functions

$$p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n;\ s)$$
$$\alpha_1, \ldots, \alpha_n \in \Gamma$$
$$\beta_1, \ldots, \beta_n \in \Gamma'$$
$$s \in S$$
$$n = 1, 2, \ldots,$$

that is, a system of functions satisfying

1. $p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n;\ s) \geq 0$ for all $n, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, s$.

2. $\sum_{\beta_1, \ldots, \beta_n} p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n;\ s) = 1$ for all $n, \alpha_1, \ldots, \alpha_n, s$.

Physically we interpret $p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n;\ s)$ as the probability that the sequence $\beta_1, \ldots, \beta_n$ will appear at the output if the input sequence $\alpha_1, \ldots, \alpha_n$ is applied and the initial state of the channel, that is, the state just prior to the appearance of α_1, is s. The state of the channel may change as the components of the input sequence are applied; our model assumes that knowledge of the initial state and the input sequence determines the distribution of the output sequence.

In this chapter we consider only a very special case of the discrete channel. However, the techniques of analysis which will be developed are basic to the analysis of more general channels. Specifically, we are going to consider the discrete channel without memory; such a channel is characterized by the requirements that successive symbols be acted on independently and that the functions p_n do not depend on the state s. Formally we have the following:

Definition. A discrete channel is *memoryless* if

1. the functions $p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n;\ s)$ do not depend on s, hence may be written $p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n)$, and

2. $p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n) = p_1(\beta_1 \mid \alpha_1)\, p_1(\beta_2 \mid \alpha_2) \cdots p_1(\beta_n \mid \alpha_n)$ for all $\alpha_1, \ldots, \alpha_n \in \Gamma$, $\beta_1, \ldots, \beta_n \in \Gamma'$, $n = 1, 2, \ldots$.

The second condition may be replaced by the conjunction of two other conditions as follows:

Lemma 3.1.1. Given probability functions satisfying condition (1), define

$$p_n(\beta_1, \ldots, \beta_{n-k} \mid \alpha_1, \ldots, \alpha_n) = \sum_{\beta_{n-k+1}, \ldots, \beta_n} p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n),$$
$$1 \leq k \leq n - 1.$$

(This quantity is interpreted as the probability that the first $n - k$ output symbols will be $\beta_1, \ldots, \beta_{n-k}$ when the input sequence $\alpha_1, \ldots, \alpha_n$ is applied.) Also define

$$p_n(\beta_n \mid \alpha_1, \ldots, \alpha_n;\ \beta_1, \ldots, \beta_{n-1}) = \frac{p_n(\beta_1, \ldots, \beta_{n-1}, \beta_n \mid \alpha_1, \ldots, \alpha_n)}{p_n(\beta_1, \ldots, \beta_{n-1} \mid \alpha_1, \ldots, \alpha_n)};$$

this expression is interpreted as the conditional probability that the nth output symbol will be β_n, given that the input sequence $\alpha_1, \ldots, \alpha_n$ is applied and the first $n - 1$ output symbols are $\beta_1, \ldots, \beta_{n-1}$. (Note that the subscript n on the functions above corresponds to the length of the input sequence.)

The functions satisfy condition (2) if and only if for all $n = 1, 2, \ldots,$ both of the following conditions are satisfied.

a. $p_n(\beta_n \mid \alpha_1, \ldots, \alpha_n;\ \beta_1, \ldots, \beta_{n-1}) = p_1(\beta_n \mid \alpha_n)$ for all $\alpha_1, \ldots, \alpha_n \in \Gamma, \quad \beta_1, \ldots, \beta_n \in \Gamma'$

b. $p_n(\beta_1, \ldots, \beta_{n-k} \mid \alpha_1, \ldots, \alpha_n) = p_{n-k}(\beta_1, \ldots, \beta_{n-k} \mid \alpha_1, \ldots, \alpha_{n-k})$ for all $\alpha_1, \ldots, \alpha_n \in \Gamma, \quad \beta_1, \ldots, \beta_{n-k} \in \Gamma', \quad 1 \le k \le n - 1.$

Condition (a) puts into evidence the memoryless feature of the channel, and condition (b) indicates the nonanticipatory behavior.

Proof. Suppose that condition (2) is satisfied. Then

$$\begin{aligned}
p_n(\beta_n \mid \alpha_1, \ldots, \alpha_n;\ \beta_1, \ldots, \beta_{n-1}) &= \frac{p_n(\beta_1, \ldots, \beta_{n-1}, \beta_n \mid \alpha_1, \ldots, \alpha_n)}{p_n(\beta_1, \ldots, \beta_{n-1} \mid \alpha_1, \ldots, \alpha_n)} \\
&= \frac{\prod_{k=1}^{n} p_1(\beta_k \mid \alpha_k)}{\sum_{\beta_n} p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n)} \\
&= \frac{\prod_{k=1}^{n} p_1(\beta_k \mid \alpha_k)}{\sum_{\beta_n} \prod_{k=1}^{n} p_1(\beta_k \mid \alpha_k)} \\
&= \frac{\prod_{k=1}^{n} p_1(\beta_k \mid \alpha_k)}{\prod_{k=1}^{n-1} p_1(\beta_k \mid \alpha_k) \sum_{\beta_n} p_1(\beta_n \mid \alpha_n)} \\
&= p_1(\beta_n \mid \alpha_n), \qquad \text{proving (a).}
\end{aligned}$$

To prove (b), note that the above argument shows that

$$p_n(\beta_1, \ldots, \beta_{n-1} \mid \alpha_1, \ldots, \alpha_n) = \prod_{k=1}^{n-1} p_1(\beta_k \mid \alpha_k)$$
$$= p_{n-1}(\beta_1, \ldots, \beta_{n-1} \mid \alpha_1, \ldots, \alpha_{n-1}).$$

An induction argument now establishes (b).

Conversely, if (a) and (b) are satisfied then

$$p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n)$$
$$= p_n(\beta_1, \ldots, \beta_{n-1} \mid \alpha_1, \ldots, \alpha_n)p_n(\beta_n \mid \alpha_1, \ldots, \alpha_n;\ \beta_1, \ldots, \beta_{n-1}).$$

By (b), the first term on the right is $p_{n-1}(\beta_1, \ldots, \beta_{n-1} \mid \alpha_1, \ldots, \alpha_{n-1})$; by (a) the second term is $p_1(\beta_n \mid \alpha_n)$. Proceeding inductively, we establish (2).

Note that we could have specified $k = 1$ in the statement of condition (b) and the lemma would still hold.

Thus a discrete memoryless channel is characterized by a matrix whose elements are $a_{\alpha\beta} = p_1(\beta \mid \alpha)$, $\alpha \in \Gamma$, $\beta \in \Gamma'$. ($a_{\alpha\beta}$ is the element in row α and column β.) The matrix $[p_1(\beta \mid \alpha)]$ is called the *channel matrix;* henceforth we shall drop the subscript and write $p(\beta \mid \alpha)$ for $p_1(\beta \mid \alpha)$. In this chapter, the word "channel" will mean "discrete memoryless channel" unless otherwise specified.

3.2. The information processed by a channel; channel capacity; classification of channels

Consider a discrete memoryless channel with input alphabet* $x_1, \ldots, x_M$, output alphabet $y_1, \ldots, y_L$ and channel matrix $[a_{ij}]$, $a_{ij} = p(y_j \mid x_i)$, $i = 1, \ldots, M$, $j = 1, \ldots, L$. If an input symbol is chosen at random, that is, if X is a random variable taking on the values $x_1, \ldots, x_M$ with probabilities $p(x_1), \ldots, p(x_M)$ respectively, then the channel output also becomes a random variable. The joint distribution of the input X and the output Y is given by $P\{X = x_i,\ Y = y_j\} = p(x_i)p(y_j \mid x_i)$, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, L$, and the distribution of Y is given by

$$P\{Y = y_j\} = \sum_{i=1}^{M} p(x_i)p(y_j \mid x_i), \qquad j = 1, 2, \ldots, L.$$

---

* In Chapter 2 we distinguished between the symbols $x_1, x_2, \ldots, x_M$, which were the building blocks for a message, and the code alphabet $a_1, a_2, \ldots, a_D$. Coding is necessary since the inputs accepted by the channel may have no relation to the symbols $x_1, x_2, \ldots x_M$. In this chapter we assume for convenience that $x_1, x_2, \ldots, x_M$ is the code alphabet. We do this because we would like to reserve the symbol $H(X)$ for the uncertainty at the input to a channel. If it becomes necessary to distinguish uncoded information from coded information, then we shall denote the original (uncoded) symbols by m_1, m_2, etc.; thus the m_i will be the components of a message. Each message will be assigned a code word, that is, a sequence of symbols from the alphabet $x_1, x_2, \ldots, x_M$.

Thus the specification of an input distribution induces in a natural way an output distribution and a joint distribution on input and output. We may therefore calculate the *input uncertainty* $H(X)$, the *output uncertainty* $H(Y)$, and the *joint uncertainty* of input and output $H(X, Y)$ as well as the conditional uncertainties $H(Y \mid X)$ and $H(X \mid Y)$. It is natural to define the *information processed by the channel* as

$$I(X \mid Y) = H(X) - H(X \mid Y). \tag{3.2.1}$$

By the discussion in Section 1.5, we have

$$I(X \mid Y) = H(Y) - H(Y \mid X) = I(Y \mid X) = H(X) + H(Y) - H(X, Y).$$

It is important to notice that the information processed by a channel depends on the input distribution $p(x)$. We may vary the input distribution until the information reaches a maximum; the maximum information is called the *channel capacity*. Specifically, we define the channel capacity as

$$C = \max_{p(x)} I(X \mid Y). \tag{3.2.2}$$

(There is a true maximum rather than just a least upper bound; see Problem 3.12.)

The significance of the channel capacity is not at all clear from what we have done so far. However, we shall prove later in this chapter that (loosely speaking) it is possible to transmit information through a channel at any rate less than the channel capacity with an arbitrarily small probability of error; completely reliable transmission is not possible if the information processed is greater than the channel capacity. The calculation of the capacity of a channel is a difficult problem in general; the solution in certain special cases will be discussed in the next section.

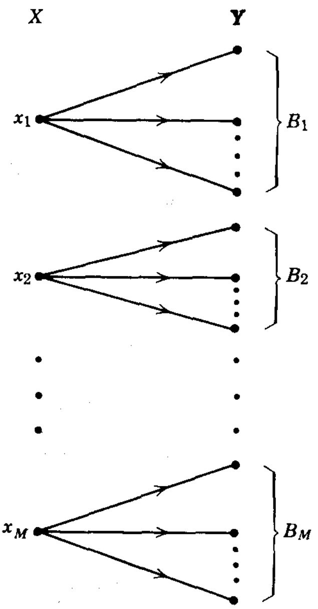


Fig. 3.2.1. Lossless channel.

It is convenient at this point to introduce certain classes of channels which are easy to analyze.

a. A channel is *lossless* if $H(X \mid Y) = 0$ for all input distributions. In other words, a lossless channel is characterized by the fact that the input is determined by the output and hence that no transmission errors can occur. Equivalently, the values of Y may be partitioned into disjoint sets

Hamming bound suggests that as the error correction requirements increase, the maximum possible number of code words decreases.

A consideration neglected so far is ease of decoding. The ideal-observer decision scheme is a "minimum-distance" decoder; that is, the received sequence $\mathbf{v}_j$ is decoded as the code word $\mathbf{w}_i$ that minimizes $d(\mathbf{w}_i, \mathbf{v}_j)$. Thus the decoder must contain a "table" or "code book" whose entries are all binary sequences of the specified code-word length. Each possible received sequence is assigned to the closest code word. The storage requirements for such a decoder are quite severe. In this section we investigate a coding scheme that improves the efficiency of the decoding process, and at the same time gives considerable insight into the structure of error correcting codes.

The earliest "parity check" codes were devised as a very simple and easily mechanized error detection scheme. Given a code consisting of all binary sequences of length n, an extra digit (the "parity digit") is added to each code word, the digit being chosen so as to make the total number of ones even (or odd). For example, if the original code is $\{00, 01, 10, 11\}$, and an "even parity" is used, that is, the parity digit is chosen so that the number of ones is even, the resulting code is $\{000, 011, 101, 110\}$. It is not difficult to see that if a single error (or in fact any e-tuple error where e is odd) is made in transmission, the received sequence will have an odd number of ones. Hence all errors involving an odd number of digits can be detected. The only operation the decoder must perform is to add the number of ones in the received sequence. This idea was generalized by Hamming and later by Slepian, who formulated the basic theory we present here.

If $r_1, r_2, \ldots, r_n$ are the digits of a code word, the fact that the number of ones is even may be expressed by the condition $r_1 + r_2 + \cdots + r_n = 0$ modulo 2.† Thus it appears that the binary sequences of length n which have an even number of ones are precisely the sequences which satisfy a certain modulo 2 linear equation. To generalize this idea we consider codes whose words satisfy a set of simultaneous linear equations.

Definition. Given a set of simultaneous linear equations of the form

$$\begin{aligned} a_{11}r_1 + a_{12}r_2 + \cdots + a_{1n}r_n &= 0 \\ &\;\vdots \\ a_{m1}r_1 + a_{m2}r_2 + \cdots + a_{mn}r_n &= 0 \end{aligned} \qquad \text{(mod 2)}. \qquad (4.3.1)$$

† Modulo 2 arithmetic is the arithmetic of the field consisting of two elements 0 and 1, with the rules $0 + 0 = 1 + 1 = 0, 0 + 1 = 1 + 0 = 1,\ 1 \cdot 1 = 1,\ 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$, $0/1 = 0$, $1/1 = 1$, $a - b = a + b$ for $a, b = 0$ or 1.

The set of solutions to (4.3.1) is called a *parity check code* (The terminology *group code* is also used, for reasons which will become clear in Section 4.4.)

The m by n matrix $A = [a_{ij}]$ is called the *parity check matrix*. If the rank of A is t, and if columns $j_1, \ldots, j_t$ of A are linearly independent, then the $n - t = k$ digits r_j, $j \neq j_1, \ldots, j \neq j_t$, may be specified arbitrarily in (4.3.1), and are called a set of *information digits* of the code. The digits $r_{j_1}, \ldots, r_{j_t}$ are then determined from (4.3.1), and are called a set of *check digits* for the code. A parity check code must contain 2^k words, where $k = n - t = n - \text{rank } A$; each assignment of values to a set of information digits determines a unique code word.

If a parity check code is to be used to process the information produced by a source which emits R binary symbols per second, then in accordance with the discussion in Chapter 3 we would set $2^{nR} = 2^k$, or $R = k/n$. Thus a high percentage of information digits corresponds to a high transmission rate; on the other hand, in general the greater the percentage of check digits the greater the error correcting ability of the code. An example of a set of parity check equations is shown below for the case $n = 6$, $t = m = 3$, $k = 3$.

$$\begin{aligned} r_1 \qquad\quad + r_4 + r_5 \qquad &= 0 \\ r_2 + r_3 + r_4 \qquad + r_6 &= 0 \\ r_1 \qquad + r_3 + r_4 \qquad + r_6 &= 0 \end{aligned} \quad \text{(mod 2).} \tag{4.3.2}$$

We may choose r_1, r_2, and r_3 as check digits and r_4, r_5, and r_6 as information digits. To form a code word, we first choose values for r_4, r_5, and r_6; for example, $r_4 = 0$, $r_5 = 1$, $r_6 = 1$. We then solve the parity check equation to obtain $r_1 = r_4 + r_5 = 1$, $r_3 = r_1 + r_4 + r_6 = 0$, $r_2 = r_3 + r_4 + r_6 = 1$. Thus the code word is 110011. Since there are three information digits, there are $2^3 = 8$ code words in the code; all eight code words are listed in Table 4.3.1.

The fact that the code words of a parity check code are determined by a set of linear equations allows us to use algebraic techniques to study the

Table 4.3.1. Example of a parity check code

	r_1	r_2	r_3	r_4	r_5	r_6
w_1	0	0	0	0	0	0
w_2	0	0	1	0	0	1
w_3	1	1	1	0	1	0
w_4	1	1	0	0	1	1
w_5	1	1	0	1	0	0
w_6	1	1	1	1	0	1
w_7	0	0	1	1	1	0
w_8	0	0	0	1	1	1

properties of codes. However, there is a basic question concerning the adequacy of parity check codes. It would be desirable to know whether or not such codes meet the specifications of the fundamental theorem. In other words, given a binary symmetric channel with capacity $C > 0$ and a positive number $R < C$, we would hope to find a sequence of parity check codes which maintain the transmission rate R with a probability of error which approaches zero with increasing code-word length. In fact we can find such a sequence, as we shall prove in Section 4.6. Thus from the point of view of the fundamental theorem, there is no loss of generality in considering parity check codes.

We now turn to the problem of relating the parity check equations to the error correcting ability of the code. If the digits $r_1, r_2, \ldots, r_n$ are transmitted over a binary symmetric channel, the received digits will be $r_1', r_2', \ldots, r_n'$ where $r_i' = r_i$ if no error has been made in r_i, and $r_i' = r_i + 1 \pmod 2$ if an error in r_i has occurred. Suppose that at the receiver we try to determine whether or not $r_1', r_2', \ldots, r_n'$ satisfy the parity check equations. In other words, we calculate

$$\begin{aligned} c_1 &= a_{11}r_1' + a_{12}r_2' + \cdots + a_{1n}r_n' \\ c_2 &= a_{21}r_1' + a_{22}r_2' + \cdots + a_{2n}r_n' \\ &\;\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \pmod 2).\dagger \\ c_m &= a_{m1}r_1' + a_{m2}r_2' + \cdots + a_{mn}r_n' \end{aligned}$$

The binary sequence

$$\mathbf{c} = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{bmatrix}$$

is called the *corrector* or *syndrome* associated with the sequence $\mathbf{v} = (r_1', \ldots, r_n')$. In matrix form,

$$\mathbf{c} = A \begin{bmatrix} r_1' \\ \cdot \\ \cdot \\ \cdot \\ r_n' \end{bmatrix} = A\mathbf{v}^T.$$

(In general, code words and received sequences will be written as row vectors; $\mathbf{v}^T$ is the transpose of $\mathbf{v}$.) If no error has occurred, then $r_i' = r_i$

† For the remainder of this chapter all arithmetic will be modulo 2 unless otherwise specified.

for all i and hence $\mathbf{c}$ is the zero vector $\mathbf{0}$. We will see in the next section that the construction of the minimum-distance decoder involves the consideration of the correctors associated with various error combinations.

Before looking at the construction of the minimum-distance decoder, we develop a few properties of correctors.

Suppose that the sequence $\mathbf{w} = (r_1, \ldots, r_n)$ is transmitted and the sequence $\mathbf{v} = (r_1', \ldots, r_n')$ is received. The sequence $\mathbf{z} = \mathbf{v} - \mathbf{w} = (r_1' - r_1, \ldots, r_n' - r_n)$ is called the *error pattern vector* corresponding to $\mathbf{w}$ and $\mathbf{v}$; the error pattern vector is thus a binary sequence with ones in the positions that are in error, and zeros elsewhere. The corrector associated with $\mathbf{v}$ is $\mathbf{c} = A\mathbf{v}^T = A(\mathbf{z}^T + \mathbf{w}^T) = A\mathbf{z}^T + A\mathbf{w}^T = A\mathbf{z}^T$ ($\mathbf{w}$ is a code word so that $A\mathbf{w}^T = \mathbf{0}$). Consequently *the corrector is determined by the error pattern vector.* Now *if* $\mathbf{z}$ *has ones in positions* $j_1, j_2, \ldots, j_e$ *and zeros elsewhere, that is, digits* $j_1, \ldots, j_e$ *are in error, then the vector* $A\mathbf{z}^T$ *is simply the* (*mod* 2) *sum of columns*, $j_1, j_2, \ldots, j_e$ *of* A. For example, in the code described by the parity check equations (4.3.2), errors in positions 2 and 4 will yield the corrector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Observe that a single error in position 1, or a triple error in positions 2, 3, and 5, will yield the same corrector; in general many different error patterns will correspond to a given corrector. (In the next section we find out how many.)

4.4. The application of group theory to parity check coding

The object of this section is to prove the following theorem.

Theorem 4.4.1. The minimum-distance decoding scheme for a parity check code may be described as follows.

1. Given a received sequence $\mathbf{v}$, calculate the corrector $\mathbf{c}$ associated with $\mathbf{v}$.

2. Examine all error patterns whose corrector is $\mathbf{c}$, and correct the error of smallest magnitude. In other words, if among all sequences $\mathbf{z}$ such that $A\mathbf{z}^T = \mathbf{c}$, $\mathbf{z}_0$ is a sequence of minimum weight, that is, minimum number of ones, $\mathbf{v}$ is decoded as $\mathbf{v} - \mathbf{z}_0$. Thus if $\mathbf{v}$ is received, we declare that the code word $\mathbf{v} - \mathbf{z}_0$ was transmitted (note that $\mathbf{v} - \mathbf{z}_0$ is a code word since $A(\mathbf{v} - \mathbf{z}_0)^T = A\mathbf{v}^T - A\mathbf{z}_0{}^T = \mathbf{c} - \mathbf{c} = \mathbf{0}$) and that error pattern $\mathbf{z}_0$ occurred.

Theorem 4.4.1 implies that the minimum-distance decoder may be regarded as a function which assigns an error pattern to each corrector. Thus instead of storing a table of all 2^n binary sequences of length n and their associated code words, we need only store a table of all possible correctors and their associated error patterns. Since the correctors are linear combinations of the columns of A and the rank of A is $t = n - k$, there are $2^t = 2^{n-k}$ distinct correctors, in general a much smaller number than 2^n.

To prove Theorem 4.4.1, as well as to gain further insight into the structure of parity check codes, it is convenient to introduce some concepts from group theory.

Definition. A *group* is a set G together with an operation,† which we refer to as "addition," satisfying

1. $a \in G$, $b \in G$ implies $a + b \in G$ (*closure*).
2. $a + (b + c) = (a + b) + c$ for all $a, b, c, \in G$ (*associativity*).
3. There is an element $0 \in G$ such that $a + 0 = 0 + a = a$ for all $a \in G$ (*identity element*).
4. For each $a \in G$ there is an element $(-a) \in G$ such that $a + (-a) = (-a) + a = 0$ (*inverse*).

A group is *abelian* or *commutative* if $a + b = b + a$ for all $a, b \in G$.

The integers under ordinary addition and the set of binary sequences of a fixed length n under modulo 2 addition are examples of abelian groups. As far as the application to parity check coding is concerned, the basic result needed is the following:

Theorem 4.4.2. Let S be the set of code words in a parity check code. Then S is a group under modulo 2 addition.

Proof. Let A be the parity check matrix of the code. Then a binary sequence $\mathbf{w}$ is a code word if and only if $\mathbf{w}$ satisfies the parity check equations, that is, if and only if $A\mathbf{w}^T = \mathbf{0}$. To prove the closure property we note that if $\mathbf{w}_1 \in S$, $\mathbf{w}_2 \in S$, then $A\mathbf{w}_1{}^T = \mathbf{0}$, $A\mathbf{w}_2{}^T = \mathbf{0}$. Hence $A(\mathbf{w}_1 + \mathbf{w}_2)^T = A(\mathbf{w}_1{}^T + \mathbf{w}_2{}^T) = A\mathbf{w}_1{}^T + A\mathbf{w}_2{}^T = \mathbf{0}$; hence $\mathbf{w}_1 + \mathbf{w}_2 \in S$. The associative property follows from the definition of modulo 2 addition. We may take the zero vector as the identity element, since the zero vector always satisfies the parity check equations and is therefore a code word. We may take each element of S as its own inverse, again by definition of modulo 2 addition. This completes the proof.

† By an operation we mean a function that assigns to every pair a, b of elements in G, another element $a + b$.

The set S, of course, is an abelian group, since modulo 2 addition is commutative. If the code words of S are of length n, S is said to be a *subgroup* of the group $\mathcal{B}_n$ of all binary sequences of length n, since S is contained in the larger group $\mathcal{B}_n$.

We now prove the converse of Theorem 4.4.2.

Theorem 4.4.3. Let S be a set of binary sequences of length n that forms a group under modulo 2 addition. Then S is a parity check code, that is, there is a parity check matrix A such that the code determined by A is S.

Proof. Let us arrange the code words in an s by n matrix M, where s is the total number of code words. An example is given in Table 4.4.1. It may be verified that the code does in fact form a group; for example, $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_4$, $\mathbf{w}_5 + \mathbf{w}_7 = \mathbf{w}_2$.

Table 4.4.1. Matrix of code words

$$\begin{array}{c|cccccc}
\mathbf{w}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{w}_1 & 1 & 0 & 1 & 0 & 0 & 1 \\
\mathbf{w}_2 & 1 & 1 & 0 & 0 & 1 & 0 \\
\mathbf{w}_3 & 0 & 1 & 0 & 1 & 0 & 1 \\
\mathbf{w}_4 & 0 & 1 & 1 & 0 & 1 & 1 \\
\mathbf{w}_5 & 1 & 1 & 1 & 1 & 0 & 0 \\
\mathbf{w}_6 & 1 & 0 & 0 & 1 & 1 & 1 \\
\mathbf{w}_7 & 0 & 0 & 1 & 1 & 1 & 0
\end{array} = M$$

Let k be the rank of M, and $m = n - k$. Then k is the maximum number of linearly independent rows of M, and also the maximum number of linearly independent columns of M. Assume for convenience that $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ are linearly independent. (If not, we may rearrange the rows.) Then every code word in S is a modulo 2 linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$. But every linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ is a member of S since S is a group. Thus S coincides with the set of all binary sequences $\mathbf{w}$ which can be expressed as

$$\mathbf{w} = \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_k\mathbf{w}_k, \qquad \lambda_i = 0 \text{ or } 1. \tag{4.4.1}$$

By the linear independence of $\mathbf{w}_1, \ldots, \mathbf{w}_k$, each distinct choice of $(\lambda_1, \ldots, \lambda_k)$ produces a different $\mathbf{w}$. Thus the number of code words in S is 2^k. Now assume for convenience that the last k columns of M (columns $m + 1, m + 2, \ldots, m + k$ where $m = n - k$) are linearly independent. Then the remaining columns $1, 2, \ldots, m$ may be expressed as linear combinations of columns $m + 1, m + 2, \ldots, m + k$. In other words, each digit $r_1, r_2, \ldots, r_m$ of the code words of S is expressible as a linear combination of digits $r_{m+1}, r_{m+2}, \ldots, r_{m+k}$. Hence we may write a set of linear equations relating $r_1, r_2, \ldots, r_m$ to $r_{m+1}, r_{m+2}, \ldots, r_{m+k}$, that is,

$$
\begin{aligned}
r_1 &= b_{11}r_{m+1} + b_{12}r_{m+2} + \cdots + b_{1k}r_{m+k} \\
r_2 &= b_{21}r_{m+1} + b_{22}r_{m+2} + \cdots + b_{2k}r_{m+k} \\
&\;\;\vdots \\
r_m &= b_{m1}r_{m+1} + b_{m2}r_{m+2} + \cdots + b_{mk}r_{m+k}.
\end{aligned}
\tag{4.4.2}
$$

The equations (4.4.2) are of the form

$$A\begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix} = 0,$$

where

$$A = \left[\begin{array}{c|cccc} & -b_{11} & -b_{12} & \cdots & -b_{1k} \\ I_m & & \vdots & & \\ & -b_{m1} & -b_{m2} & \cdots & -b_{mk} \end{array}\right] \tag{4.4.3†}$$

and I_m is an identity matrix of order m.

Thus we have shown that every code word $\mathbf{w} \in S$ satisfies $A\mathbf{w}^T = \mathbf{0}$, where A is the matrix (4.4.3). To show that every solution of $A\mathbf{w}^T = \mathbf{0}$ is in S we note that in solving the equations (4.4.2) the variables r_{m+1}, $r_{m+2}, \ldots, r_{m+k}$ may be specified arbitrarily. Thus $A\mathbf{w}^T = \mathbf{0}$ has 2^k solutions. But we have shown that S has 2^k elements, each of which satisfies $A\mathbf{w}^T = \mathbf{0}$. Hence the set of row vectors $\mathbf{w}$ satisfying $A\mathbf{w}^T = \mathbf{0}$ is identical to the set S of code words, that is, the code determined by A is S. The proof is complete.

To illustrate the process of constructing the parity check equations (4.4.2), consider the code of Table 4.4.1. Since $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ form a maximal set of linearly independent code words, any parity check equations satisfied by $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ will necessarily be satisfied by all code words. [To see this, we observe that $A\mathbf{w}_i^T = \mathbf{0}$ for $i = 1, 2, \ldots, k$ implies $A(\lambda_1\mathbf{w}_1^T + \cdots + \lambda_k\mathbf{w}_k^T) = \mathbf{0}$ for all possible choices of the λ_i.] Thus

† Addition and subtraction are the same in modulo 2 arithmetic, but the use of the minus sign in certain arguments facilitates generalizations to nonbinary coding (see Section 4.9).

we restrict our attention to the submatrix

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

formed by $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. Since the last three columns of Q are linearly independent, we should be able to express the remaining three columns in terms of these. For example, to express column 1 in terms of columns 4, 5, and 6, we have to find numbers a_1, a_2, and a_3 such that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad a_i = 0 \text{ or } 1. \tag{4.4.4}$$

Equation (4.4.4) yields $a_3 = 1$, $a_2 = 1$, $a_1 + a_3 = 0$ or $a_1 = 1$. Similarly, to express columns 2 and 3 in terms of columns 4, 5, and 6 we have to solve

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = b_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{4.4.5}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \tag{4.4.6}$$

Solving (4.4.5) and (4.4.6) we obtain $b_1 = 1$, $b_2 = 1$, $b_3 = 0$, $d_1 = 1$, $d_2 = 0$, $d_3 = 1$. From (4.4.4), (4.4.5), and (4.4.6) we may write

$$\begin{aligned} r_1 &= r_4 + r_5 + r_6 \\ r_2 &= r_4 + r_5 \\ r_3 &= r_4 \phantom{{}+r_5} + r_6. \end{aligned}$$

Hence the parity check matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Because of Theorems 4.4.2 and 4.4.3 a parity check code is often called a "group code." The term "(n, k) code" is also used to refer to a parity check code with code words of length n and k information digits.

We now turn to the problem of constructing the minimum-distance decoder for a parity check code. As an example, consider the code defined by the parity check equations

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.4.7}$$

The four code words corresponding to (4.4.7) are $\mathbf{w}_0 = 0000$, $\mathbf{w}_1 = 0101$, $\mathbf{w}_2 = 1110$, and $\mathbf{w}_3 = 1011$.

Table 4.4.2. Error pattern vectors and received sequences for a group code

$\mathbf{w}_0$	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$
0000	0101	1110	1011
0110	0011	1000	1101
1111	1010	0001	0100
0010	0111	1100	1001

Various error pattern vectors and their associated received sequences are shown in Table 4.4.2. The leftmost element in any row is an error pattern vector. The element $\mathbf{v}_i$ ($i = 0, 1, 2, 3$) in a given row represents the received sequence when code word $\mathbf{w}_i$ is transmitted and the error pattern vector is $\mathbf{v}_0$. For example, if $\mathbf{w}_2$ is transmitted and the error pattern vector is 0110, that is, errors are made in positions 2 and 3, the received sequence is 1000. The rows of Table 4.4.2 are examples of what are called *cosets* in group theory.

Definition. Given a subgroup S of a group G, the *coset* associated with an element $\mathbf{z} \in G$ is the set of all elements $\mathbf{z} + \mathbf{w}$, where $\mathbf{w}$ ranges over S. The coset associated with $\mathbf{z}$ is written $\mathbf{z} + S$.

In the example we are considering, G is the group of all sixteen binary sequences of length 4 under modulo 2 addition, S is the set of code words $\{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Various cosets are

$$\begin{aligned} 0110 + S &= \{0110, 0011, 1000, 1101\} \\ 1000 + S &= \ 0110 + S \\ 1111 + S &= \{1111, 1010, 0001, 0100\} \\ 0000 + S &= \ S. \end{aligned}$$

The key property of cosets is the following.

Lemma 4.4.4. The cosets $\mathbf{z}_1 + S$ and $\mathbf{z}_2 + S$ are either disjoint or identical. (Thus in Table 4.4.2 no sequence appears in more than one row.)

Proof. Suppose $\mathbf{z}_1 + S$ and $\mathbf{z}_2 + S$ are not disjoint. Then there is an element $\mathbf{v}'$ such that $\mathbf{v}' \in \mathbf{z}_1 + S$ and $\mathbf{v}' \in \mathbf{z}_2 + S$. Hence $\mathbf{v}' = \mathbf{z}_1 + \mathbf{w}_1$ for some $\mathbf{w}_1 \in S$ and $\mathbf{v}' = \mathbf{z}_2 + \mathbf{w}_2$ for some $\mathbf{w}_2 \in S$. To show that $\mathbf{z}_1 + S$ and $\mathbf{z}_2 + S$ are identical, suppose $\mathbf{v} \in \mathbf{z}_1 + S$. Then $\mathbf{v} = \mathbf{z}_1 + \mathbf{w}_3$ for some $\mathbf{w}_3 \in S$. Therefore $\mathbf{v} = \mathbf{v}' - \mathbf{w}_1 + \mathbf{w}_3 = \mathbf{z}_2 + (\mathbf{w}_2 - \mathbf{w}_1 + \mathbf{w}_3)$. Hence $\mathbf{v}$ is of the form $\mathbf{z}_2 + \mathbf{w}$ where $\mathbf{w} = \mathbf{w}_2 - \mathbf{w}_1 + \mathbf{w}_3 \in S$. Thus $\mathbf{v} \in \mathbf{z}_2 + S$. A symmetrical argument shows that $\mathbf{v} \in \mathbf{z}_2 + S$ implies $\mathbf{v} \in \mathbf{z}_1 + S$, completing the proof.

Each coset $\mathbf{z} + S$ has exactly as many elements as S since all the elements $\mathbf{z} + \mathbf{w}$ $(\mathbf{w} \in S)$ are distinct. (If $\mathbf{z} + \mathbf{w}_1 = \mathbf{z} + \mathbf{w}_2$, then $\mathbf{w}_1 = \mathbf{w}_2$.) Hence the number of cosets is the number of elements of G divided by the number of elements of S. If G is the group of 2^n binary sequences of length n and S is a group code with 2^k code words of length n, then the number of cosets is 2^{n-k}. Thus the number of cosets and the number of possible corrector values coincide. This is not a coincidence, as we shall see shortly.

We now establish a symmetry property of the minimum-distance decoder for a group code.

Lemma 4.4.5. Let S be a group code. Suppose that for a particular code word $\mathbf{w}_i$ and a particular error pattern vector $\mathbf{z}$,

$$d(\mathbf{w}_i + \mathbf{z}, \mathbf{w}_i) \leq d(\mathbf{w}_i + \mathbf{z}, \mathbf{w}_j) \tag{4.4.8}$$

for all code words $\mathbf{w}_j$. Then

$$d(\mathbf{w} + \mathbf{z}, \mathbf{w}) \leq d(\mathbf{w} + \mathbf{z}, \mathbf{w}') \tag{4.4.9}$$

for all code words $\mathbf{w}, \mathbf{w}'$.

The interpretation of Lemma 4.4.5 is as follows. Suppose that the error pattern $\mathbf{z}$ is corrected by the minimum-distance decoder when the code word $\mathbf{w}_i$ is transmitted, that is, the minimum-distance decoder assigns to the received sequence $\mathbf{w}_i + \mathbf{z}$ the code word $\mathbf{w}_i$, so that condition (4.4.8) is satisfied. Then by (4.4.9) we may without loss of generality assume that the error pattern $\mathbf{z}$ is corrected no matter which code word is sent. In other words, without changing the probability of error we may adjust the minimum-distance decoder so that *a given error pattern is either always corrected or never corrected.* From now on, we assume that the minimum-distance decoder always has this property; thus the statement "the error pattern $\mathbf{z}$ is corrected" is unambiguous.

Proof. First observe that if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are binary sequences of the same length, then $d(\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3) = d(\mathbf{v}_1, \mathbf{v}_2)$. For example,

$$d(0111, 1010) = 3 \quad \text{and} \quad d(0111 + 1100, 1010 + 1100) = d(1011, 0110) = 3.$$

We therefore have

$$\begin{aligned} d(\mathbf{z} + \mathbf{w}, \mathbf{w}') &= d(\mathbf{z} + \mathbf{w} + \mathbf{w}_i - \mathbf{w}, \mathbf{w}' + \mathbf{w}_i - \mathbf{w}) \\ &= d(\mathbf{z} + \mathbf{w}_i, \mathbf{w}' + \mathbf{w}_i - \mathbf{w}) \geq d(\mathbf{z} + \mathbf{w}_i, \mathbf{w}_i) \qquad \text{(by 4.4.8)} \\ &= d(\mathbf{z} + \mathbf{w}_i + \mathbf{w} - \mathbf{w}_i, \mathbf{w}_i + \mathbf{w} - \mathbf{w}_i) = d(\mathbf{z} + \mathbf{w}, \mathbf{w}). \end{aligned}$$

Another consequence of Lemma 4.4.5 is that if we wish to correct an error pattern $\mathbf{z}_1$, we cannot correct any other error pattern $\mathbf{z}_2$ in the coset $\mathbf{z}_1 + S$. For if $\mathbf{z}_2 = \mathbf{z}_1 + \mathbf{w}$ where $\mathbf{w} \in S$, $\mathbf{w} \neq \mathbf{0}$, then the transmission of $\mathbf{w}$ together with the error pattern $\mathbf{z}_1$ will yield the same received sequence as the transmission of the zero vector together with error pattern $\mathbf{z}_2$. Thus it is impossible to correct both $\mathbf{z}_1$ and $\mathbf{z}_2$ for all possible transmitted sequences. Hence *exactly one sequence in each coset can serve as a correctible error pattern.* Since the probability of an error pattern increases as the number of digits involved in the error decreases, it appears that among the sequences in a given coset, we should choose as the correctible error pattern that sequence with the fewest number of ones. This result is established by the following theorem.

Theorem 4.4.6. To construct the minimum-distance decoding scheme for a group code S, it is sufficient to choose as the correctible error pattern for each coset a sequence with minimum weight, that is, minimum number of ones. If $\mathbf{z}$ is such a sequence, then the received sequence $\mathbf{z} + \mathbf{w}$ ($\mathbf{w} \in S$) is decoded as $\mathbf{w}$.

Proof. If $\mathbf{z}$ is a sequence of minimum weight in the coset $\mathbf{z} + S$, then $d(\mathbf{z}, \mathbf{w}) = d(\mathbf{z} - \mathbf{w}, \mathbf{w} - \mathbf{w}) = d(\mathbf{z} - \mathbf{w}, \mathbf{0})$. Since $\mathbf{z}$ has minimum weight among all sequences $\mathbf{z} + \mathbf{w}$, $\mathbf{w} \in S$, we have $d(\mathbf{z} - \mathbf{w}, \mathbf{0}) \geq d(\mathbf{z}, \mathbf{0})$. Hence the sequence $\mathbf{z}$ is at least as close to zero as to any other code word, so that in the minimum-distance decoder we may correct the error pattern $\mathbf{z}$ when $\mathbf{0}$ is transmitted, and hence by Lemma 4.4.5 we may correct $\mathbf{z}$ for all possible transmitted sequences. The theorem is proved.

It follows from Theorem 4.4.6 that the minimum-distance decoder may be constructed in the following way.

Write down a table of cosets (such as Table 4.4.2); choose a sequence $\mathbf{z}$ of minimum weight in the coset $\mathbf{z} + S$; rearrange the table so that the entries in the row corresponding to $\mathbf{z} + S$ are $\mathbf{z} + \mathbf{0}$, $\mathbf{z} + \mathbf{w}_1, \ldots,$ $\mathbf{z} + \mathbf{w}_{2^k-1}$, where the $\mathbf{w}_i$ are the code words. (Thus $\mathbf{z}$ is the leftmost element of its coset, or the so-called "*coset leader.*") If a received sequence $\mathbf{v}$ is in the column of the coset table headed by $\mathbf{w}_i$, then $\mathbf{v} = \mathbf{w}_i + \mathbf{z}$ for some correctible error pattern $\mathbf{z}$; hence $\mathbf{v}$ is decoded as $\mathbf{w}_i$. The minimum-distance decoding scheme for the code defined by (4.4.7) and Table 4.4.2 is shown in Table 4.4.3. Notice that row 3 has two sequences of weight one, namely 0001 and 0100. Either sequence can be chosen as the coset leader.

Table 4.4.3. Minimum-distance decoding scheme for a group code

$\mathbf{w}_0$	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$
0000	0101	1110	1011
1000	1101	0110	0011
0001	0100	1111	1010
0010	0111	1100	1001

We now show that there is a one-to-one correspondence between the 2^{n-k} correctible error patterns of a group code and the 2^{n-k} possible values of the corrector.

Theorem 4.4.7. All sequences in the same coset of a group code S have the same corrector. Two sequences in different cosets have different correctors.

Proof. If $\mathbf{z}_1$ and $\mathbf{z}_2$ are in the same coset, then $\mathbf{z}_2 = \mathbf{z}_1 + \mathbf{w}$ for some $\mathbf{w} \in S$. Thus $A\mathbf{z}_2{}^T = A\mathbf{z}_1{}^T$ so that $\mathbf{z}_1$ and $\mathbf{z}_2$ have the same corrector. On the other hand, if $\mathbf{z}_1$ and $\mathbf{z}_2$ have the same corrector then $A\mathbf{z}_1{}^T = A\mathbf{z}_2{}^T$ or $A(\mathbf{z}_2 - \mathbf{z}_1)^T = \mathbf{0}$. But then $\mathbf{z}_2 - \mathbf{z}_1$ satisfies the parity check equations, hence $\mathbf{z}_2 - \mathbf{z}_1$ is a code word $\mathbf{w}$. Therefore $\mathbf{z}_1$ and $\mathbf{z}_2 = \mathbf{z}_1 + \mathbf{w}$ are in the same coset.

Finally, we prove the main theorem of this section.

Proof of Theorem 4.4.1. Let $\mathbf{c}$ be the corrector associated with a given received sequence $\mathbf{v}$; let $\mathbf{v}$ belong to the coset $\mathbf{z}_0 + S$ where $\mathbf{z}_0$ is a sequence of minimum weight in $\mathbf{z}_0 + S$. By Theorem 4.4.7, the sequences of the coset $\mathbf{z}_0 + S$ are precisely the sequences whose corrector is $\mathbf{c}$. It follows that among all sequences $\mathbf{z}$ such that $A\mathbf{z}^T = \mathbf{c}$, $\mathbf{z}_0$ is a sequence of minimum weight. If $\mathbf{v} = \mathbf{z}_0 + \mathbf{w}$, then, by Theorem 4.4.6, $\mathbf{v}$ is decoded as $\mathbf{w} = \mathbf{v} - \mathbf{z}_0$. This agrees exactly with the procedure given in Theorem 4.4.1.

We conclude this section with an example. Consider the code whose parity check matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The minimum-distance decoding table is shown in Table 4.4.4. Notice that all single errors are corrected, along with some (but not all) double errors and some triple errors.

Table 4.4.4. Minimum-distance decoding table for a parity check code

Code Words	000000	111010	101101	010111	(Transposed) Correctors 0000
	100000	011010	001101	110111	1000
	010000	101010	111101	000111	0100
Single	001000	110010	100101	011111	0010
errors	000100	111110	101001	010011	0001
	000010	111000	101111	010101	1110
	000001	111011	101100	010110	1011
	110000	001010	011101	100111	1100
	101000	010010	000101	111111	1010
Double	100100	011110	001001	110011	1001
errors	100010	011000	001111	110101	0110
	100001	011011	001100	110110	0011
	010100	101110	111001	000011	0101
	010001	101011	111100	000110	1111
Triple	110100	001110	011001	100011	1101
errors	110001	001011	011100	100110	0111

To calculate the probability of error for a group code, we note that correct transmission will occur if and only if the error pattern is a coset leader and hence correctible. Thus if N_i is the number of coset leaders of weight i, or equivalently the number of i-tuple errors corrected by the code, then the probability of correct transmission is

$$p(e') = \sum_{i=0}^{n} N_i \beta^i (1 - \beta)^{n-i}.$$

For the code of Table 4.4.4,

$$p(e') = (1 - \beta)^6 + 6\beta(1 - \beta)^5 + 7\beta^2(1 - \beta)^4 + 2\beta^3(1 - \beta)^3.$$

Note that the probability of correct transmission $p(e' \mid \mathbf{w})$ is the same for each transmitted code word $\mathbf{w}$; hence for a parity check code the average and maximum probability of error coincide.

4.5. *Upper and lower bounds on the error correcting ability of parity check codes*

In this section we investigate the relation between the number of code words in a parity check code and its error correcting ability. As we have seen previously, an increase in the number of words in a code will in general be accompanied by a reduction in the number of errors corrected. We will try to obtain quantitative results to this effect.

We first propose the following problem. We wish to construct a parity check code that has certain error correcting properties; for example, we might want a code that corrects e-tuple (and all smaller) errors. How do we choose the parity check matrix so that the resulting code has the desired properties? To begin with, let us consider the construction of a single-error correcting code. Since the corrector associated with an error in a single digit r_j is the jth column of the parity check matrix, we need only choose n distinct (nonzero) columns; the resulting code must be single-error correcting, since no two distinct single errors can result in the same corrector. (Also, since no column is zero, a single error cannot be confused with the "no error" condition.) For example, if $n = 7$, $k = 4$, the code defined by the following parity check equations will correct single errors.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

r_1, r_2, r_3 are check digits.
r_4, r_5, r_6, r_7 are information digits.

Note that if the parity check matrix A is m by n, we may as well assume that the rank of A is m. For if the rank is $t < m$, then $m - t$ rows are linearly dependent on the remaining t rows and may be eliminated. We may also assume for convenience that the m by m submatrix in the left-hand corner of A is nonsingular; in fact we may assume without loss of generality that the submatrix is an identity matrix. If the submatrix is not originally an identity matrix the equations may be diagonalized by elementary row transformations. For example, consider the parity check equations (4.3.2):

$$\begin{aligned}
r_1 \quad\quad\quad &+ r_4 + r_5 \quad\quad = 0 \\
r_2 + r_3 &+ r_4 \quad\quad + r_6 = 0 \\
r_1 \quad + r_3 &+ r_4 \quad\quad + r_6 = 0.
\end{aligned}$$

Let us add equation 1 to equation 3, obtaining

$$\begin{aligned}
r_1 \quad\quad\quad &+ r_4 + r_5 \quad\quad = 0 \\
r_2 + r_3 &+ r_4 \quad\quad + r_6 = 0 \\
r_3 \quad &\quad\quad + r_5 + r_6 = 0.
\end{aligned}$$

Now let us add equation 3 to equation 2, obtaining

$$\begin{aligned}
r_1 \quad\quad &+ r_4 + r_5 \quad\quad = 0 \\
r_2 \quad &+ r_4 + r_5 \quad\quad = 0 \\
r_3 &\quad\quad + r_5 + r_6 = 0.
\end{aligned}$$

The submatrix in the lower left-hand corner is now an identity matrix, and the new equations have exactly the same solution as the original equations; in other words, both sets of equations define the same code. The diagonalization process will be successful as long as the original submatrix is nonsingular.

The general relation between the parity check matrix and the error correcting ability of the code is contained in the following theorem.

Theorem 4.5.1. The parity check code defined by the matrix A will correct e-tuple (and all smaller) errors if and only if every set of $2e$ columns of A is linearly independent.

Proof. By Theorem 4.4.1, e-tuple and all smaller errors will be corrected if and only if all error patterns of weight $\leq e$ yield distinct correctors, that is, if and only if no linear combination of e or fewer columns of A equals another such linear combination. But this is precisely the condition that each set of $2e$ columns of A be linearly independent.

As an example, the following parity check matrix describes a double-error correcting code with 7 check digits and 3 information digits.

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}. \tag{4.5.1}$$

It may be verified that every set of 4 columns of the matrix (4.5.1) is linearly independent. However, we may find triple errors which are not correctible. For example,

$$\mathbf{c}(r_1) + \mathbf{c}(r_8) + \mathbf{c}(r_9) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{c}(r_3) + \mathbf{c}(r_4) + \mathbf{c}(r_6)$$

$[\mathbf{c}(r_j) = j\text{th column of } A]$.

Thus a triple error in r_1, r_8, and r_9 will be confused with a triple error in r_3, r_4, and r_6.

Next we shall consider the following problem. Suppose we wish to construct a parity check code that corrects e-tuple and all smaller errors, with a fixed code-word length n. We ask how many check digits are necessary to construct the code. We wish to use as few check digits as possible, since a small number of check digits corresponds to a large number of code words. In general, we cannot determine the minimum number of check digits exactly, but we can establish some useful lower and upper bounds, as follows.

Theorem 4.5.2. (Hamming Lower Bound on the Number of Check Digits). The number of check digits in an e-tuple-error correcting parity check code must satisfy

$$2^m \geq \sum_{i=0}^{e} \binom{n}{i} \tag{4.5.2}$$

where n = code word length
m = number of check digits $= n - k$.

Proof. In order that a parity check code be capable of correcting e-tuple and all smaller errors, each error combination involving e or fewer digits must result in a distinct corrector. The number of possible i-tuple errors in an n-digit code word is equal to the number of ways of choosing i digits out of n, namely $\binom{n}{i}$. The number of possible corrector values is 2^m; hence in order that there be a unique corrector for each error combination, we must have

$$2^m \geq 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{e},$$

which proves (4.5.2). (The term "1" appears on the right since if no error is made, the associated corrector must be the zero vector.)

The Hamming lower bound on the number of check digits is in fact identical to the Hamming upper bound on the number of code words, which we proved for general binary codes in Section 4.2. To see this, note that the number of code words is $2^k = 2^{n-m} = 2^n/2^m$. Hence (4.5.2) is equivalent to

$$2^k \le \frac{2^n}{\sum_{i=0}^{e} \binom{n}{i}},$$

which is (4.2.1) for the special case of a parity check code.

The Hamming lower bound is *necessary but not sufficient* for the constructibility of an e-tuple-error correcting parity check code. In other words, if m_0 is the smallest positive integer satisfying (4.5.2) for a given n and e, there may not be a parity check code using m_0 check digits. For example, if $n = 10$, $e = 2$, we find from (4.5.2) that $m_0 = 6$. However, there is no double-error correcting parity check code with fewer than seven check digits, as the reader may verify by trial and error.

An upper bound on the number of check digits required for the construction of a code is provided by the following theorem.

Theorem 4.5.3. (Varsharmov-Gilbert-Sacks Condition). An e-tuple-error correcting parity check code with words of length n may be constructed if the number of check digits m satisfies

$$2^m > \sum_{i=0}^{2e-1} \binom{n-1}{i}. \tag{4.5.3}$$

The condition (4.5.3) is *sufficient but not necessary* for constructibility of a code. In other words, if for a fixed n and e, m_0 is the smallest positive integer satisfying (4.5.3), then it is guaranteed that an e-tuple-error correcting parity check code with m_0 check digits exists. However, there may be an e-tuple-error correcting code with fewer check digits. For example, if $n = 10$ and $e = 2$, then (4.5.3) yields $m_0 = 8$. However, it is possible to construct a double-error correcting code with seven check digits. [See the parity check matrix (4.5.1).]

Proof. The proof is accomplished by giving a synthesis procedure for the required code. We shall choose the columns $\mathbf{c}(r_1), \mathbf{c}(r_2), \ldots, \mathbf{c}(r_n)$ of the parity check matrix successively. By Theorem 4.5.1 we must satisfy the condition that every set of $2e$ columns be linearly independent. First we choose $\mathbf{c}(r_1)$ arbitrarily. [Of course we must have $\mathbf{c}(r_1) \neq \mathbf{0}$ since the zero vector must correspond to perfect transmission.] We now choose $\mathbf{c}(r_2)$ such that $\mathbf{c}(r_2) \neq \mathbf{0}$ and $\mathbf{c}(r_2) \neq \mathbf{c}(r_1)$. This can be done provided $2^m > 2$. We now choose $\mathbf{c}(r_3)$ such that

$$\mathbf{c}(r_3) \neq \mathbf{0}, \qquad \mathbf{c}(r_3) \neq \mathbf{c}(r_1), \qquad \mathbf{c}(r_3) \neq \mathbf{c}(r_2), \qquad \mathbf{c}(r_3) \neq \mathbf{c}(r_1) + \mathbf{c}(r_2),$$

thus guaranteeing that $\mathbf{c}(r_1)$, $\mathbf{c}(r_2)$ and $\mathbf{c}(r_3)$ will be linearly independent. A choice for $\mathbf{c}(r_3)$ can be made if $2^m > 4$. Let us digress for a moment to consider the special case $e = 2$. Then every set of four columns must be linearly independent. Thus we choose $\mathbf{c}(r_4)$ such that $\mathbf{c}(r_4)$ is not equal to any of the following:

$$\mathbf{0},\ \mathbf{c}(r_1),\ \mathbf{c}(r_2),\ \mathbf{c}(r_3),\ \mathbf{c}(r_1) + \mathbf{c}(r_2),$$
$$\mathbf{c}(r_1) + \mathbf{c}(r_3),\ \mathbf{c}(r_2) + \mathbf{c}(r_3),\ \mathbf{c}(r_1) + \mathbf{c}(r_2) + \mathbf{c}(r_3).$$

Such a choice can be made if $2^m > 8$. We now choose $\mathbf{c}(r_5)$ such that

$$\begin{aligned}
&\mathbf{c}(r_5) \neq \mathbf{0}\\
&\mathbf{c}(r_5) \neq \mathbf{c}(r_i), \qquad i = 1, 2, 3, 4\\
&\mathbf{c}(r_5) \neq \mathbf{c}(r_i) + \mathbf{c}(r_j), \qquad i, j = 1, 2, 3, 4, \quad i \neq j\\
&\mathbf{c}(r_5) \neq \mathbf{c}(r_i) + \mathbf{c}(r_j) + \mathbf{c}(r_k), \qquad i, j, k, = 1, 2, 3, 4, \quad i \neq k, j \neq k.
\end{aligned}$$

Note that $\mathbf{c}(r_5)$ *can* equal $\mathbf{c}(r_1) + \mathbf{c}(r_2) + \mathbf{c}(r_3) + \mathbf{c}(r_4)$ since we do not require that every set of five columns be linearly independent. Therefore $\mathbf{c}(r_5)$ can be chosen if

$$2^m > 1 + 4 + \binom{4}{2} + \binom{4}{3} = 15.$$

Now let us return to the general case. Suppose we have determined $\mathbf{c}(r_1), \mathbf{c}(r_2), \dots, \mathbf{c}(r_{n-1})$. We then choose $\mathbf{c}(r_n)$ such that

$$\begin{aligned}
&\mathbf{c}(r_n) \neq \mathbf{0}\\
&\mathbf{c}(r_n) \neq \mathbf{c}(r_i), \qquad i = 1, 2, \dots, n - 1\\
&\mathbf{c}(r_n) \neq \mathbf{c}(r_i) + \mathbf{c}(r_j), \qquad i, j = 1, 2, \dots, n - 1, \quad i \neq j\\
&\mathbf{c}(r_n) \neq \mathbf{c}(r_i) + \mathbf{c}(r_j) + \mathbf{c}(r_k), \qquad i, j = 1, 2, \dots, n - 1, \quad i, j, k \text{ distinct}\\
&\quad\vdots\\
&\mathbf{c}(r_n) \neq \mathbf{c}(r_{i_1}) + \mathbf{c}(r_{i_2}) + \cdots + \mathbf{c}(r_{i_{2e-1}}),\\
&\qquad i_1, i_2, \dots, i_{2e-1} = 1, 2, \dots, n - 1, \quad i_1, i_2, \dots, i_{2e-1} \text{ distinct}
\end{aligned}$$

At worst, all of the above combinations are distinct.† The number of

† It is not necessary that all the combinations be distinct; for example, in the parity check matrix (4.5.1) we have $2e = 4$ but several combinations involving three or fewer columns are equal, for example $\mathbf{c}(r_1) + \mathbf{c}(r_8) + \mathbf{c}(r_9) = \mathbf{c}(r_3) + \mathbf{c}(r_4) + \mathbf{c}(r_6)$; $\mathbf{c}(r_2) + \mathbf{c}(r_6) = \mathbf{c}(r_8) + \mathbf{c}(r_9) + \mathbf{c}(r_{10})$.

combinations is

$$1 + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{2e-1};$$

hence the theorem.

In general the smallest value of m (say m_1) satisfying the Varsharmov-Gilbert-Sacks condition exceeds the minimum value of m (say m_2) satisfying the Hamming lower bound. However, it was proved by Sacks (1958) that $m_1/m_2 \leq 2$. It may also be seen from (4.5.2) and (4.5.3) that the upper and lower bounds coincide when $e = 1$. In other words, a single-error correcting code is constructible *if and only if* $2^m \geq 1 + n$. (This fact may be seen directly without reference to Theorem 4.5.3, since for single-error correction we merely need n distinct nonzero columns.)

The bounds of this section say nothing about the probability of error of a code, which is a more basic quantity than the number of errors corrected. However, the results will be useful in Section 4.7, where we give precise bounds on the probability of error for general binary codes.

4.6. Parity check codes are adequate

In this section we show that parity check codes meet the specifications of the fundamental theorem. More precisely, we will prove the following result.

Theorem 4.6.1. Given a binary symmetric channel with capacity $C > 0$ and a positive number $R < C$, there is a sequence of parity check codes (s_i, n_i, λ_i), $i = 1, 2, \ldots,$ where $n_1 < n_2 < \ldots, s_i = 2^{k_i}, R \leq k_i/n_i < C$, and $\lim_{i\to\infty} \lambda_i = 0$.

Thus for each i we can construct a parity check code with at least $2^{n_i R}$ words and a (maximum) probability of error λ_i which approaches zero as $i \to \infty$. Hence, even if we are restricted to the class of parity check codes, we can still maintain the transmission rate $R < C$ with an arbitrarily small probability of error.

Proof. First observe that given a parity check code with 2^k words, we may find a *set of generators* for the code, that is, a set consisting of k code words (say) $\mathbf{w}_1, \ldots, \mathbf{w}_k$, such that each word of the code can be written as a modulo 2 linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_k$. (See the proof of Theorem 4.4.3.) The idea of the proof is to form a parity check code by selecting generators at random, and then to show that the "random-coding" proof of Section 3.5 goes through almost word for word. In particular we consider the following random experiment:

1. Fix R' such that $R < R' < C$. Let k and n be positive integers such that $R \leq k/n \leq R'$. Independently select binary sequences $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ of length n, choosing the components of each sequence independently with 0 and 1 equally likely.

2. Form a code with $s = 2^k$ words† $\mathbf{w}_j, j = 0, 1, \ldots, 2^k - 1$, by taking all possible modulo 2 linear combinations of $\mathbf{w}_1, \ldots, \mathbf{w}_k$; in other words, set $\mathbf{w}_j = b_{1j}\mathbf{w}_1 + \cdots + b_{kj}\mathbf{w}_k \pmod 2$, where $b_{ij} = 0$ or 1, $i = 1, 2, \ldots, k$. (Let $\mathbf{w}_0$ be the zero vector, corresponding to $b_{i0} = 0$ for all i.)

Now if a_{tj} is the tth component of $\mathbf{w}_j$, then

$$a_{tj} = b_{1j}a_{t1} + b_{2j}a_{t2} + \cdots + b_{kj}a_{tk}. \tag{4.6.1}$$

Since the components of the words $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are chosen independently, the a_{tj}, $t = 1, 2, \ldots, n$, are independent. Furthermore $P\{a_{tj} = 0\} = P\{a_{tj} = 1\} = 1/2$ for all t, provided $j \neq 0$. To see this, we observe that there are as many sequences $(a_{t1}, \ldots, a_{tk})$ that produce $a_{tj} = 0$ in (4.6.1) as there are sequences that produce $a_{tj} = 1$. For if $j \neq 0$, at least one coefficient b_{rj} is not zero. If $\sum_{m=1}^{k} b_{mj}a_{tm} = 0$, then define $a_{tm}^* = a_{tm}$, $m \neq r$; $a_{tr}^* = 1 + a_{tr}$. This yields $\sum_{m=1}^{k} b_{mj}a_{tm}^* = 1$. In this way we establish a one-to-one correspondence between sequences that produce $a_{tj} = 0$ in (4.6.1) and sequences that produce $a_{tj} = 1$.

Thus the components of the words $\mathbf{w}_j$, $j \neq 0$, are (statistically) independent and have the distribution that achieves channel capacity (that is, 0 and 1 are equally probable).

By a similar argument, we may show that the words $\mathbf{w}_j$ and $\mathbf{w}_{j'}$, j, $j' \neq 0$, $j \neq j'$, are independent. To accomplish this, we will prove that

$$P\{a_{tj} = \gamma, a_{tj'} = \gamma'\} = 1/4 \quad \text{for} \quad \gamma, \gamma' = 0,1, \quad t = 1, 2, \cdots n. \tag{4.6.2}$$

† Although the words $\mathbf{w}_1, \ldots, \mathbf{w}_k$ chosen in step 1 are statistically independent, a particular choice may result in $\mathbf{w}_1, \ldots, \mathbf{w}_k$ being *linearly* dependent. In this case, the words $\mathbf{w}_j, j = 0, 1, \ldots, 2^k - 1$, will not be distinct, and we obtain a code which strictly speaking is not a parity check code as defined in Section 4.3. If, however, the number of distinct words of the code is 2^r, $r < k$, and $\mathbf{w}_1', \ldots, \mathbf{w}_r'$ is a set of r linearly independent words, we may select sequences $\mathbf{w}_{r+1}', \ldots, \mathbf{w}_k'$ such that $\mathbf{w}_1', \ldots, \mathbf{w}_k'$ are linearly independent, and in this way construct a parity check code with 2^k distinct words. It can be shown (see Problem 4.11) that the average probability of error of the new code cannot exceed the average probability of error of the original code. Hence if the random experiment described above yields a code with average probability of error ε, we may always find a parity check code with the same number of words and an average probability of error $\leq \varepsilon$.

We note also that in this proof and elsewhere, "independent" will mean "statistically independent"; the term "linearly independent" will not be abbreviated.

It then follows that $P\{a_{tj} = \gamma, a_{tj'} = \gamma'\} = P\{a_{tj} = \gamma\} P\{a_{tj'} = \gamma'\}$; hence a_{tj} and $a_{tj'}$ are independent for each t, which implies the independence of $\mathbf{w}_j$ and $\mathbf{w}_{j'}$.

To prove (4.6.2), we note that since $j \neq j'$, we have $b_{rj} \neq b_{rj'}$ for some r. Say $b_{rj} = 1$ and $b_{rj'} = 0$. Then given any sequence $(a_{t1}, \ldots, a_{tk})$ such that

$$\sum_{m=1}^{k} b_{mj} a_{tm} = 0, \qquad \sum_{m=1}^{k} b_{mj'} a_{tm} = 0,$$

we define $a_{tm}{}^* = a_{tm}$, $m \neq r$; $a_{tr}{}^* = 1 + a_{tr}$. It follows that

$$\sum_{m=1}^{k} b_{mj} a_{tm}{}^* = 1, \qquad \sum_{m=1}^{k} b_{mj'} a_{tm}{}^* = 0.$$

This establishes a one-to-one correspondence between sequences that produce $a_{tj} = a_{tj'} = 0$ in (4.6.1) and sequences that produce $a_{tj} = 1$, $a_{tj'} = 0$. Thus

$$P\{a_{tj} = 0, a_{tj'} = 0\} = P\{a_{tj} = 1, a_{tj'} = 0\}. \tag{4.6.3}$$

But

$$P\{a_{tj} = 0, a_{tj'} = 0\} + P\{a_{tj} = 1, a_{tj'} = 0\} = P\{a_{tj'} = 0\} = 1/2,$$

so that both terms of (4.6.3) are 1/4.

In a similar fashion, we establish the other relations of (4.6.2). Note that although the $\mathbf{w}_j$ are *pairwise* independent, it is not true that the entire set $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{s-1})$ is independent. However, $\mathbf{w}_1, \ldots, \mathbf{w}_k$ do form an independent set.

Now let us try to reproduce the "random-coding" proof of Section 3.5. The role of the code words $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(s)}$ in that proof is played by $\mathbf{w}_1, \ldots, \mathbf{w}_{s-1}$. The constant a in the definition of the set A is taken as $\frac{1}{2}n(R' + C)$. If $\mathbf{x}$ is any binary sequence of length n, we have shown that $P\{\mathbf{w}_j = \mathbf{x}\} = (1/2)^n$, which is the probability determined by selecting independently the components of the input sequence, each component chosen with the distribution that achieves capacity. The random-coding proof may be taken over word for word from Section 3.5, except for two difficulties. In the original proof, the code words were independent. However, as we observed at the time [see the discussion preceding (3.5.11)], only pairwise independence is required.

A slight difficulty is also caused by our leaving out the zero code word. Let S_1 be the code consisting of the words $\mathbf{w}_1, \ldots, \mathbf{w}_{s-1}$, with the decoding sets B_i, $1 \leq i \leq s - 1$, determined by the minimum-distance decoding scheme. Let $\bar{\varepsilon}_1$ be the average probability of error of S_1. Let S_2 be the code formed by adding the zero word to S_1. However, instead of using the minimum-distance decoder for S_2, take the decoding set of $\mathbf{w}_i$ to be B_i

when $i \neq 0$, and take the decoding set of $\mathbf{w}_0$ to be the empty set (so that an error is always made when $\mathbf{w}_0$ is transmitted). The average probability of error of S_2 is then

$$\bar{\varepsilon}_2 = \frac{1}{s}\sum_{i=0}^{s-1} p(e \mid \mathbf{w}_i) = \frac{1}{s} + \frac{s-1}{s}\,\bar{\varepsilon}_1.$$

Finally, let $S_2{}^*$ be the code S_2 with the minimum-distance decoder, and let ε^* be the average probability of error of $S_2{}^*$. Since the minimum-distance decoder corresponds to the ideal observer with all words equally likely, $\varepsilon^* \leq \bar{\varepsilon}_2$. The random-coding proof shows that $E[\bar{\varepsilon}_1] \to 0$ as $n \to \infty$.

Since $s = 2^k \geq 2^{nR} \to \infty$ as $n \to \infty$, $E[\varepsilon^*] \to 0$ as $n \to \infty$. Since $R < C$, we can find integers k_i and n_i, $i = 1, 2, \ldots$, such that $n_1 < n_2 < \cdots$ and $R \leq k_i/n_i < R' < C$. Thus the above procedure yields codes that maintain the transmission rate R and have an arbitrarily small probability of error. The theorem is proved.

4.7. *Precise error bounds for general binary codes*

In Chapter 3 we showed that we could construct a sequence of codes $([2^{nR}], n, \lambda_n)$ with $\lambda_n \to 0$ at an exponential rate. In this section we obtain more precise information about the rate of approach of λ_n to zero. We shall need a few combinatorial results.

Lemma 4.7.1. If $0 < p < 1$, $q = 1 - p$, and np is an integer, then

$$(8npq)^{-1/2} \cdot 2^{nH(p,q)} \leq \binom{n}{np} \leq (2\pi npq)^{-1/2} \cdot 2^{nH(p,q)}. \tag{4.7.1}$$

Proof. We use *Stirling's formula:*

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \cdots\right).$$

If no terms of the exponential series are taken, $n!$ is underestimated; if one term is taken, $n!$ is overestimated, etc. To establish the left-hand inequality, we write

$$\binom{n}{np} = \frac{n!}{(np)!\,(nq)!} \geq \frac{\sqrt{2\pi n}\, n^n e^{-n} \exp\left(-1/12np - 1/12nq\right)}{\sqrt{2\pi np}\,(np)^{np} e^{-np} \sqrt{2\pi nq}\,(nq)^{nq} e^{-nq}}.$$

[We have underestimated $n!$ and overestimated $(np)!$ and $(nq)!$.] Assume that $np \geq 1$ and $nq \geq 3$. [By the symmetry in p and q of the expression (4.7.1), the only remaining cases to consider are $np = 1$, $nq = 1$; $np = 1$,

$nq = 2$; $np = 2, nq = 2$; for these cases, the inequality may be verified by direct substitution.] Then

$$\frac{1}{12np} + \frac{1}{12nq} \leq \frac{1}{12} + \frac{1}{36} = \frac{1}{9},$$

and

$$\exp\left(-\frac{1}{12np} - \frac{1}{12nq}\right) \geq e^{-1/9} = 0.895 \geq \tfrac{1}{2}\sqrt{\pi}.$$

Hence

$$\binom{n}{np} \geq (8npq)^{-1/2} p^{-np} q^{-nq} = (8npq)^{-1/2} 2^{-n(p \log p + q \log q)}$$

which proves the left-hand inequality. To establish the right-hand side, we overestimate $n!$ and underestimate $(np)!$ and $(nq)!$, as follows:

$$\binom{n}{np} \leq \frac{\sqrt{2\pi n}\, n^n e^{-n} e^{1/12n}}{\sqrt{2\pi np}\,(np)^{np} e^{-np} \sqrt{2\pi nq}(nq)^{nq} e^{-nq}} \times \exp\left[\frac{1}{12np} - \frac{1}{360(np)^3} + \frac{1}{12nq} - \frac{1}{360(nq)^3}\right]$$

Assume $p \geq q$; then $1/360(np)^3 \leq 1/360np$ since $np \geq 1$; similarly $1/360(nq)^3 \leq 1/360nq$. Thus

$$\frac{1}{12n} - \frac{1}{12np} - \frac{1}{12nq} + \frac{1}{360(np)^3} + \frac{1}{360(nq)^3} \leq -\frac{1}{12nq} + \frac{1}{360np} + \frac{1}{360nq} \leq (\text{since } p \geq q) - \frac{1}{12nq} + \frac{1}{180nq} \leq 0.$$

Thus

$$\binom{n}{np} \leq (2\pi npq)^{-1/2} p^{-np} q^{-nq} = (2\pi npq)^{-1/2} \cdot 2^{nH(p,q)}.$$

Since the upper bound is symmetrical in p and q, the restriction $p \geq q$ may be dropped; this completes the proof.

Lemma 4.7.2. Let $\Psi_p(n, \lambda)$ be the "tail" of a binomial distribution (with parameters n and p) from λn to n, that is,

$$\Psi_p(n, \lambda) = \sum_{k=\lambda n}^{n} \binom{n}{k} p^k q^{n-k}$$

Now as in Theorems 5.3.1 and 5.3.2, $x^n - 1 = h(x)g(x)$, where $g(x) = \sum_{i=0}^{m} a_i x^i$ is the generator polynomial of the code and $h(x) = \sum_{i=0}^{k} h_i x^i$ is the polynomial whose coefficients appear in the parity check matrix A of (5.3.1). If we compare coefficients of $x^n - 1$ and $h(x)g(x)$, we obtain the identities

$$a_0 h_{n-r} + a_1 h_{n-r-1} + \cdots + a_{m-1} h_{k-r+1} + a_m h_{k-r} = 0, \qquad r = 1, 2, \ldots, n-1 \tag{5.3.2}$$

(define $a_i = 0$ if $i < 0$ or $i > m$; define $h_j = 0$ if $j < 0$ or $j > k$).
Now let

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{m-1} \end{bmatrix}$$

and take

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}.$$

We claim that the parity check matrix A of (5.3.1) is

$$[\mathbf{x} \mid T\mathbf{x} \mid T^2\mathbf{x} \mid \cdots \mid T^{n-1}\mathbf{x}].$$

We may prove this by induction. The first column of A is clearly $\mathbf{x}$. Now suppose that the rth column of A is $T^{r-1}\mathbf{x}$. We note that

$$T \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{m-2} \\ y_{m-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{m-1} \\ z \end{bmatrix} \qquad \text{where } z = a_0 y_0 + \cdots + a_{m-1} y_{m-1}.$$

By the induction hypothesis,

$$T^r\mathbf{x} = TT^{r-1}\mathbf{x} = T(\text{column } r \text{ of } A)$$

$$= T\begin{bmatrix} h_{n-r} \\ h_{n-r-1} \\ \cdot \\ \cdot \\ \cdot \\ h_{k-r+2} \\ h_{k-r+1} \end{bmatrix} = \begin{bmatrix} h_{n-(r+1)} \\ h_{n-(r+2)} \\ \cdot \\ \cdot \\ \cdot \\ h_{k-r+1} \\ a_0 h_{n-r} + a_1 h_{n-r-1} + \cdots + a_{m-1} h_{k-r+1} \end{bmatrix}.$$

By (5.3.2),

$$T^r\mathbf{x} = \begin{bmatrix} h_{n-(r+1)} \\ h_{n-(r+2)} \\ \cdot \\ \cdot \\ \cdot \\ h_{k-r+1} \\ h_{k-r} \end{bmatrix} = \text{column } r+1 \text{ of } A.$$

Finally, since $x^n - 1 = h(x)g(x)$, we have $h_0 = a_0 = 1$; thus the last column of A is

$$T^{n-1}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}.$$

Consequently, $$T^n\mathbf{x} = \text{column 1 of } T$$

$$= \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} = \mathbf{x}.$$

Since a_0 is not zero, T is nonsingular and the theorem is proved.

COROLLARY 5.3.4.1. a. The class of cyclic codes (with the exception of the code consisting of the entire space) coincides with the class of parity check codes whose parity check matrices are of the form

$$[\mathbf{x} \mid T\mathbf{x} \mid \cdots \mid T^{n-1}\mathbf{x}],$$

where T is a nonsingular matrix of the form (5.1.1),

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix},$$

and $T^n\mathbf{x} = \mathbf{x}$.

b. For any such code S in the latter class (excluding the degenerate case in which S consists of the zero vector alone), the minimal polynomial of T may be identified with the generator polynomial $g(x)$ of the code. If $x^n - 1 = h(x)g(x)$, then the parity check matrix $[\mathbf{x} \mid T\mathbf{x} \mid \cdots \mid T^{n-1}\mathbf{x}]$ coincides with the matrix (5.3.1).

Proof. Part (a) is an immediate consequence of the argument used in the proof of Theorem 5.3.4, along with Theorem 5.1.1. To prove (b), let $A = [\mathbf{x} \mid T_1\mathbf{x} \mid \cdots \mid T_1^{n-1}\mathbf{x}]$ be the parity check matrix of a cyclic code S, where T_1 is a nonsingular matrix of the form (5.1.1),

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad T_1{}^n\mathbf{x} = \mathbf{x}.$$

If T_1 is m by m, it may be verified directly that the m by m submatrix formed by the first m columns of A is triangular, and therefore A has rank m. If $n = m$, then the code consists of the zero vector alone, so assume $n > m$. Since T_1 is of the form (5.1.1), Lemma 5.2.1 implies that the minimal polynomial of T_1 is of degree m. By Lemma 5.3.3b, the minimal polynomial of T_1 may be identified with the generator polynomial $g(x)$ of the code. It follows that the matrix T which corresponds to S in the argument of the proof of Theorem 5.3.4 is the same as T_1. The final assertion of (b) now follows.

As an example, consider the code associated with Fig. 5.2.1. Since the first column of the matrix A of that figure is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

A is of the form (5.3.1). The generator polynomial of the code is $g(x) = 1 + x^3 + x^4$. Also, $x^{15} - 1 (= x^{15} + 1) = h(x)g(x)$, where $h(x)$, determined from the first row of A, is $1 + x^3 + x^4 + x^6 + x^8 + x^9 + x^{10} + x^{11}$. The code word $\mathbf{w} = (100110000000000)$ corresponding to $g(x)$, together with 10 cyclic shifts of $\mathbf{w}$, generate the code, that is, form a maximal linearly independent set of code words.

As another example, the parity check matrix

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

generated by the matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(see Fig. 5.1.2) is of the form (5.3.1). The generator polynomial of the code is $g(x) = 1 + x^2 + x^4$, and $x^6 - 1 = h(x)g(x)$ where $h(x)$, determined from the first row of A_1, is $1 + x^2$. The words (101010) and (010101), corresponding to $g(x)$ and $xg(x)$, generate the code.

As a final example, let

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

be the parity check matrix whose columns consist of two repetitions of the cycle of length 3 in Fig. 5.1.2 (see p. 138). The minimal polynomial of the matrix T of Fig. 5.1.2 is $f(\lambda) = 1 + \lambda^2 + \lambda^4$, but the rank of A_4 is only 2. By Lemma 5.3.3, the generator polynomial $g(x)$ is a proper factor of $f(x)$, necessarily of degree 2 since the degree of the generator coincides with the rank of the parity check matrix. Since

$$f(x) = (1 + x + x^2)^2,$$

we have $g(x) = 1 + x + x^2$. Hence a maximal linearly independent set of code words consists of (111000), (011100), (001110), and (000111).

Since the number of check digits is 2 and the generator polynomial is $1 + x + x^2$, a parity check matrix A may be generated by the matrix $T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since the code word length is 6, we have

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(which coincides with the first two rows of A_4). Thus $h(x) = 1 + x + x^3 + x^4$.

5.4. *Bose-Chaudhuri-Hocquenghem codes*

In this section we describe a class of multiple-error correcting codes that are very easy to construct and that perform very well at least up to word lengths of several thousand.

The procedure to be described yields, for any e and q, an e-tuple-error correcting code with word length $n = 2^q - 1$. Let T be a q by q matrix of the form (5.1.1). Assume that the minimal polynomial of T satisfies the conditions of Theorem 5.2.6 so that T has maximal period. The parity check matrix A_0 of the Bose-Chaudhuri-Hocquenghem code is shown in Table 5.4.1 ($\mathbf{x}$ is any nonzero vector).

As an example, suppose $e = 3$, $q = 4$, and the minimal polynomial of T is $f(\lambda) = \lambda^4 + \lambda + 1$. The associated Bose-Chaudhuri-Hocquenghem code is shown in Table 5.4.2.

It may be verified that the Bose-Chaudhuri-Hocquenghem codes are cyclic (see Problem 5.3). To establish that the matrix A_0 does in fact define an e-tuple-error correcting code, we prove the following theorem.

Theorem 5.4.1. Every set of $2e$ columns of the Bose-Chaudhuri-Hocquenghem matrix A_0 is linearly independent.

Table 5.4.1. **Parity check matrix of the Bose-Chaudhuri-Hocquenghem code**

$$A_0 = \left[\begin{array}{c|c|c|c|c|c}
\mathbf{x} & T\mathbf{x} & T^2\mathbf{x} & T^3\mathbf{x} & \cdots & T^{n-1}\mathbf{x} \\
\mathbf{x} & T^3\mathbf{x} & T^6\mathbf{x} & T^9\mathbf{x} & \cdots & T^{3(n-1)}\mathbf{x} \\
\mathbf{x} & T^5\mathbf{x} & T^{10}\mathbf{x} & T^{15}\mathbf{x} & \cdots & T^{5(n-1)}\mathbf{x} \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\mathbf{x} & T^{2e-1}\mathbf{x} & T^{4e-2}\mathbf{x} & T^{6e-3}\mathbf{x} & \cdots & T^{(2e-1)(n-1)}\mathbf{x}
\end{array}\right]$$

A_0 is qe by n, $\qquad n = 2^q - 1$

Proof. Suppose the columns headed by $T^{k_1}\mathbf{x}, T^{k_2}\mathbf{x}, \ldots, T^{k_r}\mathbf{x}$ add to zero, with $r \leq 2e$. Then

$$\begin{array}{llll}
T^{k_1}\mathbf{x} & + T^{k_2}\mathbf{x} & + \cdots + T^{k_r}\mathbf{x} & = \mathbf{0} \\
(T^{k_1})^3\mathbf{x} & + (T^{k_2})^3\mathbf{x} & + \cdots + (T^{k_r})^3\mathbf{x} & = \mathbf{0} \\
\cdot & & & \\
\cdot & & & \\
\cdot & & & \\
(T^{k_1})^{2e-1}\mathbf{x} & + (T^{k_2})^{2e-1}\mathbf{x} & + \cdots + (T^{k_r})^{2e-1}\mathbf{x} & = \mathbf{0}
\end{array} \tag{5.4.1}$$

Table 5.4.2. **Parity check matrix for a triple-error correcting Bose-Chaudhuri-Hocquenghem code**

$$A_0 = \begin{bmatrix}
1&0&0&0&1&0&0&1&1&0&1&0&1&1&1 \\
0&0&0&1&0&0&1&1&0&1&0&1&1&1&1 \\
0&0&1&0&0&1&1&0&1&0&1&1&1&1&0 \\
0&1&0&0&1&1&0&1&0&1&1&1&1&0&0 \\
1&0&0&0&1&1&0&0&0&1&1&0&0&0&1 \\
0&1&1&1&1&0&1&1&1&1&0&1&1&1&1 \\
0&0&1&0&1&0&0&1&0&1&0&0&1&0&1 \\
0&0&0&1&1&0&0&0&1&1&0&0&0&1&1 \\
1&0&1&1&0&1&1&0&1&1&0&1&1&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&1&0&1&1&0&1&1&0&1&1&0&1&1 \\
0&1&1&0&1&1&0&1&1&0&1&1&0&1&1
\end{bmatrix}$$

$$T = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&1&0&0 \end{bmatrix}$$

$$f(\lambda) = \lambda^4 + \lambda + 1$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Each equation of (5.4.1) is of the form $p(T)\mathbf{x} = \mathbf{0}$ where $p(T)$ is a matric polynomial. We claim that $p(T)$ must be zero, for if $p(T)$ were nonzero, then $p(T)$ would be nonsingular by Lemma 5.2.3. If the equation $p(T)\mathbf{x} = \mathbf{0}$ is premultiplied by the inverse of $p(T)$ we obtain $\mathbf{x} = \mathbf{0}$, a contradiction.

Thus we may "get rid of" the vector $\mathbf{x}$ in (5.4.1). If for the sake of abbreviation we replace T^{k_i} by y_i, we obtain

$$\begin{aligned} &y_1 + y_2 + \cdots + y_r = 0 \\ &y_1^3 + y_2^3 + \cdots + y_r^3 = 0 \\ &y_1^5 + y_2^5 + \cdots + y_r^5 = 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ &y_1^{2e-1} + y_2^{2e-1} + \cdots + y_r^{2e-1} = 0 \end{aligned} \tag{5.4.2}$$

We claim that $y_1^{2k} + y_2^{2k} + \cdots + y_r^{2k} = 0$ for $k = 1, 2, \ldots, e$. To see this, we square both sides of the first equation of (5.4.2) and note that the cross terms $y_iy_j + y_iy_j$ are zero (modulo 2). Hence

$$(y_1 + \cdots + y_r)^2 = y_1^2 + y_2^2 + \cdots + y_r^2.$$

[What we are saying here is that if $p(S)$ is any polynomial in the binary matrix S, then $[p(S)]^2 = p(S^2)$.] Similarly, to obtain $y_1^4 + \cdots + y_r^4 = 0$ we square both sides of $y_1^2 + \cdots + y_r^2 = 0$; to obtain $y_1^6 + \cdots + y_r^6 = 0$, we square both sides of $y_1^3 + \cdots + y_r^3 = 0$, etc. Thus we have

$$\begin{aligned} &y_1 + y_2 + \cdots + y_r = 0 \\ &y_1^2 + y_2^2 + \cdots + y_r^2 = 0 \\ &y_1^3 + y_2^3 + \cdots + y_r^3 = 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ &y_1^r + y_2^r + \cdots + y_r^r = 0. \end{aligned} \tag{5.4.3}$$

We write only r equations in (5.4.3) so as to obtain a square array. The equations (5.4.3) may be written in matrix form as follows:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_r \\ y_1^2 & y_2^2 & \cdots & y_r^2 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ y_1^{r-1} & y_2^{r-1} & \cdots & y_r^{r-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{5.4.4}$$

Equation (5.4.4) is of the form $B\mathbf{y} = \mathbf{0}$, where the components of the matrix B and the vector $\mathbf{y}$ are elements in the Galois field GF(2^q) (Theorem 5.2.4). The elements $y_1, \ldots, y_r$ are nonzero; hence *the equations* $B\mathbf{z} = \mathbf{0}$ *have a nontrivial solution in the Galois field* GF(2^q). Consequently the determinant of B is zero. But det B is a standard determinant called the *Vandermonde* determinant; its value is $\Pi_{i>j}(y_i - y_j)$.

[To see this, note that det B is a homogeneous polynomial $g(y_1, y_2, \ldots, y_r)$ of degree $1 + 2 + \cdots + (r - 1) = r(r - 1)/2$. If $y_i = y_j$, then columns i and j of det B are identical, hence det $B = 0$. Thus for all $i \neq j$, $y_i - y_j$ is a divisor of $g(y_1, \ldots, y_r)$. Since the degree of

$$\prod_{i>j} (y_i - y_j)$$

is $r(r - 1)/2$, the result follows, up to a constant factor.]

Now in the present case the elements $y_1, y_2, \ldots y_r$, being different powers of the matrix T, are distinct. (See the remarks after Theorem 5.2.6.) Hence none of the products $y_i - y_j$, $i > j$, is zero. The conclusion is that det $B \neq 0$, a contradiction. Thus our assumption that a set of $2e$ or fewer columns is linearly dependent is untenable. Notice that if r were greater than $2e$ the above proof would break down, since we would have no way of proving that

$$y_1^{2e+1} + y_2^{2e+1} + \cdots + y_r^{2e+1} = 0.$$

The matrix A_0 of Table 5.4.2 is 12 by 15, but the rank of A_0 is less than 12 since row 10 of the matrix consists entirely of zeros, and rows 11 and 12 are identical. It is possible to give an explicit and computationally feasible procedure for obtaining the rank of A_0. First we establish a preliminary result.

Lemma 5.4.2. Let T be a binary matrix whose minimal polynomial is irreducible. If $p(T)$ is any matric polynomial in T such that $p(T)$ is not the zero matrix, then the minimal polynomial of $p(T)$ is also irreducible.

Proof. If the minimal polynomial $f(\lambda)$ of $p(T)$ is not irreducible, then there are nonzero polynomials $g(\lambda)$ and $h(\lambda)$, of lower degree than $f(\lambda)$, such that $f(\lambda) = g(\lambda)h(\lambda)$. Replacing λ by $S = p(T)$ we obtain $g(S)h(S) = f(S) = 0$. Now $g(S) = g(p(T))$ may be expressed as a matric polynomial in T; since $g(S)$ is not zero, we may (by Lemma 5.2.3) premultiply by the inverse of $g(S)$ to obtain $h(S) = 0$, which contradicts the minimality of $f(\lambda)$. This proves the lemma. [For those familiar with algebra, what we are saying here is that the minimal polynomial of any element α of a field F_1, such that α is algebraic over a field $F \subset F_1$, is irreducible. In this case, $\alpha = p(T)$, F is the modulo 2 field, and F_1 is the Galois field GF(2^q) where q is the degree of the minimal polynomial of T. (See Theorem 5.2.4.)]

The next theorem allows the precise determination of the rank.

Theorem 5.4.3. Let A_0 be the Bose-Chaudhuri-Hocquenghem parity check matrix. If $f_i(\lambda)$ is the minimal polynomial of T^i ($i = 1, 3, 5, \ldots 2e - 1$), and $f(\lambda)$ is the least common multiple of the $f_i(\lambda)$, that is, the polynomial of lowest degree which is a multiple of each $f_i(\lambda)$, then the rank of A_0 is the degree of $f(\lambda)$. Furthermore, if $i \neq j$, then $f_i(\lambda)$ and $f_j(\lambda)$ are either relatively prime or identical, so that $f(\lambda)$ is the product of the distinct members of the set $\{f_i(\lambda), i = 1, 3, \ldots, 2e - 1\}$.

Proof. Suppose that we have a finite collection of cyclic codes with parity check matrices A_i and generator polynomials $g_i(x)$, each code having words of length n. A binary sequence $\mathbf{w}$ represented by the polynomial $b(x)$ satisfies $A_i\mathbf{w}^T = \mathbf{0}$ for all i if and only if $b(x)$ is a multiple of each $g_i(x)$. Now it is immediate from the definition of a cyclic code that the intersection of the codes defined by the matrices A_i is cyclic. Furthermore, the above argument shows that the associated generator polynomial $g(x)$ is the least common multiple of the $g_i(x)$. To obtain the Bose-Chaudhuri-Hocquenghem codes we may take

$$A_i = [\mathbf{x} \mid T^i\mathbf{x} \mid T^{2i}\mathbf{x} \mid \cdots \mid T^{(n-1)i}\mathbf{x}],$$
$$i = 1, 3, \ldots, 2e - 1;$$

by Lemma 5.4.2, the minimal polynomial $f_i(\lambda)$ of T^i is irreducible; by Lemma 5.3.3c, $f_i(x) = g_i(x)$ for each i, and consequently $f(x) = g(x)$. Since the rank of the parity check matrix of a cyclic code is the degree of the generator polynomial of the code (see Theorems 5.3.1 and 5.3.2), the first part of the theorem is proved. Now if $f_i(\lambda)$ and $f_j(\lambda)$ are not identical, they must be relatively prime, for if $f_i(\lambda)$ and $f_j(\lambda)$ were to have a nontrivial common factor, the polynomials could not be irreducible, a contradiction. The theorem is proved.

For the code of Table 5.4.2, we find (see Problem 5.9) that

$$f_1(\lambda) = 1 + \lambda + \lambda^4, \qquad f_3(\lambda) = 1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4,$$
$$f_5(\lambda) = 1 + \lambda + \lambda^2,$$

so that the rank of A_0 is 10. The rank of the submatrix A_i, which by the above argument is the degree of $f_i(\lambda)$, is 4 for $i = 1$ and 3, and 2 for $i = 5$. Thus two rows of the matrix A_5 must be linearly dependent on the other rows of A_5, which agrees with our previous observations that row 10 of A_0 contains only zeros, and row 11 is identical with row 12.

A procedure for obtaining the minimal polynomial of the matrix T^i without actually computing the powers of T is indicated in the solution to Problem 5.9.

If $2^q - 1$ is prime, then the rank of each matrix A_i is q. (As above, Lemma 5.4.2 implies that the minimal polynomial of T^i is irreducible; Theorem 5.2.5 then implies that T^i has maximal period, and consequently the columns of each A_i constitute a permutation of the set of all $2^q - 1$ nonzero q-vectors. Thus all the matrices A_i have the same rank. But the minimal polynomial of T is of degree q; therefore A_1 has rank q.) However, whether or not $2^q - 1$ is prime, it is possible for $f_i(\lambda)$ and $f_j(\lambda)$ to coincide when $i \neq j$. For example, if $q = 5$, then T^5 and T^9 have the same minimal polynomial. For if $p(S)$ is any polynomial in the binary matrix S, $p(S^2) = [p(S)]^2$. [See the discussion after (5.4.2).] Thus if $p(T^5) = 0$, then $p(T^{10}) = p(T^{20}) = p(T^{40}) = p(T^9) = 0$. (Note $T^{2^q-1} = T^{31} = I$). Similarly, if $p(T^9) = 0$, then $p(T^{18}) = p(T^{36}) = p(T^5) = 0$. Thus the set of polynomials annulled by T^5 coincides with the set of polynomials annulled by T^9. Similarly, if $q = 6$, we may verify that T^{13} and T^{19} have the same minimal polynomial.

5.5. *Single-error correcting cyclic codes; automatic decoding*

A shift register may be used to generate a cyclic code that corrects single errors and at the same time entirely eliminates the storage problem

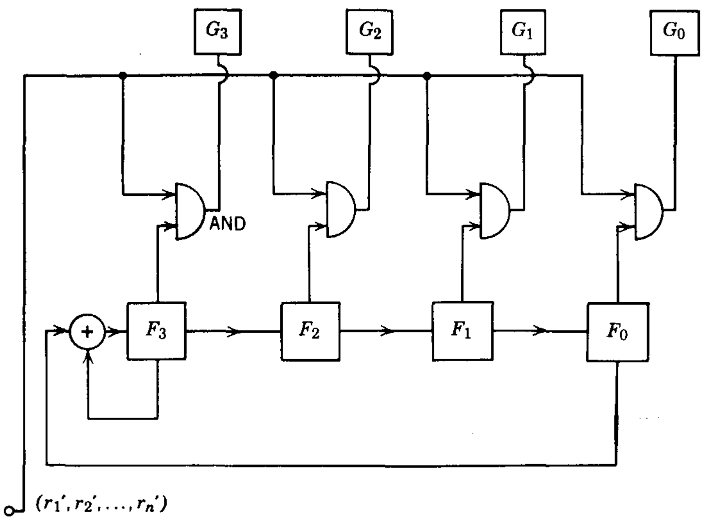


Fig. 5.5.1. Automatic decoder for the cyclic code of Fig. 5.2.1.

at the decoder. As an example, consider the shift register and associated code of Fig. 5.2.1. An automatic decoding circuit may be constructed as shown in Fig. 5.5.1. The original shift register of Fig. 5.2.1 is combined, through a number of AND gates, with another set of binary storage devices G_0, G_1, G_2, G_3. The G_i are constructed so that their state changes if a "1" input is applied. In other words the content of the G register after application of an input

$$\mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is the modulo 2 sum of $\mathbf{y}$ and the content of G before application of the input.

In order to examine the operation of the decoding circuit, suppose that the received digits $r_1', r_2', \ldots, r_n'$ arrive at times $t = 1, 2, \ldots, n$ respectively. The columns of the parity check matrix are $\mathbf{x}$, $T\mathbf{x}$, $T^2\mathbf{x}, \ldots,$ $T^{n-1}\mathbf{x}$, where $\mathbf{x}$ is some nonzero vector.

$$\left(\text{In Fig. 5.2.1, } \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.\right)$$

The F shift register is adjusted so that its content at time $t = 0$ is $\mathbf{x}$. The G register is initially set to have content zero. At time $t = 1$, digit r_1' appears on the input line. If $r_1' = 0$, the AND gates will be closed and the G register will still have content zero. If $r_1' = 1$, the AND gates will allow the content of the F register to be transferred to the G register. In short, the content of G at $t = 1$ will be $r_1'\mathbf{x}$. At $t = 2$, F has content $T\mathbf{x}$ and r_2' appears at the input line; hence the input to G is $r_2'T\mathbf{x}$. The content of G at $t = 2$ is therefore $r_1'\mathbf{x} + r_2'T\mathbf{x}$. Continuing in this fashion, the content of G at $t = n$ is

$$r_1'\mathbf{x} + r_2'T\mathbf{x} + \cdots + r_n'T^{n-1}\mathbf{x},$$

which is the corrector $\mathbf{c}$ associated with $r_1', r_2', \ldots, r_n'$. If $\mathbf{c} = \mathbf{0}$, the received sequence is accepted as the input code word. If $\mathbf{c} \neq \mathbf{0}$, we employ a comparison circuit (not shown in Fig. 5.5.1) to determine which digit is in error. The corrector is compared with the content of F as F cycles through the entire set of columns of the parity check matrix. If there is a correspondence at column j, that is, if $\mathbf{c} = T^{j-1}\mathbf{x}$, then a single error in digit j is assumed.

This brief discussion does not do justice to the subject. For further details concerning efficient encoding and decoding of cyclic codes, see Peterson (1961).

5.6. *Notes and remarks*

Cyclic codes are treated in detail by Peterson (1961), who uses the machinery of abstract algebra. Theorems 5.3.1 and 5.3.2 form the basis of Peterson's approach.

A feedback shift register is a special case of a device called a "linear sequential circuit"; see Elspas (1959), Friedland (1959), and Friedland and Stern (1959). Such devices were applied to the construction of error correcting codes by Abramson (1961), Melas (1960), and others, but Theorem 5.3.4 does not seem to have been stated explicitly anywhere.

The results of Section 5.4 are due to Bose and Chaudhuri (1960) and Hocquenghem (1959). Bose-Chaudhuri-Hocquenghem codes can be constructed in the nonbinary case, but for the proof of Theorem 5.4.1 to remain valid, additional blocks generated by $T^2\mathbf{x}$, $T^4\mathbf{x}$, . . . , $T^{2e}\mathbf{x}$ must be added to A_0. Thus in the nonbinary case the number of check digits for a given word length is greatly increased.

Very little is known about the asymptotic properties of Bose-Chaudhuri-Hocquenghem codes. According to the data of Peterson (1961), the relationship between the word length n, the number of check digits m, and the number of errors corrected e, is approximately that of the Varsharmov-Gilbert-Sacks condition for values of n up to 1000. If this behavior were to be maintained for arbitrarily large n, the asymptotic properties of the codes would be very desirable. (See the remarks at the end of Section 4.10). However, Peterson (1960) has proved that if e/n is kept constant, the transmission rate $R = k/n$ (k = number of information digits) approaches zero as n approaches infinity. But this does not necessarily imply that the asymptotic properties are bad. The difficulty is that we have proved that the errors corrected by the codes include *at least* the errors of magnitude e or smaller. It is possible that sufficiently many error patterns of weight greater than e are corrected to allow a positive transmission rate to be maintained with an arbitrarily small probability of error.

PROBLEMS

5.1 Give a parity check matrix for a cyclic code generated by a single cycle of a feedback shift register whose characteristic polynomial is

a. $\lambda^4 + \lambda + 1$

b. $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$

c. $\lambda^5 + \lambda^4 + 1$ (three possible codes).

5.2 Find the generator polynomial $g(x)$ of the code in Problem 5.1c which uses words of length 7, and find the polynomial $h(x)$ such that $h(x)g(x) = x^7 - 1$.

5.3 Show that the Bose-Chaudhuri-Hocquenghem codes are cyclic.

5.4 Give the parity check equations for a double-error correcting Bose-Chaudhuri-Hocquenghem code with a code-word length of 31.

5.5 The binary sequence $\mathbf{w} = (1010011)$ is a code word of a cyclic code with 3 information digits.

a. Find a parity check matrix for the code, and a feedback shift register that will generate the code.

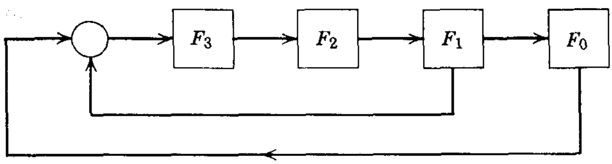
b. Find the generator polynomial $g(x)$ of the code, and the polynomial $h(x)$ such that $h(x)g(x) = x^7 - 1$.

5.6 [The application of Cyclic Codes for Burst-Error Correction; Melas (1960).] An *error burst* of width $q + 1$ beginning at digit i is an error pattern such that r_i and r_{i+q} are in error, a digit r_j $(i < j < i + q)$ may or may not be in error, and all digits before r_i or after r_{i+q} are correct. For example, the error bursts of width 4 may be represented by the following binary sequences:

1001	r_i and r_{i+3} in error
1011	r_i, r_{i+2}, r_{i+3} in error
1101	r_i, r_{i+1}, r_{i+3} in error
1111	$r_i, r_{i+1}, r_{i+2}, r_{i+3}$ in error.

Binary sequences like these, which describe the distribution of errors within a burst without regard to the position of the burst within the code word, are called *burst-error patterns*. These are 2^{q-1} burst-error patterns of width $q + 1$.

Now consider a code generated by a feedback shift register with maximal period, as shown below.



$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\varphi(\lambda) = \lambda^4 + \lambda + 1$$
$$T^4 + T + I = 0$$

$$\begin{array}{c} \begin{array}{ccccccccccccccc} \mathbf{x} & T\mathbf{x} & T^2\mathbf{x} & T^3\mathbf{x} & & & & & \cdots & & & & & T^{13}\mathbf{x} & T^{14}\mathbf{x} \end{array} \\ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Consider a typical burst-error pattern, say 1101. If the burst begins at digit 1, the associated corrector is

$$(I + T + T^3)\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = T^7\mathbf{x}.$$

If the burst begins at digit 2, the corrector is

$$(T + T^2 + T^4)\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = T^8\mathbf{x}.$$

If the burst begins at digit 5, the corrector is $(T^4 + T^5 + T^7)\mathbf{x} = T^{11}\mathbf{x}$, etc.

a. Show that a translation of an error burst by t digits results in a translation of the associated corrector by t digits. All possible burst-error patterns of width 4 or less are listed below, along with the associated correctors (assuming that the burst begins at digit 1).

Burst-Error Pattern	*Corrector*
1	$\mathbf{x}$
11	$(I + T)\mathbf{x} = T^4\mathbf{x}$
101	$(I + T^2)\mathbf{x} = T^8\mathbf{x}$
111	$(I + T + T^2)\mathbf{x} = T^{10}\mathbf{x}$
1001	$(I + T^3)\mathbf{x} = T^{14}\mathbf{x}$
1011	$(I + T^2 + T^3)\mathbf{x} = T^{13}\mathbf{x}$
1101	$(I + T + T^3)\mathbf{x} = T^7\mathbf{x}$
1111	$(I + T + T^2 + T^3)\mathbf{x} = T^{12}\mathbf{x}$

b. Show that if a cyclic code is generated by a maximal-period feedback shift register with q stages, each error burst of width q or less results in a distinct corrector (assuming all error bursts begin at the same digit).

It follows from (a) and (b) that knowledge of the corrector and the digit at which the burst begins completely specifies the error pattern. The various burst-correcting schemes are all based on the use of additional check digits in order to determine the position of the burst. Note that a burst may begin at the end of a code word and end at the beginning of the word. In the above example, the burst-error pattern 1101 beginning at digit 14 represents an error in digits 14, 15, and 2, and results in the corrector $(I + T + T^3)T^{13}\mathbf{x} = T^5\mathbf{x}$. Because of the cyclic structure of the codes, bursts of this type will also be corrected. (Note that it is sufficient to assume that T has an irreducible minimal polynomial; however, the maximal-period case gives the longest code words.)

5.7 (Abramson, 1959). A parity check matrix A is generated by a maximal-period q-stage shift register. The matrix A is then augmented by the addition of an "all-check" digit, so that the parity check equations become

$$\left[\begin{array}{cccc} & & & \\ & A & & \\ & & & \\ \hline 1 & 1 & \cdots & 1 & 1 \end{array}\right]\begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix} = 0, \qquad n = 2^q - 1$$

Show that the above code may be used to correct single errors and double adjacent errors. For example, if $q = 3$, the parity check equations might be

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix A is generated by a feedback shift register with characteristic polynomial $\lambda^3 + \lambda + 1$.

A corrector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

would be associated with a single error in r_6.

A corrector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

would be associated with a double adjacent error in r_3 and r_4. Decoding can be accomplished since the "all-check" digit distinguishes between a single error and a double error, and no two double adjacent errors result in the same corrector.

5.8 (Fire, 1959). A parity check matrix A is generated by a maximal-period q_1-stage feedback shift register. An identity matrix I_{q_2} of order $q_2 = 2q_1 - 1$ is formed. The parity check matrix for the Fire code is

$$\begin{bmatrix} I_{q_2} & | & I_{q_2} & | & I_{q_2} & | & \cdots & | & I_{q_2} \\ A & | & A & | & & \cdots & & & \end{bmatrix}$$

The length of the code words is n = least common multiple of $2^{q_1} - 1$ and q_2. The number of check digits is $q_1 + q_2$. For example, if A is the matrix of Problem 5.7 [$q_1 = 3$, $q_2 = 5$, $n = \text{lcm}(7, 5) = 35$], the parity check matrix of the associated Fire code is

$$\begin{bmatrix}
10000 & 10000 & 10000 & 10000 & 10000 & 10000 & 10000 \\
01000 & 01000 & 01000 & 01000 & 01000 & 01000 & 01000 \\
00100 & 00100 & 00100 & 00100 & 00100 & 00100 & 00100 \\
00010 & 00010 & 00010 & 00010 & 00010 & 00010 & 00010 \\
00001 & 00001 & 00001 & 00001 & 00001 & 00001 & 00001 \\
10010 & 11100 & 10111 & 00101 & 11001 & 01110 & 01011 \\
00101 & 11001 & 01110 & 01011 & 10010 & 11100 & 10111 \\
01011 & 10010 & 11100 & 10111 & 00101 & 11001 & 01110
\end{bmatrix}$$

Show that the Fire code will correct all error bursts of width q_1 or less. For example, suppose that in the above code the corrector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix}.$$

The top five digits identify the burst-error pattern as 101, and indicate that the burst starts at digit 4, 9, 14, 19, 24, 29, or 34. Now considering only the matrix A, the corrector associated with a burst-error pattern of 101 beginning at digit 1 is

$$(I + T^2)\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = T^6\mathbf{x}.$$

Since

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T\mathbf{x} = T^8\mathbf{x},$$

a burst-error pattern of 101 beginning at digit 3 will produce a corrector of

$$(I + T^2)T^2\mathbf{x} = T^6T^2\mathbf{x} = T^8\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence we conclude from the last three digits of the corrector that the burst begins at digit 3, 10, 17, 24, or 31. Thus we have identified the 101 pattern beginning at digit 24, that is, a double error in digits 24 and 26.

5.9 If

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

find the minimal polynomials of T^3 and T^5.

5.10 Show that the period of a nonsingular binary matrix cannot exceed $2^q - 1$, where q is the degree of the minimal polynomial of the matrix.

CHAPTER SIX

Information Sources

6.1. *Introduction*

We have not considered in any detail the model for a source of information. We have assumed at various times (Chapter 2) that the information to be transmitted consisted of the outcome of an experiment involving a random variable X; if successive values of X were to be transmitted, the values were assumed to be chosen independently. In other words, an information source has at times meant for us a sequence of independent, identically distributed random variables $X_1, X_2, \ldots$. At other times (Chapter 3) we have completely ignored the statistical properties of the source and focused instead on the assignment of a code word to all possible sequences the source could produce in a given interval of time; we required that the resulting code have a probability of error which was uniformly small for all possible code words so that statistical fluctuation of the source would not affect performance of the decoding scheme.

In this chapter we examine more closely the properties of information sources. We shall find that of all the sequences a source can conceivably produce in a given length of time, only a fraction are "meaningful"; the remaining sequences may be ignored when we are considering the problem of transmitting the information produced by the source. This result will allow us to take advantage of the statistical structure of a particular source to achieve a higher transmission rate for a given probability of error.

We will also try to use the concepts developed in the chapter to say something about the properties of a language as a means of communication. In addition, the results will be needed in Chapter 7 in the analysis of channels with memory. The mathematical equipment we shall use is the theory of finite Markov chains, which will be developed in this chapter.

6.2. *A mathematical model for an information source*

Let us begin by observing that whatever model we use for an information source, the source must produce a sequence of symbols

$$\cdots X_{t-2}X_{t-1}X_tX_{t+1}X_{t+2}\cdots$$

If we like, we may imagine the symbols being produced at a uniform rate, one per unit of time, with X_t the symbol produced at time t. The structure of the source may be described by giving the statistical dependence of X_t on the past symbols $X_{t-1}, X_{t-2}, \ldots$. As an example, suppose we have a primitive language with two symbols A and B. Suppose that in a sample of text in the language, the probability distribution of X_t is determined

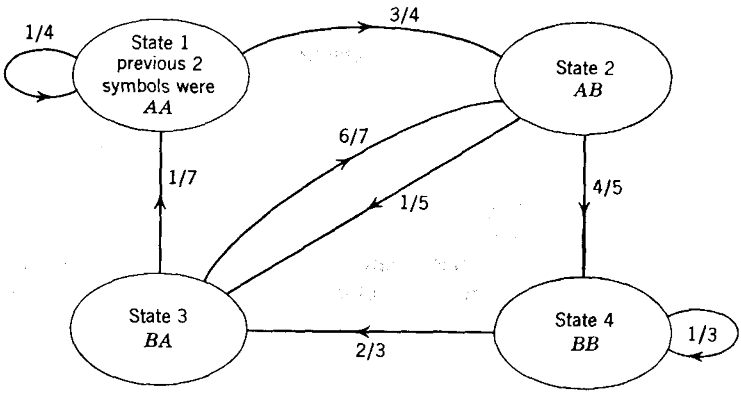


Fig. 6.2.1. Information source.

by the previous two symbols X_{t-1} and X_{t-2} but is not affected by the additional knowledge of X_{t-3}, X_{t-4}, etc. Let us assume that the conditional distribution of X_t given X_{t-1} and X_{t-2} is as follows:

$$P\{X_t = A \mid X_{t-2} = A, X_{t-1} = A\} = 1/4$$
$$P\{X_t = A \mid X_{t-2} = A, X_{t-1} = B\} = 1/5$$
$$P\{X_t = A \mid X_{t-2} = B, X_{t-1} = A\} = 1/7$$
$$P\{X_t = A \mid X_{t-2} = B, X_{t-1} = B\} = 2/3$$
$$P\{X_t = B \mid X_{t-2} = \alpha, X_{t-1} = \beta\} = 1 - P\{X_t = A \mid X_{t-2} = \alpha, X_{t-1} = \beta\} \quad (\alpha, \beta = A \text{ or } B)$$
$$P\{X_t = A \mid X_{t-3} = \alpha, X_{t-2} = \beta, X_{t-1} = \gamma\} = P\{X_t = A \mid X_{t-2} = \beta, X_{t-1} = \gamma\} \quad (\alpha, \beta, \gamma = A \text{ or } B)$$

and so on.

The source just described may be characterized by four "states" and a set of "transition probabilities." (See Fig. 6.2.1.) The source is in state 1 at time $t - 1$ if $X_{t-2} = X_{t-1} = A$, in state 2 if $X_{t-2} = A$, $X_{t-1} = B$, etc. If the source is in state 2 at time $t - 1$, then with probability 1/5 the source moves to state 3 at time t and $X_t = A$; with probability 4/5 the source moves to state 4 at time t and $X_t = B$.

output $g(1, s_{01}) = 0$. To summarize, if the source is initially in state a_1, and the channel is in state s_{10}, then with probability 3/4 the next source and channel states are (a_0, s_{00}); with probabiity $(1/4)(1/2) = 1/8$ the states are (a_1, s_{00}), and with probability 1/8 the states are (a_1, s_{01}). The corresponding inputs and outputs are determined from the states.

In general, the probability of transition from the pair of states $(a_{k_1}, s_{i_1j_1})$ to the pair (a_k, s_{ij}) is the product of the source transition probability p_{k_1k} and the channel transition probability from $s_{i_1j_1}$ to s_{ij} under the input k. The results are summarized in Table 7.1.3.

Table 7.1.3. A source-channel matrix for example 7.1.1

$$
\begin{array}{c}
(a_0, s_{00}) \\ (a_0, s_{10}) \\ (a_0, s_{01}) \\ (a_0, s_{11}) \\ (a_1, s_{00}) \\ (a_1, s_{10}) \\ (a_1, s_{01}) \\ (a_1, s_{11})
\end{array}
\overset{\begin{array}{cccccccc} (a_0, s_{00}) & (a_0, s_{10}) & (a_0, s_{01}) & (a_0, s_{11}) & (a_1, s_{00}) & (a_1, s_{10}) & (a_1, s_{01}) & (a_1, s_{11}) \end{array}}{\begin{bmatrix}
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{2}{3} \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{3}{4} & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & 0 \\
\frac{3}{4} & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & 0 \\
0 & \frac{3}{8} & 0 & \frac{3}{8} & 0 & 0 & 0 & \frac{1}{4} \\
0 & \frac{3}{8} & 0 & \frac{3}{8} & 0 & 0 & 0 & \frac{1}{4}
\end{bmatrix}}
$$

$(a_k, s_{ij}) \to$ input $f(a_k) = k$
output $g(x_k, s_{ij})$ as given in Table 7.1.1

Thus the action of the source on the channel may be described by the matrix of Table 7.1.3, which is called the "source-channel matrix" corresponding to the given source.

In the next section we are going to use the concepts developed here to define a general model for discrete channels with memory.

7.2. The finite-state channel

The model we are trying to develop is that of a channel with a finite number of internal states. The present state of the channel will in a sense represent a summary of its past history. The application of an input will result in a transition to a (possibly) different state and the production of an output. Specifically, given a finite set $S = \{s_1, \ldots, s_r\}$ called the *set of states* and two finite sets $\Gamma = \{b_1, \ldots, b_t\}$ and Δ called the *input alphabet* and the *output alphabet*, respectively, we define a *finite-state channel* as a collection of r by r transition matrices (that is, matrices of nonnegative numbers with all row sums equal to unity) $M_1, \ldots, M_t$, one for each input letter, together with a function g that assigns to each pair (b, s), $b \in \Gamma$, $s \in S$, an element $g(b, s) \in \Delta$. (The matrix M_i will also be denoted by $M_{b(i)}$ and is called the *channel matrix* corresponding to the input b_i.)

The interpretation of this definition is the following (compare Examples 7.1.1 and 7.1.2). If the channel is initially in state s_j and an input b_i is applied, the channel makes a transition governed by the matrix M_i. The probability of moving from s_j to s_k is the element $M_i(s_j, s_k)$ in row j and column k of the matrix M_i. If the new state is s_k, an output $g(b_i, s_k)$ is produced.

Notice that given an initial state and a sequence of inputs, the distribution of the corresponding output sequence is determined by the channel matrices. Thus the finite-state channel may be regarded as a special case of the general discrete channel defined in Section 3.1.

Now consider a Markov information source $\mathcal{S}$ with set of states $S_0 = \{a_1, \ldots, a_q\}$, transition matrix $\Pi = [p_{ij}]$, $i, j = 1, \ldots, q$, alphabet $\Gamma = \{b_1, \ldots, b_t\}$ = the input alphabet of the channel, and function f from S_0 to Γ. As in the discussion in Section 7.1, the operation of connecting the source to the channel is described by the *source-channel matrix* Q corresponding to $\mathcal{S}$. Q is defined as a qr by qr matrix (q = number of source states, r = number of channel states) whose rows (and columns) correspond to pairs (a, s), $a \in S_0$, $s \in S$. The element in row (a_i, s_k) and column (a_j, s_l) of Q is $p_{ij}M_{f(a_j)}(s_k, s_l)$. In other words, if initially the source is in state a_i and the channel in state s_k, then with probability p_{ij} the source moves from a_i to a_j and emits an input $f(a_j)$; the channel then moves from s_k to s_l with probability $M_{f(a_j)}(s_k, s_l)$.

For purposes of analysis we must impose restrictions on the channel. We say that a finite state channel is *regular* if every finite product $M_{i(1)}M_{i(2)} \cdots M_{i(k)}$, $i(1), \ldots, i(k) \in \{1, 2, \ldots, t\}$, $k = 1, 2, \ldots$, is regular; that is, steady state probabilities exist for the Markov chain determined by the matrix $M_{i(1)}M_{i(2)} \cdots M_{i(k)}$. (The integers $i(1), \ldots, i(k)$ are not necessarily distinct.)

We remark that the product of regular matrices need not be regular. For example, take

$$M_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & x & x \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} x & x & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(The x's stand for positive elements.) The product M_1M_2 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & x & x \end{bmatrix},$$

which is not regular; in fact the chain determined by M_1M_2 has two essential sets and is therefore not indecomposable.

The following two theorems provide the justification for the restriction of regularity. Theorem 7.2.1 implies that if a regular Markov source is connected to a finite-state regular channel, the associated input sequence, output sequence, and sequence of input-output pairs are all regular Markov sources and therefore obey the asymptotic equipartition property. This will be essential in the proof of the coding theorem for the channel. Theorem 7.2.2 provides a criterion that allows us to determine in a finite number of operations whether or not a given finite state channel is regular.

Theorem 7.2.1. If a regular Markov source is connected to a finite-state regular channel, the corresponding source-channel matrix is regular.

Proof. Consider any regular Markov source with alphabet $\Gamma = \{b_1, \ldots, b_t\}$. By Theorem 6.3.2, there is a source state a^* and an integer k such that a^* is reachable from any initial state in k steps. In particular, a^* may be reached starting from a^*; hence it is possible for the source to proceed through a sequence of states of the form $a^* a_{i(1)} \cdots a_{i(k-1)} a_{i(k)}$ with $a_{i(k)} = a^*$.

Assume that the input corresponding to $a_{i(j)}$ is $b_{i(j)} \in \Gamma$. Let $M = M_{i(1)} M_{i(2)} \cdots M_{i(k)}$; by hypothesis, M is regular, and consequently there is an integer n such that M^n has a positive column. But

$$M^n = M_{i(1)} \cdots M_{i(k)} M_{i(1)} \cdots M_{i(k)} \cdots M_{i(1)} \cdots M_{i(k)};$$

hence there is some channel state s^* such that s^* is reachable in nk steps from any initial state under the sequence of source states

$$a_{i(1)} \cdots a_{i(k)} a_{i(1)} \cdots a_{i(k)} \cdots a_{i(1)} \cdots a_{i(k)}.$$

Finally, given any source-channel state pair (a, s) we may in k steps reach (a^*, s_i) for some i; but for any i, (a^*, s^*) is reachable from (a^*, s_i) in nk steps. Thus (a^*, s^*) is reachable in $k + nk$ steps from any initial state, so that if Q is the source-channel matrix, the column of $Q^{(n+1)k}$ corresponding to (a^*, s^*) is positive. The proof is complete.

The definition of regularity may be restated as follows. Given any finite input sequence $b_{i(1)} b_{i(2)} \cdots b_{i(k)}$, there must exist a channel state s^* and an integer n such that for any initial channel state, s^* is reachable under the input sequence

$$b_{i(1)} b_{i(2)} \cdots b_{i(k)} b_{i(1)} b_{i(2)} \cdots b_{i(k)} \cdots b_{i(1)} b_{i(2)} \cdots b_{i(k)} \qquad (n \text{ times}).$$

Using this criterion, we may check that the channels of Examples 7.1.1 and 7.1.2 are regular. In Example 7.1.1, if a given input sequence contains at least one "1," then it is possible to reach and remain forever in state

s_{11}; similarly, if the input sequence contains at least one "0," it is possible to reach and remain forever in state s_{00}. In Example 7.1.2, for any input sequence we may reach and remain permanently in state $B0$ (or equally well state $B1$). Thus in both cases the channels are regular.

The definition of regularity seemingly involves the checking of an infinite number of matrices. We may however, replace the definition by a criterion involving only a finite number of steps, as shown by the following theorem.

Theorem 7.2.2. Let $M_1, M_2, \ldots, M_t$ be arbitrary r by r transition matrices. Define sets of matrices $A(k)$, $k = 1, 2, \ldots,$ as follows. Let

$$A(1) = \{M_1, M_2, \ldots, M_t\}.$$

If $k > 1$, let $A(k) =$ the set of all matrices that can be expressed as a product of at most k, not necessarily distinct, elements of $A(1)$. Also define

$$A = \bigcup_{k=1}^{\infty} A(k)$$

Let R be the set of all regular r by r transition matrices. Then if $A(2^{r^2}) \subset R$ then $A \subset R$.

Thus in the definition of regularity we may restrict k to be $\leq 2^{r^2}$, so that only a finite number of operations will be involved.

Proof. As noted in Section 6.3, in deciding whether or not a matrix has steady state probabilities, all we need to know about a given element of the matrix is whether it is zero or positive. In other words, without loss of generality we may assume that we are dealing with matrices with only two possible elements 0 and x (with $xx = x$, $x + x = x$). More precisely let $A'(k)$ be the set of matrices formed by replacing all positive elements of the matrices in $A(k)$ by x. Define A' and R' similarly from A and R. It follows that $A(2^{r^2}) \subset R$ if and only if $A'(2^{r^2}) \subset R'$. Since the number of r by r matrices with elements 0 and x is 2^{r^2}, $A'(k)$ must equal $A'(k + 1)$ for some $k \leq 2^{r^2}$. Now by definition of the sets $A(k)$, $A(k + 1) \subset A(k + 2)$ for all k, and hence $A'(k + 1) \subset A'(k + 2)$. But if $M \in A'(k + 2)$ and $M \notin A'(k + 1)$ then $M = M_a M_b$ where $M_a \in A'(k + 1) = A'(k)$, $M_b \in A'(1)$. Thus $M \in A'(k + 1)$. Consequently $A'(k + 1) = A'(k + 2)$, and similarly $A'(j) = A'(k)$ for $j \geq k$, so that $A' = A'(2^{r^2})$. Consequently

$$A(2^{r^2}) \subset R \Rightarrow A' = A'(2^{r^2}) \subset R' \Rightarrow A \subset R.$$

If we take $t = 1$ in Theorem 7.2.2, we obtain as a corollary the result that in applying the criterion of Theorem 6.3.2, we need only examine the powers Π^n, $n \leq 2^{r^2}$, of a transition matrix Π in order to determine whether or not the associated Markov chain has steady state probabilities. For if S is the set of r by r transition matrices not having a positive column, an argument identical to that of Theorem 7.2.2 shows that if $A(2^{r^2}) \subset S$, then $A \subset S$. Thus if none of the matrices Π^n, $n \leq 2^{r^2}$, has a positive column, then no power of Π has a positive column so that Π is not regular.

7.3. *The coding theorem for finite-state regular channels*

Suppose that a regular Markov source is connected to a given finite-state regular channel, and Q is the resulting source-channel matrix. Q then determines a finite Markov chain with states (a_i, s_k) where the a_i are the states of the source and the s_k are the states of the channel.

By the assumption of regularity, the chain associated with Q has steady state probabilities. Given the pair (a_i, s_k) we can determine the corresponding input $f_1(a_i, s_k) = f(a_i)$ and output $g_1(a_i, s_k) = g(f(a_i), s_k)$. Thus the chain associated with Q determines a stationary sequence of input-output pairs (X_n, Y_n), $n = 1, 2, \ldots$. We may therefore define an input uncertainty $H\{\mathbf{X}\} = \lim_{n\to\infty} H(X_n \mid X_1, \ldots, X_{n-1})$, an output uncertainty $H\{\mathbf{Y}\} = \lim_{n\to\infty} H(Y_n \mid Y_1, \ldots, Y_{n-1})$, and a joint uncertainty

$$H\{\mathbf{X}, \mathbf{Y}\} = \lim_{n\to\infty} H[(X_n, Y_n) \mid (X_1, Y_1), \ldots, (X_{n-1}, Y_{n-1})].$$

We define the *information conveyed about the process* $\{\mathbf{X}\}$ *by the process* $\{\mathbf{Y}\}$ as

$$\begin{aligned} I[\{\mathbf{X}\} \mid \{\mathbf{Y}\}] &= H\{\mathbf{X}\} + H\{\mathbf{Y}\} - H\{\mathbf{X}, \mathbf{Y}\} \\ &= I[\{\mathbf{Y}\} \mid \{\mathbf{X}\}]. \end{aligned}$$

It follows from Theorem 6.4.1 that

$$I[\{\mathbf{X}\} \mid \{\mathbf{Y}\}] = \lim_{n\to\infty} n^{-1}\, I(X_1, \ldots, X_n \mid Y_1, \ldots, Y_n)$$

and hence that $I[\{\mathbf{X}\} \mid \{\mathbf{Y}\}]$ is nonnegative. Furthermore, $H\{\mathbf{X}, \mathbf{Y}\} - H\{\mathbf{X}\}$, which may be regarded as the conditional uncertainty of $\{\mathbf{Y}\}$ given $\{\mathbf{X}\}$, is given by $\lim_{n\to\infty} n^{-1} H(Y_1, \ldots, Y_n \mid X_1, \ldots, X_n)$ and also by

$$\lim_{n\to\infty} H(Y_n \mid X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1})$$

(see Problem 7.2).

We define the *capacity* of the given channel as $C =$ the least upper bound of $I[\{\mathbf{X}\} \mid \{\mathbf{Y}\}]$, taken over all regular Markov sources. We are going to prove that it is possible to transmit information at any rate less than C with an arbitrarily small probability of error. First we define a *code* for the finite-state regular channel just as in the memoryless case; that is, a *code* (u, n, ε) is a set of input n-sequences $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(u)}$, together with a *decision scheme,* that is, a collection of disjoint sets $B_1, \ldots, B_u$ of output n-sequences, such that

$$P\{(Y_1, \ldots, Y_n) \in B_i \mid (X_1, \ldots, X_n) = \mathbf{x}^{(i)}\} \geq 1 - \varepsilon$$

for all $i = 1, 2, \ldots, u$ and any initial state.

Thus, no matter what the past history of the channel, decoding is accomplished with a probability of error not exceeding ε.

We now prove a lemma that will be useful in the proof of the coding theorem. If Q is the source-channel matrix corresponding to a regular Markov source which is connected to a finite-state regular channel, then Q has steady state probabilities $w_{(a,s)}$, $a \in S_0 =$ set of source states, $s \in S =$ set of channel states. The steady state probability of the channel's being in state s is given by

$$w_s = \sum_a w_{(a,s)}.$$

Define a *code* $(u, n, \varepsilon;\ Q)$ as a set of input n-sequences $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(u)}$, together with a decision scheme determined by sets $B_1, \ldots, B_u$, such that if the initial channel state is chosen at random, with state s assigned probability w_s $(s \in S)$, and the sequence $\mathbf{x}^{(i)}$ is transmitted, the probability of error, that is, the probability that the corresponding output sequence will not belong to B_i, is $\leq \varepsilon$ for all i.

Lemma 7.3.1. Let Q be a source-channel matrix for a finite-state regular channel. If for a given positive number R there is a sequence of codes $([2^{nR}], n, \varepsilon_n;\ Q)$ such that $\varepsilon_n \to 0$ as $n \to \infty$, then for any positive number $R' < R$ there is a sequence of codes $([2^{nR'}], n, \varepsilon_n')$ such that $\varepsilon_n' \to 0$ as $n \to \infty$.

Proof. Consider the following random experiment. Let $\mathbf{x}_i$ be any input n-sequence, and let B_i be any set of output n-sequences. Let (a_0, s_0) be any source-channel state pair. Allow the source and channel to make N transitions starting from (a_0, s_0) with the transition probabilities governed by the matrix Q. After N transitions, send the sequence $\mathbf{x}_i$ through the channel.

If (A_k, Z_k) is the source-channel state pair after k transitions, then the probability that the output sequence corresponding to $\mathbf{x}_i$ will not belong to

B_i is given by

$$P\{(Y_{N+1}, \ldots, Y_{N+n}) \notin B_i \mid (X_{N+1}, \ldots, X_{N+n}) = \mathbf{x}_i, (A_0, Z_0) = (a_0, s_0)\}$$
$$= \sum_{a,s} P\{(A_N, Z_N) = (a, s) \mid (A_0, Z_0) = (a_0, s_0)\}$$
$$\times\ P\{(Y_{N+1}, \ldots, Y_{N+n}) \notin B_i \mid (X_{N+1}, \ldots, X_{N+n}) = \mathbf{x}_i, (A_N, Z_N) = (a, s)\} \tag{7.3.1}$$
$$\xrightarrow{(N\to\infty)} \sum_{a,s} w_{(a,s)} P\{(Y_1, \ldots, Y_n) \notin B_i \mid (X_1, \ldots, X_n) = \mathbf{x}_i, (A_0, Z_0) = (a, s)\}$$
$$= \sum_s w_s P\{(Y_1, \ldots, Y_n) \notin B_i \mid (X_1, \ldots, X_n) = \mathbf{x}_i, Z_0 = s\}. \tag{7.3.2}$$

Note that the second factor of the summand of (7.3.1) is independent of N and hence coincides with the second factor of the summand of (7.3.2). Furthermore, since the number of source-channel state pairs is finite, the convergence of (7.3.1) to (7.3.2) is uniform in n, $\mathbf{x}_i$, B_i, and (a_0, s_0).

By hypothesis there are codes $([2^{nR}], n, \varepsilon_n;\ Q)$ with $\varepsilon_n \to 0$ as $n \to \infty$. But by definition of such codes, the expression (7.3.2), evaluated for a code word $\mathbf{x}_i$ and a decoding set B_i, is just the probability of error when $\mathbf{x}_i$ is sent.

For each n, consider the code $([2^{nR}], n, \varepsilon_n;\ Q)$ with code words $\mathbf{x}_i$ and decoding sets B_i. Given $\varepsilon > 0$, fix $N = N(\varepsilon)$ such that (7.3.1) and (7.3.2) differ by less than ε, uniformly in n, $\mathbf{x}_i$, B_i, and (a_0, s_0). Let n be large enough so that $\varepsilon_n < \varepsilon$ (and hence (7.3.1) is less than 2ε).

Now observe that if i and a_0 are fixed,

$$P\{(Y_{N+1}, \ldots, Y_{N+n}) \notin B_i \mid (X_{N+1}, \ldots, X_{N+n}) = \mathbf{x}_i,$$
$$(A_0, Z_0) = (a_0, s_0)\} = \sum_{\mathbf{x}} p(\mathbf{x}) g_{s_0}(\mathbf{x})$$

where

$$p(\mathbf{x}) = P\{(X_1, \ldots, X_N) = \mathbf{x} \mid A_0 = a_0\}$$

and

$$g_{s_0}(\mathbf{x}) = P\{(Y_{N+1}, \ldots, Y_{N+n}) \notin B_i \mid (X_1, \ldots, X_{N+n}) = (\mathbf{x}, \mathbf{x}_i), Z_0 = s_0\},$$

and the summation is over all input N-sequences $\mathbf{x}$.

Let r be the total number of channel states. Since $\sum_{\mathbf{x}} p(\mathbf{x}) g_{s_0}(\mathbf{x}) < 2\varepsilon$, the set of all $\mathbf{x}$ such that $g_{s_0}(\mathbf{x}) \geq 4r\varepsilon$ must have a probability $< 1/2r$. For if not, we would have

$$\sum_{\mathbf{x}} p(\mathbf{x}) g_{s_0}(\mathbf{x}) \geq \sum_{\{\mathbf{x}: g_{s_0}(\mathbf{x}) \geq 4r\varepsilon\}} p(\mathbf{x}) g_{s_0}(\mathbf{x})$$
$$\geq 4r\varepsilon P\{\mathbf{x}: g_{s_0}(\mathbf{x}) \geq 4r\varepsilon\} \geq 2\varepsilon,$$

a contradiction.

If $D_{s_0} = \{\mathbf{x}: g_{s_0}(\mathbf{x}) \geq 4r\varepsilon\}$, then $\sum_{s_0} P(D_{s_0}) \leq r(1/2r) = \frac{1}{2}$. In particular, the sets D_{s_0} cannot exhaust the entire collection of input N-sequences, so that there is an N-sequence $\mathbf{x}_i'$ belonging to none of the D_{s_0}; for such a sequence, $g_{s_0}(\mathbf{x}_i') < 4r\varepsilon$ *for all possible initial states* s_0.

It follows that if we form codes with words $\mathbf{x}^{(i)} = (\mathbf{x}_i', \mathbf{x}_i)$, $i = 1, 2, \ldots, [2^{nR}]$, and decoding sets $B_i^* = \{(\mathbf{y}_i', \mathbf{y}_i)\colon \mathbf{y}_i \in B_i\}$ (in other words we decode by applying the original decision scheme to the last n coordinates of the received sequence), we will have constructed codes $([2^{nR}], N + n, 4r\varepsilon)$. If we let $m = N + n$ and observe that $2^{nR} = 2^{m(R - m^{-1}NR)} = 2^{mR'}$ where $R' \to R$ as $m \to \infty$, we have the desired result.

Lemma 7.3.1 is analogous to Lemma 3.5.3 in that it asserts that if we can make the "average" probability of error arbitrarily small for a given transmission rate, we can also make the maximum probability of error arbitrarily small.

We may now prove the coding theorem for the finite-state regular channel.

***Theorem** 7.3.2.* Given a finite-state regular channel with capacity $C > 0$ and a positive number $R < C$, there is a sequence of codes $([2^{nR}], n, \varepsilon_n)$ with $\varepsilon_n \to 0$ as $n \to \infty$.

Proof. By definition of C, there is a regular Markov source $\mathcal{S}$ which, when connected to the channel, yields an information $I = I[\{\mathbf{X}\} \mid \{\mathbf{Y}\}]$ which is greater than R. Let Q be the corresponding source-channel matrix. Now suppose that the initial channel state is chosen at random, with state s assigned the steady state probability w_s. Let $p(\mathbf{y} \mid \mathbf{x})$ be the probability that the n-sequence $\mathbf{y}$ is received when the n-sequence $\mathbf{x}$ is transmitted. The system of probability functions $p(\mathbf{y} \mid \mathbf{x})$ forms a discrete channel as defined in Chapter 3. Thus Lemma 3.5.2 applies (recall that we observed at the time that the assumption that the channel be memoryless is not essential to the lemma). In the lemma, we need to specify a probability function $p(\mathbf{x})$ defined on the set of all input n-sequences. Let us take $p(\mathbf{x})$ to be the probability that the source $\mathcal{S}$ produces the sequence $\mathbf{x}$, that is, $p(\mathbf{x}) = P\{(X_1, \ldots, X_n) = \mathbf{x}\}$, when the initial state of the source is chosen at random in accordance with the stationary distribution induced by the matrix Q. [Specifically, the source state a is assigned probability $w_a = \sum_s w_{(a,s)}$, where $w_{(a,s)}$ is the steady state probability of the source-channel state pair (a, s).]

We conclude that for each real number b and positive integers u and n, there is a code $(u, n, \varepsilon_n;\ Q)$ such that

$$\varepsilon_n \le u \cdot 2^{-b} + P\{(\mathbf{x}, \mathbf{y})\colon \log [p(\mathbf{y} \mid \mathbf{x})/p(\mathbf{y})] \le b\}.$$

Let $b = \frac{1}{2}n(R + I)$, $u = [2^{nR}]$. Define a sequence of random variables V_n as follows. If $(X_1, \ldots, X_n) = \boldsymbol{\alpha}$ and $(Y_1, \ldots, Y_n) = \boldsymbol{\beta}$, then

$$V_n = n^{-1} \log \frac{p(\boldsymbol{\beta} \mid \boldsymbol{\alpha})}{p(\boldsymbol{\beta})} = n^{-1} \log \frac{p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p(\boldsymbol{\alpha})p(\boldsymbol{\beta})}.$$

It follows from the asymptotic equipartition property (Theorem 6.6.1) and the fact that a regular Markov source is ergodic (Theorem 6.6.2) that V_n converges in probability to $H\{\mathbf{X}\} + H\{\mathbf{Y}\} - H\{\mathbf{X}, \mathbf{Y}\} = I$. Since

$$P\{(\mathbf{x}, \mathbf{y}) : \log [p(\mathbf{y} \mid \mathbf{x})/p(\mathbf{y})] \leq b\} = P\{V_n \leq \tfrac{1}{2}(R + I)\} \to 0 \quad \text{as} \quad n \to \infty,$$

we have $\varepsilon_n \to 0$ as $n \to \infty$. The result now follows from Lemma 7.3.1.

The weak converse to the coding theorem holds, that is, any code (u, n, ε) for the finite-state regular channel must satisfy

$$\log u \leq (nC + \log 2)/(1 - \varepsilon)$$

where C is the channel capacity (Blackwell, Breiman, and Thomasian, 1958); hence, if the transmission rate is maintained above capacity, the probability of error cannot be made to approach zero (compare Theorem 3.7.3). In fact, the probability of error must approach one with increasing code-word length (that is, the strong converse to the coding theorem holds) as shown by Wolfowitz (1963b).

7.4. The capacity of a general discrete channel: comparison of the weak and strong converses

It is possible to construct a definition of the capacity of a discrete channel in such a way that the definition will not depend on the properties of any specific channel. First, a code (u, n, ε) for an arbitrary discrete channel is defined in exactly the same way as a code for the finite-state channel, that is, a collection of u input n-sequences together with a decision scheme such that the maximum probability of error is $\leq \varepsilon$, regardless of the state of the channel. A nonnegative number R is called a *permissible rate of transmission* for the channel if there exist codes $([2^{nR}], n, \varepsilon_n)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ (n is allowed to approach infinity through a subsequence of the positive integers).

We define the *channel capacity* C as the least upper bound of all permissible transmission rates. (Note that 0 is always a permissible rate, so that the set of permissible rates is not empty.) Thus if $R < C$, the rate R can be maintained with an arbitrarily small probability of error; this is not possible if $R > C$. In the case of the discrete memoryless or the finite-state channel, C agrees with the expressions given previously.

In general, we establish that a particular number C_0 is the capacity of a given channel by proving a *coding theorem* (such as Theorem 3.5.1 or Theorem 7.3.2) and a *weak converse* (such as Theorem 3.7.3). In other words, we prove first that for any $R < C_0$, there exist codes $([2^{nR}], n, \varepsilon_n)$ with $\varepsilon_n \to 0$ as $n \to \infty$, and hence that $C \geq C_0$. We then prove that given any sequence of codes $([2^{nR}], n, \varepsilon_n)$, $R > C_0$, there is an $\varepsilon > 0$ such that for sufficiently large n, $\varepsilon_n \geq \varepsilon$; consequently $C \leq C_0$.

We say that the *strong converse* holds for a given channel if any sequence of codes $([2^{nR}], n, \lambda_n)$, with $R > C$, must be such that $\lim_{n\to\infty} \lambda_n = 1$. To gain further insight into this concept, we introduce the notion of the *λ-capacity* of a discrete channel. Given any discrete channel, let $N(n, \lambda)$ be the maximum possible number of code words in a code that uses sequences of length n and that has maximum probability of error at most λ. If $0 < \lambda < 1$, the *λ-capacity* of the given channel is defined by

$$C(\lambda) = \limsup_{n\to\infty} n^{-1} \log N(n, \lambda) \tag{7.4.1}$$

If one is willing to use codes whose maximum probability of error is $\leq \lambda$ (instead of demanding that the maximum probability of error be made arbitrarily small), then $C(\lambda)$ may be regarded as the least upper bound of all permissible transmission rates. Specifically, we call a number R a *λ-permissible rate of transmission* if given any positive integer n_0, there exists a code $([2^{nR}], n, \lambda_n)$ with $n \geq n_0$ and $\lambda_n \leq \lambda$. It follows directly from the definition of the lim sup of a sequence of numbers that $C(\lambda)$ is the least upper bound of all λ-permissible transmission rates.

Since any code (u, n, λ_1) is automatically a code (u, n, λ_2) if $\lambda_1 < \lambda_2$, it follows that $C(\lambda)$ is nondecreasing. We have the following result.

Lemma 7.4.1. The channel capacity is given by

$$C = \lim_{\lambda\to 0} C(\lambda). \tag{7.4.2}$$

Proof. Let $C_0 = \lim_{\lambda\to 0} C(\lambda)$. If $R < C_0$ and $0 < \lambda < 1$ then $R < C(\lambda)$, hence R is λ-permissible. Thus given any n_0 we can find codes $([2^{nR}], n, \lambda_n)$ with $n \geq n_0$ and $\lambda_n \leq \lambda$. Since λ may be chosen arbitrarily small, it follows that $C \geq C_0$. On the other hand, if $R > C_0$ there is a $\lambda \in (0, 1)$ such that $R > C(\lambda)$. Since R is not λ-permissible, it follows that for sufficiently large n, no code $([2^{nR}], n, \lambda_n)$ can be such that $\lambda_n \leq \lambda$. Consequently no number greater than C_0 can be a permissible transmission rate, and hence $C \leq C_0$.

The strong converse has a striking interpretation in terms of the λ-capacity.

Lemma 7.4.2. The strong converse holds for a discrete channel if and only if

$$C(\lambda) = C \quad \text{for} \quad \text{all } \lambda \in (0, 1). \tag{7.4.3}$$

Proof. If the strong converse holds, then given any $\lambda \in (0, 1)$ and any $R > C$, for sufficiently large n no code $([2^{nR}], n, \lambda_n)$ can exist with $\lambda_n \leq \lambda$.

Therefore R is not λ-permissible for any λ; consequently $C(\lambda) \leq R$ for all λ. Since this holds for any $R > C$, we have $C(\lambda) \leq C$ for all λ. But since $C(\lambda)$ is nondecreasing, Lemma 7.4.1 yields $C(\lambda) = C$ for all $\lambda \in (0, 1)$. Conversely, suppose that $C(\lambda) = C$ for all $\lambda \in (0, 1)$. Given $R > C$ and $\lambda \in (0, 1)$. Since $R > C(\lambda)$, R is not λ-permissible and therefore if we have any sequence of codes $([2^{nR}], n, \lambda_n)$, we must have $\lambda_n > \lambda$ eventually. In other words $\lambda_n \to 1$ as $n \to \infty$, and the strong converse holds.

We are going to give an example of a channel for which the strong converse fails to hold. Our example will be a binary channel with the curious feature that $\lim_{\lambda \to 0} C(\lambda) = 0$, $\lim_{\lambda \to 1} C(\lambda) = 1$ [thus violating (7.4.3)]. Hence if we desire to transmit information with an arbitrarily small probability of error, the only permissible transmission rate is zero, while if we are willing to put up with an arbitrarily high probability of error, we can approach the maximum transmission rate of a noiseless binary channel.

Consider a channel whose inputs and outputs are binary sequences of length $n (n = 1, 2, \ldots)$. The action of the channel is described by specifying that if the output sequence $\mathbf{y}$ differs from the input sequence $\mathbf{x}$ in exactly i places $(i = 0, 1, \ldots, n)$ then

$$p(\mathbf{y} \mid \mathbf{x}) = \left[(n + 1)\binom{n}{i}\right]^{-1}.$$

Thus the number of transmission errors is uniformly distributed between 0 and n, and all error patterns of the same weight have the same probability. (A formal definition of the channel may be readily supplied.)

We shall establish the following result.

Theorem 7.4.3. For $\lambda > \frac{3}{4}$, $C(\lambda) \geq 1 - H(2\lambda - 1)$, where for brevity we write $H(x)$ for the uncertainty $-x \log x - (1 - x) \log (1 - x)$; consequently $\lim_{\lambda \to 1} C(\lambda) = 1$. Furthermore, for all λ, $C(\lambda) \leq 1 - H(\frac{1}{2}(1 - \lambda))$; consequently $\lim_{\lambda \to 0} C(\lambda) = 0$.

Proof. The Varsharmov-Gilbert-Sacks condition (Theorem 4.5.3) states that if

$$2^{n-k} > \sum_{i=0}^{2t-1} \binom{n-1}{i},$$

it is possible to construct a binary code with 2^k words of length n which will correct t-tuple and all smaller errors. Now

$$\sum_{i=0}^{an} \binom{n}{i} \leq 2^{nH(a)} \quad \text{if} \quad 0 \leq a < \tfrac{1}{2} \qquad \text{(Lemma 4.7.2)}$$

Thus if $2^{n-k} > 2^{(n-1)H((2t-1)/(n-1))}$, in particular if $2^{n-k} > 2^{nH(2t/n)}$ (and $2t/n < \frac{1}{2}$), we can find a binary code with 2^k words of length n which corrects all errors of weight t or less. Set $R = k/n$ and $\alpha = 2t/n$. Then the above condition implies that if $R < 1 - H(\alpha)$, for sufficiently large n we can construct a code with at least 2^{nR} words of length n which corrects all errors of magnitude $\leq n\alpha/2$. Let us use this code in the given channel. If $p(e')$ is the probability of correct decoding for any specific word of the code, and Z is the number of transmission errors, then

$$\begin{aligned} p(e') &= P\{Z \leq \tfrac{1}{2}n\alpha\}p(e' \mid Z \leq \tfrac{1}{2}n\alpha) + P\{Z > \tfrac{1}{2}n\alpha\}p(e' \mid Z > \tfrac{1}{2}n\alpha) \\ &\geq P\{Z \leq \tfrac{1}{2}n\alpha\}p(e' \mid Z \leq \tfrac{1}{2}n\alpha) = \frac{\tfrac{1}{2}n\alpha + 1}{n+1} \cdot 1 \\ &\geq \tfrac{1}{2}\alpha. \end{aligned}$$

It follows that R is $(1 - \frac{1}{2}\alpha)$-permissible, where R may be chosen arbitrarily close to $1 - H(\alpha)$. Hence $C(1 - \frac{1}{2}\alpha) \geq 1 - H(\alpha) = 1 - H(1 - \alpha)$ $(\alpha < \frac{1}{2})$, and the first part of the theorem is proved.

To prove the second part of the theorem, consider any code for the given channel, say a code with N words of length n and maximum probability of error λ. Since each error pattern of weight i or $n - i$ has probability

$$\left[(n+1)\binom{n}{i}\right]^{-1},$$

which decreases as i varies from 0 to $\frac{1}{2}n$, it follows that if r is the smallest integer such that $2r/(n+1) \geq 1 - \lambda$, each decoding set of the code has at least $2\sum_{j=0}^{r-2}\binom{n}{j}$ sequences. [The way to achieve a probability of correct transmission of $1 - \lambda$ with the smallest possible number of sequences in a decoding set is to correct the errors of weight 0 and n, 1 and $n - 1, \ldots,$ $r - 2$ and $n - (r - 2)$, and some or all of the errors of weight $r - 1$ and $n - (r - 1)$.]

Hence, as in Theorem 4.2.2,

$$N \leq \frac{2^n}{2\sum_{j=0}^{r-2}\binom{n}{j}} \leq \frac{2^{n-1}}{\sum_{j=0}^{[\frac{1}{2}(n+1)(1-\lambda)-2]}\binom{n}{j}}$$

Using the fact that

$$n^{-1}\log\left[\sum_{i=0}^{an}\binom{n}{i}\right] \to H(a) \quad \text{as} \quad n \to \infty \quad \text{if} \quad 0 \leq a < \tfrac{1}{2}$$

(Lemma 4.7.2), we obtain

$$C(\lambda) = \limsup_{n\to\infty} n^{-1} \log N(n, \lambda) \le 1 - H(\tfrac{1}{2}(1 - \lambda)),$$

and the theorem follows.

In the example just given, the variance of the number of transmission errors is of order greater than n (actually of order n^2), as opposed to the case of a binary symmetric channel, where the variance is of the order of n. Wolfowitz (1963a) suggests that this fact is essentially responsible for the nonconstant behavior of $C(\lambda)$.

7.5. *Notes and remarks*

The finite-state channel was introduced and analyzed by Blackwell, Breiman, and Thomasian (1958). These authors achieved greater generality than we have in this chapter by allowing the Markov chains involved in the definition of the channel to be indecomposable rather than simply regular. However, they used several ideas from martingale theory in their presentation, and we have avoided this here.

Theorem 7.2.2 is due to Thomasian (1963).

Just as in the case of the discrete memoryless channel (Theorem 3.6.4), there is an exponential bound on the rate of approach toward zero of the probability of error of codes with a fixed transmission rate; see Blackwell (1961b).

A channel that fits naturally into the finite-state model is the so-called *discrete finite-memory channel*. Intuitively, a channel with memory m should have the property that the distribution of the state of the channel at the end of an input sequence of length n, $n \ge m$, depends only on the last m inputs, and not on any previous inputs or states. Now if an input sequence $x_{i(1)} \cdots x_{i(m)}$ is applied, the distribution of the state at the end of the sequence may be found from the matrix $M = M_{i(1)}M_{i(2)} \cdots M_{i(m)}$, where $M_{i(k)}$ is the channel transition matrix corresponding to the input $x_{i(k)}$. To say that the state distribution at the end of the sequence is independent of the initial state is to say that the matrix M has identical rows. Thus a finite-memory channel may be regarded as a finite-state channel with the property that for some positive integer m, each matrix $M_{i(1)}M_{i(2)} \cdots M_{i(m)}$ (and hence each matrix $M_{i(1)}M_{i(2)} \cdots M_{i(n)}$, $n \ge m$) has identical rows. A transition matrix with identical rows has at least one positive column, and thus a finite-memory channel is regular.

Our approach to the discrete finite-memory channel is that of Blackwell, Breiman, and Thomasian (1958). Alternate treatments may be found in Khinchin (1957), Feinstein (1959), and Wolfowitz (1961).

The results of this chapter have given no clue as to how to compute the capacity of a given finite-state channel. Very little is known in this area. An important initial step has been taken by Wolfowitz (1963b), who showed the existence of a procedure whereby we can achieve any preassigned approximation to the capacity in a finite number of steps. However, practical algorithms do not exist at present. One of the few explicit results is that of Gilbert (1960), who found the capacity of the channel of Example 7.1.2. The capacity of the channel of Example 7.1.1 is apparently unknown.

There is a channel that in a sense combines some of the features of the discrete memoryless channel and the discrete channel with memory. Suppose we have a collection of discrete memoryless channels, one for each element s in some arbitrary set S. A code word is to be transmitted through one of the channels, but the identity of the channel to be used is unknown to the sender and may be regarded as being chosen by "nature." We wish to construct codes that maintain a given transmission rate and at the same time have a low probability of error *regardless of which channel is used.* Formally, we define *a compound channel* (the terminology is that of Wolfowitz, 1961) as a system of probability functions $p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n;\ s)$, where $\alpha_1, \ldots, \alpha_n$ belong to a specified finite set Γ called the *input alphabet,* $\beta_1, \ldots, \beta_n$ belong to a specified finite set Γ' called the *output alphabet,* s belongs to an arbitrary set S called the *set of states,* n ranges over the positive integers, and finally for each s and n we have

$$p_n(\beta_1, \ldots, \beta_n \mid \alpha_1, \ldots, \alpha_n;\ s) = p_1(\beta_1 \mid \alpha_1;\ s)p_1(\beta_2 \mid \alpha_2;\ s) \cdots p_1(\beta_n \mid \alpha_n;\ s)$$

The compound channel is thus a special case of the general discrete channel, and a code for the compound channel is therefore defined in the usual way. Notice that we require that the input alphabets be the same for all the individual memoryless channels in the collection indexed by the set S, and similarly for the output alphabets.

Now if $p(x)$ is any probability distribution on the input alphabet, then for each $s \in S$ we may calculate the information $I(s;\ p(x))$ processed by the discrete memoryless channel corresponding to s. The following theorem was proved by Wolfowitz (1960, 1961) and independently by Blackwell, Breiman, and Thomasian (1959).

The capacity of the compound channel is

$$C = \sup_{p(x)} \inf_{s \in S} I(s;\ p(x)).$$

In accordance with the discussion in Section 7.4, the proof may be accomplished by proving a coding theorem and a weak converse. The strong converse to the coding theorem also holds, as was shown by Wolfowitz (1960, 1961).

The notion of the λ-capacity of a channel and the discovery that there are channels for which the strong converse does not hold are due to Wolfowitz (1963a). The example discussed in Theorem 7.4.3 is the author's (1965).

In certain physical situations we may have a discrete channel whose state at the beginning of an input sequence is known to the sender or to the receiver, or to both. For a discussion of this type of problem, see Wolfowitz (1961).

PROBLEMS

7.1 a. Reformulate Example 7.1.1 so that the output is a function of the channel state alone rather than the input-state pair.

b. Show that any finite-state channel model may be reformulated so that the output is a function of the channel state alone, and furthermore, if the original channel is regular so is the new channel, and the capacity of the new channel is the same as that of the original.

7.2 Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of inputs produced by a regular Markov source that is connected to a finite-state regular channel, and let $\{Y_n, n = 1, 2, \ldots\}$ be the corresponding output sequence. Define the *conditional uncertainty of the sequence* $\{Y_n\}$ *given the sequence* $\{X_n\}$ as

$$H(\{\mathbf{Y}\} \mid \{\mathbf{X}\}) = \lim_{n\to\infty} H(Y_n \mid X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1}).$$

Show that

$$H(\{\mathbf{Y}\} \mid \{\mathbf{X}\}) = \lim_{n\to\infty} n^{-1}H(Y_1, \ldots, Y_n \mid X_1, \ldots, X_n).$$

Hence show that $H\{\mathbf{X}, \mathbf{Y}\} = H\{\mathbf{X}\} + H(\{\mathbf{Y}\} \mid \{\mathbf{X}\})$.

7.3 Let $X_n, n = 1, 2, \ldots$, and $Y_n, n = 1, 2, \ldots$, be stationary sequences of random variables defined on the same sample space, such that the sequence of pairs (X_n, Y_n) is also stationary. (Assume that the X_n and Y_n can take on only finitely many possible values.) Define the conditional uncertainty of the sequence $\{Y_n\}$ given the sequence $\{X_n\}$ as in Problem 7.2. Do the results of that problem still hold?

CHAPTER EIGHT

Continuous Channels

8.1. Introduction

In this chapter we consider communication channels that are "continuous" in two senses. First we allow the input and output alphabets to contain an uncountably infinite number of elements, although we require that the material to be transmitted through the channel be in the form of a discrete sequence of symbols In other words, once every second (say) we transmit a symbol from a possibly uncountable alphabet. This is the so-called *time-discrete, amplitude-continuous* channel Second, we allow the transmission of information to be continuous in time. In other words, we transmit a "time function" $f(t)$ defined on an interval of real numbers, whose range is also an interval of real numbers. This is the so-called *time-continuous, amplitude-continuous* channel

The first channel mentioned above is somewhat easier to formalize. Given an arbitrary set S called the *set of states*, we define a time-discrete, amplitude-continuous channel as a system of probability density functions $p_n(y_1, \ldots, y_n \mid x_1, \ldots, x_n, s)$, $s \in S$, $n = 1, 2, \ldots$, the x_i and y_j real; that is, a system of nonnegative (Borel measurable) functions such that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_n(y_1, \ldots, y_n \mid x_1, \ldots, x_n; s)\, dy_1 \cdots dy_n = 1 \qquad \text{for all } n, s, x_1, \ldots, x_n.$$

Thus for each input sequence and each state, we have a probability measure on the set of all output sequences (in this case, Euclidean n-space) (We could have achieved greater generality by allowing arbitrary probability measures on the set of all output sequences for each input sequence and state.) A *code* (u, n) for such a channel is defined in the now familiar way as a collection of sequences $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(u)}$ belonging to Euclidean n-space E^n, together with a *decision scheme*, that is, a partition of E^n into disjoint (Borel) sets $B_1, \ldots, B_u$. A code (u, n, λ) is a code (u, n) with maximum probability of error $\leq \lambda$, that is, a code (u, n) such that

$$P\{(Y_1, \ldots, Y_n) \in B_i \mid (X_1, \ldots, X_n) = \mathbf{x}^{(i)}\} \geq 1 - \lambda$$

for all possible code words $\mathbf{x}^{(i)}$ and all possible states s.

In a given problem, physical considerations may require that some restriction be placed on the set of allowable inputs. For example, if each component of a code word is represented by the amplitude of a pulse, we may wish to require that $|x_i| \leq$ some constant M for all i, or possibly we may have an "average power" limitation of the form $n^{-1} \sum_{i=1}^{n} x_i^2 \leq M$. Given (Borel) subsets F_n of Euclidean n-space ($n = 1\ 2, \ldots$) we may define a *restricted* time-discrete, amplitude-continuous channel corresponding to the F_n as a system of probability density functions

$$p_n(y_1, \ldots, y_n \mid x_1, \ldots, x_n;\ s),$$

defined only for $(x_1, \ldots, x_n) \in F_n$. A code (u, n) and a code (u, n, λ) are defined in exactly the same way as before, except that the code words are now required to belong to the set F_n.

8.2. The time-discrete Gaussian channel

In many physical communication situations, disturbances enter in the form of an unwanted "noise" superimposed on the signal to be transmitted, thus producing a result in the form of signal plus noise, from which the desired signal is to be extracted. The assumption that the noise is Gaussian seems to be reasonable in a wide variety of physical settings; the assumption is definitely desirable from the mathematical point of view, as we shall see in the course of the chapter.

A *time-discrete Gaussian channel* is a time-discrete, amplitude-continuous channel whose probability density functions exhibit no dependence on the state s, and are given explicitly by

$$p_n(y_1, \ldots, y_n \mid x_1, \ldots, x_n) = (2\pi N)^{-\frac{1}{2}n} \exp\left[-\sum_{j=1}^{n} \frac{(y_j - x_j)^2}{2N}\right]. \tag{8.2.1}$$

Thus the output sequence is obtained from the input sequence by the addition of a sequence of independent Gaussian random variables with mean 0 and variance N. We shall see that if no restriction is placed on the choice of code words, arbitrarily high transmission rates can be maintained with an arbitrarily small probability of error. We are going to make a restriction on the "average power" of the input sequence; in other words, we introduce a restricted channel determined by sets $F_n = \{(x_1, \ldots, x_n)\colon n^{-1} \sum_{i=1}^{n} x_i^2 \leq M\}$. The resulting channel is called a *time-discrete Gaussian channel with average power limitation M*.

We may establish a relationship between the number of words in a code and the probability of error by proving a result which is analogous to the Fundamental Lemma 3.5.2 and its extension in Problem 3.11. The result is not restricted to the Gaussian case.

Lemma 8.2.1. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(\mathbf{x})$ an arbitrary probability density function on Euclidean n-space. Write $p(\mathbf{y} \mid \mathbf{x})$ for the density $p_n(y_1, \ldots, y_n \mid x_1, \ldots, x_n)$, and write F for F_n.

For any real number a, let

$$A = \left\{(\mathbf{x}, \mathbf{y}) \colon \log \frac{p(\mathbf{y} \mid \mathbf{x})}{p(\mathbf{y})} > a\right\}.$$

Then for each positive integer u, there is a code (u, n, λ) (with all code words belonging to F), such that

$$\lambda \leq ue^{-a} + P\{(\mathbf{X}, \mathbf{Y}) \notin A\} + P\{\mathbf{X} \notin F\} \tag{8.2.2}$$

where

$$P\{(\mathbf{X}, \mathbf{Y}) \in A\} = \int \cdots \int_A p(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y}, \qquad p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x})p(\mathbf{y} \mid \mathbf{x}),$$

and

$$P\{\mathbf{X} \in F\} = \int \cdots \int_F p(\mathbf{x})\, d\mathbf{x}.$$

[For convenience we assume that *all logarithms in this chapter are to the base e:* thus we have a term ue^{-a} in (8.2.2) instead of $u \cdot 2^{-a}$.]

Proof. The argument proceeds exactly as in Lemma 3.5.2 (and Problem 3.11), with sums replaced by integrals, but we will go through the details for convenience. Let ε be the expression on the right side of (8.2.2). If $\varepsilon \geq 1$, there is nothing to prove, so assume $0 < \varepsilon < 1$. Pick (if possible) a sequence $\mathbf{x}^{(1)} \in F$ such that

$$P\{\mathbf{Y} \in A_{\mathbf{x}^{(1)}} \mid \mathbf{X} = \mathbf{x}^{(1)}\} \geq 1 - \varepsilon \quad \text{where} \quad A_{\mathbf{x}} = \{\mathbf{y} \colon (\mathbf{x}, \mathbf{y}) \in A\};$$

choose the decoding set B_1 to be $A_{\mathbf{x}^{(1)}}$. Having chosen $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k-1)}$ and $B_1, \ldots, B_{k-1}$, select $\mathbf{x}^{(k)} \in F$ such that

$$P\left\{\mathbf{Y} \in A_{\mathbf{x}^{(k)}} - \bigcup_{i=1}^{k-1} B_i \,\Big|\, \mathbf{X} = \mathbf{x}^{(k)}\right\} \geq 1 - \varepsilon;$$

set $B_k = A_{\mathbf{x}^{(k)}} - \bigcup_{i=1}^{k-1} B_i$. If the process does not terminate in a finite number of steps, then the sequences $\mathbf{x}^{(i)}$ and decoding sets B_i, $i = 1, 2, \ldots, u$, form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$.) We will show $t \geq u$ by showing that

$\varepsilon \le te^{-a} + P\{(\mathbf{X}, \mathbf{Y}) \notin A\} + P\{\mathbf{X} \notin F\}$. We proceed as follows. Let $B = \bigcup_{j=1}^{t} B_j$. (If $t = 0$, take $B = \emptyset$.) Then

$$\begin{aligned}
P\{(\mathbf{X}, \mathbf{Y}) \in A\} &= \int_{(\mathbf{x},\mathbf{y})\in A} p(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \int_{\mathbf{x}} p(\mathbf{x}) \int_{\mathbf{y}\in A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbf{x}} p(\mathbf{x}) \int_{\mathbf{y}\in B\cap A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x} \\
&\quad + \int_{\mathbf{x}} p(\mathbf{x}) \int_{\mathbf{y}\in \bar{B}\cap A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbf{x}} p(\mathbf{x}) \int_{\mathbf{y}\in B\cap A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x} \\
&\quad + \int_{\mathbf{x}\in F} p(\mathbf{x}) \int_{\mathbf{y}\in \bar{B}\cap A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x} \\
&\quad + \int_{\mathbf{x}\notin F} p(\mathbf{x}) \int_{\mathbf{y}\in \bar{B}\cap A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x}.
\end{aligned} \tag{8.2.3}$$

Since $B \cap A_{\mathbf{x}} \subset B$, the first term of (8.2.3) cannot exceed

$$\int_{\mathbf{x}} p(\mathbf{x}) \int_{\mathbf{y}\in B} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x} = P\{\mathbf{Y} \in B\}.$$

But since $B_i \subset A_{\mathbf{x}^{(i)}}$,

$$P\{\mathbf{Y} \in B\} = \sum_{i=1}^{t} P\{\mathbf{Y} \in B_i\} \le \sum_{i=1}^{t} P\{\mathbf{Y} \in A_{\mathbf{x}^{(i)}}\}.$$

Now if $\mathbf{y} \in A_{\mathbf{x}^{(i)}}$, then $(\mathbf{x}^{(i)}, \mathbf{y}) \in A$; hence $\log [p(\mathbf{y} \mid \mathbf{x}^{(i)})/p(\mathbf{y})] > a$, or $p(\mathbf{y}) < p(\mathbf{y} \mid \mathbf{x}^{(i)})e^{-a}$. Consequently

$$P\{\mathbf{Y} \in A_{\mathbf{x}^{(i)}}\} = \int_{\mathbf{y}\in A_{\mathbf{x}^{(i)}}} p(\mathbf{y})\, d\mathbf{y} \le e^{-a} \int_{\mathbf{y}\in A_{\mathbf{x}^{(i)}}} p(\mathbf{y} \mid \mathbf{x}^{(i)})\, d\mathbf{y} \le e^{-a}.$$

Therefore the first term of (8.2.3) is $\le te^{-a}$.

To estimate the second term, we claim that

$$\int_{\mathbf{y}\in \bar{B}\cap A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y} < 1 - \varepsilon \quad \text{for all } \mathbf{x} \in F. \tag{8.2.4}$$

First assume $\mathbf{x}$ = one of the code words $\mathbf{x}^{(k)}$. Then we have

$$\bar{B} \cap A_{\mathbf{x}} = \emptyset. \tag{8.2.5}$$

To establish (8.2.5), we show (as in Lemma 3.5.2) that if $\mathbf{y} \in A_{\mathbf{x}^{(k)}}$ then $\mathbf{y}$ necessarily belongs to B. Since $B_k = A_{\mathbf{x}^{(k)}} - \bigcup_{i=1}^{k-1} B_i$, if $\mathbf{y} \in A_{\mathbf{x}^{(k)}}$ and $\mathbf{y} \notin \bigcup_{i=1}^{k-1} B_i$, then $\mathbf{y} \in B_k \subset B$; on the other hand, if $\mathbf{y} \in A_{\mathbf{x}^{(k)}}$ and $\mathbf{y} \in \bigcup_{i=1}^{k-1} B_i$, then, since B is the union of all the decoding sets, we have $\mathbf{y} \in B$. Thus (8.2.4) is proved when $\mathbf{x}$ is a code word. Now assume that $\mathbf{x} \neq$ any of the $\mathbf{x}^{(i)}$. If $\int_{\mathbf{y} \in \bar{B} \cap A_{\mathbf{x}}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y} \geq 1 - \varepsilon$, then, since $\mathbf{x}$ is assumed $\in F$ and $\bar{B} \cap A_{\mathbf{x}} = A_{\mathbf{x}} - \bigcup_{i=1}^{t} B_i$, we could enlarge the code by taking $\mathbf{x}^{(t+1)} = \mathbf{x}$, $B_{t+1} = \bar{B} \cap A_{\mathbf{x}}$, contradicting the assumption that the process terminates after t steps. Thus the second term of (8.2.3) is $\leq 1 - \varepsilon$.

We estimate the third term of (8.2.3) by observing that $\bar{B} \cap A_{\mathbf{x}}$ is a subset of the entire space E^n, and hence the third term cannot exceed

$$\int_{\mathbf{x} \notin F} p(\mathbf{x}) \int_{\mathbf{y}} p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}\, d\mathbf{x} = P\{\mathbf{X} \notin F\}.$$

We conclude that $P\{(\mathbf{X}, \mathbf{Y}) \in A\} \leq te^{-a} + 1 - \varepsilon + P\{\mathbf{X} \notin F\}$, and the result follows.

As in the discussion of Section 7.4, we say that a number R is a *permissible rate of transmission* for a given time-discrete, amplitude-continuous channel (restricted or not) if there exist codes $([e^{nR}], n, \lambda_n)$ with $\lambda_n \to 0$ as $n \to \infty$ (where n is allowed to approach infinity through a subsequence of the positive integers).

The *channel capacity* is defined as the least upper bound of all permissible transmission rates.

Let C be the capacity of the time-discrete, amplitude-continuous Gaussian channel with average power limitation M and "noise" variance N [see (8.2.1)]. We are going to show that

$$C = \tfrac{1}{2} \log \left(1 + \frac{M}{N}\right). \tag{8.2.6}$$

As indicated in Section 7.4, in order to establish (8.2.6) we shall prove a *coding theorem* which states that $C \geq \frac{1}{2} \log (1 + M/N)$ [or equivalently, if $R < \frac{1}{2} \log (1 + M/N)$, we can find codes $([e^{nR}], n, \lambda_n)$ with $\lambda_n \to 0$ as $n \to \infty$], and a *weak converse* which states that $C \leq \frac{1}{2} \log (1 + M/N)$. We first prove the coding theorem.

Theorem 8.2.2. $C \geq \frac{1}{2} \log (1 + M/N)$.

Proof. In Lemma 8.2.1, take

$$p(\mathbf{x}) = (2\pi M_0)^{-\frac{1}{2}n} \exp\left(-\sum_{j=1}^{n} \frac{x_j^2}{2M_0}\right) \quad \text{where} \quad M_0 < M.$$

In other words, choose the components $X_1, \ldots, X_n$ of the input independently, each component having a Gaussian distribution with mean 0 and variance $M_0 < M$. The third term of (8.2.2) then becomes

$$P\{\mathbf{X} \notin F\} = P\left\{n^{-1}\sum_{j=1}^{n} X_j^{\,2} > M\right\}.$$

But $n^{-1}\sum_{j=1}^{n} X_j^{\,2}$ is the arithmetic average of n independent, identically distributed random variables with expected value $E(X_j^{\,2}) = M_0 < M$. By the weak law of large numbers, the arithmetic average converges in probability to M_0, hence $P\{\mathbf{X} \notin F\} \to 0$ as $n \to \infty$.

Now we note that the output sequence $\mathbf{Y}$ is given by $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$, where the components of $\mathbf{X}$ are independent Gaussian random variables with mean 0 and variance M_0, and the components of $\mathbf{Z} = \mathbf{Y} - \mathbf{X}$ are independent Gaussian random variables with mean 0 and variance N [see (8.2.1)]. By the form of (8.2.1), the conditional distribution of $\mathbf{Z}$ given $\mathbf{X}$ is the same as the (unconditional) distribution of $\mathbf{Z}$, and hence $\mathbf{X}$ and $\mathbf{Z}$ are independent. Since sums of independent normally distributed random variables are also normally distributed, the components of $\mathbf{Y}$ are independent Gaussian random variables with mean 0 and variance $N + M_0$. With this in mind, let us examine the second term of (8.2.2). We have

$$\log \frac{p(\mathbf{y} \mid \mathbf{x})}{p(\mathbf{y})} \le a$$

$$\Leftrightarrow \log\left[\frac{(2\pi N)^{-\frac{1}{2}n} \exp\left(-\sum_{j=1}^{n}(y_j - x_j)^2/2N\right)}{[2\pi(N + M_0)]^{-\frac{1}{2}n} \exp\left(-\sum_{j=1}^{n} y_j^{\,2}/2(N + M_0)\right)}\right] \le a$$

$$\Leftrightarrow \tfrac{1}{2}n \log\left(1 + \frac{M_0}{N}\right) + \frac{1}{2}\sum_{j=1}^{n}\left[\frac{y_j^{\,2}}{(N + M_0)} - \frac{(y_j - x_j)^2}{N}\right] \le a.$$

Let $W_j = Y_j^{\,2}/2(N + M_0) - Z_j^{\,2}/2N$. Since $E(Y_j^{\,2}) = N + M_0$ and $E(Z_j^{\,2}) = N$, W_j has mean 0. We may compute the variance of W_j as follows:

$$\operatorname{Var} W_j = E(W_j^{\,2}) = \frac{1}{4}\left\{E\left[\left(\frac{Y_j}{\sqrt{N + M_0}}\right)^4\right] + E\left[\left(\frac{Z_j}{\sqrt{N}}\right)^4\right] - \frac{2E(Y_j^{\,2}Z_j^{\,2})}{N(N + M_0)}\right\}.$$

Since $Y_j/\sqrt{N + M_0}$ and $Z_j/\sqrt{N}$ are each normal (0, 1),

$$E\left[\left(\frac{Y_j}{\sqrt{N + M_0}}\right)^4\right] = E\left[\left(\frac{Z_j}{\sqrt{N}}\right)^4\right] = 3.$$

Also $E(Y_j^2Z_j^2) = E[(X_j^2 + 2X_jZ_j + Z_j^2)Z_j^2]$; since X_j and Z_j are independent,

$$E(Y_j^2Z_j^2) = E(X_j^2)E(Z_j^2) + 2E(X_j)E(Z_j^3) + E(Z_j^4) = M_0N + 0 + 3N^2.$$

Thus

$$\operatorname{Var} W_j = \frac{1}{4}\left[6 - \frac{2N(M_0 + 3N)}{N(M_0 + N)}\right] = \frac{M_0}{N + M_0}.$$

Now let $V_n = \sum_{j=1}^n W_j$. Then $E(V_n) = 0$, $\operatorname{Var} V_n = \sum_{j=1}^n \operatorname{Var} W_j = nM_0/(N + M_0)$ by independence of the W_j; furthermore,

$$\log \frac{p(\mathbf{y} \mid \mathbf{x})}{p(\mathbf{y})} \le a \Leftrightarrow \tfrac{1}{2}n \log\left(1 + \frac{M_0}{N}\right) + V_n \le a.$$

Let us choose $a = \frac{1}{2}n \log(1 + M_0/N) - n\delta$, where $\delta > 0$. With this choice, and with the aid of Chebyshev's inequality, the second term of (8.2.2) becomes

$$\begin{aligned} P\{(\mathbf{X}, \mathbf{Y}) \notin A\} &= P\left\{(\mathbf{x}, \mathbf{y})\colon \log \frac{p(\mathbf{y} \mid \mathbf{x})}{p(\mathbf{y})} \le a\right\} \\ &= P\{V_n \le -n\delta\} \le \frac{E(V_n^2)}{n^2\delta^2} = \frac{M_0}{(N + M_0)\delta^2} \frac{1}{n}. \end{aligned}$$

Thus $P\{(\mathbf{X}, \mathbf{Y}) \notin A\} \to 0$ as $n \to \infty$.

Finally, in Lemma 8.2.1 let $u = [e^{nR}]$ where $R < \frac{1}{2} \log(1 + M/N)$. The first term of (8.2.2) becomes

$$u \cdot e^{-a} \le e^{nR} e^{-\frac{1}{2}n \log(1+M_0/N)} e^{n\delta}.$$

If M_0 is chosen sufficiently close to M, and δ sufficiently close to 0, we will have $R - \frac{1}{2}\log(1 + M_0/N) + \delta < 0$, and $u \cdot e^{-a}$ will approach 0 as $n \to \infty$. The theorem follows.

8.3. Uncertainty in the continuous case

Before proving the converse to the coding theorem, we must extend the notion of uncertainty to the continuous case, and examine some of the properties of the resulting uncertainty function. If X is an absolutely continuous random variable, that is, a random variable having a probability density function $p(x)$, we define the *uncertainty of* X as

$$H(X) = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx,$$

provided the integral exists.

In contrast to the situation in the discrete case, $H(X)$ may be arbitrarily large, positive or negative. To see this, let X be uniformly distributed between 0 and b. We then have $H(X) = \log b$, which takes on all real values as b ranges over $(0, \infty)$. In fact $H(X)$ may be $+\infty$ or $-\infty$. For example, let $p(x) = x^{-1}(\log x)^{-2}$, $x \geq e$; $p(x) = 0$, $x < e$. Note that

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_{e}^{\infty} \frac{d(\log x)}{(\log x)^2} = 1.$$

Then

$$H(X) = -\int_{e}^{\infty} x^{-1}(\log x)^{-2}(-\log x - 2\log\log x)\,dx$$

$$\geq \int_{e}^{\infty} \frac{dx}{x\log x} = \int_{e}^{\infty} \frac{d(\log x)}{\log x} = +\infty.$$

As another example, let I_n $(n = 2, 3, 4, \ldots)$ be any system of disjoint intervals of real numbers such that the length of I_n is $(n\log n)^{-2}$. Let $K = \sum_{n=2}^{\infty} n^{-1}(\log n)^{-2}$; note that $K < \infty$ since $\int_2^{\infty} x^{-1}(\log x)^{-2}\,dx < \infty$. Define $p(x) = n/K$ if $x \in I_n$ $(n = 2, 3, 4, \ldots)$; $p(x) = 0$ otherwise. Then

$$\int_{-\infty}^{\infty} p(x)\,dx = K^{-1}\sum_{n=2}^{\infty} n^{-1}(\log n)^{-2} = 1,$$

so that $p(x)$ is a probability density function. Furthermore,

$$H(X) = -\sum_{n=2}^{\infty} \frac{(n/K)\log(n/K)}{(n\log n)^2} = \log K - K^{-1}\sum_{n=2}^{\infty}(n\log n)^{-1}$$

But $\sum_{n=2}^{\infty}(n\log n)^{-1} = \infty$ since $\int_2^{\infty}(x\log x)^{-1}\,dx = \infty$, so that

$$H(X) = -\infty.$$

Note that in the above example, $\bigcup_{n=2}^{\infty} I_n$ may be taken to be a bounded set; if this is done, then all the moments of X are finite, and yet $H(X) = -\infty$. We shall see later in this section that the finiteness of the variance of X implies that $H(X) < \infty$. If $p(x) \leq L < \infty$ for all x then $H(X) \geq -\log L > -\infty$, so that boundedness of the density is sufficient to eliminate the case $H(X) = -\infty$.

Now if X and Y are random variables which are defined on the same sample space and which have a joint density $p(x, y)$, we may define the *joint uncertainty* of X and Y as

$$H(X, Y) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log p(x, y)\, dx\, dy$$

and the *conditional uncertainty of Y given X* as

$$H(Y \mid X) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log p(y \mid x)\, dx\, dy$$

where $p(y \mid x) = p(x, y)/p(x)$ is the conditional density of Y given X.

We may prove results that are the continuous analog of the results of Section 1.4. The following lemma is useful.

Lemma 8.3.1. Let $p(x)$ and $q(x)$ be arbitrary probability density functions.

a. If $-\int_{-\infty}^{\infty} p(x) \log q(x)\, dx$ is finite, then $-\int_{-\infty}^{\infty} p(x) \log p(x)\, dx$ exists, and furthermore

$$-\int_{-\infty}^{\infty} p(x) \log p(x)\, dx \le -\int_{-\infty}^{\infty} p(x) \log q(x)\, dx, \tag{8.3.1}$$

with equality if and only if $p(x) = q(x)$ for almost all x (with respect to Lebesgue measure).

b. If $-\int_{-\infty}^{\infty} p(x) \log p(x)\, dx$ is finite, then $-\int_{-\infty}^{\infty} p(x) \log q(x)\, dx$ exists, and (8.3.1) holds.

Proof. We will prove (a); the argument of (b) is quite similar. Since $\log b \le b - 1$ with equality if and only if $b = 1$ (see Fig. 1.4.1) we have $p(x) \log [q(x)/p(x)] \le q(x) - p(x)$, with equality if and only if $p(x) = q(x)$. (Define $q(x)/p(x) = 0$ when $p(x) = q(x) = 0$, and take $0 \cdot \infty = 0$.) Thus

$$\int_{-\infty}^{\infty} p(x) \log \frac{q(x)}{p(x)}\, dx \le \int_{-\infty}^{\infty} q(x)\, dx - \int_{-\infty}^{\infty} p(x)\, dx = 1 - 1 = 0,$$

with equality if and only if $p(x) = q(x)$ for almost all x. Now

$$-p(x) \log p(x) = p(x) \log \frac{q(x)}{p(x)} - p(x) \log q(x). \tag{8.3.2}$$

(Note that there can be no set of positive measure on which $q(x) = 0$ and $p(x) > 0$, for if there were, $-\int_{-\infty}^{\infty} p(x) \log q(x)\, dx$ would be $+\infty$. Thus the right side of the previous equation will not be of the form $-\infty + \infty$ except possibly on a set of measure 0.)

It follows that

$$-\int_{-\infty}^{\infty} p(x) \log p(x)\, dx \leq -\int_{-\infty}^{\infty} p(x) \log q(x)\, dx.$$

Finally, if equality holds in (8.3.1), the finiteness of $-\int_{-\infty}^{\infty} p(x) \log q(x)\, dx$ allows us to conclude from (8.3.2) that

$$\int_{-\infty}^{\infty} p(x) \log \frac{q(x)}{p(x)}\, dx = 0,$$

and hence that $p(x) = q(x)$ for almost all x. The lemma is proved.

Lemma 8.3.1 has the following immediate consequences.

Lemma 8.3.2. Let X and Y be random variables with a joint density $p(x, y)$. Assume that $H(X)$ and $H(Y)$ are both finite. Then:

a. $H(X, Y)$ exists, and $H(X, Y) \leq H(X) + H(Y)$, with equality if and only if X and Y are independent.

b. $H(Y \mid X)$ and $H(X \mid Y)$ exist, and $H(X, Y) = H(X) + H(Y \mid X) = H(Y) + H(X \mid Y)$.

c. $H(Y \mid X) \leq H(Y)$ [also $H(X \mid Y) \leq H(X)$], with equality if and only if X and Y are independent.

Thus if we define the *information conveyed about X by Y* by $I(X \mid Y) = H(X) - H(X \mid Y)$, we have $I(X \mid Y) = I(Y \mid X)$, and $I(X \mid Y) \geq 0$, with equality if and only if X and Y are independent.

Proof. We note that by Lemma 8.3.1(a),

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log p(x, y)\, dx\, dy \leq -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log p(x)p(y)\, dx\, dy,$$

with equality if and only if $p(x, y) = p(x)p(y)$ for almost all (x, y). The hypothesis that $H(X)$ and $H(Y)$ are finite implies that the integral on the right is finite, so that the application of Lemma 8.3.1 is legitimate. This proves (a).

To prove (b), we note that

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log p(y \mid x)\, dx\, dy = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log p(x, y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log p(x)\, dx\, dy.$$

[The first integral on the right exists and is $< \infty$ by (a); the second integral on the right is finite by hypothesis. Thus the integral on the left exists, and we have $H(Y \mid X) = H(X, Y) - H(X)$.]

Finally, (c) follows immediately from (a) and (b) since $H(X, Y) = H(X) + H(Y \mid X) \leq H(X) + H(Y)$, with equality if and only if X and Y are independent.

Just as in Section 1.4, we may extend the notion of uncertainty to random vectors. For example, if $X = (X_1, \ldots, X_n)$, and $X_1, \ldots, X_n$ have a joint density $p(x_1, \ldots, x_n)$, we define

$$H(X) = -\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_1, \ldots, x_n) \log p(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

The above results will remain valid in the more general context.

We shall now prove a series of results which will be needed in the proof of the converse to the coding theorem for the time-discrete Gaussian channel. First, we show that the Gaussian distribution has maximum uncertainty among all distributions with a given variance.

Theorem 8.3.3. Let X be an absolutely continuous random variable with density $p(x)$. If X has finite variance σ^2, then $H(X)$ exists, and $H(X) \leq \frac{1}{2} \log (2\pi e \sigma^2)$, with equality if and only if X is Gaussian (with variance σ^2).

Proof. Let X be an arbitrary random variable with density $p(x)$, mean μ, and variance σ^2. Let $q(x) = (2\pi\sigma^2)^{-1/2} e^{-(x-\mu)^2/2\sigma^2}$. But

$$-\int_{-\infty}^{\infty} p(x) \log q(x)\, dx = \int_{-\infty}^{\infty} p(x) \left[\frac{1}{2} \log (2\pi\sigma^2) + \frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \tfrac{1}{2} \log (2\pi\sigma^2) + \frac{\sigma^2}{2\sigma^2} = \tfrac{1}{2} \log (2\pi e \sigma^2).$$

The result now follows from Lemma 8.3.1a.

We are going to be interested in situations in which a code word is chosen at random and then transmitted through a Gaussian channel.

Since the number of words in a code is finite, we are applying a discrete input to a channel and receiving a continuous output. We must examine the interplay between the uncertainty functions in the discrete and continuous cases.

Let X be a discrete random variable that takes on the values $x_1, \ldots, x_M$. Suppose that Y is a random variable such that for each i, Y has a conditional density given $X = x_i$, say $p(y \mid x_i)$. (In the case that will be of most interest to us, $Y = X + Z$, where Z is a Gaussian random variable.) Then Y is absolutely continuous, with density

$$p(y) = \sum_{i=1}^{M} p(x_i)p(y \mid x_i).$$

We may define the *uncertainty of Y given that* $X = x_i$ as

$$H(Y \mid X = x_i) = -\int_{-\infty}^{\infty} p(y \mid x_i) \log p(y \mid x_i)\, dy,$$

and the *uncertainty of Y given X* as

$$H(Y \mid X) = \sum_{i=1}^{M} p(x_i)H(Y \mid X = x_i).$$

Furthermore, we may define the *uncertainty of X given that* $Y = y$ as

$$H(X \mid Y = y) = -\sum_{i=1}^{M} p(x_i \mid y) \log p(x_i \mid y),$$

where $p(x_i \mid y) = p(x_i)p(y \mid x_i)/p(y)$ is the conditional probability that $X = x_i$ given $Y = y$. The *uncertainty of X given Y* is defined as

$$H(X \mid Y) = \int_{-\infty}^{\infty} p(y)H(X \mid Y = y)\, dy.$$

[Notice that the above definitions closely parallel the corresponding definitions in the discrete case; furthermore, $H(X \mid Y)$ is always finite and in fact is bounded by $\log M$.]

Now assume for simplicity that $H(Y \mid X = x_i)$ is finite for each i, and that $H(Y)$ is finite. It follows that

$$H(Y \mid X) \leq H(Y), \tag{8.3.3}$$

with equality if and only if X and Y are independent. For

$$H(Y \mid X) = -\sum_{i=1}^{M} p(x_i) \int_{-\infty}^{\infty} p(y \mid x_i) \log p(y \mid x_i)\, dy.$$

As in the proof of Lemma 8.3.1,

$$\int_{-\infty}^{\infty} p(y \mid x_i) \log \frac{p(y)}{p(y \mid x_i)}\, dy \le \int_{-\infty}^{\infty} p(y \mid x_i)\left[\frac{p(y)}{p(y \mid x_i)} - 1\right] dy = 0,$$

with equality if and only if $p(y) = p(y \mid x_i)$ for almost all y. Thus

$$\sum_{i=1}^{M} p(x_i) \int_{-\infty}^{\infty} p(y \mid x_i) \log \frac{p(y)}{p(y \mid x_i)}\, dy \le 0,$$

with equality if and only if $p(y) = p(y \mid x_i)$ for almost all (x_i, y); in other words, $H(Y \mid X) \le H(Y)$, with equality if and only if X and Y are independent. Furthermore,

$$H(Y) + H(X \mid Y) = H(X) + H(Y \mid X). \tag{8.3.4}$$

For

$$\begin{aligned} H(X \mid Y) &= \int_{-\infty}^{\infty} p(y)\left[-\sum_{i=1}^{M} p(x_i \mid y) \log p(x_i \mid y)\right] dy \\ &= -\sum_{i=1}^{M} p(x_i) \int_{-\infty}^{\infty} p(y \mid x_i) \log \left[\frac{p(x_i)p(y \mid x_i)}{p(y)}\right] dy \\ &= H(X) + H(Y \mid X) - H(Y). \end{aligned}$$

We define the *information conveyed about X by Y*, just as in the discrete case, by

$$I(X \mid Y) = H(X) - H(X \mid Y).$$

By (8.3.4),

$$I(X \mid Y) = H(Y) - H(Y \mid X);$$

furthermore, by (8.3.3), $I(X \mid Y) \ge 0$, that is,

$$H(X \mid Y) \le H(X), \tag{8.3.5}$$

with equality if and only if X and Y are independent.

We now prove a convexity result that is analogous to Theorem 3.3.1.

Theorem 8.3.4. Let X be a discrete random variable with values $x_1, \ldots, x_M$; let Y be a random variable such that for each i, Y has a conditional density $p(y \mid x_i)$ given that $X = x_i$. Assume $H(Y \mid X = x_i)$ is finite for $i = 1, 2, \ldots, M$. Suppose that for each $k = 1, 2 \ldots, n$, $p_k(\alpha)$, $\alpha = x_1, \ldots, x_M$, is an assignment of probabilities to the values of X. Let $p_0(\alpha)$ be a convex linear combination of the $p_k(\alpha)$, that is, $p_0(\alpha) = \sum_{k=1}^{n} a_k p_k(\alpha)$, where all $a_k \ge 0$ and $\sum_{k=1}^{n} a_k = 1$.

Then, assuming $H_k(Y)$ is finite for each $k = 1, 2, \ldots, n$,

$$I_0(X \mid Y) \geq \sum_{k=1}^{n} a_k I_k(X \mid Y),$$

where the index k indicates that the quantity in question is computed under the assignment $p_k(\alpha)$.

Proof. We first compute $H_0(Y \mid X)$:

$$H_0(Y \mid X) = -\sum_{i=1}^{M} p_0(x_i) \int_{-\infty}^{\infty} p_0(y \mid x_i) \log p_0(y \mid x_i)\, dy.$$

Since $p_0(y \mid x_i)$ is just the conditional density $p(y \mid x_i)$,

$$\begin{aligned} H_0(Y \mid X) &= -\sum_{k=1}^{n} \sum_{i=1}^{M} a_k p_k(x_i) \int_{-\infty}^{\infty} p(y \mid x_i) \log p(y \mid x_i)\, dy \\ &= \sum_{k=1}^{n} a_k H_k(Y \mid X). \end{aligned} \tag{8.3.6}$$

We must show that $\sum_{k=1}^{n} a_k I_k(X \mid Y) - I_0(X \mid Y) \leq 0$. But

$$\sum_{k=1}^{n} a_k I_k(X \mid Y) - I_0(X \mid Y) = \sum_{k=1}^{n} a_k H_k(Y) - H_0(Y) \qquad \text{[by (8.3.6)]}$$

By hypothesis, $-\int_{-\infty}^{\infty} p_k(y) \log p_k(y)\, dy = H_k(Y)$ is finite, hence by Lemma 8.3.1b, $-\int_{-\infty}^{\infty} p_k(y) \log p_0(y)\, dy$ exists and

$$-\int_{-\infty}^{\infty} p_k(y) \log p_0(y)\, dy \geq -\int_{-\infty}^{\infty} p_k(y) \log p_k(y)\, dy.$$

If we multiply the above inequality by a_k and then sum over $k = 1, 2, \ldots, n$, we see that $H_0(Y)$ exists and $H_0(Y) \geq \sum_{k=1}^{n} a_k H_k(Y)$. The theorem is proved.

Note that Theorem 8.3.4 and the discussion preceding it extend immediately to the case in which X and Y are replaced by random vectors.

8.4. *The converse to the coding theorem for the time-discrete Gaussian channel*

We now apply the results of Section 8.3 to prove the converse to the coding theorem. Just as in the case of the discrete memoryless channel, the key step is *Fano's inequality* (compare Theorem 3.7.1).

Theorem 8.4.1. Given an arbitrary code (s, n) for a time-discrete Gaussian channel, consisting of words $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(s)}$. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector which takes on the value $\mathbf{x}^{(i)}$ with probability $p(\mathbf{x}^{(i)})$, $i = 1, 2, \ldots, s$, where $\sum_{i=1}^{s} p(\mathbf{x}^{(i)}) = 1$. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be the corresponding output sequence.

If $p(e)$ is the overall probability of error of the code, computed for the given input distribution, then

$$H(\mathbf{X} \mid \mathbf{Y}) \leq H(p(e), 1 - p(e)) + p(e) \log (s - 1).$$

Proof. The first part of the proof proceeds exactly as in Theorem 3.7.1. If $g(\mathbf{y})$ is the input selected by the decoder when the sequence $\mathbf{y}$ is received, then by the grouping axiom (Section 1.2, Axiom 3),

$$H(\mathbf{X} \mid \mathbf{Y} = \mathbf{y}) = H(q, 1 - q) + qH(1) + (1 - q)H(q_1, \ldots, q_{s-1})$$

where $q = P\{\mathbf{X} = g(\mathbf{y}) \mid \mathbf{Y} = \mathbf{y}\} = 1 - p(e \mid \mathbf{y})$ and the q_i are of the form

$$\frac{p(\mathbf{x} \mid \mathbf{y})}{\sum_{\mathbf{x} \neq g(\mathbf{y})} p(\mathbf{x} \mid \mathbf{y})}$$

Since $H(q_1, \ldots, q_{s-1}) \leq \log (s - 1)$,

$$H(\mathbf{X} \mid \mathbf{Y} = \mathbf{y}) \leq H(p(e \mid \mathbf{y}), 1 - p(e \mid \mathbf{y})) + p(e \mid \mathbf{y}) \log (s - 1). \tag{8.4.1}$$

We now must deviate slightly from the proof of Theorem 3.7.1. As in the discrete case, we must show that the uncertainty of a convex linear combination of probabilities is greater than or equal to the convex linear combination of the uncertainties, but in this case the convex linear combination is an integral rather than a summation. However, we may reason as follows.

Let V be a random variable that equals 1 if a decoding error occurs, and 0 otherwise. Then

$$\int_{-\infty}^{\infty} p(\mathbf{y}) H(p(e \mid \mathbf{y}), 1 - p(e \mid \mathbf{y}))\, d\mathbf{y} = \int_{-\infty}^{\infty} p(\mathbf{y}) H(V \mid \mathbf{Y} = \mathbf{y})\, d\mathbf{y} = H(V \mid \mathbf{Y}).$$

By (8.3.5),

$$H(V \mid \mathbf{Y}) \leq H(V) = H(p(e), 1 - p(e)). \tag{8.4.2}$$

Thus if we multiply (8.4.1) by $p(\mathbf{y})$ and integrate over all $\mathbf{y}$, we find using (8.4.2) that $H(\mathbf{X} \mid \mathbf{Y}) \leq H(p(e), 1 - p(e)) + p(e) \log (s - 1)$, as required. (The fact that the channel is Gaussian implies that all uncertainties appearing in the above proof are finite, and thus there is no difficulty in applying the results of Section 8.3.)

We shall need one more preliminary result.

write

$$|y_k(t_1) - y_k(t_2)|^2 = \left|\int_a^b [R(t_1, \tau) - R(t_2, \tau)]x_k(\tau)\, d\tau\right|^2;$$

by the Schwarz inequality,

$$|y_k(t_1) - y_k(t_2)|^2 \leq \left(\int_a^b |R(t_1, \tau) - R(t_2, \tau)|^2\, d\tau\right) \|x_k\|^2. \qquad (2.3)$$

Now R is uniformly continuous on $[a, b] \times [a, b]$; therefore given $\varepsilon > 0$ there is a $\delta > 0$ such that if $|t_1 - t_2| < \delta$ and $|\tau_1 - \tau_2| < \delta$, then $|R(t_1, \tau_1) - R(t_2, \tau_2)| < \varepsilon N^{-1}(b - a)^{-1/2}$. In particular, if $|t_1 - t_2| < \delta$, then $|R(t_1, \tau) - R(t_2, \tau)| < \varepsilon N^{-1}(b - a)^{-1/2}$ for all τ. Thus by (2.3), $|t_1 - t_2| < \delta$ implies $|y_k(t_1) - y_k(t_2)|^2 \leq \varepsilon^2 N^{-2}(b - a)^{-1}(b - a)\, \|x_k\|^2 < \varepsilon^2$ for all k.

Thus the functions y_n are equicontinuous. By the Arzela-Ascoli theorem, there is a subsequence $y_{n(i)}$ converging uniformly to a continuous function y. But

$$\|y_{n(i)} - y\|^2 = \int_a^b |y_{n(i)}(t) - y(t)|^2\, dt \to 0 \quad \text{as} \quad i \to \infty$$

since the uniform convergence allows the limit to be taken inside the integral sign; thus $y_{n(i)} \to y$ in L_2 and the result follows.

Theorem 2.2. Let R be a real-valued function on $[a, b] \times [a, b]$ which is *symmetric* $[R(t, \tau) = R(\tau, t)$ for all $t, \tau]$ as well as continuous, so that the associated operator A defined by (2.1) is symmetric as well as compact. Then all eigenvalues of A are nonnegative if and only if

$$\int_a^b\int_a^b R(t, \tau)x(t)x(\tau)\, dt\, d\tau \geq 0 \quad \text{for all continuous functions } x \text{ on } [a, b]. \qquad (2.4)$$

Proof. Suppose that the double integral is nonnegative for all continuous x. Let x be an eigenfunction corresponding to the nonzero eigenvalue λ; x is continuous by (2.2). The double integral (2.4) becomes $\lambda\int_a^b x^2(t)\, dt = \lambda\, \|x\|^2$. We may take $x \neq 0$; it follows that $\lambda > 0$.

Conversely, suppose that all eigenvalues of A are nonnegative. If x is continuous on $[a, b]$, then $x \in L_2[a, b]$; and hence by Theorem 1.14, $x = h + \sum_{n=1}^{\infty} c_n e_n$ where the e_n are an orthonormal basis for the space spanned by the eigenvectors corresponding to the nonzero eigenvalues, $c_n = (x, e_n)$, and $Ah = 0$. Now the double integral (2.4) may be written as (x, Ax); by Lemma 1.12 we have

$$(x, Ax) = (x, Ah) + \sum_{n=1}^{\infty} c_n(x, Ae_n)$$
$$= (x, Ah) + \sum_{n=1}^{\infty} c_n\lambda_n(x, e_n) = \sum_{n=1}^{\infty} \lambda_n c_n^2 \geq 0.$$

The theorem is proved.

Note that in Theorem 2.2, the phrase "for all continuous x on $[a, b]$" in (2.4) may be replaced by "for all $x \in L_2[a, b]$"; essentially the same proof may be used.

Theorem 2.3. Let A be the integral operator (2.1) corresponding to a function R which is *nonnegative definite* as well as continuous and symmetric. (R is said to be nonnegative definite if

$$\sum_{i,j=1}^{n} x(t_i)R(t_i, t_j)x(t_j) \geq 0$$

for all possible choices of $t_1, \ldots, t_n \in [a, b]$, and all possible real-valued functions x on $[a. b]$.) Then all eigenvalues of A are nonnegative.

Proof. If x is continuous, then

$$\int_a^b \int_a^b R(t, \tau)x(t)x(\tau)\, dt\, d\tau$$

is an ordinary Riemann integral; the approximating Riemann sums are of the form $\sum_{i,j=1}^{n} x(t_i)R(t_i, t_j)x(t_j)$. Since R is nonnegative definite, the approximating sums are all nonnegative, and hence the integral is nonnegative. The result follows from Theorem 2.2.

Theorem 2.4. (Mercer's Theorem). Let A be the integral operator (2.1) corresponding to a function R that is continuous, symmetric, and nonnegative definite. Let $\{e_n, n = 1, 2, \ldots\}$ be an orthonormal basis for the space spanned by the eigenvectors corresponding to the nonzero (hence positive) eigenvalues of A. If the basis is taken so that e_n is an eigenvector corresponding to the eigenvalue λ_n, then

$$R(t, \tau) = \sum_{n=1}^{\infty} \lambda_n e_n(t)e_n(\tau), \qquad t, \tau \in [a, b] \tag{2.5}$$

where the series converges absolutely, converges to $R(t, \tau)$ uniformly in both variables, and also converges to $R(t, \tau)$ in $L_2([a, b] \times [a, b])$, that is,

$$\int_a^b \int_a^b |R(t, \tau) - \sum_{k=1}^{n} \lambda_k e_k(t) e_k(\tau)|^2 \, dt \, d\tau \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. We break the proof into several parts.

a. $R(t, \tau) = \sum_{n=1}^{\infty} \lambda_n e_n(t) e_n(\tau)$, where the series converges in L_2 in each variable separately.

Fix t and apply Theorem 1.14 to $R(t, \tau)$. We obtain

$$R(t, \tau) = h(t, \tau) + \sum_{n=1}^{\infty} c_n(t) e_n(\tau), \tag{2.6}$$

where

$$\int_a^b R(t, \tau) h(t, \tau) \, d\tau = 0, \qquad c_n(t) = \int_a^b R(t, \tau) e_n(\tau) \, d\tau = \lambda_n e_n(t),$$

and the series converges in $L_2[a, b]$. It follows from (2.6) and Lemma 1.12 that

$$\int_a^b R(t, \tau) h(t, \tau) \, d\tau = \int_a^b h^2(t, \tau) \, d\tau + \sum_{n=1}^{\infty} c_n(t) \int_a^b h(t, \tau) e_n(\tau) \, d\tau.$$

By Lemma 1.9,

$$\int_a^b h(t, \tau) e_n(\tau) \, d\tau = 0, \qquad n = 1, 2, \ldots .$$

Hence $0 = \int_a^b h^2(t, \tau) \, d\tau$. Therefore for each t, $h(t, \cdot) = 0$ in $L_2[a, b]$ and the result (a) is proved.

b. Let $R_n(t, \tau) = R(t, \tau) - \sum_{i=1}^{n} \lambda_i e_i(t) e_i(\tau)$; then $R_n(t, t) \geq 0$ for all t. By (a),

$$R_n(t, \tau) = \sum_{i=n+1}^{\infty} \lambda_i e_i(t) e_i(\tau),$$

with L_2 convergence in each variable separately.

If $x \in L_2[a, b]$ then, using Lemma 1.12, we obtain

$$\int_a^b \int_a^b R_n(t, \tau) x(t) x(\tau) \, dt \, d\tau = \sum_{i=n+1}^{\infty} \lambda_i \, |(x, e_i)|^2 \geq 0. \tag{2.7}$$

Now if $R_n(t_0, t_0) < 0$ then by continuity, $R_n(t, \tau) \leq -\varepsilon < 0$ for some positive ε and for t, τ in some neighborhood U of (t_0, t_0) described by

$t_0 - \alpha < t < t_0 + \alpha$, $t_0 - \alpha < \tau < t_0 + \alpha$. Now define $x(t) = 1$ for $t_0 - \alpha < t < t_0 + \alpha$; $x(t) = 0$ elsewhere. Then $x \in L_2[a, b]$ and

$$\int_a^b \int_a^b R_n(t, \tau)x(t)x(\tau)\,dt\,d\tau = \iint_U R_n(t, \tau)\,dt\,d\tau \leq -4\alpha^2\varepsilon < 0,$$

contradicting (2.7).

c. The series $\sum_{n=1}^{\infty} \lambda_n e_n(t)e_n(\tau)$ converges absolutely; the convergence is uniform in τ for each fixed t, and uniform in t for each fixed τ.

By (b),

$$R_n(t, t) = R(t, t) - \sum_{i=1}^{n} \lambda_i e_i^2(t) \geq 0,$$

hence

$$\sum_{i=1}^{\infty} \lambda_i e_i^2(t) \leq R(t, t) \leq M < \infty.$$

Now by the Schwarz inequality for sums,

$$\left|\sum_{i=m}^{n} \lambda_i e_i(t)e_i(\tau)\right|^2 = \left|\sum_{i=m}^{n} \sqrt{\lambda_i}e_i(t)\sqrt{\lambda_i}e_i(\tau)\right|^2 \leq \sum_{i=m}^{n} \lambda_i e_i^2(t) \sum_{i=m}^{n} \lambda_i e_i^2(\tau)$$

$$\leq \sum_{i=m}^{n} \lambda_i e_i^2(t) \sum_{i=1}^{\infty} \lambda_i e_i^2(\tau) \leq M \sum_{i=m}^{n} \lambda_i e_i^2(t) \to 0 \quad \text{as} \quad m, n \to \infty,$$

uniformly in τ for each fixed t. The assertion (c) follows.

d. The series $\sum_{n=1}^{\infty}\lambda_n e_n(t)e_n(\tau)$ converges pointwise to $R(t, \tau)$.

Fix t. By (a), $\sum_{n=1}^{\infty}\lambda_n e_n(t)e_n(\tau) = R(t, \tau)$ with L_2 convergence. By (c), $\sum_{n=1}^{\infty}\lambda_n e_n(t)e_n(\tau) =$ some function $S(t, \tau)$ with uniform, hence L_2, convergence. It follows that for each t, $R(t, \tau) = S(t, \tau)$ for almost every τ. But $R(t, \cdot)$ is continuous and $S(t, \cdot)$, the uniform limit of continuous functions, is also continuous. Thus $R(t, \tau) = S(t, \tau)$ for *every* τ, that is, $R(t, \tau) \equiv S(t, \tau)$.

e. (Dini's Theorem). Given a sequence of real-valued functions $\{g_n, n = 1, 2, \ldots\}$ continuous on $[a, b]$ and a real-valued function g continuous on $[a, b]$ such that $g_n(x) \leq g_{n+1}(x)$ for all x and all n, and $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \in [a, b]$. Then $g_n \to g$ uniformly on $[a, b]$. The same conclusion holds if the sequence is monotone decreasing instead of increasing.

Given $\varepsilon > 0$, let $U_n = \{x \in [a, b]\colon\ |g_n(x) - g(x)| < \varepsilon\}$. Since $g_n(x) \to g(x)$ for all x, $\bigcup_{n=1}^{\infty} U_n = [a, b]$. Thus we have a family of open sets that cover $[a, b]$; by the Heine-Borel theorem, a finite subfamily covers $[a, b]$; that is, $\bigcup_{n=1}^{N} U_n = [a, b]$ for some N. By the monotonicity of the sequence $\{g_n\}$, $U_n \subset U_{n+1}$ for all n, and hence $\bigcup_{n=1}^{N} U_n = U_N$.

Consequently $U_N = [a, b]$, so that $|g_N(x) - g(x)| < \varepsilon$ for all x. By monotonicity, $|g_n(x) - g(x)| < \varepsilon$ for all x if $n \geq N$; the result follows.

Proof of Mercer's Theorem. By (d), $R(t, t) = \sum_{n=1}^{\infty} \lambda_n e_n^2(t)$ with pointwise convergence. Let $g_n(t) = \sum_{i=1}^{n} \lambda_i e_i^2(t)$. Then g_n is continuous, $g_n(t) \leq g_{n+1}(t)$ for all t, and $g_n(t) \to R(t, t)$ for all t. By (e), $g_n(t) \to R(t, t)$ uniformly on $[a, b]$. Now as in (c),

$$\left| \sum_{i=m}^{n} \lambda_i e_i(t) e_i(\tau) \right|^2 \leq M \sum_{i=m}^{n} \lambda_i e_i^2(t) \to 0 \quad \text{as} \quad m, n \to \infty,$$

uniformly in t *and* τ. The proof is complete.

3. *The Karhunen-Loeve theorem*

Let $X(t)$, $t \in [a, b]$ (a, b finite), be a continuous-parameter second-order random process with zero mean and covariance function $R(t, \tau)$. In other words, for each $t \in [a, b]$, $X(t)$ is a (real-valued) random variable; $E[X(t)] = 0$ for all $t \in [a, b]$, and $E[X(t)X(\tau)] = R(t, \tau)$ for $t, \tau \in [a, b]$. Note that R is nonnegative definite, since

$$\sum_{i,j=1}^{n} c_i R(t_i, t_j) c_j = E\left[\left(\sum_{i=1}^{n} c_i X(t_i)\right)^2\right] \geq 0.$$

If g is a real-valued function on $[a, b]$, we define $\int_a^b g(t)X(t)\, dt$ as follows. Let $\Delta: a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b]$, with $|\Delta| = \max_{1 \leq i \leq n} |t_i - t_{i-1}|$. Define

$$I(\Delta) = \sum_{k=1}^{n} g(t_k) X(t_k)(t_k - t_{k-1}).$$

If the family of random variables $\{I(\Delta)\}$ converges in mean square to a random variable I as $|\Delta| \to 0$ [that is, $E[|I(\Delta) - I|^2] \to 0$ as $|\Delta| \to 0$] then we say that $g(t)X(t)$ is *integrable* over $[a, b]$. I is called the *integral* of $g(t)X(t)$ over $[a, b]$ and we write $I = \int_a^b g(t)X(t)\, dt$.

If Ω (with an appropriate sigma-algebra of subsets and an appropriate probability measure) is the probability space on which the $X(t)$ are defined, then mean square convergence is convergence in the Hilbert space $L_2(\Omega)$ of random variables on Ω with finite second moments.

Theorem 3.1. If g is continuous on $[a, b]$ and the covariance function R is continuous on $[a, b] \times [a, b]$, then $g(t)X(t)$ is integrable over $[a, b]$.

Proof. Let

$$\Delta : a = t_0 < t_1 < \cdots < t_n = b, \qquad \Delta' : a = \tau_0 < \tau_1 < \cdots < \tau_m = b$$

be two partitions of $[a, b]$. Then

$$I(\Delta)I(\Delta') = \sum_{k=1}^{n} \sum_{j=1}^{m} g(t_k)g(\tau_j)X(t_k)X(\tau_j)(t_k - t_{k-1})(\tau_j - \tau_{j-1}).$$

Thus

$$E[I(\Delta)I(\Delta')] = \sum_{k=1}^{n} \sum_{j=1}^{m} g(t_k)g(\tau_j)R(t_k, \tau_j)(t_k - t_{k-1})(\tau_j - \tau_{j-1}). \tag{3.1}$$

The right side of (3.1) is an approximating sum to a Riemann integral. Thus

$$\lim_{|\Delta|, |\Delta'| \to 0} E[I(\Delta)I(\Delta')] = \int_a^b \int_a^b g(t)g(\tau)R(t, \tau)\, dt\, d\tau. \tag{3.2}$$

By (3.2),

$$E[(I(\Delta) - I(\Delta'))^2] = E[I^2(\Delta)] - 2E[I(\Delta)I(\Delta')] + E[I^2(\Delta')] \to 0$$
$$\text{as} \quad |\Delta|, |\Delta'| \to 0.$$

Thus $\{I(\Delta)\}$ is a generalized Cauchy sequence in $L_2(\Omega)$, and hence $I(\Delta)$ converges in mean square to a random variable I as $|\Delta| \to 0$. The theorem is proved.

Lemma 3.2. If g and h are continuous on $[a, b]$, and R is continuous on $[a, b] \times [a, b]$, then

$$E\left[\int_a^b g(t)X(t)\, dt \int_a^b h(\tau)X(\tau)\, d\tau\right] = \int_a^b \int_a^b g(t)h(\tau)R(t, \tau)\, dt\, d\tau$$

Furthermore, $E\left[\int_a^b g(t)X(t)\, dt\right] = E\left[\int_a^b h(\tau)X(\tau)\, d\tau\right] = 0.$

Proof. As in Theorem 3.1, let Δ and Δ' be two partitions of $[a, b]$. Let

$$I(\Delta) = \sum_{k=1}^{n} g(t_k)X(t_k)(t_k - t_{k-1}), \qquad J(\Delta') = \sum_{j=1}^{m} h(\tau_j)X(\tau_j)(\tau_j - \tau_{j-1}),$$

$$I = \int_a^b g(t)X(t)\, dt, \qquad J = \int_a^b h(\tau)X(\tau)\, d\tau.$$

By Theorem 3.1, $I(\Delta) \to I$, $J(\Delta') \to J$ in mean square as $|\Delta|, |\Delta'| \to 0$. It follows that $E[I(\Delta)J(\Delta')] \to E[IJ]$. For we have

$$\begin{aligned} |E[I(\Delta)J(\Delta') - IJ]| &\leq E\,[|I(\Delta)(J(\Delta') - J)|] + E[|(I(\Delta) - I)J|] \\ &\leq (E[I^2(\Delta)]E[(J(\Delta') - J)^2])^{1/2} \\ &\qquad + (E[J^2]E[(I(\Delta) - I)^2])^{1/2} \\ &\qquad\qquad \text{(by the Schwarz inequality)} \\ &\to \sqrt{E[I^2]} \cdot 0 + \sqrt{E[J^2]} \cdot 0 = 0. \end{aligned}$$

But as in (3.2),

$$E[I(\Delta)J(\Delta')] \to \int_a^b \int_a^b g(t)h(\tau)R(t, \tau)\, dt\, d\tau.$$

Similarly, $I(\Delta) \to I$ in mean square implies $E[I(\Delta)] \to E(I)$. Since $E[I(\Delta)] = 0$ for all Δ, the lemma follows.

Lemma 3.3. If h is continuous on $[a, b]$ and R is continuous on $[a, b] \times [a, b]$, then

$$E\left[X(t) \int_a^b h(\tau)X(\tau)\, d\tau\right] = \int_a^b h(\tau)R(t, \tau)\, d\tau, \qquad t \in [a, b].$$

Proof. Let $J(\Delta') = \sum_{j=1}^m h(\tau_j)X(\tau_j)(\tau_j - \tau_{j-1})$, and let

$$J = \int_a^b h(\tau)X(\tau)\, d\tau.$$

Then $J(\Delta') \to J$ in mean square as $|\Delta'| \to 0$. Just as in the argument of Lemma 3.2, $E[X(t)J(\Delta')] \to E[X(t)J]$. But [compare (3.2)]

$$E[X(t)J(\Delta')] \to \int_a^b h(\tau)R(t, \tau)\, d\tau,$$

and the lemma follows.

Theorem 3.4. (Karhunen-Loève Expansion). Let $X(t)$, $t \in [a, b]$, (a, b finite), be a continuous-parameter second-order random process with zero mean and continuous covariance function $R(t, \tau)$. Then we may write

$$X(t) = \sum_{k=1}^{\infty} Z_k e_k(t), \qquad a \leq t \leq b, \tag{3.3}$$

where the e_k are eigenfunctions of the integral operator (2.1) corresponding to R, and form an orthonormal basis for the space spanned by the eigenfunctions corresponding to the *nonzero* eigenvalues. The Z_k are given by $\int_a^b X(t)e_k(t)\,dt$, and are orthogonal random variables ($E[Z_kZ_j] = 0$ for $k \neq j$) with zero mean and variance λ_k, where λ_k is the eigenvalue corresponding to e_k.

The series $\sum_{k=1}^{\infty} Z_ke_k(t)$ converges in mean square to $X(t)$, *uniformly in* t, that is,

$$E\left[\left(X(t) - \sum_{k=1}^{n} Z_ke_k(t)\right)^2\right] \to 0 \quad \text{as} \quad n \to \infty,$$

uniformly for $t \in [a, b]$.

Proof. By Theorem 3.1, $\int_a^b X(t)e_k(t)\,dt$ exists and defines a random variable Z_k. By Lemma 3.2, $E[Z_k] = 0$ and

$$E(Z_kZ_j) = \int_a^b e_k(t)\left[\int_a^b R(t, \tau)e_j(\tau)\,d\tau\right] dt$$

$$= \lambda_j \int_a^b e_k(t)e_j(t)\,dt = \begin{matrix} 0 & \text{for} & k \neq j \\ \lambda_k & \text{for} & k = j \end{matrix}.$$

Now let $S_n(t) = \sum_{k=1}^{n} Z_ke_k(t)$. Then

$$E[(S_n(t) - X(t))^2] = E[S_n^2(t)] - 2E[S_n(t)X(t)] + E[X^2(t)]$$

$$= \sum_{k=1}^{n} \lambda_k e_k^2(t) - 2\sum_{k=1}^{n} E[X(t)Z_k]e_k(t) + R(t, t)$$

By Lemma 3.3,

$$E[X(t)Z_k] = \int_a^b R(t, \tau)e_k(\tau)\,d\tau = \lambda_k e_k(t).$$

Thus

$$E[(S_n(t) - X(t))^2] = R(t, t) - \sum_{k=1}^{n} \lambda_k e_k^2(t) \to 0 \quad \text{as} \quad n \to \infty,$$

uniformly in t, by Theorem 2.4. The result follows.

The Karhunen-Loève expansion assumes a special form when the process $X(t)$ is Gaussian.

Theorem 3.5. If $X(t)$ is a Gaussian process, then the Z_k of the Karhunen-Loève expansion of $X(t)$ are independent Gaussian random variables.

Proof. The approximating sum to $Z_k = \int_a^b X(t)e_k(t)\,dt$ is

$$I_k(\Delta) = \sum_{i=1}^{n} X(t_i)e_k(t_i)(t_i - t_{i-1}).$$

Since $X(t)$ is a Gaussian process, the $I_k(\Delta)$, $k = 1, 2, \ldots$, form a Gaussian sequence, that is, $I_{k_1}(\Delta), \ldots, I_{k_r}(\Delta)$ are jointly Gaussian for all finite sets of indices $\{k_1, \ldots, k_r\}$. Now the joint characteristic function of $I_{k_1}(\Delta), \ldots, I_{k_r}(\Delta)$ is

$$\begin{aligned} M(t_1, \ldots, t_n) &= E\left[\exp\left(i\sum_{j=1}^{r} t_j I_{k_j}(\Delta)\right)\right] \\ &= \exp\left(-\frac{1}{2}\sum_{j,m=1}^{r} t_j\sigma_{jm}(\Delta)t_m\right) \end{aligned}$$

where

$$i = \sqrt{-1} \quad \text{and} \quad \sigma_{jm}(\Delta) = E[I_{k_j}(\Delta)I_{k_m}(\Delta)].$$

Since $I_k(\Delta)$ converges in mean square to Z_k as $|\Delta| \to 0$ $(k = 1, 2, \ldots)$, it follows as in the argument of Lemma 3.2 that $E[I_{k_j}(\Delta)I_{k_m}(\Delta)] \to E[Z_{k_j}Z_{k_m}]$. Thus the joint characteristic function of $I_{k_1}(\Delta), \ldots, I_{k_r}(\Delta)$ approaches $\exp\left(-\frac{1}{2}\sum_{j,m=1}^{r} t_j\sigma_{jm}t_m\right)$ where $\sigma_{jm} = E[Z_{k_j}Z_{k_m}]$. Thus $Z_{k_1}, \ldots, Z_{k_r}$ are jointly Gaussian for each finite set of indices $\{k_1, \ldots, k_r\}$; that is, the Z_k, $k = 1, 2, \ldots$, form a Gaussian sequence. Since the Z_k are orthogonal random variables, they are independent.

Thus in the Gaussian case, the Karhunen-Loève expansion is a series of independent random variables. It follows (Loève, 1955, p. 251) that *for each fixed t*, the series (3.3) converges with probability 1, that is, for each t, the series converges pointwise to $X(t)$ for almost all points ω in the probability space Ω.

As an example of the Karhunen-Loève expansion, we take $R(t, \tau) = \min(t, \tau)$, with the expansion interval $= [0, 1]$. [If in addition we take $X(t)$ to be Gaussian, we obtain the *Brownian motion* process.] To find the eigenvalues of the integral operator associated with the covariance function, we must solve the integral equation

$$\int_0^1 \min(t, \tau)e(\tau)\,d\tau = \lambda e(t), \qquad 0 \le t \le 1,$$

or equivalently

$$\int_0^t \tau e(\tau)\,d\tau + t\int_t^1 e(\tau)\,d\tau = \lambda e(t), \qquad 0 \le t \le 1. \tag{3.4}$$

If we differentiate (3.4) with respect to t, we obtain

$$\int_t^1 e(\tau)\,d\tau = \lambda \frac{de(t)}{dt}. \tag{3.5}$$

Differentiating again with respect to t, we have

$$-e(t) = \lambda \frac{d^2e(t)}{dt^2}. \tag{3.6}$$

Thus any solution of the integral equation, that is, any eigenfunction, must be of the form

$$e(t) = A \sin \frac{t}{\sqrt{\lambda}} + B \cos \frac{t}{\sqrt{\lambda}}. \tag{3.7}$$

[Note that if $\lambda = 0$ then $e(t) \equiv 0$ by (3.6); thus 0 is not an eigenvalue.] Now if we let $t = 0$ in (3.4) we obtain $e(0) = 0$, and therefore $B = 0$ in (3.7). If we let $t = 1$ in (3.5) we obtain $de(t)/dt = 0$ when $t = 1$, and therefore

$$\cos \frac{1}{\sqrt{\lambda}} = 0, \quad \text{or} \quad \frac{1}{\sqrt{\lambda}} = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, \ldots.$$

Thus the eigenvalues are

$$\lambda_n = \frac{4}{(2n-1)^2\pi^2} \tag{3.8}$$

and the orthonormalized eigenfunctions are

$$e_n(t) = \sqrt{2} \sin (n - \tfrac{1}{2})\pi t. \tag{3.9}$$

[It may be verified by direct substitution that the functions (3.9) satisfy the integral equation (3.4).]

Finally, if we define $Z_k^* = Z_k/\sqrt{\lambda_k}$, where the Z_k are as given in Theorem 3.4, then the Karhunen-Loève expansion becomes

$$X(t) = \sqrt{2} \sum_{n=1}^{\infty} Z_n^* \frac{\sin (n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi} \tag{3.10}$$

where the Z_n^* are a sequence of orthogonal random variables with mean 0 and variance 1.

If $X(t)$ is Gaussian, we may show that for almost all ω, the series (3.10) converges *uniformly* for $0 \le t \le 1$, and therefore may be used to define a process whose sample functions are continuous. For we have

$$\sum_{n=1}^{\infty} \frac{E[(Z_n^*)^2]}{(n - \frac{1}{2})^2\pi^2} = \sum_{n=1}^{\infty} (n - \tfrac{1}{2})^{-2}\pi^{-2} < \infty.$$

It follows that the series

$$\sum_{n=1}^{\infty} \frac{|Z_n^*|}{(n - \frac{1}{2})\pi}$$

converges in mean square, and consequently, being a series of independent random variables, converges with probability 1. The result now follows from the Weierstrass criterion for uniform convergence.

For a further discussion of the theory of second-order random processes, see Loève (1955), Chapter 10.

4. *Further results concerning integral operators determined by a covariance function*

In this section we shall need some standard facts about Fourier transforms on $L_2(-\infty, \infty)$. [See, for example, Goldberg (1961), Titchmarsh (1937).] Consider the Hilbert space of all complex-valued Borel measurable functions f defined on the entire real line, such that $\int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty$; we denote this space by $L_2^*(-\infty, \infty)$. [The norm of f is

$$\|f\| = \left[\int_{-\infty}^{\infty} |f(t)|^2\, dt\right]^{1/2}$$

and the inner product of two functions f and g is

$$\int_{-\infty}^{\infty} f(t)\bar{g}(t)\, dt,$$

where the bar denotes complex conjugate.]

If $f \in L_2^*(-\infty, \infty)$, then the function F_N defined by

$$F_N(\omega) = \int_{-N}^{N} f(t)e^{-i\omega t}\, dt$$

converges in $L_2^*(-\infty, \infty)$ as $N \to \infty$ to a function F, that is, $\|F_N - F\| \to 0$. Conversely, the function f_N defined by

$$f_N(t) = (2\pi)^{-1} \int_{-N}^{N} F(\omega)e^{i\omega t}\, d\omega$$

converges in $L_2^*(-\infty, \infty)$ to f. F is called the *Fourier transform* of f, and f the *inverse Fourier transform* of F. The pair (f, F) is called a *Fourier* *transform pair.* A function in $L_2^*(-\infty, \infty)$ is uniquely determined by its Fourier transform.

If $f, g \in L_2^*(-\infty, \infty)$ with Fourier transforms F, G then the *Parseval relation* states that

$$\int_{-\infty}^{\infty} f(t)\bar{g}(t)\, dt = (2\pi)^{-1} \int_{-\infty}^{\infty} F(\omega)\bar{G}(\omega)\, d\omega;$$

consequently, $\|F\| = (2\pi)^{1/2} \|f\|$.

If $f, g \in L_2^*(-\infty, \infty)$, the *convolution* of f and g is the function h defined by

$$h(t) = \int_{-\infty}^{\infty} f(s)g(t - s)\, ds.$$

[By the Schwarz inequality, h exists but is not necessarily in $L_2^*(-\infty, \infty)$.] If F and G are the Fourier transforms of f and g respectively, and H is the product FG, then the *convolution theorem* is a name for a class of results which state that under certain conditions, (h, H) is a Fourier transform pair. In particular, if $f \in L_1^*(-\infty, \infty)$, that is, $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$, then $h \in L_2^*(-\infty, \infty)$ and the Fourier transform of h is H. Also, if $H \in L_2^*(-\infty, \infty)$, then so is h, and (h, H) is a Fourier transform pair.

Let $n(t)$, $-\infty < t < \infty$, be a second-order stationary random process with continuous covariance function $R(\tau) = E[n(t)n(t + \tau)]$. Assume that the process has a spectral density $N(\omega)$, that is, assume that $R(\tau)$ can be expressed as

$$(2\pi)^{-1} \int_{-\infty}^{\infty} N(\omega)e^{i\omega t}\, d\omega,$$

where N is a nonnegative function integrable over $(-\infty, \infty)$. Let T be a positive real number, and let A_T be the integral operator on $L_2[-T/2, T/2]$ corresponding to the covariance function; that is, the operator defined by

$$y(t) = \int_{-T/2}^{T/2} R(t - \tau)x(\tau)\, d\tau; \qquad -\frac{T}{2} \le t \le \frac{T}{2}. \tag{4.1}$$

We assume that 0 *is not an eigenvalue of* A_T. (Later we shall investigate conditions under which this result holds.) It follows (Theorems 1.14, 2.3) that the eigenfunctions corresponding to the positive eigenvalues of

A_T span the Hilbert space $L_2[-T/2, T/2]$. Let $\{e_n(t; T), -T/2 \le t \le T/2\}$ be an orthonormal basis of $L_2[-T/2, T/2]$, where $e_n(t; T)$ is taken as an eigenfunction corresponding to the positive eigenvalue $\lambda_n(T)$.

Let $G(\omega) = [N(\omega)]^{1/2}$ and let g be the inverse Fourier transform of G. (Since G is an even function, g is real valued.) Define

$$\varphi_n(t; T) = [\lambda_n(T)]^{-1/2} \int_{-T/2}^{T/2} g(t - s)e_n(s; T)\, ds, \qquad -\infty < t < \infty.$$

We shall establish several properties of the functions φ_n.

Lemma 4.1. For each T, the functions $\varphi_n(t; T)$ are orthonormal over $(-\infty, \infty)$.

Proof. For simplicity we write $\varphi_n(t)$ for $\varphi_n(t; T)$, $e_n(t)$ for $e_n(t; T)$, and λ_n for $\lambda_n(T)$. We have

$$\int_{-\infty}^{\infty} \varphi_n(t)\varphi_m(t)\, dt = \lambda_n^{-1/2}\, \lambda_m^{-1/2}$$
$$\times \int_{-\infty}^{\infty} \left[\int_{-T/2}^{T/2} g(t - s)e_n(s)\, ds \int_{-T/2}^{T/2} g(t - s')e_m(s')\, ds' \right] dt. \tag{4.2}$$

Since the Fourier transform of the function $h(t) = g(t - s)$ is $e^{-i\omega s}G(\omega)$, the Parseval relation implies that

$$\int_{-\infty}^{\infty} g(t - s)g(t - s')\, dt = (2\pi)^{-1} \int_{-\infty}^{\infty} |G(\omega)|^2\, e^{i\omega(s'-s)}\, d\omega = R(s' - s).$$

An application of the Schwarz inequality shows that

$$\int_{-\infty}^{\infty} |g(t - s)g(t - s')|\, dt \le \|g\|^2.$$

It follows that Fubini's theorem may be applied to the integral (4.2), and we obtain

$$\int_{-\infty}^{\infty} \varphi_n(t)\varphi_m(t)\, dt = \lambda_n^{-1/2}\lambda_m^{-1/2} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} R(s' - s)e_n(s)e_m(s')\, ds\, ds'.$$

Since e_n is an eigenfunction corresponding to λ_n, the integral has the value 0 if $n \ne m$, and 1 if $n = m$. The lemma follows.

If L is a subspace of $L_2(-\infty, \infty)$ and $f \in L_2(-\infty, \infty)$ we will use the notation $f \perp L$ to indicate that f is orthogonal to L, that is,

$$\int_{-\infty}^{\infty} f(t)h(t)\,dt = 0 \quad \text{for} \quad \text{all } h \in L.$$

Lemma 4.2. For each T, let L_T be the subspace of $L_2(-\infty, \infty)$ spanned by the functions $\{\varphi_n(t; T)\}$. If $f \in L_2(-\infty, \infty)$ and $T' < T$, then

a. $f \perp L_T$ implies $f \perp L_{T'}$.

b. $L_{T'} \subset L_T$.

Proof. To say that $f \perp L_T$ is to say that

$$\int_{-\infty}^{\infty} f(t)\varphi_n(t; T)\,dt = 0, \qquad n = 1, 2, \ldots,$$

or

$$[\lambda_n(T)]^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\left[\int_{-T/2}^{T/2} g(t-s)e_n(s; T)\,ds\right] dt = 0, \qquad n = 1, 2, \ldots. \tag{4.3}$$

We may reverse the order of integration (invoking Fubini's theorem as in Lemma 4.1) to obtain

$$[\lambda_n(T)]^{-\frac{1}{2}} \int_{-T/2}^{T/2} e_n(s; T)\left[\int_{-\infty}^{\infty} g(t-s)f(t)\,dt\right] ds = 0, \qquad n = 1, 2, \ldots. \tag{4.4}$$

Thus the function $h(s) = \int_{-\infty}^{\infty} g(t-s)f(t)\,dt$, $-T/2 \le s \le T/2$, is orthogonal to all the $e_n(s; T)$. Since the e_n span $L_2[-T/2, T/2]$, h is orthogonal to itself so that $h = 0$ almost everywhere, that is, $\int_{-\infty}^{\infty} g(t-s)f(t)\,dt = 0$ for almost every $s \in [-T/2, T/2]$, hence for almost every $s \in [-T'/2, T'/2]$. It follows that (4.4), hence (4.3), holds with T replaced by T', and consequently $f \perp L_{T'}$, proving part (a) of the lemma.

We now prove (b). Given any $f \in L_2(-\infty, \infty)$, by the projection theorem we may write $f = f_1 + f_2$, where $f_1 \in L_T$ and $f_2 \perp L_T$. By (a), $f_2 \perp L_{T'}$. Using the projection theorem again, we have $f_1 = f_3 + f_4$, where $f_3 \in L_{T'}$, and $f_4 \perp L_{T'}$. If P_T is the linear operator on $L_2(-\infty, \infty)$ which assigns to a function its projection on L_T, we have $P_T f = f_1$ and $P_{T'}P_T f = P_{T'} f_1 = f_3$. But $f = f_3 + (f_2 + f_4)$, where $f_3 \in L_{T'}$ and $f_2 + f_4 \perp L_{T'}$, and thus $f_3 = P_{T'} f$. It follows that

$$P_{T'}P_T = P_{T'}. \tag{4.5}$$

Now assume $f \in L_{T'}$. By the Pythagorean relation, $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$. If f_2 is not zero, then $\|P_T f\| = \|f_1\| < \|f\|$; since projection cannot increase the norm, we have $\|P_{T'} P_T f\| \leq \|P_T f\| < \|f\|$. But by (4.5), $\|P_{T'} P_T f\| = \|f\|$, a contradiction. Thus $f_2 = 0$ and therefore $f = f_1 \in L_T$, completing the proof.

Note that the above argument shows that the statements "$f \perp L_\beta$ implies $f \perp L_\alpha$" and "$L_\alpha \subset L_\beta$" are equivalent for any closed subspaces L_α, L_β of $L_2(-\infty, \infty)$.

Lemma 4.3. Let H be the subspace of $L_2(-\infty, \infty)$ consisting of those functions whose Fourier transforms vanish whenever $G(\omega)$ (equivalently, $N(\omega)$) vanishes. With L_T as in Lemma 4.2, let L_∞ be the subspace spanned by $\bigcup_T L_T$. Then $L_\infty = H$.

Proof. Fix T, and define

$$e_n^*(s) = e_n(s; T), \qquad |s| \leq \frac{T}{2}$$

$$= 0, \qquad |s| > \frac{T}{2}.$$

Then $e_n^* \in L_2(-\infty, \infty)$; let E_n^* be the Fourier transform of e_n^*. The functions φ_n may then be written as

$$\varphi_n(t; T) = [\lambda_n(T)]^{-1/2} \int_{-\infty}^{\infty} g(t - s) e_n^*(s)\, ds.$$

Since the Fourier transform of a convolution is the product of the Fourier transforms, the Fourier transform of φ_n is $[\lambda_n(T)]^{-1/2} G(\omega) E_n^*(\omega)$. It follows that $\varphi_n \in H$ for all $n = 1, 2, \ldots$, and all T, hence $L_T \subset H$ for all T, or $L_\infty \subset H$.

To show that $H \subset L_\infty$, we shall show that if $f \in L_2(-\infty, \infty)$, then $f \perp L_\infty$ implies $f \perp H$; the result then follows as in Lemma 4.2b. If $f \perp L_\infty$ then $f \perp L_T$ for every T; thus by the argument of Lemma 4.2a, $\int_{-\infty}^{\infty} g(s - t) f(t)\, dt = 0$ for almost every $s \in [-T/2, T/2]$. [Note that since $G(\omega)$ is real for all ω, g is an even function, so that $g(s - t) = g(t - s)$.] Since this holds for arbitrary T,

$$\int_{-\infty}^{\infty} g(s - t) f(t)\, dt = 0 \quad \text{for} \quad \text{almost all real numbers } s.$$

Again using the convolution theorem for Fourier transforms, we have

$$G(\omega)F(\omega) = 0 \quad \text{for} \quad \text{almost all } \omega, \tag{4.6}$$

where F is the Fourier transform of f.

Now let $f_1 \in H$, and let F_1 be the Fourier transform of f_1. By definition of H, $G(\omega) = 0$ implies $F_1(\omega) = 0$. But by (4.6), $G(\omega) \neq 0$ implies $F(\omega) = 0$. Thus $F(\omega)F_1(\omega) = 0$ for almost all ω, and consequently F and F_1 are orthogonal. By the Parseval relation, f and f_1 are orthogonal. Since f_1 is an arbitrary element of H, $f \perp H$ and the lemma is proved.

Lemma 4.4. Given a function $s \in H$ with Fourier transform S, define $F(\omega) = S(\omega)/G(\omega)$ if $G(\omega) \neq 0$; $F(\omega) = 0$ if $G(\omega) = 0$ [$= S(\omega)$]. Assume that $F \in L_2{}^*(-\infty, \infty)$, that is,

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{N(\omega)}\, d\omega < \infty.$$

Let f be the inverse Fourier transform of F (note $f \in H$). Define

$$s_n(T) = \int_{-T/2}^{T/2} s(t)e_n(t;\, T)\, dt, \quad f_n(T) = \int_{-\infty}^{\infty} f(t)\varphi_n(t;\, T)\, dt, \quad n = 1, 2, \ldots .$$

Then $[\lambda_n(T)]^{-1/2}s_n(T) = f_n(T)$, $n = 1, 2, \ldots .$

Proof. By the convolution theorem for Fourier transforms,

$$s(t) = \int_{-\infty}^{\infty} g(t - s)f(s)\, ds.$$

Thus

$$s_n(T) = \int_{-T/2}^{T/2} \left[\int_{-\infty}^{\infty} g(t - s)f(s)\, ds\right] e_n(t;\, T)\, dt.$$

Reversing the order of integration and using the fact that $g(t - s) = g(s - t)$, we obtain

$$s_n(T) = \int_{-\infty}^{\infty} f(s)\left[\int_{-T/2}^{T/2} g(s - t)e_n(t;\, T)\, dt\right] ds$$

$$= [\lambda_n(T)]^{1/2} \int_{-\infty}^{\infty} f(s)\varphi_n(s;\, T)\, ds$$

and the result follows.

Theorem 4.5. Let s be a function $\epsilon\ H$ with Fourier transform S. If $s_n(T)$ is as defined in Lemma 4.4, then

$$\lim_{T\to\infty} \sum_{n=1}^{\infty} \frac{|s_n(T)|^2}{\lambda_n(T)} = (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{N(\omega)}\, d\omega \quad \text{(assuming the integral is finite)},$$

where the convergence as $T \to \infty$ is monotone from below. [The integrand $|S(\omega)|^2/N(\omega)$ is defined to be zero if $N(\omega) = S(\omega) = 0$.]

Proof. Let F, f, and $f_n(T)$ be as in Lemma 4.4. Define

$$h(t) = \sum_{n=1}^{\infty} f_n(T)\varphi_n(t;\, T).$$

[Note that since the φ_n are orthonormal and

$$\sum_{n=1}^{\infty} |f_n(T)|^2 \leq \int_{-\infty}^{\infty} f^2(t)\, dt < \infty$$

(*Bessel's inequality*; see the argument of Theorem 1.14), the series $\sum_{n=1}^{\infty} f_n(T)\varphi_n(t;\, T)$ converges in $L_2(-\infty, \infty)$.] We claim that *h is the projection of f on L_T*. To see this, use the projection theorem to write $f = f_1 + f_2$ where $f_1 \,\epsilon\, L_T$ and $f_2 \perp L_T$. Since f_2 is orthogonal to all the functions $\varphi_n(t;\, T)$, it follows that

$$f_n(T) = \int_{-\infty}^{\infty} f_1(t)\varphi_n(t;\, T)\, dt.$$

By definition of L_T, the $\varphi_n(t;\ T)$ are an orthonormal basis for L_T; hence

$$f_1(t) = \sum_{n=1}^{\infty} f_{1n}(T)\varphi_n(t;\, T)$$

where $f_{1n}(T) = \int_{-\infty}^{\infty} f_1(t)\varphi_n(t;\, T)\, dt = f_n(T)$ and the series converges in $L_2(-\infty, \infty)$. (Again this can be justified by an argument similar to that of Theorem 1.14.) Thus f_1, the projection of f on L_T, is in fact h.

Now write $P_T f$ for the projection of f on L_T. If $T' < T$, then $L_{T'} \subset L_T$ by Lemma 4.2b; the projection theorem then implies that $\|f - P_T f\| \leq \|f - P_{T'} f\|$. Therefore $\lim_{T\to\infty} \|f - P_T f\|$ exists. If this limit is $\varepsilon > 0$, then given any T and any $f_1 \,\epsilon\, L_T$ we would have $\|f - f_1\| \geq \|f - P_T f\| \geq \varepsilon$. This contradicts Lemma 4.3, which states that any function in H can be approximated arbitrarily closely (in the L_2 sense) by functions in $\bigcup_T L_T$.

Therefore $\lim_{T\to\infty} \|f - P_T f\| = 0$, and consequently $\lim_{T\to\infty} \|P_T f\|^2 = \|f\|^2$. But

$$\|P_T f\|^2 = \sum_{n=1}^{\infty} |f_n(T)|^2 = \sum_{n=1}^{\infty} \frac{|s_n(T)|^2}{\lambda_n(T)} \qquad \text{by Lemma 4.4.}$$

Furthermore if $T' < T$, then by (4.5), $\|P_{T'} f\| = \|P_{T'} P_T f\| \leq \|P_T f\|$. The theorem follows.

We conclude with some conditions under which the zero eigenvalue is excluded.

Theorem 4.6. Let A_T be the integral operator defined by (4.1). If (a) the set on which $N(\omega) = 0$ has measure zero, or if (b) the covariance function R belongs to $L_2(-\infty,\infty)$, and the set on which $N(\omega) > 0$ is of positive measure and is contained in a finite interval, then 0 is not an eigenvalue of A_T. In particular, (a) implies that if $N(\omega)$ is a rational function of ω, then 0 is not an eigenvalue.

Proof. If $G(\omega) = [N(\omega)]^{1/2}$ and g is the inverse Fourier transform of G, then by the convolution theorem,

$$R(t) = \int_{-\infty}^{\infty} g(s)g(t - s)\, ds.$$

Hence

$$R(t - \tau) = \int_{-\infty}^{\infty} g(s)g(t - \tau - s)\, ds = \int_{-\infty}^{\infty} g(s' - \tau)g(s' - t)\, ds'.$$

If 0 is an eigenvalue then there is a function $f \in L_2[-T/2, T/2]$, $f \not\equiv 0$, such that

$$\int_{-T/2}^{T/2} R(t - \tau)f(\tau)\, d\tau = 0, \qquad t \in \left[-\frac{T}{2}, \frac{T}{2}\right];$$

consequently

$$\int_{-T/2}^{T/2} \int_{-T/2}^{T/2} R(t - \tau)f(t)f(\tau)\, dt\, d\tau = 0.$$

But this integral is

$$\int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \left[\int_{-\infty}^{\infty} g(s - t)g(s - \tau)\, ds\right] f(t)f(\tau)\, dt\, d\tau,$$

or, after the order of integration is interchanged,

$$\int_{-\infty}^{\infty}\left[\int_{-T/2}^{T/2} g(s-t)f(t)\,dt\right]^2 ds.$$

It follows that $\int_{-T/2}^{T/2} g(s-t)f(t)\,dt = 0$ for almost every s. If we define $f^*(t) = f(t)$, $-T/2 \leq t \leq T/2$, $f^*(t) = 0$ elsewhere, and if F^* is the Fourier transform of f^*, then the convolution theorem implies that $[N(\omega)]^{1/2}F^*(\omega) = 0$ for almost every ω. If N is positive almost everywhere, F^* must be zero almost everywhere, so that F^*, hence f, is 0 almost everywhere, a contradiction. This proves (a). To prove (b), again suppose that $\int_{-T/2}^{T/2} R(t-s)f(s)\,ds = 0$, $t \in [-T/2, T/2]$, for some function f, $f \not\equiv 0$. Define

$$f_1(t) = \int_{-T/2}^{T/2} R(t-s)f(s)\,ds, \qquad -\infty < t < \infty;$$

then $f_1 \in L_2(-\infty, \infty)$ and $f_1(t) = 0$, $-T/2 \leq t \leq T/2$. If F_1 and F^* are the Fourier transforms of f_1 and f^* respectively, where f^* is as defined in part (a), then the convolution theorem yields $F_1(\omega) = N(\omega)F^*(\omega)$, and hence F_1 vanishes outside a finite interval $[-B, B]$. But in this case f_1 is given explicitly by

$$f_1(t) = (2\pi)^{-1}\int_{-B}^{B} F_1(\omega)e^{i\omega t}\,d\omega. \tag{4.7}$$

We may use the formula (4.7) to define a function $\hat{f}_1$ which assigns to each *complex* number t the complex number $(2\pi)^{-1}\int_{-B}^{B} F_1(\omega)e^{i\omega t}\,d\omega$. Since the limits of integration are finite, it may be verified by direct differentiation that $\hat{f}_1$ is analytic everywhere. Since $\hat{f}_1$ agrees with f_1 on the real line and hence vanishes on the real interval $[-T/2, T/2]$, $\hat{f}_1$ is zero everywhere and hence so is f_1. Consequently $F_1(\omega) = 0$ for almost all ω, and hence $N(\omega) > 0$ implies $F^*(\omega) = 0$.

But since f^* vanishes outside a finite interval, the same argument as above shows that F^* is the restriction to the real line of a function which is analytic everywhere.

Finally, let D be the set on which $N(\omega) > 0$; by hypothesis, D has positive measure and is therefore uncountable, and consequently has a finite limit point. Since a function that is analytic everywhere and vanishes on such a set necessarily vanishes everywhere, it follows that $F^* = 0$, so that $f = 0$, a contradiction.

A result similar to Theorem 4.6 was proved by Root and Pitcher (1955); the other results of this section are due to Kelly, Reed, and Root (1960).

Values of $-\log_2 p$ and $-p \log_2 p$

$-\log_2 p$

p	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0		6.643856	5.643856	5.058894	4.643856	4.321928	4.058894	3.836501	3.643856	3.473931
0.1	3.321928	3.184425	3.058894	2.943416	2.836501	2.736966	2.643856	2.556393	2.473931	2.395929
0.2	2.321928	2.251539	2.184425	2.120294	2.058894	2.000000	1.943416	1.888969	1.836501	1.785875
0.3	1.736966	1.689660	1.643856	1.599462	1.556393	1.514573	1.473931	1.434403	1.395929	1.358454
0.4	1.321928	1.286304	1.251539	1.217591	1.184425	1.152003	1.120294	1.089267	1.058894	1.029146
0.5	1.000000	0.971431	0.943416	0.915936	0.888969	0.862496	0.836501	0.810966	0.785875	0.761213
0.6	0.736966	0.713119	0.689660	0.666576	0.643856	0.621488	0.599462	0.577767	0.556393	0.535332
0.7	0.514573	0.494109	0.473931	0.454032	0.434403	0.415037	0.395929	0.377070	0.358454	0.340075
0.8	0.321928	0.304006	0.286304	0.268817	0.251539	0.234465	0.217591	0.200913	0.184425	0.168123
0.9	0.152003	0.136062	0.120294	0.104697	0.089267	0.074001	0.058894	0.043943	0.029146	0.014500

$-p \log_2 p$

p	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0	0.066439	0.112877	0.151767	0.185754	0.216096	0.243534	0.268555	0.291508	0.312654
0.1	0.332193	0.350287	0.367067	0.382644	0.397110	0.410545	0.423017	0.434587	0.445308	0.455226
0.2	0.464386	0.472823	0.480573	0.487668	0.494134	0.500000	0.505288	0.510022	0.514220	0.517904
0.3	0.521090	0.523795	0.526034	0.527822	0.529174	0.530101	0.530615	0.530729	0.530453	0.529797
0.4	0.528771	0.527385	0.525646	0.523564	0.521147	0.518401	0.515335	0.511956	0.508269	0.504282
0.5	0.500000	0.495430	0.490577	0.485446	0.480043	0.474373	0.468441	0.462251	0.455808	0.449116
0.6	0.442179	0.435002	0.427589	0.419943	0.412068	0.403967	0.395645	0.387104	0.378347	0.369379
0.7	0.360201	0.350817	0.341230	0.331443	0.321458	0.311278	0.300906	0.290344	0.279594	0.268660
0.8	0.257542	0.246245	0.234769	0.223118	0.211293	0.199295	0.187129	0.174794	0.162294	0.149629
0.9	0.136803	0.123816	0.110671	0.097369	0.083911	0.070301	0.056538	0.042625	0.028563	0.014355

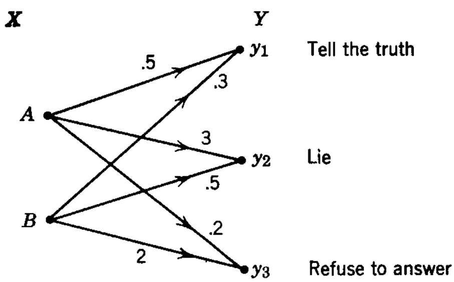
Solutions to Problems

1.1 $p(A) + p(B) = 1$

$$p(y_1) = 0.5p(A) + 0.3[1 - p(A)]$$
$$p(y_2) = 0.3p(A) + 0.5[1 - p(A)]$$
$$p(y_3) = 0.2p(A) + 0.2[1 - p(A)] = 0.2.$$
$$H(Y) = -\sum_{i=1}^{3} p(y_i) \log p(y_i) = -[0.3 + 0.2p(A)] \log [0.3 + 0.2p(A)]$$
$$-[0.5 - 0.2p(A)] \log [0.5 - 0.2p(A)] - 0.2 \log 0.2.$$
$$H(Y \mid X) = p(A)H(Y \mid X = A) + p(B)H(Y \mid X = B)$$
$$= H(0.5, 0.3, 0.2) = -0.5 \log 0.5 - 0.3 \log 0.3 - 0.2 \log 0.2.$$
$$I(X \mid Y) = H(X) - H(X \mid Y).$$

To maximize $I(X \mid Y)$, differentiate with respect to $p(A)$ and set the result equal to zero. The maximum occurs when $p(A) = 0.5$; hence

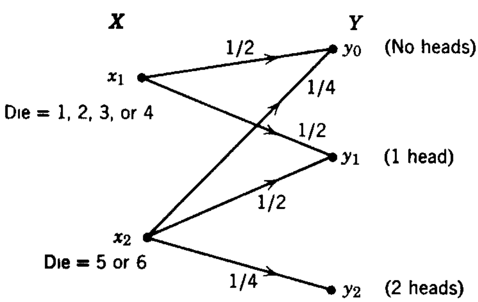
$$I_{\max} = -0.8 \log 0.4 + 0.5 \log 0.5 + 0.3 \log 0.3 = 0.036.$$



Problem 1.1

1.2 $I(X \mid Y) = H(Y) - H(Y \mid X) = H(\frac{5}{12}, \frac{1}{2}, \frac{1}{12}) - \frac{2}{3}H(\frac{1}{2}, \frac{1}{2})$

$$-\tfrac{1}{3}H(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}) = -\tfrac{5}{12} \log \tfrac{5}{12} - \tfrac{1}{12} \log \tfrac{1}{12} - \tfrac{2}{3} = 0.158.$$



$p(x_1) = \frac{2}{3}, p(x_2) = \frac{1}{3}$

$p(y_0) = \frac{5}{12}, p(y_1) = \frac{1}{2}, p(y_2) = \frac{1}{12}.$

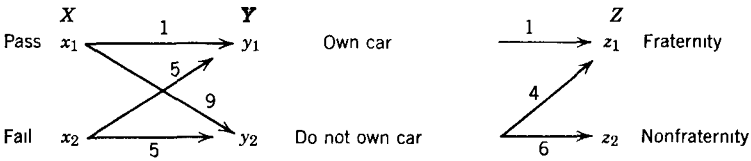
Problem 1.2

1.3 $p(x_1) = 0.75,\quad p(x_2) = 0.25 \qquad p(z_1 \mid x_1) = 0.46,\quad p(z_2 \mid x_1) = 0.54$
$p(y_1) = 0.2,\quad p(y_2) = 0.8 \qquad p(z_1 \mid x_2) = 0.7,\quad p(z_2 \mid x_2) = 0.3$
$p(z_1) = 0.52,\quad p(z_2) = 0.48$

a. $$I(X \mid Y) = H(Y) - H(Y \mid X) = H(0.2, 0.8) - 0.75H(0.1, 0.9) - 0.25H(0.5, 0.5) = 0.12$$

b. $$I(X \mid Z) = H(Z) - H(Z \mid X) = H(0.52, 0.48) - 0.75H(0.46, 0.54) - 0.25H(0.7, 0.3) = 0.03$$

c. First digit conveys $H(X) = H(0.75, 0.25) = 0.811$
Second digit conveys $H(Y \mid X) = 0.75H(0.1, 0.9) + 0.25H(0.5, 0.5) = 0.602$
Third digit conveys $H(Z \mid X, Y) = H(Z \mid Y)$ [since $p(z_k \mid x_i, y_j) = p(z_k \mid y_j)$]
$$= 0.2H(1) + 0.8H(0.4, 0.6)$$
$$= 0.8H(0.4, 0.6) = 0.777.$$



Problem 1.3

1.4 a. $H(Y \mid X) + H(Z \mid X)$
$$= -\sum_{i,j,k} p(x_i, y_j, z_k)[\log p(y_j \mid x_i) + \log p(z_k \mid x_i)]$$
$$= -\sum_i p(x_i) \sum_{j,k} p(y_j, z_k \mid x_i) \log p(y_j \mid x_i)p(z_k \mid x_i)$$

By Lemma 1.4.1, for each i,
$$-\sum_{j,k} p(y_j, z_k \mid x_i) \log p(y_j \mid x_i)p(z_k \mid x_i) \geq -\sum_{j,k} p(y_j, z_k \mid x_i) \log p(y_j, z_k \mid x_i)$$
with equality if and only if $p(y_j, z_k \mid x_i) = p(y_j \mid x_i)p(z_k \mid x_i)$ for all j, k. Thus $H(Y \mid X) + H(Z \mid X) \geq H(Y, Z \mid X)$ with equality if and only if
$$p(y_j, z_k \mid x_i) = p(y_j \mid x_i)p(z_k \mid x_i) \quad \text{for all } i, j, k.$$

b. $$H(Y, Z \mid X) = -\sum_{i,j,k} p(x_i, y_j, z_k) \log p(y_j, z_k \mid x_i)$$
$$= -\sum_{i,j,k} p(x_i, y_j, z_k) \log [p(y_j \mid x_i)p(z_k \mid x_i, y_j)]$$
$$= H(Y \mid X) + H(Z \mid X, Y)$$

c. $H(Z \mid X, Y) = H(Y, Z \mid X) - H(Y \mid X)$ [by (b)]
$\leq H(Y \mid X) + H(Z \mid X) - H(Y \mid X)$ [by (a)]

so $H(Z \mid X, Y) \leq H(Z \mid X)$ with equality if and only if
$$p(y_j, z_k \mid x_i) = p(y_j \mid x_i)p(z_k \mid x_i) \quad \text{for all } i,j,k.$$

1.5 By Lemma 1.4.1,

$$-\sum_{i=1}^{n} a_i \log a_i \le -\sum_{i=1}^{n} a_i \log\left(\frac{a_i x_i}{\sum_{j=1}^{n} a_j x_j}\right)$$

or

$$-\sum_{i=1}^{n} a_i \log a_i \le -\sum_{i=1}^{n} a_i \ \log a_i - \sum_{i=1}^{n} a_i \log x_i + \left(\sum_{i=1}^{n} a_i\right)\log\left(\sum_{j=1}^{n} a_j x_j\right).$$

Thus

$$\sum_{i=1}^{n} a_i \log x_i \le \log\left(\sum_{j=1}^{n} a_j x_j\right) \text{or } \log\left(x_1^{a_1}\cdots x_n^{a_n}\right) \le \log\left(\sum_{j=1}^{n} a_j x_j\right)$$

and the desired inequality follows. By Lemma 1.4.1, equality holds if and only if

$$a_i = \frac{a_i x_i}{\sum_{j=1}^{n} a_j x_j} \quad \text{for} \quad \text{all } i,$$

or

$$x_i = \sum_{j=1}^{n} a_j x_j = \text{constant} \quad \text{for} \quad \text{all } i.$$

1.6
$$\begin{aligned}
H(p_1', \dots, p_M') &= -(p_1 - \Delta p)\log(p_1 - \Delta p) - (p_2 + \Delta p)\log(p_2 + \Delta p) \\
&\quad - \sum_{i=3}^{M} p_i \log p_i \\
&= -p_1 \log(p_1 - \Delta p) - p_2 \log(p_2 + \Delta p) - \sum_{i=3}^{M} p_i \log p_i \\
&\quad + \Delta p \log\frac{p_1 - \Delta p}{p_2 + \Delta p} \\
&\ge -p_1 \log(p_1 - \Delta p) - p_2 \log(p_2 + \Delta p) - \sum_{i=3}^{M} p_i \log p_i \\
&\quad \text{since the last term is } \ge 0 \\
&> -\sum_{i=1}^{M} p_i \log p_i = H(p_1, \dots, p_M) \qquad \text{by Lemma 1.4.1}
\end{aligned}$$

1.7 $p_i' = \sum_{j=1}^{M} a_{ij} p_j$; hence

$$\begin{aligned}
H(p_i', \dots, p_M') &= -\sum_{i=1}^{M} p_i' \log p_i' = -\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij} p_j \log p_i' \\
&= -\sum_{j=1}^{M} p_j \left[\sum_{i=1}^{M} a_{ij} \log p_i'\right].
\end{aligned}$$

Now by problem 1.5,

$$(p_1')^{a_{1j}} \cdots (p_M')^{a_{Mj}} \le \sum_{i=1}^{M} a_{ij} p_i'. \tag{1}$$

(Note $\sum_{i=1}^{M} a_{ij} = 1$.) Hence, taking logarithms,

$$\sum_{i=1}^{M} a_{ij} \log p_i' \le \log\left(\sum_{i=1}^{M} a_{ij} p_i'\right).$$

Consequently

$$H(p_1', \ldots, p_M') \geq -\sum_{j=1}^{M} p_j \log \left(\sum_{i=1}^{M} a_{ij} p_i'\right).$$

But $\sum_{j=1}^{M}(\sum_{i=1}^{M} a_{ij} p_i') = \sum_{i=1}^{M} p_i'(\sum_{j=1}^{M} a_{ij}) = 1$ so Lemma 1.4.1 yields

$$-\sum_{j=1}^{M} p_j \log \left(\sum_{i=1}^{M} a_{ij} p_i'\right) \geq -\sum_{j=1}^{M} p_j \log p_j \tag{2}$$

so that $H(p_1', \ldots, p_M') \geq H(p_1, \ldots, p_M)$. Clearly if $(p_1', \ldots, p_M')$ is a rearrangement of $(p_1, \ldots, p_M)$, then $H(p_1', \ldots, p_M') = H(p_1, \ldots, p_M)$. On the other hand, if the uncertainties are equal, then equality must hold in both (1) and (2), the equality in (1) holding for all j. Equality in (1) implies (see Problem 1.5) that for each j, all p_i' corresponding to positive a_{ij} are equal, that is,

$$p_i' = \sum_{i=1}^{M} a_{ij} p_i' = \text{a constant } c_j \tag{3}$$

whenever $a_{ij} > 0$. (Note if $p_i' = c_j$ then $\sum_{i=1}^{M} a_{ij} p_i' = c_j \sum_{i=1}^{M} a_{ij} = c_j = p_i'$.) But equality in (2) implies

$$p_j = \sum_{i=1}^{M} a_{ij} p_i' \quad \text{for} \quad \text{all } j. \tag{4}$$

Therefore, by (3) and (4), $a_{ij} > 0$ implies $p_i' = p_j$. Since for each i, there is at least one j such that $a_{ij} > 0$, each p_i' is equal to some p_j; hence $(p_1', \ldots, p_M')$ is a rearrangement of $(p_1, \ldots, p_M)$.

In Problem 1.6 we have

$$\begin{aligned} p_1' &= p_1 - \Delta p = \alpha p_1 + \beta p_2 \\ p_2' &= p_2 + \Delta p = \beta p_1 + \alpha p_2 \\ p_i' &= p_i, \qquad i = 3, \ldots, M \end{aligned}$$

where $\alpha = (p_1 - \Delta p - p_2)/(p_1 - p_2)$, $\beta = \Delta p/(p_1 - p_2)$. Since $\alpha + \beta = 1$, the matrix

$$A = \begin{bmatrix} \alpha & \beta & 0 & \cdots & & 0 \\ \beta & \alpha & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ 0 & 0 & 0 & \cdots & & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

is doubly stochastic, and therefore Problem 1.6 is a special case of this result.

1.8 $H(X, Y) = H(X) + H(Y \mid X) = H(Y) + H(X \mid Y)$. But $H(Y \mid X) = 0$ since Y is determined by X; hence $H(Y) \leq H(X)$. $H(Y) = H(X)$ if and only if $H(X \mid Y) = 0$, that is, if and only if X is determined by Y, that is, for each y, there is an x_i such that $p(x_i \mid y_j) = 1$. Hence $H(Y) = H(X)$ if and only if g is one to one on $\{x_1, \ldots, x_M\}$, that is, $x_i \neq x_j$ implies $g(x_i) \neq g(x_j)$.

1.9 $P\{Z = z_k \mid X = x_i\} = P\{Y = z_k - x_i \mid X = x_i\}$. Given $X = x_i$, as z_k runs over all possible values of Z, $z_k - x_i$ runs over all possible values of Y; hence

$$H(Z \mid X = x_i) = H(Y \mid X = x_i) \quad \text{for} \quad \text{each } i.$$

Consequently $H(Z \mid X) = H(Y \mid X)$. The remaining conclusions of part (a) follow from Theorem 1.4.5. For part (b), take $Y = -X$; then $H(Z) = 0$ so in general

$$H(Z) < H(X), \qquad H(Z) < H(Y).$$

1.10 For $k = 2$, this is the ordinary grouping axiom 3. The general assertion follows by induction, since if the result has been established for $k - 1$, then by induction hypothesis,

$$\begin{aligned} H(p_1, \ldots, p_{r_1}; \;\cdots;\; p_{r_{k-1}+1}, \ldots, p_{r_k}) &= H(p_1 + \cdots + p_{r_1} \\ &\quad + p_{r_1+1} + \cdots + p_{r_2};\; p_{r_2+1} \cdots + p_{r_3};\; \cdots;\; p_{r_{k-1}+1} + \cdots + p_{r_k}) \\ &\quad + \left(\sum_{i=1}^{r_2} p_i\right) H\left(\frac{p_1}{\sum_{i=1}^{r_2} p_i}, \ldots, \frac{p_{r_2}}{\sum_{i=1}^{r_2} p_i}\right) + \sum_{i=3}^{k} (p_{r_{i-1}+1} + \cdots + p_{r_i}) \\ &\qquad \times H\left(\frac{p_{r_{i-1}+1}}{\sum_{j=r_{i-1}+1}^{r_i} p_j}, \ldots, \frac{p_{r_i}}{\sum_{j=r_{i-1}+1}^{r_i} p_j}\right) \end{aligned}$$

Applying the ordinary grouping axiom, the second term on the right becomes

$$\begin{aligned} &\left(\sum_{i=1}^{r_2} p_i\right)\left[H\left(\frac{\sum_{i=1}^{r_1} p_i}{\sum_{i=1}^{r_2} p_i}, \frac{\sum_{i=r_1+1}^{r_2} p_i}{\sum_{i=1}^{r_2} p_i}\right)\right. \\ &\left. + \frac{\sum_{i=1}^{r_1} p_i}{\sum_{i=1}^{r_2} p_i} H\left(\frac{p_1}{\sum_{i=1}^{r_1} p_i}, \ldots, \frac{p_{r_1}}{\sum_{i=1}^{r_1} p_i}\right) + \frac{\sum_{i=r_1+1}^{r_2} p_i}{\sum_{i=1}^{r_2} p_i} H\left(\frac{p_{r_1+1}}{\sum_{i=r_1+1}^{r_2} p_i}, \ldots, \frac{p_{r_2}}{\sum_{i=r_1+1}^{r_2} p_i}\right)\right]. \end{aligned}$$

Thus

$$\begin{aligned} &H(p_1, \ldots, p_{r_1}; \;\cdots;\; p_{r_{k-1}+1}, \ldots, p_{r_k}) \\ &\quad = H(p_1 + \cdots + p_{r_2};\; p_{r_2+1} + \cdots + p_{r_3};\; \cdots;\; p_{r_{k-1}+1} + \cdots + p_{r_k}) \\ &\qquad + \left(\sum_{i=1}^{r_2} p_i\right) H\left(\frac{\sum_{i=1}^{r_1} p_i}{\sum_{i=1}^{r_2} p_i}, \frac{\sum_{i=r_1+1}^{r_2} p_i}{\sum_{i=1}^{r_2} p_i}\right) \\ &\qquad + \sum_{i=1}^{k} (p_{r_{i-1}+1} \cdots + p_{r_i}) H\left(\frac{p_{r_{i-1}+1}}{\sum_{j=r_{i-1}+1}^{r_i} p_j}, \ldots, \frac{p_{r_i}}{\sum_{j=r_{i-1}+1}^{r_i} p_j}\right). \end{aligned}$$

Again using the induction hypothesis, the sum of the first two terms on the right in this equation is

$$H(\underbrace{p_1 + \cdots + p_{r_1},\ p_{r_1+1} + \cdots p_{r_2}}_{\substack{\text{group 1}\\ \text{(2 terms)}}},\ \underbrace{p_{r_2+1} + \cdots + p_{r_3}}_{\substack{\text{group 2}\\ \text{(1 term)}}}, \ldots, \underbrace{p_{r_{k-1}+1} + \cdots p_{r_k}}_{\substack{\text{group } k-1\\ \text{(1 term)}}})$$

and the result follows.

The above argument uses the ordinary grouping axiom but none of the other axioms for the uncertainty function. If we use all four axioms, the derivation is easier: Let $q_j = \sum_{i=r_{j-1}+1}^{r_j} p_i$. Then the right-hand side of the statement to be proved is

$$\begin{aligned}
&-\sum_{j=1}^{k} q_j \log q_j - \sum_{i=1}^{k} q_i \sum_{m=r_{i-1}+1}^{r_i} \frac{p_m}{q_i} \log \frac{p_m}{q_i} \\
&= -\sum_{j=1}^{k} q_j \log q_j - \sum_{i=1}^{k} \sum_{m=r_{i-1}+1}^{r_i} (p_m \log p_m - p_m \log q_i) \\
&= -\sum_{j=1}^{k} q_j \log q_j - \sum_{t=1}^{r_k} p_t \log p_t + \sum_{i=1}^{k} q_i \log q_i \\
&= H(p_1, \ldots, p_{r_k}).
\end{aligned}$$

1.11 Let $h'(p) = h(p) + C \log p$. Then $\sum_{i=1}^{M} p_i h'(p_i) = 0$. Taking all $p_i = 1/n$ we have $h'(1/n) = 0$, $n = 1, 2, \ldots$. If r/s is a rational number, then, taking

$$p_1 = \frac{r}{s} \qquad p_2 = p_3 = \cdots = p_{s-r+1} = \frac{1}{s}$$

we have

$$\frac{r}{s} h'\left(\frac{r}{s}\right) + \frac{s-r}{s} h'\left(\frac{1}{s}\right) = 0;$$

hence $h'(r/s) = 0$. The result now follows by continuity.

1.12 Let $f(n) = h(1/n)$. Then the requirements (a) and (b) imply $f(nm) = f(n) + f(m)$, and $n < m \Rightarrow f(n) < f(m)$, $n, m = 1, 2, \ldots$. The proof of Theorem 1.2.1 in the text shows that $f(n) = C \log_b n$ where $C > 0$, $b > 1$.

If p is a rational number r/s, then

$$h\left(\frac{1}{s}\right) = h\left(\frac{r}{s}\frac{1}{r}\right) = h\left(\frac{r}{s}\right) + h\left(\frac{1}{r}\right).$$

Thus

$$h\left(\frac{r}{s}\right) = f(s) - f(r) = C \log \frac{s}{r} = -C \log \frac{r}{s}.$$

Hence $h(p) = -C \log p$ for rational p. The general assertion follows by continuity.

2.1 a.

S_0	S_1	S_2	S_3	S_4	S_5	S_6
010	1	100	11	00	01	0
0001		1110		110	011	10
0110		01011			110	001
1100					0	110
00011						0011
00110						0110 code word
11110						
101011						

The code is not uniquely decipherable.

$A_0 = 0001$	$A_1 = 1$	$A_2 = 01011$	$A_3 = 11$	$A_4 = 110$	$A_5 = 0$	$A_6 = 0110$
$W_0 = 00011$	$W_1 = 101011$	$W_2 = 010$	$W_3 = 11110$	$W_4 = 1100$	$W_5 = 00110$	$W_6 = 0110$

Ambiguous sequence $= A_0W_1W_4W_6 = 0001 \quad 101011 \quad 1100 \quad 0110$
$= W_0W_2W_3W_5 = 00011 \quad 010 \quad 11110 \quad 00110$

S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
abc	*d*	*ba*	*ce*	*ac*	*cd*	*eac*	*ac*	*c*	*eac*	*ac*
abcd	*abd*			*ab*	*c*	*eab*	*ab*	*cd*	*eab*	*ab*
e							*d*	*ba*	*ce*	*d*
dba										
bace										
ceac										
ceab										
eabd										

$S_7 = S_{10}$; hence $S_i = S_{i+3}, \quad i \geq 7$

None of the sets S_i contains a code word, so the code is uniquely decipherable.

2.2 a. Let ω_i be the number of code words of length i ($\omega_1 = 1$, $\omega_2 = 2$, $\omega_n = 0$, $n \geq 3$). Then a message whose coded form has exactly k letters must begin with a code word of length 1 or a code word of length 2; hence

$$N(k) = \omega_1 N(k-1) + \omega_2 N(k-2), \qquad k \geq 3.$$

Now $N(2) = \omega_1 N(1) + \omega_2$; hence if we define $N(0) = 1$, the above equation is valid for $k \geq 2$. Thus we must solve the linear homogeneous difference equation $N(k+2) - \omega_1 N(k+1) - \omega_2 N(k) = 0$, $k = 0, 1, \ldots$, subject to $N(0) = 1$, $N(1) = 1$. Assuming a solution of the form $N(k) = \lambda^k$ we obtain

$$\lambda^{k+2} - \omega_1\lambda^{k+1} - \omega_2\lambda^k = 0 \quad \text{or} \quad \lambda^k(\lambda^2 - \lambda - 2) = 0.$$

The two nonzero roots are $\lambda_1 = 2$, $\lambda_2 = -1$. Hence

$$N(k) = A \cdot 2^k + B(-1)^k, \qquad k = 0, 1, \ldots.$$

Since $N(0) = A + B = 1$, $N(1) = 2A - B = 1$, we have

$$A = \tfrac{2}{3},\ B = \tfrac{1}{3}.$$

So $N(k) = (\tfrac{2}{3})2^k + (\tfrac{1}{3})(-1)^k, \qquad k = 0, 1, \ldots.$

b. As in (a), $N(k) = \omega_1 N(k-1) + \omega_2 N(k-2) + \omega_3 N(k-3)$ ($\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 2$, $\omega_n = 0$, $n \geq 4$) or

$$N(k+3) - N(k+2) - N(k+1) - 2N(k) = 0, \qquad k \geq 0.$$

The assumption $N(k) = \lambda^k$ yields the characteristic equation

$$\lambda^3 - \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda^2 + \lambda + 1) = 0$$

with roots

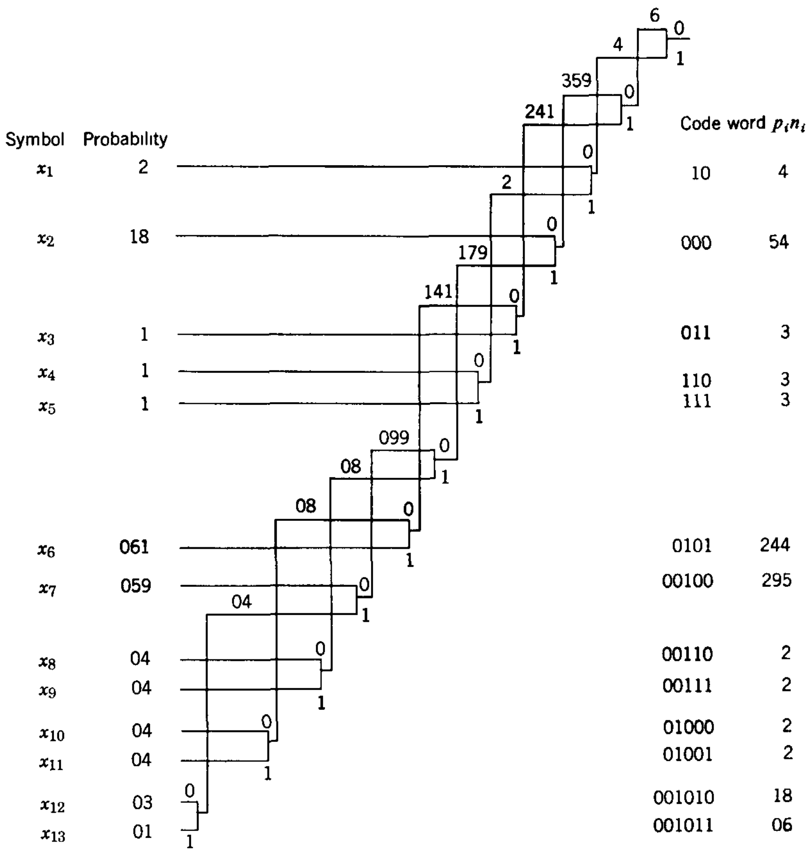
$$\lambda_1 = 2, \qquad \lambda_2 = -\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3} = e^{i\,2\pi/3}, \qquad \lambda_3 = -\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3} = e^{-i2\pi/3}.$$

Thus $N(k) = A \cdot 2^k + B\cos(2\pi/3)k + C\sin(2\pi/3)k$, $k = 0, 1, \ldots$. The boundary conditions are

$$N(0) = 1 = A + B$$
$$N(1) = 1 = 2A - \tfrac{1}{2}B + \tfrac{1}{2}\sqrt{3}\,C$$
$$N(2) = 2 = 4A - \tfrac{1}{2}B - \tfrac{1}{2}\sqrt{3}\,C.$$

These equations have the solution $A = \frac{4}{7}$, $B = \frac{3}{7}$, $C = \sqrt{3}/21$, so

$$N(k) = (\tfrac{4}{7})2^k + \tfrac{3}{7}\cos\frac{2\pi}{3}k + \frac{\sqrt{3}}{21}\sin\frac{2\pi}{3}k, \qquad k = 0, 1, \ldots.$$



Problem 2.3

2.3 The process of constructing the code is indicated schematically in the accompanying figure. The average code-word length is $\bar{n} = \sum_{i=1}^{13} p_i n_i = 3.419$. The uncertainty may be computed to be $H(X) = 3.355$.

2.4 By Theorem 2.5.2 there is a binary code whose average word length is less than $H(X) + 1$; since the Huffman code minimizes the average word length for a given set of probabilities, the result is immediate.

2.5 Add fictitious symbols with zero probability to the given set until the number of symbols is congruent to 1, modulo $(D - 1)$. An optimal code for the enlarged set will then yield an optimal code for the original set. Now we shall prove that given any optimal code we may construct another code with exactly the same word lengths such that, among the words of maximal length, at least D words agree in all places but the last. Therefore without loss of generality we may assume that an optimal code has this property. In particular there are at least D words of maximal length, and consequently (note that Lemma 2.6.2a holds just as in the binary case) the D least probable symbols have code words of equal length. The proof for the binary case now goes through word for word, except that we combine symbols D at a time instead of 2 at a time.

To prove the required assertion, pick any word W of maximal length n_M. Let Q be the sequence consisting of all digits of W except the last. Say the code alphabet is $\{0, 1, \ldots, D - 1\}$ and $W = Q0$. If there are more than D words of maximal length we may, without changing the instantaneous property of the code, take any $(D - 1)$ words of length n_M other than W and replace them by $Q1, Q2, \ldots, Q(D - 1)$; thus it suffices to consider the case in which there are at most D words of maximal length; we may assume without loss of generality that these words are $Q0, Q1, \ldots, Qi$, $i < D$. We may assume that $i > 0$ since if $i = 0$, $Q0$ may be replaced by Q.

Now as in the proof of Theorem 2.3.1, the number of terminal points on the tree of order D and size n_M excluded by a code with word lengths $n_1, \ldots, n_M$ is $T = \sum_{i=1}^{M} D^{(n_M - n_i)}$. Since $D \equiv 1 \bmod (D - 1)$ and (after the fictitious symbols are added) $M \equiv 1 \bmod (D - 1)$, it follows that $T \equiv 1 \bmod (D - 1)$. If R is any sequence of length $n_M - 1$ other than Q, then if R is not a code word and no code word is a prefix of R we may replace one of the words Qj by R. The resulting code is still instantaneous and has an average word length at most that of the original code. (Possibly Qj corresponds to a symbol with probability zero so that the average word length is unchanged.) Thus without loss of generality we may assume that every such sequence R is either a code word or has a code word as a prefix. In either case, the terminal points corresponding to the sequences Rj, $j = 0, 1, \ldots, D - 1$ are all excluded. Therefore the only terminal points not excluded are those corresponding to $Q(i + 1), \ldots, Q(D - 1)$. Since the total number of terminal points is $D^{n_M} \equiv 1 \bmod (D - 1)$, the number of terminal points not excluded is congruent to zero modulo $(D - 1)$. Hence $D - 1 - i = 0$ or $D - 1$; the second case is impossible since $i > 0$; consequently $i = D - 1$ and the proof is complete.

For the given numerical example, we first group 2 symbols at a time (or add one fictitious symbol and group 3 at a time) and afterward we combine 3 symbols at a time. The code-word lengths are 1, 2, 2, 2, 2, 2, 3, 3.

3.1 Since

$$p(y_j) = \sum_{i=1}^{3} p(x_i)p(y_j \mid x_i) \quad \text{and} \quad p(x_i \mid y_j) = \frac{p(x_i)p(y_j \mid x_i)}{p(y_j)}$$

we compute

$$\begin{array}{lll} p(x_1 \mid y_1) = 2/3, & p(x_2 \mid y_1) = 1/9, & p(x_3 \mid y_1) = 2/9 \\ p(x_1 \mid y_2) = 1/2, & p(x_2 \mid y_2) = 3/8, & p(x_3 \mid y_2) = 1/8 \\ p(x_1 \mid y_3) = 2/7, & p(x_2 \mid y_3) = 2/7, & p(x_3 \mid y_3) = 3/7 \end{array}$$

Thus the ideal observer is $g(y_1) = x_1, g(y_2) = x_1, g(y_3) = x_3$. The probability of error is

$$\sum_{j=1}^{3} p(y_j)p(e \mid y_j) = \sum_{j=1}^{3} p(y_j)P\{X \neq g(y_j) \mid y_j\}$$

$$p(e) = \frac{3}{8}\left(\frac{1}{9} + \frac{2}{9}\right) + \frac{1}{3}\left(\frac{3}{8} + \frac{1}{8}\right) + \frac{7}{24}\left(\frac{2}{7} + \frac{2}{7}\right) = \frac{11}{24}.$$

3.2 If y_j is received, the probability of correct transmission is $p(e' \mid y_j) = \sum_{i=1}^{M} q_{ji}P\{X = x_i \mid y_j\}$. (Necessarily $q_{ji} \geq 0$ for all i, j and $\sum_{i=1}^{M} q_{ji} = 1, j = 1, \ldots, L$.)

If $p(x_{i_0} \mid y_j) = \max_{1 \leq i \leq M} p(x_i \mid y_j)$, then

$$p(e' \mid y_j) \leq \sum_{i=1}^{M} q_{ji}\, p(x_{i_0} \mid y_j) = p(x_{i_0} \mid y_j);$$

equality can be achieved by taking $q_{ji_0} = 1, q_{ji} = 0, i \neq i_0$.

Thus $p(e' \mid y_j)$ is maximized for each j, and hence the overall probability of correct transmission is maximized, by the above choice of q_{ji}; however, this choice is precisely the ideal observer.

3.3 $I(X \mid Y) = H(X) - H(X \mid Y)$

$I(X \mid Z) = H(X) - H(X \mid Z)$

Now

$$p(x \mid y, z) = \frac{p(x, y, z)}{p(y, z)} = \frac{p(x, y)p(z \mid x, y)}{p(y)p(z \mid y)}.$$

The essential consequence of the cascading process is that $p(z \mid x, y) = p(z \mid y)$ for all x, y, z; in fact, this condition should be used as the definition of a cascade combination. It follows that $p(x \mid y, z) = p(x \mid y)$, hence $H(X \mid Y, Z) = H(X \mid Y)$. Thus

$$I(X \mid Y) - I(X \mid Z) = H(X \mid Z) - H(X \mid Y, Z) \geq 0$$

by Problem 1.4.

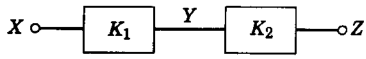
Since $I(X \mid Y) \geq I(X \mid Z)$ for any input distribution, the capacity of K_1 $\geq$ the capacity of the cascade. Now write

$$I(Y \mid Z) = H(Z) - H(Z \mid Y)$$
$$I(X \mid Z) = H(Z) - H(Z \mid X).$$

Since $p(z \mid y) = p(z \mid x, y)$, $H(Z \mid Y) = H(Z \mid X, Y)$. Thus

$$I(Y \mid Z) - I(X \mid Z) = H(Z \mid X) - H(Z \mid X, Y) \geq 0,$$

so that the capacity of $K_2 \geq$ the capacity of the cascade.



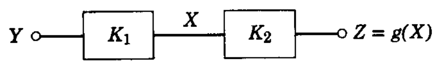
Problem 3 3

3.4 We may regard Y as the input and X the output of a discrete memoryless channel, the output X may be thought of as being applied as the input of a deterministic channel with output Z (see diagram) The conditions for cascading are satisfied, that is, $p(z \mid y, x) = p(z \mid x)$, the result then follows from the argument of Problem 3 3. Alternately,

$$\begin{aligned} H(Y \mid Z) &= E[-\log p(Y \mid Z)] = E[-\log p(Y \mid g(X))] \\ &= -\sum_{i\,j} p(x_i, y_j) \log P\{Y = y_j \mid Z = g(x_i)\} \\ &= \sum_i p(x_i)[-\sum_j p(y_j \mid x_i) \log P\{Y = y_j \mid Z = g(x_i)\}]. \end{aligned}$$

By Lemma 1.4 1,

$$H(Y \mid Z) \geq \sum_i p(x_i)[\ -\sum_j p(y_j \mid x_i) \log p(y_j \mid x_i)] = H(Y \mid X).$$



Problem 3 4 Channel matrix of $K_1 = [p(x \mid y)]$

3.5 $p_n = (1 - \beta)p_{n-1} + \beta(1 - p_{n-1})$ or $p_n - (1 - 2\beta)p_{n-1} = \beta$. The solution to this difference equation is $p_n = A(1 - 2\beta)^n + B$ where the "particular solution" B is determined by $B - (1 - 2\beta)B = \beta$ or $B = \frac{1}{2}$, and A is determined by taking $n = 0$ $A + B = p_0$ Thus

$$p_n = \tfrac{1}{2} + (p_0 - \tfrac{1}{2})(1 - 2\beta)^n \to \tfrac{1}{2} \quad \text{as} \quad n \to \infty.$$

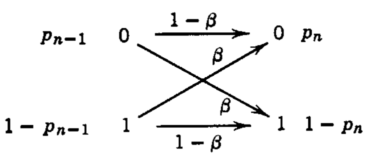
Now the channel matrix of the cascade combination of n channels is

$$\Pi_n = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{bmatrix} p_n^* & 1 - p_n^* \\ 1 - p_n^* & p_n^* \end{bmatrix} \end{array}$$

where $p_n^* = \frac{1}{2} + (p_0 - \frac{1}{2})(1 - 2\beta)^n$, evaluated when $p_0 = 1$.

The capacity of the n-stage combination is

$$C_n = 1 - H(p_n^*, 1 - p_n^*) \to 1 - H(\tfrac{1}{2}, \tfrac{1}{2}) = 0 \quad \text{as} \quad n \to \infty.$$

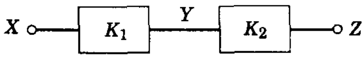


Problem 3 5

3.6 Let Π_i = channel matrix of K_i, $i = 1, 2$. We have

$$p(z \mid x) = \sum_y p(y \mid x) p(z \mid y).$$

Thus the matrix of the cascade combination is $\Pi_1 \Pi_2$; similarly if channels $K_1, K_2, \ldots, K_n$ are cascaded (in that order) the matrix of the cascade combination is the product $\Pi_1 \Pi_2 \cdots \Pi_n$.



Problem 3.6

Remark Concerning Problems 3.5 and 3.6

If X_i is the output of the ith channel [= the input of the $(i+1)$th channel] of a cascade combination, the conditions for cascading are expressed by requiring $p(x_k \mid x_0, \ldots, x_{k-1}) = p(x_k \mid x_{k-1})$ for all $x_0, \ldots, x_k$, $k = 1, 2, \ldots, n$.

3.7 Without loss of generality assume $\beta > \alpha$. We may use Theorem 3.3.3. We have

$$\Pi = \begin{bmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{bmatrix}, \qquad \Pi^{-1} = (\alpha - \beta)^{-1} \begin{bmatrix} 1 - \beta & \alpha - 1 \\ -\beta & \alpha \end{bmatrix}.$$

Let $H_\alpha = H(\alpha, 1 - \alpha) = H(Y \mid X = x_1)$, $H_\beta = H(\beta, 1 - \beta) = H(Y \mid X = x_2)$. We must show that the hypothesis of Theorem 3.3.3 is satisfied. We have [see (3.3.4)]

$$d_1 = -\frac{1 - \beta}{\beta - \alpha} \exp_2\left[\frac{(1 - \beta)H_\alpha + (\alpha - 1)H_\beta}{\beta - \alpha}\right] + \frac{\beta}{\beta - \alpha} \exp_2\left(\frac{-\beta H_\alpha + \alpha H_\beta}{\beta - \alpha}\right).$$

Thus $d_1 > 0$ if and only if

$$\log \frac{1 - \beta}{\beta} < \frac{-\beta H_\alpha + \alpha H_\beta - (1 - \beta)H_\alpha - (\alpha - 1)H_\beta}{\beta - \alpha}, \quad \text{or} \quad \log \frac{1 - \beta}{\beta} < \frac{H_\beta - H_\alpha}{\beta - \alpha}.$$

Upon expansion, this becomes $-\alpha \log \alpha - (1 - \alpha) \log (1 - \alpha) < -\alpha \log \beta - (1 - \alpha) \log (1 - \beta)$, which is true by Lemma 1.4.1. Consequently $d_1 > 0$. A similar argument shows that $d_2 > 0$.

Thus by (3.3.5),

$$C = \log\left[\exp_2\left(\frac{(1 - \beta)H_\alpha + (\alpha - 1)H_\beta}{\beta - \alpha}\right) + \exp_2\left(\frac{-\beta H_\alpha + \alpha H_\beta}{\beta - \alpha}\right)\right]$$

3.8 Let the input alphabets of K_1 and K_2 be $x_1, \ldots, x_r$ and $x_{r+1}, \ldots, x_M$ respectively; let the output alphabets of K_1 and K_2 be $y_1, \ldots, y_s$ and $y_{s+1}, \ldots,$

y_L. If $p(x)$ is any input distribution, and $p = \sum_{i=1}^{r} p(x_i)$, then

$$\begin{aligned} H(X) &= -\sum_{i=1}^{M} p(x_i) \log p(x_i) \\ &= -p \log p - (1-p)\log(1-p) + pH_1(X) + (1-p)H_2(X) \end{aligned}$$

by the grouping axiom 3 of Section 1.2

where

$$H_1(X) = -\sum_{i=1}^{r} \frac{p(x_i)}{p} \log \frac{p(x_i)}{p}$$

is the input uncertainty of K_1 under the distribution $\{p(x_i)/p,\ i = 1, \ldots, r\}$ and

$$H_2(X) = -\sum_{i=r+1}^{M} \frac{p(x_i)}{1-p} \log \frac{p(x_i)}{1-p}$$

is the input uncertainty of K_2 under the distribution $\{p(x_i)/(1-p),\ i = r + 1, \ldots, M\}$. (Alternately

$$H(X) = -\sum_{i=1}^{r} p \frac{p(x_i)}{p} \log \left[\left(\frac{p(x_i)}{p}\right) p\right] - \sum_{i=r+1}^{M} \frac{(1-p)p(x_i)}{1-p} \log \left[\frac{p(x_i)}{1-p}(1-p)\right]$$

leading to the same expression as above.) Now

$$\begin{aligned} H(X \mid Y) &= -\sum_{i=1}^{M} \sum_{j=1}^{L} p(x_i, y_j) \log p(x_i \mid y_j) \\ &= -\sum_{i=1}^{r} \sum_{j=1}^{s} p \frac{p(x_i)}{p} p(y_j \mid x_i) \log p(x_i \mid y_j) \\ &\quad - \sum_{i=r+1}^{M} \sum_{j=s+1}^{L} (1-p) \left[\frac{p(x_i)}{1-p}\right] p(y_j \mid x_i) \log p(x_i \mid y_j). \end{aligned}$$

Now if $1 \le i \le r$, $1 \le j \le s$, the conditional probability $p(x_i \mid y_j) = P\{X = x_i \mid Y = y_j\}$ is

$$\begin{aligned} p(x_i \mid y_j) &= \frac{p(x_i)p(y_j \mid x_i)}{p(y_j)} = \frac{p[p(x_i)/p]\, p(y_j \mid x_i)}{p \sum_{i=1}^{M} \frac{p(x_i)}{p} p(y_j \mid x_i)} \\ &= P\{X_1 = x_i \mid Y_1 = y_j\}, \end{aligned}$$

that is, the conditional probability that the input of K_1 is x_i given that the output of K_1 is y_j, under the distribution $\{p(x_i)/p,\ i = 1, 2, \ldots, r\}$.

Thus

$$\begin{aligned} H(X \mid Y) = pH_1(X \mid Y) + (1-p)H_2(X \mid Y),\ \text{and}\ I(X \mid Y) &= H(X) - H(X \mid Y) \\ &= H(p, 1-p) + pI_1(X \mid Y) + (1-p)I_2(X \mid Y), \end{aligned}$$

where the subscript i denotes that the indicated quantity is calculated for K_i under the appropriate input distribution. Now for a given p, we are completely free to choose the probabilities $p(x_i)/p$, $i = 1, \ldots, r$, and $p(x_i)/1-p$, $i = r+1, \ldots, M$. We can do no better than to choose the probabilities so that $I_1(X \mid Y) = C_1$, $I_2(X \mid Y) = C_2$.

Thus it remains to maximize $H(p, 1-p) + pC_1 + (1-p)C_2$. Differentiating, we obtain

$$-1 - \log p + 1 + \log(1-p) + C_1 - C_2 = 0, \text{ or } p = \frac{2^{C_1}}{2^{C_1} + 2^{C_2}}.$$

Hence

$$C = H\left(\frac{2^{C_1}}{2^{C_1} + 2^{C_2}}, \frac{2^{C_2}}{2^{C_1} + 2^{C_2}}\right) + \frac{C_1 \cdot 2^{C_1} + C_2 \cdot 2^{C_2}}{2^{C_1} + 2^{C_2}}$$

$$= \frac{2^{C_1}}{2^{C_1} + 2^{C_2}} \log(2^{C_1} + 2^{C_2}) + \frac{2^{C_2}}{2^{C_1} + 2^{C_2}} \log(2^{C_1} + 2^{C_2}).$$

Therefore $C = \log(2^{C_1} + 2^{C_2})$.

3.9 Let X_i be the input to K_i, and Y_i the corresponding output. By definition of the product channel,

$$H(Y_1, Y_2 \mid X_1, X_2) = H(Y_1 \mid X_1) + H(Y_2 \mid X_2).$$

Now $H(Y_1, Y_2) \le H(Y_1) + H(Y_2)$ with equality if and only if Y_1 and Y_2 are independent. Therefore $I(X_1, X_2 \mid Y_1, Y_2) \le I(X_1 \mid Y_1) + I(X_2 \mid Y_2) \le C_1 + C_2$.

If we take X_1 and X_2 to be independent, with X_i having the distribution that achieves channel capacity for K_i, then Y_1 and Y_2 are independent (by the same argument as in the proof of the fundamental theorem), and equality is achieved in the above expression.

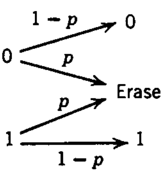
Thus $C = C_1 + C_2$.

3.10 Channel (a) is the sum of

$$\begin{bmatrix} 1-\beta & \beta \\ \beta & 1-\beta \end{bmatrix} \quad \text{and} \quad [1].$$

By Problem 3.8, $2^C = 2^{1-H(\beta,1-\beta)} + 2^0$, or $C = \log(1 + 2^{1-H(\beta,1-\beta)})$.

Channel (b) is symmetric; hence



Problem 3.10

$$C = \log 4 + p \log \tfrac{1}{2}p + (1-p)\log \tfrac{1}{2}(1-p),$$

or $C = 1 - H(p, 1-p)$ = the capacity of a binary symmetric channel with error probability p.

For channel (c), see the accompanying figure. If $P\{X = 0\} = \alpha$ then $H(X) = H(\alpha, 1-\alpha)$.

$$H(X \mid Y) = p(\text{erase})H(X \mid Y = \text{erase}) = pH(\alpha, 1-\alpha).$$

Thus $I(X \mid Y) = (1-p)H(\alpha, 1-\alpha)$, which is a maximum for $\alpha = \frac{1}{2}$. Therefore $C = 1 - p$.

3.11 a. Proceed exactly as in the proof of the fundamental lemma, with the set A replaced by $A^* = A \cap F^*$, where $F^* = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in F\}$. Observe that all code words selected in the algorithm must belong to F, since if $\mathbf{x} \notin F$, $A_{\mathbf{x}}^*(= \{\mathbf{y}:(\mathbf{x}, \mathbf{y}) \in A^*\})$ is empty. We obtain a code (s, n, λ) such that $\lambda \leq s \cdot 2^{-a} + P\{(\mathbf{X}, \mathbf{Y}) \notin A^*\}$. Since

$$P\{(\mathbf{X}, \mathbf{Y}) \notin A^*\} \leq P\{(\mathbf{X}, \mathbf{Y}) \notin A\} + P\{(\mathbf{X}, \mathbf{Y}) \notin F^*\} = P\{(\mathbf{X}, \mathbf{Y}) \notin A\} + P\{\mathbf{X} \notin F\},$$

the result follows.

b. We shall prove that the maximum value of C_0, that is, the least upper bound of all permissible rates of transmission, is

$$C_0^* = H[(1 - r)(1 - \beta) + r\beta, (1 - r)\beta + r(1 - \beta)] - H(\beta, 1 - \beta),$$

which is the information processed by the binary symmetric channel when the input distribution is $p(1) = r, p(0) = 1 - r$. Notice that as $r \to \frac{1}{2}$, $C_0 \to C = 1 - H(\beta, 1 - \beta)$.

We shall first prove a *coding theorem*, which states that given $R < C_0^*$, there exist codes $([2^{nR}], n, \lambda_n)$, with all words belonging to F_n, such that $\lim_{n\to\infty} \lambda_n = 0$. We use the results of part (a), with $F = F_n$.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ where the X_i are independent and $P\{X_i = 1\} = p < r$, $P\{X_i = 0\} = 1 - p$. Then $P\{\mathbf{X} \notin F_n\} = P\{n^{-1} \sum_{i=1}^n X_i > r\} \to 0$ as $n \to \infty$ by the weak law of large numbers. Let us take $a = kn$. Then as in the proof of the fundamental theorem, $P\{(\mathbf{X}, \mathbf{Y}) \notin A\} = P\{n^{-1} \sum_{i=1}^n U_i \leq k\}$, where the U_i are independent, identically distributed random variables with expected value

$$I(X_i \mid Y_i) = H(Y_i) - H(Y_i \mid X_i) = H[(1 - p)(1 - \beta) + p\beta,$$
$$(1 - p)\beta + p(1 - \beta)] - H(\beta, 1 - \beta) \to C_0^* \quad \text{as} \quad p \to r.$$

Given $R < C_0^*$, we take $s = [2^{nR}]$, $k = \frac{1}{2}(R + C_0^*)$, and p sufficiently close to r so that $I(X_i \mid Y_i) > k$. Then by choice of s and $a = kn$, $s \cdot 2^{-a} \to 0$ as $n \to \infty$. Also $P\{(\mathbf{X}, \mathbf{Y}) \notin A\} \to 0$ as $n \to \infty$ by the weak law of large numbers. The result follows.

We now prove a *weak converse*, which states that any code (s, n), all of whose words belong to F_n, must satisfy

$$\log s \leq \frac{nC_0^* + \log 2}{1 - \overline{p(e)}},$$

where $\overline{p(e)}$ is the average probability of error of the code. As in Theorem 3.7.3, this implies that if we require that all code words belong to F_n, it is not possible to maintain a transmission rate above C_0^* with an arbitrarily small probability of error.

Given a code (s, n) with all words in F_n. We can proceed just as in the proof of Theorem 3.7.3 (with C_0^* replacing C) except for one difficulty. Although we have $I(\mathbf{X} \mid \mathbf{Y}) \leq \sum_{i=1}^n I(X_i \mid Y_i)$ as in (3.7.4), it is not true that $I(X_i \mid Y_i) \leq C_0^*$ for all i. For example, if $s = 2$, $n = 5$, $r = \frac{2}{5}$, and the words of the code are $\mathbf{w}_1 = 11000$ and $\mathbf{w}_2 = 10000$, then $P\{X_2 = 1\} = P\{X_2 = 0\} = \frac{1}{2}$, and therefore $I(X_2 \mid Y_2) = C > C_0^*$. However, we may prove that $\sum_{i=1}^n I(X_i \mid Y_i) \leq nC_0^*$. Thus $I(\mathbf{X} \mid \mathbf{Y}) \leq nC_0^*$, so that the proof in the text will go through. To see this, let X be a symbol chosen with uniform probability from among all sn digits of the code vocabulary $\mathbf{w}_1, \ldots, \mathbf{w}_s$. Since X may be selected by first choosing a digit j at random, and then selecting a code word at random and examining the jth digit of the word, we have

$$P\{X = \alpha\} = n^{-1} \sum_{j=1}^{n} P\{X_j = \alpha\}, \qquad \alpha = 0, 1.$$

By Theorem 3.3.1, $I(X \mid Y) \geq n^{-1} \sum_{j=1}^{n} I(X_j \mid Y_j)$. But since all words belong to F_n, $P\{X = 1\} \leq r$, and hence $I(X \mid Y) \leq C_0^*$. The result follows.

3.12 The information $I(X \mid Y)$ processed by a channel may be expressed as a function of the input probabilities $p(x_1), \ldots, p(x_M)$. [See (3.3.6).] The domain of I is the subset of Euclidean M-space on which all $p(x_i) \geq 0$ and

$$\textstyle\sum_{i=1}^{M} p\,(x_i) = 1.$$

Thus the domain is a closed and bounded subset of Euclidean M-space. Since such a set is compact, and a continuous function defined on a compact set attains a maximum value on the set, the result follows.

3.13 The easiest example is that of a useless channel. In this case $Y_1, \ldots, Y_n$ are independent regardless of the input distribution. To see this, write

$$P\{Y_1 = \beta_1, \ldots, Y_n = \beta_n\} = \sum_{\alpha_1, \ldots, \alpha_n} P\{X_1 = \alpha_1, \ldots, X_n = \alpha_n\} P\{Y_1 = \beta_1, \ldots, Y_n = \beta_n \mid X_1 = \alpha_1, \ldots, X_n = \alpha_n\}.$$

Since the channel is memoryless the second factor in the summand is

$$\prod_{i=1}^{n} P\{Y_i = \beta_i \mid X_i = \alpha_i\}.$$

Since the channel is useless, $P\{Y_i = \beta_i \mid X_i = \alpha_i\} = P\{Y_i = \beta_i\}$. It follows that $P\{Y_1 = \beta_1, \ldots, Y_n = \beta_n\} = \prod_{i=1}^{n} P\{Y_i = \beta_i\}$, and therefore $Y_1, \ldots, Y_n$ are independent.

4.1 There are $2^m - 1$ nonzero binary sequences of length m. Hence for a given m, the maximum n is $2^m - 1$. Note that since $2^m = 1 + n$, the Hamming bound (Theorem 4.5.2) for a single-error correcting code is met. For $m = 4$ we have

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad n = 16 - 1 = 15.$$

4.2 a. It is possible to correct e-tuple errors and detect $(e + 1)$-tuple errors if and only if every error pattern of weight $\leq e$ yields a distinct corrector and in addition no error pattern of weight $e + 1$ has the same corrector as an error pattern of weight $\leq e$. (For then an $(e + 1)$-tuple error would be confused with an error pattern of weight $\leq e$ and would therefore not be detected.) This is precisely the condition that every set of $2e + 1$ columns of the parity check matrix be linearly independent.

b. Since the original code corrects e-tuple errors, every set of $2e$ columns of A is linearly independent. Now suppose

$$\sum_{k=1}^{2e+1} \lambda_{i_k} \mathbf{c}(r_{i_k}) = \mathbf{0} \qquad \text{(modulo 2)},$$

where $\lambda_j = 0$ or 1 and $\mathbf{c}(r_j)$ is the jth column of A_0. Since $2e + 1$ is odd, and each column of A_0 has a "1" in its bottom element, there must be an even number of nonzero λ's, in particular there are at most $2e$ nonzero λ's. This implies a linear dependence among $2e$ or fewer columns of A, a contradiction.

c. Augment a parity check matrix of a single-error correcting code to obtain

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

d. The condition is $2^m > \sum_{i=0}^{2e} \binom{n-1}{i}$. In view of part (a), the proof proceeds exactly as in Theorem 4.5.3.

4.3 a.
$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

b. Diagonalize the parity check matrix A (see Section 4.5) to obtain a new parity check matrix

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & a_{16} \\ 0 & 1 & 0 & 0 & 1 & a_{16} + a_{26} \\ 0 & 0 & 1 & 0 & 1 & a_{16} + a_{36} \\ 0 & 0 & 0 & 1 & 0 & a_{26} + a_{36} + a_{46} \end{bmatrix}.$$

If the last column of A' is

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

then every set of 3 columns of A' is linearly independent and the code will correct single errors and detect double errors. We obtain

$$\begin{bmatrix}1\\1\\0\\1\end{bmatrix} \quad \text{by taking } a_{16} = 1, \quad a_{26} = 0, \quad a_{36} = 1, \quad a_{46} = 0;$$

$$\begin{bmatrix}1\\0\\1\\1\end{bmatrix} \quad \text{by taking } a_{16} = 1, \quad a_{26} = 1, \quad a_{36} = 0, \quad a_{46} = 0;$$

$$\begin{bmatrix}0\\1\\1\\1\end{bmatrix} \quad \text{by taking } a_{16} = 0, \quad a_{26} = 1, \quad a_{36} = 1, \quad a_{46} = 1.$$

Now in A', $\mathbf{c}(r_1) + \mathbf{c}(r_2) + \mathbf{c}(r_3) + \mathbf{c}(r_5) = \mathbf{0}$ regardless of the choice of the elements of column 6; hence not every set of 4 columns of A' (hence of A) is linearly independent so that not all double errors can be corrected.

4.4 Let $\mathbf{w}$ be a nonzero code word of minimum weight, and let h be the weight of $\mathbf{w}$. Since $\text{dist}(\mathbf{w}_0, \mathbf{w}) = h$, (where $\mathbf{w}_0$ is the zero code word), the minimum distance is $\leq h$. Suppose the minimum distance is $h_1 < h$. Then there are distinct code words $\mathbf{w}_i, \mathbf{w}_j$ such that $\text{dist}(\mathbf{w}_i, \mathbf{w}_j) = h_1$. But $\text{dist}(\mathbf{w}_i, \mathbf{w}_j) = \text{dist}(\mathbf{w}_i + \mathbf{w}_i, \mathbf{w}_i + \mathbf{w}_j) = \text{dist}(\mathbf{w}_0, \mathbf{w}_i + \mathbf{w}_j) = \text{weight of } (\mathbf{w}_i + \mathbf{w}_j)$. Thus the weight of $\mathbf{w}_i + \mathbf{w}_j$ is $h_1 < h$, a contradiction.

4.5 a. An error in decoding will be made if and only if an error of magnitude $\leq e$ is confused with the "no-error" condition. This will happen if and only if some set of e or fewer columns of the parity check matrix adds to zero. Hence e-tuple (and all smaller) errors can be detected if and only if every set of e columns of the parity check matrix is linearly independent.

b. To say that columns $k_1, k_2, \ldots, k_t$ of a matrix A add to zero is to say that $A\mathbf{w}^T = \mathbf{0}$ where $\mathbf{w}$ is a row vector with ones in positions $k_1, \ldots, k_t$ and zeros elsewhere. Thus the equivalent condition on the code words is that every nonzero code word have weight $> e$. (Alternately, the minimum distance between code words is $> e$.)

c. An error pattern $\mathbf{z}$ is detectable if and only if $\mathbf{z}$ does not yield the same corrector as the zero vector, that is, if and only if $A\mathbf{z}^T \neq \mathbf{0}$, that is, if and only if $\mathbf{z}$ is not a code word. Therefore the number of detectable errors is 2^n minus the number of code words, or $2^n - 2^k$.

4.6 a. If $\mathbf{w}$ is a code word, then $\mathbf{w} = \sum_{i=1}^{k} \lambda_i \mathbf{u}_i$ (modulo 2) where the $\mathbf{u}_i$ are the rows of G^*, and $\lambda_i = 0$ or 1. Thus

$$\mathbf{w} = [\lambda_1, \ldots, \lambda_k]G^*,$$

and therefore

$$[I_m \mid B^T]\mathbf{w}^T = [I_m \mid B^T]G^{*T}\begin{bmatrix}\lambda_1\\ \cdot\\ \cdot\\ \cdot\\ \lambda_k\end{bmatrix}$$

$$= [I_m \mid B^T]\left[\frac{B^T}{I_k}\right]\begin{bmatrix}\lambda_1\\ \cdot\\ \cdot\\ \cdot\\ \lambda_k\end{bmatrix} = (B^T + B^T)\begin{bmatrix}\lambda_1\\ \cdot\\ \cdot\\ \cdot\\ \lambda_k\end{bmatrix} = \mathbf{0}$$

Thus $[I_m \mid B^T]\mathbf{w}^T = \mathbf{0}$ for every code word $\mathbf{w}$. Since rank $[I_m \mid B^T] = m$, $[I_m \mid B^T]\mathbf{w}^T = \mathbf{0}$ has $2^k = 2^{n-m}$ solutions; thus the vectors $\mathbf{w}$ satisfying $[I_m \mid B^T]\mathbf{w}^T = \mathbf{0}$ are precisely the code words of the given code, so that $[I_m \mid B^T]$ is a parity check matrix.

b. (i) Diagonalize G to obtain

$$G^* = \begin{bmatrix}1&0&1&0&0&0\\1&1&0&1&0&0\\1&1&0&0&1&0\\0&1&0&0&0&1\end{bmatrix}. \qquad \text{Hence } A = \begin{bmatrix}1&0&1&1&1&0\\0&1&0&1&1&1\end{bmatrix}$$

$$\text{(ii) } G^* = G = [1 \quad 1 \quad 1 \quad 1 \quad 1] \quad \text{so} \quad A = \begin{bmatrix}1&0&0&0&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\end{bmatrix}.$$

4.7 Let $A = [I_m \mid A_1]$. Delete row 1 and column 1 of A (or equally well, row j and column j where $1 \le j \le m$). If columns i and j of the reduced matrix are the same, then columns i and j of the original matrix A agree in all digits except the first, where they must disagree. It follows that $\mathbf{c}(r_1) + \mathbf{c}(r_i) + \mathbf{c}(r_j) = \mathbf{0}$, contradicting the hypothesis (see Problem 4.2) that every set of 3 columns of A is linearly independent. Furthermore, no column of the reduced matrix can be zero. For if column i of the reduced matrix is zero, then column i of the original matrix would be zero or else would coincide with column 1; each possibility contradicts the hypothesis.

4.8 a. Take

$$A = \begin{bmatrix}1&0&0&1&1&0&1\\0&1&0&1&0&1&1\\0&0&1&0&1&1&1\end{bmatrix},$$

$n = 7$, $k = 4$, $m = 3$; $2^m = 1 + n$, $2^k = 16$ = number of code words. The Hamming bound is met, hence the code is lossless.

b. The Hamming bound is

$$2^m = \sum_{i=0}^{e} \binom{n}{i} \quad \text{or } (e = 2), \quad 2^m = 1 + n + \frac{n(n-1)}{2}.$$

Since $m = 4$, $n^2 + n - 30 = 0$ or $n = 5$. Then $k = n - m = 1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(compare Problem 4.6b).

c. If $n = 2^m - 1$ the Hamming code meets the Hamming bound and is lossless. In general, let $2^{m-1} \leq n < 2^m$. [Note that by construction of the code (Problem 4.1), n cannot be less than 2^{m-1}.] Given any binary column vector $\mathbf{c}$ of length m, if $\mathbf{c}$ is a column of the parity check matrix A then $\mathbf{c}$ is the corrector associated with a single error. If $\mathbf{c}$ is not a column of A then $\mathbf{c}$ is the modulo 2 sum of the column vector corresponding to 2^{m-1} and a column vector $\mathbf{c}'$ corresponding to a binary number less than 2^{m-1}. For example if $m = 4$, $n = 10$ then

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The vector

$$\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

is not a column of A, but

$$\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{c}(r_8) + \mathbf{c}(r_3).$$

Thus all correctors correspond either to single or to double errors, so the code is close-packed.

4.9 $B(\lambda, \beta) = -H(\lambda, 1 - \lambda) - \lambda \log \beta - (1 - \lambda) \log (1 - \beta)$

$$\frac{\partial B}{\partial \lambda} = -\log \frac{1 - \lambda}{\lambda} + \log \frac{1 - \beta}{\beta}$$

Now $1 - R = H(\lambda, 1 - \lambda)$ so $dR/d\lambda = \log [\lambda/(1 - \lambda)]$. Thus

$$\frac{\partial B}{\partial R} = \frac{\partial B/\partial \lambda}{dR/d\lambda} = 1 - \frac{\log (1 - \beta)/\beta}{\log (1 - \lambda)/\lambda}$$

Since

$$\frac{\lambda_c}{1 - \lambda_c} = \left(\frac{\beta}{1 - \beta}\right)^{1/2}, \quad \frac{\partial B}{\partial R} = 1 - 2 = -1 \quad \text{at} \quad \lambda = \lambda_c.$$

Therefore the tangent to $B(\lambda, \beta)$ at $\lambda = \lambda_c$ has slope -1, so that the tangent must be

$$1 - R + B(\lambda_c, \beta) - H(\lambda_c, 1 - \lambda_c) = H(\lambda, 1 - \lambda) + B(\lambda_c, \beta) - H(\lambda_c, 1 - \lambda_c),$$

which is the lower bound of (4.7.28) for $R < R_c$.

4.10 Given a code (s, n) with average probability of error less than λ, choose any number $k > 1$ such that $k\lambda < 1$. Then it is not possible that as many as s/k code words **w** have $p(e \mid \mathbf{w}) \geq k\lambda$, for if this were the case, we would have

$$\sum_{i=1}^{s} p(e \mid \mathbf{w}_i) \geq \frac{s}{k}(k\lambda) = s\lambda,$$

and consequently the average probability of error would be $\geq \lambda$, a contradiction. Thus we may construct a subcode having at least $s - s/k$ words and a maximum probability of error $< k\lambda$. If $s = [2^{nR}]$, $R > C$, then there is a $\delta > 0$ such that for sufficiently large n, $s(1 - k^{-1}) \geq 2^{n(C+\delta)}$. (Compare Lemma 3.5.3.) In other words, we are able to construct a sequence of codes that allows us to maintain a transmission rate above channel capacity with a maximum probability of error $< k\lambda < 1$. This contradicts the strong converse (Theorem 4.8.1).

4.11 Let B_i be the decoding set of $\mathbf{w}_i$ ($i = 1, 2, \ldots, s$) in the original code. Since the first and second words are assigned the same probability, the average probability of error is unchanged if we replace B_2 by $B_1 \cup B_2$ and B_1 by the empty set $\emptyset$. Now consider the new code in which $\mathbf{w}_1'$ replaces $\mathbf{w}_1$. If we take the decoding set of $\mathbf{w}_1'$ to be $\emptyset$, the decoding set of $\mathbf{w}_2$ to be $B_1 \cup B_2$, and the decoding set of $\mathbf{w}_i$ to be $\mathbf{B}_i$, $i > 2$, then the average probability of error is the same as that of the original code. But the ideal-observer decision scheme has an average probability of error at least as small as the scheme just given. The result follows.

5.1 a.
$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \qquad \text{period} = 15.$$

Choosing

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

we obtain

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

b.
$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \text{period} = 5.$$

With

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{we have } A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are 3 cycles of length 5, we may get different parity check matrices for different choices of **x**. However, all such matrices define the same code (see Lemma 5.3.3).

c.
$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There is one cycle of length 21, one cycle of length 7, one cycle of length 3, and the trivial cycle of length 1. Hence the period of T is 21. If

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

then

$$A = A_1 = \begin{bmatrix} 0&0&0&0&1&1&1&1&1&0&1&0&1&0&0&1&1&0&0&0&1 \\ 0&0&0&1&1&1&1&1&0&1&0&1&0&0&1&1&0&0&0&1&0 \\ 0&0&1&1&1&1&1&0&1&0&1&0&0&1&1&0&0&0&1&0&0 \\ 0&1&1&1&1&1&0&1&0&1&0&0&1&1&0&0&0&1&0&0&0 \\ 1&1&1&1&1&0&1&0&1&0&0&1&1&0&0&0&1&0&0&0&0 \end{bmatrix}.$$

If

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{then} \quad A = A_2 = \begin{bmatrix} 0&1&1&1&0&0&1 \\ 1&1&1&0&0&1&0 \\ 1&1&0&0&1&0&1 \\ 1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&1 \end{bmatrix} \qquad (A_2 \text{ has rank 3}).$$

If

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{then} \quad A = A_3 = \begin{bmatrix} 1&1&0 \\ 1&0&1 \\ 0&1&1 \\ 1&1&0 \\ 1&0&1 \end{bmatrix} \qquad (A_3 \text{ has rank 2}).$$

5.2 By Lemma 5.3.3a, $f(x) = 1 + x^4 + x^5$ is a multiple of the generator polynomial $g(x)$. Now $x^5 + x^4 + 1 = (x^3 + x + 1)(x^2 + x + 1)$; since the rank of the parity check matrix A_2 of Problem 5.1c is 3, $g(x)$ is of degree 3, and thus $g(x) = x^3 + x + 1$. We may therefore generate the code by taking

$$T = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 1&1&0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The corresponding parity check matrix is

$$\begin{bmatrix} 0&0&1&0&1&1&1 \\ 0&1&0&1&1&1&0 \\ 1&0&1&1&1&0&0 \end{bmatrix}$$

which is of the form (5.3.1). It follows that $h(x) = 1 + x + x^2 + x^4$.

5.3 Let A be as in Table 5.4.1. If $\mathbf{w} = (r_0, r_1, \ldots r_{n-1})$ is a code word then

$$A_0\mathbf{w}^T = \begin{bmatrix} \mathbf{x} & T\mathbf{x} & \cdots & T^{n-1}\mathbf{x} \\ \mathbf{x} & T^3\mathbf{x} & \cdots & T^{3(n-1)}\mathbf{x} \\ & \cdot & & \cdot \\ & \cdot & & \\ & \cdot & & \\ \mathbf{x} & T^{2e-1}\mathbf{x} & \cdots & T^{(2e-1)(n-1)}\mathbf{x} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

If A_0 is premultiplied by the matrix

$$\begin{bmatrix} T & & & & & \\ & T^3 & & & & \\ & & T^5 & & \mathbf{0} & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & \mathbf{0} & & & T^{2e-1} \end{bmatrix} = B,$$

we obtain (noting that $T^{sn}\mathbf{x} = \mathbf{x}$, $s = 1, 2, \ldots, 2e - 1$)

$$BA_0\mathbf{w}^T = \begin{bmatrix} T\mathbf{x} & T^2\mathbf{x} & \cdots & T^{n-1}\mathbf{x} & \mathbf{x} \\ T^3\mathbf{x} & T^6\mathbf{x} & \cdots & T^{3(n-1)}\mathbf{x} & \mathbf{x} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ T^{2e-1}\mathbf{x} & T^{4e-2}\mathbf{x} & \cdots & T^{(2e-1)(n-1)}\mathbf{x} & \mathbf{x} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

Thus A_0 is obtained from BA_0 by a cyclic shift of columns. Examination of the equation $BA_0\mathbf{w}^T = \mathbf{0}$ shows that $(r_{n-1}r_0 \cdots r_{n-2})$ is a code word.

5.4 $e = 2$, $n = 31 = 2^q - 1$, so $q = 5$. From Table 5.2.1 we may take

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

with characteristic polynomial $\lambda^5 + \lambda^2 + 1$. If

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

the corresponding parity check matrix is

$$A_0 = \begin{bmatrix}
1&0&0&0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0 \\
0&0&0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0&1 \\
0&0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0&1&0 \\
0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0&1&0&0 \\
0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0&1&0&0&0 \\
1&0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&0&1&1&1&0&0&0&1&0&1&0&1&1&0 \\
0&0&0&1&0&1&0&1&1&0&1&0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&0&1&1&1 \\
0&1&1&1&1&1&0&1&1&1&0&0&0&1&0&1&0&1&1&0&1&0&0&0&0&1&1&0&0&1&0 \\
0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&0&1&1&1&0&0&0&1&0&1&0&1&1&0&1 \\
0&0&1&0&1&0&1&1&0&1&0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&0&1&1&1&0
\end{bmatrix}$$

5.5 a. There are $7 - 3 = 4$ check digits, hence the code may be generated by a nonsingular 4 by 4 matrix T of the form (5.1.1), and a vector

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(See Corollary 5.3.4.1.) If T has characteristic polynomial $a_0 + a_1\lambda + \cdots + a_{m-1}\lambda^{m-1} + \lambda^m$, then $(a_0 a_1 \cdots a_{m-1} 1 0 \cdots 0)$ together with $k - 1$ cyclic shifts generates the code. By shifting the given word $\mathbf{w} = (1010011)$ we find that (1110100) is a code word and that (1110100), (0111010), (0011101) generate the code.

It follows that an appropriate choice of T is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

With

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

the parity check matrix is

$$A = [\mathbf{x} \quad T\mathbf{x} \quad \cdots \quad T^6\mathbf{x}] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

b. By Corollary 5.3.4.1, $g(x)$ is the minimal polynomial of T, that is, $g(x) = 1 + x + x^2 + x^4$. Since the above parity check matrix is of the form (5.3.1), we obtain $h(x) = 1 + x + x^3$.

5.6 a. The corrector associated with a given error pattern is $T^i\mathbf{x}$ for some i. (By the remarks after Theorem 5.2.6, every matric polynomial in T is a power of T.) If the error burst is translated by t digits, the associated corrector is $T^tT^i\mathbf{x} = T^{t+i}\mathbf{x}$.

b. The corrector associated with a burst of length $\leq q$ beginning at digit 1 is $p(T)\mathbf{x}$ where $p(T)$ is a matric polynomial of degree $\leq q - 1$. If $p_1(T)\mathbf{x} = p_2(T)\mathbf{x}$, then if $p_1(T) \neq p_2(T), p_1(T) - p_2(T)$ is nonsingular by Lemma 5.2.3 and Theorem 5.2.6, hence $\mathbf{x} = \mathbf{0}$, a contradiction. Thus $p_1(T) = p_2(T)$ so that the corrector determines the error burst uniquely.

The above argument holds under the weaker assumption that T has an irreducible minimal polynomial.

5.7 Clearly, single errors are correctible.

Assume a double adjacent error. The associated corrector is

$$\begin{bmatrix} (I + T)T^i\mathbf{x} \\ 0 \end{bmatrix}.$$

This could not coincide with the corrector of a single error, since the last digit of the latter would be 1. If $(I + T)T^i\mathbf{x} = (I + T)T^j\mathbf{x}$, $0 \leq i, j \leq n - 1 = 2^q - 2$ then by Lemma 5.2.3, $I + T$ is nonsingular, hence $T^i\mathbf{x} = T^j\mathbf{x}$. Another application of Lemma 5.2.3 yields $T^i = T^j$, hence $i = j$. Thus distinct double adjacent errors have distinct correctors.

5.8 We must show that the corrector determines uniquely (a) the length of the burst, (b) the burst-error pattern, and (c) the starting position of the burst. Let

$$\mathbf{c} = \begin{bmatrix} \mathbf{c}_2 \\ \hline \mathbf{c}_1 \end{bmatrix} \begin{matrix} \updownarrow q_2 \\ \updownarrow q_1 \end{matrix}$$

be the corrector corresponding to an error burst of width $q_1' \leq q_1$. Since $q_2 = 2q_1 - 1$, $\mathbf{c}_2$ will contain a block of zeros of width $2q_1 - 1 - q_1' \geq q_1 - 1$ (considering the first component of $\mathbf{c}_2$ as following the last component); that is,

$$\mathbf{c}_2 = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{q_1'-2} \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \updownarrow q_2$$

Hence,

a. The length of the burst is q_2 minus the number of zeros in the block.

b. The burst pattern is $1b_1b_2 \cdots b_{q_1'-2}1$. No other allowable burst pattern could produce the same $\mathbf{c}_2$, for such a pattern would have to begin with one of the b_i and end at or after the "1" before b_1. The length of the burst would then be $\geq q_1 + 1$.

c. The starting position of the burst is the position of the first nonzero entry after the block of zeros $+ tq_2$ where t is an integer and $0 \leq t < n/q_2$.

We claim that t is determined by $\mathbf{c}_1$ and therefore the corrector specifies the error pattern uniquely. For (see Problem 5.6) the correctors $\mathbf{c}_1 = \mathbf{c}_{1j}$ produced by a fixed burst-error pattern of width $\leq q_1$ beginning at digit j ($j = 1, 2, \ldots, 2^{q_1} - 1$) are distinct. (If $\mathbf{c}_{1j}$ and $\mathbf{c}_{1k}$ are two such correctors, with $j < k$, then $\mathbf{c}_{1j} = T^a\mathbf{x}$ and $\mathbf{c}_{1k} = T^{a+k-j}\mathbf{x}$; hence $\mathbf{c}_{1j} = \mathbf{c}_{1k}$ implies $T^{k-j}\mathbf{x} = \mathbf{x}$, or $T^{k-j} = I$, with $0 < k - j \leq 2^{q_1} - 2$, a contradiction.)

Thus $\mathbf{c}_2$ tells us that the starting position of the burst is $i + tq_2$ and $\mathbf{c}_1$ tells us that the starting position is $j + u(2^{q_1} - 1)$, where i and j are known and $t < n/q_2$, $u < n/(2^{q_1} - 1)$. If there were more than one possible value for t (and hence u) we would have

$$i + tq_2 = j + u(2^{q_1} - 1)$$
$$i + t'q_2 = j + u'(2^{q_1} - 1)$$

Hence $(t - t')q_2 = (u - u')(2^{q_1} - 1) = r$. Assuming that $t > t'$ and hence $u > u'$, r is a multiple of both q_2 and $2^{q_1} - 1$ and $r < n$, contradicting the fact that n is the least common multiple. The result follows.

5.9 We illustrate the calculation for T^5. The characteristic polynomial of T^5 is of degree 4; hence the minimal polynomial is of degree ≤ 4. Let $S = T^5$. Since $T^4 + T + I = 0$,

$$S = TT^4 = T(T + I) = T^2 + T.$$

The "coordinates" of S with respect to I, T, T^2, T^3 are therefore (0110). Now

$$\begin{aligned} S^2 &= T^{10} = (T^2 + T)^2 &&= T^4 + T^2 = T^2 + T + I = (1110) \\ S^3 &= T^{15} = I &&= (1000) \\ S^4 &= T^{20} = T^5 = T^2 + T = (0110) \end{aligned}$$

We shall try to find b_0, b_1, b_2, b_3 such that

$$b_0 I + b_1 S + b_2 S^2 + b_3 S^3 + b_4 S^4 = 0;$$

that is,

$$b_0(1000) + b_1(0110) + b_2(1110) + b_3(1000) + b_4(0110) = (0000).$$

This yields

$$\begin{aligned} b_0 + b_2 + b_3 &= 0 \\ b_1 + b_2 + b_4 &= 0. \end{aligned}$$

We may take $b_3 = b_4 = 0$, $b_0 = b_1 = b_2 = 1$. Thus $S^2 + S + I = 0$. Clearly, no polynomial of degree 1 can be satisfied by S. Hence the minimal polynomial of S is $\lambda^2 + \lambda + 1$.

A similar calculation shows that the minimal polynomial of T^3 is $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$.

5.10 (Solution due to A. Wyner and B. Elspas.) Let $f(\lambda)$ be the minimal polynomial of T. Write

$$\lambda^k - 1 = q_k(\lambda) f(\lambda) + r_k(\lambda), \qquad k = 1, 2, \ldots, 2^q - 1$$

where the degree of $r_k(\lambda)$ is less than the degree of $f(\lambda) = q$. If $r_k(\lambda) = 0$ for some k we may replace λ by T to obtain

$$T^k - I = q_k(T) f(T) = 0$$

and we are finished.

If $r_k(\lambda)$ is the constant polynomial $1 (= -1)$ for some k, then

$$T^k - I = q_k(T) f(T) - I = -I, \quad \text{or} \quad T^k = 0,$$

which is impossible since T is nonsingular. Thus assume that none of the $r_k(\lambda)$ is 0 or 1. There are 2^q distinct polynomials of degree $< q$, leaving $2^q - 2$ when 0 and 1 are removed. Since there are $2^q - 1$ of the $r_k(\lambda)$, we must have $r_{k_1}(\lambda) = r_{k_2}(\lambda)$ for some $k_1 > k_2$. It follows that $T^{k_1} = T^{k_2}$, or $T^{k_1 - k_2} = I$, with $1 \leq k_1 - k_2 \leq 2^q - 1$.

6.1 a. To find the order, construct the following table:

	X_t	
Z_{t-1}	A	B
s_1	s_1	s_2
s_2	s_4	s_3
s_3	s_4	s_3
s_4	s_1	s_2

		$X_{t-1}X_t$		
Z_{t-2}	AA	AB	BA	BB
s_1	s_1	s_2	s_4	s_3
s_2	s_1	s_2	s_4	s_3
s_3	s_1	s_2	s_4	s_3
s_4	s_1	s_2	s_4	s_3

Thus X_{t-1} and X_t determine Z_t; hence the source is of order 2.

b. $P\{X_3 = A \mid Z_0 = s_1\} = P\{Z_3 = s_1 \mid Z_0 = s_1\} + P\{Z_3 = s_4 \mid Z_0 = s_1\}$
$= p_{11}^{(3)} + p_{14}^{(3)} = .512 + .208 = .72.$

c. $P\{X_{t+1} = A, X_{t+2} = B, X_{t+3} = B, X_{t+4} = A, X_{t+5} = A\}$
$=$ (since $X_{t+1} = A, X_{t+2} = B \Leftrightarrow Z_{t+2} = s_2$)
$P\{Z_{t+2} = s_2\}P\{X_{t+3} = B, X_{t+4} = X_{t+5} = A \mid Z_{t+2} = s_2\}$
$= P\{Z_{t+2} = s_2\}P\{Z_{t+3} = s_3, Z_{t+4} = s_4, Z_{t+5} = s_1 \mid Z_{t+2} = s_2\}$
$= w_2p_{23}p_{34}p_{41} = \frac{1}{3}(.1)(.8)(.7) = .0187$

6.2

	X_t		
Z_{t-1}	A	B	C
s_1	s_1	s_2	
s_2		s_2	s_3
s_3	s_1		s_3

The source is of order 1.

$$\Pi = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix}$$

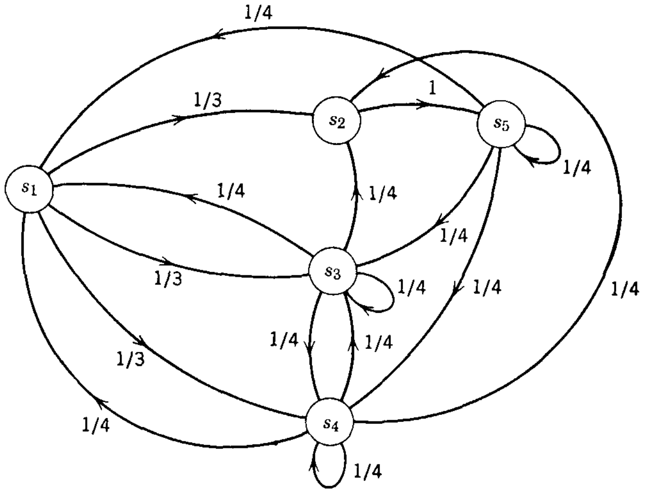
$W\Pi = W$ becomes

$$\begin{aligned} \tfrac{3}{4}w_1 \qquad\qquad + \tfrac{1}{4}w_3 &= w_1 \\ \tfrac{1}{4}w_1 + \tfrac{2}{3}w_2 \qquad\qquad &= w_2 \\ \tfrac{1}{3}w_2 + \tfrac{3}{4}w_3 &= w_3 \end{aligned}$$

and
$$w_1 + w_2 + w_3 = 1$$

Thus $w_1 = w_3 = \frac{4}{11}$, $w_2 = \frac{3}{11}$.

$$H\{\mathbf{X}\} = \tfrac{4}{11}H(3/4, 1/4) + \tfrac{3}{11}H(2/3, 1/3) + \tfrac{4}{11}H(3/4, 1/4).$$



$$
\begin{aligned}
f(s_1) &= \sigma\\
f(s_2) &= A\\
f(s_3) &= B\\
f(s_4) &= C\\
f(s_5) &= A
\end{aligned}
$$

Problem 6.3.*a*,*b*.

6.3 Let σ = space. A possible state diagram is shown in the accompanying figure. If

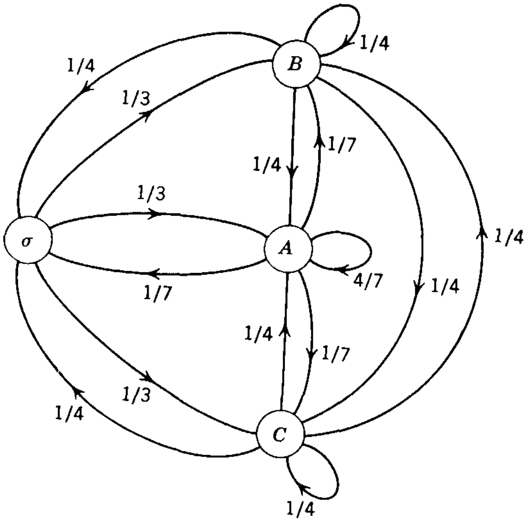
$$
\begin{array}{rl}
X_t = \sigma, & \text{then } Z_t = s_1\\
X_t = B, & \phantom{\text{then }} Z_t = s_3\\
X_t = C, & \phantom{\text{then }} Z_t = s_4\\
X_{t-1} \neq A,\ X_t = A & \phantom{\text{then }} Z_t = s_2\\
X_{t-1} = X_t = A & \phantom{\text{then }} Z_t = s_5
\end{array}
$$

Thus X_{t-1} and X_t determine Z_t and the source is of order 2. The steady state probabilities are $w_1 = w_2 = \frac{1}{6}$, $w_3 = w_4 = w_5 = \frac{2}{9}$. The source uncertainty is

$$
\begin{aligned}
H\{\mathbf{X}\} &= \tfrac{1}{6}H(1/3, 1/3, 1/3) + \tfrac{1}{6}H(1) + \tfrac{2}{3}H(1/4, 1/4, 1/4, 1/4)\\
&= \tfrac{1}{6}\log 3 + \tfrac{2}{3}\log 4 = \tfrac{1}{6}\log 3 + \tfrac{4}{3} = 1.5975.
\end{aligned}
$$

A state diagram for the first-order approximation is shown in the accompanying figure. Note

$$
\begin{aligned}
p_{AA} = P\{X_t = A \mid X_{t-1} = A\} &= \frac{P\{X_{t-1} = A,\ X_t = A\}}{P\{X_{t-1} = A\}}\\
&= \frac{w_5}{w_2 + w_5} = \frac{4/18}{7/18} = \frac{4}{7}.
\end{aligned}
$$



Problem 6.3.*c*.

The steady state probabilities are

$$\begin{aligned} w_\sigma &= w_1 = 1/6 = P\{X_t = \sigma\} \\ w_B &= w_3 = 2/9 = P\{X_t = B\} \\ w_C &= w_4 = 2/9 = P\{X_t = C\} \\ w_A &= w_2 + w_5 = 7/18 = P\{X_t = A\}. \end{aligned}$$

The uncertainty of the first-order approximation is

$$\begin{aligned} H_1\{\mathbf{X}\} &= H(X_t \mid X_{t-1}) = \tfrac{1}{6}H(1/3, 1/3, 1/3) + \tfrac{4}{9}H(1/4, 1/4, 1/4, 1/4) \\ &+ \tfrac{7}{18}H(4/7, 1/7, 1/7, 1/7) = \tfrac{1}{6}\log 3 + \tfrac{8}{9} + \tfrac{7}{18}H(4/7, 1/7, 1/7, 1/7) = 1.8004, \end{aligned}$$

which is larger than $H\{\mathbf{X}\}$, as expected.

6.4

$$\begin{aligned} P\{X_1 = \alpha_1, \ldots, X_n = \alpha_n\} &= \sum_{\alpha_0 \in \Gamma} P\{X_0 = \alpha_0, X_1 = \alpha_1, \ldots, X_n = \alpha_n\} \\ = \sum_{\alpha_0 \in \Gamma} P\{X_0 = \alpha_0, \ldots, &X_{n-1} = \alpha_{n-1}\} P\{X_n = \alpha_n \mid X_0 = \alpha_0, \ldots, X_{n-1} = \alpha_{n-1}\} \\ = \sum_{\alpha_0 \in \Gamma} P\{X_0 = \alpha_0, \ldots, &X_{n-1} = \alpha_{n-1}\} p_{(\alpha_0 \ldots \alpha_{n-1})(\alpha_1 \ldots \alpha_n)} \end{aligned}$$

Now in S_n, $p_{(\alpha_0\alpha_1' \ldots \alpha_{n-1}')(\alpha_1\alpha_2 \ldots \alpha_n)} = 0$ unless $\alpha_i' = \alpha_i$, $i = 1, 2, \ldots, n-1$, so we may write the above expression as

$$\begin{aligned} &P\{X_1 = \alpha_1, \ldots, X_n = \alpha_n\} \\ &= \sum_{\alpha_0, \alpha_1', \ldots, \alpha'_{n-1} \in \Gamma} P\{X_0 = \alpha_0, X_1 = \alpha_1', \ldots, X_{n-1} = \alpha'_{n-1}\} p_{(\alpha_0\alpha_1' \ldots \alpha_{n-1}')(\alpha_1 \ldots \alpha_n)} \end{aligned}$$

This is of the form

$$w_{(\alpha_1 \cdots \alpha_n)} = \sum_s w_s p_{s(\alpha_1 \cdots \alpha_n)}.$$

Hence the $w_{(\alpha_1 \cdots \alpha_n)} = P\{X_1 = \alpha_1, \ldots, X_n = \alpha_n\}$ form a stationary distribution.

6.5

$$P\{X_{i_1} = X_{i_2} = \cdots = X_{i_n} = 1\}$$
$$= P\{X_{i_1} = X_{i_2} = \cdots = X_{i_n} = -1\} = 1/2 \quad \text{for} \quad \text{all } i_i, \ldots, i_n$$

so the sequence is stationary.

Let α be the sequence consisting of the single digit 1. Then

$$n^{-1}N_\alpha^n(X_1, \ldots, X_n) = n/n = 1 \quad \text{with} \quad \text{probability } 1/2$$
$$= 0 \quad \text{with} \quad \text{probability } 1/2$$

Let $0 < \varepsilon < 1/2$; then $P\{|n^{-1}N_\alpha^n - P\{X_1 = \alpha\}| \geq \varepsilon\} = P\{|n^{-1}N_\alpha^n - 1/2| \geq \varepsilon\} = 1$ since $|n^{-1}N_\alpha^n - \frac{1}{2}| = \frac{1}{2}$.

Thus $n^{-1}N_\alpha^n$ cannot converge in probability to $P\{X_1 = \alpha\}$, and the sequence is not ergodic.

6.6

$$P\{X_{t-1} = A\} = P\{X_{t-1} = A, X_t = A\} + P\{X_{t-1} = A, X_t = B\}$$

and

$$P\{X_{t-1} = A\} = P\{X_t = A\} = P\{X_{t-1} = A, X_t = A\} + P\{X_{t-1} = B, X_t = A\}$$

The result follows.

6.7 The original source is of order 1, so that $H\{\mathbf{X}\} = H(X_n \mid X_{n-1})$. Now

$$H(Y_n \mid Y_0, Y_1, \ldots, Y_{n-1}) = \sum_{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}} P\{Y_0 = \alpha_0, \ldots, Y_{n-1} = \alpha_{n-1}\}$$
$$\times H(Y_n \mid Y_0 = \alpha_0, \ldots, Y_{n-1} = \alpha_{n-1}).$$

Break the summation into 3 parts:

1. $\alpha_{n-2} \neq e, \alpha_{n-1} \neq e$:

$$H(Y_n \mid Y_0 = \alpha_0, \ldots, Y_{n-1} = \alpha_{n-1}) = 0, \quad \text{and}$$
$$P\{Y_{n-2} = \alpha_{n-2}, Y_{n-1} = \alpha_{n-1}\} = P\{X_{n-2} = \alpha_{n-2}, X_{n-1} = \alpha_{n-1}\}.$$

2. $\alpha_{n-2} \neq e, \alpha_{n-1} = e$:

$$H(Y_n \mid Y_0 = \alpha_0, \ldots, Y_{n-1} = \alpha_{n-1})$$
$$= H(X_n \mid X_{n-2} = \alpha_{n-2}) \quad \text{and} \quad P\{Y_{n-2} = \alpha_{n-2}, Y_{n-1} = e\} = \tfrac{1}{3}P\{X_{n-2} = \alpha_{n-2}\}.$$

3. $\alpha_{n-2} = e, \alpha_{n-1} \neq e$:

$$H(Y_n \mid Y_0 = \alpha_0, \ldots, Y_{n-1} = \alpha_{n-1}) = H(X_n \mid X_{n-1} = \alpha_{n-1}) \quad \text{and}$$
$$P\{Y_{n-2} = e, Y_{n-1} = \alpha_{n-1}\} = \tfrac{1}{3}P\{X_{n-1} = \alpha_{n-1}\}.$$

Thus

$$H(Y_n \mid Y_0, Y_1, \ldots, Y_{n-1}) = \tfrac{1}{3}[H(X_n \mid X_{n-2}) + H(X_n \mid X_{n-1})]$$
$$= \tfrac{1}{3}[H(X_n \mid X_{n-2}) + H\{\mathbf{X}\}] = \text{constant} \quad \text{for} \quad n \geq 2.$$

Thus $H\{\mathbf{Y}\} = \frac{1}{2}[H(X_n \mid X_{n-2}) + H\{\mathbf{X}\}]$. Now

$$H\{\mathbf{X}\} = \tfrac{7}{16}H(5/7, 2/7) + \tfrac{9}{16}H(4/9, 5/9) \quad \text{and}$$

$$\begin{aligned} H(X_n \mid X_{n-2}) &= w_A H(X_n \mid X_{n-2} = A) + w_B H(X_n \mid X_{n-2} = B) \\ &\ \tfrac{7}{16}H(\tfrac{4}{49} + \tfrac{25}{63}, \tfrac{10}{49} + \tfrac{20}{63}) + \tfrac{9}{16}H(\tfrac{10}{63} + \tfrac{20}{81}, \tfrac{25}{63} + \tfrac{16}{81}). \end{aligned}$$

This determines $H\{\mathbf{Y}\}$.

6.8 a. As in the proof of (6.6.3), write

$$E(V_n) = \sum_k kP\{V_n = k\} = \sum_{k<\beta-\varepsilon} kP\{V_n = k\} + \sum_{\beta-\varepsilon\le k\le\beta+\delta} kP\{V_n = k\} + \sum_{k>\beta+\delta} kP\{V_n = k\}.$$

Thus

$$\begin{aligned} E(V_n) &\geq 0 + (\beta - \varepsilon)P\{\beta - \varepsilon \leq V_n \leq \beta + \delta\} + (\beta + \delta)P\{V_n > \beta + \delta\} \\ &= (\beta - \varepsilon)(1 - P\{V_n < \beta - \varepsilon\} - P\{V_n > \beta + \delta\}) + (\beta + \delta)P\{V_n > \beta + \delta\}, \end{aligned}$$

and the result follows.

b. Let $n \to \infty$ in (6.6.3) and part (a) of this problem. We obtain

$$-\varepsilon \leq -\beta + \liminf_{n\to\infty} E(V_n) \leq -\beta + \limsup_{n\to\infty} E(V_n) \leq \delta.$$

Since ε and δ are arbitrary positive numbers,

$$\lim_{n\to\infty} E(V_n) = \beta.$$

By part (a) of the proof of Theorem 6.6.1, $\beta = H\{\mathbf{X}\}$.

6.9 Let $\{X_n, n = 1, 2, \ldots\}$ be the given information source. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$ be any two states of the mth-order approximation $\mathcal{S}_m$. Consider the set of all sequences $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r)$, $r \geq m + 1$, such that $(\gamma_1, \ldots, \gamma_m) = \boldsymbol{\alpha}$ and $(\gamma_{r-m+1}, \ldots, \gamma_r) = \boldsymbol{\beta}$. We claim that we can find an r and a $\boldsymbol{\gamma}$ such that $p(\boldsymbol{\gamma}) = P\{X_1 = \gamma_1, \ldots, X_r = \gamma_r\} > 0$. For by ergodicity, $n^{-1}N_{\boldsymbol{\alpha}}{}^n(X_1, \ldots, X_n) \to p(\boldsymbol{\alpha})$, $n^{-1}N_{\boldsymbol{\beta}}{}^n(X_1, \ldots, X_n) \to p(\boldsymbol{\beta})$ in probability; by definition of $\mathcal{S}_m$, $p(\boldsymbol{\alpha}) > 0$, $p(\boldsymbol{\beta}) > 0$. Therefore if $0 < \delta < \min(p(\boldsymbol{\alpha}), p(\boldsymbol{\beta}))$, then $P\{n^{-1}N_{\boldsymbol{\alpha}}{}^n \geq \delta \text{ and } n^{-1}N_{\boldsymbol{\beta}}{}^n \geq \delta\} \to 1$ as $n \to \infty$. This implies that there is a sequence $\boldsymbol{\gamma}$ with positive probability such that either $\boldsymbol{\gamma}$ begins with $\boldsymbol{\alpha}$ and ends with $\boldsymbol{\beta}$, or vice versa. In the former case, the claim is established. In the latter case, we observe that by ergodicity, the relative frequency of $\boldsymbol{\gamma}$ converges in probability to $p(\boldsymbol{\gamma}) > 0$, and if $\boldsymbol{\gamma}$ occurs sufficiently many times among the symbols $X_1, \ldots, X_n$, $\boldsymbol{\alpha}$ must precede $\boldsymbol{\beta}$ somewhere in the sequence.

Now let $\mathbf{s}_1 = (\gamma_1, \ldots, \gamma_m)$, $\mathbf{s}_2 = (\gamma_2, \ldots, \gamma_{m+1})$, $\ldots$, $\mathbf{s}_{r-m+1} = (\gamma_{r-m+1}, \ldots, \gamma_r)$, and let $p(\mathbf{s}_i, \mathbf{s}_{i+1}) = P\{X_i = \gamma_i, \ldots, X_{i+m-1} = \gamma_{i+m-1}, X_{i+m} = \gamma_{i+m}\}$, $i = 1, 2 \ldots, r - m$. Since $X_1 = \gamma_1, \ldots, X_r = \gamma_r \Leftrightarrow (X_1, \ldots, X_m) = \mathbf{s}_1, (X_2, \ldots, X_{m+1}) = \mathbf{s}_2, \ldots, (X_{r-m+1}, \ldots, X_r) = \mathbf{s}_{r-m+1}$, it follows that $p(\mathbf{s}_i) > 0$ and $p(\mathbf{s}_i, \mathbf{s}_{i+1}) > 0$ for all i. But by definition of $\mathcal{S}_m$, the transition probability from $\mathbf{s}_i$ to $\mathbf{s}_{i+1}$ in $\mathcal{S}_m$ is

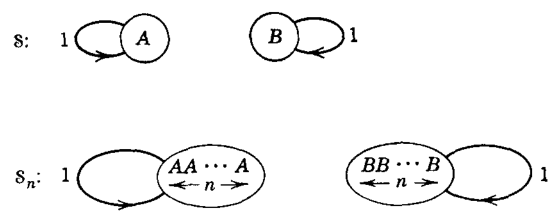
$$p_{\mathbf{s}_i\mathbf{s}_{i+1}} = \frac{p(\mathbf{s}_i, \mathbf{s}_{i+1})}{p(\mathbf{s}_i)} = P\{X_{i+m} = \gamma_{i+m} \mid X_i = \gamma_i, \ldots, X_{i+m-1} = \gamma_{i+m-1}\}$$

Since $s_1 = \alpha$, $s_{r-m+1} = \beta$, and $p_{s_i s_{i+1}} > 0$ for all i, β is reachable from α in $\mathcal{S}_m$. Thus every state in the mth-order approximation is reachable from every other state, and hence $\mathcal{S}_m$ is indecomposable with the essential class consisting of all the states.

6.10 If the set S of all states is not essential, then there are two distinct states, let us call them s_1 and s_2, such that s_2 cannot be reached from s_1. Let S_1 be the set of states reachable from s_1. Then the states of S_1 form a Markov chain (if $s \in S_1$ and s' can be reached from s, then s' can be reached from s_1 and hence $s' \in S_1$; thus the submatrix of the transition matrix corresponding to the states of S_1 is a Markov matrix). Since $s_2 \notin S_1$, S_1 is a proper subset of S. Apply the above argument to S_1, and continue in this fashion until an essential set is found. The process must terminate since the original chain has only a finite number of states; if we come down to a chain containing only one state, the set consisting of that state alone is essential.

An infinite chain need not have an essential set. For example, take the set of states to be the integers, with $p_{i,i+1} = 1$ for all integers i (that is, at each transition, move one unit to the right). If state j can be reached from state i, then state i cannot be reached from state j, so no set of states can be essential.

6.11 For the source $\mathcal{S}$ shown in the accompanying figure, the order of every approximation $\mathcal{S}_n$ is 1.

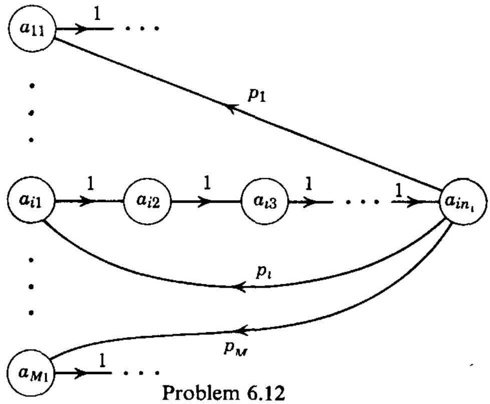


Problem 6.11. Any initial distribution $[w_A, w_B]$ is stationary; take $w_A > 0$, $w_B > 0$, $w_A + w_B = 1$.

6.12 A state diagram for the source $\{X_n\}$ is indicated in the accompanying figure. The states are a_{ij}, $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, M$, and the transition probabilities are $p(a_{ij}, a_{i,\,j+1}) = 1$ for $j = 1, 2, \ldots, n_i - 1$ and all i; $p(a_{in_i}, a_{k1}) = p_k$ for $k = 1, 2, \ldots, M$ and all i.

Since each code word begins with a different letter, we have a unifilar Markov source. Every state may be reached from every other state, and hence the source is indecomposable. Thus there is a unique stationary distribution with probabilities w_{ij} assigned to the states a_{ij}. By inspection of the state diagram, $w_{i1} = w_{i2} = \cdots = w_{in_i} =$ (say) b_i, and

$$b_k = \sum_{i=1}^{M} b_i p(a_{in_i}, a_{k1}) = p_k \sum_{i=1}^{M} b_i \quad \text{for all } k.$$



Problem 6.12

But $\sum_{i=1}^{M} b_i n_i = 1$, and hence $(\sum_{k=1}^{M} p_k n_k)(\sum_{i=1}^{M} b_i) = 1$. Consequently $b_i = p_i/\bar{n}$ for all i. It follows from Theorem 6.4.2 that

$$H\{\mathbf{X}\} = \sum_{i=1}^{M} w_{in_i} H(p_1, \ldots, p_M) = H(Y) \sum_{i=1}^{M} b_i = H(Y)/\bar{n}.$$

6.13. a. If we form a sequence of length $T + 1$ beginning at state s_i, the first transition must lead to some state s_j. If we are in state s_j after one transition, we may complete the sequence of length $T + 1$ by constructing a sequence of length T starting from s_j. Since the source is unifilar, all sequences formed in this way are distinct. Thus

$$N_i(T + 1) = \sum_{j=1}^{r} a_{ij} N_j(T), \qquad i = 1, \ldots, r.$$

This is a system of linear homogeneous difference equations with constant coefficients, and therefore the solution is a linear combination of exponentials λ^T. To find the particular λ's, we assume a solution of the form $N_i(T) = y_i \lambda^T$ to obtain $\lambda^T(\lambda y_i) = (\lambda^T) \sum_{j=1}^{r} a_{ij} y_j$, $i = 1, \ldots, r$, or $\lambda \mathbf{y} = A\mathbf{y}$. Thus the allowable λ's are the eigenvalues of A. The result (a) now follows from the observation that

$$N(T) \leq \sum_{i=1}^{r} N_i(T).$$

b. Since $H(X_1, \ldots, X_T)$ cannot exceed the logarithm of the total number of possible sequences of length T (Theorem 1.4.2), we have

$$T^{-1} H(X_1, \ldots, X_T) \leq T^{-1} \log N(T) \leq T^{-1} (\log K + T \log \lambda_0).$$

Using Theorem 6.4.1 we have

$$H\{\mathbf{X}\} = \lim_{T \to \infty} T^{-1} H(X_1, \ldots, X_T) \leq \log \lambda_0.$$

c. Since the components x_i are >0, $p_{ij} \geq 0$ for all i and j. Since $\mathbf{x}$ is an eigenvector for λ_0, $\sum_{j=1}^{r} p_{ij} = 1$, and hence the p_{ij} form a legitimate assignment of transition probabilities. Since $p_{ij} > 0$ if and only if $a_{ij} = 1$, the assignment is consistent with the given connection matrix. By Theorem 6.4.2, $H\{\mathbf{X}\} = \sum_{i=1}^{r} w_i H_i$, where H_i is the uncertainty of state s_i and $[w_1 \cdots w_r]$ is the unique stationary distribution for the source. (Note that in general, the stationary distribution will depend on the choice of the p_{ij}). Thus

$$H\{\mathbf{X}\} = -\sum_{i,j=1}^{r} w_i p_{ij} \log p_{ij} = \sum_{i,j} w_i p_{ij} (\log \lambda_0 + \log x_i - \log x_j).$$

(Note that only terms for which $a_{ij} = 1$ contribute to the summation.) Now

$$\sum_{i,j} w_i p_{ij} \log x_i = \sum_{i} w_i \log x_i$$

and

$$\sum_{i,j} w_i p_{ij} \log x_j = \sum_{j} \log x_j \sum_{i} w_i p_{ij} = \sum_{j} w_j \log x_j,$$

the last step following since the w_i form a stationary distribution. Thus

$$H\{\mathbf{X}\} = \sum_{i,j} w_i p_{ij} \log \lambda_0 = \log \lambda_0.$$

d. The connection matrix is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

To find the eigenvalues, we set the determinant

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

equal to zero, to obtain $\lambda^2 - \lambda - 1 = 0$. The largest eigenvalue is $\lambda_0 = \frac{1}{2}(1 + \sqrt{5})$ and the maximum uncertainty is $\log \lambda_0$. An eigenvector with positive components may be formed by taking $x_1 = \frac{1}{2}(1 + \sqrt{5})$, $x_2 = 1$. The corresponding transition probabilities are given by

$$p_{11} = \lambda_0^{-1} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}, \qquad p_{12} = [\tfrac{1}{4}(1 + \sqrt{5})^2]^{-1} = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5},$$
$$p_{21} = 1, \qquad p_{22} = 0.$$

Remark: Intuitively, we might expect that p_{ij} should be (roughly) the number of sequences of length $T + 1$ beginning at s_i and passing through s_j at the first transition [that is, $a_{ij}N_j(T)$], divided by the number of sequences of length $T + 1$ beginning at s_i [that is $N_i(T + 1)$]. If λ_0 is in fact greater than the magnitude of all other eigenvalues, then for large T, the term involving λ_0^T dominates the solution of the difference equation of (a). Thus asymptotically we have $N_i(T)$ proportional to $x_i \lambda_0^T$, hence

$$\frac{a_{ij}N_j(T)}{N_i(T+1)} \sim \frac{a_{ij}x_j}{\lambda_0 x_i}.$$

7.1 a. Let the channel be in state s_{kij}, $k, i, j = 0$ or 1, at time $t = n$ if the ball labeled j is in the occupied slot at $t = n$, the ball labeled i was in the occupied slot at $t = n - 1$, and an input k was applied before the transition at $t = n$. The transition probability from s_{kij} to $s_{k_1 i_1 j_1}$ under the input k_1 is given by $M_{k_1}(s_{ij}, s_{i_1 j_1})$, where the matrices M_{k_1} are given in Table 7.1.1. The transition probability from s_{kij} to $s_{k_1 i_1 j_1}$ under an input other than k_1 is zero. The output function g' is given by $g'(s_{kij}) = g(k, s_{ij})$ where g is the original output function of Table 7.1.1.

b. We proceed as in (a). Given a finite-state regular channel with states s_j, $j = 1, 2, \ldots, r$, and input alphabet $\Gamma = \{b_1, \ldots, b_t\}$, we define a new channel with the same input alphabet Γ, and states s_{ij}, $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, r$. The transition probability from s_{ij} to $s_{i_1 j_1}$ under the input b_{i_1} is given by $M_{i_1}(s_j, s_{j_1})$; the transition probability under any other input is zero. If $g(b_i, s_j)$ is the original output function, the new output function is $g'(s_{ij}) = g(b_i, s_j)$. Since the original channel is regular, for any finite input sequence $b_{i_1} \cdots b_{i_k}$ there is a channel state s_{j_0} and an integer N such that for any initial channel state, s_{j_0} is reachable under the input sequence $b_{i_1} \cdots b_{i_k} \cdots b_{i_1} \cdots b_{i_k}$ (N times). In the new channel, the state $s_{i_k j_0}$ is reachable under the same input sequence, so the new channel is regular. If a source is connected to the new channel, the distribution of input and output sequences is (by construction) exactly the same as if the source were connected to the original channel, and thus the capacity is preserved.

7.2 First observe that

$$H(Y_{n+1} \mid X_1, \ldots, X_{n+1}, Y_1, \ldots, Y_n) \le H(Y_{n+1} \mid X_2, \ldots, X_{n+1}, Y_2, \ldots, Y_n)$$

by Theorem 1.4.5 or Problem 1.4c

$$= H(Y_n \mid X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1})$$

by stationarity

Thus $\lim_{n\to\infty} H(Y_n \mid X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1})$ exists. Now $n^{-1}H(Y_1, \ldots, Y_n \mid X_1, \ldots, X_n) = n^{-1}\sum_{i=1}^{n} g_{in}$, where $g_{in} = H(Y_i \mid X_1, \ldots, X_n, Y_1, \ldots, Y_{i-1})$. Since the channel is nonanticipatory, that is, the distribution of a given output letter does not depend on future inputs,

$$g_{in} = g_i = H(Y_i \mid X_1, \ldots, X_i, Y_1, \ldots, Y_{i-1}).$$

Since $\lim_{i\to\infty} g_i = H(\{\mathbf{Y}\} \mid \{\mathbf{X}\})$, it follows that

$$\lim_{n\to\infty} n^{-1}H(Y_1, \ldots, Y_n \mid X_1, \ldots, X_n) = H(\{\mathbf{Y}\} \mid \{\mathbf{X}\}).$$

Now by Theorem 6.4.1,

$$\begin{aligned} H\{\mathbf{X}, \mathbf{Y}\} &= \lim_{n\to\infty} n^{-1}H(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \\ &= \lim_{n\to\infty} n^{-1}H(X_1, \ldots, X_n) + \lim_{n\to\infty} n^{-1}H(Y_1, \ldots, Y_n \mid X_1, \ldots, X_n) \\ &= H\{\mathbf{X}\} + H(\{\mathbf{Y}\} \mid \{\mathbf{X}\}). \end{aligned}$$

7.3 The results of Problem 7.2 do not hold in general. To see this, let $Z_0, Z_1, \ldots, Z_n, \ldots,$ be a sequence of independent, identically distributed random variables, and take $X_n = Z_{n-1}$, $Y_n = Z_n$, $n = 1, 2, \ldots$.

We then have $H(Y_n \mid X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1}) = H(Z_n \mid Z_0, \ldots, Z_{n-1}) = H(Z_n) =$ a constant H for all n. But

$$\begin{aligned} H(Y_1, \ldots, Y_n \mid X_1, \ldots, X_n) &= H(Z_1, \ldots, Z_n \mid Z_0, \ldots, Z_{n-1}) \\ &= H(Z_n \mid Z_0) = H(Z_n) = H. \end{aligned}$$

Thus $H(\{\mathbf{Y}\} \mid \{\mathbf{X}\}) = H$, but

$$\lim_{n\to\infty} n^{-1}H(Y_1, \ldots, Y_n \mid X_1, \ldots, X_n) = \lim_{n\to\infty} n^{-1}H = 0 \neq H$$

in general.

However, if we make an assumption of "nonanticipatory" behavior, that is, $H(Y_i \mid X_1, \ldots, X_n, Y_1, \ldots, Y_{i-1}) = H(Y_i \mid X_1, \ldots, X_i, Y_1, \ldots, Y_{i-1})$ for all n, and all $i = 1, 2, \ldots, n$, then the argument of Problem 7.2 will go through intact.

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