

ELECTRIC CIRCUIT ANALYSIS



K. S. Suresh Kumar

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PEARSON

Chennai • Delhi • Chandigarh

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This book is dedicated to the memory of

Mrs. Chellamma

who was my Class Teacher for Class III during the academic year 1967–68 at
Raghavamandiram Lower Primary School, Attingal, Thiruvananthapuram District,
Kerala, India.

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Preface

The field of electrical and electronic engineering is vast and diverse. However, two topics hold the key to the entire field. They are ‘Circuit Theory’ and ‘Signals and Systems’. Both these topics provide a solid foundation for later learning, as well as for future professional activities.

This undergraduate textbook, the first of a two-book series, deals with one of these two pivotal subjects in detail. In addition, it connects ‘Circuit Theory’ and ‘Signals and Systems’, thereby preparing the student-reader for a detailed study of this important subject either concurrently or subsequently.

The theory of electric circuits and networks, a subject derived from the more basic subject of electromagnetic fields, is the cornerstone of electrical and electronics engineering. Students need to master this subject and assimilate its basic concepts in order to become competent engineers.

My book *Electric Circuits and Networks* (ISBN 978-81-317-1390-7), published by Pearson Education in 2009, was an attempt to provide a solid foundation on electric circuits and networks to the undergraduate students. However, this book was perceived as being too voluminous and too comprehensive for a first-level course on Circuits. Hence, for better acceptability and better utilization of the content, I decided to rewrite the material and present it in two books. Of these, the first one on Electric Circuit Analysis has been designed to serve as a textbook for first/second level course on circuits and the second one on Electric Network Analysis and Synthesis has been envisaged for an advanced level course on network analysis and synthesis. The latter text is being augmented with additional content that caters to the needs of the advanced user. This text, *Electric Circuit Analysis*, is the first of the series, while work on the second text with new material included on network functions, electric filters and passive network design, is currently in progress.

Objectives of the Book Series

Primary Objective: To serve as textbooks that will meet students’ and instructors’ need for a one/two-semester course on electrical circuits and networks for undergraduate students of electrical and electronics engineering (EE), electronics and communications engineering (EC) and allied streams. This textbook series introduces, explains and reinforces the basic concepts of analysis of dynamic circuits in time-domain and frequency-domain.

Secondary Objective: To use circuit theory as a carrier of the fundamentals of linear system and continuous signal analysis so that the students of EE and EC streams are well-prepared to take up a detailed study of higher level subjects such as analog and digital electronics, pulse electronics, analog and digital communication systems, digital signal processing, control systems, and power electronics at a later stage.

Electric Circuits in EE and EC Curricula

The subject of Electric Circuits and Networks is currently covered in two courses in Indian technical universities. The introductory portion is covered as a part of a course offered in the first year of

undergraduate program. It is usually called ‘Basic Electrical Engineering’. About half of the course time is devoted to introductory circuit theory covering the basic principles, DC circuit analysis, circuit theorems and single frequency sinusoidal steady-state analysis using phasor theory. This course is usually a core course for *all disciplines*. Therefore, it is limited very much in its content and depth as far as topics in circuit theory are concerned. The course is aimed at giving an overview of electrical engineering to undergraduate students of all engineering disciplines.

Students of disciplines other than EE and EC need to be given a brief exposure to electrical machines, industrial electronics, power systems etc., in the third semester. Many universities include this content in the form of a course called ‘Electrical Technology’ in the third semester for students of other engineering disciplines. This approach makes it necessary to teach them AC steady-state analysis of RLC circuits even before they can be told about transient response in such circuits. EE students, however, need AC phasor analysis only from the fourth or fifth semester when they start on Electric Machines and Power Systems. But the first year course on basic electrical engineering has to be a common course and hence even EE and EC students learn AC steady-state analysis before transient response.

The second course on circuits is usually taught in the third or fourth semester and is termed ‘Electric Circuit Theory’ for EE students and ‘Circuits and Networks’ or ‘Network Analysis’ for EC students. Few comments on these different course titles and course content are in order.

Traditionally, undergraduate circuit theory courses for EE stream slant towards a ‘steady-state’ approach to teaching circuit theory. The syllabi of many universities in India contain extensive coverage on single-phase and three-phase circuits with the transients in RC and RL circuits postponed to the last module in the syllabus. The course instructor usually finds himself with insufficient contact hours towards the end of the semester to do full justice to this topic. EE stream often orients Circuits courses to serve as prerequisites for courses on electrical machines and power systems.

This led to the EC stream preparing a different syllabus for their second-level circuit theory course—one that was expected to orient the student towards the dynamic behaviour of circuits in time-domain and analysis of dynamic behaviour in the frequency domain. But, in practice, the syllabus for this subject is an attempt to crowd too many topics from Network Analysis and Synthesis into what should have been a basic course on Circuits.

Such a difference in orientation between the EE-stream syllabus for circuit theory and EC-stream syllabus for circuit theory is neither needed nor desirable. The demarcation line between EE and EC has blurred considerably over the last few years. In fact, students of both disciplines need good coverage of Linear Systems Analysis or Signals and Systems in the third or fourth semester. Unfortunately, Linear Systems Analysis has gone out of the curriculum even in those universities which were wise enough to introduce it earlier, and Signals and Systems has started making its appearance in EC curriculum in many universities. But the EE stream is yet to lose its penchant for AC steady-state in many Indian technical universities.

The subject of electrical circuit theory is as *electronic* as it is *electric*. Inductors and capacitors do not get scared and behave differently when they see a transistor. Neither do they reach sinusoidal steady-state without going through a transient state just because they happen to be part of a power system or electrical machine.

Against this background, I state the pedagogical viewpoint I have adopted in writing this textbook.

Pedagogical Viewpoint Adopted in this Book

- With a few minor changes in emphasis here and there, both EE and EC students need the same Circuit Theory course.

- Introducing time-domain response of circuits before AC steady-state response is pedagogically superior. However, curricular constraints make it necessary to introduce AC steady-state analysis first and it is done that way in this book.
- ‘Lumped Linear Electrical Circuits’ is an ideally suited subject to introduce and reinforce ‘Linear System’ concepts and ‘Signals and Systems’ concepts in the EE and EC undergraduate courses. This is especially important in view of shortage of course time, which makes it difficult to introduce full-fledged courses in these two subjects. This textbook is organized along the flow of Linear Systems Analysis concepts.
- Circuit Theory is a very important *foundation course* for EE, EC and allied disciplines. The quality of teaching and intellectual capability of students (especially the quality of teaching) varies widely in different sectors of technical educational institutions in India. Therefore, a textbook on circuit theory has to be written explaining the basic concepts thoroughly and repeatedly, with the average student in mind—not the brilliant ones who manage to get into ivy-league institutions. Such a textbook will supplement good teaching in the case of students of premier institutions and, more importantly, save the average students from life-long confusion.
- The pages of a textbook on Circuit Theory are precious due to the reasons described above. Therefore, all extraneous matter should be dispensed with. The first in this category is the so-called historical vignettes aimed at motivating the students. I have avoided them and instead, used the precious pages to explain basic concepts from different points of view.
- The pre-engineering school curriculum in India prepares the students well in mathematics and physics. Engineering students have not yet become impatient enough to demand examples of practical applications of each and every basic concept introduced in subjects like Circuit Theory or Newtonian Mechanics. There is no need to keep motivating the student by citing synthetic-looking examples of complex electrical and electronic systems when one is writing on basic topics in Circuit Theory. The pages can be used for providing detailed explanation on basic concepts. The first year or second year undergraduate student is far away from a practical engineering application! I believe that a typical Indian engineering student is willing to cover the distance patiently.
- Circuit Theory is a foundation course. It is difficult to quote a practical application for each and every concept without spending considerable number of pages to describe the application and set the background. The pedagogical impact of this wasteful exercise is doubtful. However, those applications that are within the general information level of an undergraduate student should be included. Thus, applications that require long explanations to fit them into the context must be avoided in the interest of saving pages for explanations on Circuit Theory concepts.
- Circuit Theory is a basic subject. All other topics that the students are going to learn in future semesters will be anchored on it. Hence, it should be possible to set pointers to applications in higher topics in a textbook on Circuit Theory. Such pointers can come in the form of worked examples or end-of-chapter problems that take up an idealized version of some practical application. An example would be to use an idealized form of fly-back switched mode converter and to show how the essential working of this converter can be understood from the inductance $v-i$ relationship. In fact, all well-known switched mode power converter circuits can be employed in the chapter which deals with the $v-i$ relation of an inductor. Similarly, switched capacitor circuits can be introduced in the section dealing with the $v-i$ relation of a capacitor.
- Circuit Theory can be learnt well without using simulation software. Circuit simulation packages are only tools. I am of the opinion that using simulation software becomes a source of distraction in a foundation course. A foundation course is aimed at flexing the student’s intellect in order to encourage the growth of analytical capability in him.

- An argument usually put forth in support of simulation software as an educational aid is that it helps one to study the response of circuits for various parameter sets and visualize the effect of such variations. That is precisely why I oppose it in a foundation course. Ability to visualize such things using his/her head and his/her ability for mental imagery is very much essential in an engineer. Let the student develop that first. He/she can seek the help of simulation software later when he/she is dealing with a complex circuit that goes beyond the limits of mental imagery.

After all, we do not include a long chapter on waveform generators and another one on oscilloscopes in every Circuit Theory textbook. In fact, some of the modern-day waveform generators and oscilloscopes are so complex that a chapter on each of them will not really be out of place. Yet, we do not spend pages of a Circuit Theory textbook for that. The same rule governs simulation software too.

Pedagogy

- Every chapter begins with a statement of chapter objectives and relative emphasis of topics covered in that chapter.
- Detailed summary covering all the important points made in the chapter is provided at the end of each chapter.
- Boxed entries highlight important concepts and reinforce them. The book is replete with worked examples illustrating the concepts explained in the text are included. Simple formula-substitution kind of worked examples are avoided.
- Large number of problems is included at the end of every chapter. Section-wise organization of these problems is *avoided intentionally*. I expect the student to understand the entire chapter and use all the concepts covered in that chapter (and from earlier chapters) to solve a problem if necessary. After all, no one tells him which concepts are relevant in solving a particular problem in the examination hall or in practical engineering.

Outline and Organization

This book contains 14 chapters.

The first three chapters address the basic concepts. The first chapter goes into the physics of two-terminal circuit elements briefly and deals with element relations, circuit variables, and sign convention. It also addresses the concepts of linearity, time-invariance and bilaterality properties of two-terminal elements. This chapter assumes that the reader has been introduced to the basic physics of electromagnetic fields in pre-engineering high school physics. It also attempts to explain the important assumptions underlying circuit theory from the point of view of electromagnetic fields. The treatment is qualitative and not at all intended to be rigorous.

The second chapter covers the two basic laws – Kirchhoff's voltage law and Kirchhoff's current laws – in detail. Emphasis is placed on the applicability of these two laws under various conditions.

The third chapter looks into the v - i relationship of the resistor, the inductor and the capacitor. Series-parallel equivalents are also covered in this chapter. This chapter analyses the v - i relations of inductor and capacitor in great detail. The concept of 'memory' in circuit elements is introduced in this chapter and the electrical circuits are divided into two classes – *memoryless circuits* and *circuits with memory*. Circuits with memory are termed as *Dynamic Circuits* from that point onwards.

The next two chapters deal with analysis of *memoryless circuits*. Chapter 4 takes up the analysis of memoryless circuits containing independent voltage and current sources, linear resistors and linear memoryless dependent sources using node analysis and mesh analysis methods. An argument based

on nodal admittance matrix (or mesh impedance matrix) and its cofactors is used to show that a memoryless circuit comprising memoryless linear two-terminal elements will be a *linear system* and that it will obey the superposition principle. Chapter 5 systematically develops all important circuit theorems from the properties of a linear system.

After the analysis of memoryless circuits, the book moves on to elucidate sinusoidal steady-state in dynamic circuits. This part of the book starts with a detailed look at power and energy in periodic waveforms in Chapter 6. The periodic sinusoid is introduced and the concepts of its amplitude, frequency and phase are made clear. The concept of cycle-average power in the context of periodic waveforms is covered in detail.

Chapter 7 begins with a qualitative description of transient response and forced response taking an RL circuit as an example, and illustrates how the sinusoidal steady-state can be solved by using the complex exponential function. It goes on to expound on phasor theory, transformation of the circuit into phasor domain, solving the circuit in phasor domain, and moving back to time-domain. It also introduces active power, reactive power and power factor and presents the basic ideas of frequency response.

Chapter 8 takes up three-phase balanced and unbalanced circuits and includes symmetrical components as well. Unbalanced three-phase circuits and symmetrical components may be optional in 'Basic Electrical Engineering' course.

Chapter 9 addresses the issue of expanding a periodic waveform along the imaginary axis in signal space at discrete points. Fourier series in trigonometric and exponential forms are covered in detail in this chapter. This chapter (i) explains how a periodic waveform can be expanded in terms of sinusoids and why such an expansion is necessary (ii) shows how such an expansion may be obtained for a given periodic waveform, and (iii) shows how the expansion can be used to solve for the forced response of a circuit.

Expansion of input functions along imaginary axis in signal space for aperiodic waveforms through Fourier transforms is not included in this text. It will be included in the second text of the series.

The next three chapters deal with the time-domain analysis of dynamic circuits using the differential equation approach. Chapter 10 is one of the key chapters in the book. It takes up a simple RL circuit and uses it as an example system to develop many important linear systems concepts. The complete response of an RL circuit to various kinds of inputs such as unit impulse, unit step, unit complex exponential, and unit sinusoid is fully delineated from various points of view in this chapter. The chapter further expounds on the need and sufficiency of initial current specification, the concepts of time constant, rise and fall times, and bandwidth.

The response of a circuit is viewed as the sum of *transient response* and *forced response* on the one hand and as the sum of *zero-input response* and *zero-state response* on the other. The role of various response components is clearly spelt out. The application of superposition principle to zero-state component and zero-input component is examined in detail.

Impulse response is shown to be an all-important response of a circuit. The equivalence between impulse excitation and non-zero initial conditions is established in this chapter. The chapter also shows how to derive the zero-state response to other inputs like unit step and unit ramp from impulse response in detail. The tendency of inductance to keep a circuit current smooth is pointed out and illustrated.

The notions of DC steady-state, AC steady-state and periodic steady-state are made clear and demonstrated through several worked examples. The chapter ends with a general method of solution to single time-constant RL circuits in 'transient response + forced response' format as well as in 'zero-input response + zero-state response' format. This chapter places emphasis on impulse response as the

key circuit response, keeping in perspective the discussion on convolution integral in the second text planned under this series.

Chapters 11 and 12 take up a similar analysis of RC and RLC circuits respectively. Further, these chapters gradually introduce the concept of sinusoidal steady-state frequency response curves through RC and RLC circuits and set the background for Fourier series in a later chapter. Specific examples where the excitation is in the form of a sum of harmonically related sinusoids containing three to five terms are used to illustrate the use of frequency response curves and their linear distortion. The conditions for distortion-free transmission of signals are briefly hinted at in Chapter 11. A detailed coverage on distortion-free transmission of signals will appear in a chapter on Fourier transforms in the second text planned under this series.

Inconvenient circuit problems like shorting a charged capacitor, opening a current carrying inductor, connecting two charged capacitors together, and connecting an uncharged capacitor across a DC supply require inclusion of parasitic elements for correct explanation. Parasitic elements are emphasized at various places in chapters dealing with time-domain analysis.

Chapter 13 expands an arbitrary input signal along a line parallel to the vertical axis in a signal plane i.e., in terms of damped sinusoids of different frequencies rather than in terms of undamped sinusoids of different frequencies. This expansion is illustrated graphically in the case of a simple waveshape to convince the reader that an aperiodic signal can indeed be obtained by a large number of exponentially growing sinusoids and that there is nothing special about expansion of a waveshape in terms of undamped sinusoids. This expansion of signals leads to Laplace Transform of the signal. Properties of Laplace Transform, use of Laplace Transform in solving differential equations and circuits, transfer functions, impedance functions, poles, and zeros follow. This chapter also includes a graphical interpretation of frequency response function in the s -plane. Stability criterion is re-visited and circuit theorems are generalized.

Chapter 14 is on magnetically coupled circuits. It introduces the mutual inductance element, building on the properties of perfectly coupled linear transformer and ideal transformer. Transient response of coupled coil circuits and sinusoidal steady-state in such circuits are also covered in this chapter. Applications of two-winding transformer in impedance matching, design of tuned amplifiers etc., are explained.

Prerequisites

The student-reader is expected to have gone through basic level courses in electromagnetism, complex algebra, differential calculus and integral calculus. These are covered in the pre-engineering school curricula of all boards of senior/higher secondary school education in India.

ACKNOWLEDGEMENTS

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The credit for the good things the reader finds in this book goes to my esteemed teachers at IIT Madras during 1976–83. The faults in this book, if any, are mine.

Suresh Kumar K. S.

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Circuit Variables and Circuit Elements

CHAPTER OBJECTIVES

- To introduce the concepts of electrostatic potential difference, voltage and electromotive force.
- To introduce the concepts of 'current density' and 'current intensity' and to explain the conduction process.
- To explain the 'quasi-static' approximation involved in circuit theory.
- To explain the need for and role of quasi-static charge distributions over electrical devices.
- To define and explain various idealized two-terminal element models in use in Circuit Theory.
- To explain passive sign convention for two-terminal elements.
- To relate power and energy in a two-terminal element to its terminal voltage and current variables.
- To explain what is meant by 'lumped, linear, bilateral, passive, time-invariant circuit element'.
- To define and explain four types of linear dependent sources.

INTRODUCTION

The profession of Electrical and Electronic Engineering deals with the generation, transmission and measurement of electric signals with signal power level varying from few nWs (10^{-9} Watt) to hundreds of MW (10^6 Watt) in various applications.

This textbook deals with one of the 'kingpin' subjects in the entire field of electrical and electronic engineering in detail. An *Electric Circuit* is a mathematical model of a real physical electrical system. Physical electrical systems consist of electrical devices connected together. *Electric Circuit* idealizes the physical devices and converts the real physical system into a mathematical model that is governed by a set of physical laws. *Circuit solution* is obtained by applying those physical laws on the mathematical circuit model and employing suitable mathematical solution techniques. The circuit solution approximates the actual behaviour of the physical system to a remarkable degree of accuracy in practice.

1.2 | Circuit Variables and Circuit Elements

Though the term *electric circuit* refers to the idealized mathematical model of a physical electrical system, it is used to refer to the actual electrical system too in common practice. However, when we refer to *electric circuit* in this text, we always mean a mathematical model, unless otherwise stated.

Laws of electromagnetic fields govern the electrical behaviour of an actual electrical system. These laws, encoded in the form of four Maxwell's equations, along with the constituent relations of electrical materials and boundary constraints, contain all the information concerning the electrical behaviour of a system under a set of specified conditions. However, extraction of the required information from these governing equations will turn out to be a formidable mathematical task even for simple electrical systems. The task will involve solution of partial differential equations involving functions of time and space variables in three dimensions subject to certain boundary conditions.

Circuit theory is a special kind of approximation of electromagnetic field theory. '*Lumped parameter circuit theory*' converts the partial differential equations involving time and three space variables arising out of application of laws of electromagnetic fields into ordinary differential equations involving time alone. Circuit theory approximates electromagnetic field theory satisfactorily only if the physical electrical system satisfies certain assumptions. This chapter discusses these assumptions first.

1.1 ELECTROMOTIVE FORCE, POTENTIAL AND VOLTAGE

Charge is the attribute of matter that is responsible for a force of interaction between two pieces of matter under certain conditions. Such an attribute was seen to be necessary as a result of experiments in the past which revealed the existence of a certain kind of interaction force between particles that could not be explained by other known sources of interaction forces.

Charge is bipolar. Two positively charged particles or two negatively charged particles repel each other. Two particles with charges of opposite polarity attract each other. Further, charge comes in integral multiples of a basic unit – the basic unit of charge is the charge of an electron. The value of electronic charge is -1.602×10^{-19} coulombs. The SI unit of charge – i.e., Coulomb – represents the magnitude of charge possessed by 6.242×10^{18} electrons.

1.1.1 Force Between Two Moving Point Charges and Retardation Effect

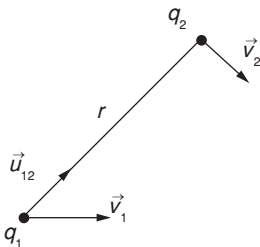


Fig. 1.1-1 Force between two point charges in motion

The force experienced by a point charge of value q_2 moving with a velocity of $\vec{v}_2$ due to another point charge of value q_1 moving with a velocity of $\vec{v}_1$ at a distance r from it contains three components in general. If the charges are moving slowly, the force components are given by approximate expressions as below.

The first component is directly proportional to the product $q_1 q_2$ and is inversely proportional to the square of distance between them. This component of force is directed along the line connecting the charges and is oriented away from q_1 . This component is governed by Coulomb's law; is termed '*the electrostatic force*' and is given by

$$\vec{F}_{12es} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \vec{u}_{12} \text{ N}$$

where ϵ_0 is the dielectric permittivity of free space ($= 8.854 \times 10^{-12}$ F/m) and $\vec{u}_{12}$ is the unit vector directed from q_1 to q_2 .

The second component of force is the *magnetic force* and arises out of motion of charges. This component is given by

$$\overline{F_{12m}} = \frac{\mu_0 q_1 q_2}{4\pi r^2} \vec{v}_2 \times (\vec{v}_1 \times \vec{u}_{12}) \text{ N}$$

where μ_0 is the magnetic permeability of free space ($= 4\pi \times 10^{-7}$ H/m) and $\vec{v}_1$ and $\vec{v}_2$ are the velocities of q_1 and q_2 respectively.

The third component of force is the *induced electric force* and depends on relative acceleration of q_1 with respect to q_2 . It is given by

$$\overline{F_{12ei}} = -\frac{\mu_0 q_1 q_2}{4\pi} \frac{\partial}{\partial t} \left(\frac{\vec{v}_1}{r} \right) \bigg|_{\vec{v}_2=0} \text{ N.}$$

Electromagnetic disturbances travel with a finite velocity – the velocity of light in the corresponding medium. Therefore, in general, the force experienced by q_2 at a time instant t depends on the position and velocity of q_1 at an earlier instant. Or, equivalently, the force that will be experienced by q_2 at $t + r/c$ depends on r and $\vec{v}_1$ at t where c is the velocity of light. This effect is called the *retardation effect*. *The expressions described above ignore this retardation effect and assume that the changes in relative position and velocity of q_1 are felt instantaneously at q_2 .*

Retardation effect can be ignored if (i) the speed of charges is such that no significant change can take place in r during the time interval needed for electromagnetic disturbance to cover the distance and (ii) the acceleration of charges is such that no significant changes in the velocity of charges take place during the time interval needed for electromagnetic disturbance to cover the distance between charges. The first condition implies that the speed of charges must be small compared to velocity of light. This condition is met by almost any circuit since the drift speed associated with current flow in circuits is usually very small compared to velocity of light. However, though the speed of charge motion is small, it is quite possible that the charges accelerate and decelerate rapidly in circuits such that the second condition is not met.

Consider the two point charges in Fig. 1.1-2. q_1 is oscillating with amplitude d and angular frequency ω rad/sec. However, let us assume that $d \ll r$. Then, neither the distance between the charges nor the unit vector between them change much with time. Therefore, the electrostatic force experienced by q_2 is more or less constant in time. q_2 is at rest and hence it does not experience any magnetic force. However, it will experience a time-varying induced electric force. Let the horizontal position of q_1 be given by $x = d \sin(\omega t)$ m. Then, velocity of q_1 is $\vec{v}_1 = \omega d \cos \omega t$ m/s. The time taken by any electromagnetic disturbance to travel r meters is r/c where c is the velocity of light in m/s. The quantity

$\frac{\partial}{\partial t} \left(\frac{\vec{v}_1}{r} \right)$ that decides the induced electric force experienced by q_2 will change by $\frac{\omega^2 d}{r} [\sin \omega(t + r/c) - \sin \omega t]$ in that time interval. Retardation effect can be ignored only if this change is negligible – i.e., only if $\frac{\omega r}{c} \ll 1$.

In the context of circuit theory, $\frac{c}{\omega}$ is the highest angular frequency of time-varying signals present in the circuit and r is the largest physical dimension measured in the physical circuit. Then, retardation effect due to finite propagation

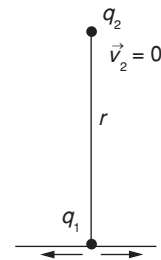


Fig. 1.1-2 Pertaining to retardation effect in a two-charge system

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speed of electromagnetic waves can be ignored in the analysis of an electrical system if the highest frequency present in the circuit and largest dimension of the system satisfy the inequality $\frac{\omega r}{c} \ll 1$.

An electrical system in which the retardation effect can be ignored is called a *quasi-static electrical system*. Electric circuit theory assumes that the electrical system that is modeled by an electrical circuit is a quasi-static system.

1.1.2 Electric Potential and Voltage

The force experienced by a charge q in arbitrary motion under quasi-static conditions due to another moving charge has three components as explained in the previous sub-section. The second component that is velocity dependent and the third component that is acceleration dependent are put together to form the *non-electrostatic component*. Thus, the force experienced by q due to another charge has an *electrostatic component* and a *non-electrostatic component*.

The force experienced by a charge q in the presence of many charges can be obtained by adding the individual forces by vector addition. Thus, the electrostatic force experienced by the charge q due to a system of charges is a superposition of electrostatic force components from the individual charges. *Electrostatic field intensity vector*, $\vec{E}_s$, at a point in space is defined as the total electrostatic force vector acting on a unit test charge (i.e., q is taken as 1C) located at that point. Then, the electrostatic force experienced by a charge q located at a point $P(x, y, z)$ is given in magnitude and direction by $q \vec{E}_s$ N. The SI unit of *electrostatic field intensity* is Newtons per Coulomb.

Electrostatic field intensity due to a point charge falls in proportion to square of distance between location of charge and location at which the field is measured. Hence, the field intensity at large distance from a system of *finite amount* of charge tends to reach zero level. Therefore, a test charge of 1 C located at infinite distance away from the charge collection will experience zero electrostatic force.

Now assume that the test charge of 1 C that was initially at infinity is brought to point $P(x, y, z)$ by moving it quasi-statically through the electrostatic field. The agent who moves the charge has to apply a force that is numerically equal to $|\vec{E}_s|$ and opposite in direction. The total work *to be done* in moving the unit test charge from infinity to $P(x, y, z)$ is obtained by integrating the quantity $-\vec{E}_s(x, y, z) \cdot \vec{dl}$ over the path of travel where $\vec{dl}$ is a small length element in the path of travel. This work is, by definition, the electric potential (*electrostatic potential* to be precise) at the point $P(x, y, z)$. It is usually designated by $V(x, y, z)$. Then,

$$V(x, y, z) = -\int_1 \vec{E}_s(x, y, z) \cdot \vec{dl} \quad (1.1-1)$$

The unit of electric potential is Newton-meter per Coulomb or Joule per Coulomb. This unit is given the name ‘Volt’. This leads to another unit for electrostatic field intensity – namely, Volt/m.

Electrostatic force field is a conservative force field. Hence the value of this work integral will depend only on the end-points and not on the particular path that was traversed. Therefore, this work integral (and hence the potential at point $P(x, y, z)$) has a unique value that depends on $P(x, y, z)$ only. Moreover, the conservative nature of electrostatic force field implies that the work done in taking a test charge around a *closed path* in that field is zero.

Eqn. 1.1-1 defines the potential at a point with respect to a point at infinity. The difference in potential at two points can be interpreted as the work to be done to carry a unit test charge from one

point to another. Specifically, let V_1 and V_2 be the electric potentials at point-1 and point-2 in space. Then, the potential of point-1 with respect to the point-2 is $V_1 - V_2$ volts and is equal to the work to be done in carrying a unit positive test charge from point-2 to point-1. This value is designated by V_{12} and is read as ‘potential of 1 with respect to 2’ or as ‘potential difference between 1 and 2’. In Circuit Analysis, this electrostatic potential difference between two points is called the ‘voltage between point-1 and point-2’. The same symbol that was used to designate the potential difference is used to designate ‘voltage’ too. Thus,

$$\begin{aligned} V_{ab} &= \text{‘Voltage between points } a \text{ and } b\text{’} \\ &= \text{‘Electrostatic Potential at } a - \text{Electrostatic Potential at } b\text{’} \\ &= \text{‘Work to be done in moving } a +1 \text{ coulomb test charge from } b \text{ to } a\text{’} \\ &= -\int_b^a \vec{E}_s(x, y, z) \cdot d\vec{l} = \int_a^b \vec{E}_s(x, y, z) \cdot d\vec{l} \text{ over any path between } a \text{ and } b. \end{aligned}$$

Obviously, $V_{ba} = -V_{ab}$

If V_{ab} is a positive quantity, we state that there is a ‘potential rise’ or ‘voltage rise’ from b to a . Equivalently, we can state that, there is a ‘potential drop’ or ‘voltage drop’ from a to b . If V_{ab} is a negative quantity, we state that, there is a ‘potential rise’ or ‘voltage rise’ from a to b . Equivalently, we can state that, there is a ‘potential drop’ or ‘voltage drop’ from b to a .

The work to be done in moving a unit positive test charge is positive when the charge is moved in a direction opposite to the direction of electrostatic field. Hence, the direction in which maximum voltage rise takes place will be opposite to the direction of electrostatic field at any point in space.

When a charge q is moved through a *voltage rise*, some non-electrostatic force has to work against the electrostatic force to effect the movement. The non-electrostatic force will do work on the charge in the process of moving it. This work done on the charge gets stored in the charge as increase in its potential energy. Hence, a charge q receives qV_{ab} Joules of potential energy when it is carried from b to a . If V_{ab} is positive – i.e., if there is a voltage rise from b to a – then, the potential energy level of charge q increases by qV_{ab} Joules in moving from b to a .

If V_{ab} is negative – i.e., if there is a voltage drop from b to a – then, the potential energy level of charge q decreases by $q|V_{ab}|$ Joules. The non-electrostatic force that maintains quasi-static condition during the movement of charge from b to a receives this energy.

Sustained and organised movement of charges through voltage rises and voltage drops in an electrostatic potential system is possible only if there are sources of non-electrostatic forces present in the electrical system. These non-electrostatic forces deliver and absorb the required amounts of energy to make movement of charges through voltage rises and voltage drops possible. Obviously, a non-electrostatic force can fulfil this role only if it has a component along the direction of motion of charge. A force that is always perpendicular to the velocity of a charge can not deliver energy to the charge or absorb energy from it. Magnetic force on a charge is always perpendicular to the velocity of the charge. Hence magnetic force can not be the non-electrostatic force that we want.

The *induced electric force*, a component of force of interaction between two charges in quasi-static motion, can deliver energy to moving charges and absorb energy from them. Thus, the induced electric force can be a source of the required kind of non-electrostatic force in an electrical system.

Electrical sources are sources of non-electrostatic forces in an electrical system. Some of the sources make use of the *induced electric force* to generate the non-electrostatic force. A DC Generator, an AC Generator etc., are examples of this kind of sources. However, there are other sources of non-electrostatic forces. A dry cell, for instance, converts the internal chemical potential energy into

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a non-electrostatic force that acts on any charge carrier that transits through the conducting material inside the dry cell. We do not concern ourselves anymore with the exact nature of the non-electrostatic force available within an electrical source. It suffices for our purpose to understand that some non-electrostatic force is available within the electrical source.

1.1.3 Electromotive Force and Terminal Voltage of a Steady Source

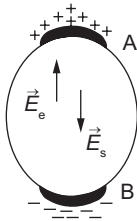


Fig. 1.1-3 A steady electrical source under open-circuit condition

Consider an electrical source on open circuit as in Fig. 1.1-3. A free charge located at a point inside the source will experience a non-electrostatic force as shown in the figure. This non-electrostatic force is expressed as a force field and the quantity $\vec{E}_e$ represents this force field. Thus, the non-electrostatic force experienced by a charge q located inside the source will be $q\vec{E}_e$ N where $\vec{E}_e$ is the non-electrostatic field intensity vector. $\vec{E}_e$ may not be constant in magnitude and direction everywhere inside the source. However, $\vec{E}_e$ will not vary with time in the case of a *steady source*.

The source contains conducting material inside. Conductors contain free electrons. The free electrons in the conducting substance will tend to move from top to

bottom (electrons have negative charge) under the influence of non-electrostatic force. The first few electrons that move so reach the bottom electrode (at B) and accumulate at that terminal. Electrons moving to B will cause an equal number of positive charges to appear at the terminal marked A. But then, such a separation of charges will result in generation of electrostatic field inside (as well as outside) the source. Thus, the remaining free electrons inside the source will experience two forces – a non-electrostatic force that tends to move them towards the lower electrode B and an electrostatic force that tends to move them towards the upper electrode A. The source reaches a steady-state soon. Under steady-state condition, the magnitude and spatial distribution of charges over the metallic electrodes at A and B are such that all the free electrons that are still within the source will experience zero net force and remain stationary (except for random thermal motion). Thus, the electrostatic field at a point inside will cancel the non-electrostatic field at that point under steady-state. The charge distribution at the terminals will arrange itself suitably such that this cancellation takes place at all points in the active region of source.

However, there is no non-electrostatic field outside the source. Therefore a test charge kept at a point outside the source will experience an electrostatic force.

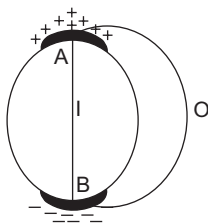


Fig. 1.1-4 Pertaining to definition of e.m.f. and open circuit voltage of a source

Refer to Fig. 1.1-4. Let a unit test charge be carried from B to A through the path BOA – i.e., over a path that is outside the source. Some work has to be done for this. The required work will be positive since we are carrying a positive test charge from a negatively charged terminal to a positively charged terminal. The work that is required to be done is the voltage of A with respect to B – i.e., V_{AB} .

However, the work required to carry a unit test charge in a closed path in electrostatic field is zero due to conservative nature of electrostatic forces. Therefore, the work to be done *against the*

electrostatic force field to carry the test charge in the path B-O-A-I-B is zero. That is, work to be done against electrostatic force to move a unit test charge from B to A over a path outside the source *plus* work to be done against electrostatic force to move a unit test charge from A to B over a path inside the source is zero. Therefore, work to be done in path B-O-A = work to be done in path B-I-A. But the electrostatic field vector inside the source is equal to $-\vec{E}_e$. Therefore,

$$V_{AB} = - \int_{B-O-A} \vec{E}_s \cdot d\vec{l} = - \int_{B-I-A} \vec{E}_s \cdot d\vec{l} = \int_{B-I-A} \vec{E}_e \cdot d\vec{l}$$

But $\int_{B-I-A} \vec{E}_e \cdot d\vec{l}$ is the work done by the non-electrostatic force generated by the source on a unit positive charge when it moves through the source from negative terminal to positive terminal. This quantity is defined as the *Electromotive force* (e.m.f.) of the source. It is usually represented by the symbol E . Therefore, the electrostatic potential difference between positive terminal and negative terminal of a source under open-circuited condition (that is, the open-circuit terminal voltage V_{AB}) is equal to the electromotive force of the source.

E of a steady source is a constant. Such a source is called a DC Voltage Source. DC stands for ‘direct current’. The *terminal voltage* (V_{AB} in Fig. 1.1-3 and Fig. 1.1-4) of a practical steady source will be equal to the electromotive force E only under open-circuit condition. Terminal voltage becomes less than E when the source is delivering some current due to the inevitably present voltage drop in the internal resistance of the source.

1.2 A VOLTAGE SOURCE WITH A RESISTANCE CONNECTED AT ITS TERMINALS

A piece of conducting substance is now connected to a steady voltage source with a steady-state static charge distribution at its terminals as shown in Fig. 1.1-3. The resulting system is shown in Fig. 1.2-1.

1.2.1 Steady-State Charge Distribution in the System

There was an electrostatic field due to the terminal charge distribution of the source in the space that is now occupied by the conducting substance. This field can change only if the terminal charge distributions change. Charges have to move and reach terminals in order to change the distribution there. That takes time. Therefore, the charge distribution at the terminals remains unaffected for a brief interval even after the conducting substance is connected across the source.

But this will result in large electrostatic forces on the free electrons in the metallic substance. These free electrons start migrating towards positive terminal (A) of the source and cancel the positive charge distribution there partially. Simultaneously, electrons are pulled from terminal B into conducting substance, thereby canceling the charge distribution in the negative terminal partially. But this change in the charge distribution will affect the balance between electrostatic force and

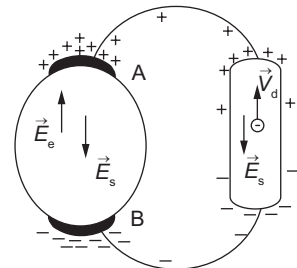


Fig. 1.2-1 Steady-state with a resistance connected across a DC voltage source

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non-electrostatic force inside the source. Now the non-electrostatic force will not be cancelled completely by the electrostatic force. The remnant non-electrostatic force will propel the free electrons present within the source towards the negative terminal (and an equal amount of positive charge gets propelled towards positive terminal). This will lead to a restoration of charge distribution at the terminals as well as creation of charge distribution on the surface of connecting wire and the conducting substance.

Soon a steady charge distribution is established as shown in Fig. 1.2-1. The non-electrostatic force and the electrostatic force at any point inside the source will cancel each other under this condition. [The conducting substance inside the source is assumed to have zero resistivity.] Therefore the terminal voltage V_{AB} will be equal to the electromotive force E of the source. The geometry of the system will decide the amount of positive and negative charges distributed on the terminal surface, wire surface and surface of the conductor. Once established, this charge distribution remains steady (in the case of a steady source) unless spatial arrangement is disturbed or the resistance is disconnected.

1.2.2 Drift Velocity and Current Density

This steady charge distribution will produce steady electrostatic field everywhere inside the conducting substance. Free electrons inside the conductor get accelerated by this electrostatic force. The velocity as well as kinetic energy of free electrons would have reached high levels in the absence of any opposing force. However, there is an opposing force. This force arises out of collisions suffered by the accelerating electrons. The electrons collide with ionized atoms in the conducting substance inelastically and lose some of the kinetic energy they acquire under the action of electrostatic force. The average effect of multitude of collisions suffered by an electron accelerating under electrostatic force is similar to that of friction. Thus, the inelastic collisions that take place between accelerating free electrons and ionized metal atoms result in a non-electrostatic force that is similar to friction. The accelerating electrons pick up kinetic energy first since they are losing potential energy by falling through a voltage drop. The kinetic energy is subsequently delivered to the lattice through inelastic collisions (or equivalently, to the non-electrostatic force that is manifest within the conducting substance as an average effect of multitude of inelastic collisions).

The free electrons inside the conducting substance reach a steady velocity in the direction opposite to that of electrostatic force under the action of electrostatic force and the non-electrostatic force arising out of collisions. This is somewhat similar to an object reaching a terminal velocity when it falls through a viscous medium. The terminal velocity attained by a free electron inside a conducting substance is called *drift velocity* and is denoted by $\bar{v}_d$. Experiments have revealed that this velocity is proportional to the electrostatic field intensity through a proportionality constant called *mobility* of the substance. Note that this is an average velocity that is super-imposed on the random thermal velocity of electrons.

Thus, the drift velocity of electrons at a point where the electrostatic field intensity is $\bar{E}_s$ is given by $\bar{v}_d = -\mu \bar{E}_s$ where μ is the mobility of the material and has m^2 per Volt-sec unit. μ of a material is temperature dependent quantity and decreases with temperature in the case of metals. The negative sign is needed to account for the fact that negatively charged electrons move in a direction opposite to that of electrostatic field.

Consider a small area element ΔA m^2 at a point inside the conductor where the drift velocity is $\bar{v}_d$ and electrostatic field intensity is $\bar{E}_s$. Let the area element be taken perpendicular to the direction of $\bar{v}_d$ (or $\bar{E}_s$). Let N be the number of free electrons available in unit volume of the conducting material. Then, there will be $N(\Delta A) v_d$ free electrons present in a volume element constituted by area ΔA m^2 and a length v_d m along the direction of drift velocity. All these electrons will cross the area element in the next second.

Negative charges crossing a surface in one direction is equivalent to positive charges crossing the same surface in the opposite direction. Hence, the total *positive* charge that crosses ΔA in one second

in the direction of $\vec{E}_s$ is $N q_e (\Delta A) v_d$ coulombs where q_e is the *magnitude* of charge of an electron and v_d is the *drift speed*.

A vector quantity called ‘*current density vector*’ (denoted by $\vec{J}$) is defined at a point inside a conductor as the total *positive* charge that crosses *unit area* in *unit time* in the direction of electrostatic field at that point with the area kept perpendicular to the direction of electrostatic field at that point. [This definition is correct only for homogeneous and isotropic materials. Metallic conductors are assumed to satisfy this requirement for all practical purposes]. It must be obvious that $\vec{J}$ at a point is proportional to $\vec{E}_s$. The dimension of current density is Coulomb per m² per second. Its unit is Ampere per m² since Coulomb/sec is called ‘Ampere’.

$\vec{J} = (N \mu q_e) \cdot \vec{E}_s = \sigma \vec{E}_s$ where σ is defined as the *conductivity* of the material and has ampere per volt per meter as its unit. Ampere per Volt is given a special name – ‘Siemens’. Therefore, the unit of conductivity will be Siemens/m.

Reciprocal of conductivity is called the *resistivity* of the material and is denoted by ρ . Its unit is Ohm-m. [Volt/Amp is given the name ‘Ohm’].

The value of conductivity (and resistivity) of a *linear, homogeneous, isotropic* material is a constant at a particular temperature. Conductivity and resistivity vary with temperature. In general, conductivity decreases with increasing temperature in the case of metallic conductors.

1.2.3 Current Intensity

Electric current intensity or, simply, *current intensity* through a surface is defined as the amount of charge that crosses the surface in unit time. ‘Current intensity’ is usually referred to as ‘current’ itself.

It is a scalar quantity. The definition begs a question – ‘charge that crosses the surface’ in which direction?

The direction of crossing implied in the definition is unambiguous in the case of a closed surface. It is in the direction of surface normal. Surface normal is drawn outwards in the case of a closed surface. Thus, the current intensity through a closed surface is the amount of charge that crosses the surface in one second *from inside to outside*. A positive current implies net positive charge flow out of the volume enclosed by the surface or net negative charge flow into the volume. A negative current implies net positive charge flow into the volume or net negative charge flow out of the volume.

However, there is ambiguity in interpreting current in the case of open surfaces if current is considered to be a pure scalar quantity. The surface normal for an open surface is not unique. The value of current through a surface in a given context can be positive or negative depending on the choice of surface normal. Therefore, the value of current intensity can be uniquely interpreted only if some direction is also associated with it. Therefore, it is customary in practice, especially in the case of wire conductors with uniform cross-section, to refer to the *direction of current*. The current density in such conductors will be approximately uniform and hence current density will have same direction at all points in the cross-section of the conductor. *The direction of current intensity in such conductors (wire conductors of uniform cross section) is taken to be same as the direction of current density itself.*

The unit of current is Coulomb/second and this unit is given a special name of ‘Ampere’. Current is denoted usually by I if it is a steady current (*i.e.*, DC current) and by $i(t)$ (or by i) if it is a time-varying quantity.

Let $q(t)$ be the *total net charge* that crossed a given cross-section in a specified direction from $t = -\infty$ to $t = t$ and let $i(t)$ be the current flowing through that cross-section in the same direction at $t = t$. Then, the relation between these two quantities are given by the following equations.

$$i(t) = \frac{dq(t)}{dt} \text{ A} \quad (1.2-1)$$

$$q(t) = \int_{-\infty}^t i(t) dt = q(0) + \int_0^t i(t) dt \text{ C}$$

1.2.4 Conduction and Energy Transfer Process

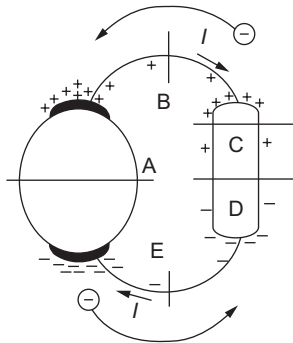


Fig. 1.2-2 A steady-current system – current through A, B, C, D and E have same value

Consider the steady voltage source with resistive load across it shown in Fig. 1.2-2. Five cross-sections (A, B, C, D and E) are marked in the figure. Also, the directions of positive current flow and electron flow are marked. Electrons flow in the counter-clockwise direction in the circuit and positive current flows in the clockwise direction.

Consider the volume between the two cross-sections marked as B and C in Fig. 1.2-2. There is a surface charge distribution on this volume. [It is possible to show by employing equations of electromagnetic fields that there will be no charge distribution inside the volume in a homogeneous conducting substance under steady current conditions as well as under quasi-static current conditions. Charges will reside only on the surface.] This charge distribution remains stationary in time since the non-electrostatic force within the source of e.m.f. is assumed to be steady. Therefore, the amount of charge that crosses into the volume through the cross-section B in unit time

has to be same as the amount of charge that crosses out of the volume through the cross-section C in unit time – otherwise the surface charge storage within this volume will change. Hence the current through B has to be the same as the current through C. Similar argument for other cross-sections will lead us to the conclusion that current through all cross-sections will have same value in this circuit.

The surface charge distribution present throughout the system is stationary. But that does not mean that the individual electrons that make this distribution stay put. For instance, a particular electron that is part of the current flow after crossing C may cancel a positive surface charge. But that will result in an unbalance in the system and another electron will move out from surface and join the current stream leaving a positive charge on the surface. Thus, though the identity of individual charges that form the surface charge may not be preserved, the surface charge will appear stationary at a macroscopic level.

Consider an electron that is part of the current flow. The electrostatic field is oriented from positive terminal to negative terminal inside the source. The non-electrostatic field is oriented from negative terminal to positive terminal. When the conduction electron travels from positive terminal to negative terminal through the source it gains electrostatic potential energy. The non-electrostatic field does positive work to impart this extra potential energy to the electron. The conduction electron then flows through connecting wire to the negatively charged terminal of resistor. The electrostatic field inside the conductor tries to accelerate it and convert its potential energy into kinetic energy. The electron soon transfers its kinetic energy to the lattice through inelastic collisions with atoms. By the time it emerges at the positively charged terminal of the resistance, it would have lost all the extra potential energy it gained earlier to the lattice. The lattice energy appears as heat in the conductor.

Thus, electrons act as a medium for transferring energy from source to conductor. The electrostatic field present everywhere in the system is a facilitator of this energy transfer process. The non-electrostatic field in the source transfers the source energy into charge carriers flowing through it in the form of potential energy of the charged particles in an electrostatic field. The charged particles carry this potential energy with them into the conductor. The non-electrostatic force (*i.e.*, the average effect of inelastic collisions) absorbs the potential energy of charges and transfers it to the lattice. The electrostatic field that is present within the conductor facilitates this process by converting the potential energy of charged particles into kinetic energy before they can deliver it to atoms through inelastic collision process.

Thus, electrostatic field permeating throughout the system is a necessary requirement for conduction and energy transfer process to take place in an electrical system. The required electrostatic field is created by surface charge distributions on conducting surfaces everywhere in the system.

1.2.5 Two-Terminal Resistance Element

Consider the steady voltage source with resistive load across it shown in Fig. 1.2-3.

Let us work out the electrostatic potential difference between *e* and *f*.

Work to be done *against electrostatic force* to carry a unit positive test charge around a closed path is zero. Therefore, the work to be done to take +1 C charge from *f* to *e* must be the same whether we move it through a path that lies inside

the conducting substance or outside. But the electrostatic

field is given by $\vec{E}_s = \frac{\vec{J}}{\sigma}$ inside the conductor. Therefore, $V_{ef} = -\frac{1}{\sigma} \int_f^e \vec{J} \cdot d\vec{l}$ with *dl* oriented from *f* to *e*. The value of

this integral will be same for any path through the conducting substance. However, evaluation of the integral to yield a closed-form result will be possible only in simple cases where the geometry of conductor has some kind of symmetry or other.

We consider a simple case of a conductor with uniform cross-section. The total current may be assumed to distribute itself uniformly throughout the cross-section in such a conductor. This results in a current density vector that has a constant magnitude of I/A (*A* is the area of cross-section) and direction parallel to the axis of conductor. This is a satisfactory assumption everywhere except at the connection ends. With this assumption, with *l* as the length of conductor and *A* as its uniform cross-sectional area, we get,

$$V_{ef} = -\frac{1}{\sigma} \int_f^e \vec{J} \cdot d\vec{l} = I \frac{l}{\sigma A} \left(\text{or } I \frac{\rho l}{A} \right) \text{ V.} \quad (1.2-2)$$

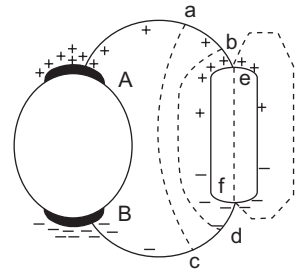


Fig. 1.2-3 Pertaining to voltage across a two-terminal resistance

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Eqn. 1.2-2 relates the electrostatic potential difference across the connection points of a piece of conductor with uniform cross-section to the current flow through it. The proportionality constant is dependent on material property (conductivity or resistivity) and geometry of the conductor. This proportionality constant is called the *resistance parameter* R .

$$R = \frac{l}{\sigma A} = \frac{\rho l}{A} \text{ Ohm} \quad (1.2-3)$$

However, actual connection point between the resistive material and external circuit may not be accessible for observation of voltage. We measure the voltage across a resistance by connecting a voltmeter to the connecting wire on either side of the element. Assume that the voltmeter is connected across a - c . Then, the voltmeter will read the electrostatic potential difference V_{ac} . But,

$$V_{ac} = V_{ef} + \underbrace{\left(-\frac{1}{\sigma} \int_d^c \vec{J} \cdot d\vec{l} - \frac{1}{\sigma} \int_a^e \vec{J} \cdot d\vec{l} \right)}_{\text{evaluated over paths through connecting wire}}.$$

Therefore, a unique voltage difference can be assigned to the conducting body only if the conductivity of connecting wires is infinitely large. However, it is to be noted that this does not imply thick connecting wires. In fact, Circuit Theory assumes that connecting wires have zero resistance and negligible thickness. The reason behind the assumption of negligible cross-section for connecting wires will be explained in a later section.

With this assumption, the electrostatic field inside connecting wires will be zero (since conductivity is infinite). Then, the electrostatic potential difference between the ends of conducting body has a unique value irrespective of which pair of points (a and b) on the connecting wire are chosen to measure it.

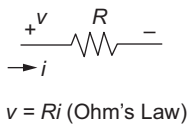


Fig. 1.2-4 Two-terminal resistance

Now a unique voltage and current variable pair can be assigned to the conducting body and its electrical behaviour can be described entirely in terms of these two variables. This model of a conducting body is called the *two-terminal resistance element* model. The symbol and element relation is shown in Fig. 1.2-4.

Ohm's Law, which is an experimental law, states that the voltage drop across a two-terminal resistance made of a linear conducting material and maintained at a constant temperature is proportional to the current entering the element at the higher potential terminal.

Resistivity and Conductivity are functions of temperature. If the temperature range considered is small, resistivity may be approximated as $\rho(T) = \rho(T_0)[1 - \alpha(T - T_0)]$ where $\rho(T_0)$ is the known resistivity at temperature T_0 and α is the *temperature coefficient of resistivity*.

1.2.6 A Time-Varying Voltage Source with Resistance Across it

Now we consider a time-varying source of non-electrostatic force with a resistance load. Refer to Fig. 1.2-5.

We assume that the conducting matter inside the source has infinite conductivity. Hence the non-electrostatic force at every point inside the source has to get cancelled by the electrostatic force at that point at all instants of time. We implicitly assumed that electromagnetic disturbances propagate with infinite speed in making this statement.

The non-electrostatic field inside the source is a function of time here. Thus $\vec{E}_e$ is a function of space and time. Therefore, $\vec{E}_s$ also has to be function of time and space in order to match $\vec{E}_e$. A time-varying $\vec{E}_s$ inside the source calls for a surface charge distribution that is time-varying. Thus the charge distributed in the system $Q(t)$ is a time-varying function.

But, now the current flowing in the circuit has two functions to perform – (i) supply the time-varying surface charge requirement at all points in the system (ii) transfer the source energy to the conducting substance. Therefore, the net charge crossing different cross-sections in unit time will not be equal. For instance, consider the two cross-sections marked C and D in Fig. 1.2-5. The volume between these two cross-sections has a certain quantity of charge distributed on its surface at t . The quantity of charge that has to get distributed in the same surface is different at $t+\Delta t$. Therefore the current crossing D can not be equal to the current crossing C since a portion of the current crossing C will get used up in supplying the required change in surface charge distribution.

Thus, the current crossing various cross-sections will be different and there is no unique single value of current in the circuit at any instant. We restore uniqueness to the circuit current by resorting to certain assumptions.

First, we assume that the connecting wires are very thin. In this case it is possible to show that the surface charge distribution on the surface of connecting wires will be extremely small in value compared to the charge distribution elsewhere. Thus, Circuit Theory assumes that (i) connecting wires are made of material with infinite conductivity so that the electrostatic field inside connecting wires is zero and voltage drop in them is zero and that (ii) connecting wires are so thin that there is virtually no surface charge distributed on their surface. With only negligible surface charge on their surface, the connecting wires will not divert any portion of current flowing through them to supply the changes in surface charge distribution. Then, the current through a section of connecting wire will be the same everywhere.

The next assumption used in Circuit Theory is that the current component that is needed to supply the changes in surface charge distribution at any point in the system is a negligible portion of the current at that point. Note that this does not amount to ignoring the electrostatic charge distribution altogether. That can not be done. Charge distribution is essential for conduction and energy transfer to take place at all. It is only that we chose to ignore the diversion of current for creating a time-varying charge distribution at various points in the system. Obviously, this assumption will be satisfactory only if the source electromotive force is a slowly varying one.

With these assumptions, the currents everywhere in the circuit in Fig. 1.2-5 will be described by a unique function of time. Then there is essentially no difference between an electrical system with a

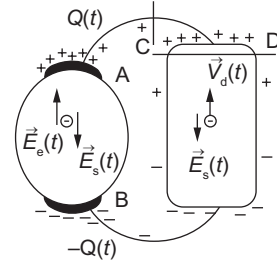


Fig. 1.2-5 A time-varying source e.m.f. with conductor across it

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steady-source and an electrical system with a time-varying source. The conductor in Fig. 1.2-5 can be modeled by a two-terminal resistance element satisfying Ohm's law on an instant to instant basis. If $v(t)$ is the electrostatic potential difference across the resistance and $i(t)$ is the current entering the higher potential terminal, then $v(t) = i(t) R$.

1.3 TWO-TERMINAL CAPACITANCE

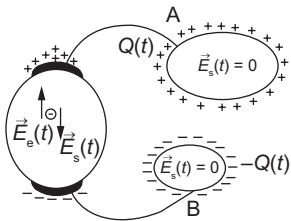


Fig. 1.3-1 A time-varying e. m. f. source with two electrodes

Consider the electrical system shown in Fig. 1.3-1. A source of time-varying electromotive force is connected to a pair of metallic electrodes A and B. We assume that the connecting wires are of infinite conductivity and near-zero cross-section. Further, we assume that the metallic electrodes are made of material with infinite conductivity. Therefore the electrostatic field inside the two electrodes will be zero at all t even when there is current flow in the electrode material.

The total surface charge distributed on the conducting surfaces in the system has two components – the charge distributed on the source terminal and the charge distributed on the electrode. The charge

distributed on the connecting wire is negligible since the wire is assumed to be very thin.

The surface charges on the source terminals and electrodes will assume suitable magnitudes and suitable distributions such that (i) the non-electrostatic field in the source is cancelled by the electrostatic field created by the charge distributions on an instant to instant basis everywhere within and (ii) the electrostatic field everywhere inside the connecting wires and electrodes is zero at all time. Thus, $Q(t)$, the total charge distributed in the electrode and the manner in which it is distributed will depend on $\vec{E}_e(x, y, z, t)$ of the source, the spatial geometry of the entire system and material/medium dielectric properties. Therefore, $Q(t)$ will change if the source is moved without affecting the relative position of electrodes. The voltage between the electrodes A and B – i.e., $V_{AB}(t)$ – will be equal to the electromotive force always; but the charge stored in the electrode system will vary with the spatial position of the source. Thus a unique ratio between $Q(t)$ and $V_{AB}(t)$ will exist only for a particular spatial arrangement of source and electrodes. The ratio will change with the position of source and can not be called a property of electrode arrangement alone.

All components in an electrical system will have static charge distributions at their terminals and on their surfaces. The electrostatic field at a point is the superposition of fields created by all these charge distributions. Thus, the voltage across terminals of one component will be decided by the work done in carrying a unit positive charge across the terminal pair against an electrostatic force that is decided by the static charge distributions in the entire electrical system. Thus, a simple ratio of the voltage across terminals of one circuit element to the value of charge distributed at its terminals and surface can not be defined in general.

Now we introduce certain assumptions so that we can ascribe the ratio $Q(t)/V_{AB}(t)$ to the electrode pair A and B without any reference to the position of other elements in the system. We assume that the distance between electrodes and the physical dimensions of the two-electrode system are very small compared to the distance between the two-electrode system and other circuit elements in the electrical system. [The reader may think of a parallel plate capacitor of large capacitance value and wonder how

such a capacitor can satisfy this requirement. That is precisely why a parallel plate capacitor is found only in the pages of textbooks. A practical ‘parallel plate capacitor’ has two aluminium foils of large length rolled into a tight cylinder shape with a pair of dielectric films between them. Such an assembly of a pair of electrodes will satisfy the assumption stated above.]

Positive and negative charge distributions of equal magnitude kept close to each other will produce only negligible electrostatic field at distant points. Therefore, the charge distribution on a pair of electrodes that satisfy the assumption stated above would not affect the electrostatic field at the locations where other circuit elements are located. And, charge distributions on other circuit elements will not affect the electrostatic field at the location where this electrode pair is located. Therefore the ratio of charge stored in the electrodes to voltage between the electrodes will depend only on the geometry of the electrode system and dielectric properties of the medium involved.

This unique and constant ratio associated with an electrode pair is defined as its *capacitance* value and the electrode system that satisfies the assumptions explained above is termed as a *two-terminal capacitor*. The magnitude of charge stored in one of the electrodes in a *linear capacitor* is proportional to the voltage across it. The symbol and variable assignment of a two-terminal capacitance is shown in Fig. 1.3-2.

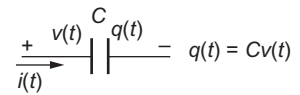


Fig. 1.3-2 A two-terminal capacitor

In fact, Circuit Theory extends the assumption of ‘*locally confined stationary electrostatic field*’ to all elements in the circuit. It assumes that the electrostatic field created by the charge distribution residing on a particular element (remember that there is no charge distribution on wires; they are of near-zero cross-section. Therefore, charge distributions can be ascribed to elements uniquely) is significant only near that element and is negligible at the location of other elements. This makes the electrostatic field around a circuit element a function of its own charge distribution alone. Therefore, the potential difference across terminals of one element will be proportional to the charge distributed on it. Thus assumption of ‘*locally confined stationary electrostatic field*’ amounts to neglecting electrostatic coupling between various elements. With this assumption, the voltage across a circuit element becomes proportional to the total charge distributed on its terminals and conducting surfaces. The proportionality constant depends on the geometry of the circuit element as well as on material dielectric properties. The fact that there has to be a certain amount of charge distributed on the surface of a circuit element for a voltage difference to exist between its terminals is equivalently described as *the capacitive effect* present in the component. *Thus every electrical element has a capacitive effect inherent in it.*

Therefore, a piece of conductor too has a capacitive effect associated with it. We ignored the current component that is required to support a time-varying charge distribution across a resistance in the previous section (Section 1.2) in order to define a two-terminal resistance. This is equivalent to neglecting the capacitive effect that is invariably present in the resistance. There is no pure resistance element in practice. All resistors come with a capacitive effect. However, if the capacitance that is present across a resistor draws only negligible current in a given circumstance, then, it may be modeled by a two-terminal resistance.

The capacitance that is present across a two-terminal resistance is called the *parasitic capacitance* associated with it. The adjective ‘*parasitic*’ gives us an impression that it is some second-order effect that has only nuisance value. That is not true – it arises out of the charge distribution that is required to make conduction possible in the resistance. Without this parasitic capacitance the resistor will not carry any current at all!

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The relation between the charge stored in a capacitor and voltage across it is given by $q(t) = Cv(t)$. C , the capacitance value has ‘Coulomb per Volt’ as its unit. This unit is given a special name – ‘Farad’. One Farad is too large a value for capacitance in practice. Practical capacitors have capacitance value ranging from few pFs ($1 \text{ pF} = 10^{-12} \text{ F}$) to few thousand μFs ($1 \mu\text{F} = 10^{-6} \text{ F}$). The value of C is a constant if the geometry of capacitor does not change with time and the material that is used as the dielectric between the metallic electrodes is linear, homogeneous and isotropic. If the value of C is a constant, it is called a *linear capacitor*.

The current that has to flow into the positively charged electrode of the capacitor is given by rate of change of the charge residing in that electrode. Therefore, the voltage across a linear capacitor is related to the current flowing into the positive electrode as below.

$$\begin{aligned} q(t) &= Cv(t) \\ i(t) &= C \frac{dv(t)}{dt} \\ v(t) &= \frac{1}{C} \int_{-\infty}^t i(t) dt = \frac{1}{C} v(0) + \frac{1}{C} \int_0^t i(t) dt \end{aligned} \quad (1.3-1)$$

The current through a capacitor depends on the first derivative of voltage appearing across it. Therefore, the current flow through the parasitic capacitance that is inevitably present across any electrical element can be neglected in the circuit model for that element only if the rate of change of electrical quantities involved in the circuit is small enough. Thus, a two-terminal resistance will model a piece of conducting substance with sufficient accuracy only if the frequency of voltage and current variables in the circuit is sufficiently small.

We have seen that there is no purely resistive two-terminal element in the physical world. A parasitic capacitance always goes along with a resistance. However, is there a pure two-terminal capacitor in real world?

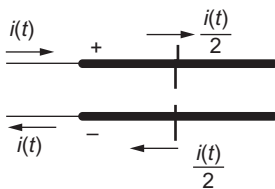


Fig. 1.3-3 Pertaining to the discussion on resistive effect in a capacitor

Consider a parallel-plate capacitor with a current $i(t)$ flowing into its positive plate as shown in Fig. 1.3-3. The current entering the positive plate from the left has to deposit charge all along the plate. Therefore the current has to flow through the cross-section of the plate from left to right. The magnitude of current comes down with length traveled towards right. Specifically, the current crossing the cross-section of the plate at mid-point will be about $0.5i(t)$. Thus, there is a linearly varying current crossing the cross-section of metallic electrode at any instant. This current flow meets with the impeding resistance of the metallic plate. Thus there will be a resistive voltage drop along the length of the plate and the plates will no longer

be equipotential surfaces. This resistive effect will produce power loss and heating in the capacitor.

There is yet another resistive effect present in a capacitor. A practical capacitor may use some dielectric material (like paper, polyester film, polypropylene film etc) between the electrodes in order to increase the capacitance value. The dielectric substance in between the electrodes has a

non-zero (though very small) conductivity; resulting in leakage current that flows from positive plate to negative plate. Thus the current entering the capacitor gets partially diverted for supplying this leakage current and only the remaining portion is available to create the charge distributions on plates.

A two-terminal capacitor model can model a physical capacitor only if these two resistive effects – one in series and one in parallel – can be neglected. The first resistive component can be made negligible by increasing the thickness of the electrodes and using a high conductivity metal to construct them. These two resistive effects are called the *parasitic resistive effects* in a capacitor.

Example: 1.3-1

The current flowing through a capacitor of 0.5F is given by $i(t) = \begin{cases} 0 \text{ A for } t < 0 \\ 2 \text{ A for } 0 < t < 2\text{s} \\ 0 \text{ A for } t > 2\text{s} \end{cases}$. Find the voltage across the capacitor at $t = 5 \text{ s}$.

Solution

Voltage across a capacitor at t is proportional to the area under the applied current waveform from $-\infty$ to t . No current was applied to the capacitor until $t = 0$. Therefore, the voltage across capacitor is 0 V for $t < 0$.

The current through the capacitor is 2A from $t = 0$. Hence, the voltage across capacitor increases at the rate of $2\text{A}/0.5\text{F} = 4\text{V/s}$ from $t = 0$. The expression for the voltage will be $i(t) = 4t \text{ V}$ as long as 2A is maintained through it.

The value of voltage across capacitor at $t = 2 \text{ s}$ is 8V. The current through the capacitor is 0A from $t = 2 \text{ s}$ onwards. There is no area addition to the area under the current waveform after $t = 2 \text{ s}$. Therefore, the capacitor voltage continues at the same value that it had when the applied current went to zero. Note that a current source of 0A is an open-circuit. A non-zero voltage can remain in an *ideal* capacitor forever.

$$\therefore v(t) = \begin{cases} 0 \text{ V for } t \leq 0 \\ 4t \text{ V for } 0 < t \leq 2\text{s} \\ 8 \text{ V for } t > 2\text{s} \end{cases}$$

Example: 1.3-2

A resistor of value 5Ω and a capacitor C are connected in parallel. The voltage applied across the parallel combination is $v(t) = V_m \sin t \text{ V}$. The current through the resistor and capacitor are seen to have equal amplitude. Find the value of C .

Solution

The current through the capacitor is given by $C \frac{dv(t)}{dt} = CV_m \cos t \text{ A}$.

The current through resistor is $0.2V_m \sin t \text{ A}$. If these two currents have same amplitude, the value of capacitance must be $C = 0.2\text{F}$.

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Example: 1.3-3

The current through a capacitor of 0.1 F is known to be $0.5\pi \sin 2\pi t\text{ A}$ in the time interval $[0, 0.5\text{ s}]$. The voltage across capacitor at $t = 0.5\text{ s}$ is seen to be 0 V . What was the voltage across capacitor at $t = 0$?

Solution

The change in voltage across a capacitor over a given time interval is $(1/C)$ times the charge delivered to it over the same time interval. Charge delivered in a given time interval is given by the area under the current waveform over the same interval.

$$\therefore \text{Change in capacitor voltage over } [0, 0.5\text{ s}] = \frac{1}{0.1} \int_0^{0.5} 0.5\pi \sin 2\pi t = \frac{5\pi}{2\pi} [-\cos 2\pi t]_0^{0.5} = 5\text{ V}$$

The voltage across capacitor at the end of a given time interval = voltage at the beginning of the interval *plus* the change in voltage over the given time interval.

The voltage across capacitor at $t = 0.5\text{ s}$ is given as 0 V . Therefore, the voltage across capacitor at $t = 0$ must have been -5 V .

1.4 TWO-TERMINAL INDUCTANCE

A moving charge that is part of a steady-current flow (*i.e.*, DC current) in a circuit is, in general, acted upon by non-electrostatic forces contributed by source regions, electrostatic forces contributed by charge distributions and frictional forces within conductors. A new force called *induced electric force* that acts on such a moving charge makes its appearance in circuits carrying unsteady (*i.e.*, time-varying) currents. This new force component gives rise to a circuit element called *inductance*.

1.4.1 Induced Electromotive Force and its Location in a Circuit

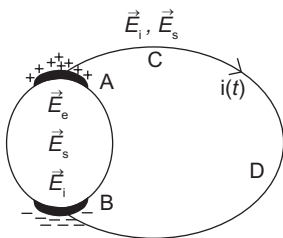


Fig. 1.4-1 A shorted source of electromotive force

Consider a source of electromotive force with short-circuit across it as in Fig. 1.4-1. The conducting material inside the source is assumed to be of infinite conductivity. The shorting wire is assumed to be of material of infinite conductivity and the cross-section of the wire is taken to be of near-zero dimension. These assumptions imply that there is no net force needed to make charged particles move inside the source as well as inside the shorting wire. Further, the static charge distribution on the surface of shorting wire is negligibly small since the wire is very thin.

Therefore, there can be no resistive voltage drops inside the source and the shorting wire. Hence, the current flow in the system will reach infinitely large level if the electromotive force of the source is a steady value – *i.e.*, if

the source is a DC voltage source. Of course, the small resistances that are inevitably present within the source and in the shorting wire will limit the current in practice.

Now consider the situation with a time-varying electromotive force in the source. The non-electrostatic field produced by the source in the source region – *i.e.*, $\vec{E}_e$ – is a time-varying quantity.

Therefore, the charge distribution on the source terminals has to be time-varying in order to generate a time-varying electrostatic field inside the source for exact cancellation of time-varying $\vec{E}_e$. This time-varying charge distribution will, in turn, produce time-varying electrostatic field inside the shorting wire, resulting in a time-varying current flow in the wire. However, the conductivity of wire material is assumed to be infinity. Hence, we should expect a time-varying current of infinite magnitude in the wire. But, the current is observed to have finite amplitude in practice. If there is no resistive effect in the wire (conductivity is taken to be infinity), then what is the mechanism responsible for preventing the current from reaching infinitely high value?

The required mechanism arises out of the third component of force of interaction between two charges in arbitrary motion. We had observed in Section 1.1 that this component is dependent on relative acceleration of interacting charges and is given by $\overline{F_{12ei}} = -\frac{\mu_0 q_1 q_2}{4\pi} \frac{\partial}{\partial t} \left(\frac{\vec{v}_1}{r} \right) \Big|_{\vec{v}_2=0}$ N where $\overline{F_{12ei}}$ is the component of force experienced by charge q_2 due to charge q_1 and $\vec{v}_1$ is the velocity of q_1 . We had termed this component of force as the *induced electric force*.

Thus, a moving charge can experience four kinds of force in general – (i) the force due to the non-electrostatic field inside a source acting on it (ii) force due to electrostatic field (iii) magnetic force due to other moving charges (iv) induced electric force from other charges which are accelerating with respect to the location of this charge. The first kind of force will be present only if the charge is inside a source region.

Magnetic force on a moving charge is in a direction perpendicular to the velocity of charge. Hence magnetic force can not change the energy of a charged particle. Therefore magnetic force can not affect the current flow in a circuit though it may produce mechanical forces in current carrying systems. Hence, we need to consider only the remaining three forces on a moving charge in circuit analysis.

The net induced electric force experienced by a charge located at a certain point in a circuit carrying *steady current* (i.e., a DC current) due to all the other moving charges in the circuit will be zero. We accept this statement without proof.

Hence, moving charges in a DC circuit do not experience any induced electric force provided there are no other circuits carrying time-varying currents in its vicinity.

However, the net induced electric force experienced by a charge located at a certain point in a circuit carrying *time-varying current* due to all the other moving charges in the circuit *will not be* zero. It will experience an induced electric force that will be proportional to its value. Thus, we can define induced electric field $\vec{E}_i$ at a point as the net induced electric force experienced by +1 C charge kept at that point. This field exists everywhere in space (including the source region) unlike the non-electrostatic field that is present only within the source.

Now, the force balance on charges inside the source requires that $\vec{E}_e + \vec{E}_s + \vec{E}_i = 0$ since the material inside the source has infinite conductivity. Similarly the force balance condition inside the shorting wire requires that $\vec{E}_s + \vec{E}_i = 0$ since the shorting wire is of infinite conductivity.

Electrostatic field is a conservative field. Hence the work to be done against the electrostatic force in carrying a unit test charge around a closed loop is zero. Induced electric field is non-conservative. Hence the work to be done against the induced electric force in carrying a unit test charge around a closed loop is non-zero.

The electrostatic field inside the source and the shorting wire can be expressed in terms of induced electric field and the non-electrostatic field generated by the source as follows.

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$$\text{Inside the source } \vec{E}_s = -(\vec{E}_e + \vec{E}_i)$$

$$\text{Inside the shorting wire } \vec{E}_s = -\vec{E}_i$$

It must be noted that all the field quantities appearing in these equations are functions of space as well as time and that these equations are valid at all points inside the source and wire. Strictly speaking, they should have been expressed as below.

$$\text{Inside the source } \vec{E}_s(x, y, z, t) = -[\vec{E}_e(x, y, z, t) + \vec{E}_i(x, y, z, t)]$$

$$\text{Inside the shorting wire } \vec{E}_s(x, y, z, t) = -\vec{E}_i(x, y, z, t)$$

Let a +1 C charge be taken around the circuit from B to A through the source and from A to B through the shorting wire. Then,

$$\oint -\vec{E}_s(x, y, z, t) \cdot d\vec{l} = 0 \text{ since electrostatic field is conservative. Therefore,}$$

$$\int_{\substack{\text{B to A} \\ \text{through the source}}} -\vec{E}_s(x, y, z, t) \cdot d\vec{l} + \int_{\substack{\text{A to B} \\ \text{through the wire}}} -\vec{E}_s(x, y, z, t) \cdot d\vec{l} = 0$$

But $-\vec{E}_s(x, y, z, t) = [\vec{E}_e(x, y, z, t) + \vec{E}_i(x, y, z, t)]$ inside the source and $-\vec{E}_s(x, y, z, t) = \vec{E}_i(x, y, z, t)$ inside the shorting wire. Therefore,

$$\int_{\substack{\text{B to A} \\ \text{through the source}}} [\vec{E}_e(x, y, z, t) + \vec{E}_i(x, y, z, t)] \cdot d\vec{l} + \int_{\substack{\text{A to B} \\ \text{through the wire}}} \vec{E}_i(x, y, z, t) \cdot d\vec{l} = 0$$

$$\therefore \int_{\substack{\text{B to A}}} \vec{E}_e(x, y, z, t) \cdot d\vec{l} = \oint \vec{E}_i(x, y, z, t) \cdot d\vec{l}$$

The quantity on the left-hand side is the electromotive force of the source – that is, it is the work done *by* the non-electrostatic force provided by the source when +1 C is taken through it from its negative terminal to the positive terminal. Similarly, the quantity on the right-hand side is the *negative* of work done *by* the non-conservative induced electric force when +1 C is taken through the loop in clockwise direction. That is, it is the *negative* of electromotive force due to the induced electric field in clockwise direction in the loop. We will term this electromotive force as the *induced electromotive force*. Obviously, the current in this circuit will have suitable magnitude at all instants such that the source electromotive force and induced electromotive force meet each other without leaving any net electromotive force in the circuit loop.

1.4.2 Relation between induced electromotive force and current

The induced electric field at a point in a circuit is the superposition of terms of the form $-\frac{\mu_0 q}{4\pi r^2} \frac{\partial}{\partial t} \left(\frac{\vec{v}}{r} \right)$

where q is the charge per carrier and $\vec{v}$ is the carrier velocity and r is the distance between the carrier and the point. All moving carriers in the circuit are to be considered in the vector summation. Since current in the circuit is related to carrier velocity, we expect the summation to turn out to be proportional

to $\frac{di(t)}{dt}$ in the circuit. Thus, induced electric field at all points in space will be proportional to $\frac{di(t)}{dt}$. Electromotive force of a force field is defined as the *work done by the field* when a unit test charge is taken through a closed path lying in that force field. Hence the induced electromotive force in any closed path will be proportional to $\frac{di(t)}{dt}$. The proportionality constant will depend on the spatial geometry of the circuit and the magnetic properties of the medium involved. This proportionality constant is termed as the *inductance* of that closed path. Inductance is designated by the symbol L . If the geometry of the circuit does not vary with time, the value of L will be a constant.

1.4.3 Faraday's Law and Induced Electromotive Force

Faraday's law of electromagnetic induction states that the induced electromotive force in a closed path in a circuit is equal to the time rate of change of flux linkage through that closed path. If the closed path is traversed in counter-clockwise direction and positive flux linkage is defined according to right-hand screw rule, then, this law states that induced electromotive force = $\frac{d\psi(t)}{dt}$ where $\psi(t)$ is the flux linkage through the closed path in 'Weber-turns' unit.

Faraday's law gives the total induced electromotive force in a closed path. *However, Faraday's law can not tell us where exactly this electromotive force is located.* The discussion in the previous sub-section has shown that the induced electromotive force is distributed all around the closed path.

Determining the polarity of induced electromotive force by using Faraday's law can be confusing at times for a beginner in circuit analysis. Lenz's law is a better option for this purpose. Lenz's law, in effect, states that the induced electric field will be in such a direction that it opposes the change in current that is the cause for appearance of the induced electric field. Refer to Fig. 1.4-1. Assume that the current $i(t)$ is increasing at some time instant. This means that all the charge carriers are accelerating in the direction of current flow at that instant. This acceleration is the cause of induced electric field in the wire and elsewhere. The direction of induced electric field inside the wire will be such that the induced electric force on a positive charge will tend to *decelerate* it. Thus the induced electromotive force will work against the source electromotive force.

Thus, the induced electromotive force in a closed loop in a circuit with time-varying current is given by $L \frac{di(t)}{dt}$ (for static circuits; see the previous sub-section) as well as by $\frac{d\psi(t)}{dt}$ (by Faraday's law) with the direction of electromotive force as per Lenz's law. Therefore,

$$\begin{aligned} L \frac{di}{dt} &= \frac{d\psi(t)}{dt} \\ \frac{d(Li(t))}{dt} &= \frac{d\psi(t)}{dt} \\ \therefore \psi(t) &= Li(t) \end{aligned}$$

Thus, inductance of a closed path is the flux linkage in that closed path for unit current. The unit of inductance is Weber-turns/ampere. This unit is given a special name – 'Henry' and is represented by 'H'. Since $L \frac{di(t)}{dt}$ yields an electromotive force, inductance gets another unit – volt-sec per amp. It follows that volt-sec and weber-turns refer to the same physical quantity.

1.4.4 The Issue of a Unique Voltage Across a Two-Terminal Element

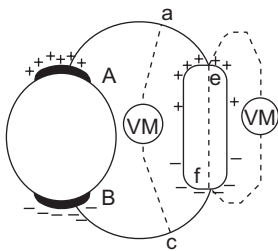


Fig. 1.4-2 Pertaining to uniqueness of terminal voltage of a two-terminal element

Refer to Fig. 1.4-2. A time-varying source of electromotive force is connected to a conductor by thin connecting wires of infinite conductivity. The charge distributions at the source terminals and load terminals produce electrostatic field everywhere in space. The electrostatic field inside the source cancels the non-electrostatic field available inside the source and the induced electric field inside the source exactly. (The conductivity inside the source is assumed to be infinity). Electrostatic field inside the connecting wires cancels the induced electric field inside them. Electrostatic field inside the conductor meets the frictional force arising out of collisions of charge carriers with atoms in the lattice and the induced electric force manifesting inside the conductor.

Three issues arise in this context.

- (i) The voltage across two points is the electrostatic potential difference between the two points. The voltage across the resistance is given by the potential difference between e and f . This voltage can be obtained by calculating the work to be done in carrying a $+1$ C charge from f to e through the inside of the conductor. But, the electrostatic field inside the conductor is equal to $-(\text{frictional force field} + \text{induced electric field})$. Therefore the terminal voltage of resistance will contain a resistive voltage drop plus a term that depends on $\frac{di(t)}{dt}$ (i.e., an inductive voltage drop). Therefore, the conductor can no longer be modeled as a pure two-terminal resistance.
- (ii) The voltmeter connected on the right of the conductor attempts to measure the terminal voltage of the resistance right across its terminals. However, the voltmeter connection creates a closed path comprising the resistance element, connecting leads and the meter. This closed path will have induced electric field everywhere inside the connecting leads as well as within the meter. Thus the meter ends up reading the terminal voltage plus the induced electromotive force in the voltmeter leads and in the meter internal circuit. Thus, the reading is in error. The amount of error will keep changing with geometry of voltmeter connection – that is, the reading will be different when the leads are disturbed into a new spatial configuration. The amount of error is dependent on the time- rate of change of flux linkage of the voltmeter loop.
- (iii) The voltmeter connected on the left of the conductor reads the actual terminal voltage of the resistance element plus the induced electromotive force in the path f - c -VM- a - e . Thus, the reading includes the induced electromotive force in the voltmeter leads and portions of connecting wire in the circuit.

Thus, no unique voltage can be assigned to the resistance by measurement. Therefore, we bring in certain assumptions. The first assumption is that the induced electric field (and, hence, the induced electromotive force too) inside the connecting wires everywhere in the circuit is negligible. The second assumption is that the induced electric field inside the conductor (or inside a capacitor) is negligible. Obviously, this is equivalent to ignoring the inductive effect present everywhere in the circuit.

No circuit can satisfy these assumptions exactly (except in DC circuits). Induced electric field is proportional to rate of change of current. If the rate of change of current is low, then, the strength of induced electric field inside the sources, resistances, capacitors and connecting wires will be low

compared to electrostatic field. Thus, the assumptions stated above can be employed if the rate of change of current in the circuit is sufficiently small.

With these assumptions, it becomes possible to model a physical resistor by an ideal two-terminal resistance model and a physical capacitor by an ideal two-terminal capacitance model. Further, voltages across a source, resistor and capacitor become unique even with time-varying current in the circuit.

But, this does not mean that we will not make use of induced electromotive force in a circuit at all!

1.4.5 The Two-Terminal Inductance

An electrical device, in general, can have four kinds of force fields that can affect current flow at every point inside the device. They are:

- (i) Some non-electrostatic field arising out of some kind of potential energy stored within the device – for instance, the non-electrostatic field generated by chemical potential energy in a dry cell.
- (ii) Electrostatic field created by charge distributions on this device as well as other devices nearby.
- (iii) Induced electric field created by time-varying current flowing in the circuit containing this device as well as in neighboring circuits.
- (iv) Non-electrostatic force field due to the collisions between moving charged particles and lattice atoms during conduction.

The model used in circuit theory for a device will depend on which of these are strong and which are negligible.

Electrostatic field will be present in all devices in an electrical system and can not be ignored in any device. Electrostatic field inside *any* device is a function of charge distributions on *all* devices in the system. However, if the physical dimensions of the devices are small compared to spatial distance between the devices, then, the electrostatic field inside a particular device is determined uniquely by the charge distribution on its surface alone. Then, there will exist a unique ratio between the electrostatic potential difference across its terminals and the total charge stored on its surface. This is how a two-terminal capacitance can be defined at all.

Thus, a two-terminal capacitance is a model for an electrical device that has only electrostatic field inside in it and the electrostatic field inside depends only on its own charge distribution. The non-electrostatic field existing in the metallic electrodes when current flows in them is ignored in an ideal two-terminal capacitance. The induced electric field that exists inside the device due time-varying currents everywhere is also ignored in an ideal two-terminal capacitance.

A piece of conductor with finite conductivity carrying a current will have electrostatic field, non-electrostatic field arising out of frictional forces and induced electric field due to time-varying currents in the circuit as well as in other circuits. An ideal two-terminal resistance models this piece of conductor by ignoring (i) the current component that is needed to build a time-varying charge distribution on its surface and (ii) the induced electric field inside the conductor.

Circuit Theory models a piece of connecting wire by ignoring all fields that exist within the wire and taking all of them to be zero at all instants. Thus, Circuit Theory assumes that there is no resistive

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drop across connecting wire; there is no induced electromotive force in connecting wire and there are no charges distributed on the connecting wire. Such an element is called the *ideal short-circuit element*.

An electrical source will have all the four kinds of fields inside. However, the *ideal two-terminal source model* of Circuit Theory attempts to model such a source by (i) ignoring the non-electrostatic field arising out of friction within conductor (ii) ignoring the induced electric field inside in comparison with electrostatic field and (iii) ignoring the component of current needed to build a time-varying charge distribution at its terminals.

And, the *ideal two-terminal inductance model* of Circuit Theory is a model for an electrical device in which there are only two fields – the induced electric field and the electrostatic field. It is not a source and hence there is no source field. It uses conducting substance and hence there is a non-electrostatic field arising out of collisions of charge carriers with lattice atoms when a current flows through it. But this field is ignored in comparison with the other fields. Further, the component of current needed to build a time-varying charge distribution on its surface is assumed to be negligibly small.

Consider a long piece of round conductor carrying a time-varying current as shown in (a) of Fig. 1.4-3. This wire is not a connection wire. It has a non-zero cross-sectional area. But it is indicated by a line in Fig. 1.4-3. The current entering the conductor is $i(t)$ and the same current leaves the conductor at far end. The value of current crossing any cross-section at a particular instant will be the same everywhere since we neglect retardation effect as well as the current that is required to build the surface charge distribution.

There is induced electric field at all points within this conductor. The induced electric field at a point inside is the sum of terms of the form $-\frac{\mu_0 q}{4\pi} \frac{\partial}{\partial t} \left(\frac{\vec{v}(t)}{r} \right)$ where q is the charge per carrier and $\vec{v}(t)$ is the carrier velocity and r is the distance between the carrier and the point – as many terms as there are moving carriers in the conductor. All the charge carriers will be moving with same instantaneous velocity that is proportional to $i(t)$. But the distance between the point at which the induced electric field is calculated and the location of carrier (*i.e.*, r) will be large for all those carriers that are moving at a far away location at the instant under consideration. Therefore, only those carriers that are presently moving within the immediate vicinity of the point at which field is being calculated will contribute to the induced electric field significantly. Thus the induced electric field will be relatively low everywhere, and, correspondingly the total induced electromotive force in the long conductor will be relatively low. The induced field as well as the total induced electromotive force will be proportional to $\frac{di(t)}{dt}$ since $\frac{\partial}{\partial t} \left(\frac{\vec{v}(t)}{r} \right)$ that appears in the equation for induced electric field due to a moving charge is directly related to $\frac{di(t)}{dt}$.

The conductor is assumed to be of large conductivity. Then, the net force experienced by a charge carrier inside must be zero. Therefore, the induced electric field at every point within the conductor will be cancelled exactly by the electrostatic field created by the surface charge distribution. This charge distribution is shown in Fig. 1.4-3 assuming that $\frac{di(t)}{dt}$ is positive at the instant under consideration.

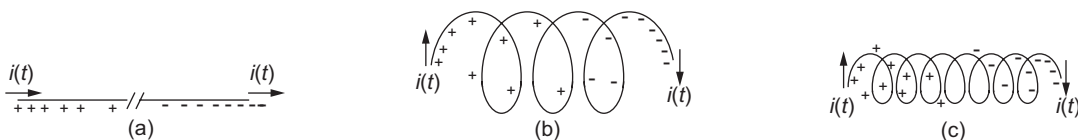


Fig. 1.4-3 Towards a two-terminal inductance

A physical inductor is constructed so as to strengthen the induced electric field and the induced electromotive force inside the conductor forming the inductor.

Refer to (b) of Fig. 1.4-3. The same conductor is wound into a coil of 4 turns. Now the relative distances between moving carriers in various sections of the wire are reduced considerably. Hence the induced electric field at any point in the conductor will have a value greater than the value when the entire conductor was stretched out in a straight-line as in (a). Therefore, the total induced electromotive force will also be higher. Obviously, the value of induced electromotive force will go up further if the turns can be kept closer.

Refer to (c) of Fig. 1.4-3. The same conductor is wound into a coil of lower diameter and higher turns. And the turns are kept closer. This structure will have still higher induced electric field everywhere. The total induced electromotive force will also be higher. If the wire has an insulation cover the turns can touch each other.

Thus, winding a long length of wire into an optimally sized and layered coil with turns touching each other will result in large induced electric field everywhere in the wire and large induced electromotive force over the length of the wire when the current through the coil is time-varying. The induced electric field everywhere inside will be cancelled by the electrostatic field created by the surface charge distribution all along the wire surface. (The conductivity of wire material is assumed to be very large). Therefore, the electrostatic potential difference between the ends of the coil – i.e., the voltage difference between coil terminals – will be equal to the total induced electromotive force in the coil. The polarity of voltage will follow Lenz's law.

What we have described here is an air-cored coil. Air-cored inductor is essentially a long piece of wire that is arranged to occupy a small region of space of dimensions that are very small compared to its length. Such a spatial confinement of a long wire results in strengthening of induced electromotive force in it. Further strengthening of induced electric field inside the wire can be attained by winding it around a core made of magnetic material (usually iron). If the core made of magnetic material is a closed structure, the induced electric field will be enhanced further. Moreover, a closed core structure confines the time-varying magnetic field to the core itself and reduces the magnetic flux linking rest of the circuit to negligible levels.

A physical inductor that is designed to strengthen the induced electric field within itself, while confining the time-varying flux-linkage to predominantly within itself, can be modeled by an ideal *two-terminal inductance model* provided the resistive voltage drop in the coil can be neglected and the capacitive effect due to surface charge distribution over the coil surface can be neglected.

The value of inductance depends on the geometry of the coil and core assembly and the magnetic properties of the core. *Inductance of a coil is proportional to the square of number of turns of the coil, area of a turn and magnetic permeability of the core material.*

The symbol and variable assignment for an ideal two-terminal inductance is shown in Fig. 1.4-4.

The governing equations of a linear two-terminal inductance are:

$$\psi(t) = Li(t)$$

$$v(t) = L \frac{di(t)}{dt} \quad (1.4-1)$$

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt = \frac{1}{L} \int_{-\infty}^0 v(t) dt + \frac{1}{L} \int_0^t v(t) dt = i(0) + \frac{1}{L} \int_0^t v(t) dt$$

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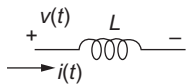


Fig. 1.4-4 A two-terminal inductance

where $\psi(t)$ is the flux-linkage at t in Weber-turns, $v(t)$ is the voltage across the inductance and $i(t)$ is the current entering the higher potential terminal. $i(0)$ is the current in the inductor at $t = 0$.

A coil can have induced electric field and induced electromotive force present in it due to accelerated motion of charges (*i.e.*, time-varying current) in the circuit in

which it is connected and/or due to accelerated motion of charges taking place in another physically separated circuit. The electromotive force induced in the coil due to its own time-varying current is termed as *self-induced electromotive force* and the electromotive force induced in it due to current in another circuit is termed as *mutually induced electromotive force*. Self-induced electromotive force is associated with an inductance value called *self-inductance*. Eqn. 1.4-1 describes the governing equations of self-inductance.

There is no region without induced electric field and induced electromotive force in any circuit carrying time-varying current. All devices and components of such a circuit are affected by electromagnetic induction. Thus, all devices have inductive effect associated with them. The associated inductance will be called the *parasitic inductance* of the two-terminal element (unless it is a two-terminal inductance). Ideal two-terminal element models ignore the parasitic inductance in a resistor, capacitor, source and connecting wire.

Example: 1.4-1

The voltage applied across an inductor of 2 H is given by $i(t) = \begin{cases} 0 \text{ V for } t < 0 \\ 2 \text{ V for } 0 < t < 2\text{s.} \\ 0 \text{ V for } t > 2\text{s} \end{cases}$. Find the current in the inductor at $t = 5 \text{ s}$.

Solution

Current in an inductor at t is proportional to the area under the applied voltage waveform from $-\infty$ to t . No voltage was applied to the inductor till $t = 0$. Therefore, the current in the inductor is 0 A for $t < 0$.

The voltage applied across the inductor is 2 V from $t = 0$. Hence, the inductor current increases at the rate of $2 \text{ V} / 2 \text{ H} = 1 \text{ A/s}$ from $t = 0$. The expression for the current will be $i(t) = t\text{A}$ as long as 2 V is maintained across it.

The value of current at $t = 2 \text{ s}$ is 2 A. The voltage applied across the inductor is 0 V from $t = 2 \text{ s}$ onwards. There is no area addition to the area under the applied voltage waveform after $t = 2 \text{ s}$. Therefore, the inductor current continues at the same value that it had when the applied voltage went to zero. Note that a voltage source of 0 V is a short-circuit. A non-zero current can circulate in a shorted *ideal* inductor forever.

$$\therefore i(t) = \begin{cases} 0 \text{ A for } t \leq 0 \\ t \text{ A for } 0 < t \leq 2\text{s.} \\ 2 \text{ A for } t > 2\text{s} \end{cases}$$

Example: 1.4-2

A 1 H inductor and a 10 Ω resistor are connected in series. The current that flows through the series combination is such that the voltage across the resistor is equal to the voltage across the inductor for all t . The voltage across the inductor at $t = 0$ is seen to be 1 V. (i) Find circuit current at $t = 1$ s. (ii) Find the energy dissipated in the resistor in the time interval $[0, 1$ s].

Solution

- (i) Let $i(t)$ be the circuit current. Then, the voltage across resistor is $10i(t)$ and the voltage across the inductor is $\frac{di(t)}{dt}$. These are stated to be equal for all t . i.e., $\frac{di(t)}{dt} = 10i(t)$ for all t .

The only function that can satisfy this requirement is an exponential function since only exponential function has the property that the waveshape remains the same on differentiation and integration. (Note – Sinusoidal functions are special cases of a generalised complex exponential function). Hence, we try the solution $Ae^{\alpha t}$. Substituting this function in the above equation, we get,

$$A\alpha e^{\alpha t} = 10 A e^{\alpha t} \quad \text{for all } t \quad \therefore \alpha = 10$$

Inductor voltage at $t = 0$ is $A\alpha e^{\alpha t}|_{t=0} = A\alpha = A$. This value is 1 V. Therefore $A = 1$.

$\therefore i(t) = e^t$ A for all t is the current through the series combination.

Hence, current through the circuit at $t = 1$ s is 2.7318 A.

- (ii) Energy dissipated in the resistor over $[0, 1$ s] = $\int_0^1 10 \times [i(t)]^2 dt = \int_0^1 10 \times e^{2t} dt = 31.95$ J

1.5 IDEAL INDEPENDENT TWO-TERMINAL ELECTRICAL SOURCES

Electrical sources are devices that are capable of applying a non-electrostatic force on a charge that moves through the source region. They can deliver energy to the charged particle or absorb energy from it.

1.5.1 Ideal Independent Voltage Source

A two-terminal voltage source will have non-electrostatic field at every point inside the source region. The charge distribution on the terminal surfaces of the source will create an electrostatic field at all points inside the source. The two fields cancel each other at all points at all instants under all conditions if the material inside the source is of infinite conductivity. The terminal voltage (which is an electrostatic potential difference) will always be equal to the internal electromotive force in that case.

The conducting material inside the source will have finite conductivity in practice. Charge carriers moving inside such material require net non-zero force to work against collisions with lattice atoms. This will call for a difference between the internal non-electrostatic field and the electrostatic field. Then, the terminal voltage will be different from the internal electromotive force. It will be less than the internal electromotive force if the source is delivering positive current out of its positive terminal and it will be more than internal electromotive force if it is absorbing positive current at its positive terminal. The difference between terminal voltage and internal electromotive force is termed as the voltage developed across the *internal resistance* of the source.

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A practical voltage source with time-varying internal electromotive force will require a time-varying current flow component to support the time-varying surface charge distribution on its terminals. That is, a practical voltage source has a parasitic capacitance right across its terminals.

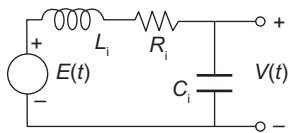


Fig. 1.5-1 Approximate equivalent circuit of a practical voltage source

A practical voltage source will have induced electric field inside due to its own time-varying current as well as due to time-varying currents elsewhere in the circuit and in neighboring circuits. This will affect the voltage appearing at its terminals. That is, a practical voltage source has internal parasitic inductance too. Thus a detailed circuit model for a practical voltage source will be as shown in Fig. 1.5-1. L_i is a lumped parameter approximation for the inductive effect distributed within the source. R_i is a lumped resistance parameter that approximates the distributed resistive effect within the source. C_i is a lumped capacitance

parameter that approximates the distributed capacitive effect within the source and at its terminals. $E(t)$ is the internal electromotive force of the source. The + and – signs do not signify the polarity of charges at the terminals. Rather, the + sign indicates the point at which the potential difference is specified and – sign indicates the reference point for specifying the potential difference. Thus $V(t)$ is the voltage of the terminal marked with + with respect to the point marked with – sign at the time instant t in Volts.

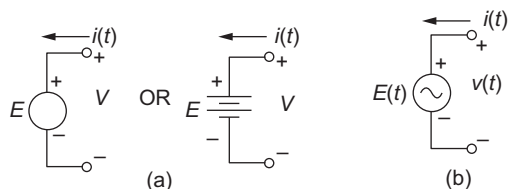


Fig. 1.5-2 Ideal independent voltage source

An ideal voltage source is one in which all the three elements R_i , L_i and C_i are assumed to be negligible. Thus the terminal voltage of an ideal voltage source is always equal to its internal electromotive force quite independent of magnitude or waveshape of current delivered or absorbed by it. Such an ideal voltage source is called an *ideal independent voltage source* if the electromotive force is a function of time only and does not depend on any other electrical or non-electrical variable. An ideal independent

voltage source is specified by the following terminal equations.

$$v(t) = E(t), \text{ a specified function of time}$$

$$i(t) = \text{Arbitrary, decided by the rest of the circuit in which this source is connected.}$$

The symbol of a constant ideal independent voltage source (that is, a DC source) is shown in (a) of Fig. 1.5-2 and that of a time-varying ideal independent voltage source is shown in (b) of Fig. 1.5-2.

1.5.2 Ideal Independent Current Source

An ideal independent current source delivers or absorbs a current at its terminals that is a specified function of time. Rest of the circuit in which it is connected decides its terminal voltage. The current delivered or absorbed by it does not depend on the voltage that appears across its terminals.

Practical current sources will have a parallel resistance and capacitance at its terminals representing the effect of finite conductivity within the source and charge distribution on its surface and terminals. These parasitic components are neglected in the ‘ideal independent current source’ model.

The symbol of a constant ideal independent current source (that is, a DC source) is shown in (a) of Fig. 1.5-3 and that of a time-varying ideal independent current source is shown in (b) of Fig. 1.5-3.

An ideal independent current source is specified by the following equations:

$$i(t) = I_s(t), \text{ a specified function of time.}$$

$v(t)$ = Arbitrary, decided by the rest of the circuit in which this source is connected.

There are no ideal independent voltage sources and ideal independent current sources in practice. These are only models of practical sources that give reasonably accurate results provided they are not applied under extreme loading conditions.

The ideal model will undoubtedly fail for a practical voltage source that is shorted or for a practical current source that is open-circuited. The short-circuit current in a DC voltage source is limited by its internal resistance while that of a time-varying voltage source is limited by internal resistance and internal inductance. Similarly, the open-circuit voltage that appears across a practical current source is limited by its internal resistance in the case of a DC source. It is limited by internal capacitance and resistance in the case of a time-varying current source.

In fact, ideal model for a voltage source models a practical voltage source accurately only when the current delivered/absorbed by it is a small fraction of its short-circuit current. Similarly, ideal model for a current source models a practical current source accurately only when the voltage appearing across its terminals is a small fraction of its open-circuit voltage.

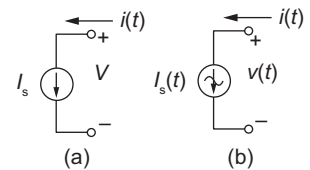


Fig. 1.5-3 Ideal Independent Current Source

1.5.3 Ideal Short-Circuit Element and Ideal Open-Circuit Element

Ideal two-terminal short-circuit element is the element that is used to model a piece of connecting wire in Circuit Theory. It is also used to model an ideal switch in closed condition. It has no resistance, no inductance and no charges distributed on it. The voltage across its terminals is constrained to remain at zero. It can carry an arbitrary current that is decided by rest of the circuit. Thus, definition of ideal short-circuit element parallels that of an ideal independent voltage source. *Hence an ideal short-circuit element may be thought of as a special case of an ideal independent voltage source with $E(t) = 0$ for all t .* It is described by the following equations.

$$v(t) = 0 \text{ V,}$$

$$i(t) = \text{Arbitrary, decided by the rest of the circuit in which this source is connected.}$$

Similarly, *an ideal open-circuit element* is equivalent to *an ideal independent current source with $I_s(t) = 0$.*

It is described by the following equations.

$$i(t) = 0 \text{ A,}$$

$$v(t) = \text{Arbitrary, decided by the rest of the circuit in which this source is connected.}$$

In practice, a short-circuit element has a little resistance and inductance in series. A practical open-circuit has a small capacitance shunting its terminals.

1.6 POWER AND ENERGY RELATIONS FOR TWO-TERMINAL ELEMENTS

An ideal two-terminal circuit element has a unique voltage variable assigned at its terminals and a unique current variable assigned to its terminals. The electrical behaviour of such an element can be described in terms of these two variables at all instants. Electromagnetic disturbances are assumed to travel instantaneously to all parts of such an element. This results in an electrical description that is

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independent of space variables for the element. Such an electrical description for an element is termed as *lumped parameter description*.

Further, ideal two terminal elements have only one kind of electrical phenomena taking place inside them. The capacitive and inductive effects in a practical resistance are neglected in order to arrive at an ideal two-terminal resistance model. The resistive and inductive effects in a physical capacitor are neglected to model it by an ideal two-terminal capacitor. The capacitive and resistive effects in a physical inductor are neglected to arrive at the ideal two-terminal inductance model.

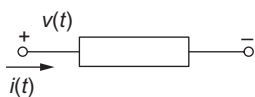


Fig. 1.6-1 An ideal two-terminal element

Moreover, lumped two-terminal elements confine the electromagnetic fields associated with them to the space inside them and in the immediate vicinity.

Such a two-terminal element can be represented in general by the symbol below in Fig. 1.6-1. The variable assignment for the element is also shown in the figure.

1.6.1 Passive Sign Convention

Current in a wire has a direction associated with it. The actual direction associated with a current – i.e., the *current direction* – is the direction in which positive charges move.

Consider a wire section A – B with a cross-section identified at C as in Fig. 1.6-2. Four possible kinds of charge motion are depicted in (a) to (d) in this figure. The *direction of current* in (a) and (d) is from left to right (from A to B). This is so since negative charge crossing a cross-section in one direction is equivalent to positive charge crossing the same cross-section in opposite direction. The *direction of current* in (b) and (c) is from right to left (from B to A) for the same reason.

However, it is not possible to decide the direction in which current will flow in an element that is a part of a circuit before we actually solve the circuit analysis problem. The actual direction in which positive charge flow through the element can be ascertained only after the circuit solution is obtained. But, despite this we need to assume some direction for current flow in each and every element in a circuit so that we can prepare the circuit equations needed for solving the circuit.

This is where the ‘*reference direction for current*’ comes in. We assign a particular direction along the element as the reference direction for current. That is, we assume that positive charge moves through the element in the direction chosen as the reference direction even before we arrive at the circuit solution. The circuit solution will either confirm our assumption or reveal to us that actual current direction is opposite to the direction we assumed. If the circuit solution returns a positive value for the element current, then positive charge flows through the element in the assumed reference direction (or negative charge flows in opposite direction). If the circuit solution results in a negative value for the element current, then positive charge flows in a direction opposite to reference direction (or negative charge flows in the reference direction).

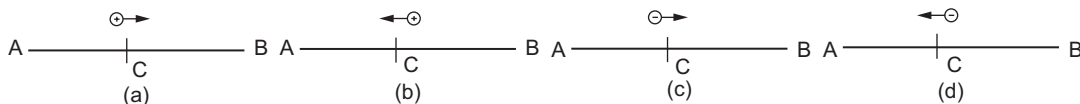


Fig. 1.6-2 Pertaining to the discussion on direction of current

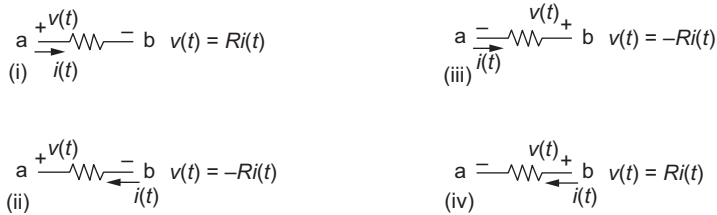


Fig. 1.6-3 Four different choices for reference directions and corresponding statements of Ohm's law

A similar issue comes up in the case of voltage across a two-terminal element. We can not determine which terminal is the higher potential terminal before we actually solve the circuit. Hence, we choose one of the two terminals to be the higher potential terminal prior to solving the circuit. The circuit solution will either confirm it by returning a positive value for element voltage or correct us by returning a negative value for that element voltage.

But, the circuit solution can reveal the correct state of affairs for an element only if the chosen reference directions for the element variables are consistent with the element voltage-current relationship that was used in solving the circuit. Consider the four different ways of selecting the reference directions for current and voltage of a two-terminal resistance as shown in Fig. 1.6-3. The correct statement for Ohm's Law is shown by the side of each choice in Fig. 1.6-3. We can use any one of these four reference direction choices provided we employ the correct statement for Ohm's Law.

However, we would like to avoid the confusion that may result from multiple choices available for reference polarities. We settle the matter once and for all by choosing one set of current and voltage reference directions for all two-terminal elements. The chosen reference directions will be as per the scheme marked as (i) in Fig. 1.6-3. *That is, we assign positive polarity of voltage variable to one of the two terminals and then assign positive current flowing into that terminal from outside. This choice of reference directions for current and voltage of a two-terminal element is called the 'Passive Sign Convention'.* See Fig. 1.6-4.

The choice shown in (iv) in Fig. 1.6-3 is also per passive sign convention. *The choice of the terminal to assign positive polarity of voltage variable is arbitrary in passive sign convention.*

The $v-i$ relation for a two-terminal resistance with passive sign convention is $v(t) = R i(t)$. It is $v(t) = L \frac{di(t)}{dt}$ for a two-terminal inductance and $i(t) = C \frac{dv(t)}{dt}$ for a two-terminal capacitance with passive sign convention.

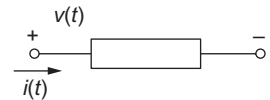


Fig. 1.6-4 Passive sign convention

1.6.2 Power and Energy in Two-Terminal Elements

Consider a general two-terminal element shown in Fig. 1.6-4. Assume that at the instant t both $v(t)$ and $i(t)$ are positive.

This implies that positive charges are flowing into the element at that instant. Positive charges moving through a voltage drop will lose their potential energy to the element. If the element is a resistance, then, the energy lost by positive charges moving from higher potential end to lower potential end will appear as heat in the resistive element. If the element is a capacitor, then, the energy

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lost by these charges will get stored in the capacitor as electrostatic energy storage. If the element is an inductance, the energy lost by charges will appear as increase in energy stored in the magnetic field in the element. If the element is a source, the energy lost by the charges will be absorbed by the source element and stored inside in some other form of energy.

Let us assume that Δq coulombs of positive charge crossed the left terminal into the element in a time interval Δt centered around the time instant t . Then, the energy lost by these charges will be $\Delta E = v(t) \times \Delta q$ Joules since 1 C passing through a voltage drop of $v(t)$ V will lose $v(t)$ Joules of energy. *Energy lost by the charge is equal to energy delivered to the element.* Therefore, the energy delivered to the element over the time interval Δt is $v(t) \times \Delta q$ Joules. The average rate at which this energy is delivered to the element is given by $v(t) \frac{\Delta q}{\Delta t}$ Joules/second. The instantaneous rate at which energy is delivered to an element is defined as the *instantaneous power delivered to the element* and is denoted by $p(t)$. Therefore,

$$p(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta E}{\Delta t} = v(t) \times \lim_{\Delta t \rightarrow 0} \frac{\Delta q}{\Delta t} = v(t) \times i(t) \text{ Joules/sec.}$$

The unit Joules/sec is given the name ‘Watts’ and denoted by ‘W’.

Thus, the instantaneous power delivered to a two-terminal element is given by $p(t) = v(t) i(t)$ Watts where $v(t)$ and $i(t)$ are the voltage across the element and current through the element as per passive sign convention.

Since instantaneous power $p(t)$ is the instantaneous rate at which energy is delivered to the element, total energy $E(t)$ that was delivered to the element from $t = -\infty$ to the current instant t is given by

$E(t) = \int_{-\infty}^t p(t) dt$. [Note: We are using the symbol $E(t)$ to denote the time-varying electromotive force of a voltage source as well as the total energy delivered to a two-terminal element. The symbol will have to be interpreted contextually.] The relation between energy delivered to a two-terminal element and power delivered to it is summarized below.

$$\begin{aligned} p(t) &= \frac{dE(t)}{dt} = v(t)i(t) \text{ W} \\ E(t) &= \int_{-\infty}^t p(t) dt = \int_{-\infty}^t v(t)i(t) dt \text{ J} \end{aligned} \tag{1.6-1}$$

Power delivered *by* a two-terminal element is obviously the negative of power delivered *to* it. Therefore,

Instantaneous power delivered by a two-terminal element = $-v(t) i(t)$ W, where $v(t)$ and $i(t)$ are instantaneous voltage and instantaneous current of the element as per passive sign convention.

A circuit can get coupled to the surroundings by electrostatic/electromagnetic coupling with other physically separate circuits in the vicinity or by mechanical, thermal or optical interaction with the environment. Consider an *isolated circuit* that has no energy coupling of any kind with the surroundings. Obviously, the total energy in that circuit has to remain constant in time. That is, the sum of energy delivered to all the elements in the circuit must remain constant. Let there be n two-terminal elements connected in such a circuit. Some of them may be electrical sources. Then,

$$E_1(t) + E_2(t) + \cdots + E_n(t) = \text{Constant}$$

Differentiating this equation both sides with respect to time, we get,

$$\frac{dE_1(t)}{dt} + \frac{dE_2(t)}{dt} + \cdots + \frac{dE_n(t)}{dt} = 0$$

But each term in this equation is nothing but the instantaneous power delivered to the corresponding two-terminal element. Therefore,

$$\sum_{\substack{\text{Over all the} \\ \text{elements in an} \\ \text{isolated circuit}}} p_i(t) = 0 \quad (1.6-2)$$

Thus, the sum of instantaneous power *delivered* to all elements in an *isolated* circuit is always zero. Or equivalently, the sum of instantaneous power *delivered* by all elements in an *isolated* circuit is always zero. This implies that total power *delivered* by the elements that deliver positive power at t must be equal to the total power *absorbed* by the elements that absorb positive power at that instant. This principle can be employed to check the solution of a circuit analysis problem.

Note that ‘power delivered to an element’ and ‘power absorbed by an element’ mean the same.

The instantaneous power delivered to a two-terminal element does not have to be positive at all instants of time. Neither does it have to be negative at all instants. It is always positive in the case of a resistance. But in all other cases, it can be positive or negative depending on the relative polarity of voltage and current in the element.

Example: 1.6-1

The current through a two-terminal element is given by $i(t) = 10(1 - e^{-1000t})$ mA for all $t \geq 0$ and $= 0$ for $t \leq 0$.

(i) Find the amount of charge that went through the element in $[0, 5 \text{ ms}]$. (ii) Find an expression for the charge that went through the element up to the time instant t . (iii) If the voltage across the element is a constant at 10 Volts and the current $i(t)$ flows out of positive terminal find and plot the energy delivered by the element as a function of time.

Solution

Refer to Fig. 1.6-5. $i(t)$ as per passive sign convention is $-10(1 - e^{-1000t})$ mA.

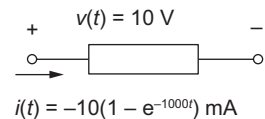


Fig. 1.6-5 The two-terminal element in Example: 1.6-1

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- (i) Charge that went through the element in a time interval $[t_1, t_2]$ is given by $\Delta q = \int_{t_1}^{t_2} i(t) dt$. Substituting the time-function for $i(t)$ and using limits $t_1 = 0$ and $t_2 = 5$ ms, we get,

$$\begin{aligned}\Delta q &= \int_0^{0.005} -10(1 - e^{-1000t}) \times 10^{-3} dt \text{ C} \\ &= -10^{-2} (t + 10^{-3} e^{-1000t}) \Big|_0^{0.005} \text{ C} \\ &= -10^{-2} [0.005 + 10^{-3} e^{-5} - 10^{-3}] \text{ C} \\ &= -40 \mu\text{C}.\end{aligned}$$

- (ii) The required expression is obtained by

$$\begin{aligned}q(t) &= \int_{-\infty}^t i(t) dt = \int_{-\infty}^0 i(t) dt + \int_0^t i(t) dt = \int_{-\infty}^0 0 dt + \int_0^t -10 \times 10^{-3} \times (1 - e^{-1000t}) dt \text{ C} \\ &= -0.01t - 10^{-5} e^{-1000t} \Big|_0^t = -0.01t - 10^{-5} e^{-1000t} + 10^{-5} \text{ C} \\ &= 10 [1 - t - e^{-t}] \mu\text{C} \text{ with } t \text{ in ms}.\end{aligned}$$

- (iii) The instantaneous power delivered to the element is $p(t) = v(t) i(t)$ where $v(t)$ and $i(t)$ are as per passive sign convention. Therefore, the power delivered by the element is given by $-v(t) i(t)$. The energy delivered by the element is obtained by integrating this quantity as below.

$$\begin{aligned}\text{Energy delivered} &= \int_{-\infty}^t -v(t) i(t) dt = \int_{-\infty}^0 -10 \times 0 dt + \int_0^t -10 \times -10(1 - e^{-1000t}) dt \text{ mJ} \\ &= (100t + 0.1e^{-1000t} - 0.1) \text{ mJ} \\ &= (0.1t + 0.1e^{-t} - 0.1) \text{ mJ} \text{ with } t \text{ in ms}.\end{aligned}$$

Example: 1.6-2

The voltage across a two-terminal element and current through it are given in Fig. 1.6-6. Passive sign convention may be assumed. Obtain the instantaneous power delivered to the element and the energy delivered to the element as functions of time.

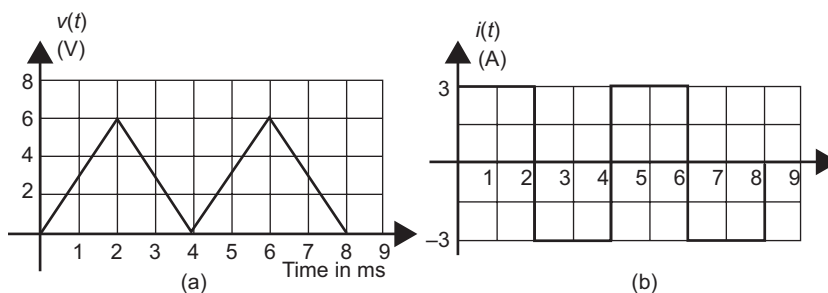


Fig. 1.6-6 Voltage and current waveform for Example: 1.6-2

Solution

Instantaneous power delivered to the element is obtained by $p(t) = v(t) i(t)$. This waveform will contain straight-line segments since the current waveform contains straight-line segments and voltage waveform is a symmetric rectangular pulse waveform. The power waveform is shown in (a) of Fig. 1.6-7.

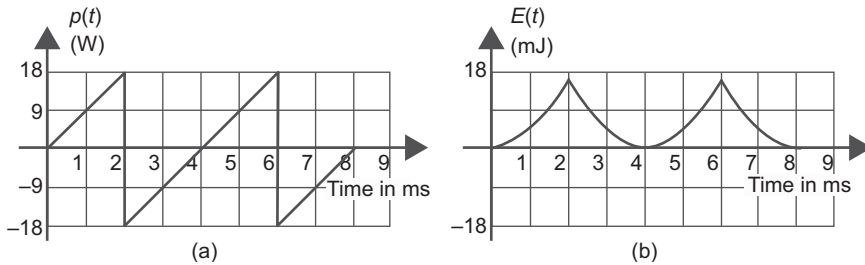


Fig. 1.6-7 (a) Waveform of instantaneous power and (b) Waveform of energy in Example: 1.6-2

The energy delivered to the element is obtained by integrating the power delivered to the element from $t = -\infty$ to $t = t$. The equation of $p(t)$ in the interval $[0, 2 \text{ ms}]$ is that of a straight-line of slope 18 W/ms . Integrating this straight-line equation results in a parabolic curve for energy in that interval. The parabolic curve reaches 18 mJ value at 2 ms (since area of the triangle in $p(t)$ curve is $18 \text{ W} \times 2 \text{ ms} \times 0.5 = 18 \text{ mJ}$.) Then $p(t)$ reverses polarity and remains negative and linear in the interval $[2 \text{ ms}, 4 \text{ ms}]$. This means that the element *delivers* power during this interval. The area of triangle in the power curve in the interval $[2 \text{ ms}, 4 \text{ ms}]$ is again 18 mJ ; but with a negative sign. Therefore, the total energy delivered to the element at the end of 4 ms period must be $18 \text{ mJ} - 18 \text{ mJ} = 0 \text{ mJ}$ and the energy curve between 2 ms and 4 ms must be parabolic again. The variation of energy delivered to the element is shown in (b) of Fig. 1.6-7.

Note that the net energy delivered to the element at the end of 8 ms is zero. The element received a total of 36 mJ of energy during the intervals $[0, 2 \text{ ms}]$ and $[4 \text{ ms}, 6 \text{ ms}]$. The element delivered a total of 36 mJ of energy during the two intervals $[2 \text{ ms}, 4 \text{ ms}]$ and $[6 \text{ ms}, 8 \text{ ms}]$.

Example: 1.6-3

In charging a storage battery, it is found that energy of 2 watt-hour is expended in 30 minutes in sending 200 C through the battery. (i) What is the terminal voltage of the battery assuming that this voltage remains constant during the charging process? (ii) What is the magnitude of average charging current?

Solution

- (i) 200 C of charge went through the battery. Energy delivered to the battery is given by $\int_0^{30 \times 60} v(t) i(t) dt$.

The battery voltage is stated to be a constant during the charging process. Let this constant voltage be V volts. Then, the energy delivered over 1800 seconds is $\int_0^{1800} V i(t) dt = V \int_0^{1800} i(t) dt = VQ$

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where Q is the charge that went through the battery in the same time interval. Therefore, $VQ = 2$ watt-hour $= 2 \times 3600$ watt-sec $= 7200$ joules. Since Q is 200 C, $V = 7200/200 = 36$ Volts.

- (ii) The average charging current is the value of a constant current that will result in same charge flow over the same time interval. Therefore, the average charging current is $200 \text{ C}/1800 \text{ sec} = 1/9$ Amps.

Example: 1.6-4

Find the current I in the direction marked in Fig. 1.6-8.

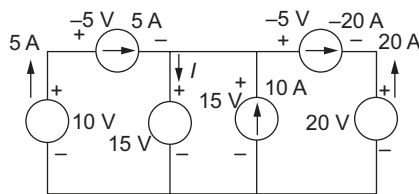


Fig. 1.6-8 Circuit for Example: 1.6-4

Solution

The sum of power delivered by all elements in an isolated circuit must be zero at all instants. Power delivered by an element in a DC circuit $= -VI$ where V and I are its voltage and current variables as per passive sign convention.

The values of V and I for 10 V source $= 10 \text{ V}$ and -5 A

$\therefore$ Power delivered by 10 V source $= 50 \text{ W}$

The values of V and I for 20 V source $= 20 \text{ V}$ and -20 A

$\therefore$ Power delivered by 20 V source $= 400 \text{ W}$

The values of V and I for 15 V source $= 15 \text{ V}$ and $I \text{ A}$

$\therefore$ Power delivered by 15 V source $= -15I \text{ W}$

The values of V and I for 5 A source $= -5 \text{ V}$ and 5 A

$\therefore$ Power delivered by 5 A source $= 25 \text{ W}$

The values of V and I for 10 A source $= 15 \text{ V}$ and -10 A

$\therefore$ Power delivered by 10 A source $= 150 \text{ W}$

The values of V and I for -20 A source $= -5 \text{ V}$ and -20 A

$\therefore$ Power delivered by -20 A source $= -100 \text{ W}$

Sum of power delivered by all elements $= (50 + 400 - 15I + 25 + 150 - 100) \text{ W}$
 $= (525 - 15I) \text{ W}$

This has to be equal to zero. Therefore, the value of I is 35 Amps.

1.7 CLASSIFICATION OF TWO-TERMINAL ELEMENTS

Circuit elements can be classified based on different criteria. Classifying elements based on the physical dimensions of the element results in two broad classes of circuits – *lumped parameter circuits* and *distributed parameter circuits*.

1.7.1 Lumped and Distributed Elements

Electromagnetic effects propagate within the circuit in the form of waves with a finite velocity. Hence, the time-variation of electromotive force taking place within electrical sources will be felt at different points in the circuit with different time delays. Therefore, the description of electrical phenomena in circuit elements, in general, will involve time and space variables. A circuit element can not be described by a unique voltage and current variable pair in that case.

However, if the circuit dimensions and element dimensions are such that the time taken by electromagnetic waves to propagate over the largest dimension in the circuit is small compared to *the characteristic time of variation* of the electromotive forces acting in the circuit, then, the retardation effect due to finite velocity of electromagnetic waves can be ignored and a simple circuit model for elements can be used.

Assume that the circuit contains many sources of sinusoidal nature and the maximum angular frequency of such source functions is ω_0 rad/sec. That is, there is some voltage or current variable of the form $X \sin(\omega_0 t)$ present in one of the sources in the circuit. Then, this variable will complete one cycle of oscillation in $2\pi/\omega_0$ seconds. The *characteristic time of variation* in this circuit is then $2\pi/\omega_0$ seconds. That is, this is a measure of the minimum time-interval over which significant changes in circuit variables will take place. Now let us assume that the largest dimension of any element in the circuit (including connecting wires) is d meters. Then, electromagnetic waves will take d/c seconds to cover this distance where c is the velocity of light in free space. If d/c is much less than $2\pi/\omega_0$, we may ignore the travel time of electromagnetic disturbances and model all the elements in the circuit by terminal voltage-current relationships. Note that this conclusion is valid only for operation at $\leq \omega_0$ rad/sec.

The ‘characteristic time of variation’ of a circuit depends on the waveshape of source functions present in the circuit. The source functions need not be sinusoidal always. However, it is possible to expand arbitrary time-functions in terms of sinusoidal functions under certain conditions. The highest frequency that appears in such expansions will have to be used to decide whether the circuit can be modeled by ignoring retardation effect. There is one kind of source function, which if present in a circuit, will not permit us to ignore retardation effect. That is a source function that contains sudden, instantaneous changes in values – that is, a function that has step discontinuities. Obviously, the characteristic time of variation of such a function is zero.

An element is classified as a *lumped element* if the net effect of electrical phenomena taking place within that element can be described in terms of only its terminal voltage and current variables, irrespective of its internal details and geometry. This amounts to neglecting the retardation effect in the element. If the electrical description of an element calls for voltage and current variables that are functions of space variables over the element (in addition to time variable), the element is called a *distributed element*.

An electrical device can be modeled by a *lumped model* only for a range of frequencies in the source functions in the circuit. The same electrical device may call for a *distributed model* if the source functions in the circuit vary rapidly enough to make retardation time within the device significant.

For instance, consider a solenoid coil of length 5 cm and diameter 1 cm with 100 turns of wire. One may be tempted to assume that the largest dimension of the coil is its length- *i.e.*, 5 cm. It is not. The largest dimension that we need here is the length of the wire and that is about 314 cm. The retardation time over this length $= 3.14/3 \times 10^8 \approx 10$ ns. If ω_0 is the highest frequency of sinusoidal components present in the sources within the circuit, then, the ‘characteristic time of variation’ is $2\pi/\omega_0$ sec. If this time is 10ns then ω_0 is 628 Mrad/sec. The corresponding cyclic frequency will be 100 MHz. Thus, this coil can be modeled as a two-terminal lumped inductance with good accuracy if the circuit contains

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source sinusoidal components at 1 MHz or below. However, it will call for a distributed model if the sources contain > 10 MHz sinusoidal components.

Consider a power transmission line of length 300 km. The retardation time over the length of the line is 1 ms. 50 Hz sinusoidal source functions have a waveform period of 20 ms. Hence, a lumped parameter model for this line amounts to ignoring 1 ms in comparison with 20 ms. But 20 ms is the time required for one full oscillation of source function. Significant change in function value takes place within a quarter cycle – *i.e.*, in 5 ms. Obviously, this power line requires a distributed model even at 50 Hz.

A 1 nF ceramic capacitor typically has two leads of 1.5 cm each. The retardation time over 3 cm is 100 ps ($1 \text{ ps} = 10^{-12} \text{ sec}$). This corresponds to a frequency value of 10 GHz. Therefore, a lumped parameter model will be satisfactory for frequencies below 10 MHz. A distributed model will be necessary for frequencies > 50 MHz.

All circuit elements of arbitrary dimensions can be modeled by lumped elements if all the sources are DC sources. But, no element, of any dimension whatsoever, can be modeled by lumped parameter model to obtain *detailed* circuit solution at and around the instants at which such DC sources are either switched into the circuit or switched out of the circuit. Such switching operations represent very rapid changes in circuit variables and retardation time can not be ignored in comparison with infinitesimal intervals.

1.7.2 Linear and Non-Linear Elements

Two-terminal elements are classified as *linear* or *nonlinear* based on whether the voltage–current relationship of the element satisfies the *linearity property*.

Two variables of time – $x(t)$ and $y(t)$ – satisfy the *property of linearity* if the relation between them is *homogeneous* and *additive* at all t .

Let the relation between the variables be represented by $y(t) = f[x(t)]$.

$y(t) = f[x(t)]$ is *homogeneous* if $f[ax(t)] = af[x(t)]$ for any t where a is any real number. That is, scaling the variable $x(t)$ by a real number a results in the scaling of the variable $y(t)$ by the same real number a .

$y(t) = f[x(t)]$ is *additive* if $f[x_1(t) + x_2(t)] = f[x_1(t)] + f[x_2(t)]$ for any t . That is, $y(t)$ corresponding to sum of two variables $x_1(t)$ and $x_2(t)$ is equal to the sum of $y(t)$ corresponding to $x_1(t)$ and $y(t)$ corresponding to $x_2(t)$ at any time instant.

We may combine the requirements of *homogeneity* and *additivity* into a composite requirement called *superposition property*.

$y(t) = f[x(t)]$ satisfies *superposition property* if

$f[a_1x_1(t) + a_2x_2(t)] = a_1f[x_1(t)] + a_2f[x_2(t)]$ for any combination of real numbers a_1 and a_2 and for any t .

Thus, a two-terminal element is linear if its $v-i$ relationship satisfies the principle of superposition.

The simplest case of a *linear* relationship between two variables occurs when $y(t)$ is proportional to $x(t)$.

Let $y(t) = kx(t)$ where k is a real number.

Then, $f[a_1x_1(t) + a_2x_2(t)] = k \times [a_1x_1(t) + a_2x_2(t)] = ka_1x_1(t) + ka_2x_2(t) = a_1 \times kx_1(t) + a_2 \times kx_2(t) = a_1f[x_1(t)] + a_2f[x_2(t)]$.

Therefore, $y(t) = kx(t)$ is a *linear relation* for any real k .

But, a relation does not have to be algebraic for it to be a linear relation. Consider the relation

$y(t) = k \frac{dx(t)}{dt}$. Then,

$$\begin{aligned}
 f[a_1x_1(t) + a_2x_2(t)] &= k \times \left(\frac{d(a_1x_1(t) + a_2x_2(t))}{dt} \right) \\
 &= a_1 \left(k \frac{dx_1(t)}{dt} \right) + a_2 \left(k \frac{dx_2(t)}{dt} \right) \\
 &= a_1 f[x_1(t)] + a_2 f[x_2(t)]
 \end{aligned}$$

Therefore, $y(t) = k \frac{dx(t)}{dt}$ is a *linear relation*.

Similarly, it can be shown that $y(t) = k \int_{-\infty}^t x(t)dt$ too is a *linear relation*.

Beginners in Circuit Analysis often tend to equate the property of linearity to straight-line nature of functional relationship between the concerned variables. Consider the following relationship.

$y(t) = mx(t) + c$ where m and c are two real numbers. Obviously, the graph of this function will be a straight-line with c as its vertical-axis intercept. But this is not a *linear relation* in the sense of linearity as defined in Circuit Theory.

$$\begin{aligned}
 f[a_1x_1(t) + a_2x_2(t)] &= m[a_1x_1(t) + a_2x_2(t)] + c \\
 &= a_1mx_1(t) + a_2mx_2(t) + c \\
 a_1f[x_1(t)] + a_2f[x_2(t)] &= a_1mx_1(t) + a_2mx_2(t) + (a_1 + a_2)c \\
 &\neq f[a_1x_1(t) + a_2x_2(t)]
 \end{aligned}$$

Therefore, $y(t) = mx(t) + c$ is not a *linear relation* in Circuit Theory. It does not satisfy the property of homogeneity. It does not satisfy the property of additivity too.

Let us examine the linearity property of various two-terminal elements we have discussed so far.

Consider a two-terminal resistance element. Its $v-i$ relation is $v(t) = Ri(t)$. It is a linear element if the R parameter is a real constant or a function of time alone. The resistance of a piece of conductor is temperature dependent. It may depend on current level in certain cases. Thus a two-terminal resistance is linear if the temperature is constant and the R parameter is either a constant or is an independent function of time alone.

A two-terminal inductance is described by $v(t) = \frac{d[Li(t)]}{dt}$ in general. If the inductance parameter L is a constant, then two-terminal inductance is a linear element. L can vary with time if the physical geometry of the device changes with time (but independent of electrical variables). The element is linear in that case too. But if L varies as a function of the current in it, then, the element is a nonlinear one.

A two-terminal capacitance is described by $i(t) = \frac{d[Cv(t)]}{dt}$ in general. If the inductance parameter C is a constant, then two-terminal inductance is a linear element. C can vary with time if the physical geometry of the device changes with time (but independent of electrical variables). The element is linear in that case too. A tuning capacitor in a radio receiver is an example. But if C varies as a function of the charge in it, then, the element is a nonlinear one.

A two-terminal ideal independent voltage source is described by the relations $v(t) = E(t)$ (an independently specified function of time) and $i(t) = \text{arbitrary}$. Obviously, this is a non-linear relationship. Thus an ideal independent voltage source is a nonlinear element. Similarly, an ideal independent current source is a nonlinear element.

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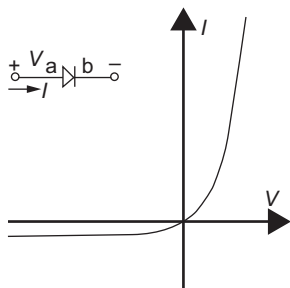


Fig. 1.7-1 Voltage–current relationship

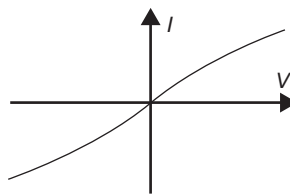


Fig. 1.7-2 $V - I$ Curve for a nonlinear, bilateral resistance of a diode – non-bilateral

1.7.3 Bilateral and Non-Bilateral Elements

Some elements have a $v - i$ relation that depends on the direction of current flow in them. A diode is an example. See Fig. 1.7-1. The $v - i$ relation of this two-terminal element is not symmetrical about the vertical axis. This is a non-bilateral element. The current that will flow in the device when it is connected across a battery of V volts will depend on how it is connected. If the terminal marked a is connected to positive terminal of the battery the resulting current will be large. If the terminal marked b is connected to positive terminal of the battery, the resulting current flow will be small.

An element with a voltage–current relation that is odd-symmetric about the vertical axis in the $v - i$ plane is called a bilateral element. A linear resistor is a bilateral element. A linear inductor is a bilateral element. A linear capacitor is bilateral.

A linear two-terminal element will always be bilateral too. However, a multi-terminal element (with more than two terminals) can be non-bilateral even if it is a linear element.

A two-terminal element is non-linear if it is non-bilateral. However, the converse is not true. Consider the $v - i$ curve of a nonlinear resistor shown in Fig. 1.7-2. The element is nonlinear; but bilateral.

1.7.4 Passive and Active Elements

The energy delivered to a two-terminal element from $t = -\infty$ to $t = t$ is given by $E(t) = \int_{-\infty}^t v(t)i(t)dt$. An element is called a *passive element* if the energy delivered to it is always non-negative for any t and for any possible terminal voltage – current conditions of the device. That is, an element is passive if $E(t) \geq 0$ for all t and for all permissible $(v(t), i(t))$ combinations.

Consider a linear resistance with constant R . Then,

$$E(t) = \int_{-\infty}^t v(t)i(t)dt = R \int_{-\infty}^t [i(t)]^2 dt = \text{a positive number for any } t.$$

Therefore, a resistance is a passive element. Consider a linear inductance element. Then,

$$E(t) = \int_{-\infty}^t v(t)i(t)dt = L \int_{-\infty}^t [i(t)] \frac{di(t)}{dt} dt = L \int_0^{i(t)} di(t) = \frac{L[i(t)]^2}{2} = \text{a positive number or zero. (We assume}$$

that L is positive). This energy is stored in the magnetic field inside the device. We have assumed that the

energy storage inside the inductor at $t = -\infty$ is zero. Thus, an inductor is a passive element. Similarly, it may be shown that $E(t) = \frac{C[v(t)]^2}{2}$ for a capacitor and that a capacitor is also a passive element.

A positive-valued resistance is a dissipating element. It can not deliver energy even for a short interval. This can be seen from the equation for power delivered to the resistor. $p(t) = v(t) i(t) = R [i(t)]^2$ = a positive number or zero. Therefore, the power delivered to a resistance can not be negative at any t and hence resistance will always consume power.

A positive-valued inductance or capacitance is an energy-storing element. It can deliver the energy back to other elements. But it can deliver only as much energy as that was given to it earlier. It can not generate energy and deliver it. Thus, the energy storage in an inductance or capacitance can not be employed to deliver energy to other elements indefinitely. The instantaneous power delivered to such an element can be negative; but the area under $p(t)$ waveform from $-\infty$ to any t will be non-negative. In other words, the energy storage in an inductance or in a capacitance can only be zero or positive at any instant.

An independent source is an active element (since it is not a passive element). An independent voltage source can *deliver* any amount of current for any duration. Similarly, an independent current source too can do that. Therefore, independent sources are *active elements*.

1.7.5 Time-Invariant and Time-Variant Elements

An element is ‘time-invariant’ if the values of parameters that characterize it are independent of time. Therefore, a two-terminal resistance is time-invariant if R is a constant; a two-terminal inductance is time-invariant if L is a constant and a two-terminal capacitance is time-invariant if C is a constant.

A synchronous generator driven at constant speed by some prime-mover is an example system that contains time-variant elements. The inductance value of various coils in the machine varies with time due to rotation.

In this book, we deal only with circuits comprising a *finite* number of ‘lumped, linear, bilateral, time-invariant’ two-terminal as well as multi-terminal elements interconnected.

1.8 MULTI-TERMINAL CIRCUIT ELEMENTS

Many electrical devices that are characterized by voltage and current variables at more than two terminals are in common use in Electrical engineering. Transformers in power engineering, transistors in electronics engineering etc., are some examples.

1.8.1 Ideal Dependent Sources

Ideal dependent sources form a category of multi-terminal elements that we employ in circuit analysis. These models are used extensively in analysis of electronic circuits to model devices like transistors, amplifiers etc.

They have two terminal pairs. The first terminal pair *senses* either a voltage variable or a current variable at the location where this terminal pair is connected in the circuit. The second pair of

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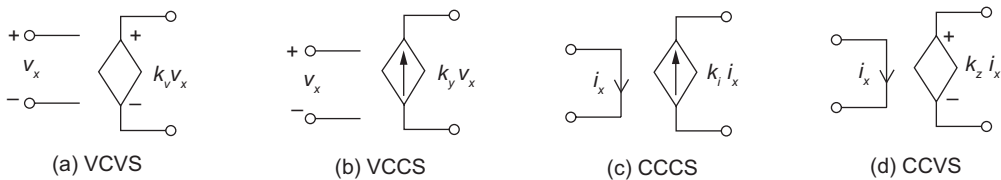


Fig. 1.8-1 Ideal dependent sources

terminals delivers either a voltage or a current to the location at which this terminal pair is connected in the circuit. However, the source function delivered is a function of the variable sensed by the first terminal pair. That is why they are called *dependent sources*. They are ideal in the sense that (i) the first terminal pair does not affect the circuit variables in any way (ii) the source function delivered by second terminal pair depends only on the variable sensed by the first terminal pair and on nothing else.

There are four dependent sources depending on the nature of circuit variable sensed by the first terminal pair and the nature of source function delivered by the second terminal pair.

A *Voltage-Controlled Voltage-Source (VCVS)* senses a voltage variable at some location in the circuit and delivers a source voltage that depends on the sensed voltage at some other location in the circuit. If the source voltage delivered is a linear function of the controlling voltage, the dependent source will be called a *linear VCVS*.

A *Voltage-Controlled Current-Source (VCCS)* senses a voltage variable at some location in the circuit and delivers a source current that depends on the sensed voltage at some other location in the circuit. If the source current delivered is a linear function of the controlling voltage, the dependent source will be called a *linear VCCS*.

A *Current-Controlled Current-Source (CCCS)* senses a current variable at some location in the circuit and delivers a source current that depends on the sensed current at some other location in the circuit. If the source current delivered is a linear function of the controlling current, the dependent source will be called a *linear CCCS*.

A *Current-Controlled Voltage-Source (CCVS)* senses a current variable at some location in the circuit and delivers a source voltage that depends on the sensed current at some other location in the circuit. If the source voltage delivered is a linear function of the controlling current, the dependent source will be called a *linear CCVS*.

The symbols used for the four *linear dependent sources* are shown in Fig. 1.8-1. k_v , k_i , g_m and r_m are real numbers. k_y is a trans-conductance, k_z is a trans-resistance, k_v and k_i are dimensionless.

1.9 SUMMARY

- *Electric Circuit* is a mathematical model of a real physical electrical system. It is an approximation of Electromagnetic Field Theory.
- Electromagnetic disturbances travel with a finite speed in electrical systems. Electric circuit theory assumes that the largest dimension in the circuit is so small that electromagnetic disturbances take negligible time to cover that distance compared to the time interval required for source quantities in the circuit to change significantly.

- In addition, *lumped parameter circuit theory* assumes the following:
 - a. The electrostatic field created by charge distribution on an electrical device is confined to space within the device and in the immediate vicinity of the device predominantly. Thus, the terminal voltage across a device and the charge stored in that device can be related through a unique ratio.
 - b. The induced electric field inside connecting wires and outside the devices is negligible. This makes it possible to assign a unique voltage variable to a device.
 - c. The connecting wires are of infinite conductivity and near-zero thickness. Thus, there are no charges distributed on their surface and no current component is needed to create charge distribution on them. This makes it possible to assign a unique current variable to an electrical device.
 - d. The component of current needed to create the time-varying charge distribution on the surface of the device is negligible except in those devices (capacitors) that are designed to make such current flow the dominant electrical phenomenon in them.
 - e. The induced electric field inside devices is negligible except in those devices (inductors) where induced electric field is the dominant electrical phenomenon by design. Such devices are designed to confine the time-varying flux linkage to space within them.
 - f. The conductivity of metallic conductors employed in capacitors and inductors is infinity. Hence, there is no resistive effect in them.
- A two-terminal element is a mathematical model of an electrical device with a terminal voltage variable $v(t)$ and current variable $i(t)$ assigned to it. These are functions of time only. The entire electrical behaviour of the device can be characterized by these two variables and a relation between them.
- Passive sign convention assigns reference polarity for $v(t)$ with + at one end and – at other end of the element. Reference direction for $i(t)$ is such that it flows into the + polarity of $v(t)$ *from outside the element*. The power delivered to the element is given by $p(t) = v(t) i(t)$ with this sign convention.
- The symbols and $v - i$ relations for the three passive two-terminal elements are shown in Fig. 1.9-1.
- Ideal independent voltage source is a two-terminal element with its voltage variable specified as a function of time and current variable as a free variable.
- Ideal independent current source is a two-terminal element with its current variable specified as a function of time and voltage variable as a free variable.
- Circuit elements are classified into *linear* and *nonlinear* elements depending on whether their $v - i$ relationship satisfy the *principle of superposition* or not. Independent sources are nonlinear elements.
- An element is a *passive element* if the energy delivered to it from $-\infty$ to t is non-negative for all t and for all permissible $(v(t), i(t))$ combinations. R , L and C are passive elements.
- A circuit element is a *time-invariant element* if the parameter of the element is a constant.
- Ideal dependent sources are four-terminal elements. The first terminal pair is connected at some location in the circuit to sense a voltage variable or a current variable there. The second terminal pair delivers either a voltage or a current at the location where it is connected. The value of voltage or current delivered is a function of *controlling variable* that is sensed by the first terminal pair.

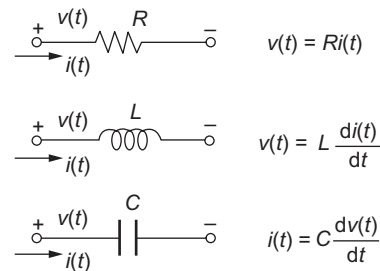


Fig. 1.9-1 Passive two-terminal elements

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- There are four kinds of dependent sources. They are Voltage-Controlled Voltage Source, Voltage-Controlled Current Source, Current-Controlled Voltage Source and Current-Controlled Current Source. A dependent source is a *linear* element if the source quantity is a linear function of the controlling-variable.

1.10 PROBLEMS

1. A fully charged lead-acid battery contains 120 Ampere-hours of charge in it. The terminal voltage of the battery is a function of charge remaining in it and is given by $V = 11(1 + 0.001C)$ V, where C represents the charge that remains in the battery in Ampere-hour units. The battery delivers a current of 10 A to a load. (a) Express the initial charge storage in the battery in Coulombs. (b) What is the terminal voltage of the battery when it has been discharged to 50% level? (c) A voltage-sensing relay cuts out the battery from the load when the terminal voltage falls below 11.35 V. How long can the battery power the 10A load? (d) What is the energy delivered by the battery to 10A load in Joules and kW-h units if the load is kept powered till cut out takes place?
2. A fully charged battery contains 100 AH charge in it. The terminal voltage of battery is a function of the charge that remains in it and the current that is delivered by it. It is given as $V = 11 + 0.001C - 0.02I$ V, where C is the AH (Ampere-hours) remaining in it and I is the current delivered by the battery in Amps. This battery is delivering current to a pulsed load that draws current with a period of 5 sec as shown in Fig. 1.10-1. The load is kept powered for 8 hours. (a) What is the charge that remains in the battery after the load is switched off? (b) What is the value of constant load current that would have resulted in same charge consumption? (c) What is the energy consumed by the load (in Joules and kW-h) in 8 hours? (d) What is the value of constant load current that would have consumed same energy in 8 hours? What would have been the charge consumption with this value of load current?

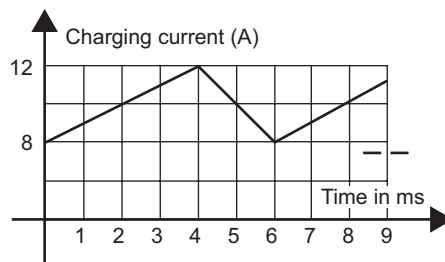


Fig. 1.10-1

3. The waveshape of current that is delivered to a 12 V lead-acid battery to charge it is shown in Fig. 1.10-2. The initial charge in the battery is 36000 C. The voltage across the battery is given by $V = 11.5 + 0.01C$ V where C is the charge stored in the battery in AH units. The charging current is applied to the battery for 5 hours. (a) Express the initial charge in the battery in AH. (b) What is the charge stored in the battery at the end of charging? (c) What is the constant charging current value that would have delivered the same amount of charge to the battery in the same time interval? (d) How much is the energy consumed in charging? (Hint: Assume that the battery voltage remains constant during a cycle of charging current.)

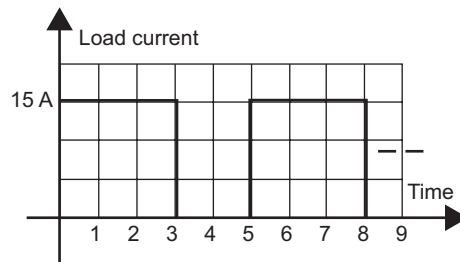


Fig. 1.10-2

4. The current through a two-terminal element is given by

$$i(t) = \begin{cases} k + 2 \cos 100\pi t + 0.5 \sin 200\pi t \text{ A for } 0 \leq t \leq 0.035 \text{ s} \\ 0 \text{ for all other } t \end{cases}$$

that went through the element in the interval $[0, 0.05 \text{ s}]$ is zero.

5. The voltage across an ideal two-terminal passive element is $v(t) = 10e^{-100t} \text{ V}$ for $t \geq 0$ and zero for $t < 0$. The current through the element is $i(t) = 0.1e^{-100t} \text{ A}$ for $t \geq 0$ and zero for $t < 0$. (a) Identify the element and its parameter value. (b) What is the amount of charge that went through the element in the time interval $[0.01 \text{ sec}, 0.05 \text{ sec}]$? (c) What is the amount of charge that went through the element in $[0, \infty]$ time-interval? (d) What is the ratio of instantaneous power delivered to it at 0.01 sec to the corresponding value at 0 sec? (e) What is the total energy delivered to the element? (f) What is the time at which the energy delivered to it reached 99% of total energy delivered to it? Assume that $v(t)$ and $i(t)$ given are as per passive sign convention.
6. The voltage across an ideal two-terminal passive element is $v(t) = 10(1 - e^{-1000t}) \text{ V}$ for $t \geq 0$ and zero for $t < 0$. The current through the element is $i(t) = 0.001e^{-1000t} \text{ A}$ for $t \geq 0$ and zero for $t < 0$. (a) Identify the element and its parameter value. (b) What is the amount of charge that went through the element in $[0, \infty]$ time-interval? (c) What is the total energy delivered to the element? (d) What is the time instant at which the power delivered to the element is a maximum? What is the value of this maximum power? What is the value of energy delivered to the element till that instant? Assume that $v(t)$ and $i(t)$ given are as per passive sign convention.
7. The voltage across an ideal two-terminal passive element is $v(t) = 600 \cos(100t) \text{ V}$ for $t \geq 0$ and zero for $t < 0$. The current through that element as per passive sign convention is $i(t) = 10 \sin(100t) \text{ A}$ for $t \geq 0$ and zero for $t < 0$. (a) Identify the element and its parameter value. (b) Find an expression for the charge that goes through the element as a function of time. (c) Find an expression for instantaneous power delivered to the element as a function of time. (d) Show that the energy delivered to the element till t is non-negative for all t .
8. The current that flows through an ideal independent voltage source with $v(t) = 12 \text{ V}$ is $i(t) = 10 + 10 \cos 100\pi t \text{ A}$ for $t \geq 0$ and 0 A for $t < 0$. Assume passive sign convention. (a) What is the power delivered by the source at $t = 1 \text{ sec}$? (b) What is the change in energy storage in the source between $t = 0$ and $t = 1 \text{ sec}$? Does the energy storage in source increase with time or decrease with time?

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9. The voltage across a two-terminal element is $v(t) = 10 \sin 1000 \pi t$ V. The current that flows into the element as per passive sign convention is

$$i(t) = \begin{cases} k + 1.5 \cos(100\pi t + \frac{\pi}{3}) \text{ A} & \text{for } 0 \leq t \leq 0.05 \text{ s} \\ 0 \text{ A} & \text{for all other } t \end{cases}$$

Find k if the energy delivered to the element in the time interval $[0, 0.1 \text{ s}]$ is zero.

10. The voltage across a two-terminal element is $v(t) = 10 \sin 1000 \pi t$ V and current in that element is

$$i(t) = \begin{cases} k + 1.5 \cos(100\pi t + \frac{\pi}{3}) \text{ A} & \text{for } 0 \leq t \leq 0.05 \text{ s} \\ 0 \text{ A} & \text{for all other } t \end{cases}$$

Assume passive sign convention. Find the average power delivered to the element over any time interval of width equal to the period of the voltage and current waveforms.

11. The voltage across a two-terminal resistor is $v(t) = 10 \sin 1000 \pi t$ V. Find the value of a DC voltage that will deliver a power that is equal to the average power delivered by this voltage source over any time interval equal to the period of the voltage waveform.
12. The value of resistance of a resistor is measured to be 10Ω at room temperature of 35°C . Temperature coefficient of this resistance is 0.004. A constant current source of 0.25 A is connected across the resistance. The resistance attains a steady temperature after some time. The *temperature rise* in the resistor after the temperature has reached a steady-state is given by $100p$ where p is the power dissipated in the resistor in Watts. (i) Find the steady-state temperature, corresponding resistance value and the power dissipated in the resistor under steady-state condition. (ii) Find the critical value of current at which the temperature of the resistor increases without any limit and it burns out.
13. A DC voltage source of 2.5 V is connected across the resistor in Problem 12. (i) Find the steady-state temperature, corresponding resistance value and the power dissipated in the resistor under steady-state condition. (ii) Find the critical value of applied DC voltage (if such a value exists) at which the temperature of the resistor increases without any limit and it burns out.
14. There are only three elements in an isolated circuit. Assume passive sign convention. The terminal voltage and current of first element are given by

$$v_1(t) = \begin{cases} 5 + 5(1 - e^{-100t}) \text{ V} & \text{for } t \geq 0 \\ 0 \text{ V} & \text{for } t < 0 \end{cases} \quad \text{and} \quad i_1(t) = \begin{cases} e^{-100t} \text{ A} & \text{for } t \geq 0 \\ 0 \text{ A} & \text{for } t < 0 \end{cases}$$

Corresponding variables for the second element are $v_2(t) = v_1(t)$ and $i_2(t) = -2 \text{ A}$. The voltage across the third element is $v_3(t) = v_1(t)$. Identify the third element assuming that it is a passive element, find its parameter value and the current through the third element as a function of time. [Hint: Sum of power delivered by all elements in a circuit is zero.]

15. The $v - i$ characteristic of a passive two-terminal element as per passive sign convention is $v(t) = 100i(t) + 20i(t)|i(t)|$ V. (a) Show that this element is nonlinear. (b) Show that this element is a passive element. (c) Show that it is a bilateral element. (d) Find the current flow through the element when the voltage across it is a constant at 100 V. There are two possible values for the current. How do you choose the correct one?

16. The current $i(t)$ through a passive two-terminal element is a single pulse as shown in Fig. 1-10-3. Plot the voltage across the element if the device is (a) a resistance of $10\ \Omega$ (b) an inductance of $0.5\ \text{H}$ with zero initial energy storage and (c) a capacitance of $10000\ \mu\text{F}$ with zero initial energy storage.

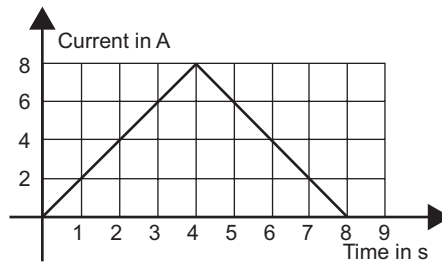


Fig. 1.10-3

17. A $1000\ \mu\text{F}$ two-terminal linear capacitor had a charge storage of $10\ \text{mC}$ across it at $t = 0$. The current delivered by the capacitor out of its positive terminal is given by $i(t) = 2 \cos(1000\pi) \text{ A}$ for $t \geq 0$. Find and plot the voltage across the capacitor terminals as a function of time.
18. The voltage across a $0.2\ \text{F}$ capacitor was 10V at $t = 0$. A current source $i_s(t) = Ae^t \text{ A}$ was applied across the capacitor during the time interval $[0, 1\ \text{s}]$ with $i_s(t)$ flowing into the positive plate. Find A if the voltage across capacitor is seen to be 0V at $t = 1\ \text{s}$.
19. A resistor of $10\ \Omega$ and a capacitor of $0.5\ \text{F}$ are connected in parallel across a time-varying voltage source. The current drawn by the resistor and the current drawn by the capacitor are seen to be equal at all t . The current delivered by the voltage source at $t = 0\ \text{s}$ is found to be 6A . Find the current through the resistor and the capacitor at $t = 1\ \text{s}$.
20. The voltage across a $0.2\ \text{H}$ two-terminal inductance is $v(t) = 10e^{-10t} \text{ V}$ for $t \geq 0$. It was kept shorted for $t < 0$ with $i(t) = 0.5\ \text{A}$ circulating in it. Assume passive sign convention and find out the current in the inductance, flux linkage in it and energy storage in it at $t = 0.5\ \text{sec}$.
21. A current source with $i(t) = 5 \sin \omega t \text{ A}$ flows through series combination of $4\ \text{H}$ inductor and $1\ \text{F}$ capacitor. The voltage across the current source is seen to be $0\ \text{V}$ for all t . What is the value of ω ?
22. (a) List the voltage and current values for all the elements in the circuit in Fig. 1.10-4 as per passive sign convention. (b) Find the unknown voltage V_x . (c) The circuit is known to be a DC circuit. Can the nature of the two-terminal element across which V_x be identified?

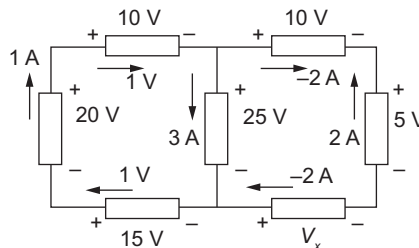


Fig. 1.10-4

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23. An isolated circuit contains four elements. The $v - i$ values at a particular time instant for three of them as per passive sign convention are (5 V, 2 A), (15 V, 1 A) and (10 V, 2 A). The voltage across the fourth element at the same instant is seen to be 15V. (a) Find all possible $v - i$ value combinations for the fourth element. (b) If the circuit is known to be a DC circuit that has been in the present state for a long time, identify whether the fourth element is a passive element or active element. (c) Can the nature of fourth element be identified if the circuit is known to be a circuit with time-varying voltages and currents?
24. The voltage that appears across two terminals of a circuit is $v(t) = [10 + 5 \sin 100\pi t]$ V for all t . Current delivered to the circuit (as per passive sign convention) is seen to be

$$i(t) = \left\{ 1 + 6 \sin \left(100\pi t - \frac{3\pi}{4} \right) \right\} \text{A}$$

Are all the elements in the circuit passive elements?

25. A circuit is known to contain only linear resistances. Two terminals are brought out from this circuit and a voltage source with $v_s(t) = [10 + 10\sin t]$ is connected across the terminals. If the current into the circuit is $i(t) = [2 + A\sin(t - \theta)]$ A find A and θ .

Chapter 2

Basic Circuit Laws

CHAPTER OBJECTIVES

- To state and explain Kirchhoff's Current Law and Kirchhoff's Voltage Law.
- To explain the different ways of stating these laws.
- To illustrate the application of these laws in Circuit Analysis by examples.
- To introduce Operational Amplifier and to illustrate application of Kirchhoff's laws in analysis of Operational Amplifier circuits.

INTRODUCTION

The electrical devices in an electrical system under quasi-static conditions are modelled by ideal two-terminal elements or multi-terminal elements. The 'circuit model' of the system is obtained by interconnecting the elements by means of connecting wires that are assumed to be of infinite conductivity and near-zero cross-sectional area. Interconnecting electrical elements into a 'circuit' will result in 'junctions' at which the connecting wire-ends of two or more two-terminal elements or multi-terminal elements will join together. Such junctions are called *nodes* in Circuit Analysis. Further, interconnection of elements will result in one or more closed paths involving two or more elements. Such closed paths comprising elements and nodes are called *loops* in Circuit Analysis.

Each two-terminal element is completely described by two variables – one terminal voltage variable $v(t)$ and one element current variable $i(t)$. Passive sign convention is assumed in assigning reference directions for these variables. The *element relations* of two-terminal elements are known. The relation is in the form of an equation relating $v(t)$ and $i(t)$ in the case of passive elements. It is in the form of constraints on $v(t)$ or $i(t)$ in the case of ideal independent sources.

Each four-terminal element will be described by two voltage variables and two current variables – one voltage variable and one current variable per terminal pair. Two relations tying up these variables or constraining them will be available for each such four-terminal element. For instance, consider a voltage-controlled voltage source. There is an open-circuit across its first terminal pair and there is a voltage source across its second terminal pair.

2.2 Basic Circuit Laws

We treat each four-terminal element as two two-terminal elements with some relation between their voltage and current variables, as far as element count in a circuit is concerned. Let there be b -elements, n -nodes and l -loops in a lumped parameter circuit. Then there are $2b$ variables – b terminal voltage variables and b element current variables – to be solved for in the circuit. We call these variables the *element variables*.

Each element contributes either an equation relating its voltage variable to its current variable or a constraint equation which imposes a constraint on either its current variable or voltage variable. Thus, we get b equations in $2b$ variables from *element relations* alone. These equations are independent of the manner in which the circuit elements are interconnected. They depend only on the nature and parameter value of the individual elements. We call this set of b equations involving $2b$ element variables the *element equation set*.

We need another set of b independent equations on $2b$ element variables to solve for all the element variables. These equations will have to be independent of the *element equation set*. They come from the interconnection details of the circuit. *They depend only on how the elements are interconnected and will not depend on the nature or parameter value of elements*. That is, they depend only on the *topology* of the circuit. This set of b independent equations that summarises the constraints imposed on $2b$ element variables by the interconnection is called the *interconnection equation set*. ‘Element equation set’ and ‘Interconnection equation set’ provide the complete set of equations needed to solve for all the element variables in a circuit.

The interconnection equation set is obtained by applying two basic conservation laws of physics to the circuit. The laws of conservation of energy and charge have been restated in a form suitable for applying to lumped parameter circuits. Gustav Robert Kirchhoff arrived at the required restatements of these conservation laws in 1857 and they are called *Kirchhoff’s Voltage Law (KVL)* and *Kirchhoff’s Current Law (KCL)*.

Kirchhoff’s Voltage Law imposes a constraint on the voltage variables appearing in a loop in the circuit. Applying this law to a loop in the circuit results in a single constraint equation involving an algebraic sum of all the voltage variables that appear in the loop. Kirchhoff’s Current Law imposes a constraint on the current variables appearing at a node in the circuit. Applying this law to a node in the circuit results in a single constraint equation involving an algebraic sum of all the current variables that appear at the node. These constraint equations are algebraic in form.

2.1 KIRCHHOFF’S VOLTAGE LAW (KVL)

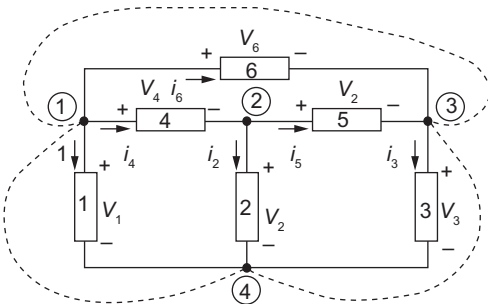


Fig. 2.1-1 A DC circuit with 6-elements, 4-nodes and 7-loops

Consider a DC circuit with many loops as shown in Fig. 2.1-1. Six two-terminal elements are interconnected in this 4-node circuit. The interconnection results in 7 loops in the circuit. The loops are 1–4–2, 2–5–3, 4–6–5, 1–6–3, 1–4–5–3, 2–4–6–3 and 1–6–5–2. The elements are numbered and encircled numbers label the nodes.

The circuit is assumed to be in DC steady-state. That is, all the sources in the circuit are assumed to be constants and all the circuit variables are assumed to be constants in time.

Thus, the charge distribution at the terminals and on the surface of each two-terminal element in the circuit is steady in time. The charge distribution

produces electrostatic field everywhere. The electrostatic field generated within an element and in the immediate vicinity of an element is proportional to the charge stored on that element. (This is a standard assumption in lumped parameter circuit theory as pointed out in Chap. 1.) The voltage variables marked in Fig. 2.1-1 are the electrostatic potential differences that exist between the terminals of elements. The connecting wires have zero resistance. Moreover, there is no charge distribution on the surface of connecting wires.

Imagine that we are carrying a unit positive test charge from node-4 back to the same node by moving it along the path shown by dotted curve in Fig. 2.1-1 in the counter-clockwise direction. The path of travel touches node-1 and node-3. Electrostatic field is a conservative field. Only electrostatic field is present at points lying on the path of travel of unit positive test charge. Therefore, the work to be done in moving the unit positive test charge around this closed path must be zero. v_1 J is the work to be done in moving a unit positive test charge from node-4 to node-1. v_2 J is the work to be done in moving a unit positive test charge from node-3 to node-1. And, v_3 J is the work to be done in moving a unit positive test charge from node-4 to node-3. Therefore, the work to be done in moving a unit positive test charge from node-4 to node-4 by moving in the dotted path in counter-clockwise direction = (The work to be done in moving a unit positive test charge from node-4 to node-1) + (The work to be done in moving a unit positive test charge from node-1 to node-3) + (The work to be done in moving a unit positive test charge from node-3 to node-4) = $v_1 - v_6 - v_3$. This has to be zero. Therefore, conservative nature of electrostatic field leads to the following equation involving the three voltage variables appearing in the loop formed by element-1, element-6 and element-3:

$$v_1 - v_6 - v_3 = 0 \quad (2.1-1)$$

If we had taken a unit positive test charge around the same path in clockwise direction, we would have obtained the following equation:

$$v_3 + v_6 - v_1 = 0 \quad (2.1-2)$$

Obviously, Eqn. 2.1-2 must be Eqn. 2.1-1 multiplied by -1 . Thus, the direction of traverse of the loop does not matter when we prepare the equation involving voltage variables appearing in that loop.

Now we dispense with the dotted path altogether. Instead, we traverse the loop formed by element-1, element-6 and element-3 in the counter-clockwise direction, starting from node-4. We collect the *voltage rise amounts across each element* as we go along and enter these quantities into a sum. Obviously, in this process we are calculating the total work to be done in carrying a unit positive test charge through a path outside the elements, but touching the nodes. Therefore this sum must be zero. The *voltage rise* across the element-1 in the direction of traverse (counter-clockwise direction) is v_1 . The *voltage rise* across the element-6 in the direction of traverse is $-v_6$. The *voltage rise* across the element-3 in the direction of traverse is $-v_3$. Therefore, the sum of voltage rises encountered in traversing the loop formed by element-1, element-6 and element-3 in the counter-clockwise direction is $v_1 - v_6 - v_3$. We have already verified that this sum must be equal to zero due to conservative nature of electrostatic field.

If we collect the *voltage drop amounts across each element* as we traverse the loop in counter-clockwise direction and enter them in a sum, we get $-v_1 + v_6 + v_3$. This sum is equal to zero since this is the work to be done in taking a unit positive test charge around the dotted path in clockwise direction.

Similarly, we could have traversed the loop in clockwise direction and collected the *voltage rises*. The sum of *voltage rises* encountered will be $v_3 + v_6 - v_1$ and will be equal to zero. If *voltage drops* are collected instead, the sum of *voltage drops* will be $-v_3 - v_6 + v_1$ and will be equal to zero.

2.4 | Basic Circuit Laws

Or, we could have entered the element voltages that we encounter when we traverse the loop in *counter-clockwise* direction in a sum, with the sign for a particular element voltage variable same as the *polarity of the variable that we meet first when we reach that element*. $-v_1 + v_6 + v_3$ is the result and it is equal to zero. The sum that is formed in this case is called the *algebraic sum of voltages*.

Or we could have formed the *algebraic sum of voltages* encountered when we traverse the loop in *clockwise* direction. $v_3 + v_6 - v_1$ is the result and it is equal to zero.

Hence, the constraint appearing among voltage variables of elements in a loop can be obtained by any of the following methods:

- (i) Traversing the loop in *clockwise* direction and equating the sum of *voltage rises* encountered to zero.
- (ii) Traversing the loop in *clockwise* direction and equating the sum of *voltage drops* encountered to zero.
- (iii) Traversing the loop in *counter-clockwise* direction and equating the sum of *voltage rises* encountered to zero.
- (iv) Traversing the loop in *counter-clockwise* direction and equating the sum of *voltage drops* encountered to zero.
- (v) Traversing the loop in *counter-clockwise* direction and equating the *algebraic sum of voltages* encountered to zero.
- (vi) Traversing the loop in *clockwise* direction and equating the *algebraic sum of voltages* encountered to zero.

All the six methods will lead to the same constraint equation. However, in the interest of systematic formulation of circuit equations, it is imperative that we adhere to any one method consistently. We choose the last method in this book. *Hence, in this book, voltage constraint equations are written by traversing the loop in clockwise direction and equating the algebraic sum of voltages encountered to zero.*

There was nothing special about the particular loop that was chosen to demonstrate the implications arising out of conservative nature of electrostatic field as far as voltage variables in a circuit are concerned. Hence, the same line of reasoning is applicable to all loops in the circuit. Therefore, at least for a DC circuit under steady-state, we can generalise the aforementioned conclusions into the following law:

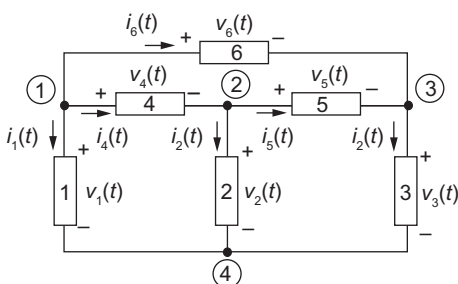


Fig. 2.1-2 A 4-node, 6-element, 7-loop circuit with time-varying voltages and currents

‘The algebraic sum of voltages in any closed path in a circuit is zero.’ This is called Kirchhoff’s Voltage Law.

Will this law hold good for circuits with time-varying voltage variables too? Consider the same circuit with time-varying voltage variables now as in Fig. 2.1-2.

Lumped parameter circuit theory assumes that the induced electric field component caused by time-varying currents in the circuit is negligible everywhere in the space surrounding the devices. Thus, the only force field that is present in the space outside circuit elements is the field generated by the coulomb force (i.e., the force that depends only on charges and the

distance between charges as per inverse square law) arising out of charge distributions on the circuit elements. The quantity of charge stored in each element in a circuit will change with time in a circuit containing time-varying sources. Therefore, the force field in the space outside the elements also will vary with time. However, at any instant t , the force field is dependent only on the charges stored in elements at that instant and the spatial distances involved. The term 'electrostatic field' does not imply that the value and direction of this field are constant in time. Rather, it means that the field arises out of 'coulomb force term'. Thus, we can use the term 'electrostatic field' to represent the force field arising out of 'coulomb force terms' even when the charge distributions on the elements vary with time.

The conservative nature of a force field arising out of 'coulomb force' is a direct result of inverse square dependence on distances displayed by such forces. Therefore, if only coulomb force field is present in the space surrounding a circuit, the work integral $-\oint \vec{E}_s \cdot d\vec{l}$ where $\vec{E}_s$ is the 'coulomb force field' or electrostatic field], evaluated at any instant t over any closed path lying outside the circuit elements, will be zero quite irrespective of whether the 'coulomb force field' is time-varying or not. Therefore, the algebraic sum of instantaneous value of voltage variables in any loop in the circuit must be zero.

However, we have been accustomed to interpret the work integral $-\oint \vec{E}_s \cdot d\vec{l}$ as the work to be done in carrying a unit positive test charge around a closed path in the $\vec{E}_s$ field quasi-statically. We face a problem in carrying over this interpretation to a time-varying situation. We have to move this unit test charge slowly around the loop. But then, the work we calculate will be the work done against $\vec{E}_s$ at different instants at different locations since we cannot be moving that test charge with infinite speed in the closed path. That is not the same as the value of integral $-\oint \vec{E}_s \cdot d\vec{l}$ at a particular time instant t .

However, the only field that is present in the space around elements in a time-varying circuit is the electrostatic field. Therefore, the field outside the elements at any instant t will be the same as the *electrostatic field* that would exist in a *DC circuit* with same elements and same geometry, but with all charge variables, voltage variables and current variables in the circuit frozen at the values they had at the time instant t . Imagine that we are carrying a unit positive test charge around a closed loop in this *frozen circuit*. The work to be done in this process will be zero since the charge was taken around a closed path in a *steady conservative field*. Therefore, the algebraic sum of voltage variables in any loop in this *frozen circuit* must be zero. But, this will imply that the algebraic sum of instantaneous value of voltage variables, at any t , in any loop in the circuit with time-varying voltages, must be equal to zero.

Therefore, KVL is valid also under time-varying conditions. The complete statement of KVL follows:

Kirchhoff's Voltage Law states that the algebraic sum of voltages in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.

It may alternatively be stated in terms of *voltage rises* or *voltage drops* as follow:

Kirchhoff's Voltage Law states that the sum of 'voltage rises' in any closed path in a lumped parameter circuit is zero on an instant-to-instant to basis.

Kirchhoff's Voltage Law states that the sum of 'voltage drops' in any closed path in a lumped parameter circuit is zero on an instant-to-instant to basis.

The circuit in Fig. 2.1-2 has 7 loops. The loops are 1-4-2, 2-5-3, 4-6-5, 1-6-3, 1-4-5-3, 2-4-6-3 and 1-6-5-2, where the numbers refer to the element labels.

2.6 Basic Circuit Laws

The KVL equations for these loops are derived in the following

$$\text{Loop 1-4-2 : } -v_1(t) + v_4(t) + v_2(t) = 0$$

$$\text{Loop 2-5-3 : } -v_2(t) + v_5(t) + v_3(t) = 0$$

$$\text{Loop 4-6-5 : } -v_4(t) + v_6(t) - v_5(t) = 0$$

$$\text{Loop 1-6-3 : } -v_1(t) + v_6(t) + v_3(t) = 0$$

$$\text{Loop 1-4-5-3 : } -v_1(t) + v_4(t) + v_5(t) + v_3(t) = 0$$

$$\text{Loop 2-4-6-3 : } -v_2(t) - v_4(t) + v_6(t) + v_3(t) = 0$$

$$\text{Loop 1-6-5-2 : } -v_1(t) + v_6(t) - v_5(t) + v_2(t) = 0$$

We observe that the first three equations will form an independent set of three equations. But the fifth equation can be obtained by adding the first two equations together. When Loop 1-4-2 equation is added to Loop 2-5-3 equation, the term $v_2(t)$ appears twice with opposite signs. Thus the resulting equation must be that of a loop formed by 1-4-5-3. Similarly, Loop 2-5-3 equation added to Loop 4-6-5 equation should result in Loop 2-4-6-3 equation. Sum of the first three equations must be same as the fourth equation.

Thus, not all the seven equations are independent. In fact, in a non-degenerate circuit containing b - elements, n -nodes and l -loops there will be exactly $(b - n + 1)$ loop equations that are independent. l will be more than or equal to $(b - n + 1)$. These statements come from a branch of study called Network Topology. We accept these statements without proof at this point.

This does not mean that a random selection of $(b - n + 1)$ loop equations from the set of l loop equations will be an independent set of loop equations. For instance, in the present example, there must be $(6 - 4 + 1) = 3$ independent loop equations. However, the loop equations for Loop 1-4-2, Loop 2-5-3 and Loop 1-4-5-3 are not independent. The third can be obtained by adding the first two. Note that the first two loops are completely contained by the third loop. A *planar circuit* is one that can be drawn on paper without any crossing of connection wires. A *basic window* in a *planar circuit* is a loop that does not contain any other loop within it. It must be intuitively clear that the loop equations for the *basic windows* of the *planar circuit* will form an independent set of loop equations. These *basic windows* of a *planar circuit* are called its '*meshes*'.

Example: 2.1-1

The source voltages of four independent voltage sources in the circuit in Fig. 2.1-3 are given as $v_{s1} = 10 \text{ V}$, $v_{s2} = 10\sin 100t \text{ V}$, $v_{s3} = 10\cos 100t \text{ V}$ and $v_{s4} = 10 \text{ V}$. v_3 is observed to have a zero average value. The time-varying component of v_1 is seen to be $10\sin(100t - 30^\circ) \text{ V}$. Find v_1 , v_2 and v_3 as functions of time.

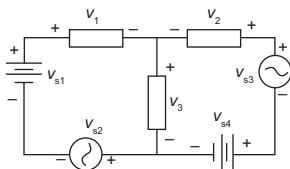


Fig. 2.1-3 Circuit for Example: 2.1-1

Solution

v_3 is stated to have a zero average. This implies that v_3 has a zero DC content. Thus, v_3 can be written as $v_3 = A \sin(100t - \theta) \text{ V}$ where A and θ are to be found. v_1 is stated to have a time-varying component of $10\sin(100t - 30^\circ) \text{ V}$. It may have a DC content too. Thus, $v_1 = B + 10 \sin(100t - 30^\circ) \text{ V}$, where B is to be found. v_2 may contain both DC and time-varying components. Thus, $v_2 = C + D \sin(100t - \phi) \text{ V}$, where C , D and ϕ have to be found.

Apply KVL in the first loop. The KVL equation is:

$$-v_{s1} + v_1 + v_3 + v_{s2} = 0$$

$$\text{i.e., } -10 + (B + 10\sin(100t - 30^\circ)) + A\sin(100t - \theta) + 10\sin 100t = 0$$

This equation involves some constants and some sinusoidal functions. This equation is the result of applying KVL to a loop in a circuit. Therefore, this equation has to be true at all t . Therefore, the equation can be split into two equations that have to be satisfied simultaneously. This is because a constant can not be met by a sinusoidal function in an equation for all t .

$$-10 + B = 0 \text{ and } 10\sin(100t - 30^\circ) + A\sin(100t - \theta) + 10\sin 100t = 0$$

First equation yields $B = 10$ V. Second equation is simplified by employing trigonometric identities as below:

$$5\sqrt{3}\sin 100t - 5\cos 100t + (A\cos\theta)\sin 100t - (A\sin\theta)\cos 100t + 10\sin 100t = 0$$

This equation can be true for all t only if the coefficient of $\sin 100t$ is zero and the coefficient of $\cos 100t$ is zero independently.

$$\therefore 5\sqrt{3} + A\cos\theta + 10 = 0 \text{ and } -5 - A\sin\theta = 0$$

$$\therefore A\cos\theta = -10 - 5\sqrt{3} = -18.66; A\sin\theta = -5$$

$$\therefore A = 19.32 \text{ and } \theta = \tan^{-1} \frac{-5}{-18.66} = 180 + \tan^{-1} \frac{5}{18.66} = 195^\circ$$

$$\therefore v_1 = 10 + 10\sin(100t - 30^\circ) \text{ V and } v_3 = 19.32\sin(100t - 195^\circ) \text{ V.}$$

Now apply KVL in the outer loop to get,

$$-v_{s1} + v_1 - v_2 + v_{s3} + v_{s4} + v_{s2} = 0$$

$$\text{i.e., } -10 + (10 + 10\sin(100t - 30^\circ)) - C - D\sin(100t - \phi) + 10\cos(100t) + 10 + 10\sin 100t = 0$$

$$(10 + 10\sin(100t - 30^\circ)) - C - D\sin(100t - \phi) + 10\cos(100t) + 10\sin 100t = 0$$

$$\therefore C = 10 \text{ and }$$

$$10\sin(100t - 30^\circ) - D\sin(100t - \phi) + 10\cos 100t + 10\sin 100t = 0$$

$$8.66\sin 100t - 5\cos 100t - (D\cos\phi)\sin 100t + (D\sin\phi)\cos 100t + 10\cos 100t + 10\sin 100t = 0$$

$$\therefore D\cos\phi = 18.66; D\sin\phi = -5 \Rightarrow D = 19.32 \text{ and } \phi = \tan^{-1} \frac{-5}{18.66} = -15^\circ$$

$$\therefore v_2 = 10 + 19.32\sin(100t + 15^\circ) \text{ V.}$$

Therefore, $v_1 = 10 + 10\sin(100t - 30^\circ)$ V, $v_2 = 10 + 19.32\sin(100t + 15^\circ)$ V and $v_3 = 19.32\sin(100t - 195^\circ)$ V is the required answer.

The key to the solution of this problem is the point that KVL has to be satisfied at all time instants.

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Example: 2.1-2

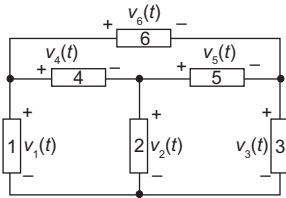


Fig. 2.1-4 Circuit for Example 2.1-2

Express the terminal voltages of elements 2, 3 and 5 in terms of terminal voltages of elements 1, 4 and 6 in the circuit in Fig. 2.1-4.

Solution

Applying KVL in the loop 1–4–2, we get, $-v_1(t) + v_4(t) + v_2(t) = 0$.

$$\therefore v_2(t) = v_1(t) - v_4(t)$$

Applying KVL in the loop 4–5–6, we get, $v_4(t) + v_5(t) - v_6(t) = 0$.

$$\therefore v_5(t) = v_6(t) - v_4(t)$$

Applying KVL in the loop 1–6–3, we get, $-v_1(t) + v_6(t) + v_3(t) = 0$

$$\therefore v_3(t) = v_1(t) - v_6(t)$$

2.2 KIRCHHOFF'S CURRENT LAW

Conservation law of charge states that charges can neither be created nor destroyed in a given volume. Hence, if the positive charge that flows into a volume at any instant t exceeds the positive charge that flows out of the volume at the same instant, then the net charge stored inside the volume must be increasing at that instant. Similarly, if the positive charge that flows out of the volume exceeds the positive charge that flows into the volume at that instant, then the net charge stored within must be decreasing at that instant. Therefore, the *net positive current* that flows into the volume at an instant t must be equal to the *rate of change of net charge* stored within that volume at that instant. A mathematical statement of this fact is called the *continuity equation for currents*.

If, for some reason, the net charge inside the volume is either constrained to remain at zero at all instants or is constrained to remain at some constant value at all instants, then the *net positive current* that flows into the volume must be zero at all instants.

Lumped parameter circuit theory assumes that the surface charge distribution on the surface of connecting wires is negligible at all instants of time. There is surface charge distribution on all circuit elements other than the connecting wires. In general, these surface charge distributions are time-varying too. However, the positive and negative charges distributed on the surface of any two-terminal or four-terminal element are equal in magnitude at all time instants under quasi-static conditions. Therefore, if we consider a volume that contains some circuit elements completely within, those elements will contribute only zero net charge to the net charge storage within the volume. The situation would, however, be different if the volume intersects some element. For instance, consider a volume that encloses only one of the plates of a capacitor. Then, there will be net charge storage within the volume and that may change with time too.

Therefore, we restrict ourselves to a volume that intersects connecting wires at many places without enclosing or intersecting even a single circuit element or a volume that intersects connecting wires at many places and completely encloses one or more circuit elements. We do not permit the volume to intersect any circuit element.

Since an element completely enclosed within a volume does not contribute to net charge within the volume and since the connecting wires have only negligible surface charge distributions on them, it follows that the net charge contained in a volume chosen the way suggested in the previous paragraph will be zero at all t . Therefore, the rate of change of net charge will also be zero at all t . Then, by continuity equation for currents, the net positive current that flows into the volume through the wires must be zero at all time-instants.

A *node* in a circuit is a part of the connecting wire. Therefore, there is no charge storage at a *node* in a circuit as per the assumptions employed by lumped parameter circuit theory. Therefore, there is no rate of change of charge storage too. Consider a special volume – a volume that encloses a node in a circuit and intersects all the wires connected at that node. Then, the reasoning outlined above leads us to the conclusion that the *net positive current* that flows into the volume through all the connecting wires that were intersected by the volume (i.e., all the wires connected together at that node) should be zero at all t . Equivalently, we may state that, *net positive current* that flows out of the volume must be zero at all t .

Consider a volume denoted by the dotted circle around node-2 in the circuit in Fig. 2.2-1. This volume intersects three wires and encloses the node-2. It does not enclose any circuit element nor does it intersect any circuit element other than the connecting wires. Therefore, the *net positive current flowing out of the volume* must be zero. This fact leads to the following equation:

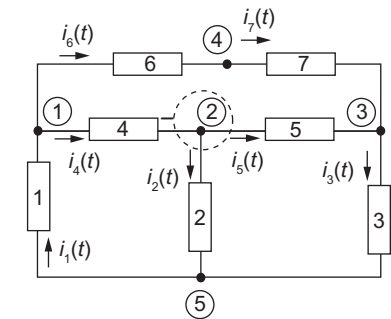


Fig. 2.2-1 Circuit for illustrating kirchhoff's current law

$$-i_4(t) + i_2(t) + i_5(t) = 0 \quad (2.2-1)$$

where $i_4(t)$ is a current that flows *into* the volume. We need a minus sign to make it a current that flows *out of* the volume. Hence the minus sign in front of $i_4(t)$ in Eqn. 2.2-1.

We could have arrived at an equation containing the same information contained in Eqn. 2.2-1 by stipulating that the *net positive current flowing into the volume* must be equal to zero. The resulting equation will be:

$$i_4(t) - i_2(t) - i_5(t) = 0 \quad (2.2-2)$$

Eqn. 2.2-2 is, obviously, Eqn. 2.2-1 multiplied by -1 and contains the same information.

We could have arrived at Eqn. 2.2-1 by stipulating that the *algebraic sum* of currents *leaving* a node must be equal to zero. 'Algebraic sum' in this case implies that if a particular current variable has its *reference direction* pointing *towards* the node, it has to enter the equation with *negative* sign. If a particular current variable has its *reference direction* pointing *away* from node, it has to be entered in the equation with *positive* sign.

Similarly, we could have arrived at Eqn. 2.2-2 by stipulating that the *algebraic sum* of currents *entering* a node must be equal to zero. 'Algebraic sum' in this case implies that if a particular current variable has its *reference direction* pointing *towards* the node, it has to enter the equation with a *positive* sign. If a particular current variable has its *reference direction* pointing *away* from node, it has to be entered in the equation with *negative* sign.

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Obviously, all the four methods of arriving at the node equation are equivalent. However, in the interest of a systematic procedure, we use the stipulation that the *algebraic sum* of currents *leaving* a node must be equal to zero. We are ready to state the Kirchhoff's current law now.

Kirchhoff's Current Law (KVL) states that the algebraic sum of currents leaving a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.

KCL at a node can be stated in alternative ways.

Kirchhoff's Current Law (KVL) states that the algebraic sum of currents entering a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.

Kirchhoff's Current Law (KVL) states that the sum of currents entering a node in a lumped parameter circuit through some wires must be equal to the sum of currents leaving the same node through the remaining wires on an instant-to-instant basis.

KCL equations at all nodes of the circuit shown in Fig. 2.2-1 are derived in the following:

$$\text{Node - 1 : } -i_1(t) + i_4(t) + i_6(t) = 0$$

$$\text{Node - 2 : } -i_4(t) + i_2(t) + i_5(t) = 0$$

$$\text{Node - 3 : } i_3(t) - i_5(t) - i_7(t) = 0$$

$$\text{Node - 4 : } -i_6(t) + = 0$$

$$\text{Node - 5 : } i_1(t) - i_2(t) - i_3(t) = 0$$

Note that the sum of these equations will be of $0 = 0$ form. This indicates that these five equations do not form an independent set of equations. If we add all KCL equations derived for all the nodes of a circuit, a particular current variable that enters some equation with a positive sign will necessarily enter some other equation in the set with a negative sign. After all, an element has to get connected to two nodes. Therefore, all terms on the left-hand side of the sum will get cancelled. Suppose we discard the KCL equation at any one node. At least two elements must be connected to any node. Therefore, the sum of four of the five KCL equations will have at least two current variables present on the left-hand side. Therefore, any set of four KCL equations will be an independent set of equations. *Thus, in general, there will be $(n-1)$ independent KCL equations in an n -node circuit.*

We had earlier accepted the fact that there will be $(b - n + 1)$ independent KVL equations for a b -element, n -node, l -loop lumped parameter circuit. $(b - n + 1)$ independent KVL equations together with $(n-1)$ independent KCL equations make the required b interconnection equations needed to solve the circuit.

It is possible to arrive at a more general form of KCL applicable to lumped parameter circuits by considering a closed surface that encloses more than one node along with one or more elements. We have reasoned earlier in this section that the net charge contained inside such a closed surface must be equal to zero. Therefore, the algebraic sum of currents leaving such a closed surface must be equal to zero on an instant-to-instant basis. Such a closed surface will contain two or more nodes, and all the elements that are connected between the nodes are within the closed surface. Such a closed surface is called a *supernode*.

How many nodes can a circuit with n nodes have? Taking two at a time, there are ${}^nC_2$ supernodes that contain two nodes each. Similarly, there are ${}^nC_3$ supernodes that contain three nodes each. See the circuit in Fig. 2.2-2.

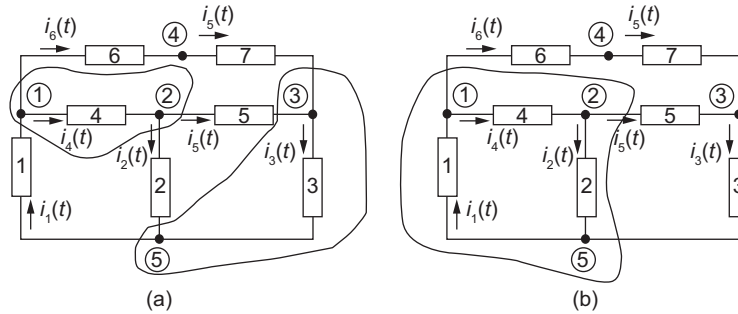


Fig. 2.2-2 (a) Circuit showing two supernodes with two nodes each
(b) Circuit showing a supernode that contains three nodes

Two supernodes, each containing two nodes, are shown in the circuit in Fig. 2.2-2(a). One supernode containing three nodes is shown in the circuit in Fig. 2.2-2(b).

Kirchhoff's Current Law is applicable to supernodes too

Therefore, the KCL equation for the supernode containing node-1 and node-2 is obtained as $-i_1(t) + i_6(t) + i_2(t) + i_5(t) = 0$. Obviously, this must be the sum of KCL equations written for node-1 and node-2. Similarly, the KCL equation for the supernode in the circuit in Fig. 2.2-2(b) must be the sum of KCL equations written for node-1, node-2 and node-5. This may be verified.

The total number of KCL equations that can be written for an n -node circuit is equal to ${}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$. This series has a sum equal to $2^n - 2$.

Only $(n - 1)$ independent equations from these $2^n - 2$ KCL equations can be used for solving the circuit.

Example: 2.2-1

Find the power delivered by all the sources in the circuit in Fig. 2.2-3.

Solution

Currents through the voltage sources and voltage across the current sources have to be obtained first. The circuit with all nodes and reference directions for variables identified is shown in Fig. 2.2-4.

Applying KCL at node-A, we get $-5 - (-2) - i_1 = 0 \Rightarrow i_1 = -3$ A

Applying KCL at node-B, we get $(-2) + (-2) - i_2 = 0 \Rightarrow i_2 = -4$ A

Applying KCL at node-C, we get $5 - (-2) + i_3 = 0 \Rightarrow i_3 = -7$ A

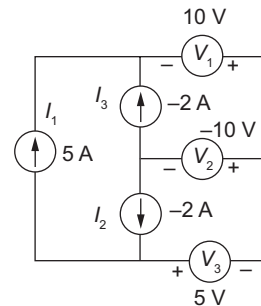


Fig. 2.2-3 Circuit for Example 2.2-1

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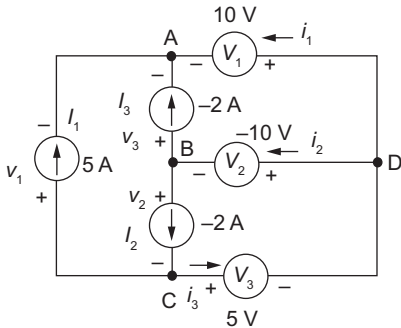


Fig. 2.2-4 Circuit with nodes and reference directions identified

Applying KVL in the loop $I_1-V_1-V_3$, we get $v_1 - 10 - 5 = 0 \Rightarrow v_1 = 15 \text{ V}$

Applying KVL in the loop $I_3-V_1-V_2$, we get $v_3 - 10 + (-10) = 0 \Rightarrow v_3 = 20 \text{ V}$

Applying KVL in the loop $I_2-V_2-V_3$, we get $-v_2 - (-10) - 5 = 0 \Rightarrow v_2 = 5 \text{ V}$

Power delivered by an element is given by $-vi$, where v and i are its voltage and current variables, respectively, as per passive sign convention.

$$\therefore \text{Power delivered by } I_1 \text{ source} = -v_1 \times 5 \text{ A} = -75 \text{ W}$$

$$\text{Power delivered by } I_2 \text{ source} = -v_2 \times (-2) \text{ A} = 10 \text{ W}$$

$$\text{Power delivered by } I_3 \text{ source} = -v_3 \times (-2) \text{ A} = 40 \text{ W}$$

$$\text{Power delivered by } V_1 \text{ source} = -10 \text{ V} \times i_1 = 30 \text{ W}$$

$$\text{Power delivered by } V_2 \text{ source} = -(-10 \text{ V}) \times i_2 = -40 \text{ W}$$

$$\text{Power delivered by } V_3 \text{ source} = -5 \text{ V} \times i_3 = 35 \text{ W}$$

Example: 2.2-2

Express $i_2(t)$, $i_4(t)$ and $i_5(t)$ in terms of $i_1(t)$, $i_3(t)$ and $i_6(t)$ in the circuit shown in Fig. 2.2-5.

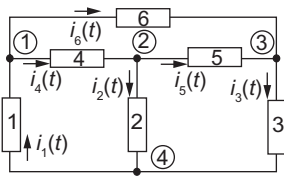


Fig. 2.2-5 Circuit for Example 2.2-2

Solution

Applying KCL at node-1, we get $-i_1(t) + i_4(t) + i_6(t) = 0 \Rightarrow i_4(t) = i_1(t) - i_6(t)$

Applying KCL at node-3, we get $-i_5(t) + i_3(t) - i_6(t) = 0 \Rightarrow i_5(t) = i_3(t) - i_6(t)$

Applying KCL at node-2, we get $-i_4(t) + i_2(t) + i_5(t) = 0 \Rightarrow i_2(t) = i_4(t) - i_5(t) = i_1(t) - i_3(t)$

Example: 2.2-3

Fig. 2.2-6 shows the connecting wires in a part of a circuit. Some of the currents are specified for $t \geq 0$ in Fig. 2.2-6. The current i_2 is seen to be a constant in time. The current i_3 is seen to approach zero as $t \rightarrow \infty$. Find i_1 , i_2 and i_3 for $t \geq 0$.

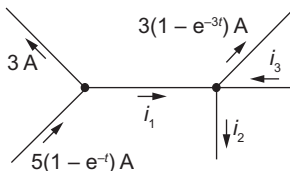


Fig. 2.2-6 Part of a circuit referred to in Example: 2.2-3

Solution

Applying KVL at first node, we get $3 + i_1 - 5(1 - e^{-t}) = 0 \Rightarrow i_1 = 2 - 5e^{-t} \text{ A}$. i_2 is stated to be a constant in time. Let $i_2 = A$. i_3 is stated to approach zero as $t \rightarrow \infty$. This implies that there is no DC component in i_3 . It may contain both e^{-t} and e^{-3t} components. Let $i_3 = Ce^{-t} + De^{-3t} \text{ A}$. Then, applying KCL at the second node, we get

$$-i_1 + 3(1 - e^{-3t}) - i_3 + i_2 = 0$$

$$\text{i.e., } -2 + 5e^{-t} + 3 - 3e^{-3t} - Ce^{-t} - De^{-3t} + A = 0$$

$$\text{i.e., } (A+1) + e^{-t}(5-C) + e^{-3t}(-3-D) = 0$$

KCL is true at all t . Hence, the last equation must be valid for all $t \geq 0$. No time-varying function can be equal to a constant unless that function itself is a constant. Thus, $(A + 1)$ term in the last equation cannot be equalled by e^{-t} and e^{-3t} terms for all $t \geq 0$. Therefore, $(A + 1)$ has to be zero.

$$\therefore A = -1 \Rightarrow i_2 = -1 \text{ A.}$$

A term involving e^{-t} cannot get cancelled by another term that involves e^{-3t} for all $t \geq 0$. Therefore, coefficient of e^{-t} must be zero and coefficient of e^{-3t} also must be zero. Therefore, $(5 - C) = 0$ and $(3 + D) = 0$.

$$\therefore C = 5 \text{ and } D = -3 \Rightarrow i_3 = (5e^{-t} - 3e^{-3t}) \text{ A.}$$

The solution is marked in Fig. 2.2-7.

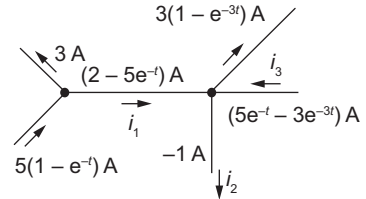


Fig. 2.2-7 Solution for Example 2.2-3

2.3 INTERCONNECTIONS OF IDEAL SOURCES

Interconnecting two or more ideal voltage sources in a loop may lead to a degenerate circuit. Similarly, interconnecting two or more ideal current sources in series may lead to a degenerate circuit.

Consider the interconnection of two ideal independent voltage sources as shown in Fig. 2.3-1. KVL requires that $-v_{s1}(t) + v_{s2}(t) = 0$. Therefore, $v_{s1}(t)$ has to be equal to $v_{s2}(t)$ at all time instants. If they are not equal to each other, then either KVL has to yield or the ideal sources have to yield. KVL cannot yield since it is only a disguised form of law of conservation of energy. Therefore, KVL has to be obeyed by a circuit at all instants.

There are two ways out of this impasse for a case where $v_{s1}(t) \neq v_{s2}(t)$. The first is to declare that *two ideal independent voltage sources cannot be connected in parallel unless their terminal voltages are equal to each other at all instants of time*. That is, we call such connections *illegal* if $v_{s1}(t) \neq v_{s2}(t)$. But, that is a kind of escaping the issue!

The correct way to resolve the issue is to recognise that *ideal independent voltage source* is a model for a practical electrical device, and, as in the case of any model, this model too has its range of applicability. Connecting a practical voltage source in parallel with another practical voltage source is a context in which they cannot be modelled by *ideal independent voltage source model* satisfactorily. In fact, the model is not satisfactory even when $v_{s1}(t) = v_{s2}(t)$. This is because there is no way to find out the current that will flow in the circuit. Any current can flow at any instant in such a circuit.

Thus, the only correct way to model a circuit that involves parallel connections of voltage sources (more generally, loops comprising only voltage sources) is to take into account the parasitic elements that are invariably associated with any practical voltage source. A somewhat detailed model for the two-source system is shown in Fig. 2.3-2.

L_{i1} and L_{i2} represent the internal inductance of the sources, C_{i1} and C_{i2} represent the terminal capacitance of the sources and R_{i1} and R_{i2} represent the internal resistance of the sources. L_c and R_c represent the inductance and resistance of connecting wires. Obviously, two practical voltage sources can be connected in parallel even if their open-circuit electromotive forces are not equal at all t ; only that they cannot be modelled by *ideal independent voltage source model*.

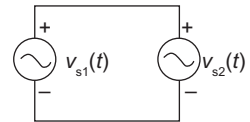


Fig. 2.3-1 Two ideal independent voltage sources in parallel

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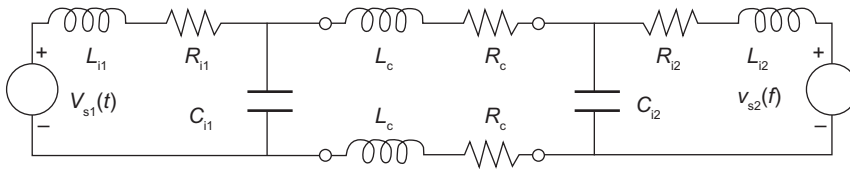


Fig. 2.3-2 A detailed model for a circuit with two voltage sources in parallel

Two ideal independent current sources in series raise a similar issue (see Fig. 2.3-3). KCL requires that $i_{s1}(t) = i_{s2}(t)$ for all t . Even if this condition is satisfied, there is no way to obtain the voltages appearing across the current sources. Therefore, the correct model to be employed for practical current sources that appear in series in a circuit is a detailed model that takes into account the

parasitic elements associated with any practical device. More generally, if there is a node in a circuit where only current sources are connected, then those current sources cannot be modelled by *ideal independent current source model*.

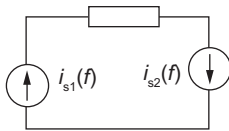


Fig. 2.3-3 Two ideal independent current sources in series with another element

Similar situations may arise in modelling practical dependent sources by *ideal dependent source models*. In all such cases we have to make the model more detailed in order to resolve the conflict that arises between Kirchhoff's laws and *ideal* nature of the model.

2.4 ANALYSIS OF A SINGLE-LOOP CIRCUIT

Circuit analysis problem involves finding the voltage and current variables of every element as functions of time, given the source functions. *Source functions* are the time-functions describing the electromotive force of independent voltage sources and source currents of independent current sources. They are also called the *excitation functions*. If the circuit contains b elements, there will be $2b$ variables to be solved for. Some of them will be known in the form of source functions. Others have to be solved for.

Element relation of each element gives us one equation per element. Thus there are b equations arising out of element relations. The remaining b equations are provided by the interconnection constraints. These equations are obtained by applying KCL at all nodes except one and KVL in all meshes (in the case of a planar circuit).

Theoretically speaking, that is all there to circuit analysis. However, systematic procedures for applying element relations, KVL equations and KCL equations would be highly desirable when it comes to analysis of complex circuits. Moreover, the fact that there are $2^n - 2$ KCL equations for a n -node circuit and only $(n - 1)$ of them are independent calls for a systematic procedure for writing KCL equations. Similarly, there will be l KVL equations for a circuit with l -loops and only $(b - n + 1)$ of them will be independent. This, again, calls for some systematic procedure for extracting a set of $(b - n + 1)$ *independent* KVL equations.

Such systematic procedures are indeed available in Circuit Theory. We will look at such procedures in detail in later chapters. However, we try to gain some experience in applying Kirchhoff's laws and element relations to simple circuits in this section.

Example: 2.4-1

Refer to Fig. 2.4-1. $V_1 = 20\text{ V}$, $V_2 = 5\text{ V}$, $R_1 = 5\ \Omega$ and $R_2 = 2.5\ \Omega$. Find all element voltages and element currents. Also find the power *delivered* to all elements.

Solution

The first step in analysis is to assign reference directions for variables. Passive sign convention is employed to decide the reference direction for current after reference polarity for voltage is decided arbitrarily. Or, reference polarity for voltage can be decided in compliance with passive sign convention after deciding reference direction for current arbitrarily. One has to start from some point. Let us start at the first source terminal and move through the circuit in a clockwise direction.

The source function is already assigned polarity. This does not prevent us from assigning another voltage variable to the first source with any polarity that we may decide. However, there is simply no reason to do so. Therefore, in the case of a voltage source we accept the polarity of source function itself as the reference polarity for element voltage.

Obviously, a new voltage variable is also not required. The source function itself is the value of the voltage variable for an ideal independent voltage source.

Now we have to assign a current variable with its reference direction entering the positive polarity from outside. This current is called i_3 (see Fig. 2.4-2).

Now we reach the resistor R_1 . We assign positive polarity of its voltage variable at the first terminal that we come across – that is, the left terminal of the resistor. Then, its current variable must enter from left as per passive sign convention.

Variable assignment and reference direction assignment for remaining elements is completed in a similar manner and the final variable assignment is shown in Fig. 2.4-2. The nodes in the circuit are also identified in the circuit and labelled.

The next step in the analysis is to apply KCL at any three nodes. Let us apply KCL at node-a, node-b and node-c. The result will be that $-i_3 = i_1 = i_4 = i_2$. Next, we apply KVL in the loop to get $-V_1 + v_1 + V_2 + v_2 = 0$.

The next step in the analysis is to make use of the element relations. We have already made use of the element relations of the voltage sources to set the voltage variables for the sources at their source function values themselves. The remaining elements are two resistors. The element relation for a resistor is given by Ohm's law. The relations are

$$v_1 = R_1 i_1$$

$$v_2 = R_2 i_2$$

Now, these relations and the fact that $-i_3 = i_1 = i_4 = i_2$ are made use of in the KVL equation to simplify it as $-V_1 + R_1 i_1 + R_2 i_2 + V_2 = 0$. Note that we have eliminated i_2 by using $i_1 = i_2$. Solving this

equation, we get $i_1 = \frac{V_1 - V_2}{R_1 + R_2}\text{ A}$.

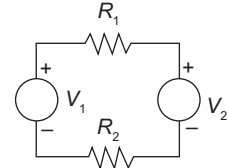


Fig. 2.4-1 Single-loop circuit in Example 2.4-1

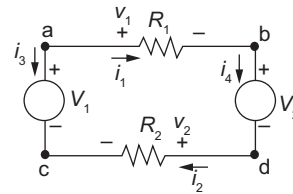


Fig. 2.4-2 Variable assignment and reference directions in the circuit in Example 2.4-1

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Now the voltage across each resistor can be worked out by using its element relation once again.

$$v_1 = R_1 i_1 = \frac{R_1(V_1 - V_2)}{R_1 + R_2} \text{ V and } v_2 = R_2 i_2 = R_2 i_1 = \frac{R_2(V_1 - V_2)}{R_1 + R_2} \text{ V.}$$

The currents through voltage sources can be noted as $i_3 = -i_1 = \frac{V_2 - V_1}{R_1 + R_2} \text{ A}$ and $i_4 = i_1 = \frac{V_1 - V_2}{R_1 + R_2} \text{ A}$.

Substituting the numerical values in the example, we get

$$i_1 = i_2 = -i_3 = i_4 = 2 \text{ A}$$

$$v_1 = 10 \text{ V}; v_2 = 5 \text{ V}$$

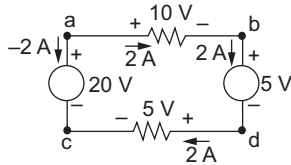


Fig. 2.4-3 Circuit solution in Example 2.4-1

This solution is marked in Fig. 2.4-3. Note that the -2 A flowing *into* the first source can be marked as $+2 \text{ A}$ if the reference direction is reversed. The value of -2 A implies that *positive current flows out from the positive terminal of that source*.

The *power delivered to an element* is given by v_i , where v and i are its voltage variable and current variable, respectively, as per passive sign convention.

$$\text{Power delivered to } 20 \text{ V source} = (-2 \text{ A}) \times (+20 \text{ V}) = -40 \text{ W}$$

$$\text{Power delivered to } 5 \text{ V source} = (+2 \text{ A}) \times (+5 \text{ V}) = +10 \text{ W}$$

$$\text{Power delivered to } 5 \Omega \text{ resistance} = (+2 \text{ A}) \times (+10 \text{ V}) = +20 \text{ W}$$

$$\text{Power delivered to } 2.5 \Omega \text{ resistance} = (+2 \text{ A}) \times (+5 \text{ V}) = +10 \text{ W}$$

Note that the sum of power delivered to all elements is zero. The 20 V source delivers 40 W of power which is shared by the two resistors and the second source. Second source receives 10 W power from the first source and absorbs it.

Four elements were connected in this circuit with no two elements sharing the *same pair* of nodes. Two elements have only one common node. This kind of connection is called a *series connection of elements*. Obviously, in a *series connection* of elements, same current will flow in the same direction in all elements.

Example: 2.4-2

Solve the circuit in Fig. 2.4-4 completely.

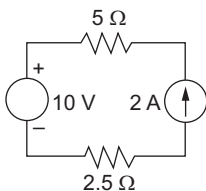


Fig. 2.4-4 Circuit for Example 2.4-2

Solution

This is a series connected, single-loop circuit. Therefore, same current should flow through all the elements and it is constrained to be equal to 2 A in counter-clockwise direction in the circuit by the independent current source that comes in series with other elements. Other elements have no role in deciding the circuit current.

However, the current source will have to pay a price for deciding the circuit current in an autocratic manner. The price it has to pay

is the amount of voltage that it has to support across its terminals and the amount of power it has to deliver or absorb. The current source will have no control over these quantities; they will be decided by rest of the elements in the circuit.

The circuit after variable assignment and reference direction assignment is shown in Fig. 2.4-5(a).

Obviously, we do not need to assign any current variable for any element in the circuit if we choose the reference directions for all element currents to coincide with the actual direction of current flow in the circuit. The polarity of element voltage variables is assigned as per passive sign convention.

Applying KVL in the loop, we get, $-10 - (5 \times 2) - v_3 - (2.5 \times 2) = 0$

Therefore, $v_3 = -25$ V. The circuit solution is marked in Fig. 2.4-5(b).

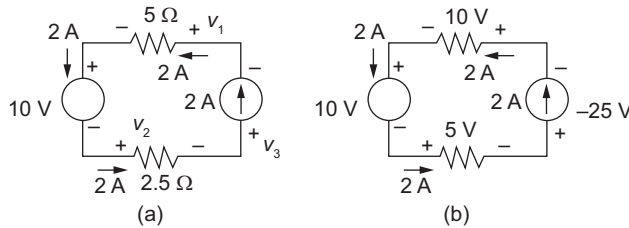


Fig. 2.4-5 (a) Variable and reference direction assignment in the circuit in Example 2.4-2
(b) Circuit solution

Note that actually the upper terminal of current source is at a higher potential compared to the lower terminal. This indicates that the current source is delivering power. However, 2 A flows into the positive polarity terminal of the 10 V voltage source. Therefore, the voltage source is absorbing power from the current source. Thus, the current source must be delivering all the power that is absorbed by the two resistors and the voltage source. Let us verify this.

$$\text{Power delivered by current source} = -(-25 \text{ V}) \times (2 \text{ A}) = 50 \text{ W}$$

$$\text{Power absorbed by } 5 \Omega \text{ resistor} = 10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$$

$$\text{Power absorbed by } 2.5 \Omega \text{ resistor} = 5 \text{ V} \times 2 \text{ A} = 10 \text{ W}$$

$$\text{Power absorbed by the voltage source} = 10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$$

Total power absorbed by power absorbing elements is 50 W and that is equal to the power delivered by current source.

Example: 2.4-3

Solve the circuit in Fig. 2.4-6 completely.

Solution

This single-loop circuit contains a voltage-controlled voltage-source (VCVS) in series with other elements. The controlling variable of this VCVS is the voltage across the independent current source with positive polarity placed at source current delivery point.

A VCVS is a four-terminal element. The output terminal pair is connected in series with other elements in Fig. 2.4-6. But where is the input terminal pair?

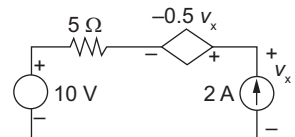


Fig. 2.4-6 Circuit for Example 2.4-3

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Fig. 2.4-7 shows the actual connections involved in the circuit. The input terminal pair of the VCVS is connected across the terminals of the independent current source and the output terminal pair is connected between right-side terminal of the resistor and left-side terminal of current source. The source function value of VCVS is $-0.5v_x$, where v_x is the voltage sensed by its input terminal pair.

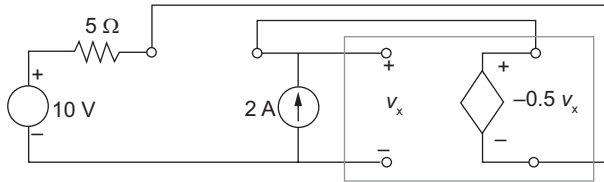


Fig. 2.4-7 Circuit in Example 2.4-3 redrawn to show the VCVS terminal connections

The input terminal pair of an ideal VCVS is an open-circuit and does not affect the circuit behaviour in any manner. Therefore, it is sufficient to identify the controlling variable of a VCVS as in Fig. 2.4-6.

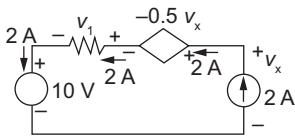


Fig. 2.4-8 Circuit variable and reference direction assignment in Example 2.4-3

The variable and reference direction assignment is shown in Fig. 2.4-8. Note that we have not assigned any new voltage variable across the current source. The voltage variable v_x itself is taken as its voltage variable. Therefore, the reference direction of current in current source and reference polarity for its voltage is not according to passive sign convention. This will not cause any problem in applying the element equation of KVL. This is because the voltage across a current source is independent of its current. However, the fact that we are not using passive sign convention for this element

has to be kept in mind when calculating the power delivered by this source. Adhering to passive sign convention is important in the case of passive elements since the element relation of a passive element depends on the relative polarities of voltage and current in the element.

Applying KVL in the loop, we get

$$\begin{aligned} -10\text{V} - v_1 - (-0.5v_x) + v_x &= 0 \\ \text{i.e., } -10\text{V} - (5\Omega \times 2\text{A}) + 1.5v_x &= 0. \\ \therefore v_x &= \frac{40}{3}\text{V} = 13.33\text{V} \end{aligned}$$

The complete solution is marked in Fig. 2.4-9.

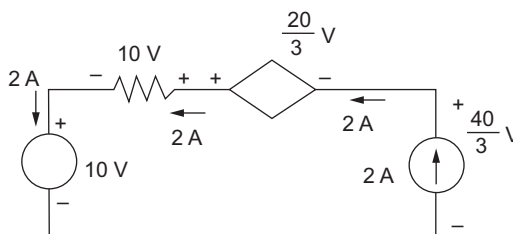


Fig. 2.4-9 Circuit solution in Example 2.4-3

Power absorbed by the 10 V voltage source	$= 10 \text{ V} \times 2 \text{ A}$	$= 20 \text{ W}$
Power absorbed by the 5Ω resistance	$= 10 \text{ V} \times 2 \text{ A}$	$= 20 \text{ W}$
Power delivered by the VCVS	$= (20/3) \text{ V} \times 2 \text{ A}$	$= 40/3 \text{ W}$
Power delivered by the 2 A current source	$= (40/3) \text{ V} \times 2 \text{ A}$	$= 80/3 \text{ W}$
Total power absorbed	$= 20 \text{ W} + 20 \text{ W}$	$= 40 \text{ W}$
Total power delivered	$= (40/3) \text{ W} + (20/3) \text{ W}$	$= 40 \text{ W}$

2.5 ANALYSIS OF A SINGLE-NODE-PAIR CIRCUIT

A set of circuit elements is said to be connected in *parallel* if they have two nodes in common. Fig. 2.5-1 shows three circuit elements connected in parallel. All the three elements have one of their terminals connected at node-A and the other terminal connected at node-B. Circuit in Fig. 2.5-1(a) shows reference polarity assignment for the element voltage variables for the three elements. The terminal connected to the node-A is assigned the positive polarity in all the elements in this case. Circuit in Fig. 2.5-1(b) shows another possible polarity assignment. Here, the terminal that is connected to node-B is assigned the positive polarity of voltage variable in the case of second element.

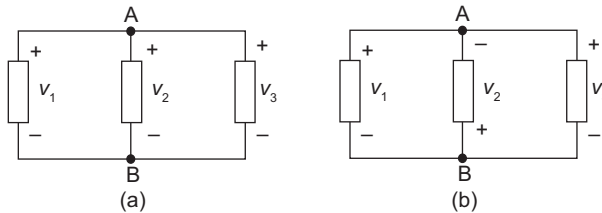


Fig. 2.5-1 Parallel connection of elements

This circuit has two meshes. We can apply KVL in those meshes. KVL applied to meshes in circuit in Fig. 2.5-1(a) will show that $v_1 = v_2 = v_3$ in the circuit. However, KVL applied to meshes in circuit in Fig. 2.5-1(b) will show that $v_1 = -v_2 = v_3$. Thus, the terminal voltages of elements connected in parallel will have same value at all t if same reference polarity assignment is used for all of them. That is, parallel-connected elements have a *common terminal voltage* if reference polarity is same for all of them. Therefore, it is a standard practice in circuit analysis to assign positive polarity to terminals connected to a common node in the case of a set of parallel-connected elements.

A set of parallel-connected elements result in one node-pair. There will only be one independent KCL equation in a circuit containing just one node-pair. However, such a circuit may contain many meshes. Applying KVL in all those meshes will result in an already-noted conclusion that all the elements in parallel will have same terminal voltage.

Example: 2.5-1

Solve the circuit in Fig. 2.5-2 completely.

Solution

The 10 V independent voltage source across the node-pair fixes the terminal voltage of all elements at 10V. The current through 10 Ω

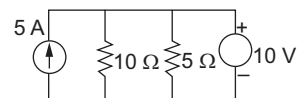


Fig. 2.5-2 Circuit for Example 2.5-1

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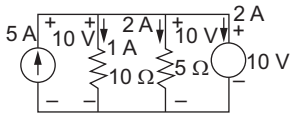


Fig. 2.5-3 Circuit solution in Example 2.5-1

$\therefore$ Current flowing into the positive polarity of 10 V source = 2 A.

The circuit solution is marked in Fig. 2.5-3.

Power delivered by 5 A current source = $10 \text{ V} \times 5 \text{ A} = 50 \text{ W}$

Power absorbed by 10Ω resistor = $10 \text{ V} \times 1 \text{ A} = 10 \text{ W}$

Power absorbed by 5Ω resistor = $10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$

Power absorbed by 10 V current source = $10 \text{ V} \times 2 \text{ A} = 20 \text{ W}$

Total power delivered = Total power absorbed

will then be $10 \text{ V}/10 \Omega = 1 \text{ A}$ and the current through 5Ω will be $10 \text{ V}/5 \Omega = 2 \text{ A}$. These currents flow from top node to bottom node.

Now we apply KCL at the top node.

Current flowing into the positive polarity of 10V source = $5 \text{ A} + 1 \text{ A} + 2 \text{ A} = 0$.

Example: 2.5-2

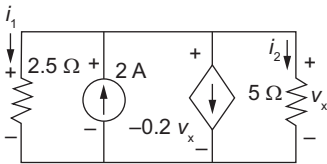


Fig. 2.5-4 Circuit for Example 2.5-2

Find the circuit solution for the circuit in Fig. 2.5-4.

Solution

The variable assignment and reference polarity assignment is shown in Fig. 2.5-5.

The terminal voltage of all elements will be v_x with this polarity assignment. Now, $i_1 = v_x/2.5 \text{ A}$ and $i_2 = v_x/5 \text{ A}$. Applying KCL at the top node, we get

$$i_1 - 2(-0.2v_x) + i_2 = 0$$

$$\text{i.e., } \frac{v_x}{2.5} - 2 + (0.2v_x) + \frac{v_x}{5} = 0 \Rightarrow 0.4v_x = 2 \Rightarrow v_x = 5 \text{ V}$$

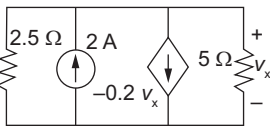


Fig. 2.5-5 Reference polarity assignment in the circuit in Example 2.5-2

Now, $i_1 = 2 \text{ A}$ and $i_2 = 1 \text{ A}$. The circuit solution is marked in Fig. 2.5-6.

Power absorbed by 2.5Ω resistor = $5 \text{ V} \times 2 \text{ A} = 10 \text{ W}$

Power absorbed by 5Ω resistor = $5 \text{ V} \times 1 \text{ A} = 5 \text{ W}$

Power delivered by 2 A current source = $5 \text{ V} \times 2 \text{ A} = 10 \text{ W}$

Power delivered by the dependent current source = $5 \text{ V} \times 1 \text{ A} = 5 \text{ W}$

Total power delivered = Total power absorbed = 15 W

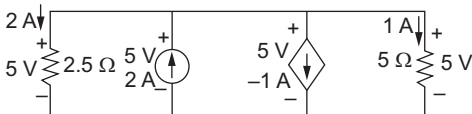


Fig. 2.5-6 Circuit solution in Example 2.5-2

2.6 ANALYSIS OF MULTI-LOOP, MULTI-NODE CIRCUITS

The method of analysis of multi-loop, multi-node circuits, using element relations along with KCL and KVL equations, is illustrated through worked examples in this section.

Example: 2.6-1

Find the voltage across the dependent current source (v_y) in the circuit in Fig. 2.6-1.

Solution

This circuit has six elements, three nodes and three meshes. We need to find out only v_y . Let us try to solve the circuit in terms of v_x and v_y without using any other new variables.

$$\begin{aligned}\text{We note that } v_{AD} &= v_{CD} + v_{AC} = (10 - v_x) \text{ V.} \\ \therefore v_{AB} &= v_{AD} - v_{BD} = (10 - v_x) - v_y \text{ V.} \\ v_{BC} &= v_{BD} - v_{CD} = (v_y - 10) \text{ V.} \\ v_{CA} &= v_x \text{ V.}\end{aligned}$$

Note that we have essentially employed KVL in order to arrive at these relations. Now, we can express all the currents at node-A and node-B in terms of v_x and v_y . The currents *going away from* node-A are $(v_{AB}/3)$ A, $(v_{AD}/8)$ A and $(-v_{CA}/1)$ A. The sum of these three terms must be zero.

$$\therefore \frac{10 - v_x - v_y}{3} + \frac{10 - v_x}{8} - \frac{v_x}{1} = 0$$

The currents *going away from* node-B are $(-v_{AB}/3)$ A, $(v_{BC}/5)$ A and v_x . Sum of these terms must be zero.

$$\therefore \frac{-10 + v_x + v_y}{3} + \frac{v_y}{5} + v_x = 0$$

Simplifying these two equations, we get

$$35v_x + 8v_y = 110$$

$$20v_x + 8v_y = 80$$

Solving these two equations, we get $v_x = 2$ V; $v_y = 5$ V

Therefore, the voltage across the dependent current source = 5 V.

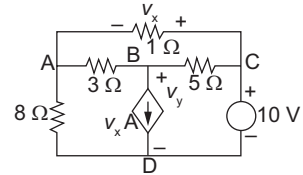


Fig. 2.6-1 Circuit for Example 2.6-1

Example: 2.6-2

Find the power delivered by the voltage and current sources in the circuit shown in Fig. 2.6-2.

Solution

We need to find out the current through the 20 V voltage source and the voltage across the 5 A current source. Refer to Fig. 2.6-3.

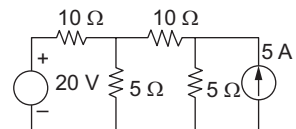


Fig. 2.6-2 Circuit for Example 2.6-2

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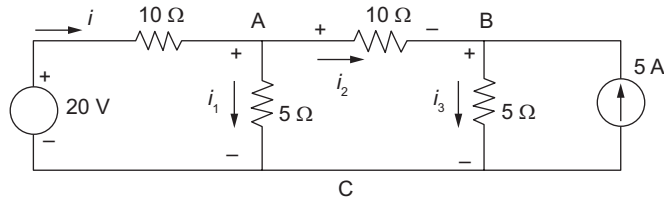


Fig. 2.6-3 Circuit in Example 2.6-2 with variables assigned

KVL in the first mesh gives

$$\begin{aligned} -20 + 10i + V_{AC} &= 0 \\ \Rightarrow V_{AC} &= (20 - 10i) \text{ V.} \\ \therefore i_1 &= \frac{20 - 10i}{5} = 4 - 2i \text{ A.} \end{aligned}$$

KCL has to be satisfied at node-A.

$$\therefore i_2 = i - i_1 = i - (4 - 2i) = 3i - 4 \text{ A.}$$

Applying KCL at node-B, we get, $-i_2 + i_3 - 5 = 0 \Rightarrow i_3 = i_2 + 5 = (3i + 1) \text{ A}$

Therefore, $V_{BC} = 5 \times (3i + 1) = 15i + 5 \text{ V.}$

Now we apply KVL on the outer loop of the circuit to get,

$$\begin{aligned} -20 + 10i + 10i_2 + V_{BC} &= 0 \\ \text{i.e., } -20 + 10i + (30i - 40) + (15i + 5) &= 0 \\ \text{i.e., } 55i &= 55 \\ \therefore i &= 1 \text{ A} \end{aligned}$$

And, $V_{BC} = 15i + 5 = 20 \text{ V.}$

Therefore, the current delivered by the voltage source is 1 A and the voltage appearing the current source is 20 V.

Therefore, the power delivered by the voltage source = $20 \text{ V} \times 1 \text{ A} = 20 \text{ W}$

The power delivered by the current source = $20 \text{ V} \times 5 \text{ A} = 100 \text{ W}$

Example: 2.6-3

Find i_x in the circuit shown in Fig. 2.6-4.

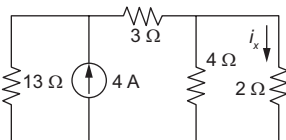


Fig. 2.6-4 Circuit for Example 2.6-3

Solution

Voltage across 2Ω resistance is $2i_x \text{ V}$ with positive polarity at the top terminal. Therefore, the current through 4Ω is $2i_x/4 = 0.5i_x \text{ A}$ from the top terminal to the bottom terminal.

Therefore, the current through 3Ω must be $1.5i_x \text{ A}$ from left to right by KCL applied to the node at which the three resistors are connected.

Therefore, the voltage across 13Ω must be $2i_x + 3 \times 1.5i_x = 6.5i_x \text{ V.}$

Therefore, the current through $13\ \Omega$ must be $6.5i_x/13 = 0.5i_x$ A from the top terminal to the bottom terminal. Now, apply KCL at the current source node to get $1.5i_x + 0.5i_x = 4$.
 $\therefore i_x = 2$ A.

Example: 2.6-4

Find the ratio $\frac{v_o}{v_s}$ in the circuit shown in Fig. 2.6-5.

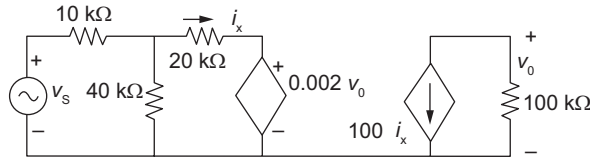


Fig. 2.6-5 Circuit for Example 2.6-4

Solution

The current that flows through $100\ \text{k}\Omega$ at output side is $100i_x$ from the bottom terminal to the top terminal. Therefore, $v_o = -10^7 i_x$ V. Therefore, the voltage generated by the VCVS at input side is $-0.002 \times 10^7 i_x = -2 \times 10^4 i_x$ V with polarity as shown in Fig. 2.6-5.

Therefore, voltage across $40\ \text{k}\Omega$ resistance is $= 20 \times 10^3 i_x - 2 \times 10^4 i_x = 0$ V.

Therefore, current through $40\ \text{k}\Omega$ resistance is 0 A.

Applying KVL in the first mesh, we get, voltage drop across $10\ \text{k}\Omega = v_s$ V.

Therefore, current through $10\ \text{k}\Omega$ resistance $= 10^{-4} v_s$ A

Current through $40\ \text{k}\Omega$ is zero. Therefore, by KCL, $i_x = 10^{-4} v_s$ A.

We know that $v_o = -10^7 i_x$ V.

$$v_o = -10^7 i_x = -10^7 \times 10^{-4} v_s = -1000 v_s$$

$$\therefore \frac{v_o}{v_s} = -1000$$

2.7 KVL AND KCL IN OPERATIONAL AMPLIFIER CIRCUITS

Linear dependent sources are used to model electronic amplifiers. For instance, the VCVS is the model of an ideal voltage-to-voltage amplifier. Ideal amplifiers are physically unrealisable. Operational Amplifiers try to approach the ideal.

Operational Amplifier is a multi-stage high gain voltage-to-voltage amplifier with differential input and single-ended output. It is an integrated circuit package. We use the short name ‘Opamp’ for Operational Amplifier in this book.

It is a *differential amplifier* and has two signal input terminals. It amplifies the difference between the voltages applied at input terminals. These two input voltages as well as the output voltage are referred to a common *ground* terminal.

Ground in an Opamp is not a terminal or pin of the Opamp. It is a node outside the Opamp. In Opamps working from a single DC power supply, the *ground* is commonly assigned to the negative of

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the DC source. In Opamps working *from a balanced dual DC power supply*, the *ground* is commonly taken as the midpoint of the dual supply. The output voltage is measured between a single output terminal from the Opamp and the *ground* terminal that is outside the Opamp package.

Various integrated circuit design techniques are employed to minimise the non-linear distortion in output signal arising out of non-linear transfer characteristics of transistors. However, even an Opamp cannot do away with voltage, current and rate limits. The non-linear distortion arising out of *clipping* due to one or more of these limits takes place in Opamp circuits when they are overdriven or overloaded. Thus, linear models are applicable to Opamps as long as they are not operating in one of the *limited modes*.

An ideal voltage amplifier is expected to have infinite input resistance, zero output resistance and infinite bandwidth, *i.e.*, it does not differentiate between two sinusoidal signals of same amplitude and different frequencies and provides same gain to both. *Bandwidth is a measure of variation of gain with frequency of an applied sinusoid.*

A practical Opamp has a very large input resistance (in $M\Omega$) and small output resistance (in Ω) and it has finite bandwidth. Therefore, if the input applied to it is a mixture of sinusoids, it will offer different gains for different sinusoidal components at different frequencies. It will delay the output by a time-delay that will depend on the frequency. These two kinds of differential treatment to sinusoids of different frequencies lead to a difference in waveshape of output compared to that of the input. That is *distortion*. It is due to the gain of Opamp becoming a function of frequency and that happens because of capacitance of transistors. This distortion takes place even when the Opamp is in the *linear range* of operation. Therefore, it is called *linear distortion*.

Non-linear distortion changes the waveshape of output even when the input is a single frequency sinusoid. But linear distortion does not do that. Linear distortion changes the waveshape of output only when the input is *not* a single frequency sine wave, but is a mixture of sine waves of different frequencies. An ideal Opamp does not have any linear distortion.

The voltage gain of a practical Opamp is very large – typically hundreds of thousands. We hardly ever need that kind of gain in any practical application. Thus, we rarely find an Opamp being used as an amplifier without some other components (usually resistors) limiting the gain of the overall amplifier circuit to the required value that is likely to be in the range 1–100. But then, why make a circuit with a huge gain and kill its gain when it is used for amplification purposes? The simple answer is that the huge gain of Opamp is the currency that we pay for improvements in other performance measures of the overall amplifier circuit. We gain on other performance parameters by paying out gain.

For instance, the input resistance of overall amplifier can be increased and its output resistance decreased by sacrificing the gain. Its bandwidth can be increased and non-linear distortion can be decreased by sacrificing the Opamp gain.

The circuit technique that we use in order to bring about this trade-off between gain of Opamp and performance of overall amplifier circuit in which the Opamp is embedded is called *negative feedback*. We will take it up in a later section. But we note here that higher the Opamp gain, better the advantages that accrue from employing negative feedback around it.

An Opamp is expected to produce zero output when both its input terminals are connected to same voltage with respect to ground. However, practical Opamps do produce a small output under this condition. The corresponding gain is called the *common-mode gain*. The gain registered by the Opamp when it is driven by two equal but opposite sources at its input terminals is called the *differential gain*. The ratio of differential mode gain to common mode gain is defined as its '*Common Mode Rejection Ratio*' and is usually quoted in decibels (dB). Decibel value of a quantity is obtained by calculating 20 times the logarithm of the quantity with 10 as the base of logarithm.

An Opamp is expected to produce zero output when both its input terminals are connected to ground. However, practical Opamps do produce non-zero output under this condition. Thus practical Opamps exhibit *output offset*.

We now arrive at a conceptual idealisation and define an Ideal Operational Amplifier (IOA) and list its features. Ideal Opamp, obviously, cannot be made. But it provides a benchmark for evaluating a practical Opamp.

An Ideal Operational Amplifier is a voltage-to-voltage differential amplifier with infinite input resistance, infinite gain, infinite bandwidth and zero output resistance. An IOA has the following features:

Input resistance, $R_{in} \rightarrow \infty$

Output resistance, $R_o \rightarrow 0$

Voltage gain, $A_v \rightarrow \infty$

Bandwidth, $bw \rightarrow \infty$

Common mode rejection ratio (CMRR) $\rightarrow \infty$

No voltage, current and slope limits

Zero offsets and zero input bias currents

2.7.1 The Practical Operational Amplifier

The most popular general purpose Opamp, the μA 741, is available as a DIP (Dual-in-Line) package of $\frac{1}{4}$ inch $\times$ $\frac{3}{8}$ inch size. It has 8 pins. The pin details and circuit symbol of Opamp are shown in Fig. 2.7-1. It is a dual supply Opamp and the midpoint of the supply connection is taken as ground usually.

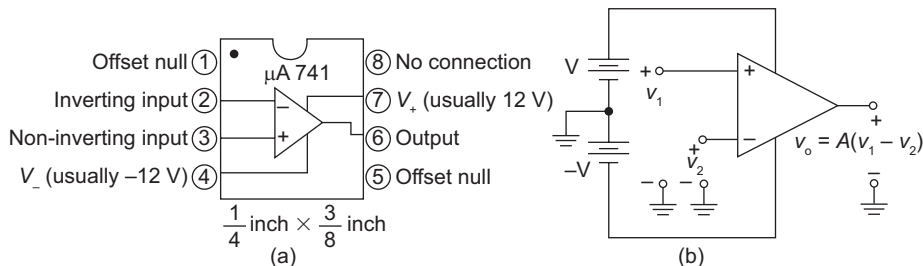


Fig. 2.7-1 (a) Top view of μA 741 IC (b) Circuit symbol and voltage polarities for an Opamp

This Opamp has a differential gain of 250,000 and CMRR of 80 dB. This implies that its common mode gain is indeed negligible. Its input resistance is about $2\text{ M}\Omega$ and output resistance is about $75\ \Omega$.

With a supply voltage of ± 12 V, its output saturation limits are 10.6 V and -11 V. It has a bandwidth of ≈ 4 Hz (very small) and a slew rate of $0.5\text{ V}/\mu\text{s}$. Its output current is limited at ± 20 mA with a supply voltage of ± 12 V.

Its input bias currents, called I_{B+} and I_{B-} , are ≈ 100 nA and the difference ($I_{B+} - I_{B-}$) between them can be ± 20 nA. This difference is called *input bias offset current*.

The output of this Opamp will be at one of the saturation limits when both inputs are grounded. The offset is so high. Hence the offset voltage for Opamps is specified at input indirectly rather than at output directly. Applying a differential voltage across the inverting and non-inverting inputs of the Opamp can null the output offset. The differential voltage that has to be applied across the two

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input terminals in order to bring the output to zero with respect to ground is defined as the *input offset voltage* and is denoted by v_{io} . The value for $\mu A741$ is about 2 mV. The polarity of this voltage can be decided only by experiment on the particular piece of IC that is being used.

The circuit in Fig. 2.7-1(b) shows the power supply connections in detail. However, these connections are usually suppressed in circuit diagrams involving Opamps. It is understood implicitly that the Opamps are powered properly in circuits. This is permissible since we seldom apply KCL at power supply nodes of Opamps in analysing a circuit containing Opamps. We do not apply KCL at ground node too. But that is because of the fact that we always assign the role of reference node to the ground node in Opamp circuits when we prepare the node equations of such circuits.

2.7.2 Negative Feedback in Operational Amplifier Circuits

Consider the circuit in Fig. 2.7-2(a). We ignore the offsets in output in our analysis in this section. Therefore, the output in the circuit in Fig. 2.7-2(a) is given by $v_o = A(v_1 - v_2)$, where A is the differential mode voltage gain of the Opamp. The value of A for $\mu A741$ Opamp is around 250,000. Let us assume that the power supply used is ± 12 V in all the three circuits. The voltage saturation levels will be taken approximately equal to the supply voltage of 12 V. Actually, it is 10.6 V and -11 V; we ignore the difference.

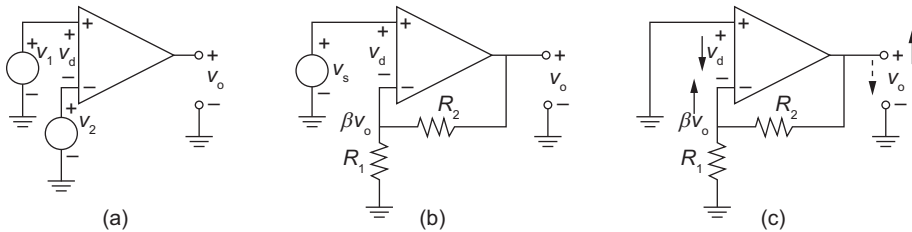


Fig. 2.7-2 (a) Opamp on open loop (b) Opamp embedded in a resistive network (c) Opamp circuit illustrating negative feedback

Hence, the first circuit will saturate when the differential input voltage $v_d = v_1 - v_2$ goes out of the range $(-48 \mu V, +48 \mu V)$. We usually do not need this much of gain. If $v_d = v_1 - v_2$ is in mV or V range, the output will spend most of the time in non-linear range under saturated condition. For instance, say $v_1 = 0.0048 \sin 2\pi t$ V and $v_2 = 0$. Then the output will be an almost clean square wave of period 1 s and amplitude of 12 V. The Opamp will be in the linear range of operation only for about 6.4 ms in 1 s. It will be in saturated condition for the remaining duration. Thus, this mode of using Opamp, called the *open-loop* mode, is not really useful for linear amplification purposes.

Consider the second circuit in Fig. 2.7-2.

Here the Opamp is embedded in a resistive network that generates interaction between the input side of Opamp with its own output side. This interaction takes place through the resistor chain R_1 and R_2 . They loop back the output of Opamp to its own input. When there is such ‘looping back’ of output of an Opamp to its input side in a circuit, we state that *the circuit employs feedback*. The *sense of feedback* can be negative or positive. We decide the *sense of feedback* in the circuit by analysing the circuit after assuming that its source is reduced to zero. The resulting circuit is shown in Fig. 2.7-2(c). We expect the output to be at zero since the Opamp was assumed to be free of offset.

Now, let us analyse the process that takes place in the system when the output is momentarily disturbed by some kind of electromagnetic pickup from some neighbouring circuit. Assume that the

output voltage increased from zero level. This increase is felt at the inverting input of Opamp through the $R_1 - R_2$ chain. The current drawn by the inverting input of Opamp loads this potential divider. However, the Opamp has very large input resistance and hence the $R_1 - R_2$ potential divider may be considered unloaded. Therefore, the potential that appears at inverting input is βv_o , where β is the feedback factor and is equal to $R_1/(R_1 + R_2)$. Therefore, when the output increases, the inverting input potential with respect to ground node also increases. This leads to a reduction in the differential voltage v_d that appears across the Opamp input terminals. A reduction of differential input of Opamp is followed by a reduction of its output voltage itself. Thus, we see that an inadvertent increase in Opamp output goes through the feedback loop to generate a corrective action that tends to restore the output to its pre-disturbed condition. When feedback results in this kind of corrective action, we term it as *negative feedback* or *degenerative feedback*.

Note that if we had connected the resistor chain at non-inverting input of Opamp and the input source to the inverting input terminal, a regenerative action, instead of a corrective action, would have taken place. An increase in output would have resulted in an increase in differential input voltage and that would have resulted in further increase in Opamp output. All circuits are under constant disturbance from other circuits. Therefore, an Opamp circuit with this kind of feedback connection will find itself going to one of the saturation levels due to some initial pickup voltage at output getting encouragement from the feedback to grow further in the same direction that it started with. This kind of feedback is called *positive feedback* or *regenerative feedback*. Positive feedback usually takes the Opamp output to saturation condition as soon as it is switched on. Obviously, positive feedback Opamp circuits cannot function as linear amplifiers.

A simple interchange of the roles of inverting input terminal and non-inverting input terminal of an Opamp in circuit changes the nature of feedback in the circuit and affects the function and operation of the circuit significantly. One has to be very careful in drawing the circuit diagrams containing Opamp. The inverting and non-inverting terminals of Opamps have to be marked properly without fail.

2.7.3 The Principles of 'Virtual Short' and 'Zero Input Current'

We continue with the analysis of circuit in Fig. 2.7-2(b). We have ascertained that the feedback involved in this circuit is negative in sense. Hence we expect the output to be zero when the input is zero. Now, we set out to find the output in terms of input when the input is non-zero. Assume that the Opamp is ideal. Hence, its input resistance is an open-circuit, output resistance is a short-circuit and its gain is infinity. The gain is taken as A first and is sent to infinity at the end of circuit solution. The circuit is redrawn with the Opamp replaced by its ideal equivalent circuit as in the circuit in Fig. 2.7-3(b). We derive the equation for output as follows:

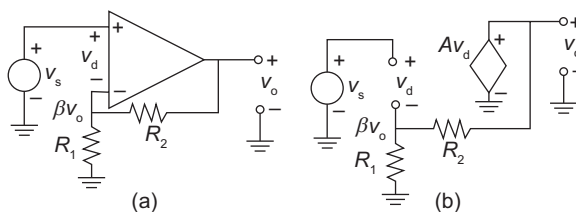


Fig. 2.7-3 (a) A single Opamp Amplifier circuit (b) Circuit with Opamp replaced by its equivalent circuit

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$$v_d = v_s - \beta v_o \text{ where } \beta = \frac{R_1}{R_1 + R_2}$$

$$v_o = A v_d = A(v_s - \beta v_o)$$

$$\therefore v_o = \frac{A}{1 + A\beta} v_s \text{ and } v_d = \frac{1}{1 + A\beta} v_s$$

We observe that the differential input v_d is only $1/(1+A\beta)$ times that of what it would have been had the source been applied directly to Opamp as in the circuit in Fig. 2.7-2(a). Since the Opamp gain is very large for a practical Opamp and is infinity for an ideal Opamp, we evaluate the limit of these expressions as $A \rightarrow \infty$. We get,

$$v_o = \lim_{A \rightarrow \infty} \frac{A}{1 + A\beta} v_s = \frac{1}{\beta} v_s = \left[1 + \frac{R_2}{R_1} \right] v_s$$

$$\text{and } v_d = \lim_{A \rightarrow \infty} \frac{1}{1 + A\beta} v_s = 0$$

Hence, the gain of overall amplifier goes to $1+R_2/R_1$. It is decided by external components entirely. And the differential input voltage goes to zero.

Why does the differential input voltage go to zero? If the Opamp is in linear range, its differential input voltage has to be equal to its output divided by the gain. The negative feedback present in the circuit resists any change in the output. Consider the situation when a certain voltage is suddenly applied to the input. Then the differential voltage increases suddenly since the Opamp will take a little time to adjust its output. The large differential voltage causes the Opamp output to increase. Increasing Opamp output reduces the differential input voltage through the feedback mechanism. Finally, a steady state comes up in the circuit when the output is such a value that the difference between the source voltage and the fed back voltage is exactly equal to the output divided by gain. The circuit attains equilibrium under that condition. Any deviation from this equilibrium condition will be corrected by negative feedback action. Since the gain is large, it requires only a small differential voltage to remain at this equilibrium. For instance, let v_s be 0.1 V, $A = 250,000$ and $\beta = 0.1$. Then $v_d = 0.1 \times 1/(1 + 25000) = 4 \mu\text{V}$ and $v_o = 0.1 \times 250000/(1 + 25000) = 999960 \mu\text{V} \approx 1 \text{ V}$. It requires only $4 \mu\text{V}$ of v_d to justify $\approx 1 \text{ V}$ of output since the gain is 250,000. Now if the gain is increased further, the value of v_d goes down further and v_o approaches closer to 1 V. In the limit when gain goes to infinity, v_d goes to zero and v_o goes to 1 V.

But will v_d be zero if v_s is 10 V? No, since the amplifier will saturate and will be in the non-linear range of operation. The large gain that is effective in linear range of operation is not available when the Opamp is operating in voltage-limited or current-limited or slope-limited modes of operation. Hence, we may conclude that the differential voltage across the non-inverting input terminal and the inverting input terminal of an Opamp is arbitrarily close to zero if the Opamp is under negative feedback and is in its linear range of operation. Thus, the two input terminals, though are not connected together, are virtually at the same potential under these conditions and behave as if they are shorted together. Therefore, there is a *virtual short* across the input terminals of an Opamp working in its linear range of operation in a negative feedback circuit. We emphasise this principle in the following.

The virtual short principle

The input terminals of an ideal Opamp in a negative feedback circuit behave as if there is short-circuit across them.

The input terminals of a practical Opamp with large gain behave as if there is short-circuit across them provided the Opamp is in a negative feedback circuit and it is operating in its linear range.

The input resistance of an ideal Opamp is infinite and hence the Opamp does not draw any current at its input terminals. The input resistance of a practical Opamp is large and the current drawn by the input terminals is usually negligible compared to currents elsewhere in the circuit. This remains true even when the Opamp is in its non-linear range of operation. We state this principle in the following.

The zero input current principle

The input terminals of an ideal Opamp do not draw any current from the circuit in which the Opamp is embedded.

Thus, from the point of view of input currents drawn by the ideal Opamp, its input terminals represent an open-circuit, and, from the point of view of differential input voltage, the same two terminals represent a short-circuit. This model for an Opamp is called the *Ideal Opamp Model* (IOA Model). *It is emphasised again that IOA model will lead to correct analysis only if the Opamp is in a negative feedback circuit and is working in its linear range of operation.*

2.7.4 Analysis of Operational Amplifier Circuits Using the IOA Model

The principles enunciated in the preceding section can be used to develop a much-simplified analysis procedure for circuits containing Opamps. The procedure is outlined in the following:

1. Ascertain whether the circuit has negative feedback or positive feedback. If it is a positive feedback circuit, the IOA model cannot be used for its analysis. Only the principle of zero input current will be applicable to such circuits. Other analysis procedures using nodal analysis or mesh analysis along with zero input current principle will then be needed.
2. Prepare KCL equations at all nodes except ground node. Ground node is taken as the reference node. Use the principle of zero input current in writing the KCL equations.
3. Apply the principle of virtual short on all Opamps in the circuit to reduce the number of KCL equations and solve the reduced set of equations.

This procedure is illustrated in the case of various Opamp circuits in the remainder of this section. These circuits not only serve as illustrations for the technique of analysis but also introduce the reader to Opamp circuits that are frequently employed in analog signal processing applications. In fact, the popularity of these circuits provides motivation for inclusion of Opamp circuits in a book on basic circuit theory.

The Non-Inverting Amplifier Circuit

The first circuit is called *Non-Inverting Amplifier* or *Amplitude Scalar*. We have already used this circuit in the preceding sections. It is shown in the circuit in Fig. 2.7-4 (a). The circuit in Fig. 2.7-4 (b) shows the application of the principles of zero input current and virtual short.

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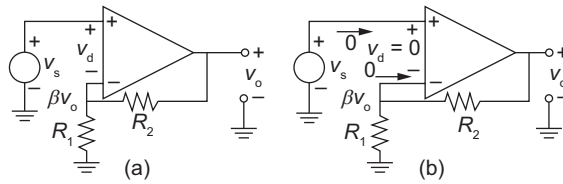


Fig. 2.7-4 (a) The non-inverting amplifier circuit (b) Application of IOA model to the amplifier circuit

We do not need any KCL equations here. The potential divider formed by R_1 – R_2 is unloaded at its output and is driven by an ideal dependent voltage source with zero series resistance. Therefore potential at inverting input is βv_o , where $\beta = R_1/(R_1 + R_2)$. The potential at non-inverting input is v_s . It will be v_s even if the source has series resistance. This is due to the fact that the current in input line is zero. Now, we apply the principle of virtual short to equate these two potentials. Both potentials are referred to the ground node.

$\therefore v_s = \beta v_o \Rightarrow v_o = \frac{v_s}{\beta} = \left[1 + \frac{R_2}{R_1}\right] v_s$. Therefore, the gain of non-inverting amplifier = $1 + \frac{R_2}{R_1}$. It has infinite input resistance and zero output resistance as per IOA model. Practically, input resistance and output resistance of Opamp will limit these.

The Voltage Follower Circuit

This circuit is a special case of the non-inverting amplifier circuit. R_2 is kept at zero value and R_1 is made an open-circuit. The resulting gain will be unity. The unity gain amplifier is used as a buffer amplifier to interconnect circuits without interfering with each other. When a second circuit is connected as a load to the first circuit, the performance of first circuit is affected by the input characteristics of the second circuit and the performance of second circuit is affected by the output characteristics of the first circuit. A unity gain buffer amplifier interfaces the two circuits by providing an infinite load resistance to the first circuit and an ideal voltage source with zero series resistance to the second circuit. Voltage follower circuit shown in Fig. 2.7-5 is used in buffering applications.

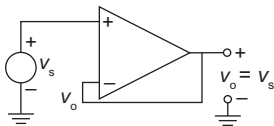


Fig. 2.7-5 The voltage follower circuit

The Inverting Amplifier Circuit

This amplifier is shown in Fig. 2.7-6.

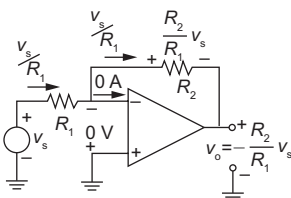


Fig. 2.7-6 The inverting amplifier circuit

Assume that input is grounded and the output voltage deviates upwards from zero due to some temporary disturbance. Then the voltage fed back to the inverting terminal increases, thereby effecting a decrease in the differential input voltage. This leads to corrective action at the output. Hence, there is negative feedback in this circuit.

The non-inverting input is grounded. Therefore, by virtual short principle, the potential at the inverting input is also zero. That is, the inverting terminal is *virtually grounded*. Hence, the

current in R_1 has to be v_s/R_1 A. Thus, the Opamp with grounded non-inverting terminal converts the input voltage source into a current source with the conversion factor decided by R_1 .

The input current into the inverting input terminal is zero by zero input current principle. Therefore, the current in R_2 has to be v_s/R_1 A by KCL at inverting input. This current flow results in a voltage drop of $v_s R_2/R_1$ V with the polarity shown in Fig. 2.7-6. Applying KVL in a loop starting at non-inverting input terminal and ending at the ground node, we get

$$0 + \frac{R_2}{R_1} v_s + v_o = 0.$$

Therefore, $v_o = -\frac{R_2}{R_1} v_s$. The gain of the amplifier is $-R_2/R_1$. The output undergoes a sign change with respect to input and hence the name – *Inverting Amplifier*. The input resistance of the amplifier is R_1 . The output resistance of this amplifier is zero as per IOA model and is small-valued in practice.

The Inverting Summer

The circuit of *Inverting Summer* is shown in Fig. 2.7-7.

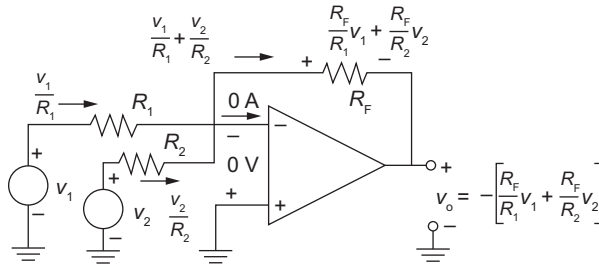


Fig. 2.7-7 The inverting summer amplifier circuit

The inverting input terminal is at virtual ground. Therefore, the currents through R_1 and R_2 are v_1/R_1 and v_2/R_2 , respectively. These two currents add at the inverting node and flow into the feedback resistor R_F , producing a voltage drop of $v_1(R_F/R_1) + v_2(R_F/R_2)$ across it. Therefore, the output voltage is $-[v_1(R_F/R_1) + v_2(R_F/R_2)]$.

The circuit acts as an inverter summer if $R_1 = R_2 = R_F = R$. Then the output voltage will be $-(v_1 + v_2)$.

More than two sources can be connected to summer in the same manner. Essentially, the virtual ground at inverting input terminal results in a voltage to current conversion in all source lines. The inverting input acts as a *summing node* where all the input line currents get added. The sum current gets pushed into the feedback resistor since Opamp input terminal does not draw any current.

The Non-Inverting Summer Amplifier

The non-inverting summer amplifier with three inputs is shown in Fig. 2.7-8.

The gain from non-inverting input terminal to the output terminal will be $1+(k-1)R/R = k$. We need to find the potential at non-inverting input terminal in terms of v_1 , v_2 and v_3 in order to obtain an expression for v_o in terms of the source voltages. The node potential at non-inverting input is marked as v . Applying KCL at this node and using the principle of zero input current, we get the following node equation, where the G 's represent the conductance values.

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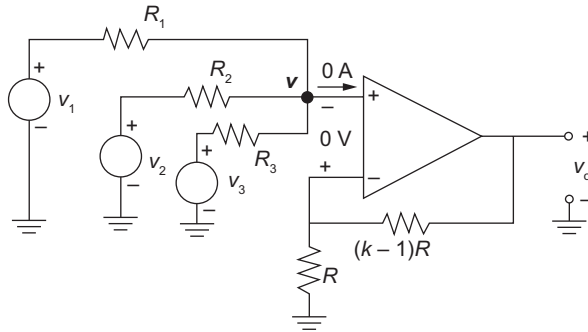


Fig. 2.7-8 The non-inverting summer amplifier circuit

$$G_1(v - v_1) + G_2(v - v_2) + G_3(v - v_3) = 0$$

$$\text{i.e., } v[G_1 + G_2 + G_3] = G_1v_1 + G_2v_2 + G_3v_3$$

$$\therefore v = \frac{G_1v_1 + G_2v_2 + G_3v_3}{G_1 + G_2 + G_3}$$

$$\text{Therefore, the output, } v_o = \frac{kG_1}{G_1 + G_2 + G_3}v_1 + \frac{kG_2}{G_1 + G_2 + G_3}v_2 + \frac{kG_3}{G_1 + G_2 + G_3}v_3$$

If there are n inputs connected at the input, the equation for v_o gets generalised to

$$v_o = \frac{k}{\sum_{j=1}^n G_j} \sum_{i=1}^n G_i v_i \text{ V.}$$

If all the input resistors are equal and $k = n$, then the circuit performs the addition function without any gain,

$$\text{i.e., } v_o = \sum_{i=1}^n v_i \text{ V.}$$

It may be observed that the gain for each source voltage is dependent on the conductance values of resistors in *all* source lines. Hence, addition of an extra source will affect the gain for all other sources. The inverting summer circuit does not suffer from this shortcoming, thanks to the inverting input terminal acting as a *summing node for currents*.

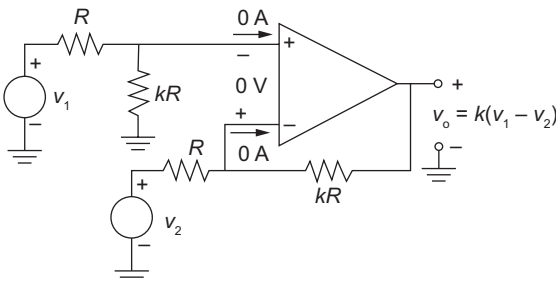


Fig. 2.7-9 The subtractor circuit using a single Opamp

If one can spend two Opamps, an inverting summer followed by an inverting amplifier with a gain of -1 is a better solution than the non-inverting summer.

The Subtractor Circuit

The *Subtractor Circuit* is shown in Fig. 2.7-9.

The principle of zero input current is used to arrive at the conclusion that the potential dividers connected at both input terminals are unloaded. Therefore, the potential of non-inverting input terminal is $kv_1/(k+1)$.

The potential at the inverting input terminal is the same as the potential at non-inverting input terminal by the principle of virtual short. Therefore, the potential at inverting terminal is $kv_1/(k+1)$ V with respect to ground node.

Now, we write the KCL at inverting input terminal and make use of principle of zero input current while writing down the equation. The KCL equation is obtained as

$$G\left(\frac{kv_1}{k+1} - v_2\right) + \frac{G}{k}\left(\frac{kv_1}{k+1} - v_o\right) = 0. \text{ Solving for } v_o, \text{ we get,}$$

$$v_o = k(v_1 - v_2). \quad k = 1, \text{ if all resistors chosen are equal.}$$

The circuit is expected to give zero output if $v_1 = v_2$. That is, its common mode gain is expected to be zero. However, in practice it isn't. The reason is the invariably present deviations in the resistance value of resistors from their nominal values. Resistors have a tolerance factor. If the actual values of resistance of resistors is such that the ratio between the two resistors connected to inverting terminal is different from the ratio of resistors connected at the non-inverting terminal, the common mode gain of the circuit will not be zero. This circuit is also called *differential amplifier*. The input resistance offered to the sources depends on the value of R used in the circuit. The performance of this circuit in amplifying voltage difference is satisfactory for less stringent applications.

Example: 2.7.1

Find the value of V in the circuit in if the output is observed to be 0 V.

Solution

This is a non-inverting amplifier of gain $= 1 + \frac{9R}{R} = 10$. The output voltage will be 10 times the voltage at non-inverting pin with respect to ground. If the output is seen to be zero, then the voltage at non-inverting input must be 0 V.

Input terminals of Opamp do not draw current.

Hence, the 2 mA delivered by the current source will flow through 10 k resistor. Therefore, the voltage at the non-inverting pin is $V + 10 \text{ k} \times 2 \text{ mA} = (V + 2) \text{ V}$. V has to be -2 V for this to become 0 V.

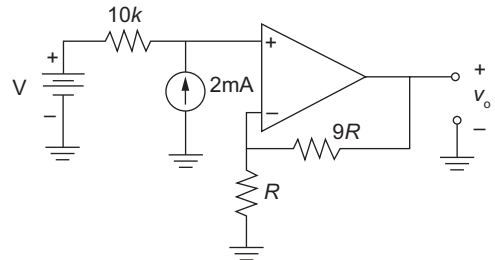


Fig. 2.7-10 Circuit for Example 2.7.1

Example: 2.7.2

Find the output voltage in the Opamp circuit in Fig. 2.7-11.

Solution

Inverting pin is at the same potential as that of the non-inverting pin by virtual short principle. Therefore, inverting pin is at 2 V. Hence, the voltage drop across 1 k resistor is $(v_s(t) - 2) \text{ V}$. Therefore, the current that comes to inverting pin from 1 k resistor line is $(v_s(t) - 2) \text{ mA}$. The current source delivers 4 mA to the same pin. Inverting pin of Operational Amplifier does not draw any

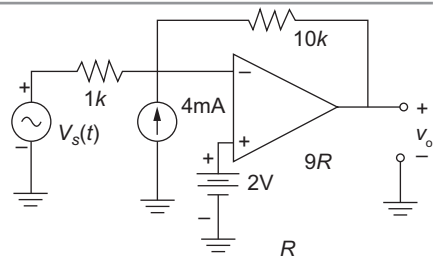


Fig. 2.7-11 Circuit for Example 2.7.2

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current. Therefore, the current that goes into 10 k will be $[4 + (v_s(t) - 2)] = (v_s(t) + 2)$ mA. Therefore, drop across 10 k will be $(10v_s(t) + 2)$ V. The inverting pin is at 2 V. Therefore, the output voltage = $2\text{ V} - (10v_s(t) + 2)\text{ V} = -10v_s(t)\text{ V}$.

2.8 SUMMARY

- Let a circuit contain b -elements, l -loops and n -nodes, and let it have a unique solution. Then the ‘circuit analysis problem’ involves solving for $2b$ variables – one voltage variable and one current variable per two-terminal element. b equations are obtained by using the element relation of b elements. The remaining b equations come from applying Kirchhoff’s Current Law and Kirchhoff’s Voltage Law. KVL and KCL equations depend only on the topology of the circuit and do not depend on the nature of elements.
- A *node* in a circuit is a junction point at which the connection leads of two or more points join together. The *node* is part of the connecting wire. Hence, according to the assumptions involved in lumped parameter circuit theory, there is negligible charge storage and negligible rate of change of charge storage at a node.
- Therefore, the net positive current that enters (or leaves) a node in a circuit through all the connecting wires connected at that node has to be zero. This is the Kirchhoff’s Current Law.
- Kirchhoff’s Current Law can be stated in three forms as follows:
 - Kirchhoff’s Current Law (KCL) states that the algebraic sum of currents leaving a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.
 - Kirchhoff’s Current Law (KCL) states that the algebraic sum of currents entering a node in a lumped parameter circuit is equal to zero on an instant-to-instant basis.
 - Kirchhoff’s Current Law (KCL) states that the sum of currents entering a node in a lumped parameter circuit through some wires must be equal to the sum of currents leaving the same node through the remaining wires on an instant-to-instant basis.
- Kirchhoff’s Current Law is also valid for any closed surface that intersects connecting wires and encloses more than one node and one or more elements without intersecting any element. Such a closed surface is called a *supernode* of the circuit.
- KCL equations for any $(n-1)$ nodes in a n -node circuit will form an independent set of equations.
- A *loop* in a circuit is *closed path* traced through nodes, connecting wires and elements such that no node is visited more than once in one complete traversal of the closed path.
- Kirchhoff’s Voltage Law for such a loop in a circuit can be stated in three equivalent forms as follows:
 - Kirchhoff’s Voltage Law states that the algebraic sum of voltages in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.
 - Kirchhoff’s Voltage Law states that the sum of ‘voltage rises’ in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.
 - Kirchhoff’s Voltage Law states that the sum of ‘voltage drops’ in any closed path in a lumped parameter circuit is zero on an instant-to-instant basis.
- A *planar* circuit is one that can be represented on a paper without crossing of connecting wires. Those closed loops that do not contain other loops within them in a planar circuit are called its *meshes*.
- The KVL equations for meshes in a planar circuit will be an independent set of equations.
- A set of two-terminal elements is said to be *series-connected* if a common current can flow through them.
- A set of two-terminal elements is said to be *parallel-connected* if they share the same node-pair. In this case, it is possible to make the terminal voltage variables of all such elements a common variable by suitable reference polarity assignment for voltage variables.
- An *Operational Amplifier* is a *multi-stage high gain voltage-to-voltage amplifier with differential input and single-ended output*. It is a *differential amplifier* and has two signal input terminals.

It amplifies the difference between the voltages applied at input terminals. These two input voltages as well as the output voltage are referred to a common *ground* terminal.

- An Ideal Operational Amplifier (IOA) is a voltage-to-voltage differential amplifier with infinite input resistance, zero output resistance, infinite differential gain, zero common mode gain, infinite bandwidth, no output current and rate restrictions, no offsets and no input bias currents.
- A simple interchange of the roles of inverting input terminal and non-inverting input terminal of an Opamp in circuit changes the nature of feedback in the circuit and affects the function and operation of the circuit significantly.
- *The principle of virtual short* – The input terminals of an ideal Opamp in a negative feedback circuit behave as if there is a short-circuit across them. The input terminals of a practical Opamp with large gain behave as if there is short-circuit across them provided the Opamp is in a negative feedback circuit and it is operating in its linear range.
- *The principle of zero input current* – The input terminals of an ideal Opamp do not draw any current from the circuit in which the Opamp is embedded.
- These two principles can be used to analyse Opamp circuits quickly, provided the Opamp is in a negative feedback circuit and is in its linear range of operation.

2.9 PROBLEMS

1. Some voltage and current variables are specified at an instant in the circuit in Fig. 2.9-1. Find the remaining variables at that instant as per the reference directions marked.

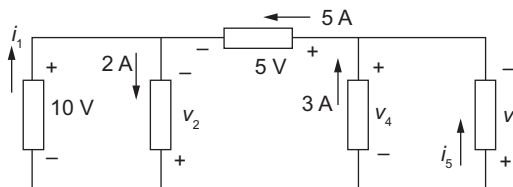


Fig. 2.9-1

2. The circuit shown in Fig. 2.9-2 is a DC circuit containing only resistors and independent sources. Some voltage and current variables are specified in the circuit. (i) Find the remaining variables with reference directions as shown. (ii) Identify those elements that *have to be* sources. (iii) Identify those elements that *can be* resistors. (iv) Identify the elements that receive positive power and calculate the total power absorbed by them. (v) Identify the elements that deliver positive power and calculate the total power absorbed by them.

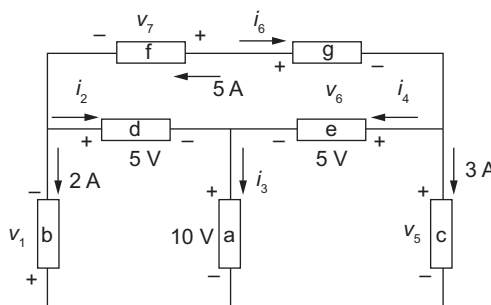


Fig. 2.9-2

2.36 | Basic Circuit Laws

3. (i) Express all element currents in terms of i_3 , i_4 , i_7 and i_8 in the circuit in Fig. 2.9-3. (ii) Express all element voltage variables in terms of v_3 , v_4 , v_7 and v_8 .

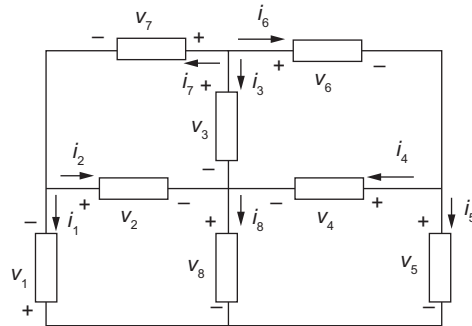


Fig. 2.9-3

4. The circuit shown in Fig. 2.9-4 is known to contain only resistors and independent sources. (i) Find all the voltage and current values in the circuit. (ii) What is the minimum number of independent sources that the circuit must contain? (iii) Draw the circuit configuration that contains minimum number of independent sources assuming that all sources are voltage sources. (iv) Draw the circuit configuration that contains minimum number of independent sources assuming that all sources are current sources.

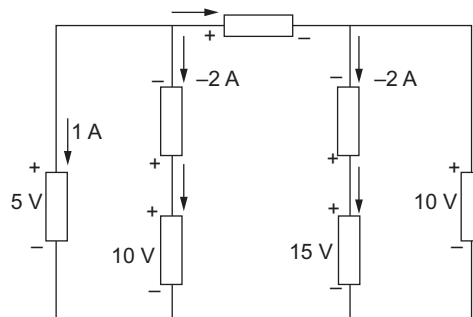


Fig. 2.9-4

5. (i) Show that the circuit solution for the circuit shown in Fig. 2.9-5 is the same as the solution for the circuit in Fig. 2.9-4. (ii) Calculate the power *delivered* by all current sources and voltage sources in the circuit and verify that the total power absorbed in the circuit is equal to the total power delivered in the circuit.

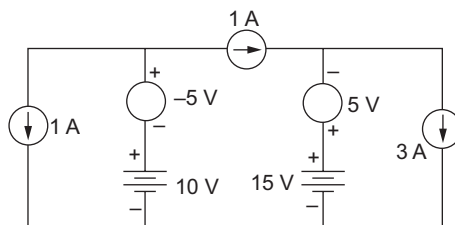


Fig. 2.9-5

6. (i) Complete the DC circuit solution for the circuit in Fig. 2.9-6. (ii) Construct *three* different circuits that have same circuit solution as in the circuit in Fig. 2.9-6 using three independent current sources and three independent voltage sources in all cases. Specify the source function values in each case.

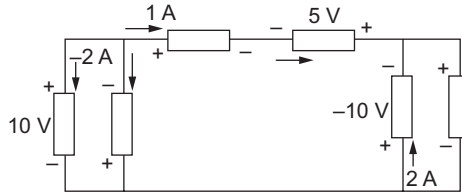


Fig. 2.9-6

7. Two identical DC practical voltage sources with internal electromotive force of E V and internal resistance of R ohms each are paralleled to meet a steady load current demand of I A represented by an independent current source in the circuit shown in Fig. 2.9-7. (i) Derive expressions for the currents delivered by the sources (I_1 and I_2) and the terminal voltage V_t in terms of E , I and R . (ii) What is the ratio of I_1 to I_2 for an arbitrary value of R ? (iii) What is the value of this ratio when $R \rightarrow 0$ in an identical manner in both sources? (iv) What is the difference between I_1 and I_2 if $R = 0$ in both sources? (v) What is the ratio of I_1 to I_2 if $R = 0$? Can it be determined uniquely. [Hint: There is a difference between ‘a quantity that tends to approach zero value’ and ‘a quantity that has a zero value’.]

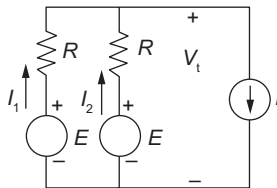


Fig. 2.9-7

8. Two identical DC practical current sources with a source current value of I A and internal resistance of R ohms each are connected in series to meet a steady load voltage demand of V V represented by an independent voltage source in the circuit shown in Fig. 2.9-8. (i) Derive expressions for the voltages appearing across the sources (V_1 and V_2) and the terminal current I_t in terms of I , V and R . (ii) Show that the two current sources share the load voltage equally for any finite value of R provided the resistance value remains equal for both sources. (iii) Show that the manner in which two current sources share the load voltage cannot be determined if the two current sources are *ideal*.

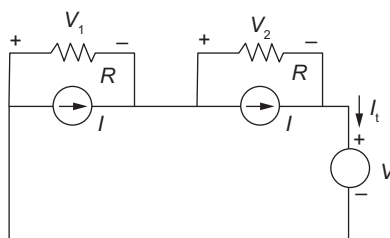


Fig. 2.9-8

2.38 | Basic Circuit Laws

9. Find the range of R such that the voltage V_o remains between 9 V and 9.5 V in the circuit in Fig. 2.9-9.

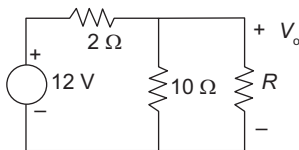


Fig. 2.9-9

10. Find the range of R such that the current I_o remains between 0.3 A and 0.35 A in the circuit in Fig. 2.9-10.

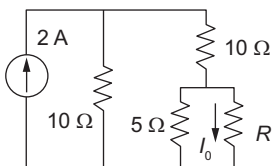


Fig. 2.9-10

11. Find the value of all resistors in the circuit in Fig. 2.9-11 and calculate the total power dissipated in them.

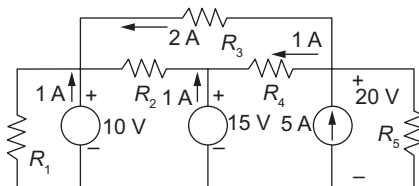


Fig. 2.9-11

12. Refer to Fig. 2.9-12. The voltage across 0.01 F capacitor is 30 V at $t = 0$. The voltage across 0.02 F capacitor is 60 V at $t = 0$. (i) Find the value of v_x at $t = 0$. (ii) Find the rate of change of capacitor voltage variables with respect to time at $t = 0$.

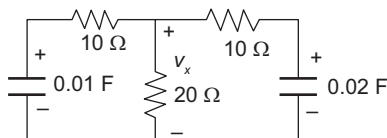


Fig. 2.9-12

13. Refer to Fig. 2.9-13. The currents flowing in the inductors at $t = 0$ are as marked in Fig. 2.9-13. (i) Find the value of i_x at $t = 0$. (ii) Find the time-rate of change of currents in the inductors at $t = 0$.

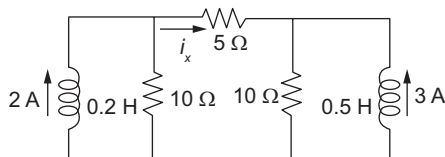


Fig. 2.9-13

- terms of voltage variables V_{AD} , V_{BD} and V_{CD} and resistance values.]

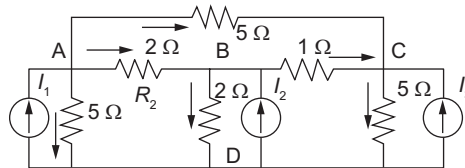


Fig. 2.9-14

- delivered by the independent source and power delivered by the dependent source.

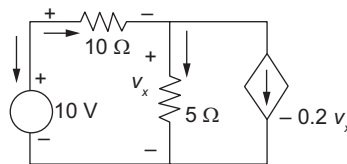


Fig. 2.9-15

- dissipated in the circuit and the power delivered by independent and dependent sources.

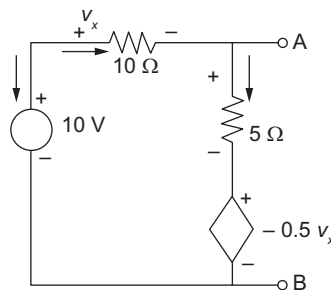


Fig. 2.9-16

- Fig. 2.9-17.

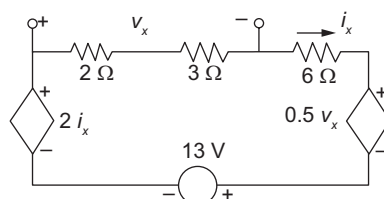


Fig. 2.9-17

2.40 | Basic Circuit Laws

18. Find the voltage across the parallel combination in the circuit shown in Fig. 2.9-18. Also find the power absorbed by all the elements in the circuit.

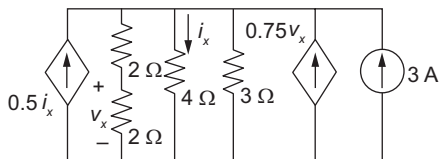


Fig. 2.9-18

19. Find the energy delivered to the $9\ \Omega$ resistor and dissipated in the $2\ \Omega$ resistors in the circuit shown in Fig. 2.9-19 during $[0, 2\text{ s}]$ if $v_{s1} = 10 \sin 100\pi t\text{ V}$ and $v_{s2} = 10\text{ V}$.

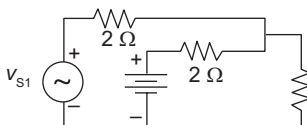


Fig. 2.9-19

20. Find the charge delivered to the 6 V voltage source from $t = 0$ to $t = 2\text{ s}$ in the circuit in Fig. 2.9-20. $i_{s1} = 2 + e^{-t}\text{ A}$ for $t \geq 0$ and 0 A for $t < 0$. $i_{s2} = te^{-2t}\text{ A}$ for $t \geq 0$ and 0 A for $t < 0$.

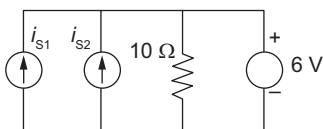


Fig. 2.9-20

21. Find the coefficients k_1 and k_2 for the dependent sources in the circuit in Fig. 2.9-21.

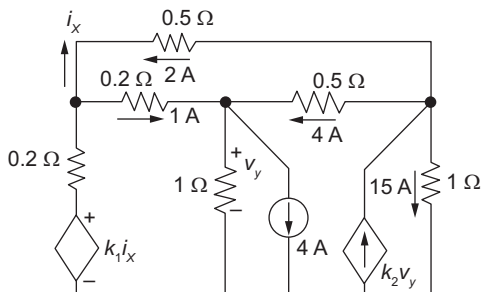


Fig. 2.9-21

22. Find α and β in the circuit in Fig. 2.9-22.

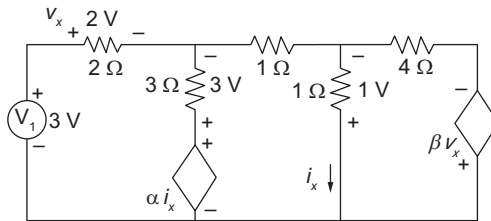


Fig. 2.9-22

23. Design an Opamp circuit to produce $v_o(t) = 3 + 7 \sin 200\pi$ V using ± 12 V power supply and a signal source $v_s(t) = 0.5 \sin 200\pi$ V.
24. Find i_L , i_s , i_o and v_o in the voltage to current converter in Fig. 2.9-23 designed to produce a constant current in a $10\ \Omega$ load.

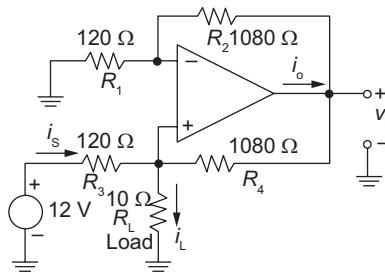


Fig. 2.9-23

25. (i) Show that the current in the load resistance R_L in the circuit in Fig. 2.9.24 is independent of R_L if $(R_3 + R_4) \gg R_5 // R_L$. (ii) Find i_L , i_s , i_o and v_o with the component values shown.

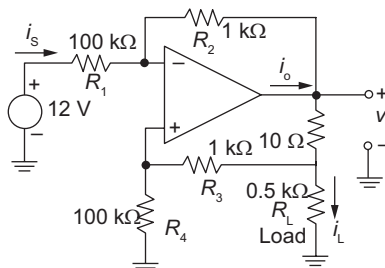


Fig. 2.9.24

2.42 | Basic Circuit Laws

26. Show that $v_o(t) = (1 + R_2/R_1)(v_1 - v_2)$ in the circuit in Fig. 2.9-25.

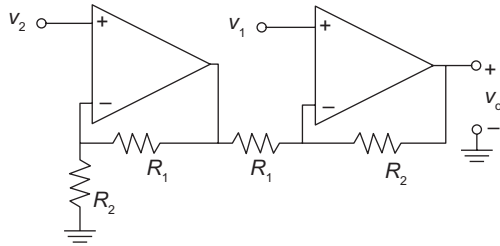


Fig. 2.9-25

27. (i) Show that the overall feedback is of degenerative nature in the circuit in Fig. 2.9-26. (ii) Derive expressions for gains at the outputs of both Opamps. (iii) Evaluate the gain v_o/v_s with $R_4 = 99R$, $R_3 = 100R$ and $R_2 = 9R_1$. (iv) Design the circuit using $\mu A741$ IC with the above constraints on resistances.

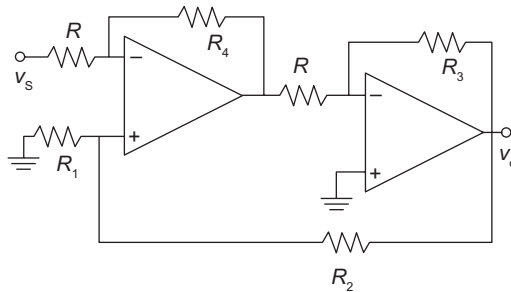


Fig. 2.9-26

Chapter 3

Single Element Circuits

CHAPTER OBJECTIVES

- *Voltage–current relation of a resistor.*
- *Voltage, current and power division principle in series and parallel resistor combinations.*
- *Voltage–current relation of an Inductor and its various implications.*
- *Initial current in an inductor and its significance.*
- *Series and parallel combination of inductors.*
- *Voltage, current and power sharing in series and parallel connection of inductors.*
- *Voltage–current relation of a capacitor and its various implications.*
- *Initial voltage across a capacitor and its significance.*
- *Series and parallel combination of capacitors.*
- *Voltage, current and power sharing in series and parallel connection of capacitors.*
- *Simple single element circuits with independent voltage source and current source excitation.*
- *Unit impulse function and unit step function.*

INTRODUCTION

In this chapter, we study simple circuits containing one type of element – resistor, inductor or capacitor – driven by one source, either an independent voltage source or independent current source. These circuits may contain more than one element, but all of them will be of same type except the source. The aim of this study will be to understand the behaviour of each element type thoroughly. We will also deal with ‘*series and parallel equivalents*’ that can be used to replace series or parallel connection of multiple elements of same type by one equivalent element of that type. Moreover, we will meet with two very interesting source functions – *unit impulse function* $\delta(t)$ and *unit step function* $u(t)$. These functions are extremely important in Circuit Analysis.

3.2 | Single Element Circuits

3.1 THE RESISTOR

The physical basis for the two-terminal resistor was dealt with in detail in Chapter 1. The graphic symbol of a linear resistor and its element relationship is shown in the following:

$$v(t) = Ri(t) \text{ or } i(t) = Gv(t) \text{ for all } t$$

$$p(t) = v(t)i(t) = R[i(t)]^2 = \frac{[v(t)]^2}{R} = \frac{[i(t)]^2}{G} = G[v(t)]^2$$

where $p(t)$ is the power delivered to the resistor in watts, R is the resistance of the resistor and G is the conductance of the resistor. R and G are reciprocals of each other.

The current response in a resistor at a particular instant depends only on the voltage applied across it at that instant. Therefore, a resistor is a *memoryless* element. The waveshape of voltage and current have to be the same in such an element. A resistor cannot change the waveshape of a signal.

A resistor can only dissipate energy. Therefore, the power delivered to a positive resistor is always positive or zero.

3.1.1 Series Connection of Resistors

Consider the series connection of n resistors $R_1, R_2, \dots, R_n$ as in Fig. 3.1-1.

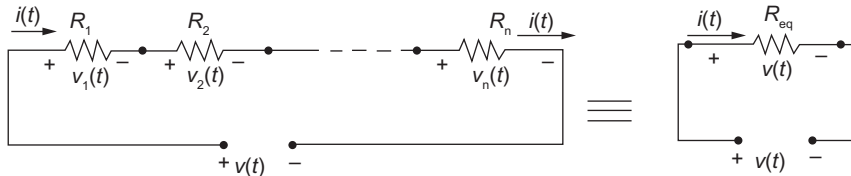


Fig. 3.1-1 Series connection of resistors and its equivalent

The current in a series circuit has to be a common variable. This is evident if we apply KCL at the junction between one resistor and the next. Therefore, there is only one current variable and that is $i(t)$ as shown in Fig. 3.1-1. Applying KVL in the loop and employing the element relationship of a linear resistor, we get

$$v(t) = v_1(t) + v_2(t) + \dots + v_n(t) \text{ for all } t$$

$$\text{i.e., } v(t) = R_1 i(t) + R_2 i(t) + \dots + R_n i(t) \text{ for all } t$$

$$\text{i.e., } v(t) = [R_1 + R_2 + \dots + R_n] i(t) \text{ for all } t \quad (3.1-1)$$

$$\text{i.e., } v(t) = R_{eq} i(t) \text{ for all } t \text{ where } R_{eq} = \sum_{k=1}^n R_k$$

Thus, the entire series connection can be replaced by a single resistor of resistance value equal to sum of resistance values of all the resistors in series. Further,

$$v(t) = v_1(t) + v_2(t) + \cdots + v_n(t) \text{ for all } t$$

and

$$v_1(t) : v_2(t) : \cdots : v_n(t) = R_1 : R_2 : \cdots : R_n$$

i.e.,

$$v_j(t) = \frac{R_j}{\sum_{k=1}^n R_k} v(t) = \frac{R_j}{R_{eq}} v(t) \text{ for } j=1 \text{ to } n \quad (3.1-2)$$

Moreover,

$$\begin{aligned} p(t) &= [v_1(t) + v_2(t) + \cdots + v_n(t)] i(t) \text{ for all } t \\ &= v_1(t)i(t) + v_2(t)i(t) + \cdots + v_n(t)i(t) \text{ for all } t \\ &= p_1(t) + p_2(t) + \cdots + p_n(t) \end{aligned}$$

and

$$p_1(t) : p_2(t) : \cdots : p_n(t) = R_1 : R_2 : \cdots : R_n$$

i.e.,

$$p_j(t) = \frac{R_j}{\sum_{k=1}^n R_k} p(t) = \frac{R_j}{R_{eq}} p(t) \text{ for } j=1 \text{ to } n \quad (3.1-3)$$

and

$$p(t) = \sum_{k=1}^n p_k(t) = [i(t)]^2 \sum_{k=1}^n R_k = R_{eq} [i(t)]^2 = \frac{[v(t)]^2}{R_{eq}}$$

The total voltage in a series combination divides in proportion to resistance value across various resistors. Since current is common and voltage is proportional to resistance value, the power delivered to individual resistor is also in proportion to its resistance value.

3.1.2 Parallel Connection of Resistors

Consider the parallel connection of n resistors $R_1, R_2, \dots, R_n$ in Fig. 3.1-2.

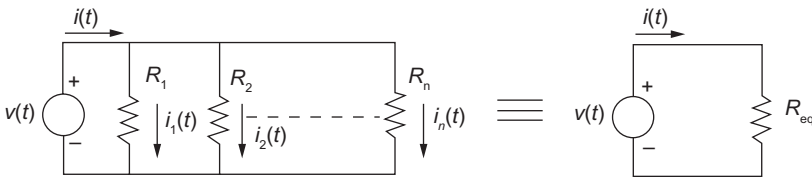


Fig. 3.1-2 Parallel connection of resistors and its equivalent

The voltage across the parallel combination is the common variable. This follows from applying KVL in the loops formed by parallel connection. Applying KCL at the positive node of voltage source and using element equations of resistors, we get,

3.4 | Single Element Circuits

$$\begin{aligned}
 & i(t) = i_1(t) + i_2(t) + \dots + i_n(t) \text{ for all } t \\
 \text{i.e.,} \quad & i(t) = G_1 v(t) + G_2 v(t) + \dots + G_n v(t) \text{ for all } t \\
 \text{i.e.,} \quad & i(t) = [G_1 + G_2 + \dots + G_n] i(t) \text{ for all } t \\
 \text{i.e.,} \quad & v(t) = G_{\text{eq}} i(t) \text{ for all } t \text{ where } G_{\text{eq}} = \sum_{k=1}^n G_k \\
 \text{or equivalently} \quad & \frac{1}{R_{\text{eq}}} = \sum_{k=1}^n \frac{1}{R_k}
 \end{aligned} \tag{3.1-4}$$

Hence, the entire parallel combination can be replaced by a resistor with a conductance value equal to sum of conductance values of all resistors in the parallel combination as far as the v - i relationship is concerned. Further,

$$\begin{aligned}
 & i(t) = i_1(t) + i_2(t) + \dots + i_n(t) \text{ for all } t \\
 \text{and} \quad & i_1(t) : i_2(t) : \dots : i_n(t) = G_1 : G_2 : \dots : G_n \\
 \text{i.e.,} \quad & i_j(t) = \frac{G_j}{\sum_{k=1}^n G_k} i(t) = \frac{G_j}{G_{\text{eq}}} i(t) \text{ for } j=1 \text{ to } n
 \end{aligned} \tag{3.1-5}$$

The total current in a parallel combination divides among the various resistors in proportion to their conductance values. Since voltage is common and current is proportional to conductance value, the power delivered to individual resistor is also in proportion to its conductance value.

$$\begin{aligned}
 & p(t) = [i_1(t) + i_2(t) + \dots + i_n(t)] v(t) \text{ for all } t \\
 & \quad = i_1(t)v(t) + i_2(t)v(t) + \dots + i_n(t)v(t) \text{ for all } t \\
 & \quad = p_1(t) + p_2(t) + \dots + p_n(t) \\
 \text{and} \quad & p_1(t) : p_2(t) : \dots : p_n(t) = G_1 : G_2 : \dots : G_n \\
 \text{i.e.,} \quad & p_j(t) = \frac{G_j}{G_{\text{eq}}} p(t) \text{ for } j=1 \text{ to } n \\
 \text{and} \quad & p(t) = \sum_{k=1}^n p_k(t) = [v(t)]^2 \sum_{k=1}^n G_k = G_{\text{eq}} [v(t)]^2 = \frac{[v(t)]^2}{R_{\text{eq}}} = R_{\text{eq}} [i(t)]^2
 \end{aligned} \tag{3.1-6a}$$

A frequently occurring important special case in parallel combination is the one involving two resistors. The equivalent resistance and conductance as well as the current division relations are marked in Fig. 3.1-3 for this important special case.

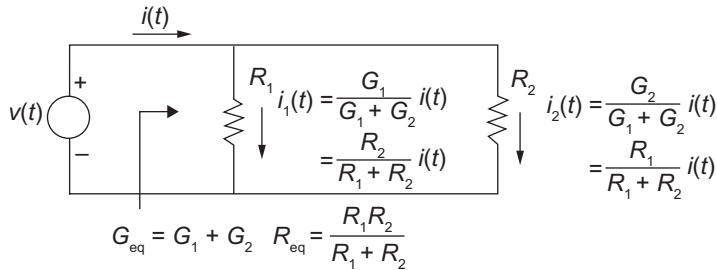


Fig. 3.1-3 Current division in a two-resistor parallel connection

Example: 3.1-1

Find R in the circuit in Fig. 3.1-4 such that v_o is zero.

Solution

This circuit has two series combinations of resistors connected across the DC source. We take the negative of the DC source as the common reference point. The potential of node between $10\ \Omega$ and $25\ \Omega$ resistors will be given by voltage division principle in series combination.

The potential of the node marked $+$ $= 24 \times 25 / (25 + 10)$ V. But we do not really need to calculate this.

We want the potential difference v_o to be zero. Therefore, the potential of the junction between $30\ \Omega$ and R with respect to the negative of the DC source must be same as the potential of the junction between $10\ \Omega$ and $25\ \Omega$. Therefore, $R / (30 + R) = 25 / 35 \Rightarrow R = 75\ \Omega$. Note that the value of R is independent of the DC source value.

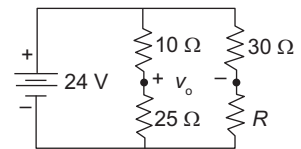


Fig. 3.1-4 Circuit for Example 3.1-1

Example: 3.1-2

Two resistors in series combination can be used to create a voltage lower than the supply voltage by voltage division. Such an arrangement is a potential divider. This arrangement is used for making a variable DC voltage from a fixed DC supply by making one of the resistors a variable one. This arrangement is widely employed in electronic circuits. One such *fixed potential divider* is shown in Fig. 3.1-5. (i) What is the output voltage when there is no load connected across the output, *i.e.*, $R = \infty$? (ii) What is the minimum value of R if the output voltage is not to fall below 95% of the value it has when there is no load connected?

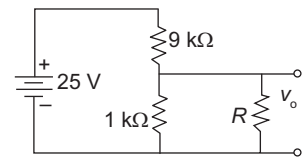


Fig. 3.1-5 Circuit for Example 3.1-2

Solution

- (i) The output when no load is connected $= 25\text{ V} \times 1\text{ k}\ \Omega / 10\text{ k}\ \Omega = 2.5\text{ V}$

3.6 | Single Element Circuits

- (ii) When R is connected, the parallel combination of R and $1\text{ k}\Omega$ will be less than $1\text{ k}\Omega$ and the voltage division ratio will be less than 0.1. This results in an output lower than 2.5 V. We want it to be more than $0.95 \times 2.5 = 2.375\text{ V}$.

$$\frac{25X}{9+X} \geq 2.375 \Rightarrow 22.625X \geq 21.375 \Rightarrow X \geq 0.945\text{ k}\Omega$$

But X is the parallel combination of R and $1\text{ k}\Omega$

$$\therefore \frac{R}{R+1} \geq 0.945 \Rightarrow R \geq 0.945R + 0.945 \Rightarrow R \geq 17.2\text{ k}\Omega$$

Therefore, the load resistor must be greater than $17.2\text{ k}\Omega$. This implies that we can draw $\approx 140\text{ }\mu\text{A}$ before potential divider output falls by more than 5% of its open-circuit output value.

Example: 3.1-3

The power dissipated in the resistor marked with * in the circuit in Fig. 3.1-6 is found to be 12.5 W. (i) Find the value of R (ii) Find the total power delivered by the source. (iii) Find the voltage, current and power dissipated for all the resistors.

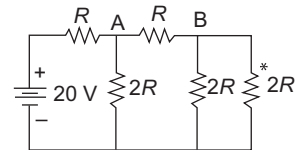


Fig. 3.1-6 Circuit for Example 3.1-3

Solution

- (i) We need to find an expression for power dissipated in the marked resistor in terms of R . We need an expression for current in it for that. We use series-parallel reductions successively as shown in Fig. 3.1-7.

Therefore, the current delivered by the voltage source is $20/2R = 10/R$. The series-parallel reduction stages show that this current divides into two equal components at the node A since equivalent resistance to the right of A is $2R$. Therefore, $5/R$ A reaches node B and splits into $2.5/R$ A into the two equal resistors connected in parallel there. Therefore, the current in the marked resistor is $2.5/R$. Therefore, power dissipated in it is $2R \times 6.25/R^2 = 12.5/R$. Since this is stated to be 12.5 W, the value of $R = 1\text{ }\Omega$.

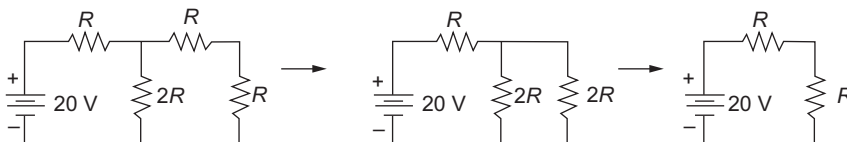


Fig. 3.1-7 Series – parallel reduction for Example 3.1-3

- (ii) The equivalent resistance across the 20V source is $2R = 2\text{ }\Omega$. Therefore, the current delivered by the source is 10 A and the power delivered by the source is $20 \times 10 = 200\text{ W}$.

(iii) The various currents, voltages and power values are marked in Fig. 3.1-8.

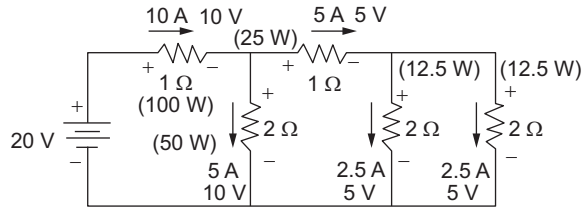


Fig. 3.1-8 Circuit solution for Example 3.1-3

Example: 3.1-4

All resistors in the circuit in Fig. 3.1-9 are in $k\Omega$. Solve the circuit for currents, voltages and power dissipated in all resistors. Also find the power delivered by the current source.

Solution

The two $20k$ resistors connected between A and C can be replaced by $10k$ (n equal resistors of $R\Omega$ in parallel can be replaced by $R/n\Omega$). Similarly, the $10k$ in parallel with $30k$ between B and C can be replaced by $(10 \times 30)/(10 + 30) = 7.5k$. The resulting circuit is shown in Fig. 3.1-10 (b).

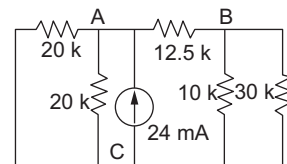


Fig. 3.1-9 Circuit for Example 3.1-4

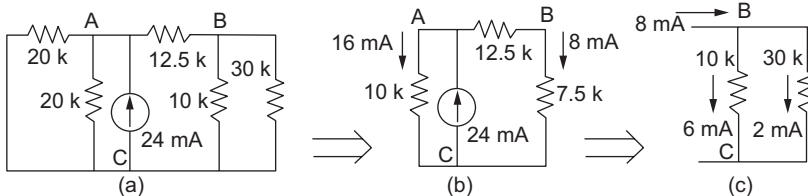


Fig. 3.1-10 Circuit reduction in Example 3.1-4

Now there is $12.5k + 7.5k = 20k$ on the right side of current source and $10k$ on the left side of current source. The total current of 24 mA is divided in the ratio $0.05:0.1$ between the right side and left side since the ratio of currents in a parallel combination is the conductance ratio. Therefore, current to the right is $1/3$ of $24\text{ mA} = 8\text{ mA}$ and that to the left is $2/3$ of $24\text{ mA} = 16\text{ mA}$ as shown in the circuit of Fig. 3.1-10 (b). The 8 mA to the right flows through $12.5k$ and reaches node B where it gets divided into $10k$ and $30k$. The conductance ratio here is $3:1$ and hence 75% of $8\text{ mA} = 6\text{ mA}$ goes into $10k$ and 25% of $8\text{ mA} = 2\text{ mA}$ goes into $30k$ as shown in the circuit of Fig. 3.1-10 (c). Similarly, the 16 mA flowing to the left side from current source node will divide into 8 mA each in the $20k$ resistors on the left in Fig. 3.1-10 (a).

Now all resistor voltages may be obtained by multiplying resistor current by resistance value. Note that $k\Omega \times \text{mA}$ will give volts. Power delivered to resistor is given by product of voltage and current (since passive sign convention is followed). The complete solution for the circuit is shown in Fig. 3.1-11.

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The power *delivered* to current source is negative-valued since the current entering the positive polarity of voltage is -24mA . This means that current source is delivering 3840 mW , which has to be equal to the sum of power dissipated in all resistors.

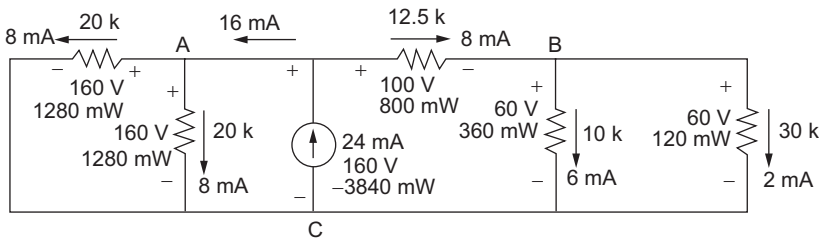


Fig. 3.1-11 Complete circuit solution for Example 3.1-4

Example: 3.1-5

- (i) Find V_{dc} in the circuit in Fig. 3.1-12 if i_x is to be 2 mA . (ii) What is V_{dc} if i_x is to be 1.5 mA ? (iii) What is i_x if V_{dc} is 128 V ? (iv) Obtain the complete solution for the circuit with V_{dc} is 96 V .

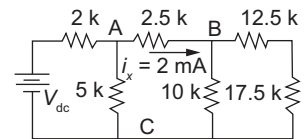


Fig. 3.1-12 Circuit for Example 3.1-5

Solution

- (i) We try to develop an expression for V_{dc} in terms of i_x . The 12.5k and 17.5k in series is replaced by 30k as in circuit of Fig. 3.1-13 (a). Further reduction is carried out by replacing 10k and 30k in parallel by its equivalent value of 7.5k to arrive at the circuit in Fig. 3.1-13 (b).

Now, i_x flows through 2.5k and 7.5k in series and develops a voltage drop of $10i_x$ (i_x is in mA) across the 5k resistor. This results in a current of $2i_x$ in the 5k resistor by Ohm's law. Applying KCL at node A in Fig. 3.1-13 (b) we get the current in 2k resistor as $3i_x$. Therefore voltage drop across the 2k resistor is $6i_x$. Applying KVL in the loop containing the DC voltage source, 2k resistor and 5k resistor, we get V_{dc} as $16i_x$.

Therefore, if i_x is 2 mA , then V_{dc} is 32 V .

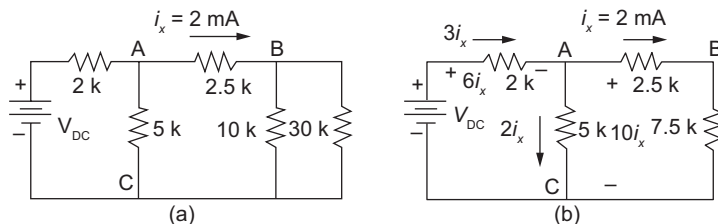


Fig. 3.1-13 Different stages of circuit reduction in Example 3.1-5

- (ii) If i_x is to be 1.5 mA , V_{dc} must be $16\text{ k}\Omega \times 1.5\text{ mA} = 24\text{ V}$
 (iii) If V_{dc} is 128 V , then i_x must be $128/16 = 8\text{ mA}$.

- (iv) If V_{dc} is 96 V, i_x will be $96/16 = 6$ mA. This 6 mA is shared by 10k and 30k in the circuit of Fig. 3.1-13 (a) in the ratio $1/10 : 1/30$, i.e., 3:1. Therefore, the current in 10k is 4.5 mA and current in 30k is 1.5 mA. But the 30k is actually series combination of 12.5k and 17.5k in the original circuit. Therefore, the current in 12.5k and 17.5k is 1.5 mA.

The voltage v_{AC} is $10i_x$ and hence it is 60V. 60V across 5k produce 12 mA current in it. This results in $12 + 6 = 18$ mA in 2k resistor in the original circuit in Fig. 3.1-12.

Now all the currents are known. Voltages across the resistors may be obtained by multiplying the current value by resistance value. Power delivered to the resistors may be obtained by multiplying the resistor voltage by current through it. The complete solution with all variable values marked is given in Fig. 3.1-14.

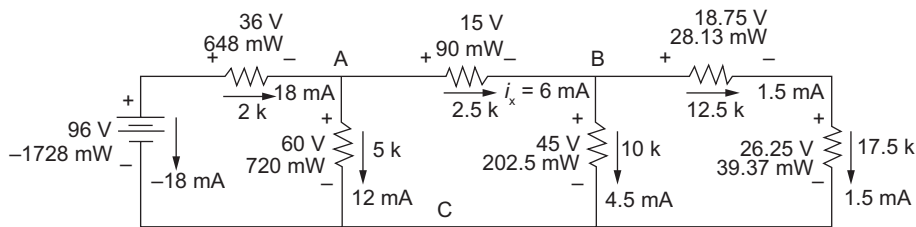


Fig. 3.1-14 Complete solution with 96 v source in Example 3.1-5

Example: 3.1-6

- (i) Find the currents in all the resistors in the circuit in Fig. 3.1-15. (ii) Find the voltage appearing across the current source and the power delivered by it.

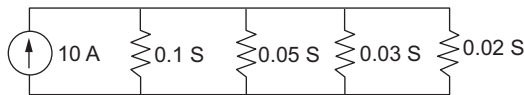


Fig. 3.1-15 Circuit for Example 3.1-6

Solution

- (i) Total current in a parallel combination gets distributed in resistors as per the conductance ratio. The relevant ratio here is $0.1:0.05:0.03:0.02$, i.e., $10:5:3:2$. Therefore, the currents are 5 A, 2.5 A, 1.5 A and 1 A in 0.1 S, 0.05 S, 0.03 S and 0.02 S resistors, respectively.
- (ii) The equivalent conductance of a parallel combination is the sum of conductance values of the participating resistors. Hence, the equivalent conductance here is 0.2 S. Therefore equivalent resistance is 5Ω . Therefore, the voltage appearing across the current source is 50 V and the power delivered by it is 500 W.

3.2 THE INDUCTOR

The physical basis for the two-terminal element called inductor has been dealt with in Chapter 1. Its unit will be V-s/A or Wb-T/A, which is given the name 'Henry' and abbreviated as 'H'. The element relation for inductor as per passive sign convention is

at the time instants marked as t_b and t_d , the $v(t)$ waveform crosses zero in the upward direction, and, $i(t)$ in all the three cases reach local minima at those time instants. Further, the polarity of $v(t)$ is negative for all time instants between t_a and t_b , and, we observe that all the three current waveforms decrease in that interval. Similarly, polarity of $v(t)$ is positive for all time instants between t_b and t_c , and, we observe that all the three current waveforms increase in that interval.

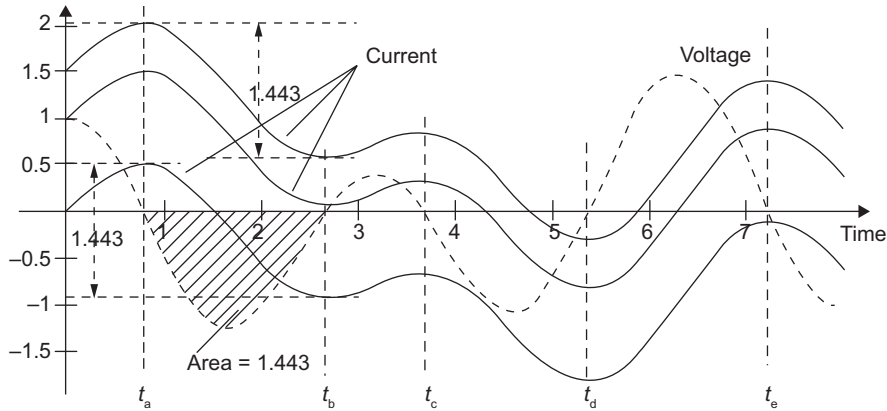


Fig. 3.2-1 Voltage-current relation in a 1 H inductor

Instantaneous current in an inductor cannot be predicted from instantaneous value of voltage across it.

If instantaneous value of voltage is positive, the inductor current will be increasing at that instant, and, if it is negative the current will be decreasing at that instant.

When voltage across an inductor crosses zero in the downward direction its current attains a local maximum and when it crosses zero in the upward direction the inductor current attains a local minimum.

Voltage across an inductor carrying a constant current is zero.

3.2.2 Change in Inductor Current Function versus Area under Voltage Function

The relation between current and voltage of an inductor is reproduced in the following:

$$v(t) = L \frac{di(t)}{dt} \text{ and } i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt \quad (3.2-1)$$

Consider two time instants t_1 and t_2 . Applying the integral form of v - i relationship we get,

$$i(t_2) - i(t_1) = \frac{1}{L} \int_{-\infty}^{t_2} v(t) dt - \frac{1}{L} \int_{-\infty}^{t_1} v(t) dt = \frac{1}{L} \int_{t_1}^{t_2} v(t) dt \quad (3.2-2)$$

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Thus the *change* in inductor current is given by $(1/L) \times$ area under voltage function between the two instants under consideration. This is also expressed as $\Delta i = \frac{V-s}{L}$ where Δi is the increase in inductor current $i(t)$ over $[t_1, t_2]$ and V-s is the area under $v(t)$ in the same interval. Therefore, $i(t_2) = i(t_1) + \Delta i = i(t_1) + \frac{V-s}{L}$. We can also relate the V-s product to *change* in flux linkage in the inductor. *In fact, the V-sec product itself is the change in flux linkage since $\Delta\psi = L\Delta i =$ area under voltage function (V-s).* Therefore, V-s and Wb-T are two units for the same quantity.

We can calculate only *change* in $i(t)$ given the $v(t)$ unless $v(t)$ is given for $(-\infty, t]$ interval. We can find the absolute instantaneous value of $i(t)$ if we know all the voltage applied to inductor from infinite past to the present instant. However, we need not insist on being given the $v(t)$ from $-\infty$ itself. It is enough that we know the area under $v(t)$ from $-\infty$ to some instant, say $t = t_0$, and $v(t)$ itself from that instant onwards. This is so because we can split the integral in Eqn. 3.2-2 as shown in the following:

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt = \underbrace{\frac{1}{L} \int_{-\infty}^{t_0} v(t) dt}_{I_{t_0}} + \frac{1}{L} \int_{t_0}^t v(t) dt \quad (3.2-3)$$

Obviously, the first term on the right-hand side is the inductor current value at t_0 . Therefore, we can work out inductor current at an instant if we know its value at some reference instant and the voltage function applied to it from that reference instant onwards. This reference instant is usually set as $t = 0$ in analysis of circuits and the value of inductor current at $t = 0$ is termed as initial condition of inductor.

Change in inductor current over $[t_1, t_2]$, $\Delta i = (\text{Area under inductor voltage over } [t_1, t_2])/L$.
 $i(t)$ at $t = t_2$ is $(i(t)$ at $t = t_1) + \Delta i$
 $i(t) = I_0 + (\text{Area under inductor voltage over } [0, t])/L$, where I_0 is the current in the inductor at $t = 0$ and is called initial condition of the inductor.

With reference to Fig. 3.2-1, the area under $v(t)$ between t_a and t_b is 1.443 V-s. The inductance value is 1H. And, it is shown in the figure that all the three possible $i(t)$ waveforms undergo a change by -1.443 A in that interval, clearly demonstrating the relation between *change* in inductor current and V-s product dumped into the inductor during the relevant time interval. The voltage waveform in Fig. 3.2-1 is known only for $t \geq 0$. The three current curves shown in the figure represent three possible initial values for the inductor current at $t = 0$. The respective initial current values can be read off the curves at $t = 0$.

3.2.3 Average Applied Voltage for a Given Change in Inductor Current

Let us assume that we want to increase the current in an inductor L from I_1 to I_2 ($I_2 > I_1$) in a time interval of Δt . This change may be accomplished by applying *any* waveform for voltage provided the area under that waveform over Δt is $L(I_2 - I_1)$ V-s. This implies that irrespective of the exact waveform of voltage applied, its average value over Δt has to be $L(I_2 - I_1)/\Delta t$ V.

Now as Δt decreases, *i.e.*, when we try to accomplish the required current change in shorter time interval, the average voltage to be applied increases. Thus, fast current change in inductor requires higher voltage to be applied across it.

The amount of current change required in an inductor decides the area-content under voltage waveform to be applied to it to bring about the change and the time allowed to bring about it decides the average voltage to be applied.

3.2.4 Instantaneous Change in Inductor Current

It follows from the last sub-section that the average voltage to be applied to cause a finite amount of change in inductor current increases to infinite value when we try to accomplish the change in current in zero time interval. We cannot bring about instantaneous change in inductor current unless we apply or support such an infinite voltage across the inductor.

Let us say we want to change the current in a 0.5 H inductor from 0 to 2 A by applying a rectangular pulse voltage from $t = 0$. The voltage area content required is $0.5 \text{ H} \times 2 \text{ A} = 1 \text{ V-s}$. The height of pulse will depend on the width of pulse. Three cases are shown in Fig. 3.2-2.

When 2.5V pulse lasting for 0.4 s is applied, the inductor current ramps up linearly from 0 to 2A in 0.4 s with a slope of 5 A/s. When 5V pulse lasting for 0.2 s is applied, the inductor current ramps up linearly from 0 to 2A in 0.2 s with a slope of 10 A/s. When 10 V pulse lasting for 0.1 s is applied, the inductor current ramps up linearly from 0 to 2A in 0.1 s with a slope of 20 A/s. We have kept the area under the voltage waveform at 1V-s in all the three cases. Now consider further shortening of pulse duration, taking it to near-zero width. The change in inductor current will be 2A if the pulse height is increased to maintain the area under the voltage waveform at 1V-s. However, the inductor current waveform will become steeper and steeper until it becomes a straight-edged waveform as pulse width $\rightarrow 0$ and pulse height $\rightarrow \infty$. Notice that though pulse height $\rightarrow \infty$ as width $\rightarrow 0$, its area is constrained to remain 1V-s. Such an idealised waveform with zero width, undefined height and finite area-content of unity is called a *unit impulse function* and denoted by the symbol $\delta(t)$. Its formal definition is

$$\delta(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined at } t=0 & \text{and } \int_{-\infty}^{\infty} \delta(t) dt = 1 \\ 0 & \text{for } 0^+ \leq t < \infty \end{cases}$$

where the time instant $t = 0^-$ is an instant which is arbitrarily close to $t = 0$ but on its left side and time instant $t = 0^+$ is an instant which is arbitrarily close to $t = 0$ but on its right side. Thus the interval $[0^-, 0^+]$ is of infinitesimal width; but $t = 0$ comes in the middle of this interval.

The graphical symbol used for $\delta(t)$ is shown in Fig. 3.2-2 (b). The height of the arrow-terminated vertical line representing $\delta(t)$ is *not* the amplitude of the function (amplitude is *undefined* at that point); rather it indicates the area-content of the waveform. The instantaneous change in inductor current from 0 to 2A is also shown in the same figure. Now look at the current waveform in the inductor. It is zero till 0^- and 2A after 0^+ and a discontinuous jump at $t = 0$. This must be 2 times the integral of impulse function ($1/L = 2$ in this case). Let us verify this.

$$\int_{-\infty}^t \delta(t) dt = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined at } t=0; & \text{This function is called unit step function } u(t) \\ 1 & \text{for } t \geq 0^+ \end{cases} \quad (3.2-4)$$

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This function is defined as a *unit step function* and is denoted by $u(t)$. Thus, when we apply a $\delta(t)$ voltage waveform to an inductor of inductance value L , the current in the inductor jumps up instantaneously by $1/L$ A. *Unit impulse voltage source* will dump 1 V-s of voltage area content into the inductor instantaneously. Equivalently, *unit impulse voltage source* will dump 1 Weber-turns (Wb-T) of flux linkage into the inductor instantaneously. The result will be a change in its current by $1/L$ A.

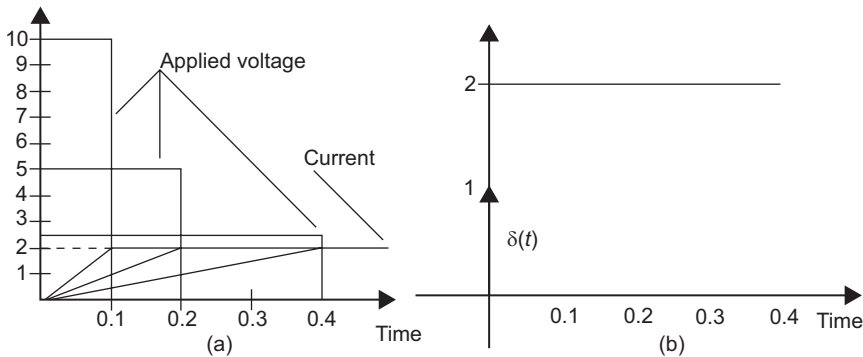


Fig. 3.2-2 Rectangular pulse application and impulse voltage

Current in an inductor cannot change instantaneously unless an impulse voltage is applied or supported in the circuit.

The current in an inductor L changes instantaneously by $1/L$ A when the circuit applies or supports a unit impulse voltage across it.

Therefore, if a circuit does not apply or support impulse voltage, the currents in inductors in that circuit will be continuous functions of time.

Strictly speaking, it is the flux linkage in an inductor that can not be changed instantaneously. However, in the case of an inductor that is not magnetically coupled to other inductors, this will amount to what we have stated above since flux linkage in such an inductor is proportional to its current. We will modify this statement suitably when we take up the study of coupled circuits later in the book.

3.2.5 Inductor with Alternating Voltage Across it

We consider the application of alternating voltage (AC voltage) across an inductor in this sub-section. Alternating voltage is a voltage waveform that alternates between positive and negative voltages *periodically* and has a zero cyclic average. This means that the area under the voltage waveform during positive half-cycle and the area under the voltage waveform during the negative half-cycle are equal. The two half-cycles need not be equal in length. However, the net area in a cycle has to be zero. This is equivalent to a *zero DC content* since the DC content of a cyclic waveform is its area-content over a cycle divided by the cycle period. It is possible to express a periodic waveform as a DC term plus a pure alternating term if there is a non-zero DC content in it.

Figure 3.2-3 shows the results of applying an alternating voltage waveform to an inductor with two values of inductance (1 H and 5 H) considered.

The dotted curve shows the applied voltage and solid curves show the current in the inductor. The integral of applied voltage is also shown in the figure. Both current curves show local maxima and minima at voltage zero-crossing points. The area under one half-cycle of voltage is 1 V-s and the current in 1 H should change by 1 A over a half-cycle and current in 5 H should change by 0.2 A over a half-cycle. Figure 3.2-3 shows that the current in 1 H inductor varies between 1.4 A and 0.4 A with the initial condition of 0.4 A. The current varies between 0.6 A and 0.4 A in the case of a 5 H inductor with same initial condition.

We need to appreciate three following points in this context.

With a specific area under a half-cycle of voltage waveform, the current in the inductor will change by an amount equal to that area value divided by L . In the next half-cycle it will vary by the same amount, but in opposite direction. *Thus, the peak-to-peak value of alternating component of inductor current will be equal to the area of one half-cycle of voltage waveform divided by L .* Therefore, higher the inductance, lower the peak-to-peak ripple current in the inductor. This conclusion is independent of the exact shape of voltage waveform.

If the frequency of voltage waveform is increased without changing its amplitude and waveshape, the half-cycle area decreases due to reduction in the half-cycle duration. Then, the peak-to-peak ripple current will also decrease. *Therefore, higher the frequency of alternating voltage applied to an inductor, lower the peak-to-peak amplitude of the alternating component of inductor current.*

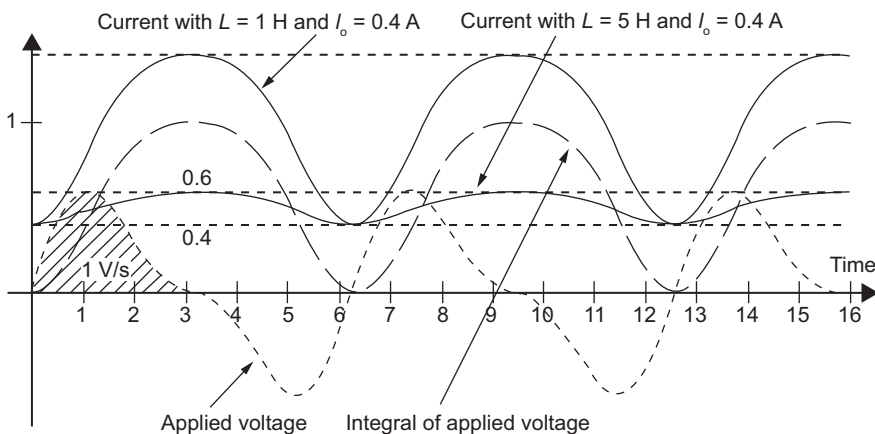


Fig. 3.2-3 Inductor with alternating voltage across it

The DC content in inductor current is decided by two factors – the initial condition and the instant of application of the alternating voltage. Examine the integral of voltage waveform in Fig. 3.2-3. The voltage waveform was applied to the inductor at its zero-crossing. Therefore, its integral goes to a maximum value of 1 V-s in the first half-cycle and then returns to zero at the end of second half-cycle. It does not go negative. This area waveform divided by L will give us the current in the inductor with zero initial condition. Notice that that current will have a DC content since the voltage area waveform has a DC content. *Thus, the net DC content in the inductor current will be its initial condition value plus cyclic average of voltage area waveform divided by L .* Notice that the second contribution to DC content in the inductor current will depend on at which point in the voltage waveform, we start applying it to the inductor. There will exist one particular waveform position in any periodic voltage such that the DC contribution to inductor current will be zero if switching is done at that position.

There can be a DC current through an inductor even when the applied voltage waveform is a pure alternating one. The amount of DC content depends upon the initial condition of the inductor and the instant at which the voltage waveform is switched on to the inductor.

When the applied voltage across an inductor is a periodic alternating waveform, the current in the inductor will contain an alternating component with the same period. The peak-to-peak amplitude of this alternating component will be directly proportional to half-cycle area of voltage waveform and inversely proportional to inductance value. It decreases with increase in frequency of the voltage.

Therefore, a large-valued inductor in a circuit can absorb alternating voltages in the circuit without contributing significant amount of alternating currents to the circuit.

3.2.6 Inductor with Exponential and Sinusoidal Voltage Input

Consider an inductor of 1 H with a voltage $v_s(t) = e^{-t}$ V switched on to it from $t = 0$ onwards. We do not know what voltage was applied to inductor till $t = 0$. But the net effect of all that voltage which may have been applied to the inductor is condensed in the initial condition available to us. We assume an initial current of zero in Fig. 3.2-4 (a) and initial current of -0.5 A in Fig. 3.2-4 (b). Straightforward integration gives us $i(t) = I_0 + (1 - e^{-t})$ A as the current in 1H inductor. The applied voltage and current for the two initial condition values are shown in Fig. 3.2-4.

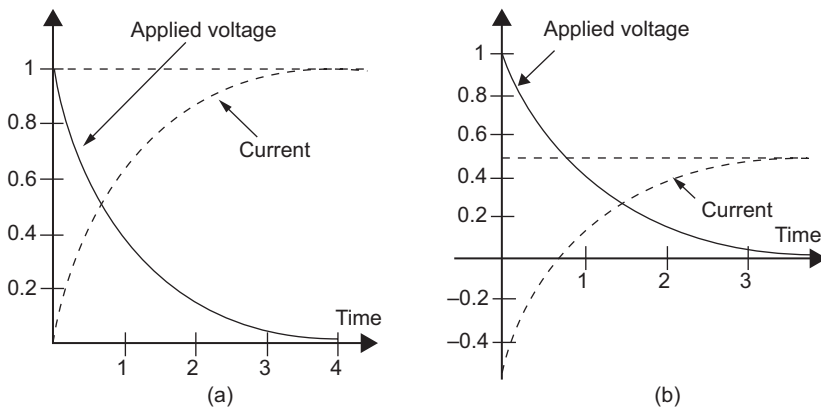


Fig. 3.2-4 Inductor with exponential voltage applied to it

The inductor preserves the waveshape of the input except for a DC offset, *i.e.*, the current in the inductor is an exponential of same index as that of the applied voltage. This is due to the fact that exponential function does not change its shape on differentiation and integration. A sinusoidal function too has that property. Therefore, we expect the inductor current to be a sinusoid of same frequency as that of input when a sinusoidal voltage is applied to it. There may be DC offset in the current as in Fig. 3.2-5 that shows the current in a 1H inductor with zero initial current when (a) $\sin t$ is applied, (b) $\sin(t + \pi/4)$ is applied and (c) $\sin(t + \pi/2)$ is applied.

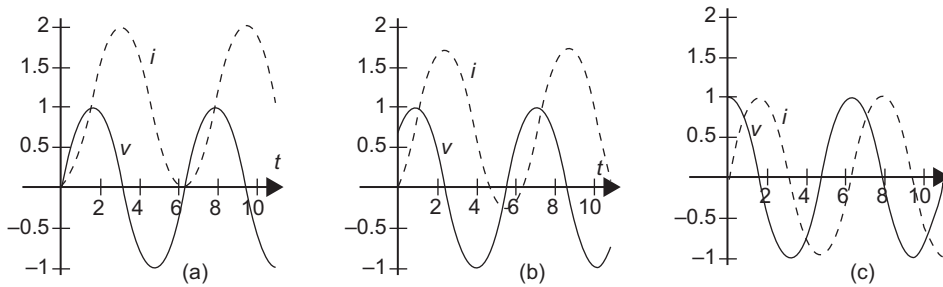


Fig. 3.2-5 Inductor with sinusoidal voltage applied to it

The alternating component of current indeed preserves the waveshape of input. The DC offset is entirely due to the input since initial current was stated to be zero. Notice that when a sinusoidal voltage is switched on at its zero-crossing, the resulting current is unipolar and reaches a peak which is twice that of its AC component amplitude, *i.e.*, it has a DC content equal to AC amplitude. This is of course clearly seen by simple integration of $v_s(t)$ as below shown in the following:

$$\begin{aligned}
 i(t) &= I_0 + \frac{1}{L} \int_0^t \sin(\omega t + \theta) dt \\
 &= I_0 + \frac{1}{\omega L} [-\cos(\omega t + \theta)]_0^t \\
 &= I_0 + \frac{1}{\omega L} [\cos \theta - \cos(\omega t + \theta)]
 \end{aligned}$$

$I_0 = 0$ A, $\omega = 1$ rad/s and $L = 1$ H for the waveforms shown Fig. 3.2-5. Both the waveforms and the above equation make it clear that DC offset in the inductor current will be zero (if initial condition is zero) if the sinusoidal voltage is switched on at its peak.

Sinusoidal voltage is a special case of a general periodic alternating waveform. We expect the current peak-to-peak amplitude to go down with the frequency. The above equation shows that it does so. Moreover, when voltage across the inductor is a sine wave, its current waveform will be an inverted cosine wave. Corresponding positions in an inverted cosine wave will take place after $T/4$ s with respect to the sine wave.

Inductor preserves the waveshape for exponential and sinusoidal inputs. The amplitude of current sinusoid in an inductor is inversely proportional to the product of frequency of applied voltage sinusoid and inductance value.

3.2.7 Linearity of Inductor

The flux linkage in an inductor and the current through it are related by a simple proportionality relationship. Hence, ψ versus i curve for a linear inductor is a straight line passing through origin. In that sense inductor is linear.

However, that is not what we mean when we say an electrical element is a linear element. *We call an electrical element linear when its voltage – current relationship satisfies two principles – the principle*

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of homogeneity and the principle of additivity. Principle of homogeneity requires that when input is scaled by a real constant, the output also must get scaled by the same constant.

We will treat $i(t)$ as input and $v(t)$ as output first. This implies that we are applying a current source across inductor and observing the voltage appearing across the combination as output. The governing equation then is $v(t) = L di(t)/dt$. Obviously, when $i(t)$ is multiplied (scaled) by a real constant α , $v(t)$ also gets scaled by same number. Hence, principle of homogeneity is satisfied.

Principle of additivity requires that when two inputs are applied simultaneously, the output observed is the sum of individual outputs observed when these inputs are applied individually. Let us say the voltage across inductor is $v_1(t)$ when a current source of $i_1(t)$ is applied to it and voltage across inductor is $v_2(t)$ when a current source of $i_2(t)$ is applied to it. Then the voltage will be $v_1(t) + v_2(t)$ when $i_1(t) + i_2(t)$ is applied if principle of additivity is satisfied. Obviously, this is also true in the present case.

Therefore, the relation $v(t) = L \frac{di(t)}{dt}$ satisfies both principles.

Now, we consider voltage as input and current as output. This implies that we are applying a voltage source across the inductor and observing its current as output. The governing relationship in this case is

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt$$

It may easily be verified that this relationship satisfies both the requirements. However, there is a caveat here. *We do not know* $v(t)$ *for* $t < 0$. Therefore, we write this relationship as

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) dt = \frac{1}{L} \int_{-\infty}^0 v(t) dt + \frac{1}{L} \int_0^t v(t) dt = I_0 + \frac{1}{L} \int_0^t v(t) dt$$

thereby absorbing all of $v(t)$ for $t < 0$ into a single number I_0 .

Now, if the portion of $v(t)$ that we apply, *i.e.*, for $t \geq 0^+$ is multiplied by a real constant α , the inductor current will be

$$i(t) = I_0 + \frac{\alpha}{L} \int_0^t v(t) dt$$

and that is not α times the earlier current. Hence, principle of homogeneity is not satisfied by the total current unless the initial condition is zero. Therefore, we will have to make a qualified statement that the principle of homogeneity is satisfied by the component of current contributed by the applied voltage function. Similarly,

$$i(t) \text{ when } v_1(t) \text{ is applied } i_1(t) = I_0 + \frac{1}{L} \int_0^t v_1(t) dt$$

$$i(t) \text{ when } v_2(t) \text{ is applied } i_2(t) = I_0 + \frac{1}{L} \int_0^t v_2(t) dt$$

$$\begin{aligned} i(t) \text{ when } v_1(t) + v_2(t) \text{ is applied } i_{12}(t) &= I_0 + \frac{1}{L} \int_0^t (v_1(t) + v_2(t)) dt \\ &= I_0 + \frac{1}{L} \int_0^t v_1(t) dt + \frac{1}{L} \int_0^t v_2(t) dt \neq i_1(t) + i_2(t) \end{aligned}$$

Hence the principle of additivity is not satisfied by the total current unless the initial condition is zero. Therefore, we will have to make a qualified statement that principle of additivity is satisfied by the component of current contributed by applied voltage function.

These two principles put together is called *superposition principle*. A linear element is one that satisfies superposition principle. Inductor is a linear element if it is understood that the superposition principle has to be applied to the current component which is produced by the applied voltage from $t = 0$ onwards. The initial current has to be excluded from the purview of superposition principle.

An inductor with zero initial current is a linear electrical element. An inductor with non-zero initial current is a linear element as far as the current component caused by applied voltage is concerned.

3.2.8 Energy Storage in an Inductor

An arbitrary $v(t)$ is applied across an inductor from $t = 0$ causing a current $i(t)$ through it. The source delivers power and energy to the inductor in this process. We will derive an expression for energy delivered to the inductor as a function of time, i.e., $E_L(t)$ and show that this energy is not dissipated by the inductor but stored by it.

$$E_L(t) - E_L(0) = \int_0^t v(t)i(t) dt = L \int_0^t i(t) \frac{di(t)}{dt} dt = L \int_0^t i(t) di(t) = \frac{L}{2} \int_0^t d[i(t)]^2 = \frac{[i(t)]^2 - [i(0)]^2}{2L}$$

$$\therefore E_L(t) = \frac{1}{2} L [i(t)]^2$$

Therefore, the change in energy delivered to an inductor over a time interval is the difference between the quantity $(1/2)Li^2$ evaluated at the end of the interval and at the beginning of the interval. The energy delivered till t is given by $(1/2)Li^2$ where i is the value of current at t .

Assume that we applied some $v(t)$ to L from $t = 0$ to t_0 and that initial condition of inductor was 0 A. At t_0 the current is I . Energy delivered to the inductor up to t_0 is $LI^2/2$ and all of it was delivered by the source connected since inductor had zero energy initially. At t_0 we remove the voltage source and short the inductor. That makes the voltage across inductor zero and therefore its current will continue at I . The power into or out of the inductor is zero since voltage across it is zero. Therefore, it will neither deliver nor take energy as long as it is kept shorted.

After some time we remove the short-circuit and connect the inductor to a DC voltage source of value V as shown in third circuit in Fig. 3.2-6. Notice that the polarity of applied voltage V is such that the current in the inductor goes down linearly with a slope of V/L A/s starting from I . It will take LI/V s for the current to reach zero. The switch S is a zero-current sensing switch and goes open at the instant the circuit current touches zero thereby isolating the inductor from any further change in current. Let us calculate the energy delivered to the DC source in this process.

$$i(t) = I - \frac{V}{L}t ; t \text{ measured from the instant at which } V \text{ was connected to } L$$

$$\text{Energy delivered to DC Source} = \int_0^{LI/V} Vi(t)dt = \int_0^{LI/V} (VI - \frac{V^2t}{L})dt = \frac{1}{2} LI^2$$

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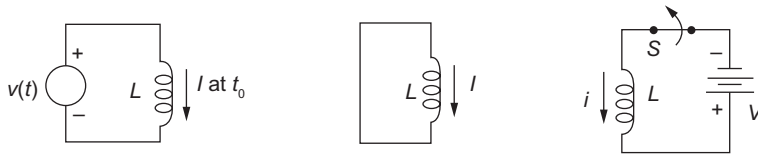


Fig. 3.2-6 On energy storage in an inductor

Hence, we see that the energy *delivered* by inductor *to* the DC source in this circuit is exactly equal to the energy *delivered* by the voltage source $v(t)$ to the inductor in the first circuit. Where was this energy when the inductor was in the second circuit? Therefore,

The total energy delivered to an inductor carrying a current i is $(1/2)Li^2$ J and this energy is stored in its magnetic field. The inductor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.

Example: 3.2-1

The current in an inductor of 2H is shown in Fig. 3.2-7. $v(t)$ does not contain any impulse. (i) What is the initial condition for inductor? (ii) Obtain $v(t)$ for 0 to 9 s and plot it. (iii) Obtain the function $p(t)$ – the power delivered to the inductor – and plot it. (iv) Obtain the function $E(t)$ – the energy stored in the inductor – and plot it. Identify the time intervals during which the voltage source charges the inductor and discharges the inductor.

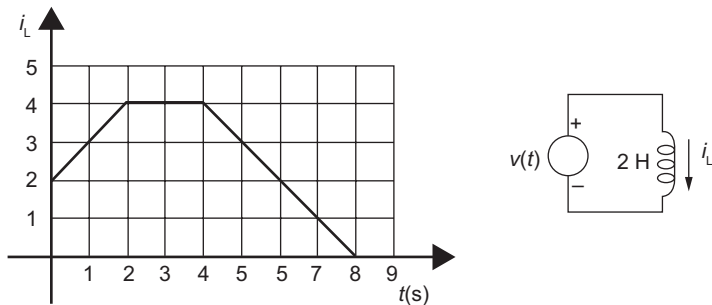


Fig. 3.2-7 Circuit and waveform for Example 3.2-1

Solution

Only an impulse voltage can change the inductor current over $[0^-, 0^+]$. The current at $t = 0^+$ is read from the given i_L waveform as 2 A.

$\therefore$ Initial current in 2H inductor at $t = 0^- = 2$ A

We use $v(t) = L(di/dt)$ to work out the $v(t)$ waveform. $i(t)$ is a piecewise linear function. The slope of current is 1 A/s in the (0 s, 2 s) interval, 0 A/s in (2 s, 4 s) interval and -1 A/s in the (4 s, 8 s) interval. Note that all intervals are open intervals. This is so because the current waveform is not differentiable at the end points of the intervals and hence the endpoints will have to be excluded from the domain of

the derivative function. We expect to observe jump discontinuities at those points in the $v(t)$ function. Using the slope values we can plot the function $v(t)$ as in Fig. 3.2-8.

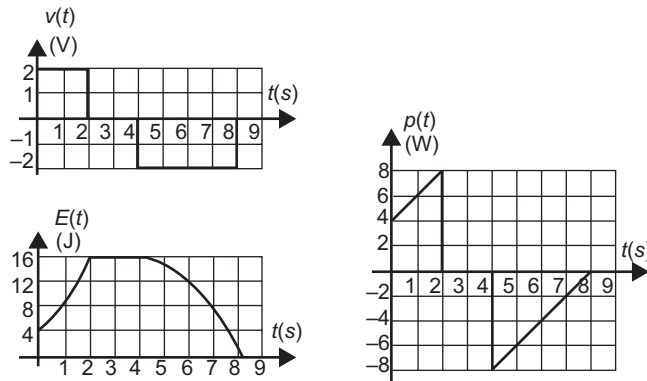


Fig. 3.2-8 Voltage, power and energy waveforms in 2H inductor in Example 3.2-1

The $v(t)$ waveform is discontinuous pulse waveform as expected. Notice that the inductor accepts a *discontinuous voltage input* and generates a *continuous current* in response. This illustrates the ability of the inductor to keep a circuit current smooth.

The power and energy waveforms are calculated by

$$p(t) = v(t)i_L(t) \quad \text{and} \quad E(t) = E(0^+) + \int_{0^+}^t p(t) dt$$

0^+ , 0^- and 0 are to be treated differently if there is an impulse voltage at $t = 0$. Generally speaking, the right instant to use in the energy equation is 0^+ . Impulse, if present, will result in a sudden change in initial condition over $[0^-, 0^+]$. We calculate the new initial condition at $t = 0^+$ and then evaluate the initial energy $E(0)$ as $0.5Li(0^+)^2$ in that case.

The power waveform in Fig. 3.2-8 shows that the source delivers power to the inductor during (0s,2s) interval and source accepts power from inductor during (4s, 8s) interval. Inductor can not dissipate energy. It can only store it. Therefore, when source delivers power to it, the energy storage in the inductor increases – this is what we call *charging* an inductor. *Discharging* an inductor is the process in which the inductor delivers energy to some other element and decreases its stored energy level. Thus, in this example, the voltage source charges the inductor during (0s,2s) interval and discharges the inductor during (4s,8s) interval.

The inductor had 4J of energy to begin with. The source delivered 12J of energy to it over (0s, 2s) interval, raising its energy storage to 16J. The inductor gave all of 16J to the source over (4s, 8s) interval and settled down at zero energy level. Therefore, the source received a net energy of 4J from the inductor.

Example: 3.2-2

The initial current in the 0.5 H inductor in Fig. 3.2-9 was 1 A at $t = 0^-$. Find the applied voltage $v(t)$ for 0 to 9s if the i_L waveform is as shown in Fig. 3.2-9.

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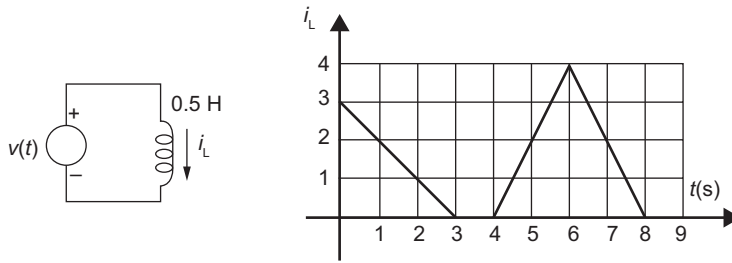


Fig. 3.2-9 Circuit and waveform for Example 3.2-2

Solution

The value of current at $t = 0^+$ is different from the given initial condition of 1 A at $t = 0^-$. The current at $t = 0^+$ is seen to be 3 A from the given data. Hence, the flux linkage of inductor changes instantaneously from 0.5×1 Wb-T to 0.5×3 Wb-T, *i.e.*, a change by + 1 Wb-T. This can happen only if 1V-s gets dumped into the inductor at $t = 0$ instantaneously. Only a voltage impulse function of magnitude equal to 1V-s can do this. Remember that *magnitude of impulse* is its area-content. Therefore, $v(t)$ should contain $\delta(t)$.

The current starts decreasing after $t = 0^+$ at the rate of -1 A/s and continues to fall at that rate in the interval (0s,3s) till it reaches zero value. After 3 s it remains quiescent at 0A. After that it rises with a rate of 2A/s in (4s,6s) interval and then falls at a rate of -2 A/s in (6s,8s) interval. Multiplying the various values of slope of current with inductor value, we get the voltage waveform as in Fig. 3.2-10. Notice the impulse function at $t = 0$.

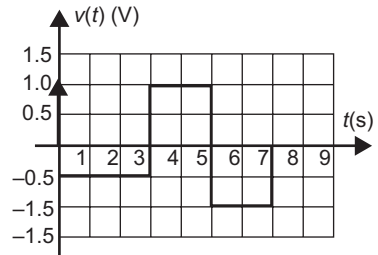


Fig. 3.2-10 Voltage waveform across 0.5 H inductor in Example 3.2-2

Example: 3.2-3

The current through a 1H inductor is shown in Fig. 3.2-11 for 0s to 18 s interval. The applied voltage is known to be impulse-free. (i) What is the initial current in the inductor at $t = 0^-$? (ii) Obtain and plot the applied voltage $v(t)$ and the power delivered to the inductor, $p(t)$? (iii) What is the net energy delivered by the voltage source to the inductor?

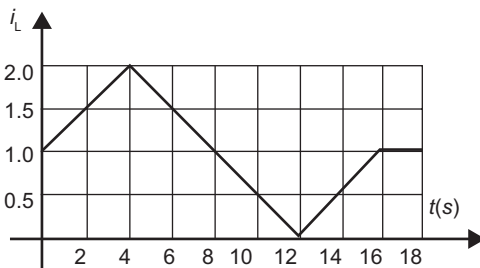
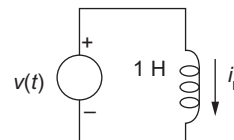


Fig. 3.2-11 Circuit and waveform for Example 3.2-3



Solution

The initial current at $t = 0^-$ is 1A since only an impulse voltage can change the initial current instantaneously.

The value of di/dt is 0.25A/s in (0,4s) interval, -0.25 A/s in (4s,12s) interval, 0.25A/s in (12s,16s) interval and 0A/s in (16s,18s) interval. Therefore $v(t)$ is 0.25V in (0s,4s) interval, -0.25 V in (4s,12s) interval, 0.25V in (12s,16s) interval and 0V in (16s,18s) interval. The $v(t)$ waveform is shown in Fig. 3.2-12.

The power waveform is obtained by forming $v(t) i_L(t)$ product and plotting it. This is also shown in Fig. 3.2-12.

The inductor started with 1A at $t = 0^+$ and ended with 1A at 16 s. Therefore, the net change in stored energy of the inductor is zero. Hence, the net energy delivered by the voltage source to inductor must also be zero.

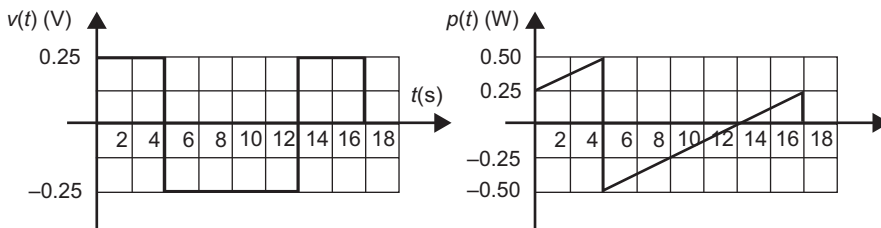


Fig. 3.2-12 Voltage and power waveforms for Example 3.2-3

Example: 3.2-4

The voltage waveform applied to an inductor of 2H is shown in Fig. 3.2-13. (i) Find the magnitude of current in the inductor at $t = 9$ s if the initial current is zero. (ii) What must have been the initial current if the current at 9 s is found to be 0A? (iii) Plot current in the inductor in the first case.

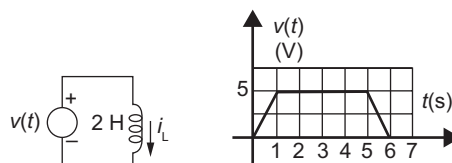


Fig. 3.2-13 Circuit and waveform for Example 3.2-4

Solution

- Inductor current at any instant is given by its initial current plus the V-s product divided by inductance value where V-s is the area under the voltage curve from initial instant to the instant at which the current is calculated. The initial instant is 0 s here. Final instant is 9 s. The V-s available in this range is 25V-s. Inductance value is 2H. Initial current is 0. Therefore, inductor current at $t = 9$ s is 12.5A. In fact it is 12.5A from $t = 6$ s onwards.
- 25Wb-T of additional flux linkage will change the current of a 2H inductor by 12.5A. Hence if current at $t = 9$ s is observed to be 0A, then the initial current must have been -12.5 A.
- The equations for $v(t)$ in various time intervals are shown in the following:

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$$v(t) = \begin{cases} 5t & \text{for } 0 \leq t < 1 \\ 5 & \text{for } 1 \leq t < 5 \\ 5 - 5(t-5) & \text{for } 5 \leq t < 6 \\ 0 & \text{for } 6 \leq t \leq 9 \end{cases}$$

The inductor current can be found by integrating $v(t)$ and dividing the integral by L .

$$i(t) = \begin{cases} 0 + 1.25t^2 & \text{for } 0 \leq t < 1 \\ 1.25 + 2.5(t-1) & \text{for } 1 \leq t < 5 \\ 11.25 + 2.5(t-5) - 1.25(t-5)^2 & \text{for } 5 \leq t < 6 \\ 12.5 & \text{for } 6 \leq t \leq 9 \end{cases}$$

This is plotted in Fig. 3.2-14.

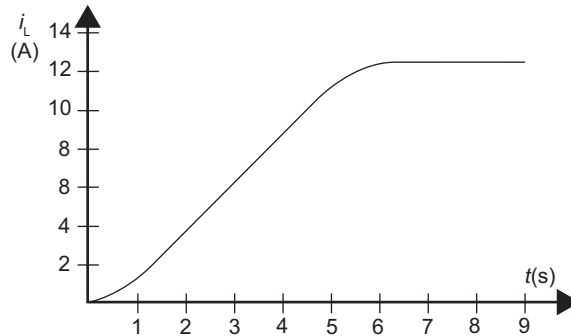


Fig. 3.2-14 Plot of current in 2H inductor in Example 3.2-4

Example: 3.2-5

The pulse voltage waveform shown in Fig. 3.2-15 is applied to an inductor of 0.5 H with initial current of 1 A. (i) Find the inductor current and energy stored in the inductor at (a) $t = 1$ s (b) $t = 2$ s and (c) $t = 3$ s. (ii) What is the net energy delivered to inductance in the time interval (a) 0 s to 1 s (b) 1 s to 2 s and (c) 0 s to 2 s? (iii) What is the maximum value of inductor current and when does it occur?

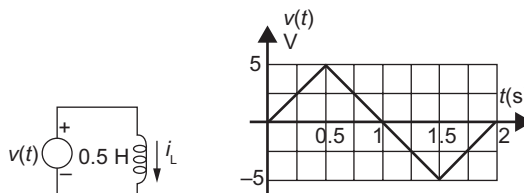


Fig. 3.2-15 Circuit and waveform for Example 3.2-5

Solution

- (i) The area under voltage waveform from 0 s to 1 s is 2.5 V-s. The value of inductance is 0.5H. Therefore, the *change* in current over this 1 s interval is $2.5/0.5 = 5\text{A}$. Initial current was 1A. There is no impulse voltage present at $t = 0$. Therefore there is no instantaneous change in current at $t = 0$. Hence, the value of current at $t = 1\text{ s}$ is $1 + 5 = 6\text{A}$.

The area under voltage waveform from 0 s to 2 s is zero. Therefore the net *change* in inductor current over this interval is zero. Hence current at $t = 2\text{ s}$ is $1 + 0 = 1\text{A}$.

Voltage applied after 2 s is zero. Hence the area under voltage waveform from 0 s to 3 s is zero and hence current at $t = 3\text{ s}$ is 1A. In fact, current is at 1A from 2 s onwards.

Energy storage in an inductor is given by $0.5Li^2\text{ J}$. Hence energy storage in 0.5H inductor at $t = 1\text{ s}$ is 9J, at $t = 2\text{ s}$ is 0.25J and at $t = 3\text{ s}$ is 0.25J.

- (ii) Net energy delivered to inductor in the first 1 s = Energy storage in the inductor at $t = 1\text{ s}$ minus energy storage in it at $t = 0\text{ s} = 9 - 0.25 = 8.75\text{ J}$

Net energy delivered to inductor in the interval between 1 s and 2 s = Energy storage in the inductor at $t = 2\text{ s}$ minus energy storage in it at $t = 1\text{ s} = 0.25 - 9 = -8.75\text{ J}$

Net energy delivered to inductor in the interval between 0 s and 2 s = Energy storage in the inductor at $t = 2\text{ s}$ minus energy storage in it at $t = 0\text{ s} = 0.25 - 0.25 = 0\text{ J}$

- (iii) The area under voltage waveform keeps increasing in the interval $[0,1]$ and hence inductor current keeps increasing in this interval. The area under voltage waveform starts decreasing after 1 s since voltage becomes negative from that point. Hence $t = 1\text{ s}$ is a time point at which the inductor current ceases to increase and starts to decrease. Hence, it must reach a local maximum there. It is a global maximum as well since the voltage does not become positive again. Value of this maximum current is 6A. It occurs at $t = 1\text{ s}$.

Example: 3.2-6

The initial current in the inductor in the circuit in Fig. 3.2-16 is 0.2A in the direction shown. What should be V if the current in the inductor is to become 0A at $t = 5\text{ ms}$?

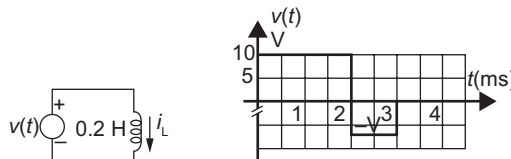


Fig. 3.2-16 Circuit and waveform for Example 3.2-6

Solution

The area under voltage waveform from 0 s to 2 ms is 0.02 V-s. The value of inductance is 0.2H. Hence, the *change* in inductor current over the first 2 ms will be $0.02/0.2 = 0.1\text{A}$. Therefore, the current in 0.2H inductor at $t = 2\text{ ms}$ will be $0.2 + 0.1 = 0.3\text{A}$.

$-V\text{ V}$ is applied for 1 ms and 0 V is applied thereafter. Therefore, V-s added to the inductor will be $-0.001V$ for all $t \geq 3\text{ ms}$. We want the current to become zero at 5 ms. This is possible only if it

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already becomes zero at 3 ms due to the $-0.001\text{ V}\cdot\text{s}$ dumped into the inductor during 2 ms to 3 ms interval.

The required flux linkage *change* is $-0.2\text{H} \times 0.3\text{A} = -0.06\text{Wb}\cdot\text{T}$. *Change* in flux linkage in an inductor is equal to the $\text{V}\cdot\text{s}$ dumped into it. Therefore, V has to be $0.06/0.001 = 60\text{V}$.

Example: 3.2-7

An arbitrary time-varying voltage waveform is applied across 1.5H with zero initial conditions as in Fig. 3.2-17. The current through the inductance is found to be 7.5A at 7 s . (i) What must be the DC voltage that should be used to replace this source such that the current through the inductor will be the same at 7 s ? (ii) If this replacement is carried out will the current in the inductor be same in the two cases at $t = 9\text{ s}$?

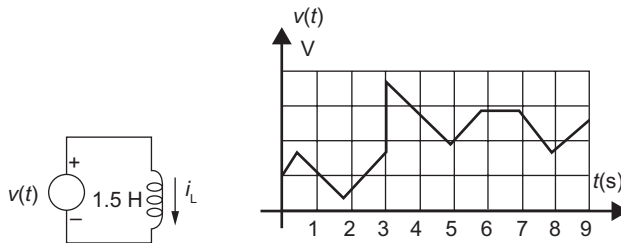


Fig. 3.2-17 Circuit and waveform for Example 3.2-7

Solution

- (i) The *change* in flux linkage over any time interval is equal to the area under voltage waveform during that interval. The *change* in flux linkage over 7 s interval is $1.5\text{H} \times (7.5 - 0) = 11.25\text{Wb}\cdot\text{T}$. Therefore, the $\text{V}\cdot\text{s}$ during that period must be $11.25\text{V}\cdot\text{s}$. If a constant voltage is to provide this $\text{V}\cdot\text{s}$ in 7 s its value must be $11.25/7 = 1.607\text{V}$.
- (ii) If this replacement is carried out, the inductor current will be same in the two cases at 7 s . That is all we can assert from the data provided. Since the $v(t)$ waveform is unknown we can not expect its area at any particular time instant to be equal to the area under a constant function. There is no reason why it cannot be so too. In short, in the absence any additional data on $v(t)$ we cannot make any predictions about equality of currents at 9 s in the two cases.

Example: 3.2-8

A periodic voltage waveform is applied across an inductor of value 0.1 H from $t = 0\text{ s}$ as in Fig. 3.2-18. The current in the inductor is found to vary periodically between 1A and 5A . (i) What is the full-cycle average value of the applied voltage waveform? (ii) What is the half-cycle average of the applied voltage waveform? (iii) What was the initial current in the inductor? (iv) Find V_p .

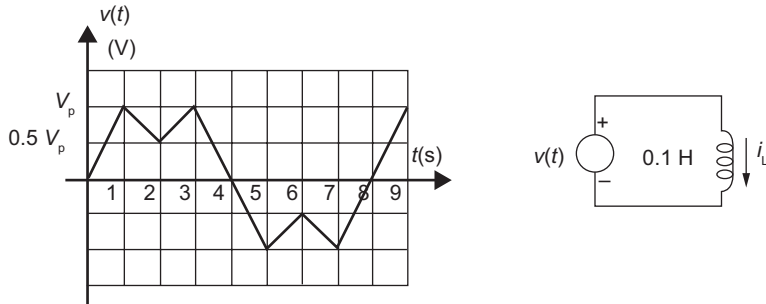


Fig. 3.2-18 Circuit and waveform for Example 3.2-8

Solution

- (i) The current is stated to be periodic. Therefore, it must be either a pure alternating waveform or such an alternating waveform plus a DC offset. Differentiation of a pure alternating waveform gives another pure alternating waveform. Differentiation of a DC term can give only zero. Hence, the derivative of inductor current will not contain DC term. Derivative of current multiplied by inductance value is the voltage across the inductor. Therefore, voltage across the inductor will not have a DC value. But the DC content in a periodic waveform is nothing but its average over a cycle period. Therefore, this voltage waveform has a full-cycle average of 0V.
- (ii) Half-cycle average of alternating voltage = Half-cycle area/half the time period. Half-cycle area of the alternating voltage is given by *change* in flux linkage of inductor between the maximum and minimum current values. It is $(5-1)\text{A} \times 0.1\text{H} = 0.4\text{Wb-T}$ in this case. Therefore, the half-cycle area of voltage waveform is 0.4 V-s and its half-cycle average value is $0.4\text{V-s} / 4\text{ s} = 0.1\text{V}$.
- (iii) The inductor current is periodic between 1A and 5A. There is no impulse content in the applied voltage. Hence its initial current must have been 1A.
- (iv) The half-cycle area in terms of V_p is $= (0.5 V_p + 0.5 V_p + 0.25 V_p) \times 2 = 2.5 V_p$ V-s. This must be equal to 0.4V-s.
 $\therefore V_p = 0.4/2.5 = 0.16\text{ V}$.

3.3 SERIES CONNECTION OF INDUCTORS

A single equivalent inductor can replace many inductors connected in series for specific analysis purposes. We consider series connection of n inductors *that have no mutual magnetic coupling* among them. Let the inductance values be $L_1, L_2, \dots, L_n$. We assume that they have the same initial current in the same direction at $t = 0^-$. Let the applied voltage be $v(t)$ and the current in the series combination be $i(t)$. See Fig. 3.3-1.

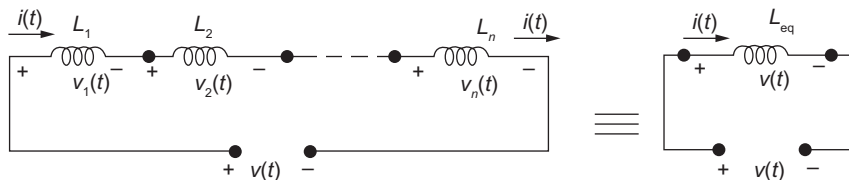


Fig. 3.3-1 Series connection of ' n ' inductors

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Applying KVL along with the element equation of inductor, we get

$$\begin{aligned}v(t) &= v_1(t) + v_2(t) + \dots + v_n(t) \\&= L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + \dots + L_n \frac{di(t)}{dt} \\&= (L_1 + L_2 + \dots + L_n) \frac{di(t)}{dt} \\&= L_{eq} \frac{di(t)}{dt} \text{ where } L_{eq} = L_1 + L_2 + \dots + L_n\end{aligned}$$

Thus, a series connection of n inductors may be replaced by an equivalent inductor with an inductance value equal to sum of the inductance values of n inductors as far as the v - i relationship is concerned. The total applied voltage across the combination is shared by the various inductors in direct proportion to inductance value. *i.e.*,

$$\begin{aligned}v(t) &= v_1(t) + v_2(t) + \dots + v_n(t) \\v_1(t) : v_2(t) : \dots : v_n(t) &= L_1 : L_2 : \dots : L_n \\v_j(t) &= \frac{L_j}{L_{eq}} v(t) \text{ for } j=1 \dots n\end{aligned}$$

The sum of flux linkages in the individual inductors is same as the flux linkage of the equivalent inductor. Flux linkage is shared in proportion to the inductance value. Similarly the total energy stored in all inductors put together is the same as the energy storage calculated using equivalent inductance value.

Thus, series equivalent of n inductors with no mutual coupling and with same initial current is 'equivalent' with respect to v - i relation, flux linkage and stored energy. The voltage across the series combination, total flux linkage and total stored energy are shared among the inductors in direct proportion to their inductance value.

Connecting two or more inductors with *unequal initial currents* is a situation in which the idealised model of an inductance does not work for a physical inductor. A physical inductor has series resistance representing its winding resistance, a parallel resistance representing losses in its core (if an iron core is used) and a parallel capacitance representing the aggregate effect of inter-winding capacitance. Connecting two inductors with different initial currents in series will result in high frequency oscillating currents and voltages everywhere in the circuit due to the winding capacitance of the inductors.

3.4 PARALLEL CONNECTION OF INDUCTORS

We develop the equivalent of n inductors in parallel in this section. We consider only *magnetically uncoupled* inductors in this section.

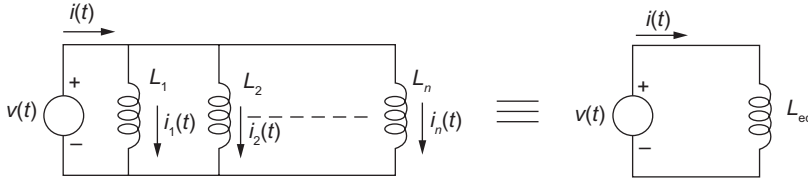


Fig. 3.4-1 Parallel connection of 'n' inductors

The *change* in inductor current over a time interval is given by the V-sec applied to it during that time interval divided by value of inductance. We assume zero initial currents at $t = 0$ and hence

$$i_k(t) = 0 + \frac{1}{L_k} \int_{0^-}^t v(t) dt \text{ for } k = 1 \text{ to } n$$

The V-sec product applied to all the inductors will be equal since they all share a common voltage from $t = 0^-$ onwards. Hence, the currents in inductors will be in *inverse proportion* to the inductance values. *i.e.*,

$$i_1(t) : i_2(t) : \dots : i_n(t) = \frac{1}{L_1} : \frac{1}{L_2} : \dots : \frac{1}{L_n}$$

Further, applying KCL at the +ve node of voltage source in the circuit in Fig. 3.4-1, we get,

$$\begin{aligned} i(t) &= i_1(t) + i_1(t) + \dots + i_n(t) \\ &= \left[\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right] \int_{0^-}^t v(t) dt = \frac{1}{L_{eq}} \int_{0^-}^t v(t) dt \end{aligned}$$

Therefore, we see that a single inductor L_{eq} can represent the n inductors in parallel as far as the $v-i$ relationship is concerned. This implies that the source will not be able to distinguish between the parallel combination of n inductors and a single inductor that is the parallel equivalent.

All the inductors had zero initial flux linkage and zero initial energy. The *change* in flux linkage in an inductance over a time interval is same as the V-sec applied to it during that time interval. Since all of them started at zero flux linkage as per our assumption, it follows that all of them will have equal flux linkage at all t .

Now, we look at the stored energy picture. We know that the stored energy function $E(t)$ of an inductor $= 0.5Li^2$, where i is the instantaneous current. However, we need another formulation for the same function now. We proceed as below shown in the following:

$$\psi(t) = \psi(0^-) + \int_{0^-}^t v(t) dt = \text{initial flux linkage} + \text{V-s added}$$

$$\text{Also, } \psi(t) = Li(t)$$

$$\therefore E(t) = \frac{1}{2} L [i(t)]^2 = \frac{1}{2} \psi(t) i(t) = \frac{1}{2L} (\text{volt-sec})^2 \text{ (if initial flux linkage is zero)}$$

Hence the stored energy at any instant in an inductor is proportional to square of V-s product dumped into it till that instant provided the inductor was initially relaxed. Therefore, in the parallel

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connection of inductors under consideration, the stored energy in the inductors will be in inverse proportion to their inductance values. They all share the same V-sec product at the same instant.

$$\begin{aligned}
 E_{\text{total}}(t) &= E_1(t) + E_2(t) + \cdots + E_n(t) \\
 &= \frac{1}{2} \left[\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n} \right] [\text{volt-sec}]^2 \\
 &= \frac{1}{2L_{\text{eq}}} [\text{volt-sec}]^2 \\
 &= \text{Energy stored in an initially relaxed inductor } L_{\text{eq}}
 \end{aligned}$$

Therefore, the equivalent inductance L_{eq} gives the correct value for stored energy. Note that we have shown this only for a parallel connection of initially relaxed inductors.

A single inductor $L_{\text{eq}} = \frac{1}{\sum_{i=1}^n 1/L_i}$ can represent a set of n inductors in parallel. Voltage, V-sec and flux linkage are common variables in parallel connection of initially relaxed inductors. Total current and total stored energy will be shared by the inductors in inverse proportion to their inductance value.

It can be shown that the conclusions arrived above are valid for inductors with non-zero initial current connected in parallel provided they have equal initial flux linkages. In the case of parallel connection of inductors with unequal initial flux linkages, circulating currents in local loops formed by the inductors can appear under certain conditions, thereby trapping a portion of total initial energy and hiding that portion of initial energy from rest of the circuit forever. Example 3.4.2 illustrates this situation.

Example: 3.4-1

Three inductors of 0.1 H, 0.05 H and 0.15 H with equal initial current of 1.5A are connected in series at $t = 0$. $v(t)$ from $t = 0^-$ is $0.3\delta(t) + v_s(t)$ where $v_s(t)$ is not known. $i(t)$ is found to be 3A in the same direction as that of initial current flow at $t = 5$ s and $v(t)$ at that instant is found to be 6V. (i) Find the voltage across each inductor and flux linkages in them at $t = 5$ s (ii) The average value of $v_s(t)$ in the first 5 s (iii) Stored energy in the circuit and each inductor at $t = 5$ s (iv) Total energy delivered by the applied voltage source in 5 s average power delivered by the source over 5 s?

Solution

The inductors are in series and they have same initial current. Hence, they can be replaced by a single inductor of $0.1 + 0.05 + 0.15 = 0.3$ H inductor as shown in Fig. 3.4-2.

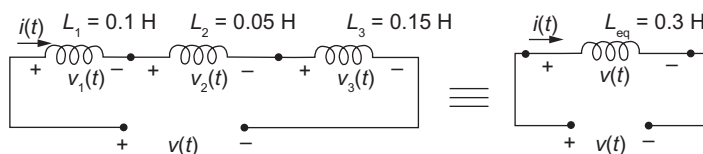


Fig. 3.4-2 Circuit for solving Example 3.4-1

- (i) Inductors in series share instantaneous voltage in proportion to their inductance values. Hence, 6 V gets distributed in the ratio 0.1:0.05:0.15, *i.e.*, 2:1:3. Therefore, 0.1 H gets 2 V across it, 0.05 H gets 1 V across it and 0.15 H gets 3 V across it at $t = 5$ s.
- Total flux linkage in the combination at $t = 5$ s is $L_{eq}i = 0.3 \text{ H} \times 3 \text{ A} = 0.9 \text{ Wb-T}$. Total flux linkage gets distributed in proportion to inductance value in a series combination. Hence, 0.9 Wb-T gets distributed in the ratio 2:1:3 in the three inductors. Hence, at $t = 5$ s the 0.1 H inductor will have 0.3 Wb-T flux linkage in it, 0.05 H will have 0.15 Wb-T flux linkage in it and 0.15 H will have 0.45 Wb-T in it.
- (ii) The total V-sec dumped into effective inductance 0.3 H in 5 s = 0.3 V-sec from the impulse component + area of $v_s(t)$ from $t = 0$ s to $t = 5$ s. But total V-s dumped must be equal to flux linkage change. This change in flux linkage is $0.3 \text{ H} \times (3 \text{ A} - 1.2 \text{ A}) = 0.54 \text{ Wb-T}$. Therefore, $0.54 = 0.3 + \text{area of } v_s(t) \text{ in } [0, 5\text{s}]$. Therefore, area of $v_s(t)$ in $[0, 5\text{s}] = 0.24 \text{ V-s}$. Therefore, average value of $v_s(t)$ over $[0, 5\text{s}] = 0.24 \text{ V-s} / 5 \text{ s} = 0.48 \text{ V}$.
- (iii) The total stored energy in a series combination will be distributed among the inductors in proportion to inductance value. The total energy storage at $t = 5$ s will be $0.5 \times 0.3 \times 3^2 = 1.35 \text{ J}$. Stored energy will be 0.45 J in 0.1 H inductor, 0.225 J in 0.05 H inductor and 0.675 J in 0.15 inductor.
- (iv) Initial stored energy in the system was $0.5 \times 0.3 \times 1.2^2 = 0.216 \text{ J}$. Total stored energy at $t = 5$ s is 1.35 J. Therefore, total energy delivered by applied voltage source in the first 5 s = $1.35 - 0.216 = 1.134 \text{ J}$. Average power delivered in the first 5 s = $1.134 \text{ J} / 5 \text{ s} = 226.8 \text{ mW}$.

Example: 3.4-2

The three inductors – 0.02H, 0.05H and 0.0333H –with same initial current of 1.5A are connected in parallel. All the three inductors have initial current of 1.5 A from top to down at $t = 0^-$. The applied voltage is $v_s(t)$ and has no impulse content. The total current into the parallel combination is found to be 9A in the same direction as the initial current at $t = 5$ ms. (i) Find the trapped energy in the parallel combination. (ii) Find the current through each inductor and flux linkages in them at $t = 5$ ms. (iii) The average value of $v_s(t)$ in the first 5 ms. (iv) Stored energy in the circuit and in each inductor at $t = 5$ ms. (v) Total energy delivered by the applied voltage source in 5 ms and average power delivered by the source over 5 ms?

Solution

The circuit containing three inductors and its equivalent circuit are shown in Fig. 3.4-3. The initial currents in various paths at $t = 0^-$ are marked within brackets. The initial values remain the same at $t = 0^+$ too since applied voltage has no impulse content.

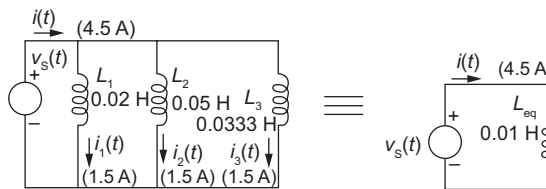


Fig. 3.4-3 Circuits for solving Example 3.4-2

- (i) Initial stored energy = Sum of initial energy in the three inductors = $0.5 \times 0.1033 \times 1.5^2 = 116.21 \text{ mJ}$. But the initial energy as per the equivalent circuit is $0.5 \times 0.01 \times 4.5^2 = 101.25 \text{ mJ}$. The

The current through a capacitor is proportional to the rate of change of voltage across it. The voltage across the capacitor is proportional to the area under the current waveform, *i.e.*, the A-s product (or C) applied through it from $t = -\infty$ where $t = -\infty$ has to be understood as the moment this capacitor came into being.

We notice that the element relationship of capacitor is similar to that of an inductor. Only that the role of voltage and current in the relationship has been interchanged.

Inductor is an element that accumulates flux linkage (V-sec or Wb-T) and makes its response variable, *i.e.*, current, proportional to the accumulated flux linkage. Capacitor is an element that accumulates charge (A-s or C) and makes its response variable, *i.e.*, voltage, proportional to the accumulated charge.

Therefore we need not enter into a detailed discussion on the implications of the element relationship of capacitor. Such a discussion will be analogous to the line of reasoning we employed in the case of inductor. Hence, we list the implications without detailed explanation.

$$i(t) = C \frac{dv(t)}{dt} \quad \Rightarrow$$

- Instantaneous voltage across a capacitor can not be predicted from instantaneous value of current through it.
- If instantaneous value of current is positive, the capacitor voltage will be increasing at that instant, and, if it is negative the voltage will be decreasing at that instant.
- When current through a capacitor crosses zero in the downward direction, its voltage attains a local maximum. When the current crosses zero in the upward direction, the capacitor voltage attains a local minimum.
- Current through a capacitor with a constant voltage across it is zero.
- Capacitor preserves the waveshape for exponential and sinusoidal inputs.

$$v(t) = \frac{1}{C} \int_{-\infty}^t i(t) dt = \frac{1}{C} \underbrace{\int_{-\infty}^0 i(t) dt}_{V_0} + \frac{1}{C} \int_0^t i(t) dt = V_0 + \frac{1}{C} \int_0^t i(t) dt \quad \Rightarrow$$

Change in capacitor voltage over $[t_1, t_2]$, $\Delta v = (\text{Area under capacitor current over } [t_1, t_2])/C$.

$(v(t) \text{ at } t = t_2) \text{ is } (v(t) \text{ at } t = t_1) + \Delta v$

$v(t) = V_0 + (\text{Area under capacitor current over } [0, t])/C$, where V_0 is the voltage across the capacitor at $t = 0$ and is called initial condition of the capacitor.

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$$C \times \Delta v = \int_{t_1}^{t_2} i(t) dt \text{ and } i_{av} \text{ in } [t_1, t_2] = \frac{\int_{t_1}^{t_2} i(t) dt}{(t_2 - t_1)} = \frac{C \Delta v}{(t_2 - t_1)} \Rightarrow$$

The amount of voltage change required in a capacitor decides the area-content under current waveform to be applied to it to bring about the change. The time allowed to bring about it decides the average current to be applied. Thus, rapid change in capacitor voltage calls for large amplitude current through it.

$$C \times \Delta v = \int_{t_1}^{t_2} i(t) dt \text{ and } i_{av} \text{ in } [t_1, t_2] = \frac{\int_{t_1}^{t_2} i(t) dt}{(t_2 - t_1)} = \frac{C \Delta v}{(t_2 - t_1)} \Rightarrow$$

- Voltage in a capacitor can not change instantaneously unless an impulse current is applied or supported in the circuit.
- Unit Impulse Current will have an area-content of unity since it is a unit impulse. Thus, unit impulse current will deposit 1 C of charge in a capacitor over $[0-, 0+]$, i.e., instantaneously. Therefore, the voltage across a capacitor C changes instantaneously by $1/C$ V when the circuit applies or supports a unit impulse current through it.
- Therefore, if a circuit does not apply or support impulse current, the voltage across capacitors in that circuit will be continuous functions of time. Capacitors absorb rapid variations in circuit currents and tend to keep circuit voltages smooth.
- The amplitude of voltage sinusoid in a capacitor is inversely proportional to the product of frequency of applied current sinusoid and capacitance value.
- There can be a DC voltage across a capacitor even when the applied current waveform is a pure alternating one. The amount of DC content depends upon the initial condition of the capacitor and the point at which the current waveform is switched on to the capacitor.
- When the applied current through a capacitor is a periodic alternating waveform, the voltage across the capacitor will contain an alternating component with the same period. The peak-to-peak amplitude of this alternating component will be directly proportional to half-cycle area of current waveform and inversely proportional to capacitance value. It decreases with increase in frequency of the current.

- Therefore, a large valued capacitor in a circuit can absorb alternating currents in the circuit without contributing significant amount of alternating voltages to the circuit. A large valued capacitor can hold the potential difference across two points in a circuit at a reasonably constant level even when large amplitude alternating currents flow through it.
- A capacitor with zero initial voltage is a linear electrical element.
- A capacitor with non-zero initial voltage is a linear element as far as the voltage component caused by applied current is concerned.
- The total energy delivered to a capacitor carrying a voltage V across it is $(1/2)CV^2$ J and this energy is stored in its electric field.
- Stored energy in a capacitor is also given by $(1/2C)Q^2$ J and $QV/2$ J.
- The capacitor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.

Example: 3.5-1

The voltage observed across the $10\mu\text{F}$ capacitor in Fig. 3.5-1 is shown in the same figure. Find the current source function $i_s(t)$ if the initial voltage across the capacitor was zero.

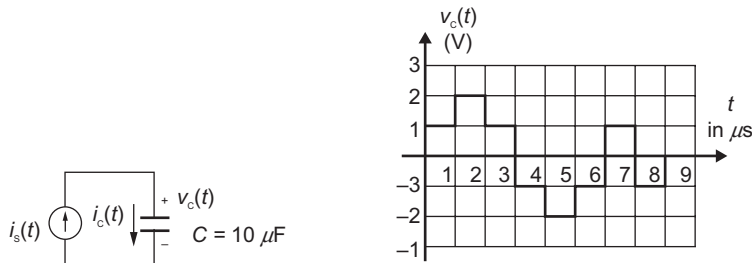


Fig. 3.5-1 Circuit and waveform for Example 3.5-1

Solution

The voltage across capacitor undergoes a sudden jump by 1V at $t = 0$. This is possible only if a charge of $10\mu\text{F} \times 1\text{V} = 10\mu\text{C}$ is dumped on the capacitor instantaneously at $t = 0$. Therefore, $i_s(t)$ must contain $10^{-5} \delta(t)$.

Similarly, at $t = 1\mu\text{s}$ the capacitor voltage again jumps up by 1V. This calls for another $10\mu\text{C}$ to be dumped on it at $t = 1\mu\text{s}$. Therefore $i_s(t)$ must contain $10^{-5} \delta(t-10^{-5})$.

The capacitor voltage jumps by -1V at $t = 2\mu\text{s}$. This calls for $-10^{-5} \delta(t-2 \times 10^{-5})$ in $i_s(t)$. Proceeding this way we get the following waveform for $i_s(t)$.

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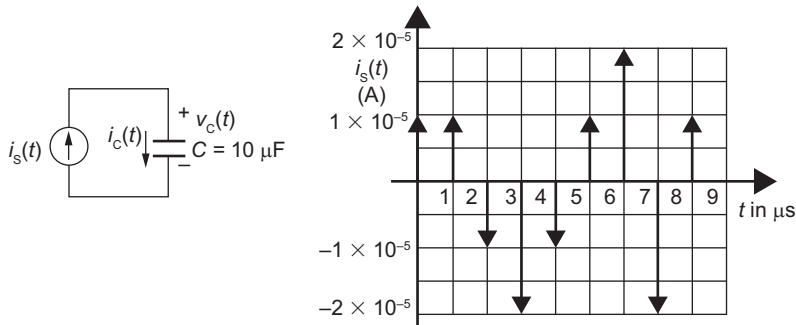


Fig. 3.5-2 Solution for current source function in Example 3.5-1

The unit given in the vertical axis is Amperes. However, when impulse content is indicated in a waveform, the value read from vertical axis must be interpreted as magnitude of area-content and unit must be suitably re-interpreted.

Example: 3.5-2

The voltage across a $1000\mu\text{F}$ capacitor with zero initial voltage at $t = 0^-$ is given in Fig. 3.5-3. Find (i) the applied current waveform, (ii) waveforms of power and energy delivered by the current source, (iii) time intervals during which the capacitor is delivering energy to the source and (iv) the net energy delivered by capacitor to the source. Also, explain why the energy delivered by the source never becomes negative in this example.

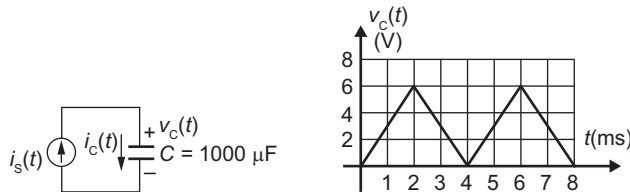


Fig. 3.5-3 Circuit and waveform for Example 3.5-2

Solution

- (i) The current through a capacitor is given by the first derivative of voltage scaled by the capacitance value of the capacitor. The voltage waveform contains four straight-line segments followed by zero value. The slopes of voltage in various intervals are as follows.

Value of slope of capacitor voltage in $[0^+, 2^-] = 3 \text{ V/ms}$

Value of slope of capacitor voltage in $[2^+, 4^-] = -3 \text{ V/ms}$

Value of slope of capacitor voltage in $[4^+, 6^-] = 3 \text{ V/ms}$

Value of slope of capacitor voltage in $[6^+, 8^-] = -3 \text{ V/ms}$

Value of slope of capacitor voltage in $[8^+, 9] = 0 \text{ V/ms}$

Multiplying these values by 1mF we get 3A , -3A , 3A , -3A and 0A as the value of current in the five intervals. Hence, the current source function will be a rectangular pulse waveform as shown in Fig. 3.5-4 (b).

- (ii) The power delivered by the current source will be given by the product of $v_C(t)$ and $i_S(t)$. It will have a waveform containing straight-line segments since $v_C(t)$ contains straight-line segments and the waveform of $i_S(t)$ is a symmetric rectangular pulse waveform. The power waveform is shown in Fig. 3.5-4 (c). The power delivered by the source alternates between positive and negative values.

Energy delivered by the current source is given by the running integral of power waveform from 0^+ . The waveform of delivered energy is shown in Fig. 3.5-4 (d). It is always positive.

- (iii) The capacitor is delivering energy to the current source when the power delivered by the current source shows a negative value.

Hence, during $[2^+, 4^-]$ and $[6^+, 8^-]$ (values indicating time in ms) time intervals the capacitor delivers energy to the current source.

- (iv) The capacitor had an initial voltage of 0V and it ends up with 0V. Therefore, the net change in stored energy of capacitor is zero. There is no other element in the circuit that can store or dissipate energy. Hence, the net energy delivered by the current source also must be zero.

The capacitor in this example started with zero voltage initially. Hence, the initial energy stored in it is zero. Capacitors can only store energy and they can not generate or dissipate energy. They can store energy temporarily and give it back to other elements later. Therefore, energy function of a capacitor is always zero or positive-valued. An electrical element with an energy function, $E(t) = \int_{-\infty}^t v(t)i(t)dt$ that is ≥ 0 for all t is called a *passive element* and capacitor is one such passive element.

A capacitor can give back more energy to a source than it received from it, even temporarily, only if it already had some energy in store before the source started acting on it. In this example, C had no such initial energy. Hence, the source can not receive more than what it gave. Therefore, the value of energy delivered by the source will never be negative in this circuit.

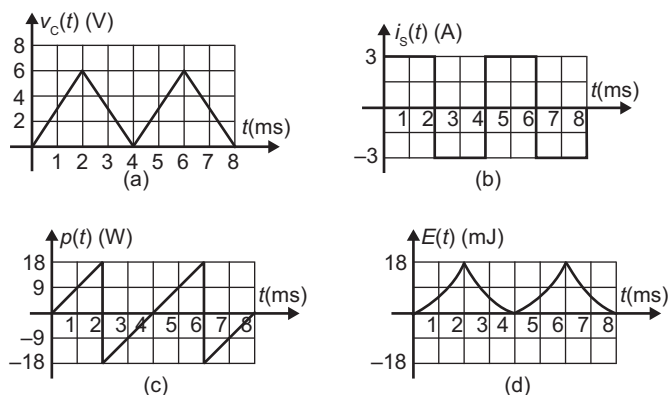


Fig. 3.5-4 Waveforms for Example 3.5-2: (a) capacitor voltage (b) capacitor current (c) power delivered by source (d) energy delivered by source

Example: 3.5-3

A pulse current waveform shown in Fig. 3.5-5 is used to charge a capacitor of 0.5 mF with zero initial voltage. $i_S(t)$ is zero after 2 ms. (i) Find voltage developed across the capacitor at (a) $t = 1$ ms, (b) $t = 2$ ms and (c) $t = 4$ ms. (ii) What is the maximum voltage across the capacitor and when does it occur? (iii) What is the net energy delivered by source to the capacitor?

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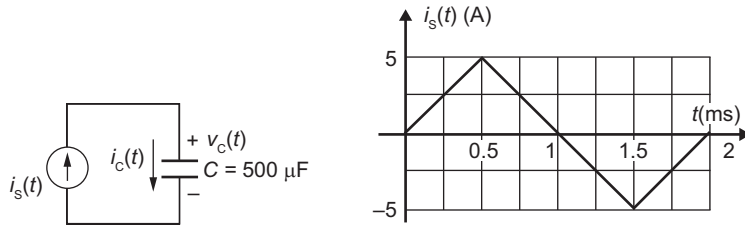


Fig. 3.5-5 Circuit and waveform for Example 3.5-3

Solution

- (i) The *change* in voltage across a capacitor over a time interval is equal to the A-sec product dumped into it during that time interval divided by capacitance value.

The area under $i_s(t)$ for $[0^+, 1\text{ms}] = 2.5 \text{ mC}$

The area under $i_s(t)$ for $[0^+, 2\text{ms}] = 0 \text{ mC}$

The area under $i_s(t)$ for $[0^+, 4\text{ms}] = 0 \text{ mC}$ since $i_s(t)$ is zero after 2 ms.

Initial voltage across capacitor is given to be zero.

Therefore, voltage across capacitor at 1 ms = $0 + 2.5 \text{ mC} / 0.5 \text{ mF} = 5 \text{ V}$

Voltage across capacitor at 2 ms = $0 + 0 = 0 \text{ V}$

Voltage across capacitor at 4 ms = $0 + 0 = 0 \text{ V}$

- (ii) The current employed to charge the capacitor is positive in the interval $[0^+, 1\text{ms}]$ and hence the voltage across capacitor increases in this interval. The current changes polarity at 1 ms and becomes a discharging current from then on. Therefore, 1 ms is the time instant at which the capacitor voltage reaches a local maximum. In this case it is a global maximum as well since the current never becomes positive after 1 ms. Hence, the maximum value of capacitor voltage is 5 V and it occurs at 1 ms.
- (iii) The capacitor started with zero voltage and ends up with zero voltage. Therefore change in stored energy in the capacitor is zero. Therefore, the net energy delivered by the source to capacitor is zero.

Example: 3.5-4

Initial voltage at $t = 0^-$ across the $200\mu\text{F}$ capacitor in Fig. 3.5-6 is -50 V . The current source is zero after 3 ms. Find (i) the voltage across the capacitor at 2 ms, energy stored in it at 2 ms and the energy delivered by the current source in the first 2 ms, (ii) the value of I if the capacitor voltage is to become zero at 4 ms and (iii) the net energy delivered by current source with the above value of I .

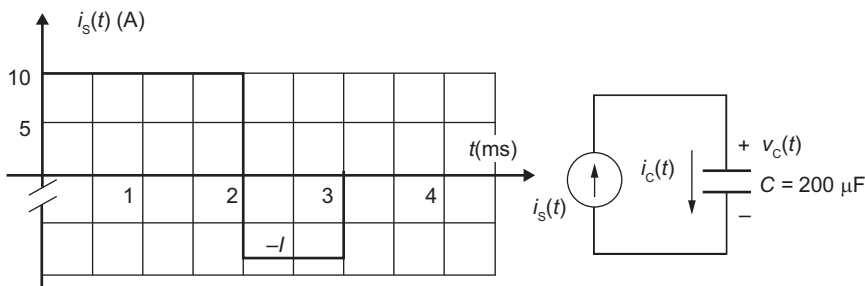


Fig. 3.5-6 Circuit and waveform for Example 3.5-4

Solution

- (i) The area under current waveform over 0ms to 2ms interval is 20mC. Hence, the *change* in capacitor voltage is $20\text{mC}/0.2\text{mF} = 100\text{V}$. Initial voltage is -50V . Therefore, the voltage across capacitor at $t = 2\text{ ms}$ is $-50 + 100 = 50\text{V}$.

Energy stored in a capacitor is given by $0.5CV^2$ J. Therefore the energy stored in capacitor at $t = 2\text{ ms}$ is 250 mJ.

Energy delivered by current source in the first 2 ms = *change* in stored energy in the capacitor over first 2 ms. Initial stored energy in the capacitor is $0.5 \times 0.2\text{mF} \times -50\text{V} \times -50\text{V} = 250\text{ mJ}$. Stored energy at 2 ms = $0.5 \times 0.2\text{mF} \times 50\text{V} \times 50\text{V} = 250\text{mJ}$. Therefore *change* in stored energy over first 2 ms is 0. Therefore energy delivered by current source over this time interval is zero.

The capacitor voltage starts at -50V and increases linearly with a slope of 50V/ms for the first 2 ms. Therefore the voltage is linear from -50V to $+50\text{V}$, crossing 0 at 1 ms. The power delivered by current source is negative in the first 1 ms and positive in the second 1 ms. The capacitor empties its initial energy into the current source in the first 1 ms and the current source puts it back on the capacitor over the next 1 ms duration.

- (ii) The capacitor has 50 V across it at 2 ms. Hence, a charge of $0.2\text{mF} \times 50\text{V} = 10\text{mC}$ has to be removed at a constant rate of $I\text{ C/s}$ over [2ms,3ms]. Therefore, I has to be $10\text{mC}/1\text{ ms} = 10\text{C/s} = 10\text{A}$. With this value of I the capacitor voltage becomes zero at 3 ms and remains at zero thereafter since $i_s(t)$ is given to be zero after 3 ms.
- (iii) With $I = 10\text{A}$ the capacitor voltage becomes 0V at 3ms. Therefore, final stored energy is zero and initial stored energy is 250mJ. Thus, the *change* in stored energy is -250mJ . There are no other energy storage elements or energy dissipating elements in the circuit. Hence this value must be the net energy delivered by the current source. It is negative implying that the source received 250mJ of energy from the capacitor. It is indeed so since the capacitor started with 250mJ in it and ended with 0 J.

Example: 3.5-5

The switch S in the circuit in Fig. 3.5-7 (a) is ideal and the capacitor is initially uncharged. The switch S is closed at $t = 0$. (i) What is the current delivered by the source? (ii) Express the voltage across capacitor as a function of time. (iii) What is the energy delivered by the voltage source? (iv) What is the energy stored in the capacitor? (v) Are the two energy values equal? If not, what happens to the mismatch energy?

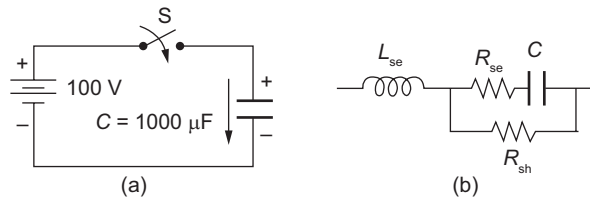


Fig. 3.5-7 (a) Circuit for Example 3.5-5 (b) A more accurate model for a capacitor

Solution

- (i) The voltage across the capacitor has to *change instantaneously* in this circuit since KVL has to be obeyed at $t = 0^+$. But instantaneous change in capacitor voltage will require an impulse current

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with an area content equal to the *change* in charge required. The change in stored charge required here is $100\text{V} \times 1000\mu\text{F} = 100\text{mC}$. Therefore, the current that flows in the circuit will be $0.1\delta(t)$.

- (ii) The voltage across capacitor was zero till $t = 0^-$. It will be 100V at and after $t = 0^+$. It is discontinuous at $t = 0$.

$$v_C(t) = \begin{cases} 0 & \text{for } t = 0^- \\ \text{undefined} & \text{for } t = 0 \\ 100 & \text{for } t \geq 0^+ \end{cases}$$

We will be tempted to write this as $v_C(t) = 100 u(t)$ where $u(t)$ is the unit step function. But we do not know the capacitor voltage before $t = 0^-$. So we can not express it that way.

- (iii) Energy delivered by the source = Voltage $\times$ Charge transported across the voltage = $100\text{V} \times 0.1\text{C} = 10\text{ J}$. It can also be calculated as the area under the power waveform. The power delivered as a function of time is $100 \times 0.1 \delta(t) = 10\delta(t)$. Area under this waveform is 10 J .
- (iv) Energy stored in the capacitor = $0.5 \times 0.001\text{F} \times (100\text{V})^2 = 5\text{ J}$.
- (v) Energy delivered by the source and energy stored in the capacitor are not equal. The source delivered 5J extra. This is a circuit in which the ‘capacitance’ model for a physical capacitor is not adequate. A physical capacitor is a physical device that is represented by the mathematical model of ‘capacitance’. This model is only a first level model that ignores the second level details of the physical device. Any electrical element will have all the three effects – resistance, capacitance and inductance – involved in it. Therefore, a capacitor has resistance and inductance associated with it. The resistance of metal foil used in the capacitor and the resistance of connecting leads contribute a series resistor to the model of capacitor. Leakage of current through the imperfect insulator used as the dielectric in the capacitor contributes a shunt resistor in the second level model for a capacitor. And the magnetic flux produced within the capacitor when current flows in its leads and foil contribute an inductance too. Thus a more accurate model for a physical capacitor is as shown in Fig. 3.5-7 (b). R_{se} is usually in $\text{m}\Omega$ range, R_{sh} is usually in $\text{k}\Omega - \text{M}\Omega$ range and L_{se} is in $\text{nH} - \mu\text{H}$ range.

We will see in a later chapter that this more detailed model will predict transient currents and voltages which oscillate at a high frequency when a DC voltage is suddenly switched on to an uncharged capacitor. The two resistors that are invariably present in any physical capacitor will take care of the extra energy delivered by the voltage source – they dissipate it.

Example: 3.5-6

A periodic current waveform is applied to a capacitor of value 0.1 F from $t = 0\text{ s}$ as in Fig. 3.5-8. The voltage across the capacitor is found to vary periodically between 1 V and 5 V . (i) What is the full-cycle average value of the applied current waveform? (ii) What is the half-cycle average of the applied current waveform? (iii) What was the initial voltage in the capacitor? (iv) Find I_p .

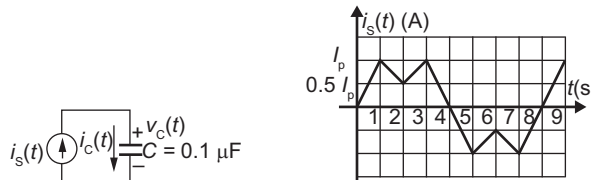


Fig. 3.5-8 Circuit and waveform for Example 3.5-6

Solution

- (i) The voltage is stated to be periodic. Therefore, it must either be a pure alternating waveform or such an alternating waveform plus a DC offset. Differentiation of a pure alternating waveform gives another pure alternating waveform. Differentiation of a DC term can give only zero. Hence the derivative of capacitor voltage will not contain a DC term. Derivative of voltage multiplied by capacitance value is the current through the capacitor. Therefore, current through the capacitor will not have a DC value. But the DC content in a periodic waveform is nothing but its average over a cycle period. Therefore, this current waveform has a full-cycle average of 0V.
- (ii) Half-cycle average of alternating current = Half-cycle area / half the time period. Half-cycle area of the alternating current is given by *change* in charge of capacitor between the maximum and minimum voltage values. It is $(5-1)\text{V} \times 0.1\text{F} = 0.4\text{C}$ in this case. Therefore, the half-cycle area of current waveform is 0.4C and its half-cycle average value is $0.4\text{C}/4\text{ s} = 0.1\text{A}$.
- (iii) The capacitor voltage is periodic between 1V and 5V. There is no impulse content in the applied current. Hence its initial voltage must have been 1V.
- (iv) The half-cycle area in terms of I_p is $= (0.5 I_p + 0.5 I_p + 0.25 I_p) \times 2 = 2.5 I_p \text{ C}$. This must be equal to 0.4 C. Therefore, $I_p = 0.4/2.5 = 0.16\text{A}$.

3.6 SERIES CONNECTION OF CAPACITORS

Practical capacitors come in standard sizes with standard voltage, current and capacitance ratings. Series or parallel connection of such capacitors is almost always required in practical applications in order to meet the application requirements on voltage rating or current rating or capacitance rating. Such series or parallel connections of capacitors can often be represented as a single equivalent capacitor for analysis purposes provided we pay careful attention to initial conditions of the capacitors involved in the connection. We look at series connection of capacitors in this section.

3.6.1 Series Connection of Capacitors with Zero Initial Energy

Let n capacitor with capacitance values $C_1, C_2, \dots, C_n$ be with zero initial voltage and no mutual electrostatic coupling among them be connected in series as shown in Fig. 3.6-1.

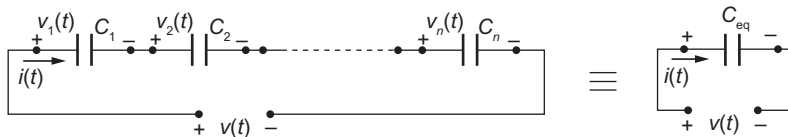


Fig. 3.6-1 Series connection of capacitors and equivalent capacitor

The *change* in the voltage across a capacitor is equal to the charge dumped into it divided by capacitance value. Capacitors that have been connected in series have to have a common current and hence all of them will get only the same charge dumped on them subsequently. *Therefore, the change in capacitor voltages across the capacitors in a series combination due to $i(t)$ will be in proportion to reciprocal of their capacitance values.* If all the capacitors had zero initial voltage

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at the instant of connection, then capacitor voltages themselves will be in proportion to reciprocal of capacitance values. Thus, $v(t)$ gets distributed among the capacitors as per the following relation.

$$\text{Ratio of distribution of } v(t) = \frac{1}{C_1} : \frac{1}{C_2} : \frac{1}{C_3} : \dots : \frac{1}{C_n}$$

$$\therefore v_j(t) = \frac{1/C_j}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n}} v(t) = \frac{1/C_j}{\sum_{k=1}^n 1/C_k} v(t)$$

$$\text{If there are only two capacitors } v_1(t) = \frac{C_2}{C_1 + C_2} v(t) \text{ and } v_2(t) = \frac{C_1}{C_1 + C_2} v(t)$$

Note that smaller capacitors will take larger share of voltage in a series connection of capacitors.

Assume that this series connection is being driven by a current source $i(t)$ across its terminals from $t = 0$ onwards. The voltage $v(t)$ developed across the current source terminals will be

$$\begin{aligned} v(t) &= \frac{1}{C_1} \int_{-\infty}^t i(t) dt + \frac{1}{C_2} \int_{-\infty}^t i(t) dt + \dots + \frac{1}{C_n} \int_{-\infty}^t i(t) dt \\ &= \left[\frac{1}{C_1} \int_{-\infty}^{0^-} i(t) dt + \frac{1}{C_2} \int_{-\infty}^{0^-} i(t) dt + \dots + \frac{1}{C_n} \int_{-\infty}^{0^-} i(t) dt \right] + \left[\sum_{k=1}^n \frac{1}{C_k} \right] \int_{0^-}^t i(t) dt \\ &= \sum_{k=1}^n V_k(0^-) + \left[\sum_{k=1}^n \frac{1}{C_k} \right] \int_{0^-}^t i(t) dt \\ &= V_{eq}(0^-) + \left[\frac{1}{C_{eq}} \right] \int_{0^-}^t i(t) dt \end{aligned} \quad (3.6-1)$$

$$\text{where } \frac{1}{C_{eq}} = \sum_{k=1}^n \frac{1}{C_k} \text{ and } V_{eq}(0^-) = \sum_{k=1}^n V_k(0^-)$$

Hence, we can replace the series connection by a single capacitor of value C_{eq} with an initial voltage of $V_{eq}(0^-)$ defined as in Eqn. 3.6-1. When all the capacitors are initially relaxed, the effective capacitor will also be initially relaxed. Otherwise, the algebraic sum of initial voltages across the capacitors will give the initial voltage across the effective capacitor.

In summary, when n capacitors with *zero initial condition* are connected in series, one capacitor with its value as defined in Eqn. 3.6-1 can be used to replace the series combination as far as $v-i$ relationship, stored charge and stored energy are concerned. All capacitors as well as effective capacitor will have equal charge at all t . Individual capacitor voltage will be as per the ratio of reciprocal of capacitance values. Total stored energy will be distributed in various capacitors as per the ratio of reciprocal of capacitance values.

3.6.2 Series Connection of Capacitors with Non-zero Initial Energy

We identify two cases in this situation. In the first case we consider a situation in which all the capacitors had *same initial charge in polarity and magnitude* at the instant they were put in series.

If they had same initial charge to begin with, they will continue to have equal charge even after they have been connected in series. This is so because current is a common variable in a series connection. Therefore, the individual capacitor voltages will be in inverse proportion to capacitance values initially and subsequently. Therefore, the series connection can be replaced by an effective capacitor of values as per Eqn. 3.6-1 and with an initial voltage as per Eqn. 3.6-1. The equivalent capacitor will be equivalent in $v-i$ relationship, in charge and in *total* stored energy. Total voltage and *total* stored energy will get distributed in individual capacitors in inverse proportion to capacitance values.

In the second case, we consider a set of capacitors with arbitrary initial charges. If they had unequal initial charges they will continue to have unequal charges subsequently too. Only *change* in charge for various capacitors will be equal in this case.

Therefore, in this case the effective capacitor will describe only the *change* in capacitor voltages correctly. The equivalent capacitor is equivalent with respect to $\Delta v(t)-i(t)$ relationship, *change* in charge and *change* in total stored energy. *Change* in charge will be equal in all capacitors. *Change* in total voltage is distributed as *changes* in individual capacitor voltages in inverse proportion to capacitance values. *Change* in total stored energy calculated from equivalent circuit will be correct. This *change* in total stored energy is equal to sum of *changes* in stored energy of individual capacitors. But the distribution *is not* in inverse proportion to capacitance values.

The initial voltage across the equivalent capacitor will be given by the algebraic sum of initial voltages of individual capacitors in this case too. But the initial stored energy that is calculated from equivalent capacitor will be less than the actual initial stored energy in the system. The difference will represent the portion of initial energy that gets trapped and hidden in the system. This portion is trapped forever in the series connection and can not be taken out by other elements.

A simple example will show that there can be trapped energy in a series connection. Consider two capacitors with initial voltages that are equal in magnitude and opposite in polarity. The initial voltage observed at terminals of series combination will be zero when we connect them in series. We can not take out the initial energy though we know that it is there. In this case the entire initial energy becomes trapped energy.

The amount of trapped energy in a series connection is given by Trapped Energy

$$= \sum_{k=1}^n \frac{1}{2} C_k [V_k(0^-)]^2 - \frac{1}{2} C_{eq} \left[\sum_{k=1}^n V_k(0^-) \right]^2 \text{ J.}$$

A single capacitor C_{eq} can replace a set of n capacitors connected in series as far as changes in charge, changes in voltage and changes in total stored energy are concerned.

$$\frac{1}{C_{eq}} = \left[\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n} \right]$$

However, the total charge in each capacitor and total charge in C_{eq} will not be the same unless all capacitors had the same charge at $t = 0^-$.

Similarly, the total stored energy in the system will not be the same as the total stored energy in C_{eq} unless all capacitors had same charge at $t = 0^-$.

If the capacitors started out with different charges at $t = 0^-$, a portion of total initial energy is trapped in the circuit and gets hidden from rest of the circuit forever.

Example: 3.6-1

Charging a large number of capacitors in parallel from a low voltage DC source and later reconnecting them in series after removing the source is a technique that is employed in High Voltage Engineering for producing high DC voltages. Solid state switches are used for forming parallel and series combinations of capacitors for this purpose. Usually capacitors with same capacitance value are employed. Assume that the capacitors used in such a system have a tolerance band of $\pm 20\%$ in their capacitance value. Consider such a system employing 8 capacitors of $2\mu\text{F}$ each. The charging voltage is 400 V. Assume that four of them had capacitance value at lower end of tolerance band and the remaining four had capacitance value at higher end of the band. (i) What is the initial energy content in the series combination? (ii) If the high voltage obtained from the series combination is used to deliver a pulse of energy to a test specimen, what is the maximum energy that is available for this purpose?

Solution

Let the nominal value of capacitors be represented by C . Then four capacitors are of $0.8C$ farads and remaining four are of $1.2C$ farads. The effective capacitor is then $0.8C/4$ and $1.2C/4$ in series $= 0.12C$ farads. The initial voltage across the effective capacitor $= 8 \times 400\text{V} = 3200\text{ V}$.

- (i) Initial energy of the series combination $= 0.5 C (4 \times 0.8 \times 400^2 + 4 \times 1.2 \times 400^2) = 640000C\text{ J}$.
Substituting $C = 2\mu\text{F}$, initial energy $= 1.28\text{J}$
- (ii) Initial energy calculated from equivalent circuit $= 0.5 C \times 0.12 \times 3200^2 = 614400C\text{ J}$. Substituting for C , $= 1.229\text{ J}$. Therefore the maximum energy available for specimen testing is 1.229 J , i.e., about 96% of total stored energy. The remaining 4% is trapped in the series combination and will not be available to external elements.

3.7 PARALLEL CONNECTION OF CAPACITORS

A single equivalent capacitor can replace many capacitors connected in parallel for specific analysis purposes. We look into parallel equivalent and constraints on it in this section.

We consider parallel connection of n capacitors *that have no mutual electrostatic coupling* among them. Let the capacitance values be $C_1, C_2 \dots C_n$. We assume that they have the same initial voltage with the same polarity at $t = 0^-$. Let the applied current be $i(t)$ and the voltage across the parallel combination be $v(t)$. See Fig. 3.7-1.

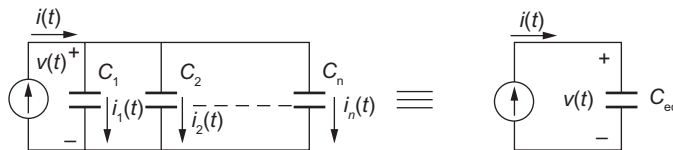


Fig. 3.7-1 Parallel connection of ' n ' capacitors

Applying KCL along with the element equation of capacitor, we get

$$\begin{aligned}
 i(t) &= i_1(t) + i_2(t) + \cdots + i_n(t) \\
 &= C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + \cdots + C_n \frac{dv(t)}{dt} \\
 &= (C_1 + C_2 + \cdots + C_n) \frac{dv(t)}{dt} \\
 &= C_{eq} \frac{dv(t)}{dt} \text{ where } C_{eq} = C_1 + C_2 + \cdots + C_n
 \end{aligned}$$

Thus, a parallel connection of n capacitors may be replaced by an equivalent capacitor with a capacitance value equal to sum of the capacitance values of n capacitors as far as the v - i relationship is concerned. The total applied current into the combination is shared by the various capacitors in direct proportion to capacitance value. *i.e.*,

$$\begin{aligned}
 i(t) &= i_1(t) + i_2(t) + \cdots + i_n(t) \\
 i_1(t) : i_2(t) : \cdots : i_n(t) &= C_1 : C_2 : \cdots : C_n \\
 i_j(t) &= \frac{C_j}{C_{eq}} i(t) \text{ for } j=1 \cdots n
 \end{aligned}$$

It can be seen that sum of charges in the individual capacitors is same as the charge of the equivalent capacitor. Thus, charge is shared in proportion to the capacitance value. Similarly, the total energy stored in all capacitors put together is the same as the energy storage calculated using equivalent capacitance value.

Thus, parallel equivalent of n capacitors is 'equivalent' with respect to v - i relation, charge and stored energy. But keep in mind that the participating capacitors should not have mutual electrostatic coupling. Moreover, that they must have the same initial voltage.

If they have unequal initial voltage, high frequency oscillating currents and voltages appear in the local loops formed when they are put in parallel due to the inductive elements that are always present in any circuit. We need better model for a physical capacitor to handle this kind of problems analytically.

Example: 3.7-1

$C_1 = 6 \mu\text{F}$, $C_2 = 3 \mu\text{F}$ and $C_3 = 3 \mu\text{F}$ in the circuit in Fig. 3.7-2. C_1 has 5V across it at $t = 0^-$. C_2 and C_3 have 3V across them at $t = 0^-$. The polarity of initial voltages is same as that of voltage variables identified in the circuit. The current source function $i_1(t)$ is zero after $3 \mu\text{s}$. (i) Find the voltage across all the capacitors at $5 \mu\text{s}$. (ii) Find the stored energy in the individual capacitors and total stored energy at $4 \mu\text{s}$. (iii) Is there any energy trapped in the system? If yes, how much?

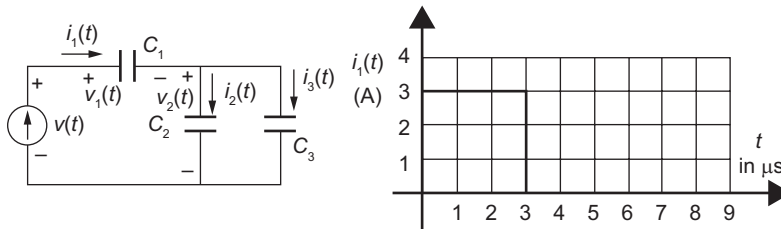


Fig. 3.7-2 Circuit and waveform for Example 3.7-1

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Solution

- (i) The effective value of C_2 and C_3 in parallel is $6\mu\text{F}$ and this $6\mu\text{F}$ is in series with C_1 which is also of $6\mu\text{F}$. Therefore, the effective capacitance of the entire circuit is $3\mu\text{F}$. The initial voltage across the equivalent capacitor is $5\text{V} + 3\text{V} = 8\text{V}$. The area under input current is $9\mu\text{C}$ for all $t \geq 3\mu\text{s}$. Therefore, the net A-s dumped into the equivalent capacitor at $5\mu\text{s}$ is $9\mu\text{C}$. This produces a change in voltage by $9\mu\text{C}/3\mu\text{F} = 3\text{V}$.

Change in voltage in a series combination of capacitors is shared by the capacitors in inverse proportion to capacitance values. Therefore, change in voltage of $C_1 = (1/C_2/C_3)/(1/C_1 + 1/C_2/C_3)$ times $3\text{V} = 1.5\text{V}$. Similarly change in $v_2(t) = 1.5\text{V}$. Therefore, $v_1(t)$ at $5\mu\text{s} = 5\text{V} + 1.5\text{V} = 6.5\text{V}$ and $v_2(t)$ at $5\mu\text{s} = 3\text{V} + 1.5\text{V} = 4.5\text{V}$.

- (ii) The capacitor voltages at $4\mu\text{s}$ are the same as at $5\mu\text{s}$ since the current source value is zero after $3\mu\text{s}$. Therefore, the stored energy in various capacitors can be calculated using the voltage values.

$$\text{Stored energy in } C_1 = 0.5 \times 6 \times 6.5^2 \mu\text{J} = 126.75 \mu\text{J}$$

$$\text{Stored energy in } C_2 = 0.5 \times 3 \times 4.5^2 \mu\text{J} = 30.375 \mu\text{J}$$

$$\text{Stored energy in } C_3 = 0.5 \times 3 \times 4.5^2 \mu\text{J} = 30.375 \mu\text{J}$$

$$\text{Total stored energy} = 187.5 \mu\text{J}$$

- (iii) The initial charge in C_1 was $30 \mu\text{C}$ and in $C_2 // C_3$ was $18 \mu\text{C}$. Thus initial charges in the two capacitors in series combination were not equal. Hence there will be trapped energy in the system.

$$\text{Initial stored energy} = 0.5 \times 6 \times 5^2 + 0.5 \times 3 \times 3^2 + 0.5 \times 3 \times 3^2 = 102 \mu\text{J}.$$

$$\text{Initial energy in the equivalent capacitor} = 0.5 \times 3 \times 8^2 = 96 \mu\text{J}.$$

$$\text{Therefore, trapped energy} = 102 - 96 = 6 \mu\text{J}.$$

3.8 SUMMARY

- A linear resistor obeys Ohm's law and is a memory-less element. The power delivered to a positive resistor is a non-negative function of time. Resistors in series have a common current and share the total voltage and power in proportion to their resistance values. Resistors in parallel have a common voltage and share the total current and power in proportion to their conductance values.
- Inductor is an element that *accumulates* flux linkage (V-s or Wb-T) and makes its response variable, *i.e.*, current, proportional to the accumulated flux linkage. Capacitor is an element that *accumulates* charge (A-s or C) and makes its response variable, *i.e.*, voltage, proportional to the accumulated charge.
- The voltage across an inductor at t is proportional to the rate of change of current through it at t . The current through the inductor is proportional to the area under the voltage waveform, *i.e.*, the V-s product (or Wb-T) applied across its terminals from $t = -\infty$ to t .
- Instantaneous current in an inductor cannot be predicted from instantaneous value of voltage across it. If instantaneous value of voltage is positive the inductor current will be increasing at that instant and if it is negative the current will be decreasing at that instant.
- When voltage across an inductor crosses zero in the downward direction its current attains a local maximum and when it crosses zero in the upward direction the inductor current attains a local minimum. Voltage across an inductor carrying a constant current is zero.
- Current in an inductor cannot change instantaneously unless an impulse voltage is applied or supported in the circuit. The current in an inductor L changes instantaneously by $1/L$ A when the circuit applies or supports a unit impulse voltage across it.
- An inductor with a large inductance value can absorb alternating voltages in a circuit without contributing significant amount of alternating currents to the circuit.

- The total energy delivered to an inductor carrying a current I is $(1/2)LI^2$ Joules and this energy is stored in its magnetic field. The inductor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.
- A single inductor $L_{eq} = L_1 + L_2 + \dots + L_n$ can replace a set of n inductors connected in series. The total applied voltage, total flux linkage and total stored energy are shared by the various inductors in direct proportion to inductance values
- A single inductor L_{eq} can replace a set of n inductors connected in parallel as far as *changes* in flux linkages, *changes* in currents and *changes* in stored energy are concerned.

$$\frac{1}{L_{eq}} = \left[\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right]$$

- The current through a capacitor at any instant is proportional to the rate of change of voltage across it at that instant. The voltage across the capacitor at any instant is proportional to the area under the current waveform, *i.e.*, the A-s product (or C) applied through it from $t = -\infty$ to that instant.
- Instantaneous voltage across a capacitor cannot be predicted from instantaneous value of current through it. If instantaneous value of current is positive the capacitor voltage will be increasing at that instant and if it is negative the voltage will be decreasing at that instant.
- When current through a capacitor crosses zero in the downward direction its voltage attains a local maximum and when it crosses zero in the upward direction the capacitor voltage attains a local minimum. Current through a capacitor with a constant voltage across it is zero.
- Voltage in a capacitor can not change instantaneously unless an impulse current is applied or supported in the circuit. Unit impulse current will deposit 1 C of charge in a capacitor instantaneously. Therefore, the voltage across a capacitor C changes instantaneously by $1/C$ V when the circuit applies or supports a *unit impulse current* through it.
- A large capacitor can absorb alternating currents in a circuit without contributing significant amount of alternating voltages in the circuit.
- The total energy delivered to a capacitor carrying a voltage V across it is $(1/2)CV^2$ J and this energy is stored in its electric field. Stored energy in a capacitor is also given by $(1/2C)Q^2$ J and $QV/2$ J. The capacitor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.
- A single capacitor C_{eq} can replace a set of n capacitors connected in series as far as *changes* in charge, *changes* in voltage and *changes* in total stored energy are concerned.

$$\frac{1}{C_{eq}} = \left[\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n} \right]$$

- A single capacitor $C_{eq} = C_1 + C_2 + \dots + C_n$ can replace a set of n capacitors connected in parallel. The total charge, total current and total stored energy are shared by the various capacitors in direct proportion to capacitance value in a parallel connection of capacitors.

3.9 PROBLEMS

(Assume zero initial condition unless specified otherwise. Passive sign convention is assumed.)

1. Two resistors are connected in series with a DC voltage source of value 15V. One of the resistors is a $10\ \Omega$. The power dissipated in the unknown resistor is 5W and the power delivered by the source is 15W. Find the unknown resistor and circuit current.

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- Three equal resistors are connected across a voltage source in series first and in parallel later. Find the ratio of power delivered by the source in the two cases.
- Two resistors R and $3R$ are connected in series with a DC voltage source of 10V. The energy dissipated in $3R$ during $[2s, 6s]$ interval is 3 J. Find the resistance values.
- Two resistors $2R$ and $3R$ are connected in parallel across a 5A DC current source. The voltage that appears across the current source is 30V. Find R and the power dissipated in each resistor.
- Three resistors – R , $2R$ and $4R$ – are connected across a 3.5A DC current source. The source is seen to deliver 35W power. Find R , currents through the resistors and the power dissipated in each resistor.
- Three resistors – R , $2R$ and $3R$ – are connected in parallel across a voltage source. The current in $3R$ is seen to be $0.6e^{-2t}u(t)$ A. Total energy dissipated in R is 8.1 J. Find the voltage source function and the current delivered by the voltage source.
- A voltage source is first connected across A–B in the circuit in Fig. 3.9-1. Later it is moved and connected across C–D. Find the ratio of power delivered by the voltage source in the two cases.

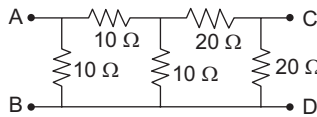


Fig. 3.9-1

- Find i_x in Fig. 3.9-2.

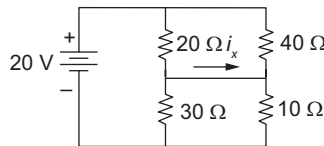


Fig. 3.9-2

- The power dissipated in the $4\ \Omega$ resistor is 1W in the circuit in Fig. 3.9-3. Find the power dissipated in the $3\ \Omega$ resistor, power delivered by the DC source and the value of source voltage.

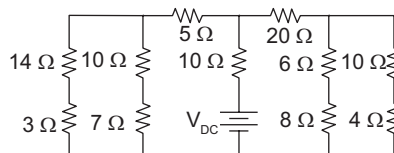
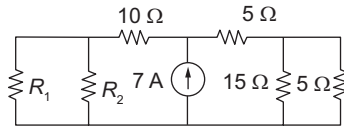
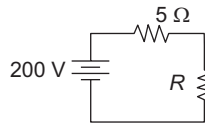


Fig. 3.9-3

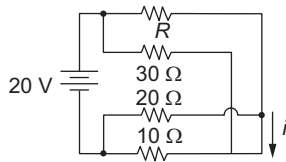
- A voltage of 5V is to be produced from a DC voltage source of 24V by a potential divider arrangement. The 5V output will be loaded by a resistance in the range $1\text{ k}\Omega$ – $10\text{ k}\Omega$. The output voltage should not vary by more than 2% when the load varies in this range. Design the potential divider such that the no-load power dissipation in it is at minimum possible value.
- The power dissipated in $15\ \Omega$ resistor is 15W and power dissipated in R_2 is 5W in the circuit in Fig. 3.9-4. (i) Find R_1 and R_2 . (ii) Solve the circuit completely and mark voltage, current and power dissipated for all elements.

**Fig. 3.9-4**

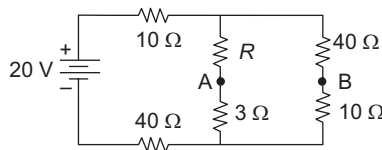
12. Find the value of R in the circuit in Fig. 3.9-5 such that R dissipates 200W of power. What is the additional resistor to be connected in parallel to R such that total power dissipated by R and the additional resistor will be 400W?

**Fig. 3.9-5**

13. What must be the value of R such that i_x is zero in the circuit in Fig. 3.9-6 ?

**Fig. 3.9-6**

14. What must be the value of R in Fig. 3.9-7 such that shorting A to B will not affect the current through R ?

**Fig. 3.9-7**

15. The voltage across a 0.1H inductor is seen to be a constant at 10V between 10 ms and 15 ms. The current through the inductor was 0.3A at 12 ms. What is the current at 13.5 ms?
16. The area under voltage waveform applied to a 10 mH inductor is 5 mV-s between 7 ms and 9 ms. If the current at 7 ms was 1A how much is it at 9 ms?
17. An inductor of 0.2 H has an initial current of 2A at $t = 0^-$. The voltage applied across it is $3\delta(t-2)$. Find the current in it (a) at 1 s (b) at 3 s.
18. An inductor of 2 H undergoes a flux linkage change of 7 Wb-T between 15 s and 17 s. What is the average voltage applied to the inductor during that interval?
19. Two identical inductors L_1 and L_2 undergo a flux linkage change by 10 Wb-T. L_1 takes 2 s for this change and L_2 takes 20 s. What is the ratio of average voltage applied to the inductors during the relevant intervals?
20. A 10H has an initial energy equivalent to the energy consumed by a 40W lamp in 1 h. Find the initial current in the inductor.

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21. A DC voltage source of 24V is switched on to an initially relaxed inductor of 4H through a 48A fuse. Assume that the fuse acts instantaneously when current through it touches 48A. How much time do we have to open the switch before the fuse blows?
22. A DC source of 12V is switched on to an inductor of 0.5H at $t = 0$. The current in it is found to be 0A at 5 s. Was there any initial stored energy in the inductor? If yes, how much?
23. A symmetric triangular voltage waveform with a peak-to-peak value of 20V and frequency 1 kHz is applied to an inductor from 0 s onwards. The inductor was carrying an initial current of 10A. The inductor current is found to vary within $\pm 3\%$ of its initial current subsequently. What is the value of inductance?
24. Two inductors with zero initial energy were paralleled at $t = 0$ and a voltage source was applied across them. The rate of change of source current at 2 s is 5A/s and the source voltage at that time was 2.5V. It was also found that the first inductor had a stored energy that is twice that of the second inductor. Find the inductance values.
25. The *unit step function* was described in Eqn. 3.2-4. The function $u(t-t_0)$ is termed as a *delayed unit step function*.

$$u(t-t_0) = \begin{cases} 0 & \text{for } (t-t_0) \leq 0^- \\ \text{undefined} & \text{for } 1 \text{ for } (t-t_0) = 0 \\ 1 & \text{for } (t-t_0) \geq 0^+ \end{cases}$$

Therefore, the jump from 0 to 1 takes place at t_0 , i.e., after a delay of t_0 s. What is the applied voltage function across an inductor of 0.5H if its current is $2u(t-2)$?

26. Another standard test signal frequently used in Electrical and Electronics Engineering is the *unit ramp function* defined as

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

Show that unit ramp is the integral of unit step function. (ii) Find the voltage across a 0.25H inductor when its current is $(5 + 25t)$ for $t \geq 0^+$ and 0 A for $t \leq 0^-$.

27. The voltage applied across an inductor of 0.3H is a rectangular pulse of height 10V and duration 30 ms. The pulse starts at $t = 20$ ms. (i) Express the voltage waveform in terms of scaled and delayed unit step functions. (ii) Obtain an expression for the current in the inductor as a function of time for $t \geq 10$ ms if the initial current at 10 ms is 1A. (iii) Plot the current in inductor, power delivered to the inductor and energy storage in inductor as functions of time.
28. The value of $i_L(t)$ is found to be 10A at 18 s in the circuit in Fig. 3.9-8. Find the ratio of initial energy storage in the inductor to stored energy in it at 17 s.

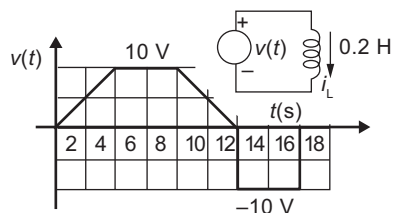


Fig. 3.9-8

29. A periodic ramp voltage waveform applied across the 2 mH inductor in Fig. 3.9-9 from $t = 0$. (i) What must be the initial current and initial energy storage in the inductor such that there is no DC component in the inductor after $t = 0$? (ii) Calculate and plot the inductor current and stored energy for one period with initial current at the value calculated above. (iii) Find the average power delivered by the source, the average being taken over a cycle.

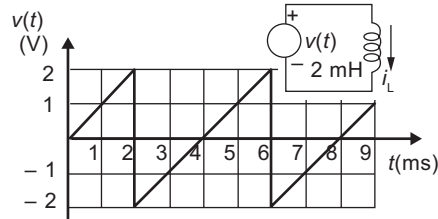


Fig. 3.9-9

30. A sinusoidal voltage $v_s(t) = 10 \sin(400t + \pi/3)$ V is applied to a 25 mH inductor from $t = 0$. (i) Plot the current, power and stored energy in the inductor as functions of time for one period of input voltage. (ii) What is the DC content in the inductor current? (iii) What is the initial current to be specified at $t = 0^-$ such that the DC content in the inductor current will be zero? (iv) What is the frequency of power variation with this value of initial current?
31. A voltage source $v_s(t) = 5e^{0.5t} [u(t) - u(t-2)]$ V is connected across an inductor of 0.5H. The initial current specified at $t = 0^-$ is 2A. (i) Plot the applied voltage and inductor current for $t = 0$ to 3 s. (ii) What is the value of current and stored energy in the inductor at $t = 4$ s?
32. An arbitrary voltage is applied across a 0.4H inductor from $t = 0$. The current in inductor was observed to be 3A at 1.5 s. The stored energy in it was found to quadruple in the next 0.2 s. Explain why the applied voltage could not have been less than 6V in the entire interval [1.5s, 1.7s].
33. An inductor used as a smoothing inductor in a DC power supply is expected to carry a DC current of 10A. At the same time a periodic voltage with a waveform shown in Fig. 3.9-10 will get applied across it. Find the minimum value of inductance such that the current in inductor will not get perturbed by more than $\pm 2\%$ of its DC value. Assume that the value of DC current given includes the DC component due to this periodic voltage also.

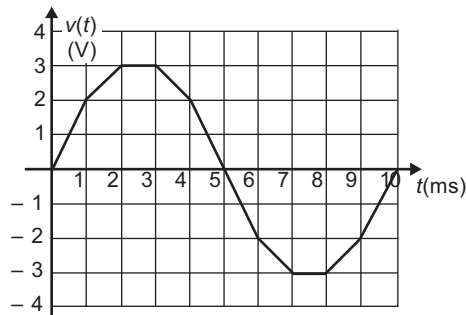


Fig. 3.9-10

34. The switch S in Fig. 3.9-11 starts at position-1 at $t = 0$ with initial current in 1 mH at zero. It remains at position-1 for 1 ms and then goes to position-2 and remains there for 0.1 ms. After that

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it is brought back to position-1 and the entire cycle is repeated. (i) Find the value of V_{dc} such that the inductor current reaches zero just prior to the switch going back to position-1. (ii) With this value of V_{dc} , calculate and plot v_L , i_L , i_S and i_{dc} for two switching cycles. (iii) Calculate the average values of v_L , i_L , i_S , i_{dc} and power flow from 10 V source to the second source. Averaging is to be done over a switching cycle. (iv) Assume that the first switching cycle was 1.5 ms/0.1 ms and all the subsequent cycles were 1ms/0.1 ms. Repeat step (iii) for a cycle after the first one. (v) Suggest a method to control the power flow to the second source.

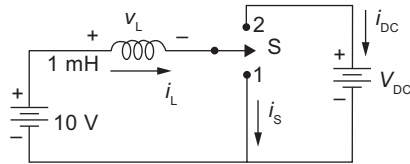


Fig. 3.9-11

35. The switch S in Fig. 3.9-12 starts at position-1 at $t = 0$ with initial current in 1mH at zero. It remains at position-1 for 1 ms and then goes to position-2 and remains there for 0.02 ms. After that it is brought back to position-1 and the entire cycle is repeated. (i) Find the value of V_{dc} such that the inductor current reaches zero just prior to the switch going back to position-1. (ii) With this value of V_{dc} , calculate and plot v_L , i_L , i_{s1} and i_{s2} for two switching cycles. (iii) Calculate the average values of v_L , i_L , i_{s1} , i_{s2} and power flow from 10 V source to the second source. Averaging is to be done over a switching cycle.

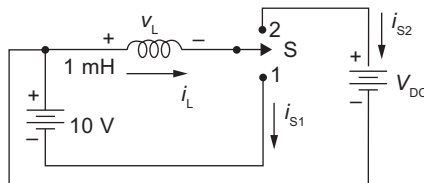


Fig. 3.9-12

36. All the inductors in the circuit in Fig. 3.9-13 had zero initial energy. The applied source voltage is $v_s(t) = (10 \sin 2000\pi t)u(t)$ V. Find and plot the waveform of current drawn from the source, current in L_4 and voltage across L_4 .

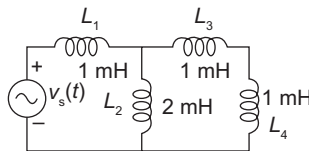


Fig. 3.9-13

37. Three inductors of 0.1H, 0.05H and 0.15H with equal initial current of 3A are connected in series at $t = 0$. $v(t)$ from $t = 0^-$ is $0.5\delta(t-2) + v_s(t)$ where $v_s(t)$ is not known. The total flux linkage in the circuit is found to double in the first 3 s. and $v(t)$ at $t = 3$ s is found to be 12V. (i) Find the voltage across each inductor and flux linkages in them at $t = 3$ s (ii) The average value of $v_s(t)$ in the first

- 3 s (iii) Stored energy in the circuit and each inductor at $t = 3$ s (iv) Total energy delivered by the applied voltage source in 3 s and average power delivered by the source over 3 s?
38. How much time is required to charge a 10mF capacitor with an initial voltage of -100V to $+100\text{V}$ using a DC current source of value 10mA ?
39. The voltage rating of a $10\mu\text{F}$ capacitor is 100V . It is being charged by a $100\mu\text{A}$ pulse current source. Its initial voltage was -75V . What is the maximum pulse width the current source can have if we do not want to end up with a blown capacitor?
40. A 5mF capacitor undergoes a change in its voltage by 25V in 10ms . What is the average value of the current source used to charge this capacitor during that interval?
41. A symmetric triangular current waveform with a peak-to-peak value of 20mA and frequency 10kHz is applied to a capacitor from 0 s onwards. The capacitor was carrying an initial voltage of 10V . The capacitor voltage is found to vary within $\pm 5\%$ of its initial voltage subsequently. What is the value of capacitance?
42. A DC current source of 12mA is switched on to a capacitor of 0.5mF at $t = 0$. The voltage in it is found to be 0V at 5 s. Was there any initial stored energy in the capacitor? If yes, how much?
43. A sinusoidal current source $2 \sin 200t$ A is applied to three capacitors -0.1mF , 0.2mF and 0.05mF in series. What is the peak-to-peak voltage developed across the combination? Which capacitor has highest peak-to-peak voltage across it?
44. Three capacitors $-10\mu\text{F}$, $22\mu\text{F}$ and $33\mu\text{F}$ are in parallel. The circuit is driven by a AC current source $10 \cos 300t$ A. What is the peak-to-peak voltage developed across the combination?
45. What is the applied current into a capacitor of 0.02F if its voltage is $5u(t-3)$?
46. What must be the charging current function if the voltage across an initially uncharged $10\mu\text{F}$ capacitor is to vary as $(5 + 2t)u(t)$ V?
47. The current applied into a capacitor of 10mF is a rectangular pulse of height 10A and duration 25ms . The pulse starts at $t = 10\text{ms}$. (i) Express the current waveform in terms of scaled and delayed unit step functions. (ii) Obtain voltage across the capacitor as a function of time for $t \geq 10\text{ms}$ if the initial voltage at 10ms is -10V . (iii) Plot the voltage across capacitor, power delivered to the capacitor and energy storage in it as functions of time.
48. The value of $v_C(t)$ is found to be 10V at 18ms in the circuit in Fig. 3.9-14. Find the ratio of initial energy storage in the capacitor to stored energy in it at 17ms .

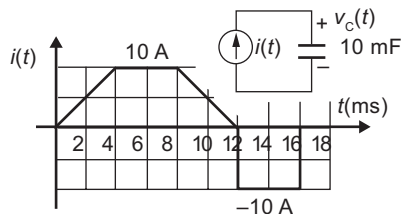


Fig. 3.9-14

49. The periodic ramp current waveform in Fig. 3.9-15 is applied to a 100pF capacitor from $t = 0$. (i) What must be the initial voltage and initial energy storage in the capacitor such that there is no DC component in the capacitor voltage after $t = 0$? (ii) Calculate and plot the capacitor voltage and stored energy for one period with initial voltage at the value calculated above. (iii) Find the average power delivered by the source, the average being taken over a cycle.

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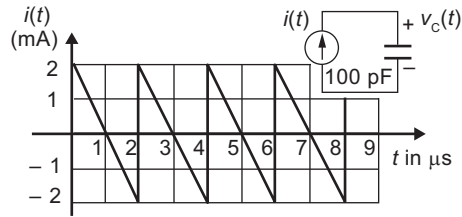


Fig. 3.9-15

50. A sinusoidal current $i_s(t) = 2 \sin(400t + \pi/3)$ A is applied to a $20 \mu\text{F}$ capacitor from $t = 0$. (i) Plot the voltage, power and stored energy in the capacitor as functions of time for one period of input current. (ii) What is the DC content in the capacitor voltage? (iii) What is the initial voltage to be specified at $t = 0^-$ such that the DC content in the capacitor voltage will be zero? (iv) What is the frequency of power variation with this value of initial voltage?
51. A current source $i_s(t) = 5e^{1.5t} [u(t) - u(t-1)]$ mA is connected across capacitor of $470 \mu\text{F}$. The initial voltage specified at $t = 0^-$ is 2 V. (i) Plot the applied current and capacitor voltage for $t = 0$ to 2 s. (ii) What is the value of voltage and stored energy in the capacitor at $t = 4$ s?
52. An arbitrary current is applied to a 0.1 F capacitor from $t = 0$. The voltage across capacitor was observed to be 10 V at 2 s. The stored energy in it was found to quadruple in the next 0.5 s. Explain why the applied current could not have been less than 2 A in the entire interval $[2 \text{ s}, 2.5 \text{ s}]$.
53. The circuit in Fig. 3.9-16 shows an idealised model for the output of a DC power supply. The output is taken across the bulk capacitor C . Load on the supply is modelled as a current source of value i_L . i_L has a DC component and pure AC component. Its AC component is shown in the same figure. The voltage across the capacitor is found to have a DC component of 12 V and rest of the power supply circuit is in effect delivering a constant current of 5 A to the capacitor node. (i) What must be the value of DC component in load current i_L ? (ii) What must be the minimum value of C such that the peak-to-peak ripple voltage at output will be less than 2% of its DC value?

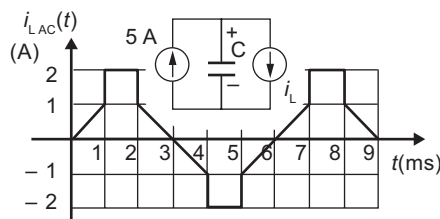


Fig. 3.9-16

54. $C_1 = 20 \mu\text{F}$, $C_2 = 10 \mu\text{F}$, $C_3 = C_4 = 20 \mu\text{F}$ in Fig. 3.9-17. They had zero initial energy at $t = 0^-$. The applied source is $(200 \sin 300t)u(t)$ V. Find and plot (i) the current drawn from the source (ii) the voltage across C_4 and current through it.

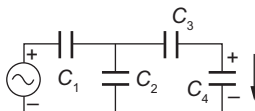


Fig. 3.9-17

55. The source in the circuit in Fig. 3.9-18 is $(2 \cos 300t)u(t)$ A. Find and plot (i) the voltage across the source (ii) the voltage across C_4 and current through it. $C_1 = C_3 = C_4 = 10 \mu\text{F}$, $C_2 = 5 \mu\text{F}$.

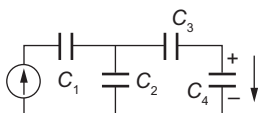


Fig. 3.9-18

56. All the three capacitors in Fig. 3.9-19 had equal initial voltages at $t = 0^-$. The source current is zero for $t > 8$ ms. The voltage across the $10 \mu\text{F}$ capacitor is observed to be 4V at $t = 2$ ms. (i) What was the initial voltage across the capacitors and what were the initial stored energy in them? (ii) Calculate and plot the voltage across the capacitors and the voltage across the current source as functions of time for 0 to 9 ms range. (iii) Calculate the total energy delivered by the current source, total energy dissipated in the resistor and change in stored energy of all the capacitors.

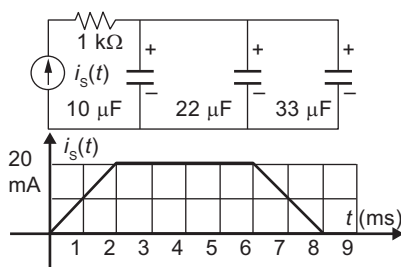


Fig. 3.9-19

57. Three capacitors – 0.02mF , 0.05mF and 0.0333mF – with same initial voltage of 10V are connected in series. The applied current source is $i_s(t)$ mA where $i_s(t)$ is not known. The charge in the 0.05mF capacitor is found to be 0.3 mC at $t = 3 \text{ s}$. (i) Find the trapped energy in the series combination (ii) Find the voltage across each capacitor and charges in them at $t = 3 \text{ s}$. (iii) The average value of $i_s(t)$ in the first 3 s. (iv) Stored energy in the circuit and in each capacitor at $t = 3 \text{ s}$. (v) Total energy delivered by the applied current source in 3 s and average power delivered by the source over 3 s?

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Chapter 4

Nodal Analysis and Mesh Analysis of Memoryless Circuits

CHAPTER OBJECTIVES

- To introduce the circuit analysis problem and explain the constraints that exist in various equation sets.
- To define node voltage variable and develop nodal analysis technique for memoryless circuits containing resistors, dependent voltage and current sources and independent voltage and current sources.
- To introduce Nodal Conductance Matrix $\mathbf{Y}$ and its properties.
- To illustrate nodal analysis technique through a series of solved examples.
- To define mesh current variable and develop mesh analysis technique for memoryless circuits containing resistors, dependent voltage and current sources and independent voltage and current sources.
- To introduce Mesh Resistance Matrix $\mathbf{Z}$ and its properties.
- To illustrate mesh analysis technique through a series of solved examples.
- To show that any voltage variable or current variable in a memoryless circuit can be expressed as a linear combination of independent voltage and current source functions.

INTRODUCTION

A memoryless circuit contains a set of memoryless elements interconnected and driven by independent voltage sources and/or current sources. We deal with two systematic procedures for analysing such circuits in this chapter. These analysis procedures are called *nodal analysis* and *mesh analysis*. Nodal analysis procedure is applicable to any circuit. However, mesh analysis procedure is applicable to only a subclass of circuits called *planar networks*.

4.2 Nodal Analysis and Mesh Analysis of Memoryless Circuits

These two analysis procedures are developed and illustrated in the context of memoryless circuits in this chapter. However, they are very well applicable to dynamic circuits as well. We will extend these procedures to dynamic circuits in later chapters.

A linear time-invariant memoryless circuit can contain linear resistors (*i.e.*, resistors that obey Ohm's law) and linear dependent sources. Linear dependent sources can be of four kinds. A *Voltage-Controlled Voltage Source* (VCVS) outputs a voltage equal to $k_v v_x$ V where k_v is a dimensionless constant and v_x is the voltage across *some other element* or *combination of elements* in the circuit. A *Current-Controlled Voltage Source* (CCVS) outputs voltage equal to $k_z i_x$ V where k_z is a *transfer-resistance* and i_x is the current through *some other element* in the circuit. A *Current-Controlled Current Source* (CCCS) produces a current equal to $k_i i_x$ A where k_i is a dimensionless constant and i_x is the current through *some other element* in the circuit. A *Voltage-Controlled Current Source* (VCCS) produces a current equal to $k_y v_x$ A where k_y is a *transfer-conductance* and v_x is the voltage across *some other element* in the circuit. In addition to linear resistors and linear dependent sources, the circuit will also contain independent voltage sources and independent current sources. Independent voltage sources and current sources are *non-linear elements*.

4.1 THE CIRCUIT ANALYSIS PROBLEM

Let such a memoryless circuit contain b elements interconnected to form n nodes and l loops. Each two-terminal element has two variables associated with it – a voltage and a current variable. Thus, there are $2b$ variables to be solved for in the circuit analysis problem. Each element is also associated with a terminal $v - i$ relationship that yields an equation called the *element equation*. Ohm's law gives the element equation of a resistor. The element equation of a linear dependent source is given in the form indicated in the last paragraph.

Thus, we have b element equations available and we need b more equations to solve for $2b$ variables.

These b equations are obtained by applying KVL and KCL constraints in the circuit. We get a KCL equation at each node and hence there are n KCL constraint equations available. However, we had noted earlier that only $n - 1$ of them will be independent. Thus, we need another set of $b - n + 1$ independent equations to solve the circuit. These are provided by the KVL constraints. There are l KVL constraint equations available where l is the number of loops in the circuit. The number of loops in the circuit will invariably be greater than $b - n + 1$. Not all of the l KVL equations are useful. Some of them can be obtained from others by forming linear combinations. However, it will be possible to find an independent set of $b - n + 1$ equations from l equations. Moreover, the maximum number of equations that can be there in an independent set of equations drawn from these l loop KVL equations will be exactly $b - n + 1$. However, the choice of equations is not unique. That is, it will be possible to find many sets of independent equations, each set containing $b - n + 1$ equations from the set of l KVL equations. Consider the circuit shown in Fig. 4.1-1.

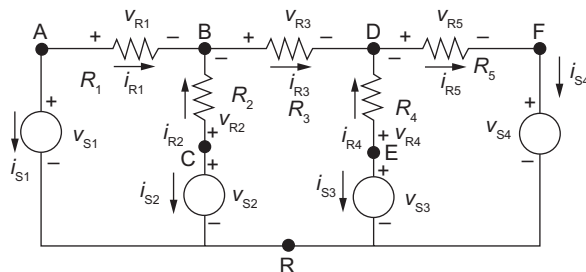


Fig. 4.1-1 A memoryless circuit with all element variables identified

This circuit has 9 elements, 7 nodes and 6 loops. The element variables are marked in the circuit diagram. The direction of current in each resistor is arbitrary. However, once the current direction is chosen, the voltage polarity will follow from the passive sign convention that is being followed. In the case of voltage sources, the polarity of voltage variable is usually selected to coincide with the stated polarity of the source and the current direction is set as per the passive sign convention.

The KCL equations at all the seven nodes are listed below. We follow a certain sign convention in writing the KCL equation at a node. We write the KCL equation as *sum of currents leaving the chosen node is equal to zero*.

$$\left. \begin{array}{l} \text{Node A } i_{S1} + i_{R1} = 0 \\ \text{Node B } -i_{R1} - i_{R2} + i_{R3} = 0 \\ \text{Node C } i_{S2} + i_{R2} = 0 \\ \text{Node D } -i_{R3} - i_{R4} + i_{R5} = 0 \\ \text{Node E } i_{S3} + i_{R4} = 0 \\ \text{Node F } i_{S4} - i_{R5} = 0 \\ \text{Node R } -i_{S1} - i_{S2} - i_{S3} - i_{S4} = 0 \end{array} \right\} \text{ for all time } t$$

We will call the KCL equation at a node as the *node equation* at that node from this point onwards. Obviously, this set of seven node equations do not form an independent set since the KCL equation at the Node R can be obtained by adding the KCL equations at all the other nodes. Any set of node equations containing six equations will be an independent set of equations in this circuit.

We usually assign one of the nodes in the circuit as a *reference node* in the sense that voltages of various other nodes will be defined and measured with respect to this node. Any node can be set as reference node in theory. However, in practice, the choice will be obvious since there will be such a node which forms a common point of reference for applying inputs and measuring outputs. If such a choice is not obvious, the practical convention is to set that node which has maximum elements connected to it as the reference node. *The KCL equation for reference node is dropped and the KCL equations at the remaining nodes are chosen as a set of $(n - 1)$ independent equations in circuit analysis.*

Now, we shift our attention to the KVL equations written for six loops. Here too, we follow a certain convention in writing the KVL equations. We start at the leftmost corner of a loop and traverse it in the clockwise direction until we get back to the starting point. As we go along, we enter the element voltages in the equation with the polarity that we see first. That is, if we meet an element voltage at its positive polarity first, we enter that voltage variable with positive sign, and, if we meet an element voltage at its negative polarity first, we enter that variable with negative sign.

$$\left. \begin{array}{l} \text{Loop RABCR } -v_{S1} + v_{R1} - v_{R2} + v_{S2} = 0 \\ \text{Loop RCBDE } -v_{S2} + v_{R2} + v_{R3} - v_{R4} + v_{S3} = 0 \\ \text{Loop REDFR } -v_{S3} + v_{R4} + v_{R5} + v_{S4} = 0 \\ \text{Loop RABDE } -v_{S1} + v_{R1} + v_{R3} - v_{R4} + v_{S3} = 0 \\ \text{Loop RCBDFR } -v_{S2} + v_{R2} + v_{R3} + v_{R5} + v_{S4} = 0 \\ \text{Loop RABDFR } -v_{S1} + v_{R1} + v_{R3} + v_{R5} + v_{S4} = 0 \end{array} \right\} \text{ for all time } t$$

4.4 | Nodal Analysis and Mesh Analysis of Memoryless Circuits

Obviously, these six equations do not form an independent set. For example, the first three will add up to the last one. The first two will add up to the fourth one. Second and third will add up to the fifth. Thus, the last three are not independent equations. The first three are independent since each contains at least one voltage variable that does not figure in the other two. Hence, we may accept the first three as the independent set of three KVL equations. However, there are other possible choices too. For instance, the first two and the last will form an independent set of three equations.

Thus, we have six independent KCL equations and three independent KVL equations making up nine equations involving 18 variables – 9 current variables and 9 voltage variables. The remaining nine equations come from element equations. The complete set of 18 equations needed to solve for 18 variables are listed below.

$$\begin{aligned}
 i_{S1} + i_{R1} &= 0; -i_{R1} - i_{R2} + i_{R3} = 0; i_{S2} + i_{R2} = 0 \\
 -i_{R3} - i_{R4} + i_{R5} &= 0; i_{S3} + i_{R4} = 0, i_{S4} - i_{R5} = 0 \\
 -v_{S1} + v_{R1} - v_{R2} + v_{S2} &= 0 \\
 -v_{S2} + v_{R2} + v_{R3} - v_{R4} + v_{S3} &= 0 \\
 -v_{S3} + v_{R4} + v_{R5} + v_{S4} &= 0 \\
 v_{R1} = R_1 i_{R1}; v_{R2} = R_2 i_{R2}; v_{R3} = R_3 i_{R3}; v_{R4} = R_4 i_{R4}; v_{R5} = R_5 i_{R5} \\
 v_{S1} = v_1(t); v_{S2} = v_2(t); v_{S3} = v_3(t); v_{S4} = v_4(t)
 \end{aligned}$$

$v_1(t)$, $v_2(t)$, $v_3(t)$ and $v_4(t)$ are the time-functions which describe the voltage delivered by the independent voltage sources.

Thus, we have 18 equations in 18 unknowns. They come in three sets – the first set consists of $(n - 1)$ KCL equations, the second set contains $(b - n + 1)$ KVL equations and the third set contributes b element equations. Can we simplify this problem and reduce the number of variable we have to deal with? This is where the systematic procedures we set out to develop in this chapter come into focus.

Nodal Analysis uses the second and third set of equations (KVL and element equations) to eliminate variables, reduces the number of pertinent variables to $(n - 1)$ node voltage variables and uses the first set of equations (KCL equations) to solve for these variables.

Mesh Analysis uses the first and third set of equations (KCL and element equations) to eliminate variables, reduces the number of pertinent variables to $(b - n + 1)$ mesh current variables and uses the second set of equations (KVL equations) to solve for these variables.

We develop the method of Nodal Analysis first through a series of examples that follow.

4.2 NODAL ANALYSIS OF CIRCUITS CONTAINING RESISTORS AND INDEPENDENT CURRENT SOURCES

The example circuit that we employ to discuss nodal analysis is shown in Fig. 4.2-1. It has 9 elements and 4 nodes. It is driven by three independent current sources I_1 , I_2 and I_3 . All the element voltage and

current variables adhering to passive sign convention have been identified, though, not labelled. The labelling scheme is the same as the one we employed in the previous section.

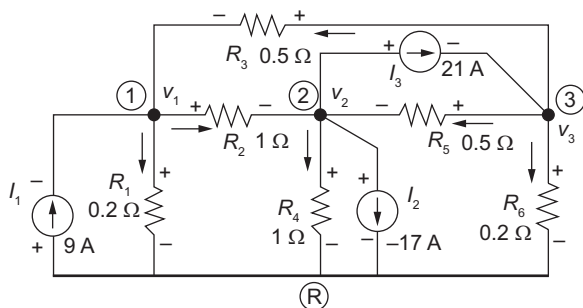


Fig. 4.2-1 Example circuit with resistors and independent current sources for nodal analysis

The node that has largest number of elements connected to it is taken as reference node R and is indicated by a thick line in the diagram.

This circuit has a solution. Certain finite voltages will exist across elements and certain finite currents will flow through them. We can measure voltage across elements by connecting a voltmeter across them. Now, assume that the resistor R_6 is removed. Obviously, the circuit will have a different solution. However, note that with R_6 removed, there is no element connected directly from node-3 to reference node. We can still connect a voltmeter between node-3 and reference node and get a finite reading that indicates the voltage of node-3 with respect to the reference node. This measured voltage is *not* the voltage across any element (we are assuming that R_6 is removed). Thus, we see that each node in the circuit will have a voltage difference with respect to the reference node quite irrespective of whether that voltage can be identified as the voltage across some element or other. These voltages are called *node voltages*. Node voltage is the voltage of a node in a circuit with respect to a chosen reference node in the circuit.

We observe that all the three node voltage variables are identifiable as element voltages in this circuit. However, this need not be the case always.

If the node voltages are known, the element voltages may be calculated as linear combination of node voltages. After all, an element has to be connected between two nodes. If one of them is reference node itself, then the element voltage is equal the node voltage of the other node or negative of that. If both nodes are different from reference node, then the element voltage will be the difference between the node voltages at those two nodes.

Three node voltage variables are identified and labelled as v_1 , v_2 and v_3 in the circuit. Since reference node is the node with respect to which the other node potentials are defined, no node voltage variable needs to be assigned for it. Its node voltage is zero by definition.

Now, apply KVL to get equations relating the element voltages to node voltages. For example, consider the loop formed by R-Node-1-Node-2-R. KVL in this loop gives,

$$\begin{aligned} -v_1 + v_{R2} + v_2 &= 0 \\ \therefore v_{R2} &= v_1 - v_2 \end{aligned}$$

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All element voltages may be related to the three node voltages in this manner by inspection.

$$\begin{aligned}
 v_{R1} &= v_1 \\
 v_{R2} &= v_2 - v_1 \\
 v_{R3} &= v_3 - v_1 \\
 v_{R4} &= v_2 \\
 v_{R5} &= v_3 - v_2 \\
 v_{R6} &= v_3 \\
 v_{I_1} &= -v_1 \\
 v_{I_2} &= v_2 \\
 v_{I_3} &= v_2 - v_3
 \end{aligned} \tag{4.2-1}$$

We observe that all element voltages can be obtained either as some node voltage straightaway or as difference between two node voltages. Hence, a set of $(n - 1)$ node voltages, defined with respect to the reference node, is a sufficient set of voltage variables for determining b element voltages. Note that we are using KVL equations to reduce the number of pertinent voltage variables to $(n - 1)$ from b .

The KCL equations remain. We use them to solve for these node voltages. But KCL equations are written in terms of currents. This is where the element equations come in. We substitute element equations in KCL equations as and when we write KCL equations in order to substitute for element currents in terms of element voltages. Of course, element voltages will be expressed in terms of node voltages only. Thus, writing node equations at the $n - 1$ nodes (because only $n - 1$ KCL equations are independent) involves two mental operations for each element – obtain element voltage in terms of node voltages and replace current variable by voltage variable with the help of element equation. We illustrate this for node-1 in the circuit in Fig. 4.2-1.

We write KCL at this node by equating the sum of currents *going away from the node* to zero.

$$\begin{aligned}
 i_{R1} + i_{R2} - i_{R3} - I_1 &= 0 \\
 \text{i.e.,} \quad \frac{v_{R1}}{R_1} + \frac{v_{R2}}{R_2} - \frac{v_{R3}}{R_3} - I_1 &= 0 \\
 \text{i.e.,} \quad G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_3) &= I_1
 \end{aligned}$$

where G represents the conductance value $(1/R)$ of corresponding resistance. The third equation, which is the final form of node equation, can be written by inspection without writing out the first two explicitly. With our convention of positive sign for current flowing away from the node, the node voltage variable at the node where KCL is being written will appear with positive sign and other node voltage variables will appear with negative sign in the equation. Moreover, the *net current delivered by current sources* to that node will appear with *positive sign on the right side* of equation. The remaining two node equations for this circuit are

$$G_2(v_2 - v_1) + G_5(v_2 - v_3) + G_4v_2 = -I_2 - I_3$$

$$G_3(v_3 - v_1) + G_5(v_3 - v_2) + G_6v_3 = I_2$$

We can solve for v_1 , v_2 and v_3 using the three equations listed below:

$$\begin{aligned} G_1v_1 + G_2(v_1 - v_2) + G_3(v_1 - v_3) &= I_1 \\ G_4v_2 + G_2(v_2 - v_1) + G_5(v_2 - v_3) &= -I_2 - I_3 \\ G_6v_3 + G_3(v_3 - v_1) + G_5(v_3 - v_2) &= I_2 \end{aligned} \quad (4.2-2)$$

The element voltages may be found out subsequently by using Eqn. 4.2-1 and the element currents may be obtained by using element equations in the last step.

We followed a certain convention in writing the node equations in Eqn. 4.2-2. Adhering to such a convention has yielded certain symmetry in these equations. Let us express these equations in matrix notation to see the symmetry clearly.

$$\begin{aligned} (G_1 + G_2 + G_3)v_1 - G_2v_2 - G_3v_3 &= I_1 \\ -G_2v_1 + (G_2 + G_4 + G_3)v_2 - G_5v_3 &= -I_2 - I_3 \\ -G_3v_1 - G_5v_2 + (G_3 + G_5 + G_6)v_3 &= I_2 \end{aligned}$$

$$\text{i.e.,} \quad \begin{bmatrix} (G_1 + G_2 + G_3) & -G_2 & -G_3 \\ -G_2 & (G_2 + G_4 + G_3) & -G_5 \\ -G_3 & -G_5 & (G_3 + G_5 + G_6) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad (4.2-3)$$

$$\text{i.e.,} \quad \mathbf{YV} = \mathbf{CU}$$

$\mathbf{Y}$ is called the *Nodal Conductance Matrix*, $\mathbf{V}$ is called the *Node Voltage Vector*, $\mathbf{U}$ is called the *Input Vector* and $\mathbf{C}$ is called the *Input Matrix*.

Note that the order of Nodal Conductance matrix is $(n - 1) \times (n - 1)$ and that it is *symmetric*. The diagonal element of $\mathbf{Y}$ matrix, y_{ii} , is the *sum of conductances connected at the node-i*. The off-diagonal element of $\mathbf{Y}$ matrix, y_{ij} , is the *negative of sum of all conductances connected between node-i and node-j*. There can be more than one conductance connected between two nodes. Then, they will be in parallel and they will add in y_{ij} . That is why y_{ij} should be the negative of *sum* of all conductances connected between node-i and node-j. The right-hand side product $\mathbf{CU}$ is a column vector of *net current injected by the current sources at the corresponding nodes*.

Now, we can write down this matrix equation by inspection after skipping all the intermediate steps. The following matrix equation results in the case of the example we considered in this section.

$$\begin{aligned} G_1 &= \frac{1}{0.2\Omega} = 5S; \quad G_2 = \frac{1}{1\Omega} = 1S; \quad G_3 = \frac{1}{0.5\Omega} = 2S; \\ G_4 &= \frac{1}{1\Omega} = 1S; \quad G_5 = \frac{1}{0.5\Omega} = 2S; \quad G_6 = \frac{1}{0.2\Omega} = 5S; \end{aligned}$$

$$\begin{bmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -21 - (-17) \\ 21 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 21 \end{bmatrix} \quad (4.2-4)$$

Solving the matrix equation by Cramer's rule, we get,

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$$v_1 = \begin{vmatrix} 9 & -1 & -2 \\ -4 & 4 & -2 \\ 21 & -2 & 9 \end{vmatrix} \div \begin{vmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{vmatrix} = \frac{9(36-4)+1(-36+42)-2(8-84)}{8(36-4)+1(-9-4)-2(2+8)} = \frac{446}{223} = 2 \text{ V}$$

$$v_1 = \begin{vmatrix} 8 & 9 & -2 \\ -1 & -4 & -2 \\ -2 & 21 & 9 \end{vmatrix} \div \begin{vmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{vmatrix} = \frac{8(-36+42)-9(-9-4)-2(-21-8)}{223} = \frac{223}{223} = 1 \text{ V}$$

$$v_1 = \begin{vmatrix} 8 & -1 & 9 \\ -1 & 4 & -4 \\ -2 & -2 & 21 \end{vmatrix} \div \begin{vmatrix} 8 & -1 & -2 \\ -1 & 4 & -2 \\ -2 & -2 & 9 \end{vmatrix} = \frac{8(84-8)+1(-21-8)+9(2+8)}{8(36-4)+1(-9-4)-2(2+8)} = \frac{669}{223} = 3 \text{ V}$$

The element voltages can be calculated by using Eqn. 4.2-1 or by inspection. Similarly, the element currents may be obtained by inspection. The complete solution is marked in the circuits appearing in Fig. 4.2-2.

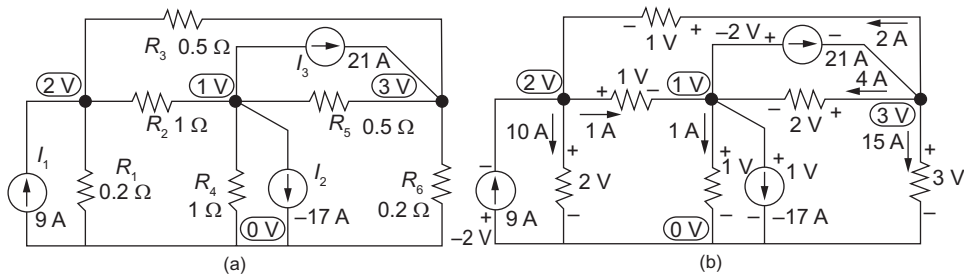


Fig. 4.2-2 Nodal analysis solution for circuit in Fig. 4.2-1

This section has shown that an n -node circuit containing only linear resistors and independent current sources will have a nodal representation given by $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{Y}$ is the nodal conductance matrix of order $(n-1) \times (n-1)$, $\mathbf{V}$ is the node voltage column vector of order $(n-1) \times 1$, $\mathbf{U}$ is the source current column vector of order $n_{cs} \times 1$ and $\mathbf{C}$ is the input matrix of order $(n-1) \times n_{cs}$. n_{cs} is the number of independent current sources in the circuit.

y_{ii} = sum of all conductances connected at i th node

y_{ij} = negative of sum of all conductances connected between i th node and j th node

$$c_{ij} = \begin{cases} 0 & \text{if } j\text{th current source is not connected at } i\text{th node} \\ 1 & \text{if } j\text{ current source is delivering current into } i\text{th node} \\ -1 & \text{if } j\text{th current source is drawing current from } i\text{th node} \end{cases}$$

Equivalently, the matrix product $\mathbf{CU}$ may be replaced by a column vector which contains the *net current delivered to a node by all current sources connected at that node*. The nodal conductance matrix will be symmetric for this kind of circuits. The node voltage vector is obtained by Cramer's rule or by Matrix inversion as $\mathbf{V} = \mathbf{Y}^{-1} \mathbf{CU}$.

Element voltages and currents may be obtained in terms of node voltages by inspection subsequently. For instance, left end of R_2 is at 2 V with respect to reference node and right end is at 1 V. Hence, v_{R_2} is $2 - 1 = 1$ V and its current is $1 \text{ V} / 1 \Omega = 1 \text{ A}$.

4.3 NODAL ANALYSIS OF CIRCUITS CONTAINING INDEPENDENT VOLTAGE SOURCES

We extend the nodal analysis technique we developed in the previous section to circuits containing independent voltage sources in addition to independent current sources in this section.

An independent voltage source can appear in three positions in a circuit. It may appear from a node to the reference node. Secondly, it may appear between two nodes at which more than two elements are incident. Thirdly, it may appear in series with some resistor. We deal with the first two cases in this section and take up the third case in the next section.

The circuit in Fig. 4.3-1 shows the first case.

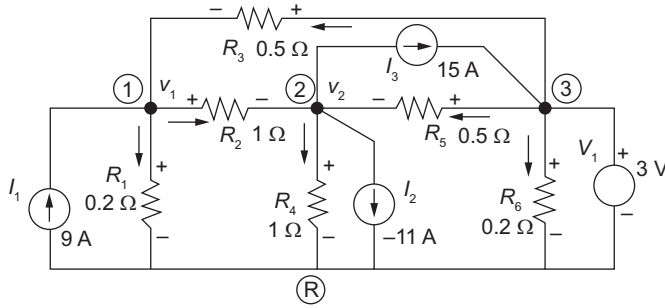


Fig. 4.3-1 Circuit showing a node voltage getting fixed by a voltage source

We identify the reference node and assign node voltage variables at other nodes as the first step in nodal analysis of circuits. The reference node and the remaining three nodes are identified in the circuit in Fig. 4.3-1. However, only two node voltage variables are assigned; the third node voltage variable is not shown assigned. The reason is obvious. The independent voltage source connected directly from that node to reference node *fixes* or *constrains* that node voltage to be at the source value. Thus, the number of node variables to be solved for has come down to 2 from 3. This, in general, is the effect of voltage sources in a circuit as far as nodal analysis is concerned. Each such voltage source imposes one constraint on the degree of freedom the circuit has and reduces the number of node voltage variables by one.

Thus, we have only two unknown node voltage variables in this 3-node + reference node problem. We write the two node equations as below.

$$\begin{aligned} G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - V_1) &= I_1 \\ G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - V_1) &= -I_2 - I_3 \end{aligned} \quad (4.3-1)$$

We do not have to write the third node equation since we know that $v_3 = V_1$. Let us cast these equations in matrix form.

$$\begin{bmatrix} (G_1 + G_2 + G_3) & -G_2 \\ -G_2 & (G_2 + G_4 + G_5) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 + G_3 V_1 \\ -I_2 - I_3 + G_5 V_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & G_3 \\ 0 & -1 & -1 & G_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_1 \end{bmatrix} \quad (4.3-2)$$

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The matrix equation has come out in the form of $\mathbf{YV} = \mathbf{CU}$ again. We note that the $\mathbf{Y}$ matrix is symmetric. However, the $\mathbf{U}$ vector now has four entries – three current source values and one voltage source value. We note carefully that the $\mathbf{C}$ matrix has suitable entries that convert the V units into A. Thus, the matrix product $\mathbf{CU}$ is a column vector of currents since a voltage multiplied by conductance is a current.

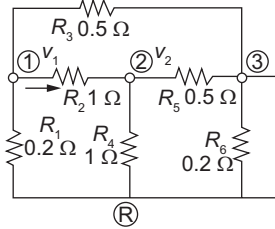


Fig. 4.3-2 Circuit in Fig. 4.3-1 with all independent sources deactivated

Moreover, we observe that, the form and entries of $\mathbf{Y}$ matrix on the left-hand side of equation cannot depend on the particular values of I_1 , I_2 , I_3 and V_1 . The matrix must be the same for any numerical values for these four inputs. Therefore, $\mathbf{Y}$ matrix entries must be the same for a case where all these inputs are zero-valued. However, an independent current source of zero value is an open circuit and an independent voltage source with zero value is a short-circuit ! Therefore, it should be possible to write down that matrix by inspection after we deactivate all the sources in the circuit. The deactivated circuit is shown in Fig. 4.3-2.

The resistor R_6 simply goes out of picture due to the short-circuit across it. R_5 gets connected between node-2 and reference node. R_3 gets connected between node-1 and reference node. We write the $\mathbf{Y}$ -matrix of this circuit by using the rules we developed in the last section.

$$\begin{aligned} y_{11} &= G_1 + G_2 + G_3 \\ y_{12} &= y_{21} = -G_2 \\ y_{22} &= G_2 + G_4 + G_5 \end{aligned}$$

We find that this is the same as the $\mathbf{Y}$ -matrix in Eqn. 4.3-2. Thus, we conclude that *an independent voltage source directly at a node in a circuit results in a reduction of node voltage variables by one and in a reduction of order of nodal conductance matrix by one. The nodal conductance matrix remains symmetric.* The nodal conductance matrix may be found by using the deactivated circuit, if necessary. However, the node equations will have to be written in order to get the right-hand side of Eqn. 4.3-2.

The resistor R_6 disappearing altogether in the analysis is an interesting aspect. We observe that the value of conductance of this resistor does not appear anywhere in Eqn. 4.3-2. Thus, it has no effect on the node voltages. We may even remove it when we write the equations for node voltage variables. Why is it so? For the simple reason that the voltage source has fixed the node potential at the third node and presence or absence of R_6 has no effect on that fact. Since node potentials decide the voltage across all elements and currents through them, the rest of the circuit is totally unaffected by the value of R_6 . The only element that gets affected by R_6 (apart from itself!) is the voltage source. Its current will have a component due to R_6 . *We may treat a resistor across an independent voltage source as a trivial element.*

Substituting the values for conductances and source functions in Eqn. 4.3-2 we get,

$$\begin{bmatrix} 8 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 9 + 2 \times 3 \\ -(-11) - 15 + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \end{bmatrix}$$

and the solution is $v_1 = 2\text{V}$ and $v_2 = 1\text{V}$.

Now, the voltages across all the resistors and the current sources may be found by inspection. Similarly, the currents through resistors can be found by applying Ohm's law to them. Currents through

current sources are already known. What remains is the current in the voltage source. An extra step has to be carried out for finding out the current in V_1 . This is the price that the voltage source demands from us for fixing the node-3 potential for us!

The extra step involves applying KCL at node-3. Note that we did not use the KCL equation at node-3 in the process of solving for node voltage variables. We use it now. All the other currents leaving or entering node-3 are known. Only the current through V_1 is unknown. Hence, it can be found.

$$G_3(V_1 - v_1) + G_5(V_1 - v_2) + G_6V_1 - I_3 + i_{V_1} = 0$$

Substituting values,

$$2(3 - 2) + 2(3 - 1) + 5 \times 3 - 15 + i_{V_1} = 0$$

$$\therefore i_{V_1} = -6 \text{ A}$$

Note that the trivial element, R_6 , is accounted now. Thus, the current in the 3V source is -6 A into the positive terminal, *i.e.*, 6 A out of the positive terminal. Fig. 4.3-3 shows the complete solution of the circuit.

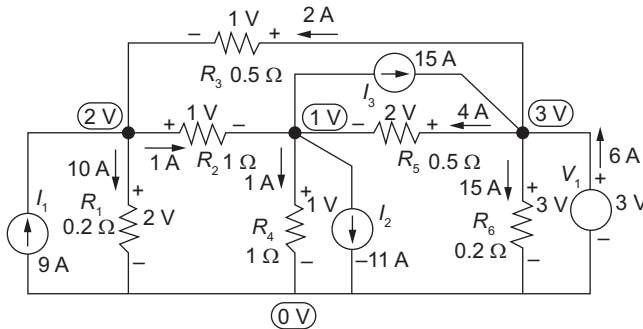


Fig. 4.3-3 Complete solution for the circuit in Fig. 4.3-1

Next, we consider a case in which an independent voltage source appears across two nodes other than the reference node. See the circuit in Fig. 4.3-4.

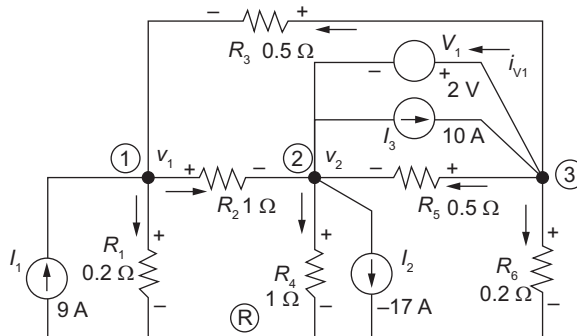


Fig. 4.3-4 Circuit with an independent voltage source across two nodes

The independent voltage source V_1 constrains the voltage at node-3 to be above the voltage at node-2 by $V_1 \text{ V}$. If we know one of them, we can find the other. Hence, only one of them – either v_2

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or v_3 – needs be assigned. Node-1 is not constrained in any way and hence needs to be assigned a node voltage variable. Hence, v_1 and v_2 are assigned as shown. It could have been v_1 and v_3 with no difference to the final solution. Thus, here too, we find that an independent voltage source imposes a constraint on node voltage variables and reduces the number of node voltage variable by one.

We write the KCL equation (or node equation) at all the nodes except at reference node. We make use of the constraint equation $v_3 = v_2 + V_1$ wherever we need v_3 in the process of writing node equations at node-1 and node-2.

The resulting equations are

$$\begin{aligned}\text{Node-1} \quad & G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_2 - V_1) = I_1 \\ \text{Node-2} \quad & G_4 v_2 + G_2 (v_2 - v_1) - G_5 V_1 - i_{V_1} = -I_2 - I_3 \\ \text{Node-3} \quad & G_6 (v_2 + V_1) + G_3 (v_2 + V_1 - v_1) + G_5 V_1 + i_{V_1} = I_3\end{aligned}$$

We get rid of i_{V_1} by adding the last two equations and end up with two equations in two unknowns.

$$\begin{aligned}\text{Node-1} \quad & G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_2 - V_1) = I_1 \\ \text{Node-2} + \text{Node-3} \quad & G_4 v_2 + G_2 (v_2 - v_1) + G_6 (v_2 + V_1) + G_3 (v_2 + V_1 - v_1) = -I_2 \quad (4.3-3)\end{aligned}$$

Note that the resistor R_5 and current source I_3 which are directly across the voltage source disappear from the equations. They become *trivial elements*. They do not affect the node voltages and hence voltage and current of other elements. They affect only the current through the voltage source.

The Eqn. 4.3-3 can be expressed in matrix form as

$$\begin{aligned}\begin{bmatrix} (G_1 + G_2 + G_3) & -(G_2 + G_3) \\ -(G_2 + G_3) & (G_2 + G_3 + G_4 + G_6) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} G_3 V_1 + I_1 \\ -(G_6 + G_3) V_1 - I_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & G_3 \\ 0 & -1 & 0 & -(G_6 + G_3) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_1 \end{bmatrix}\end{aligned}$$

The matrix equation has come out in the form of $\mathbf{YV} = \mathbf{CU}$ again. We note that the $\mathbf{Y}$ matrix is symmetric. Moreover, the $\mathbf{Y}$ -matrix can be obtained from the deactivated circuit as in the earlier case.

Substituting values for conductances and the source functions, we get,

$$\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 9 + 2 \times 2 \\ -(-17) - 7 \times 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}$$

The solution is $v_1 = 2\text{V}$ and $v_2 = 1\text{V}$. The current through the voltage source can be obtained by applying KCL at node-2 or node-3. Using node-3,

$$\begin{aligned}G_6 (v_2 + V_1) + G_3 (v_2 + V_1 - v_1) + G_5 V_1 + i_{V_1} &= I_3 \\ 5(1+2) + 2(1+2-2) + 2 \times 2 + i_{V_1} &= 10 \Rightarrow i_{V_1} = -11\text{A}\end{aligned}$$

Note that the trivial elements R_5 and I_3 came back in this equation. The complete solution is shown in Fig. 4.3-5.

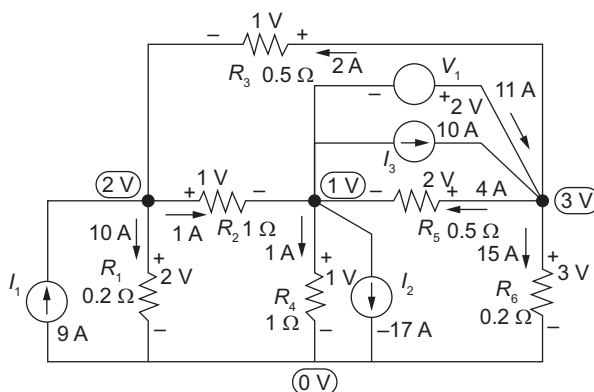


Fig. 4.3-5 Complete solution for the circuit in Fig. 4.3-4

The last two examples have demonstrated the following:

- (i) An independent voltage source imposes a constraint on node voltage variables and reduces their number by one.
- (ii) A node voltage variable *need not be* assigned at a node if an independent voltage source determines that node voltage directly or indirectly through another node voltage variable assigned to another node.
- (iii) Node equation at a node where the node voltage is fixed directly by an independent voltage source connected from that node to the reference node *is not required for solving other node voltage variables*.
- (iv) Node equations at two nodes with an independent voltage source between them have to be added to get a combined node equation that will be useful in solving the circuit analysis problem.
- (v) The nodal analysis formulation results in an equation $\mathbf{YV} = \mathbf{CU}$ where $\mathbf{Y}$ is the nodal conductance matrix of a reduced order circuit resulting from deactivating all independent sources in it. The input vector $\mathbf{U}$ contains all the independent current source functions and independent voltage source functions. The solution for the node voltage vector $\mathbf{V}$ can be written as $\mathbf{Y}^{-1}\mathbf{CU}$. This indicates that each node voltage (and hence all element voltages and currents) can be expressed as a linear combination of input source functions – i.e., $x = a_1I_1 + a_2I_2 + \dots + b_1V_1 + b_2V_2 + \dots$ where x is some node voltage variable or element current/voltage variable and a 's and b 's are coefficients decided by circuit conductances and connection details. Some of the a 's and b 's may turn out to be zero for certain choices of x . For instance a_3 is zero if $x = v_1$ in the example we just concluded, but it is non-zero if $x = i_{V_1}$.
- (vi) The current through independent voltage sources can be found out only after the node voltage variables are solved. Finding the current through an independent voltage source requires the application of KCL at one of the end nodes of that voltage source.

We know an n node circuit has only $n - 1$ node voltage variables. Each independent voltage source reduces the number of node voltage variables to be solved for by one. Then, what happens if there are $n - 1$ independent voltage sources in the circuit? This leads us to our next example. Consider the following example circuit in Fig. 4.3-6 that has all the three node voltage variables constrained by three independent voltage sources.

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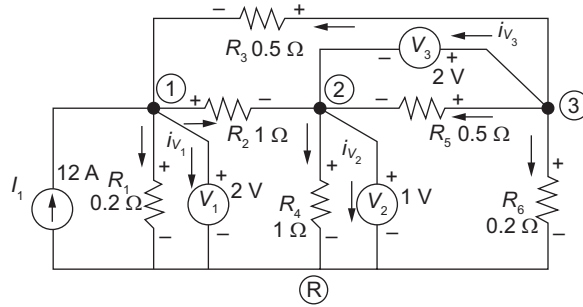


Fig. 4.3-6 A fully constrained nodal analysis example circuit

Although all the three nodes are identified in the diagram, no node voltage variable is assigned to them. This is so since the voltage source V_1 fixes the first node potential, the voltage source V_2 fixes the second node potential and sources V_2 and V_3 together fix the third node potential. Hence, the circuit solution is $v_1 = 2V$, $v_2 = 1V$ and $v_3 = 3V$. Now, all resistor voltages/currents and voltage across current sources can be worked out by inspection. Further, i_{V_1} , i_{V_2} and i_{V_3} can be found out by applying KCL at the three nodes.

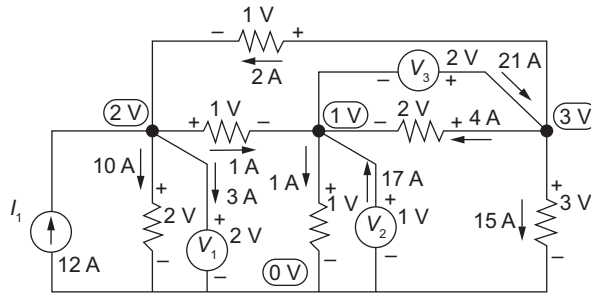


Fig. 4.3-7 Solution for circuit shown in Fig. 4.3-6

KCL at Node -1 $\rightarrow G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_3) + i_{V_1} = I_1$. Substituting numerical values and solving for i_{V_1} , we get $i_{V_1} = 3A$. Further,

$$\text{KCL at Node -2} \rightarrow G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_3) + i_{V_2} - i_{V_3} = 0$$

$$\text{KCL at Node -3} \rightarrow G_6 v_3 + G_3 (v_3 - v_1) + G_5 (v_3 - v_2) + i_{V_3} = 0$$

We eliminate i_{V_3} by adding these two equations to get $G_4 v_2 + G_2 (v_2 - v_1) + G_6 v_3 + G_3 (v_3 - v_1) + i_{V_2} = 0$. Substituting numerical values and solving for i_{V_2} , we get $i_{V_2} = -17A$. Substituting this value in the node equation at node-2, we solve for i_{V_3} to get $i_{V_3} = -21A$. The complete circuit solution is shown in Fig. 4.3-7.

This circuit had no free node voltage variable. The three independent voltage sources suitably connected had fixed the node voltage variables already. Thus, there was no circuit analysis problem to begin with.

Three independent voltage sources in a four-node circuit need not result in a fully constrained circuit necessarily. For instance, assume that the source V_3 is shifted and connected between node-1

and node-2. We note that the three independent voltage sources will then form a closed loop in which KVL equation will lead to an inconsistent equation $-2 - 2 + 1 = 0$. Practically, this means that all the three sources are being shorted together with large currents in them. However, if we insist on modelling the sources as independent ones, the circuit can have no solution when the KVL equation in a loop leads to inconsistent equation. Thus, the value of V_3 has to be changed to -1 V if it has to be connected between node-1 and node-2. KVL will be satisfied then. Now, the node voltage variable at node-3 is not constrained by these three sources and hence will have to be obtained by nodal analysis. However, the reader may verify that there is no way to determine the currents in the three voltage sources uniquely! There are infinite possible sets of values for these three currents. Thus, the circuit cannot be solved uniquely. Similar conclusions will follow for a case with more than $(n - 1)$ independent voltage sources in an n node circuit – either the circuit cannot be solved due to inconsistencies in KVL equations or the circuit can not be solved uniquely.

Thus, the maximum number of independent voltage source that can be there in a n -node circuit with unique solution is $(n - 1)$.

4.4 SOURCE TRANSFORMATION THEOREM AND ITS USE IN NODAL ANALYSIS

We take up the case where an independent voltage source is connected in series with a resistor. This case can indeed be solved by the procedure developed in the last section. However, there is a better way. We need the *Source Transformation Theorem* to understand this better method.

4.4.1 Source Transformation Theorem

It was pointed out in Chapter 1 that practical voltage sources can often be modelled as ideal independent voltage source in series with a resistance. Similarly, practical current sources can be modelled by an ideal independent current source in parallel with a resistance. Consider a pair of such practical sources delivering power to identical load resistors as shown in the circuits in Fig. 4.4-1(a) and (b). $v(t)$ is the voltage appearing across the load resistor and $i(t)$ is the current through it in both cases.

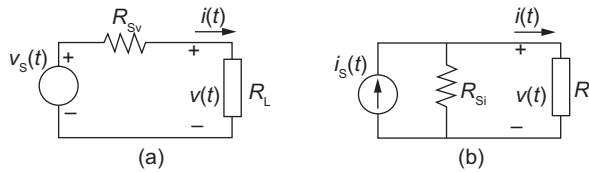


Fig. 4.4-1 Practical voltage and current sources supplying power to identical load resistors

Applying Ohm's law and KVL, we get $v(t) = v_s(t) - R_{sv}i(t)$ in the circuit in Fig. 4.4-1(a) and applying current division principle and Ohm's law we get

$$v(t) = \frac{R_L R_{si}}{R_{si} + R_L} i_s(t) = R_{si} i_s(t) \left[1 - \frac{R_{si}}{R_{si} + R_L} \right] = R_{si} i_s(t) - R_{si} i(t) \text{ in the circuit in Fig. 4.4-1 (b). What}$$

are the conditions such that the voltage developed across the load resistor is the same in both cases?

The required conditions are (i) $R_{sv} = R_{si} = R_s$ and (ii) $v_s(t) = R_s i_s(t)$. The sources are indistinguishable from their terminal behaviour if these two conditions are satisfied. That is, if the source is put inside

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a black box and an effort is made to determine whether it is a voltage source or a current source by measuring $v(t)$ and/or $i(t)$ for various values of R_L , such an attempt will fail. The two sources in Fig. 4.4-1 are completely equivalent as far as their effect on external element is concerned and one may replace the other provided they satisfy the two conditions listed above. Note that the equivalence is only with respect to what happens to the external element. They are *not equivalent* as far as what happens inside the source is concerned. For instance, the power dissipation in R_S is not the same in the two sources for the same value of R_L .

Source Transformation Theorem states that a voltage source with source function $v_S(t)$ in series with a resistance R_S can be replaced by a current source with source function $i_S(t) = v_S(t) / R_S$ in parallel with a resistance R_S without affecting any voltage/current/power variable external to the source. The direction of current source is such that current flows out of the terminal at which the positive of the voltage source is presently connected.

Similarly, a current source with source function $i_S(t)$ in parallel with a resistance R_S can be replaced by a voltage source with source function $v_S(t) = R_S i_S(t)$ in series with a resistance R_S without affecting any voltage/current/power variable external to the source. The polarity of voltage source is such that it tends to establish a current in the external circuit in the same direction as in the case when the current source is acting.

The reasoning employed in arriving at this theorem is equally valid in the case of dependent sources too. Hence, Source Transformation Theorem is applicable to dependent sources also.

Fig. 4.4-2 states the Source Transformation Theorem graphically.

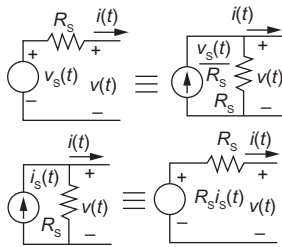


Fig. 4.4-2 Source equivalence between voltage and current sources

4.4.2 Applying Source Transformation in Nodal Analysis of Circuits

The application of this theorem in nodal analysis of circuits containing independent voltage sources is illustrated now. Consider the circuit (a) in Fig. 4.4-3. The circuit has five nodes including the reference node. All the nodes are identified. We observe that the voltage source V_3 is in series with resistor R_6 . This combination may be replaced by a current source $I_2 = V_3 / R_6$ in parallel with R_6 as shown in (b) of Fig. 4.4-3. This results in elimination of node-4. The circuit in (b) is a 4-node circuit with two independent voltage sources (V_1 and V_2) constraining its node voltage variables. Thus, there is only one node voltage variable.

We start assigning node voltage variables from the leftmost node. The first node in circuit (b) is unconstrained and therefore we assign the node voltage variable v_1 to that node. That fixes the node voltage at node-3 as $v_1 + V_2$. Hence, a new node voltage variable is not needed at node-3. Node-2 potential is directly constrained by the source V_1 and hence a node voltage variable is not needed there.

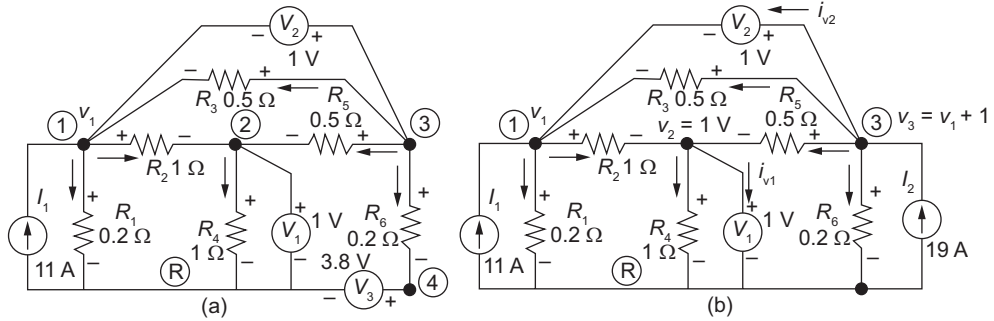


Fig. 4.4-3 (a) Nodal analysis example circuit (b) Circuit after node reduction by source transformation

We need to write the node equations at node-1 and node-3 and combine them to get an equation in the single variable v_1 . The node equation at node-2 is not needed for solving node potentials since it is a directly-constrained node.

$$\text{KCL at Node -1} \rightarrow G_1 v_1 + G_2 (v_1 - V_1) - G_3 V_2 - i_{v_2} = I_1$$

$$\text{KCL at Node -3} \rightarrow G_6 (v_1 + V_2) + G_5 V_2 + G_5 (v_1 + V_2 - V_1) + i_{v_2} = I_2$$

Adding these two equations, we get,

$$G_1 v_1 + G_2 (v_1 - V_1) + G_6 (v_1 + V_2) + G_5 (v_1 + V_2 - V_1) = I_1 + I_2 = I_1 + G_6 V_3$$

Expressing this in matrix form,

$$\left[(G_1 + G_2 + G_5 + G_6) \right] [v_1] = \begin{bmatrix} I_1 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

We note that the *trivial element* R_3 has no role in deciding the node voltages. Further, we note that the node equation has the format $\mathbf{YV} = \mathbf{CI}$ where the $\mathbf{Y}$ -matrix is of 1×1 and is the nodal conductance matrix of the deactivated circuit. On deactivating the circuit in Fig. 4.4-3(a) by replacing voltage sources with short-circuits and current sources with open circuits, a simple circuit containing four resistors – R_1 , R_2 , R_5 and R_6 – will be the result.

Substituting the numerical values and solving for v_1 , we get, $13v_1 = 26 \Rightarrow v_1 = 2\text{V}$. Therefore $v_3 = 3\text{V}$.

Now, we fit these values of node voltages into the circuit in Fig. 4.4-3(a) and obtain the voltage across the resistors and current sources and currents through resistors by inspection.

i_{v_1} is obtained by applying KCL at node-2 of the original circuit.

$$G_2 (v_2 - v_1) + G_4 v_2 + G_5 (v_2 - v_3) + i_{v_1} = 0$$

$$\text{i.e., } 1(1 - 2) + 1(1) + 2(1 - 3) + i_{v_1} = 0$$

$$\Rightarrow i_{v_1} = 4\text{ A}$$

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i_{v_2} is obtained by applying KCL either at node-1 or at node-3 of the original circuit. Choosing node-1,

$$\begin{aligned} G_2(v_1 - v_2) + G_1 v_1 + G_3(v_1 - v_3) - i_{v_2} &= I_1 \\ \text{i.e., } 1(2 - 1) + 5(2) + 2(2 - 3) - i_{v_2} &= 9 \\ \Rightarrow i_{v_2} &= -2 \text{ A} \end{aligned}$$

The current delivered by V_3 is the same as the current in R_6 . Hence, $i_{v_3} = -4 \text{ A}$. The complete solution is marked in Fig. 4.4-4.

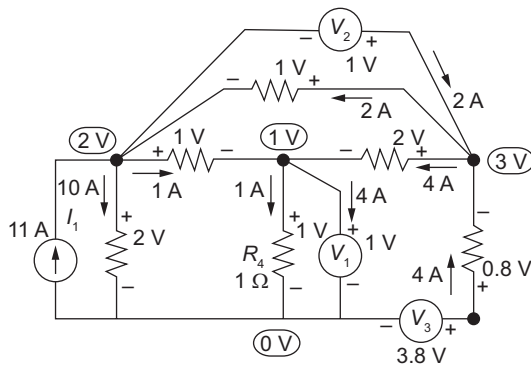


Fig. 4.4-4 Complete solution for circuit in Fig. 4.4-3(a)

4.5 NODAL ANALYSIS OF CIRCUITS CONTAINING DEPENDENT CURRENT SOURCES

Dependent current sources do not pose any problem for nodal analysis. Independent current sources appear in the right-hand side of node equations. However, dependent current sources affect the coefficients of node voltage variables in the left-hand side of node equations.

The controlling variable of a linear dependent current source will be a voltage or current existing elsewhere in the circuit. However, any voltage or current variable in the circuit can be expressed in terms of node voltage variables. Hence, the dependent current source function can be expressed in terms of node voltage variables. Therefore, dependent current sources will affect the coefficients of node equation, *i.e.*, they will change the nodal conductance matrix. We will see that they *can* destroy the symmetry of the nodal conductance matrix.

We develop the nodal analysis procedure for this kind of circuits through two examples. The first example has node voltages that are not constrained by independent voltage sources and the second one has node voltage variables constrained by independent voltage source.

Example: 4.5-1

Solve the circuit in Fig. 4.5-1(a) completely.

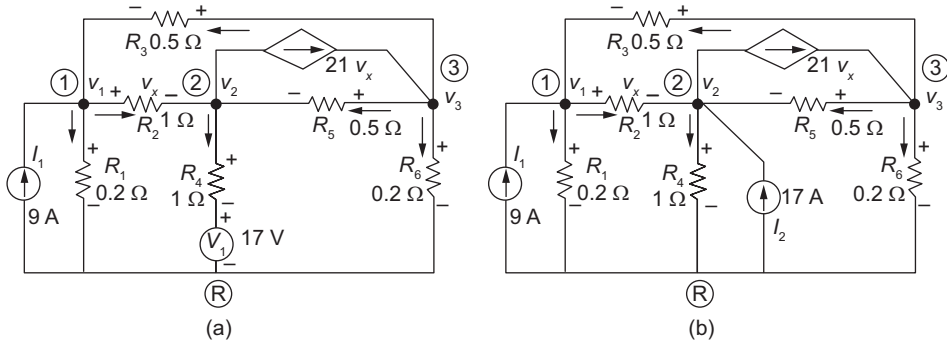


Fig. 4.5-1 (a) Circuit for Example 4.5-1 (b) Circuit after node reduction by source transformation

Solution

Step-1: Look for independent voltage sources in series with resistors and apply source transformation on such combinations.

There is one such combination in this circuit. It is V_1 in series with R_4 . Applying source transformation on this combination results in an independent current source of 17A in parallel with R_4 as shown in circuit (b) of Fig. 4.5-1.

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

Now, all the three non-reference nodes in circuit (b) are unconstrained nodes and hence we assign three node voltage variables v_1 , v_2 and v_3 as shown in the figure.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the last step and rewrite the source functions of dependent sources in terms of node voltage variables.

v_x is the controlling variable in this circuit. However, v_x is the voltage across R_2 and $= v_1 - v_2$. Therefore, the current source function is $k(v_1 - v_2)$ with $k = 21$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables.

The node equations are listed below.

$$\text{Node-1} \quad G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_3) = I_1$$

$$\text{Node-2} \quad G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_3) + k(v_1 - v_2) = I_2$$

$$\text{Node-3} \quad G_6 v_3 + G_3 (v_3 - v_1) + G_5 (v_3 - v_2) - k(v_1 - v_2) = 0$$

Casting these equations in matrix form,

$$\begin{bmatrix} (G_1 + G_2 + G_3) & -G_2 & -G_3 \\ -G_2 + k & (G_2 + G_4 + G_5 - k) & -G_5 \\ -G_3 - k & -G_5 + k & (G_3 + G_5 + G_6) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & G_4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V_1 \end{bmatrix} \quad (4.5-1)$$

Eqn. 4.5-1 is in the form $\mathbf{YV} = \mathbf{CI}$ where $\mathbf{Y}$ is the nodal conductance matrix. However, the nodal conductance matrix is now *asymmetric* and cannot be written down easily by inspection. However,

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the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as a linear combination of independent source functions.

Substituting the numerical values,

$$\begin{bmatrix} 8 & -1 & -2 \\ 20 & -17 & -2 \\ -23 & 19 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ 0 \end{bmatrix}$$

Solving for the voltage vector by Cramer's rule, $v_1 = 2\text{V}$, $v_2 = 1\text{V}$ and $v_3 = 3\text{V}$.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents.

Now, the voltage across elements and current through them can be obtained by inspection. The complete solution is shown in Fig. 4.5-2.

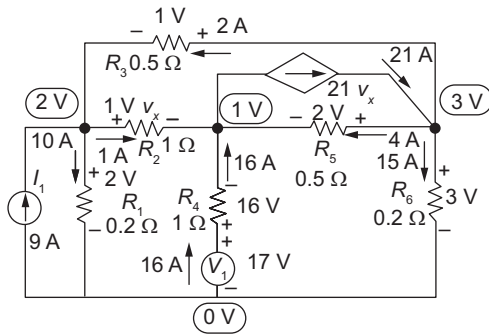


Fig. 4.5-2 Complete circuit solution in Example 4.5-1

Example: 4.5-2

Solve the circuit (a) in Fig. 4.5-3 completely.

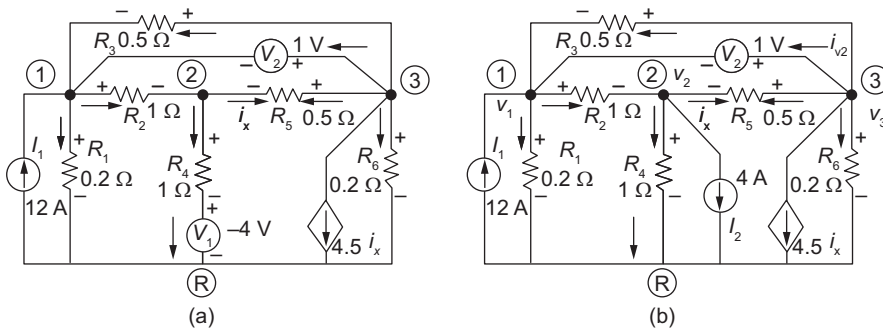


Fig. 4.5-3 (a) Circuit for Example 4.5-2 (B) Circuit after node reduction by source transformation

Solution

Step-1: Look for independent voltage sources in series with resistors and apply source transformation on such combinations.

There is one such combination in this circuit. It is V_1 in series with R_4 . Applying source transformation on this combination results in an independent current source of 4A in parallel with R_4 as shown in the circuit in Fig. 4.5-3(b).

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

We start at left-most node of the circuit in Fig. 4.5-3(b) and assign a node voltage variable v_1 there since that node is not directly constrained by a voltage source. Moving to node-2, we see that the node voltage at that node cannot be obtained from the already assigned variable v_1 and that there is no direct constraint at that node. Hence, we assign a node voltage variable v_2 at that node. Now, the node voltage at node-3 can be obtained as $v_1 + V_2$ and a node voltage variable is not needed at that node. Therefore, there are only two node voltage variables in this circuit.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the last step and rewrite the source functions of dependent sources in terms of node voltage variables.

i_x is the controlling variable in this circuit. However, $i_x = G_5 [v_2 - (v_1 + V_2)]$. Therefore, the current source function is $kG_5[v_2 - (v_1 + V_2)]$, with $k = 4.5$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables. Ignore node equation at nodes where voltage sources are connected directly to reference node. Combine the node equations at the end nodes of voltage sources connected between two non-reference nodes.

The node equations are listed below.

$$\text{Node-1} \quad G_1 v_1 + G_2 (v_1 - v_2) - G_3 V_2 - i_{V_2} = I_1$$

$$\text{Node-2} \quad G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_1 - V_2) = -I_2 = G_4 V_1$$

$$\text{Node-3} \quad G_6 (v_1 + V_2) + G_3 V_2 + G_5 (v_1 + V_2 - v_2) + kG_5 (v_2 - v_1 - V_2) + i_{V_2} = 0$$

Combining the node equations at node-1 and node-3 to eliminate i_{V_2} ,

$$\text{Node-1} + \text{Node-3} \quad G_1 v_1 + G_2 (v_1 - v_2) + G_6 (v_1 + V_2) + G_5 (v_1 + V_2 - v_2) + kG_5 (v_2 - v_1 - V_2) = I_1$$

$$\text{Node-2} \quad G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_1 - V_2) = -I_2 = G_4 V_1$$

Casting these equations in matrix form,

$$\begin{bmatrix} (G_1 + G_2 + G_6 + (1-k)G_5) & -(G_2 + (1-k)G_5) \\ -(G_2 + G_5) & (G_2 + G_4 + G_5) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(G_6 + (1-k)G_5) \\ 0 & G_4 & G_5 \end{bmatrix} \begin{bmatrix} I_1 \\ V_1 \\ V_2 \end{bmatrix}$$

This equation is in the form $\mathbf{YV} = \mathbf{CU}$ where $\mathbf{Y}$ is the nodal conductance matrix. However, the nodal conductance matrix is now *asymmetric* and cannot be written down easily by inspection. However, the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as a linear combination of independent source functions.

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Substituting the numerical values,

$$\begin{bmatrix} 4 & 6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -2 \end{bmatrix}$$

Solving for the voltage vector by Cramer's rule, $v_1 = 2\text{V}$, $v_2 = 1\text{V}$ and $v_3 = v_1 + V_2 = 3\text{V}$.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and current sources.

The voltage across resistive elements and current sources and currents through resistive elements can be obtained by inspection. The currents through independent voltage sources in series with resistors can also be obtained at this stage.

Step-6: Use appropriate node equations to solve for currents through the remaining independent voltage sources.

The current through the independent voltage source V_2 has to be determined. We use the node equation at node-1 for this purpose.

$$G_1 v_1 + G_2 (v_1 - v_2) - G_3 V_2 - i_{V_2} = I_1$$

$$10 + 1 - 2 - i_{V_2} = 12 \Rightarrow i_{V_2} = -3 \text{ A}$$

The complete solution is marked in Fig. 4.5-4.

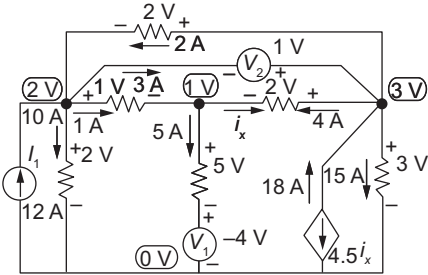


Fig. 4.5-4 Complete solution for circuit in Example 4.5-2

4.6 NODAL ANALYSIS OF CIRCUITS CONTAINING DEPENDENT VOLTAGE SOURCES

A dependent voltage source can appear in three positions in a circuit. It may appear from a node to the reference node. Secondly, it may appear between two nodes at which more than two elements are incident. Thirdly, it may appear in series with some resistor. The following example shows a circuit that has a dependent voltage source in series with a resistor.

Example: 4.6-1

Solve the circuit in Fig. 4.6-1(a).

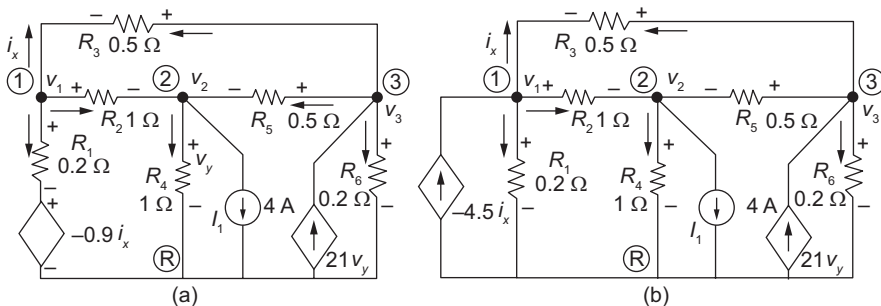


Fig. 4.6-1 (a) Circuit for Example 4.6-1 (b) Reduced circuit after source transformation

Solution

Step-1: Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation on such combinations.

There is one such combination in this circuit. It is $-0.9i_x$ in series with R_1 . Applying source transformation on this combination results in a dependent current source of $-0.9G_1i_x$ A in parallel with R_1 as shown in the circuit in Fig. 4.6-1(b).

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by a voltage source or indirectly by already assigned node voltage variables and voltage source functions.

We start at left-most node of the circuit in Fig. 4.6-1(b) and assign a node voltage variable v_1 there since that node is not directly constrained by a voltage source. Moving to node-2, we see that the node voltage at that node cannot be obtained from the already assigned variable v_1 and that there is no direct constraint at that node. Hence, we assign a node voltage variable v_2 at that node. The node voltage at node-3 cannot be obtained from v_1 and v_2 . Hence, we assign a node voltage variable v_3 at that node. Therefore, there are three node voltage variables in this circuit.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the last step and rewrite the source functions of dependent sources in terms of node voltage variables.

i_x is the controlling variable for the dependent current source at node-1 in the circuit (b) of Fig. 4.6-1. However, $i_x = G_3 [v_1 - v_3]$. Therefore, the current source function is $k_1 G_1 G_3 [v_1 - v_3]$ with $k_1 = -0.9$.

v_y is the controlling variable for the dependent current source at node-3. However, $v_y = v_2$. Therefore, the current source function at node-3 is $k_2 v_2$ with $k_2 = 21$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables. Ignore node equation at nodes where voltage sources are connected directly to reference node. Combine the node equations at the end nodes of voltage sources connected between two non-reference nodes.

The node equations are listed below.

$$\text{Node-1} \quad G_1 v_1 + G_2(v_1 - v_2) + G_3(v_1 - v_3) + k_1 G_1 G_3(v_1 - v_3) = 0$$

$$\text{Node-2} \quad G_4 v_2 + G_2(v_2 - v_1) + G_5(v_2 - v_3) = -I_1$$

$$\text{Node-3} \quad G_6 v_3 + G_3(v_3 - v_1) + G_5(v_3 - v_2) + k_2 v_2 = 0$$

Substituting the numerical values and casting these equations in matrix form,

$$\begin{bmatrix} 17 & -1 & -11 \\ -1 & 4 & -2 \\ -2 & -23 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} [4]$$

This equation is in the form $\mathbf{YV} = \mathbf{CU}$ where $\mathbf{Y}$ is the nodal conductance matrix. However, the nodal conductance matrix is now *asymmetric* and cannot be written down easily by inspection. However, the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as a linear combination of independent source functions.

4.24

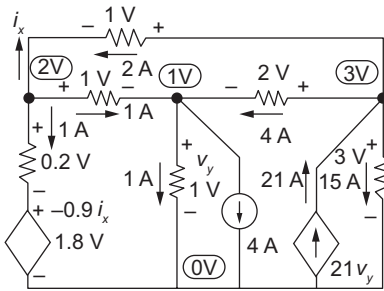


Fig. 4.6-2 Complete solution for circuit in Fig. 4.6-1 (a)

Solving for the voltage vector by Cramer's rule, $v_1 = 2\text{V}$, $v_2 = 1\text{V}$ and $v_3 = 3\text{V}$.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and current sources.

The voltage across resistive elements and current sources and currents through resistive elements can be obtained by inspection. The currents through independent voltage sources in series with resistors can also be obtained at this stage.

The complete solution is marked in Fig. 4.6-2.

Example: 4.6-2

Solve the circuit in Fig. 4.6-2 (a) in Example 4.6-2 by nodal analysis.

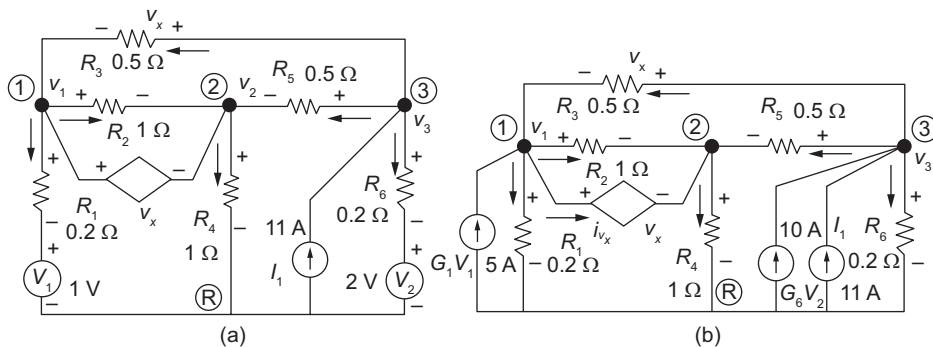


Fig. 4.6-3 (a) Circuit for Example 4.6-2 (b) Circuit after node reduction by source transformation

Solution

Step-1: Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation on such combinations.

There are two such combinations in this circuit. They are V_1 in series with R_1 and V_2 in series with R_6 . Applying source transformation on these combinations results in circuit (b) of Fig. 4.6-3.

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by a voltage source or indirectly by already assigned node voltage variables and voltage source functions.

We start at left-most node of the circuit (b) and assign a node voltage variable v_1 there since that node is not directly constrained by a voltage source to reference node. Moving to node-2, we see that the node voltage at that node can be obtained from the already assigned variable v_1 by

adding v_x . Hence, we do not assign a node voltage variable at that node. The node voltage at node-3 cannot be obtained from v_1 and there is no constraining voltage source connected from that node to reference node. Hence, we assign a node voltage variable v_3 at that node. Therefore, there are only two node voltage variables in this circuit.

Step-3: Identify the controlling variables of dependent sources in terms of the node voltage variables assigned in the last step and rewrite the source functions of dependent sources in terms of node voltage variables.

v_x is the controlling variable for the dependent voltage source between node-1 and node-2 in the circuit (b) of Fig. 4.6-1. However, $v_x = v_3 - v_1$. Therefore, the voltage source function is $k(v_3 - v_1)$ with $k = 1$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables. Ignore node equation at nodes where voltage sources are connected directly to reference node. Combine the node equations at the end nodes of voltage sources connected between two non-reference nodes.

The node equations are listed below.

$$\text{Node-1} \quad G_1 v_1 + G_2 (v_1 - (v_1 - k(v_3 - v_1))) + G_3 (v_1 - v_3) + i_{v_x} = G_1 V_1$$

$$\text{Node-2} \quad G_4 (v_1 - k(v_3 - v_1)) + G_2 (-k(v_3 - v_1)) + G_5 (v_1 - k((v_3 - v_1) - v_3)) - i_{v_x} = 0$$

$$\text{Node-3} \quad G_6 v_3 + G_3 (v_3 - v_1) + G_5 (v_3 - (v_1 - k(v_3 - v_1))) = I_1 + G_6 V_2$$

We eliminate the current through the dependent voltage source from the equations by adding the first two equations.

$$\text{Node-1} + \text{Node-2} \quad G_1 v_1 + G_3 (v_1 - v_3) + G_4 (v_1 - k(v_3 - v_1)) + G_5 (v_1 - k((v_3 - v_1) - v_3)) = G_1 V_1$$

$$\text{Node-3} \quad G_6 v_3 + G_3 (v_3 - v_1) + G_5 (v_3 - (v_1 - k(v_3 - v_1))) = I_1 + G_6 V_2$$

Substituting the numerical values and casting these equations in matrix form,

$$\begin{bmatrix} 16 & -9 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$$

Solving for the voltage vector by Cramer's rule, $v_1 = 2\text{V}$, $v_3 = 3\text{V}$. Then, $v_x = 1\text{V}$ and therefore $v_2 = v_1 - v_x = 1\text{V}$.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and current sources.

The voltage across resistive elements and current sources and currents through resistive elements can be obtained by inspection. The currents through voltage sources in series with resistors can also be obtained at this stage.

Step-6: Use appropriate node equations to solve for currents through the remaining voltage sources.

We have to find the current through the dependent voltage source by employing KCL equation at node-1 or node-2. Choosing node-2,

$$1 + 1(1 - 2) + 2(1 - 3) = i_{v_x} \Rightarrow i_{v_x} = -4 \text{ A. The complete solution is marked in Fig. 4.6-4.}$$

4.26

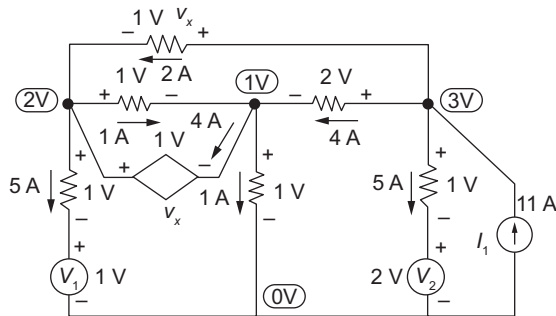


Fig. 4.6-4 Complete solution for circuit in Fig. 4.6-3(a)

Example: 4.6-3

Solve the circuit in Fig. 4.6-5 by nodal analysis.

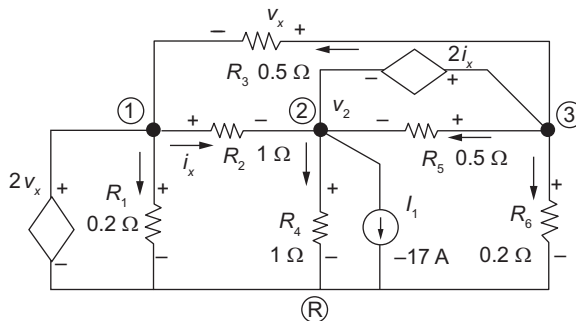


Fig. 4.6-5 Circuit for nodal analysis in Example 4.6-3

Solution

Node-1 is constrained by the dependent source to reference node and hence no node voltage variable can be assigned there. Node-2 is assigned a node voltage variable v_2 . Then, the node voltage variable at node-3 gets fixed as $v_2 + 2i_x$ through the dependent voltage source connected between node-2 and node-3. Thus, this circuit has only one node voltage variable to be solved for.

Node equation at node-1 is not needed for determining node voltages. However, it will be needed later for determining the current through the dependent voltage source connected at that node.

$$v_x = v_2 + 2i_x - 2v_x \text{ and } i_x = 2v_x - v_2$$

Solving these two equations, we get $v_x = v_2$ and $i_x = v_2$.

The KCL equations at node-2 and node-3 can be combined to form a single equation in v_3 .

This combined equation will be $v_2 + (v_2 - 2v_x) + 5(v_2 + 2i_x) + 2(v_2 + 2i_x - 2v_x) = 17$. Substituting for v_x and i_x in terms of v_2 , we get, $17v_2 = 17 \Rightarrow v_2 = 1\text{V}$.

Now, $v_x = v_2 = 1\text{V}$ and $i_x = 1\text{A}$. Therefore, the node voltages are 2V, 1V and 3V respectively at node-1, node-2 and node-3. Then, the node equation at node-1 can be employed to find current into

the positive terminal of dependent source as -9 A. Node equation at node-3 is used to determine the current into the positive terminal of second dependent source as -21 A. The complete solution is marked in Fig. 4.6-6.

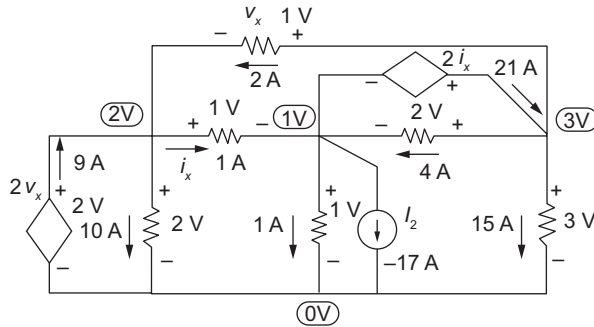


Fig. 4.6-6 Solution for circuit in Example 4.6-3

4.7 MESH ANALYSIS OF CIRCUITS WITH RESISTORS AND INDEPENDENT VOLTAGE SOURCES

Mesh Analysis uses the KCL equations and element equations to eliminate variables, reduces the number of pertinent variables to $(b - n + 1)$ mesh current variables and uses the second set of equations (KVL equations) to solve for these variables.

Mesh analysis is applicable only to *planar networks*. A planar network is one that can be drawn on a plane without any component crossing over another component. Consider the two circuits shown in Fig. 4.7-1.

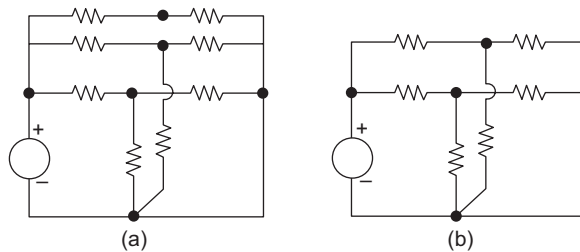


Fig. 4.7-1 (a) A non-planar circuit (b) A planar circuit that appears to be non-planar

The circuit in Fig. 4.7-1(a) is non-planar since it cannot be drawn on a plane surface without crossovers. The circuit (b) is planar though there appears to be a crossover the way it is drawn in Fig. 4.7-1. However, it can be redrawn to avoid the crossover.

4.7.1 Principle of Mesh Analysis

A circuit with n nodes, b elements and l loops will have n KCL equations, l KVL equations and b element equations involving $2b$ element variables. Only $(n - 1)$ KCL equations out of n will

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be linearly independent. Only $(b - n + 1)$ KVL equations out of l such equations will be linearly independent. Any set of KCL equations written at $(n - 1)$ nodes of the circuit will form a linearly independent set. However, any set of $(b - n + 1)$ equations drawn from the set of l voltage equations need not form a linearly independent set.

In Node Analysis, KVL equations are used to show that all element voltages can be expressed in terms of $(n - 1)$ node voltages. KCL equations along with element relations are used subsequently to set up $(n - 1)$ node equations needed for determining the node voltages.

Analogously, we try to use the KCL equations to show that all element currents can be expressed in terms of a reduced set of $(b - n + 1)$ specially defined currents called *mesh currents*. Subsequently, we set up $(b - n + 1)$ KVL equations involving these currents to determine them.

Which $(b - n + 1)$ loops do we choose for writing these KVL equations? The loops have to be chosen in such a way that the KVL equations will form an independent set. Two equations are necessarily independent if both equations contain terms that belong to only one of them. Consider the following equations.

$$v_1 - v_2 + v_3 - 2 = 0 \text{ and } -v_1 + v_2 + v_4 - 7 = 0$$

Obviously, no combinations like $a(v_1 - v_2 + v_3 - 2) + b(-v_1 + v_2 + v_4 - 7)$ can become equal to zero for all time for any combination of values for a and b for the simple reason that v_3 and v_4 cannot be got rid of. v_3 is present in only one equation and v_4 too is present only in one equation.

Thus, a sufficient but not necessary condition for a set of linear equations to form an independent set is that each equation should have at least one variable that does not appear in any other equation in the set.

Refer to the circuit in Fig. 4.7-2 .

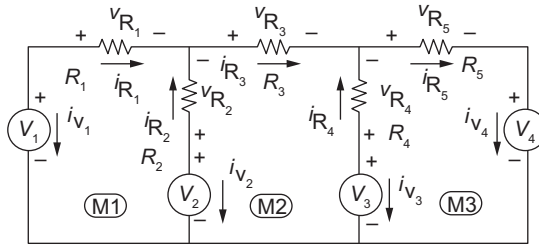


Fig. 4.7-2 Circuit for illustrating mesh analysis

The KVL equations for the three *windows* designated by M_1 , M_2 and M_3 in this circuit are given below.

$$\text{Window } M_1 : -V_1 + v_{R_1} - v_{R_2} + V_2 = 0$$

$$\text{Window } M_2 : -V_2 + v_{R_3} - v_{R_4} + V_3 = 0$$

$$\text{Window } M_3 : -V_3 + v_{R_4} + v_{R_5} + V_4 = 0$$

These equations form an independent set of $(b - n + 1) = (9 - 7 + 1) = 3$ KVL equations for the circuit. They are independent since each equation contains at least one element voltage variable that is not contained in the other two. This is so since the element R_1 is *completely owned* by first window, the element R_3 is *completely owned* by second window and the element R_5 is *completely owned* by third window. If an element is not *shared* among many windows and is *completely owned* by a particular

window, its voltage variable will appear only in the KVL equation of that particular window. However, that KVL equation cannot be generated by linearly combining KVL equations for other windows.

Thus, the KVL equations for the windows of a circuit will be the required set of independent equations provided each window contains at least one element that is totally owned by it. The windows in a planar circuit are called *Meshes*. It is possible to prove that in a planar connected network containing b elements and n nodes there will be exactly $(b - n + 1)$ meshes.

However, is it not possible for a mesh to have no element that is completely owned by it? It is possible. See Fig. 4.7-3. The mesh M_2 does not own any element. However, the four mesh KVL equations will be independent in this case too. This is so since no linear combination of three equations for the other three meshes can get rid of v_{R_6} or V_1 or V_4 .

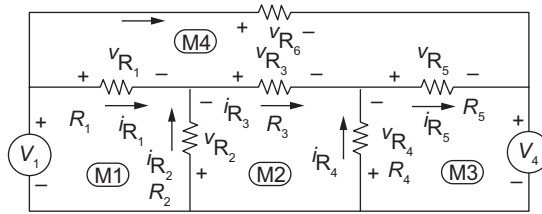


Fig. 4.7-3 Circuit with a mesh that does not own any element

Hence, we accept the fact that KVL equations for $(b - n + 1)$ meshes in a planar circuit will form a linearly independent set of equations without further discussion.

Let us take up the issue of defining $(b - n + 1)$ special currents that can be used to express all the element currents in the circuit. We can think of categorising all the elements that participate in a mesh into two groups – the first group contains those elements which appear only in that mesh and the second group contains those elements shared by this mesh with other meshes. Obviously, same current flows through all the elements belonging to first group. They are in series combination. Thus, each mesh will have a clearly identifiable current that flows in all the elements completely owned by that mesh. This current, *with clockwise direction assumed*, is defined as *mesh current* for that mesh and is used as the describing variable in Mesh Analysis just as *node voltage* was used as the describing variable in Nodal Analysis.

Refer to Fig. 4.7-2. We can derive the following equations by applying KCL at various nodes in this circuit.

$$\begin{aligned} i_{v_1} &= -i_{R_1} \\ i_{R_2} &= i_{R_3} - i_{R_1} \\ i_{v_2} &= i_{R_1} - i_{R_3} \\ i_{R_4} &= i_{R_5} - i_{R_3} \\ i_{v_3} &= i_{R_3} - i_{R_5} \\ i_{v_4} &= i_{R_5} \end{aligned}$$

Thus, all the remaining currents can be expressed in terms of three currents $-i_{R_1}$, i_{R_3} and i_{R_5} . These three currents will be the mesh currents in this circuit.

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However, a separate symbolic representation of mesh current is used in circuit analysis in order to emphasize the clockwise direction of flow in the definition of mesh current and to highlight the point that mesh current is a current that is common to all elements in the mesh. This is shown in Fig. 4.7-4.

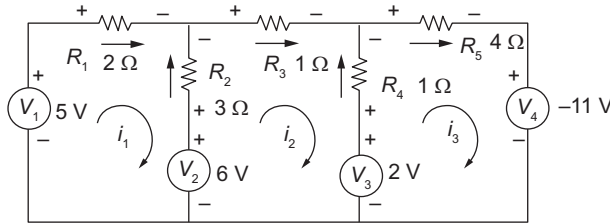


Fig. 4.7-4 Circuit for mesh analysis

The clockwise arrow and the symbol nearby in every mesh stand for the mesh current variable. Mesh current magnitude itself gives the magnitude of current in all the elements owned by that mesh. If the assumed current direction in such an element coincides with that of mesh current, element current is same as mesh current. If the assumed current direction in such an element is opposite to that of mesh current, element current is same as negative of mesh current. If an element is shared by two meshes, its current is given by the difference between the two mesh currents with due attention to be placed on current directions.

The procedure of Mesh Analysis is now illustrated using the circuit in Fig. 4.7-4 as an example. Three mesh currents i_1 , i_2 and i_3 are assigned in the three meshes in clockwise direction as shown. The KVL equations for the three meshes are written now with element equations employed to convert the voltage variables into mesh current variables. We follow a convention in writing these KVL equations. We start at the leftmost corner of the mesh and traverse the mesh in clockwise direction. We enter the voltages we meet with in a sum. A voltage is entered in the sum with the same polarity as its first polarity marking that we meet during our traversal- if we meet its positive polarity first we enter it with positive sign and if we meet its negative polarity first we enter it with negative sign.

The mesh equation for first mesh is derived below.

The first voltage that we meet is that of V_1 . We meet its negative polarity first. Therefore, $-V_1$ gets into the equation. Then, we see the voltage across R_1 with positive polarity first. The current through it is same as the mesh current i_1 and hence $R_1 i_1$ enters the equation. The next voltage we meet with is that of R_2 with its negative polarity first. The current through R_2 is $i_2 - i_1$ in the direction assumed for it in the diagram. Hence, $-R_2(i_2 - i_1)$ enters the equation. The last voltage we meet with in first mesh is that of V_2 , positive polarity coming first. Hence, $+V_2$ enters the equation. Thus, the mesh equation for the first mesh is

$$\begin{aligned} -V_1 + R_1 i_1 - R_2(i_2 - i_1) + V_2 &= 0 \\ \text{i.e., } (R_1 + R_2)i_1 - R_2 i_2 &= V_1 - V_2 \end{aligned}$$

With our convention of traversing a mesh in clockwise direction, the mesh current variable in the mesh where KVL is being applied will appear with positive sign and other mesh current variables will appear with negative sign in the equation. And the *net rise in voltage contributed by all the independent voltage sources* in that mesh will appear with *positive sign on the right side* of equation. The remaining two mesh equations for this circuit are

$$\begin{aligned}
 &R_2(i_2 - i_1) + R_3i_2 - R_4(i_3 - i_2) = V_2 - V_3 \\
 \text{i.e.,} \quad &-R_2i_1 + (R_2 + R_3 + R_4)i_2 - R_4i_3 = V_2 - V_3 \\
 &R_4(i_3 - i_2) + R_5i_3 = V_3 - V_4 \\
 \text{i.e.,} \quad &-R_4i_2 + (R_4 + R_5)i_3 = V_3 - V_4
 \end{aligned}$$

We can solve for i_1 , i_2 and i_3 using the three equations listed below.

$$\begin{aligned}
 (R_1 + R_2)i_1 - R_2i_2 + 0i_3 &= V_1 - V_2 \\
 -R_2i_1 + (R_2 + R_3 + R_4)i_2 - R_4i_3 &= V_2 - V_3 \\
 0i_1 - R_4i_2 + (R_4 + R_5)i_3 &= V_3 - V_4
 \end{aligned}$$

We followed a certain convention in writing these mesh equations in Eqn. 4.2-2. Adhering to such a convention has resulted in certain symmetry in these equations. Let us express these equations in matrix notation to see the symmetry clearly.

$$\begin{bmatrix} (R_1 + R_2) & -R_2 & 0 \\ -R_2 & (R_2 + R_3 + R_4) & -R_4 \\ 0 & -R_4 & (R_4 + R_5) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

$$\text{i.e., } \mathbf{ZI} = \mathbf{DU}$$

where $\mathbf{Z}$ is called the *Mesh Resistance Matrix*, $\mathbf{I}$ is called the *Mesh Current Vector*, $\mathbf{U}$ is called the *Input Vector* and $\mathbf{D}$ is called the *Input Matrix*.

Note that the order of Mesh Resistance Matrix is $(b - n + 1) \times (b - n + 1)$ and that it is *symmetric*. The diagonal element of $\mathbf{Z}$ -matrix, z_{ii} , is the *sum of resistances appearing in the mesh-i*. The off-diagonal element of $\mathbf{Z}$ -matrix, z_{ij} , is the *negative of sum of all resistances appearing in common between mesh-i and mesh-j*. There can be more than one resistance shared between two meshes. Then, they will be in series and they will add in z_{ij} . That is why z_{ij} should be the negative of *sum* of all resistances shared by mesh-i and mesh-j. The right-hand side product $\mathbf{DU}$ is a column vector of *net voltage rise imparted in the mesh by the independent voltage sources in the corresponding meshes*.

Now, we can write down this matrix equation by inspection after skipping all the intermediate steps. The following matrix equation results in the case of the example we considered in this section.

$$\begin{aligned}
 &R_1 = 2\Omega; R_2 = 3\Omega; R_3 = 1\Omega; R_4 = 1\Omega; R_5 = 4\Omega; V_1 = 5\text{ V}; V_2 = 6\text{ V}; V_3 = 2\text{ V}; V_4 = -11\text{ V} \\
 &\begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 13 \end{bmatrix}
 \end{aligned}$$

Solving the matrix equation by Cramer's rule, we get, $i_1 = 1\text{ A}$, $i_2 = 2\text{ A}$ and $i_3 = 3\text{ A}$. The element currents can be obtained by applying KCL at various nodes in the circuit. This can be done by inspection. Resistor voltages may then be obtained by applying Ohm's law. The complete solution of the circuit is shown in Fig. 4.7-5.

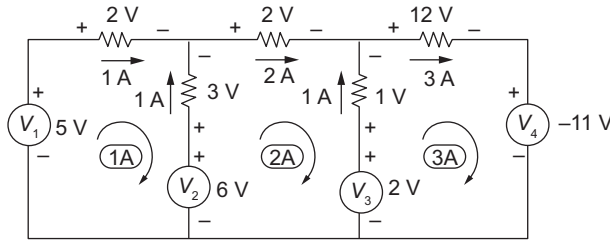


Fig. 4.7-5 Complete mesh analysis solution for the circuit in Fig. 4.7-4

This section has shown that an n -node, b -element circuit containing only linear resistors and independent voltage sources will have a Mesh representation given by $\mathbf{ZI} = \mathbf{DU}$ where $\mathbf{Z}$ is the Mesh Resistance Matrix of order $(b - n + 1) \times (b - n + 1)$, $\mathbf{I}$ is the mesh current column vector of order $(b - n + 1) \times 1$, $\mathbf{U}$ is the source voltage column vector of order $n_{vs} \times 1$ and $\mathbf{D}$ is the input matrix of order $(b - n + 1) \times n_{vs}$. n_{vs} is the number of independent voltage sources in the circuit.

z_{ii} = sum of all resistances appearing in the i^{th} mesh

z_{ij} = negative of sum of all resistances common to i^{th} mesh and j^{th} mesh

$$d_{ij} = \begin{cases} 0 & \text{if } j^{th} \text{ voltage source is not present in the } i^{th} \text{ mesh} \\ 1 & \text{if } j^{th} \text{ voltage source provides voltage rise in the } i^{th} \text{ mesh} \\ -1 & \text{if } j^{th} \text{ voltage source provides voltage drop in the } i^{th} \text{ mesh} \end{cases}$$

Equivalently, the matrix product $\mathbf{DU}$ may be replaced by a column vector that contains the *net voltage rise contributed to a mesh by all voltage sources participating in that mesh*. The Mesh Resistance Matrix will be symmetric for this kind of circuits. The mesh current vector is obtained by Cramer's rule or by Matrix inversion as $\mathbf{I} = \mathbf{Z}^{-1} \mathbf{DU}$.

4.7.2 Is Mesh Current Measurable?

Mesh current of a mesh in a planar circuit is related to the current that flows in the series combination of all those elements that participate only in that mesh *if such elements are present in that mesh*. In such cases, a mesh current is indeed a physical quantity and it can be measured. One can always introduce an ammeter in series with an element that appears only in the concerned mesh and measure the mesh current flowing in that mesh.

However, what if there is no wholly owned element in a particular mesh? For instance, consider the mesh marked as M_k in part of a large circuit, shown in Fig. 4.7-6.

This mesh in circuit Fig. 4.7-6(a) has no element wholly owned by it. The mesh current i_k assigned to this mesh cannot be identified as the current flowing through any of the circuit elements appearing in the mesh. However, let us try to create a *wholly owned* element in this mesh without affecting the circuit solution in any manner. Assume that R_2 is a member of only one mesh. Then, nothing prevents us from changing our viewpoint to that expressed by the circuit in Fig. 4.7-6(b). Here, we have introduced an additional node at the junction between R_2 and R_1 and introduced a short-circuit element in between the new node and the old one. Introduction of a shorting link causes no change in the circuit variables anywhere in the circuit. However, now we identify this newly introduced short-circuit element as the

element *exclusively owned* by mesh M_k and identify the mesh current variable i_k as the current that flows in this element. We can introduce an ammeter there as shown in Fig. 4.7-6(b) and measure i_k .

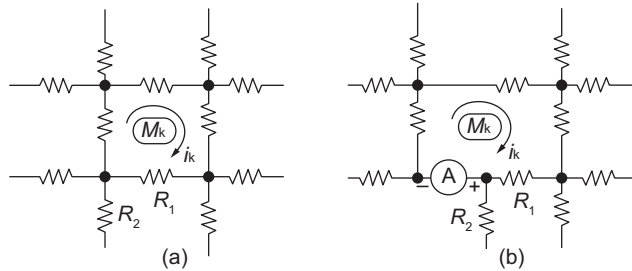


Fig. 4.7-6 Circuit pertaining to measurability of mesh currents

However, even this technique will fail to make the mesh current variable i_k measurable if R_2 is a member of yet another mesh. If the entire periphery of the mesh M_k is shared by some mesh or other, then, a short-circuit element introduced anywhere in the periphery will be shared by some other mesh.

Thus, we conclude that there can be meshes in which mesh current cannot be identified as the current flowing in any element in that mesh. Therefore, in general, mesh current is a ‘fictitious current’ that is not measurable directly. It is a ‘fictitious current’ that can be thought of as ‘flowing around the periphery of the mesh’. Element currents are measurable. Each element current is a combination of two ‘peripheral currents’ or mesh currents. However, these peripheral currents are not always measurable.

However, the KVL equations written for meshes in a planar circuit will form a set of $(b - n + 1)$ independent equations quite independent of whether the mesh currents are measurable or not.

4.8 MESH ANALYSIS OF CIRCUITS WITH INDEPENDENT CURRENT SOURCES

An independent current source that appears in parallel with a resistor can be converted into an independent voltage source in series with a resistor by applying source transformation theorem. This measure will reduce the number of meshes in the circuit by one and increase the number of nodes in the circuit by one. Such current sources in a circuit do not pose any special problem for mesh analysis procedure.

An independent current source may also appear in series with another element. Such a current source cannot be converted into an equivalent voltage source. This kind of current source will impose constraints on the mesh current variables in the circuit.

If an independent current source that cannot be equivalenced to a voltage source participates in only one mesh, then, that mesh current is *fixed* by that current source function. That is, that mesh current is no more a variable. Thus, the degree of freedom of the circuit decreases by one and number of mesh current variables to be determined comes down by one. Moreover, one does not need the mesh equation for that mesh to solve for the remaining mesh current variables. However, that mesh equation will have to be used for finding out the voltage across the current source after all the other mesh current variables have been obtained.

If a current source that cannot be equivalenced to a voltage source is shared by two meshes in a circuit, it constrains the mesh currents in those two meshes to have a difference decided by the current source function. This constraint equation of the form $i_i - i_j = i_s(t)$, where $i_s(t)$ is the value of current

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source, supplies one equation for finding out the two mesh currents. Hence, we will not need both the two mesh equations obtained by applying KVL in the two meshes. In fact, both of them will contain the voltage across the current source as a term and we will have to eliminate that voltage term by adding the two mesh equations to obtain a combined-mesh equation. Thus, for this kind of current source connection, we get one equation from the constraint imposed by the current source and a second one by adding the two mesh equations for the meshes. Finally, one of those two mesh equations will have to be used to determine the voltage across the current source after all mesh current variables have been solved for.

These aspects of Mesh Analysis are illustrated through a series of examples that follow.

Example: 4.8-1

Apply mesh analysis procedure on the circuit in Fig. 4.8-1 (a).

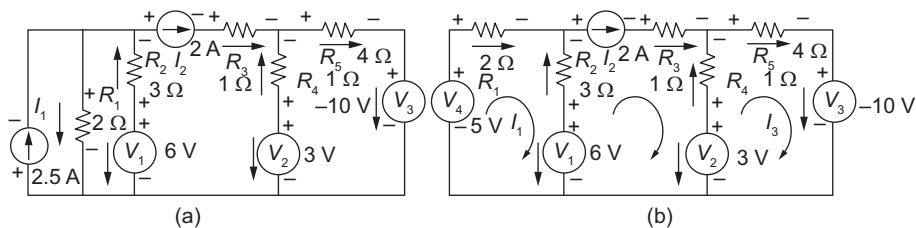


Fig. 4.8-1 (a) Circuit for (b) Circuit after mesh reduction by source transformation

Solution

All the circuit variables are identified following passive sign convention. The labelling of variables will be v or i with the name of element as subscript.

Step-1: If there are current sources that appear directly across resistors convert them into equivalent voltage sources in series with resistors. This is called ‘mesh reduction’.

There is one such combination in circuit of Fig. 4.8-1 (a). It is I_1 (2.5 A) in parallel with R_1 ($2\ \Omega$). We replace these with a voltage source of value $R_1 I_1$ (5 V) in series with R_1 ($2\ \Omega$) with positive polarity at top as shown in circuit of Fig. 4.8-1 (b).

The second current source I_2 in circuit (a) cannot be reduced this way.

Step-2: Assign mesh current variables in the reduced circuit starting with leftmost mesh. Assign a mesh current variable to a mesh only if its mesh current is not constrained directly by a current source and its mesh current is not decided by other mesh current variables already assigned along with current source functions.

There are three meshes in the reduced circuit. The first mesh has no current source directly constraining its mesh current. No other mesh was assigned mesh current variable and hence this mesh current cannot get decided by other already assigned mesh current variables. Therefore, we assign a mesh current variable to this mesh and call it i_1 .

Moving to the second mesh we observe that the independent current source I_2 directly constrains its mesh current to be I_2 A (2 A). Therefore, we do not assign any mesh current variable to this mesh.

There is no current source in the third mesh. Its mesh current cannot be obtained from the already assigned mesh current variables. Therefore, we assign a mesh current variable to this mesh and call it i_3 .

Thus, this circuit has two mesh current variables – i_1 and i_3 – to be solved for.

Step-3: Prepare the mesh equations by applying KVL in meshes, starting at left bottom corner and traversing the mesh in clockwise direction. Ignore the meshes that are directly constrained by current sources. Further, if two meshes share a current source, then, add the mesh equations for those two meshes to generate a new equation that will be used in solution process. The number of equations at the end of this step will be equal to number of meshes – number of irreducible independent current sources.

The two mesh equations are:

$$-V_4 + R_1 i_1 - R_2 (i_2 - i_1) + V_1 = 0 \text{ for mesh-1}$$

$$\text{i.e., } (R_1 + R_2) i_1 - R_2 i_2 = V_4 - V_2 = R_1 I_1 - V_2$$

$$-V_2 + R_4 (i_3 - I_2) + R_5 i_3 + V_3 = 0 \text{ for mesh-3}$$

$$\text{i.e., } (R_4 + R_5) i_3 = R_4 I_2 + V_2 - V_3$$

These mesh equations are expressed in matrix form below.

$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ 0 & R_4 + R_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_3 \end{bmatrix} = \begin{bmatrix} R_1 I_1 - V_2 & 0 \\ 0 & R_4 I_2 + V_2 - V_3 \end{bmatrix}$$

Step-4: Solve the mesh equations by Cramer's rule or by matrix inversion.

Substituting numerical values and simplifying,

$$\begin{bmatrix} 5 & -3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 15 \end{bmatrix}$$

Solving for i_1 and i_3 by Cramer's rule, $i_1 = 1\text{A}$ and $i_3 = 3\text{A}$. i_2 is already known to be 2A .

Step-5: Use the mesh current values and apply KCL at various nodes in the original circuit to obtain element currents and voltages for all resistive elements and current through voltage sources.

Refer to circuits in Fig. 4.8-1. The resistor currents and voltages as per direction and polarity marked in Fig. 4.8-1 is calculated below.

Current through $R_1 = I_1 - i_1 = 1.5\text{A}$, $\therefore$ Voltage across $R_1 = 3\text{V}$

Current through $R_2 = i_2 - i_1 = 1\text{A}$, $\therefore$ Voltage across $R_2 = 3\text{V}$

Current through $R_3 = I_2 = 2\text{A}$, $\therefore$ Voltage across $R_3 = 2\text{V}$

Current through $R_4 = i_3 - I_2 = 1\text{A}$, $\therefore$ Voltage across $R_4 = 1\text{V}$

Current through $R_5 = i_3 = 3\text{A}$, $\therefore$ Voltage across $R_5 = 12\text{V}$

Now, the currents through the voltage sources are

Current through $V_1 = -1\text{A}$, current through $V_2 = -1\text{A}$ and current through $V_3 = 3\text{A}$

Step-6: Use the element voltages calculated in the above step and apply KVL in various meshes in the original circuit to obtain element voltages for all current sources.

Applying KVL in the mesh formed by first current source and the resistor R_1 in the original circuit, we get the voltage across I_1 as -3V .

Applying KVL in the mesh in which the second current source appears, we get the voltage across the current source I_2 as -1V .

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Note that these voltages follow passive sign convention.

The complete circuit solution is shown in Fig. 4.8-2.

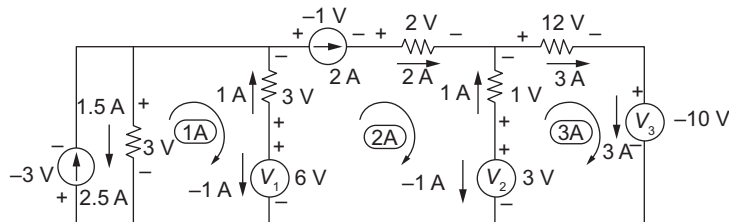


Fig. 4.8-2 Complete mesh analysis solution for circuit in Fig. 4.8-1(a)

Example: 4.8-2

Find the total power dissipated in the circuit and power delivered by each source in Fig. 4.8-3.

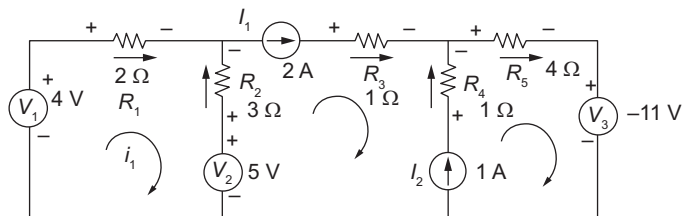


Fig. 4.8-3 Circuit for Example 4.8-2

Solution

There is no current source that can be transformed into a voltage source in this circuit. However, the current source I_1 directly constrains the second mesh current to be 2A. The third mesh current gets constrained to be $I_1 + I_2 = 3A$. Therefore, there is no need to assign mesh current variables in second and third meshes. The mesh current variable i_1 is assigned to first mesh as shown in Fig. 4.8-3.

The mesh equation for the first mesh is $5i_1 = 4V + 3\Omega \times 2A - 5V = 5V \times i_1 = 1A$. The complete solution is shown in Fig. 4.8-4.

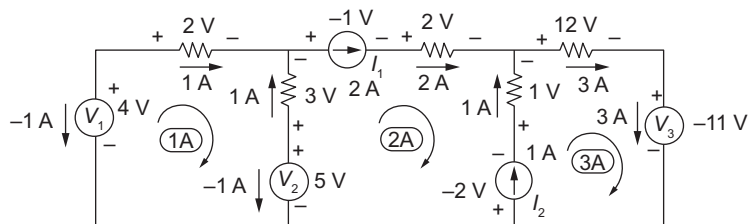


Fig. 4.8-4 Circuit solution for Example 4.8-2

We have followed passive sign convention throughout. Hence, the power dissipated in each resistor is the product of voltage across it and current through it. Total power dissipated in the circuit is found by adding up this product for all the resistors.

$\therefore$ Total power dissipated in the circuit $= 2V \times 1A + 3V \times 1A + 2V \times 2A + 1V \times 1A + 12V \times 3A = 46 \text{ W}$

Power delivered by an element is equal to negative of power dissipated in it. Power dissipated in an element is the product of voltage and current as per passive sign convention. Hence, the power delivered by a source is negative of vi product with v and i marked according to passive sign convention.

$\therefore$ Power delivered by $V_1 = -(4V \times -1A) = 4 \text{ W}$

Power delivered by $V_2 = -(5V \times -1A) = 5 \text{ W}$

Power delivered by $V_3 = -(-11V \times 3A) = 33 \text{ W}$

Power delivered by $I_1 = -(-1V \times 2A) = 2 \text{ W}$

Power delivered by $I_2 = -(-2V \times 1A) = 2 \text{ W}$

The total delivered power is equal to the total dissipated power.

The last two examples have demonstrated that:

- An independent current source imposes a constraint on mesh current variables and reduces their number by one.
- A mesh current variable *need not be* assigned for a mesh if an independent current source determines that mesh current directly or indirectly through another mesh variable assigned to another mesh.
- Mesh equation for a mesh in which the mesh current is fixed directly by an independent current source appearing only in that *is not required for solving other mesh current variables*.
- Mesh equations for two meshes that share an independent current source among them have to be *added* to get a combined mesh equation that will be useful in solving the circuit analysis problem.
- The mesh analysis formulation results in an equation $\mathbf{ZI} = \mathbf{DU}$ where $\mathbf{Z}$ is the Mesh Resistance Matrix of a reduced order circuit resulting from deactivating all independent sources in it. The input vector $\mathbf{U}$ contains all the independent current source functions and independent voltage source functions. The solution for the mesh current vector $\mathbf{I}$ can be written as $\mathbf{Z}^{-1}\mathbf{DU}$. This indicates that each mesh current (and hence all element voltages and currents) can be expressed as a linear combination of input source functions- *i.e.*, $x = a_1I_1 + a_2I_2 + \dots + b_1V_1 + b_2V_2 + \dots$ where x is some mesh current variable or element current/voltage variable and a 's and b 's are coefficients decided by circuit resistances and connection details. Some of the a 's and b 's may turn out to be zero for certain choices of x .
- The voltage across independent current sources can be found out only after the mesh current variables are solved. Finding the voltage across an independent current source requires the application of KVL in a mesh containing that current source.

We had observed that an n -node b -element circuit has only $(b - n + 1)$ mesh current variables. Further, each independent current source reduces the number of mesh current variables to be solved for by one. Then, what happens if there are $(b - n + 1)$ independent current sources in the circuit? Further, what if there are more than that many independent current sources? This leads us to our next example. Consider the following example circuit that has all the three mesh current variables constrained by three independent current sources.

Example: 4.8-3

Solve the circuit in Fig. 4.8-5 by Mesh Analysis.

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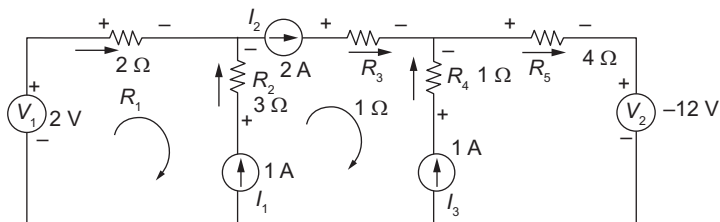


Fig. 4.8-5 Circuit for Example 4.8-3

Solution

The independent current source I_2 constrains the second mesh current to be equal to 2 A. This results in first mesh current getting constrained by the equation $i_1 - i_2 = -1\text{A}$ resulting in $i_1 = 1\text{A}$. The independent current source I_3 along with the current source I_2 imposes a constraint on i_3 resulting in $i_3 = 3\text{A}$. Thus, all the three mesh current variables are constrained by the independent current sources. There is no mesh current variable to be determined.

If the sources in this circuit are deactivated, the resulting network will have *no closed loops*. This is yet another feature of a fully constrained circuit.

Elements other than the current sources have nothing to do with mesh currents in this circuit. They affect only the voltages that appear across the current sources. These voltages can be found by applying KVL in the meshes.

Applying KVL to the first mesh,

$$-2\text{V} + 2\Omega \times 1\text{A} - 3\Omega \times (2\text{A} - 1\text{A}) - v_{I_1} = 0 \Rightarrow v_{I_1} = -3\text{V}$$

Applying KVL to the third mesh,

$$v_{I_3} + 1\Omega \times (3\text{A} - 2\text{A}) + 4\Omega \times 3\text{A} - 12\text{V} = 0 \Rightarrow v_{I_3} = -1\text{V}$$

Applying KVL to the second mesh,

$$v_{I_1} + 3\Omega \times (2\text{A} - 1\text{A}) + v_{I_2} + 1\Omega \times 2\text{A} - 1\Omega \times (3\text{A} - 2\text{A}) - v_{I_3} = 0 \Rightarrow v_{I_2} = -2\text{V}$$

The currents through resistors, voltages across them and currents through voltage sources can be found by inspection. The complete solution is marked in Fig. 4.8-6.

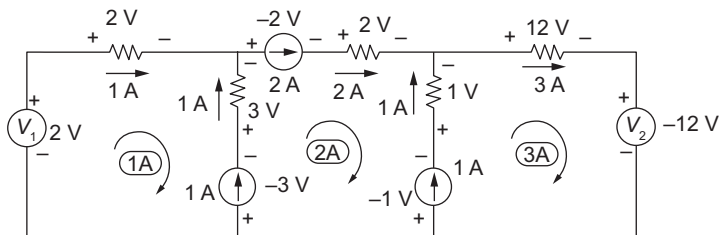


Fig. 4.8-6 Complete solution for circuit in Example 4.8-3

Three independent current sources in a three-mesh circuit need not result in a fully constrained circuit necessarily. For instance, assume that the source I_3 is shifted and connected in series with R_1 in Fig. 4.8-5. We note that the three independent current sources will have to satisfy the condition that $I_3 + I_1 - I_2 = 0$ due to KCL constraint at the node between R_1 and R_2 . This constraint is satisfied with the values used in the present example. However, the reader may verify that it will be impossible to determine the voltages that appear across the current sources in this case though the mesh currents can be determined. The circuit will have many solutions.

The constraint above need not be satisfied by any three arbitrary current sources. If they do not, then they are trying to violate KCL. That will imply that circuit has no solution. In practice, there will be a solution since each practical current source will have some finite resistance across it. But then, the circuit becomes a different one.

Similar conclusions will follow for a case with more than $(b - n + 1)$ independent current sources in an n -node b -element circuit – either the circuit cannot be solved due to inconsistencies in KCL equations or the circuit cannot be solved uniquely.

Thus, the maximum number of independent current source that can be there in an n -node b -element circuit with unique solution is $(b - n + 1)$.

4.9 MESH ANALYSIS OF CIRCUITS CONTAINING DEPENDENT SOURCES

Dependent voltage sources do not pose any problem for mesh analysis. Independent voltage sources appear on the right-hand side of mesh equations. However, dependent voltage sources affect the coefficients of mesh current variables in the left-hand side of mesh equations.

The controlling variable of a linear dependent current source will be a voltage or current existing elsewhere in the circuit. But, any voltage or current variable in the circuit can be expressed in terms of mesh current variables. Hence, the dependent current source function can be expressed in terms of mesh current variables. Therefore, dependent voltage sources will affect the coefficients of mesh equation, *i.e.*, they will change the Mesh Resistance Matrix. We will see that they can destroy the symmetry of the Mesh Resistance Matrix.

Dependent current sources place constraints on mesh current variables the same way independent current sources do. They cause a reduction in the number of mesh currents to be determined in the circuit. They too affect the coefficients of mesh equations and make Mesh Resistance Matrix asymmetric.

Example: 4.9-1

Apply mesh analysis on the circuit in Fig. 4.9-1.

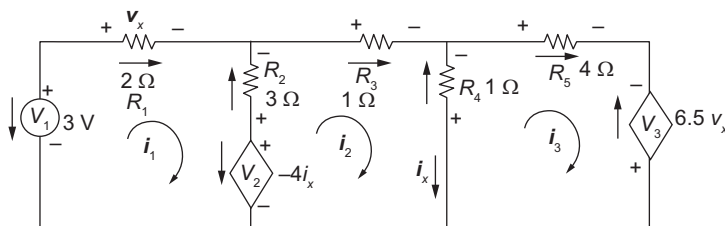


Fig. 4.9-1 Circuit for Example 4.9-1

Solution

Step-1: Carry out mesh reduction by employing source transformation, if relevant.

There are no current sources appearing in parallel with any resistor. Hence, no mesh reduction is possible.

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Step-2: Identify meshes and assign mesh current variables.

There are three meshes and there is no current source to impose any constraints on them. Hence, all the three meshes are assigned mesh current variables.

Step-3: Identify the controlling variable of dependent sources in terms of mesh current variables and express dependent source functions in terms of mesh current variables.

i_x is the controlling variable of V_2 . i_x is the current flowing in R_4 from top to bottom and hence it is equal to $(i_2 - i_3)$. Therefore, the source function of V_2 is $k_1(i_2 - i_3)$ with $k_1 = -4$.

v_x is the controlling variable for V_3 . v_x is the voltage across R_1 . Therefore, $v_x = R_1 i_1$ and the source function of V_3 is $k_2 R_1 i_1$ with $k_2 = 6.5$.

Step-4: Prepare the mesh equations and solve them.

The mesh equations are written with the dependent source functions expressed in terms of mesh current variables.

$$\begin{aligned} \text{Mesh-1} \quad & -V_1 + R_1 i_1 - R_2(i_2 - i_1) + k_1(i_2 - i_3) = 0 \\ \text{i.e.,} \quad & (R_1 + R_2)i_1 - (R_2 - k_1)i_2 - k_1 i_3 = V_1 \\ \text{Mesh-2} \quad & -k_1(i_2 - i_3) + R_2(i_2 - i_1) + R_3 i_2 - R_4(i_3 - i_2) = 0 \\ \text{i.e.,} \quad & -R_2 i_1 + (R_2 + R_3 + R_4 - k_1)i_2 - (R_4 - k_1)i_3 = 0 \\ \text{Mesh-3} \quad & R_4(i_3 - i_2) + R_5 i_3 - k_2 R_1 i_1 = 0 \\ \text{i.e.,} \quad & -k_2 R_1 i_1 - R_4 i_2 + (R_4 + R_5)i_3 = 0 \end{aligned}$$

These equations are expressed in matrix form below.

$$\begin{bmatrix} (R_1 + R_2) & -(R_2 - k_1) & -k_1 \\ -R_2 & (R_2 + R_3 + R_4 - k_1) & -(R_4 - k_1) \\ -k_2 R_1 & -R_4 & (R_4 + R_5) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [V_1]$$

Note the asymmetry in Mesh Resistance Matrix.

Substituting the numerical values and solving for mesh currents by Cramer's rule,

$$\begin{bmatrix} 5 & -7 & 4 \\ -3 & 9 & -5 \\ -13 & -1 & 5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [3] \Rightarrow i_1 = 1\text{A}, i_2 = 2\text{A} \text{ and } i_3 = 3\text{A}$$

Step-5: Apply KCL at various nodes of the circuit to find all the element currents and resistor voltages.

Consider R_4 . Applying KCL at the node formed by R_3 , R_4 and R_5 , we get the current flowing from top to bottom in R_4 as $i_2 - i_3$. However, the reference direction that was chosen for current in R_4 is from bottom to top. Hence, current in $R_4 = -(i_2 - i_3)$ in the direction marked in Fig. 4.9-1. The value is 1A.

This makes the value of i_x equal to -1A and hence the dependence voltage source V_2 source function becomes $-4 \times -1 = 4\text{V}$.

Consider R_1 . The reference direction for its current as marked in Fig. 4.9-1 is from left to right and is in the same direction as the first mesh current. R_1 is wholly owned by first mesh and hence current through it is i_1 itself. The value is 1A. That makes the voltage across R_1 equal to 2V. Moreover, the value of v_x is also 2V. Therefore, the source function of the dependent voltage V_1 is $6.5 \times 2 = 13\text{V}$.

The remaining voltage and current variables also may be evaluated similarly. The complete circuit solution is shown in Fig. 4.9-2.

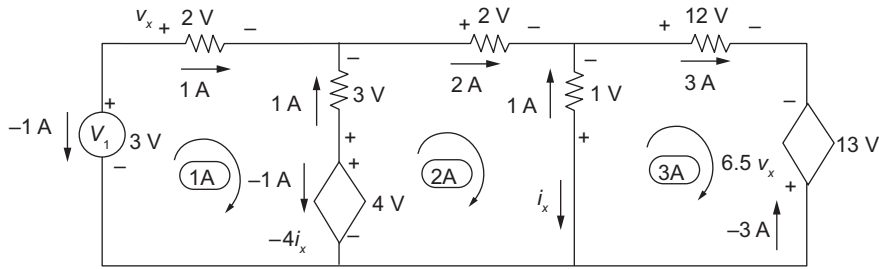


Fig. 4.9-2 Complete mesh analysis solution for Example 4.9-1

Example: 4.9-2

Apply mesh analysis on the circuit in Example 4.9-2.

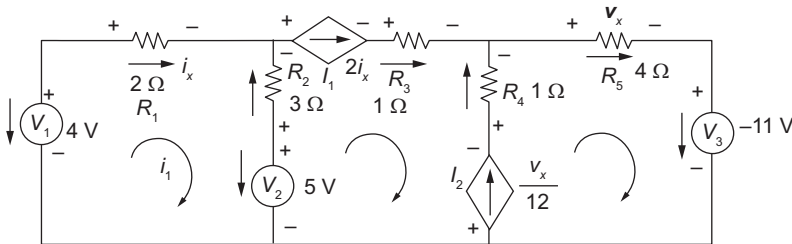


Fig. 4.9-3 Circuit for Example 4.9-2

Solution

Step-1: Carry out mesh reduction by employing source transformation, if relevant.

There are no current sources appearing in parallel with any resistor. Hence, no mesh reduction is possible.

Step-2: Identify meshes and assign mesh current variables.

The second mesh current is directly constrained by dependent current source I_1 and the third mesh current is indirectly constrained by $i_3 = I_2 + I_1$. Hence, there is only one mesh current variable and that is i_1 in the first mesh.

Step-3: Identify the controlling variable of dependent sources in terms of mesh current variables and express dependent source functions in terms of mesh current variables.

i_x is the controlling variable of I_1 . i_x is the current flowing in R_1 from left to right and hence it is equal to i_1 . Therefore, the source function of I_1 is $2i_1$ A.

v_x is the controlling variable for I_2 . v_x is the voltage across R_5 . Therefore, $v_x = R_5(I_2 + I_1) = R_5(I_2 + i_1) = 2i_1 + v_x/3$.

$\therefore v_x = 12i_1$ and the source function of $I_2 = i_1$ A.

Step-4: Prepare the mesh equations and solve them.

The mesh equation for the first mesh is written with the dependent source functions expressed in terms of mesh current variables.

$$-4 + 2i_1 - 3(2i_1 - i_1) + 5 = 0$$

i.e.,

$$i_1 = 1$$

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Therefore, the mesh currents are $i_1 = 1\text{ A}$, $i_2 = 2\text{ A}$ and $i_3 = 3\text{ A}$.

Step-5: Apply KCL at various nodes of the circuit to find all the element currents and resistor voltages.

Consider R_4 . Applying KCL at the node formed by R_3 , R_4 and R_5 , we get the current flowing from top to bottom in R_4 as $i_2 - i_3$. However, the reference direction that was chosen for current in R_4 is from bottom to top. Hence, current in $R_4 = -(i_2 - i_3)$ in the direction marked in The value is 1 A.

Currents through other resistors and voltage sources may be obtained in a similar manner.

Step-6: Apply KVL in various meshes to obtain the voltage across the current sources.

Apply KVL to the third mesh first.

$$v_{i_2} + 1 \times (3 - 2) + 4 \times 3 + (-11) = 0 \Rightarrow v_{i_2} = -2\text{ V}$$

Now, apply KVL to the second mesh.

$$-5 + 3 \times (2 - 1) + v_{i_1} + 1 \times 2 - 1(3 - 2) - v_{i_2} = 0 \Rightarrow v_{i_1} = -1\text{ V}$$

The complete circuit solution is shown in Fig. 4.9-4.

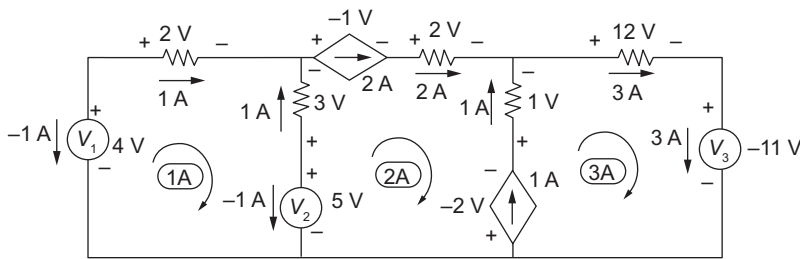


Fig. 4.9-4 Complete mesh analysis solution for the circuit in Example 4.9-2

4.10 SUMMARY

- This chapter dealt with two systematic procedures for solving the circuit analysis problem in the case of memoryless circuits. Memoryless circuits contain linear resistors, linear dependent sources and independent sources.
- Circuit analysis problem for an n -node, b -element circuit involves the determination of b element voltage variables and b element current variables, given the source functions of all independent sources present in the circuit.
- *Node voltage is the voltage of a node in a circuit with respect to a chosen reference node in the circuit.* In Nodal Analysis, KVL equations are used to show that all element voltages can be expressed in terms of $(n - 1)$ node voltages and KCL equations along with element relations are used subsequently to set up the $(n - 1)$ node equations needed for determining the node voltages. Nodal Analysis is applicable to any circuit that has a unique solution.
- *Mesh current is a fictitious current that flows in clockwise direction in the internal periphery of a mesh.* In Mesh Analysis, the KCL equations are used to show that all element currents can be expressed in terms of a reduced set of $(b - n + 1)$ specially defined currents called *mesh currents*. Subsequently, $(b - n + 1)$ KVL equations involving these currents are set up to determine them. Mesh Analysis is applicable only to circuits that are planar.

- Source Transformation Theorem is a valuable aid in both analysis procedures. It states that a voltage source $v_S(t)$ in series with a resistance R_S can be replaced by a current source $i_S(t) = v_S(t) / R_S$ in parallel with R_S without affecting any voltage/current/power variable external to the source. The direction of current source is such that current flows out of the terminal at which the positive of the voltage source is presently connected.
- The procedure for *Nodal Analysis* of a circuit containing linear resistors, linear dependent sources and independent sources is summarized below.

Nodal Analysis Procedure

Step-1: Assign reference current directions and reference polarities for voltages of all elements as per passive sign convention. Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation on such combinations to convert them into current sources in parallel with resistors. This is called 'node reduction'. The resulting circuit is referred to as the reduced circuit.

Step-2: Select a reference node. Assign node voltage variables at those nodes where the node voltage variable is not decided directly by a voltage source (independent or dependent source) or indirectly by already assigned node voltage variables and voltage source functions.

Step-3: Identify the controlling variables of dependent sources in terms of the node voltage variables assigned in the last step and express the source functions of dependent sources in terms of node voltage variables.

Step-4: Prepare the node equations for the reduced circuit. Ignore node equation at nodes where voltage sources (independent or dependent sources) are connected directly to reference node. Combine (add) the node equations at the end nodes of voltage sources (independent or dependent sources) connected between two non-reference nodes. The number of equations at the end of this step will be equal to number of nodes minus number of irreducible *independent* voltage sources.

Step-5: Solve for node voltage variables by elimination technique. The equations may also be expressed as a matrix equation and solved by using Cramer's rule or matrix inversion.

Step-6: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and voltage across current sources. This is done by applying KVL in various loops in the circuit along with Ohm's law for linear resistors.

Step-7: Use appropriate node equations to solve for currents through the independent and dependent voltage sources.

- The procedure for *Mesh Analysis* of a circuit containing linear resistors, linear dependent sources and independent sources is summarized below.

Mesh Analysis Procedure

Step-1: Assign reference current directions and reference polarities for voltages of all elements as per passive sign convention. If there are current sources that appear directly across resistors convert them into equivalent voltage sources in series with resistors. This is called 'mesh reduction'. The resulting circuit is referred to as the reduced circuit.

Step-2: Assign mesh current variables in the reduced circuit starting with leftmost mesh. Assign mesh current variables to a mesh only if its mesh current is not constrained directly by a current source (independent or dependent) and its mesh current is not decided by other mesh current variables already assigned along with current source functions.

Step-3: Prepare the mesh equations by applying KVL in meshes, starting at left bottom corner and traversing the mesh in clockwise direction. Ignore the meshes that are directly constrained by current sources (independent or dependent). Further, if two meshes share a current source, then, add the mesh equations for those two meshes to generate a new equation that will be used in solution process. The number of equations at the end of this step will be equal to number of meshes minus number of irreducible *independent* current sources.

Step-4: Identify the controlling variable of dependent sources in terms of mesh current variables and express dependent source functions in terms of mesh current variables.

Step-5: Solve the mesh equations by elimination technique. The equations may also be expressed as a matrix equation and solved by using Cramer's rule or matrix inversion.

Step-6: Use the mesh current values and apply KCL at various nodes in the original circuit to obtain element currents and voltages for all resistive elements (use Ohm's law) and current through voltage sources.

Step-7: Use the element voltages calculated in the above step and apply KVL in various meshes in the original circuit to obtain element voltages for all current sources.

- An n -node circuit containing only linear resistors and independent current sources will have a Nodal representation given by $\mathbf{YV} = \mathbf{CU}$ where $\mathbf{Y}$ is the nodal conductance matrix of order $(n - 1) \times (n - 1)$, $\mathbf{V}$ is the node voltage column vector of order $(n - 1) \times 1$, $\mathbf{U}$ is the source current column vector of order $n_{cs} \times 1$ and $\mathbf{C}$ is the input matrix of order $(n - 1) \times n_{cs}$. n_{cs} is the number of independent current sources in the circuit. The matrix product $\mathbf{CU}$ may be replaced by a column vector which contains the *net current delivered to a node by all independent current sources connected at that node*. The Nodal Conductance Matrix will be symmetric for this kind of circuits.

y_{ii} = sum of all conductances connected at i^{th} node

y_{ij} = negative of sum of all conductances connected between i^{th} node and j^{th} node

$$c_{ij} = \begin{cases} 0 & \text{if } j^{th} \text{ current source is not connected at } i^{th} \text{ node} \\ 1 & \text{if } j^{th} \text{ current source is delivering current into } i^{th} \text{ node} \\ -1 & \text{if } j^{th} \text{ current source is drawing current from } i^{th} \text{ node} \end{cases}$$

- The nodal analysis formulation results in an equation $\mathbf{YV} = \mathbf{CU}$ where $\mathbf{Y}$ is the nodal conductance matrix of the reduced order circuit (if node reduction is possible) resulting from deactivating all independent sources in it. $\mathbf{Y}$ will be *symmetric* if there are no dependent sources in the circuit. Dependent sources *can* make $\mathbf{Y}$ *asymmetric*. The input vector $\mathbf{U}$ contains all the independent current source functions and independent voltage source functions. The solution for the node voltage vector $\mathbf{V}$ can be written as $\mathbf{Y}^{-1}\mathbf{CU}$.

- Each node voltage (and hence all element voltages and currents) can be expressed as a linear combination of input source functions- i.e., $x = a_1 I_1 + a_2 I_2 + \dots + b_1 V_1 + b_2 V_2 + \dots$ where x is some node voltage variable or element current/voltage variable and a 's and b 's are coefficients decided by circuit conductances and connection details. Some of the a 's and b 's may turn out to be zero for certain choices of x .
- An n -node, b -element circuit containing only linear resistors and independent voltage sources will have a Mesh representation given by $\mathbf{ZI} = \mathbf{DU}$ where $\mathbf{Z}$ is the Mesh Resistance Matrix of order $(b - n + 1) \times (b - n + 1)$, $\mathbf{I}$ is the mesh current column vector of order $(b - n + 1) \times 1$, $\mathbf{U}$ is the source voltage column vector of order $n_{vs} \times 1$ and $\mathbf{D}$ is the input matrix of order $(b - n + 1) \times n_{vs}$. n_{vs} is the number of independent voltage sources in the circuit. The matrix product $\mathbf{DU}$ may be replaced by a column vector which contains the *net voltage rise contributed to a mesh by all voltage sources participating in that mesh*. The Mesh Resistance Matrix will be symmetric for this kind of circuits.

z_{ii} = sum of all resistances appearing in the i^{th} mesh

z_{ij} = negative of sum of all resistances common to i^{th} mesh and j^{th} mesh

$$d_{ij} = \begin{cases} 0 & \text{if } j^{\text{th}} \text{ voltage source is not present in the } i^{\text{th}} \text{ mesh} \\ 1 & \text{if } j^{\text{th}} \text{ voltage source provides voltage rise in the } i^{\text{th}} \text{ mesh} \\ -1 & \text{if } j^{\text{th}} \text{ voltage source provides voltage drop in the } i^{\text{th}} \text{ mesh} \end{cases}$$

- The mesh analysis formulation results in an equation $\mathbf{ZI} = \mathbf{DU}$ where $\mathbf{Z}$ is the Mesh Resistance Matrix of a reduced order circuit (if mesh reduction is possible) resulting from deactivating all independent sources in it. $\mathbf{Z}$ will be *symmetric* if there are no dependent sources in the circuit. Dependent sources can make $\mathbf{Z}$ *asymmetric*. The input vector $\mathbf{U}$ contains all the independent current source functions and independent voltage source functions. The solution for the mesh current vector $\mathbf{I}$ can be written as $\mathbf{Z}^{-1}\mathbf{DU}$.
- Each mesh current (and hence all element voltages and currents) can be expressed as a linear combination of input source functions- i.e., $x = a_1 I_1 + a_2 I_2 + \dots + b_1 V_1 + b_2 V_2 + \dots$ where x is some mesh current variable or element current/voltage variable and a 's and b 's are coefficients decided by circuit resistances and connection details. Some of the a 's and b 's may turn out to be zero for certain choices of x .

4.11 PROBLEMS

- Find the power delivered by the -7A current source in the circuit in Fig. 4.11-1 by nodal analysis.

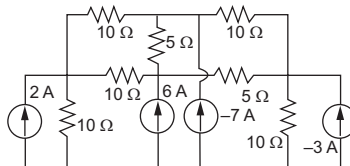
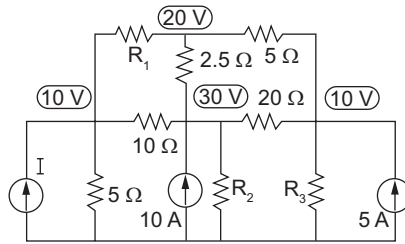


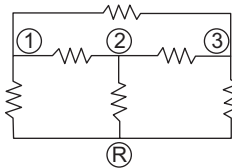
Fig. 4.11-1

- The node voltages in the circuit in Fig. 4.11-2 are marked in the figure. Find the values for R_1 , R_2 , R_3 and I .

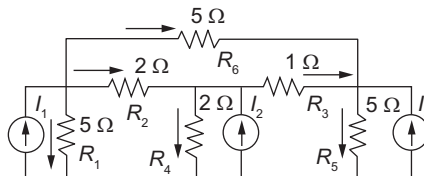
**Fig. 4.11-2**

3. The nodal conductance matrix of the circuit in Fig. 4.11-3 is given below. Find the values of all resistances in the circuit.

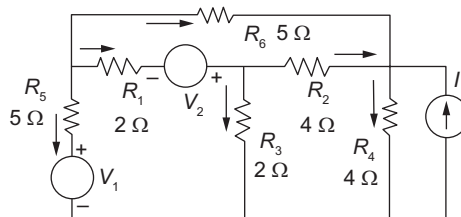
$$Y = \begin{bmatrix} 0.1 & -0.02 & -0.04 \\ -0.02 & 0.06 & -0.02 \\ -0.04 & -0.02 & 0.12 \end{bmatrix} \text{S}$$

**Fig. 4.11-3**

4. (i) Express all the currents marked in the circuit in Fig. 4.11-4 as linear combinations of I_1 , I_2 and I_3 by nodal analysis. (ii) Solve for I_1 , I_2 and I_3 such that currents through R_2 , R_3 and R_6 are zero. With these values of current sources, find the currents through remaining resistors.

**Fig. 4.11-4**

5. (i) Express all the currents marked in the circuit in Fig. 4.11-5 as linear combinations of V_1 , V_2 and I by nodal analysis. (ii) Solve for V_1 , V_2 and I such that currents through R_1 , R_5 and R_6 are zero. With these values of current sources, find the currents through remaining resistors and the node voltages with respect to bottommost node.

**Fig. 4.11-5**

6. Find V_1 , V_2 and V_3 such that $v_1 = 10$ V, $v_2 = 10$ V and $v_3 = 20$ V in the circuit in Fig. 4.11-6. With these values of V_1 , V_2 and V_3 , find the power delivered by all voltage sources and power dissipated by all resistors. Use nodal analysis.

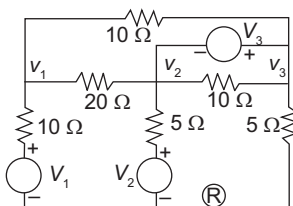


Fig. 4.11-6

7. (i) The nodal conductance matrix of the circuit in Fig. 4.11-7 is given as

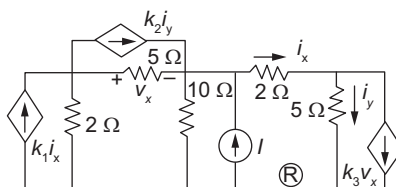


Fig. 4.11-7

$$Y = \begin{bmatrix} 0.1 & -0.3 & 0.13 \\ -0.2 & 0.8 & 0.47 \\ 0.1 & -0.6 & 0.7 \end{bmatrix} \text{ S. Find } k_1, k_2 \text{ and } k_3 \text{ and solve the circuit completely by nodal analysis if } I = 1 \text{ A.}$$

8. Find k_1 , k_2 and k_3 such that the nodal conductance matrix for the circuit in Fig. 4.11-8 is lower triangular. Find the power delivered by independent sources and dependent sources. Use nodal analysis.

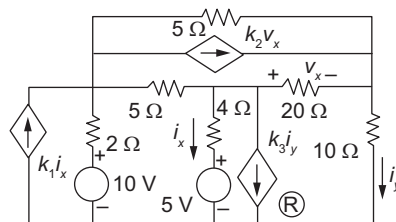


Fig. 4.11-8

9. Find all dependent source coefficients such that the Y -matrix of the circuit in Fig. 4.11-9 is diagonal.

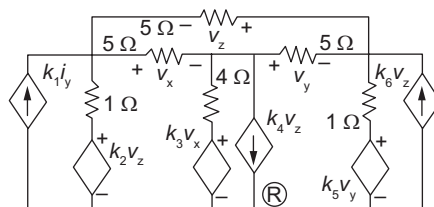


Fig. 4.11-9

4.48 | Nodal Analysis and Mesh Analysis of Memoryless Circuits

10. Find k such that v is zero in the circuit in Fig. 4.11-10. Solve the circuit completely with this value of k . Use nodal analysis.

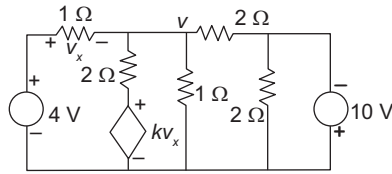


Fig. 4.11-10

11. Find k such that v is zero in the circuit in Fig. 4.11-11. Solve the circuit completely for this value of k . Use nodal analysis.

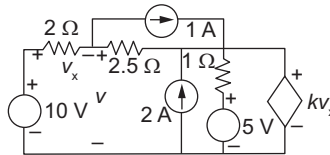


Fig. 4.11-11

12. Find the node voltages and resistor currents in the circuit in Fig. 4.11-12 by nodal analysis.

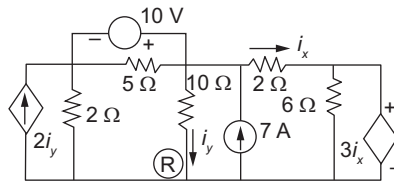


Fig. 4.11-12

13. The nodal conductance matrix of the circuit in Fig. 4.11-13 is given below. Find the values of all resistances in the circuit.

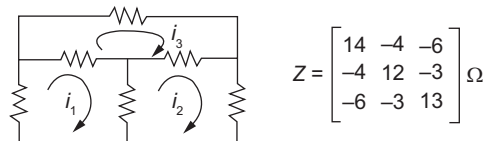


Fig. 4.11-13

14. Express all the resistor currents in the directions as marked in the form linear combinations of V_1 , V_2 and V_3 for the circuit shown in Fig. 4.11-14. Use Mesh Analysis.

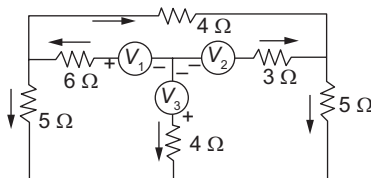


Fig. 4.11-14

15. (i) Express v in the circuit in Fig. 4.11-15 as a linear combination of V_1 , V_2 and I . (ii) Find I such that $v = 0$ if $V_1 = V_2 = 5$ V. (iii) Solve the circuit completely with these source values. Use Mesh Analysis.

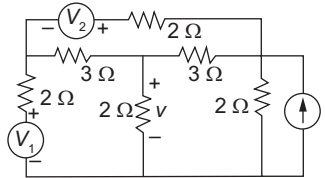


Fig. 4.11-15

16. All resistors in the circuit in Fig. 4.11-16 are of 2Ω . Find currents in all resistors and voltage across current sources by mesh analysis.

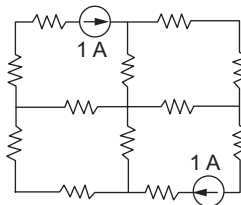


Fig. 4.11-16

17. Find the current delivered to the 12.5 V source, power delivered by all voltage sources and power dissipated in all resistors by Mesh Analysis on the circuit in Fig. 4.11-17.

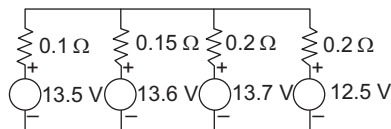


Fig. 4.11-17

18. Repeat the above problem (Problem 17) by nodal analysis.
 19. Repeat Problem 5 by using mesh analysis.
 20. Solve the circuit in Fig. 4.11-18 completely.

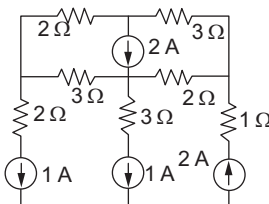
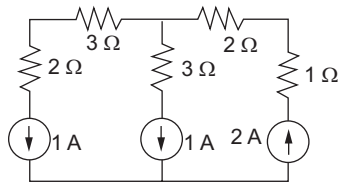
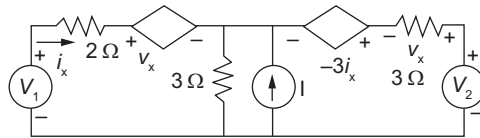


Fig. 4.11-18

21. Can the circuit in Fig. 4.11-19 be solved uniquely? If yes, find the solution. If no, find at least two solutions.

**Fig. 4.11-19**

22. Solve Problem 11 by mesh analysis.
23. Express the resistor currents as linear combinations of V_1 , V_2 in the circuit in Fig. 4.11-20. Use mesh analysis.

**Fig. 4.11-20**

Chapter 5

Circuit Theorems

CHAPTER OBJECTIVES

- To derive Superposition Theorem from the property of linearity of elements.
- To explain the two key theorems – Superposition Theorem and Substitution Theorem in detail.
- To derive other theorems like Compensation Theorem, Thevenin's Theorem, Norton's Theorem, Reciprocity Theorem and Maximum Power Transfer Theorem from these two key principles.
- To provide illustrations of applications of circuit theorems in circuit analysis through solved examples.
- To emphasise the use of Compensation Theorem, Thevenin's Theorem and Norton's Theorem in circuits containing dependent sources as a pointer to their applications in the study of electronic circuits.

INTRODUCTION

Circuit Analysis involves determination of element voltages and currents for all elements of the circuit using element equations and interconnection equations. Kirchhoff's Current Law equations at all nodes and Kirchhoff's Voltage Law equations in all loops along with element v - i relationship equations will yield the necessary set of equations.

However, we need systematic procedures for exploiting these equations. Node analysis and mesh analysis were two such systematic procedures we took up for detailed study in the last chapter. In this chapter, we discuss some circuit theorems and circuit transformations that increase our efficiency in solving circuits and that render further insight into certain features of a *linear* circuit. These theorems constitute a basic set of tools that enhance the analyst's efficiency in solving circuits.

The previous chapter showed the following:

- (1) All the element voltages and element currents in a circuit can be obtained from its node voltages that are governed by a matrix equation $\mathbf{YV} = \mathbf{CU}$, where $\mathbf{V}$ is the node voltage column

5.2 | Circuit Theorems

vector, $\mathbf{Y}$ is the nodal conductance matrix of the circuit, $\mathbf{U}$ is the input column vector containing source functions of all *independent* voltage sources and current sources in the circuit and $\mathbf{C}$ is the input matrix. The elements of $\mathbf{Y}$ -matrix are decided by the values of conductances in the circuit and values of coefficients of linear dependent sources in the circuit. It is a symmetric matrix if there are no dependent sources in the circuit. Dependent sources *can* render $\mathbf{Y}$ -matrix asymmetric. The $\mathbf{C}$ matrix will, in general, contain 0, 1, -1 and conductance values as well as dependent source coefficients.

- (2) An alternative formulation is given by a matrix equation $\mathbf{ZI} = \mathbf{DU}$, where $\mathbf{I}$ is the mesh current column vector, $\mathbf{Z}$ is the mesh resistance matrix of the circuit, $\mathbf{U}$ is the input column vector containing source functions of all *independent* voltage sources and current sources in the circuit and $\mathbf{D}$ is the input matrix. All the element voltages and element currents in a circuit can be obtained from its mesh currents. The elements of $\mathbf{Z}$ -matrix are decided by the values of resistances in the circuit and values of coefficients of linear dependent sources in the circuit. It is a symmetric matrix if there are no dependent sources in the circuit. Dependent sources *can* render $\mathbf{Z}$ -matrix asymmetric. The $\mathbf{D}$ matrix will, in general, contain 0, 1, -1 and resistance values as well as dependent source coefficients.
- (3) Any response variable in a circuit (*i.e.*, any element voltage or current or combination thereof) can be expressed as a *linear combination* of source functions of independent voltage sources and current sources present and active in the circuit, *i.e.*, $x = a_1 I_1 + a_2 I_2 + \dots + b_1 V_1 + b_2 V_2 + \dots$, where x is some chosen response variable and a 's and b 's are coefficients decided by circuit conductances/resistances, dependent source coefficients and interconnection details. Some of the a 's and b 's may turn out to be zero for certain choices of x .

These three basic properties of a circuit comprising linear elements are recast into various useful theorems that simplify the circuit analysis procedure in practical contexts. That is, the so-called circuit theorems are more or less restatements of these basic facts. These three observations were arrived at by analysis of memoryless circuits. However, we will show in later chapters that they are true for dynamic circuits too. Therefore, all important circuit theorems that we arrive at in this chapter will be also valid for dynamic circuits.

The crucial fact the reader should keep in mind is that we are stating nothing more in almost all circuit theorems (in many of them anyway) than what is already contained in the three properties described earlier.

5.1 LINEARITY OF A CIRCUIT AND SUPERPOSITION THEOREM

Consider a purely resistive circuit driven by two independent current sources and an independent voltage source shown in Fig. 5.1-1.

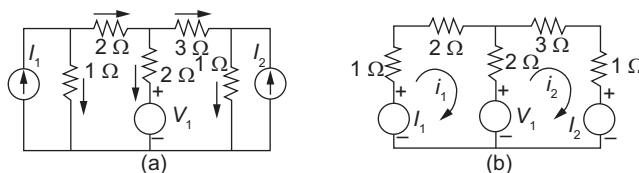


Fig. 5.1-1 (a) Circuit with three independent sources (b) Circuit after source transformation

The mesh equations for the circuit in Fig. 5.1-1 (b) in matrix form can be derived as follows:

$$\begin{bmatrix} 5 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V_1 \end{bmatrix}$$

Solving the matrix equation, we get

$$\begin{aligned} i_1 &= \frac{3}{13} I_1 - \frac{1}{13} I_2 - \frac{2}{13} V_1 \\ i_2 &= \frac{1}{13} I_1 - \frac{5}{26} I_2 + \frac{3}{26} V_1 \end{aligned}$$

Now, any element voltage or current can be expressed in terms of these two mesh currents. For example, consider the current in resistor in the central limb in the direction shown in circuit of Fig. 5.1-1 (a). This current is obtained by applying KCL at the junction between the three resistors and is $= i_1 - i_2 = \frac{2}{13} I_1 + \frac{3}{26} I_2 - \frac{7}{26} V_1$.

Consider the current in the 1Ω across the current source I_1 . This is obtained by applying KCL at the node where I_1 , 2Ω and 1Ω are connected together. The current in 2Ω is same as the first mesh current in circuit of Fig. 5.1-1 (b) and hence the current in $1\Omega = I_1 - i_1 = \frac{10}{13} I_1 + \frac{1}{13} I_2 + \frac{2}{13} V_1$.

Currents in all elements can be worked out in a similar manner. These currents are marked in Fig. 5.1-2 using a notation where the three numbers in brackets show the coefficients of I_1 , I_2 and V_1 , respectively. Or, they can be interpreted as the current components when all the three sources have *unit values*. Consider the numbers marked for the voltage source V_1 as $\left(\frac{2}{13}, \frac{3}{26}, -\frac{7}{26}\right)$. This means that I_1 contributes $\frac{2}{13}$ A per unit A to this current, I_2 contributes $\frac{3}{26}$ A per unit A to this current and V_1 contributes $-\frac{7}{26}$ A per unit V to this current.

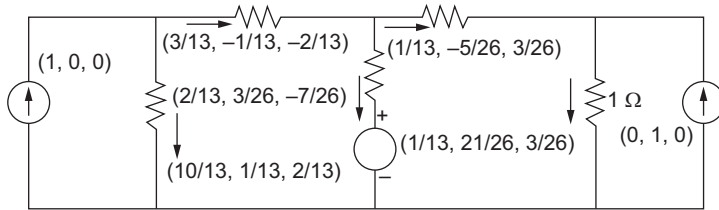


Fig. 5.1-2 Currents in various elements in the circuit of Fig. 5.1-1 (a)

Thus, each element current (and element voltage too) is made up of three components. Each independent source contributes one component to each element current and voltage. The components contributed by all the independent sources add together to form the total response.

That the total currents in elements and total voltage across them will satisfy KCL and KVL respectively is only to be expected. In fact, that was the basis for node analysis and mesh analysis. However, what is not so obvious is that *components contributed by a particular independent source to all element currents will satisfy KCL at all nodes without depending in any way on the components*

5.4 | Circuit Theorems

provided by other independent sources. Similarly, components contributed by a particular independent source to all element voltages will satisfy KVL in all loops without depending in any way on the components provided by other independent sources. This may be verified easily in the example that was analysed in this section.

The implication from this observation is that the components contributed by a particular independent source to circuit variables are the same as the solution of the circuit when that source is acting alone without the other sources present. In other words, the contribution of a particular source to a circuit variable does not change when some other source(s) is acting simultaneously.

This can also be understood in another way. We had seen that all mesh current variables and node voltage variables (and hence all voltage variables current variables in the circuit) for a circuit can be expressed as linear combinations of independent source functions. The solution for mesh currents in the circuit in Fig. 5.1-1 was

$$\begin{aligned} i_1 &= \frac{3}{13} I_1 - \frac{1}{13} I_2 - \frac{2}{13} V_1 \\ i_2 &= \frac{1}{13} I_1 - \frac{5}{26} I_2 + \frac{3}{26} V_1 \end{aligned}$$

Thus, there are three contributions in each variable. The coefficient involved in each contribution is a constant. Its value does not depend on the particular values that the sources happen to assume. It depends only on the resistance values, structure of the circuit and the location where the particular independent source is connected. Therefore, we expect the coefficients of I_1 to remain at $\frac{3}{13}$ and $\frac{1}{13}$ in i_1 and i_2 , respectively, irrespective of the values I_2 and V_1 happen to have. And, we choose to think of the situation when I_2 and V_1 are zero-valued. Then the circuit has only one source and it will produce $i_1 = \frac{3}{13} I_1$ and $i_2 = \frac{1}{13} I_1$, since the coefficients are independent of source values. Similarly, the circuit will have $i_1 = -\frac{1}{13} I_2$ and $i_2 = -\frac{5}{26} I_2$ when $I_1 = V_1 = 0$, and $i_1 = -\frac{2}{13} V_1$ and $i_2 = \frac{3}{26} V_1$ when $I_1 = I_2 = 0$.

Thus, we can view the solution of the circuit, when I_1 , I_2 and V_1 are simultaneously acting, as the sum or *superposition* of solution of three identical circuits with only one of the sources taking a non-zero value in each circuit. *A current source with zero-value is an open-circuit and a voltage source with zero value is a short-circuit.* Hence, in the first circuit we replace I_2 by an open-circuit and V_1 by a short-circuit and solve it to get the first contribution due to I_1 . In the second circuit, we replace I_1 by an open-circuit and V_1 by a short-circuit and solve it to get the second contribution due to I_2 . And in the third circuit we replace I_1 by an open-circuit and I_2 by an open-circuit and solve it to get the third contribution due to V_1 . We add the contributions to get the solution for the original circuit in which all the three sources were acting simultaneously. *We can do this only because the contributions from various sources stand segregated in the form of a linear combination with no interaction among them.*

Thus, a multi-source circuit problem can be split into many single-source circuit problems, and the response for any circuit variable in the multi-source circuit can be found as the *superposition* (i.e., sum) of responses for same circuit variable in all those single-source circuits. Hence, we will be constructing the final solution by piecing together the individual contributions from independent sources.

We systematise this further. We note that the contribution from any particular independent source to a particular response variable is proportional to the source function value. We have been terming the proportionality constant as a *coefficient of contribution*. In $x = a_1 I_1 + a_2 I_2 + \dots + b_1 V_1 + b_2 V_2 + \dots$, where x is some circuit response variable and $I_1, I_2, \dots$ are the independent current source functions and $V_1, V_2, \dots$ are the independent voltage source functions, the a 's and b 's are the proportionality constants or the so-called coefficients of contribution. They are the ones that matter, not the particular values of source functions. Each coefficient can be interpreted as the *contribution to the circuit variable due to*

unit value of a particular input source, i.e., contribution per unit input. If we know the *contribution per unit input* for each source, we can find out the contribution due to that source by a simple *scaling* operation that involves multiplying *contribution per unit input* by the source function value.

Now we are ready to state the different forms of *Superposition Theorem*.

The response of any circuit variable in a multi-source *linear memoryless* circuit containing n independent sources can be obtained by adding the responses of the same circuit variable in n single-source circuits with i^{th} single-source circuit formed by keeping only i^{th} independent source active and all the remaining independent sources deactivated.

Deactivation of an independent current source is achieved by replacing it with an open-circuit and deactivation of an independent voltage source is achieved by replacing it with a short-circuit. *Dependent sources are not to be treated as sources while applying Superposition Theorem. They will be present in all the single-source component circuits.*

The response of any circuit variable x in a multi-source *linear memoryless* circuit containing n independent sources can be expressed as $x(t) = \sum_{i=1}^{i=n} a_i U_i(t)$, where $U_i(t)$ is the source function of i^{th} independent source (can be a voltage source or current source) and a_i is its *coefficient of contribution*. The coefficient of contribution has the physical significance of *contribution per unit input*.

The principle embodied in the above can also be stated in the following manner:

The coefficient of contribution, a_i , which is a constant for a time-invariant circuit, can be obtained by solving for $x(t)$ in a single-source circuit in which all independent sources other than the i^{th} one are deactivated by replacing independent voltage sources with short-circuits and independent current sources with open-circuits.

But, why should a linear combination $x = a_1 I_1 + a_2 I_2 + \dots + b_1 V_1 + b_2 V_2 + \dots$ be found term by term always? Can't we get it in subsets that contain more than one term? The third form of *Superposition Theorem* states that it can be done.

The response of any circuit variable in a multi-source *linear memoryless* circuit containing n independent sources can be obtained by adding responses of the same circuit variable in two or more circuits with each circuit keeping a subset of independent sources active in it and remaining sources deactivated such that there is no overlap between the such active source subsets among them.

5.1.1 Linearity of a Circuit

A circuit is called *linear* if its solution follows superposition principle. This is why we stated the Superposition Theorem with the adjective *linear* preceding 'circuit'. Whether we view the boxed statements on Superposition as a definition of linearity of a circuit or as a statement of an important property of linear circuits is matter of viewpoint. There is indeed a bit of circularity in *Linearity* and *Superposition Principle*.

5.6 | Circuit Theorems

Why did the memoryless circuits we have been dealing with till now obey superposition principle? The elements of memoryless circuits were constrained to be linear time-invariant elements (except the independent sources). We used only *linear* resistors and *linear* dependent sources. The v - i relationship of all those elements obey superposition principle. Therefore, all KCL and KVL equations in nodal analysis and mesh analysis had the form of linear combinations. Such KVL and KCL equations lead to nodal conductance matrix (and mesh resistance matrix) that contain only constants in the case of a time-invariant circuit (*i.e.*, resistances are constants and coefficients of dependent sources are also constants). Similarly, the input matrix ($\mathbf{C}$ in nodal analysis and $\mathbf{D}$ in mesh analysis) will contain only constants in the case of circuits constructed using linear time-invariant elements. Thus, the solution for node voltage variables and mesh current variables will come out in the form of linear combination of independent source functions. And, after all, Superposition Theorem is only a restatement of this fact. Therefore, Superposition Theorem holds in the circuit since we used only linear elements in constructing it except for independent sources that are non-linear. Hence, we conclude that *a memoryless circuit constructed from a set of linear resistors, linear dependent sources and independent sources (they are non-linear elements) results in a circuit which obeys Superposition Theorem and hence, by definition, is a linear circuit.*

Linearity of a circuit element and linearity of a circuit are two different concepts. An element is linear if its v - i relationship obeys principle of homogeneity and principle of additivity. A circuit is linear, if all circuit variables in it, without any exception, obey principle of homogeneity and principle of additivity, *i.e.*, the principle of superposition. It may appear intuitively obvious that a circuit containing only linear elements will turn out to be a linear circuit. However, remember that we did use non-linear elements – independent sources are non-linear elements. Then, it is not so obvious. The earlier discussion offers a plausibility reasoning to convince us that a circuit containing linear elements and independent sources will indeed be a linear circuit. However, the mathematical proof for this apparently straightforward conclusion is somewhat formidable.

Linearity and superposition appear so natural to us. But the fact is that most of the practical electrical and electronic circuits are non-linear in nature. Linearity, at best, is only an approximation that circuit analysts employ to make the analysis problem more tractable. We illustrate why Superposition Theorem does not hold for a circuit containing a non-linear element by an example. See circuit in Fig. 5.1-3 (a). The resistor R in it is a non-linear one with a v - i relationship given by $v = 2i^2$, for $i \geq 0$, and $-2i^2$, for $i < 0$.

The circuit is solved by writing the KVL equation in the first mesh. We first make use of KCL at the current source node to obtain the current through the 1Ω resistor as $i - I$ A. Then, KVL in the first mesh gives

$$-V + (i - I) + 2i^2 = 0 \Rightarrow i = 0.25[\sqrt{1 + 8(V + I)} - 1] \text{ A}$$

The value of this current for $V = 1 \text{ V}$ and $I = 1 \text{ A}$ is 0.78 A . Corresponding voltage across the non-linear resistor is $2i^2 \approx 1.22 \text{ V}$ and the remaining circuit variables can now be obtained easily. The complete solution is marked in circuit of Fig. 5.1-3 (b).

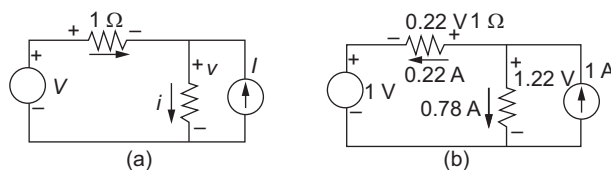


Fig. 5.1-3 (a) A circuit containing a non-linear resistor (b) Circuit solution for $V = 1 \text{ V}$ and $I = 1 \text{ A}$

We find out the circuit solution when the independent sources are acting one by one. Fig. 5.1-4 show the relevant sub-circuits and solution.

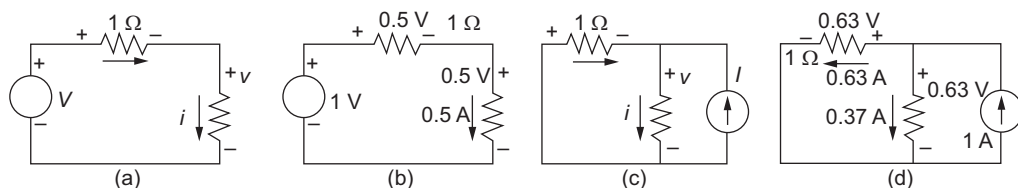


Fig. 5.1-4 Circuits with one independent source acting at a time and circuit solution

The circuit in Fig. 5.1-4 (a) is solved by using the KVL equation $-V + i + 2i^2 = 0$. The solution for i will be $i = 0.25[\sqrt{1+8V} - 1]$ A. The solution for a case with $V = 1$ V and $I = 1$ A is marked in circuit of Fig. 5.1-4 (b).

The circuit in Fig. 5.1-4 (c) is solved by using the KCL equation at the current source node $2i^2 + 2i - I = 0$. The solution for i will be $i = 0.5[\sqrt{1+2I} - 1]$ A. The solution for a case with $V = 1$ V and $I = 1$ A is marked in circuit of Fig. 5.1-4 (d).

We observe that the current through the non-linear resistor when both sources are acting simultaneously is 0.78 A, whereas the sum of responses from two circuits (circuit of Fig. 5.1-4 (a) and (c)) is $0.5 \text{ A} + 0.37 \text{ A} = 0.87 \text{ A}$. Thus Superposition does not work in this circuit.

In general, $0.25[\sqrt{1+8(V+I)} - 1] \neq 0.25[\sqrt{1+8V} - 1] + 0.5[\sqrt{1+2I} - 1]$, and hence this circuit does not obey Superposition Theorem. We also note that it is not possible to identify the contributions from the independent voltage source and independent current source separately when the two sources are acting simultaneously. We may try expanding the $\sqrt{1+8(V+I)}$ term in the solution for i in binomial series. Then we get,

$$i = [(V+I) - 0.25(V+I)^2 + \dots] = V + I - 0.25V^2 - 0.25I^2 - 0.5VI + \dots$$

Thus, i is decided by V and I through their higher powers along with first power terms. Higher power terms cannot satisfy superposition principle. Moreover, there are cross product terms such as VI , V^2I , VI^2 , etc. in the expression. We cannot ascribe such terms to voltage source or current source exclusively. We may take the view that they are the contributions from current source. In that case we have to admit that the contribution from the current source to the current i depends on whether the other source is active or not. That kind of dependence results in non-adherence to superposition principle. Thus, we conclude that non-linear elements in a circuit results in the circuit response failing to meet superposition principle due to (i) sources contributing to response variables through their higher powers and (ii) sources contributing jointly to response variables through cross product terms.

Example: 5.1-1

An independent voltage source and an independent current source from outside drive a memoryless circuit containing no independent sources within it as shown in Fig. 5.1-5. The current delivered by the voltage source is found to be 1 A when the current source is disconnected, 2 A when the current source is

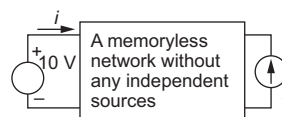


Fig. 5.1-5 Circuit for Example 5.1-1

5.8 | Circuit Theorems

delivering 10 A into the circuit and 0.5 A when the current source is taking out 10 A from the circuit. Is the memoryless circuit a linear one?

Solution

Let us assume that the circuit is linear. Then the current delivered by the 10 V source can be expressed as a linear combination of the two source functions.

i.e., $i = a \times 10 + bI$. The source function value of voltage source has been substituted in this equation. i is given to be 1 A when $I = 0$, and 2 A when $I = 10$ A. Therefore, $a = 0.1$ A/V and $b = 0.1$ A/A if the circuit is linear.

Then, when $I = -10$ A, the current delivered by the voltage source must be $0.1 \times 10 - 0.1 \times 10 = 0$ A. But it is stated that the current observed under this condition is 0.5 A.

Hence, the circuit within the box is not linear.

Example: 5.1-2

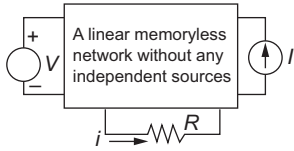


Fig. 5.1-6 Circuit for Example 5.1-2

A certain resistor R in a linear memoryless network driven by two independent sources as shown in Fig. 5.1-6 is found to dissipate 36 W when only the voltage source is acting and 64 W when only the current source is acting. Find the power dissipated in the resistor when both sources are acting simultaneously. Is the answer unique?

Solution

Since the network is linear, the current through the resistor R can be expressed as a linear combination of V and I .

$$\therefore i = aV + bI$$

The power dissipated, *i.e.*, $i^2 R$, is given as 36 W when $I = 0$ and 64 W when $V = 0$.

$$\therefore P_v = (aV)^2 R = 36 \text{ and } P_i = (bI)^2 R = 64$$

$$\therefore (aV) = \sqrt{\frac{P_v}{R}} \text{ and } (bI) = \sqrt{\frac{P_i}{R}}$$

The power that will be dissipated when both sources are acting simultaneously is given by

$$\begin{aligned} P_{vi} &= (aV + bI)^2 R \\ &= (aV)^2 R + (bI)^2 R + 2(aV)(bI)R \\ &= P_v + P_i + 2\sqrt{\frac{P_v}{R}}\sqrt{\frac{P_i}{R}}R \\ &= P_v + P_i + 2\sqrt{P_v P_i} = 36 + 64 + 2\sqrt{36 \times 64} = 196 \text{ W} \end{aligned}$$

Note that the power dissipated when both sources are acting is not the sum of powers dissipated when one source is acting at a time, *i.e.*, power is not a superposable quantity. The reason is very simple – $(i_1 + i_2)^2 \neq i_1^2 + i_2^2$ and $(v_1 + v_2)(i_1 + i_2) \neq v_1 i_1 + v_2 i_2$.

The power calculated as 196 W is not a unique answer. Power dissipated in a resistor when a certain current is flowing through it is independent of direction of the current since power depends on square of the resistor current. We have unconsciously assumed that both a and b are positive or negative. But we have to account for the possibility of a and b having opposite signs, *i.e.*, the possibility of two current contributions canceling each other partially. This possibility is taken into account by modifying the total power equation as $P_{vi} = P_v + P_i \pm 2\sqrt{\frac{P_v}{R}}\sqrt{\frac{P_i}{R}}R$. Hence, the second possible value of power when both sources are acting simultaneously is 4 W.

Additional information in the form of current values or voltage values will be needed to decide between 196 and 4 W.

Example: 5.1-3

The source function values for the three independent sources in Fig. 5.1-7 are $V = 10$ V, $I_1 = 1$ A and $I_2 = 2$ A. The current in the resistor R is seen to be 1 A when the two current sources are switched off and 1.5 A when only I_2 is switched off and 2 A when all the three sources are active. Find what voltage must be applied by the voltage source if the current in R is to become zero with no change in current source values?

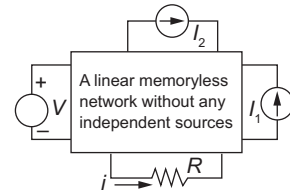


Fig. 5.1-7 Circuit for Example 5.1-3

Solution

The current through R can be written as a linear combination of three source functions.

$$i.e., \quad i = aV + bI_1 + cI_2$$

Substituting the data stated in the problem we get three equations in three unknowns as follows:

$$1 = 10a$$

$$1.5 = 10a + b$$

$$2 = 10a + b + 2c$$

Solving this system of equations, $a = 0.1$, $b = 0.5$ and $c = 0.25$.

$$\therefore i = 0.1V + 0.5I_1 + 0.25I_2$$

The required voltage to make i zero with $I_1 = 1$ A and $I_2 = 2$ A is obtained by

$$0 = 0.1V + 0.5 + 0.5 \Rightarrow V = -10 \text{ V}$$

Example: 5.1-4

Find the value of V in the circuit in Fig. 5.1-8, such that the voltage source delivers zero power to the circuit by using Superposition Theorem.

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Solution

The circuit is a linear one. Therefore, the current delivered by the voltage source can be expressed as a linear combination of the three source functions. Power delivered by the voltage source will be zero if the current delivered by it is zero. Thus, we want a value of V such that $aV + bI_1 + cI_2 = 0$, where a , b and c are the per unit contributions to the current delivered by the voltage source from the three source functions.

We determine the three contributions first by solving three single-source circuits shown in Fig. 5.1-9.

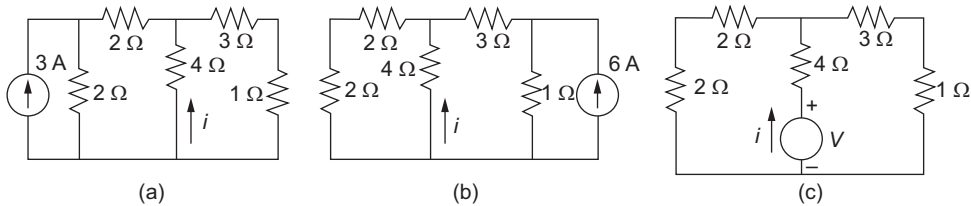


Fig. 5.1-9 Sub-circuits for applying Superposition Theorem in Example 5.1-4

The value of i in circuit of Fig 5.1-9 (a) is found to be $-3A \times \frac{2}{2 + (2 + 4/(3+1))} \times \frac{3+1}{4 + (3+1)} = -0.5A$

The value of i in circuit of Fig 5.1-9 (b) is found to be $-6A \times \frac{1}{1 + (3 + 4/(2+2))} \times \frac{2+2}{4 + (2+2)} = -0.5A$

The value of i in circuit of Fig 5.1-9 (c) is found to be $\frac{V}{4 + (2 + 2)/(3+1)} = \frac{V}{6} A$

Therefore, current delivered by voltage source when all the three sources are active = $-0.5 - 0.5 + \frac{V}{6} = \frac{V-6}{6} A$.

Therefore, the value of V such that the current (and power) delivered by the voltage source will be zero = 6 V.

We note that we did not have to resort to node analysis or mesh analysis in solving the three single-source circuits shown in Fig. 5.1-9.

Example: 5.1-5

Find the power dissipated in the resistor R_2 in the circuit in Fig. 5.1-10 by applying Superposition Theorem.

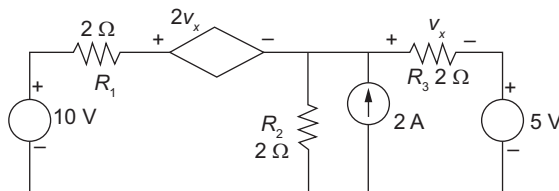


Fig. 5.1-10 Circuit for Example 5.1-5

Solution

We find the current through R_2 by applying Superposition Theorem first. The single-source circuit when the first voltage source is acting is shown in Fig. 5.1-11.

This circuit is solved by KVL in the first mesh. Let the first mesh current be i_1 . Then $v_x = 2//2 i_1 = i_1$.

$-10 + 2i_1 + 2i_1 + i_1 = 0 \Rightarrow i_1 = 2\text{A}$, and hence $i = 1\text{A}$

The single-source circuit when the second voltage source is acting alone is shown in circuit of Fig. 5.1-12 (a).

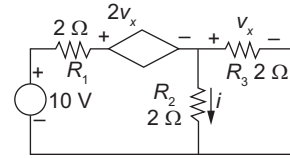


Fig. 5.1-11 Circuit with only the first voltage source acting in Example 5.1-5

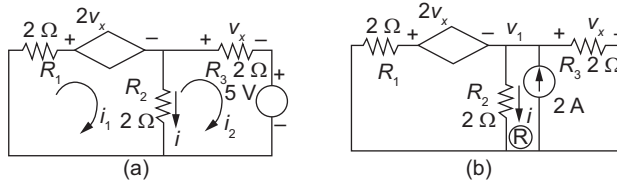


Fig. 5.1-12 Circuits with only (a) the second voltage source acting and (b) only the current source is acting in Example 5.1-5

This circuit is solved by mesh analysis. The controlling variable v_x of the dependent source is $2i_2$. The mesh equations are

$$\begin{aligned} -2i_1 + 4i_2 + 2(i_1 - i_2) &= 0 \\ 2(i_2 - i_1) + 2i_2 + 5 &= 0 \end{aligned}$$

Simplifying the equations lead to $2i_1 + i_2 = 0$ and $-2i_1 + 4i_2 = -5$. Solving these we get $i_1 = 0.5\text{A}$ and $i_2 = -1\text{A}$. Therefore, $i = i_1 - i_2 = 1.5\text{A}$.

The circuit in Fig. 5.1-12 (b) shows the single-source circuit when only the current source is acting. This circuit is solved by nodal analysis. The node voltage v_1 is assigned as shown in circuit diagram. The controlling variable of dependent source is same as v_1 . Therefore, the potential at the right end of R_1 is $3v_1$ with respect to the reference node. Writing KCL at the top node, $0.5v_1 + 0.5v_1 + 0.5 \times 3v_1 = 2 \Rightarrow 2.5v_1 = 2 \Rightarrow v_1 = 0.8\text{V}$. Therefore, $i = 0.4\text{A}$.

Therefore the current in R_2 when all the three sources are acting simultaneously is $= 1 + 1.5 + 0.4 = 2.9\text{A}$

Therefore, the power dissipated in $R_2 = 2 \times 2.9^2 = 16.82\text{W}$.

5.2 STAR-DELTA TRANSFORMATION THEOREM

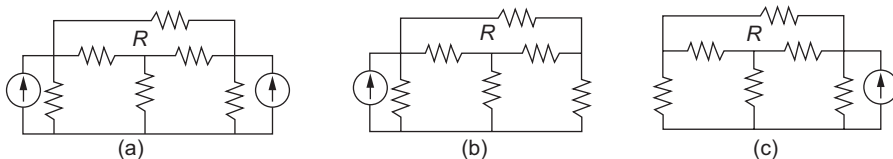


Fig. 5.2-1 Circuit in which Start-Delta Transformation will be helpful

5.12 | Circuit Theorems

We observe from the examples on application of Superposition Theorem in the last section that we may have to use nodal analysis and mesh analysis often to solve the single-source circuits. However, we can expect to avoid these procedures in the case of circuits involving only resistors and independent sources. We will be able to solve the single-source circuits by employing series and parallel equivalents repeatedly. However, there is one pair of resistor connections that will not yield to this kind of approach. For instance, consider the problem of finding the current through the resistor R in circuit of Fig. 5.2-1 (a) by applying superposition principle. The relevant single-source circuits are shown in Fig. 5.2-1 (b) and (c).

This problem cannot be solved by series–parallel equivalents. The T-shaped (also called Y-shaped or Star-connected) resistor network containing three resistors makes it impossible to apply series–parallel reduction. Equivalently, the three outer resistors which are connected in Π form (also called Delta-connected, mesh-connected etc.) makes it impossible to apply series–parallel reduction.

It turns out that a Y-connected set of three resistors can be replaced with a Δ -connected set of three resistors without any circuit variable outside these three resistors getting affected. Similarly, a set of three resistors connected in Δ can be replaced with a set of resistors connected in Star without affecting the circuit solution in the remaining portion of the circuit. We develop equations for this transformation in this section.

First, we consider Star–Delta Transformation. We want the two resistor networks shown in Fig. 5.2-2 to be equivalent with respect the external network.

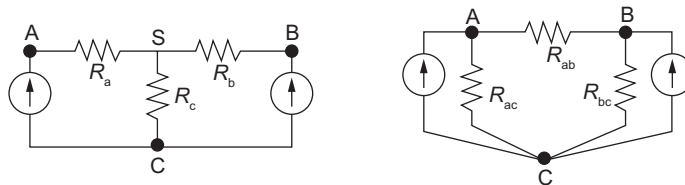


Fig. 5.2-2 Circuits related to Star–Delta Transformation

The net effect of the external network on the star connected resistor may be modelled by two current sources driving it as shown. If the second network that is connected in delta produces same node voltages at node-A and node-B with respect to node-C when driven by the same two current sources as the star network, the external circuit solution will not be affected in any sense. This is so since node voltage variables in a circuit decide all other voltages and currents in a circuit. If the node voltage variables do not get affected, then no voltage or current in the circuit gets affected. Therefore, we can derive the values for resistors in delta network in terms of resistor values in star network by imposing the condition that v_A and v_B with respect to node-C must be the same in both circuits when driven by same current sources.

Let I_1 and I_2 be the current source functions. Then v_A and v_B in star circuit is given by

$$\begin{bmatrix} G_a & 0 & -G_a \\ 0 & G_b & -G_b \\ -G_a & -G_b & G_a + G_b + G_c \end{bmatrix} \begin{bmatrix} v_A \\ v_B \\ v_S \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix}$$

We do not need the node voltage at the node-S. We eliminate it easily since there is no current source injection at that node.

$$-G_a v_A - G_b v_B + (G_a + G_b + G_c) v_S = 0$$

$$\therefore v_S = \frac{G_a}{G_a + G_b + G_c} v_A + \frac{G_b}{G_a + G_b + G_c} v_B$$

Substituting the expression for v_S in the first two node equations, rearranging terms and expressing the final equations in matrix form, we get

$$\begin{bmatrix} \frac{G_a(G_b + G_c)}{(G_a + G_b + G_c)} & \frac{-G_a G_b}{(G_a + G_b + G_c)} \\ \frac{-G_a G_b}{(G_a + G_b + G_c)} & \frac{G_b(G_a + G_c)}{(G_a + G_b + G_c)} \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (5.2-1)$$

Now we write the node equation for the delta-connected network:

$$\begin{bmatrix} G_{ac} + G_{ab} & -G_{ab} \\ -G_{ab} & G_{ab} + G_{bc} \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (5.2-2)$$

Equations 5.2-1 and 5.2-2 should result in same v_A and v_B for network equivalence. This requires that the two nodal conductance matrices be equal. Therefore,

$$\begin{aligned} G_{ab} &= \frac{G_a G_b}{G_a + G_b + G_c} \\ G_{ac} &= \frac{G_a(G_b + G_c)}{G_a + G_b + G_c} - \frac{G_a G_c}{G_a + G_b + G_c} = \frac{G_a G_c}{G_a + G_b + G_c} \\ G_{bc} &= \frac{G_b(G_a + G_c)}{G_a + G_b + G_c} - \frac{G_a G_b}{G_a + G_b + G_c} = \frac{G_a G_c}{G_a + G_b + G_c} \end{aligned} \quad (5.2-3)$$

Expressing this in terms of resistances,

$$\begin{aligned} R_{ab} &= \frac{R_a R_b + R_b R_c + R_c R_a}{R_c} \\ R_{ac} &= \frac{R_a R_b + R_b R_c + R_c R_a}{R_b} \\ R_{bc} &= \frac{R_a R_b + R_b R_c + R_c R_a}{R_a} \end{aligned} \quad (5.2-4)$$

Equation 5.2-4 shows how the resistance values for equivalent delta may be calculated from resistance values used in star network.

Figure 5.2-3 shows the Star-Delta Transformation in a way that makes the symmetry in the equations evident.

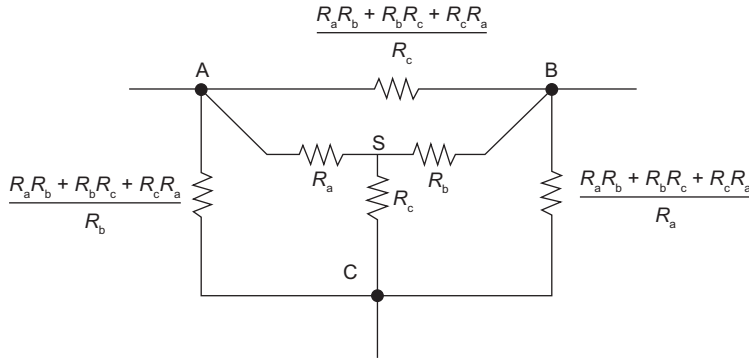


Fig. 5.2-3 Star-Delta Transformation equations

The Delta-Star Transformation may similarly be derived using an approach based on mesh analysis. Assume that a pair of independent voltage sources drive both circuits. Then the currents drawn from the sources must be the same in both circuits. The delta network will have three meshes; however, the central mesh has no voltage source in it and hence its mesh current can be eliminated by using the technique we employed in this section. The details of this derivation are skipped and the result is given as

$$R_a = \frac{R_{ab} R_{ac}}{R_{ab} + R_{bc} + R_{ac}}, R_b = \frac{R_{ab} R_{bc}}{R_{ab} + R_{bc} + R_{ac}}, R_c = \frac{R_{ac} R_{bc}}{R_{ab} + R_{bc} + R_{ac}} \quad (5.2-5)$$

Equation 5.2-5 shows how the resistance values of the equivalent star network may be obtained from the delta network resistances and Fig. 5.2-4 shows this in a manner that makes the symmetry in these equations evident.

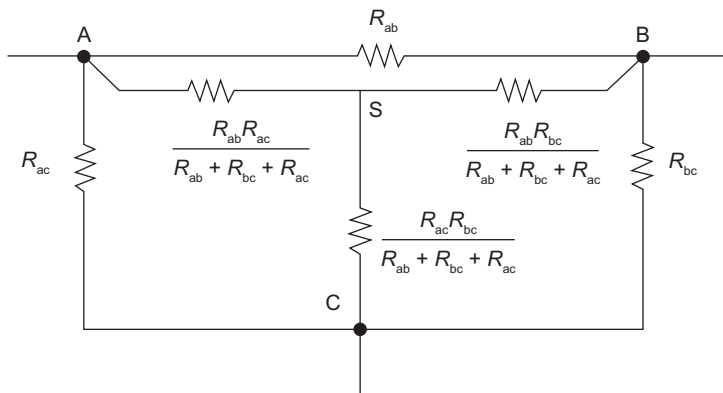


Fig. 5.2-4 Delta-star Transformation equations

Star-Delta (also called $Y - \Delta$ or $T - \Pi$ transformation) Transformation is widely employed in analysis of three-phase ac circuits. A case of special interest is that of equal resistors in all limbs of star or delta. The equations for this special case are shown in Fig. 5.2-5.

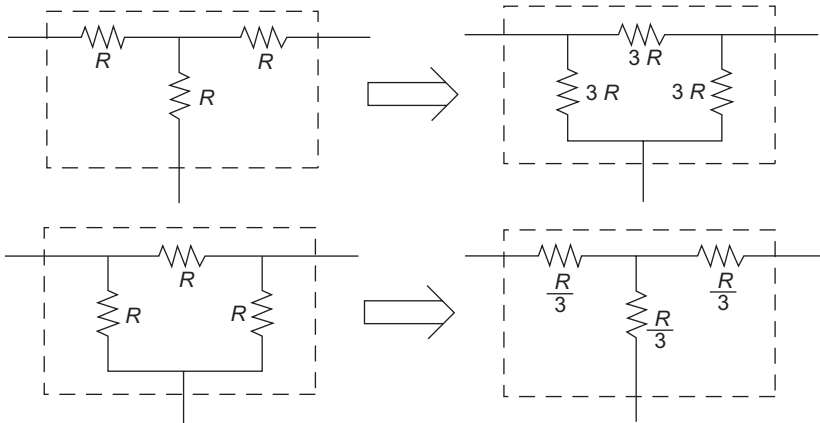


Fig. 5.2-5 A special case of Star-Delta Transformation and Delta-Star Transformation

Example: 5.2-1

The power dissipated in the resistor R in circuit of Fig. 5.2-6 (a).

Solution

The single-source circuits required for applying superposition principle are shown in Fig. 5.2-7 (a) and (c).

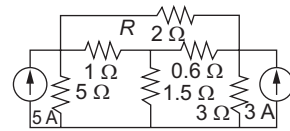


Fig. 5.2-6 Circuit for Example 5.2-1

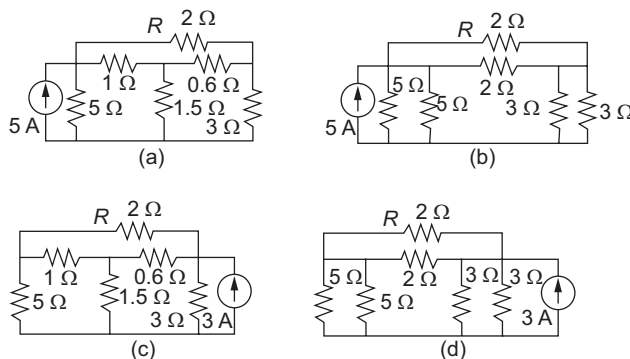


Fig. 5.2-7 Single-source circuits for applying Superposition Theorem in Example 5.2-1

Circuits in Fig. 5.2-7 (b) and (d) show the circuits in Fig. 5.2-7 (a) and (b) respectively, after Star-Delta Transformation on the inner star-connected resistors of 1Ω , 0.6Ω and 1.5Ω . Taking positive direction of current flow to be from left to right in R , the current flowing in R in circuit in Fig. 5.2-7 (b)

$$\text{is } 5A \times \frac{5/5}{5/5 + (2/2 + 3/3)} \times \frac{2}{2+2} = 1.25A. \text{ Similarly the current flowing in it in the circuit in Fig. 5.2-7 (d)}$$

$$\text{is } -3A \times \frac{3/3}{3/3 + (2/2 + 5/5)} \times \frac{2}{2+2} = -0.45A.$$

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Therefore, the current through R when both sources are acting will be $1.25 - 0.45 = 0.8$ A.
Therefore power dissipation in $R = 0.8^2 \times 2 = 1.28$ W

5.3 SUBSTITUTION THEOREM

Consider the three-mesh circuit shown in Fig. 5.3-1. Mesh analysis reveals that the mesh currents are 1 A, 2 A and 3 A as shown in the figure. Two nodes **a** and **a'** have been identified in the circuit and the current crossing the node **a** from left to right is marked as 2 A. The voltage of **a** with respect to **a'** is calculated to be 1 V and is marked in the figure.

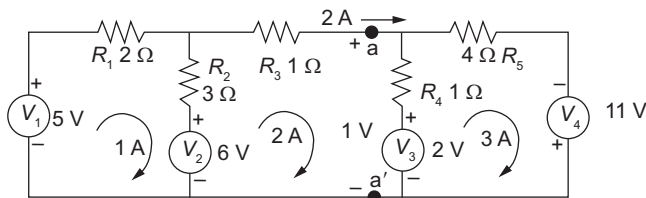


Fig. 5.3-1 Three-mesh circuit with two nodes – **a** and **a'** – identified

Now we add two current sources between the two nodes, **a** and **a'**, as shown in Fig. 5.3-2. The current sources have equal and opposite currents of 2 A magnitude.

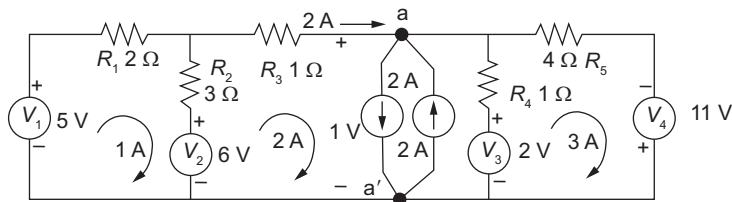


Fig. 5.3-2 Circuit in Fig. 5.3-1 with two current sources added

We have not changed the KCL equation at node **a** and node **a'**. The mesh introduced in this step is a trivial mesh. Hence the circuit solution everywhere will remain the same as before. Now introduce a pair of nodes **b** and **b'** that are connected to **a** and **a'**, respectively, by shorting links as in Fig. 5.3-3.

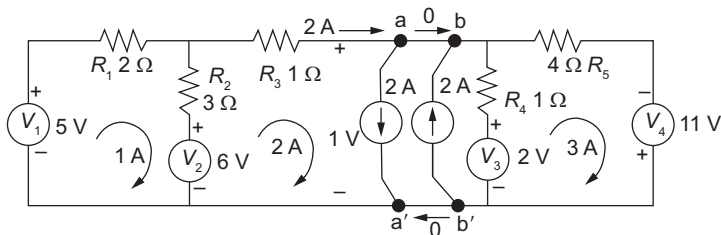


Fig. 5.3-3 Additional node pair **b** and **b'** identified in circuit of Fig. 5.3-1

We note that the current flows in the shorting links **a–b** and **a'–b'** are zero. Therefore, breaking the shorting links should not affect the circuit solution in both parts of the original circuit (see Fig. 5.3-4).

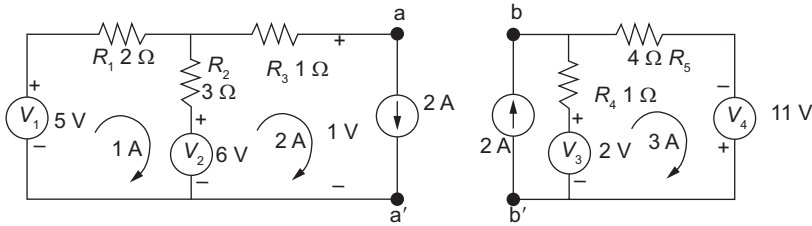


Fig. 5.3-4 The original circuit separated into two parts without the solution in either part getting affected

Thus, as far as the first part is concerned, we have been able to replace or *substitute* the second part with a current source without any circuit variable in the first part undergoing any change. This current source has a value exactly equal to the current drawn by the second part of the circuit from first part of the circuit. Similar statement can also be framed for second part of the circuit.

Now, add two independent voltage sources to the circuit in Fig. 5.3-1 instead of current sources as shown in Fig. 5.3-5. We have not affected the KVL in second mesh in any way and hence circuit solution remains the same as before throughout. However, we have reduced the voltage between **a** and **a'** to zero.

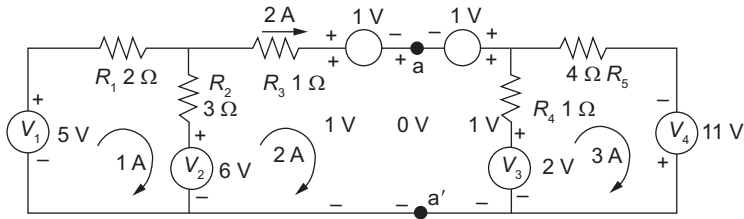


Fig. 5.3-5 Circuit in Fig. 5.3-1 with two equal and opposite voltage sources introduced in series at node **a**

If **a** and **a'** are at same potential, they can be joined together. If they can be joined together, the two parts of the circuit are connected at only one point and hence they cannot affect each other in any way. We can very well draw them as separate circuits without a common touch point as shown in Fig. 5.3-6.

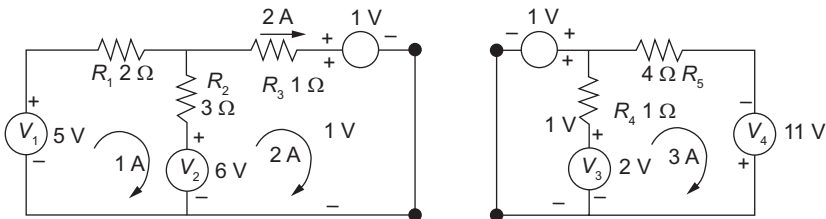


Fig. 5.3-6 Original circuit is separated into two parts without the solution in either part getting affected

Thus, as far as the first part is concerned, we have been able to replace or *substitute* the second part with a voltage source without any circuit variable in the first part undergoing any

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change. This voltage source has a value exactly equal to the voltage that was impressed on the second part of the circuit by the first part. Similar statement can be framed for second part of the circuit too.

We go one step further and end up in trouble! We extract the first part from Fig. 5.3-4 and apply the same reasoning we employed to arrive at the two parts in that figure to arrive at the circuit shown in Fig. 5.3-7 (b).

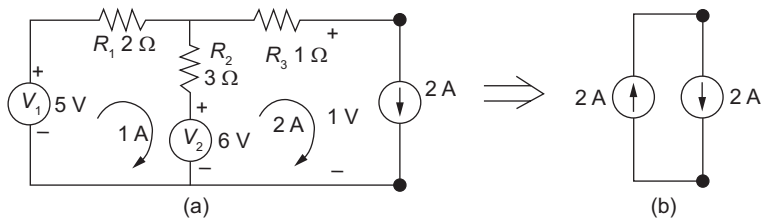


Fig. 5.3-7 The result of stretching an idea too much!

That circuit in Fig. 5.3-7 (b) has no unique solution since the voltage across the current sources can now be any value without violating any circuit law.

Thus, there must be some constraints to be satisfied by a circuit if *substitution* of a part of the circuit by a current source of value equal to the current drawn by it (or by a voltage source of value equal to voltage appearing across it) is not to affect the circuit solution in the remaining part. There are such constraints. *The constraint is that the original circuit and the circuit after substitution must have unique solution.* Linear circuits usually have unique solution – i.e., the currents and voltages everywhere are *uniquely* decided by values of independent sources and the circuit structure – except in some trivial and avoidable situations like ideal independent voltage sources in parallel or ideal independent current sources in series etc.

However, note that we used only KCL- and KVL-based arguments to arrive at the validity of *substitution*. We did not make use of element relations at all. Hence, the arguments are valid for any circuit – linear or non-linear. *Substitution Theorem* is more general than *Superposition Theorem*. The constraint of *unique solution* assumes particular significance in the case of non-linear circuits since there are non-linear that have multi-valued v - i relationships. A tunnel diode, an uni-junction transistor etc. are some examples.

There is another constraint to be satisfied before *substitution* can be done in a circuit. Consider the situation where the controlling variable of a dependent source is in the part that was subjected to substitution with the dependent source output connected in the other part. Obviously, *substitution* will not work in this case. Therefore, if there are dependent sources in the part of the circuit that is being substituted by an independent current source or voltage source, both the controlling variable and the dependent source must be within that part of the circuit. Similarly, if there is magnetic coupling in the part of the circuit being substituted, all coils belonging to the magnetically coupled system must be within that part of the circuit. This constraint may alternatively be stated as – *there should not be any interaction between the part of the circuit that is being substituted and the remaining circuit except through the pair of terminals at which they are interconnected.*

Subject to the constraints on *unique solution* and *interaction only through the connecting terminals*, we state the *Substitution Theorem* as follows.

Substitution Theorem

Let a circuit with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 and N_2 may be linear or non-linear. Let $v(t)$ be the voltage that appears at the terminals between N_1 and N_2 and let $i(t)$ be the current flowing into N_2 from N_1 . Then, the network N_2 may be replaced by an independent current source of value $i(t)$ connected across the output of N_1 or an independent voltage source of value $v(t)$ connected across the output of N_1 without affecting any voltage or current variable within N_1 provided the resulting network has unique solution.

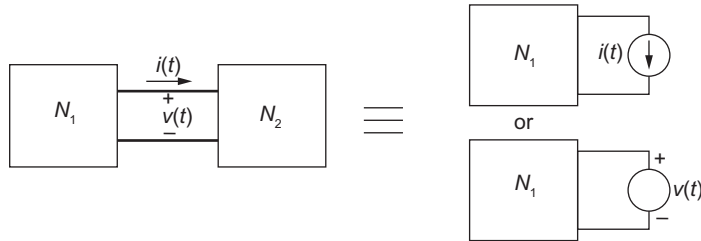


Fig. 5.3-8 The Substitution Theorem

However, what is the use of a theorem that wants us to solve a circuit first and then replace part of the circuit by a source that has a value depending on the solution of the circuit? Obviously, such a theorem will not help us to solve circuits directly. The significance of this theorem lies in the fact that it can be used to construct theoretical arguments that lead to other powerful circuit theorems that indeed help us to solve circuit analysis problems in an elegant and efficient manner. Moreover, it does find application in circuit analysis in a slightly disguised form. We take up that disguised form of Substitution Theorem now.

5.4 COMPENSATION THEOREM

The circuit in Fig. 5.4-1 (a) has a resistor marked as R . Its nominal value is 2Ω . Mesh analysis was carried out to find the current in this resistor and the current was found to be 1 A as marked in circuit of Fig. 5.4-1 (a).

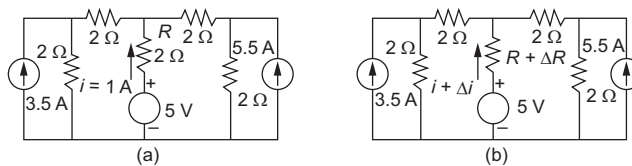


Fig. 5.4-1 Circuit to illustrate Compensation Theorem

Now assume that the resistor value changes by ΔR to $R + \Delta R$. Correspondingly, all circuit variables change by small quantities. The current through that resistor will also change to $i + \Delta i$. We can conduct

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a mesh analysis again and get the new solution. However, we can do better than that. We can work out *changes* in variables everywhere by solving a single-source circuit and then construct the circuit solution by adding *change* to the initial solution value.

We apply Substitution Theorem on the first circuit with R as the element that is being substituted and on the second circuit with $R + \Delta R$ as the part that is being substituted by an independent voltage source. The voltage source in the first circuit must be Ri V and the voltage source in the second circuit must be $(R + \Delta R)(i + \Delta i)$ V.

$$(R + \Delta R)(i + \Delta i) = Ri + (R + \Delta R)\Delta i + i\Delta R.$$

See Fig. 5.4-2.

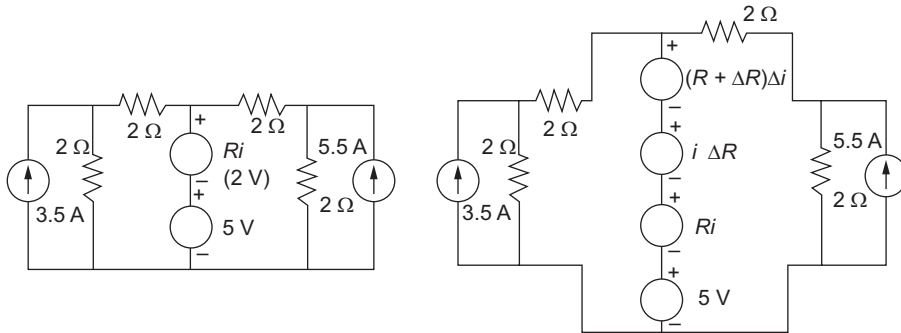


Fig. 5.4-2 Circuits after applying Substitution Theorem

Now we solve the second circuit by applying superposition principle by taking the 5 V and Ri V sources along with the two current sources together first and deactivating the remaining two voltage sources. The solution we get will be the same as the solution of the original circuit since the second circuit with the $i\Delta R$ V source and the $(R + \Delta R)\Delta i$ V source deactivated is the same as the first circuit. We already know the solution. It is the initial solution.

We have to solve the circuit with the two sources – the $i\Delta R$ V source and the $(R + \Delta R)\Delta i$ V source – to get the second component of complete solution for second circuit. This circuit is shown in Fig. 5.4-3 (a). The solution of this circuit must give the *changes* in all circuit variables due the change in R since the initial values of variables are given by the solution contributed by the other sources. Therefore, the current through central branch in the circuit of Fig. 5.4-3 (a) must be Δi .

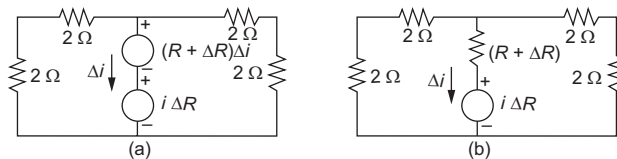


Fig. 5.4-3 (a) Circuit for obtaining changes in variables (b) After replacing voltage source by resistor

We note that the voltage of the voltage source $(R + \Delta R)\Delta i$ in circuit of Fig. 5.4-3 (a) is exactly the same as the voltage drop that will be produced by a resistor of value $(R + \Delta R)$ since the current in that branch is Δi . That is, the voltage source of value $(R + \Delta R)\Delta i$ can be thought of as the result of a substitution operation on a resistor of value $(R + \Delta R)$ in that path. We reverse this substitution and

replace the voltage source by the resistor in circuit of Fig. 5.4-3 (b). Solving circuit in (b) will give us the *change* in all circuit variables due to a change in R . Adding the initial values to change values will give us the final solution. The circuit in Fig. 5.4-3 (b) is a single-source circuit with only one voltage source of value = (change in component value) $\times$ (initial current through that component).

Let us assume that $\Delta R = 0.1\Omega$. Then the source value is $0.1\Omega \times 1\text{A} = 0.1\text{V}$. Therefore,

$$\Delta i = -\frac{0.1}{2.1 + (2 + 2)/(2 + 2)} = -0.0244\text{A} \text{ and } (i + \Delta i) = 0.9756\text{A}.$$

Reader may note that we used Superposition Theorem along with Substitution Theorem to arrive at this result. Hence Compensation Theorem is a specialised form of Substitution Theorem for a Linear Circuit.

Compensation Theorem

In a linear memoryless circuit, the change in circuit variables due to change in one resistor value from R to $R + \Delta R$ in the circuit can be obtained by solving a single-source circuit analysis problem with an independent voltage source of value $i\Delta R$ in series with $R + \Delta R$ where i is the current flowing through the resistor before its value changed. See Fig. 5.4-4.

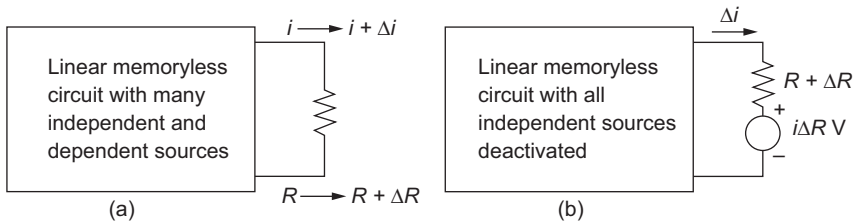


Fig. 5.4-4 The Compensation Theorem

The theorem can be extended to include dependent sources too. Changes in circuit variables due to simultaneous changes in many circuit parameters can be obtained by repeated application of Compensation Theorem or as a solution of a multi-source *change* circuit in which each parameter change is taken into account by a voltage source of suitable value.

5.5 THEVENIN'S THEOREM AND NORTON'S THEOREM

The problem of solving a circuit with different load networks connected to same delivery network occurs in electrical and electronics engineering quite often. We do not want to write the same node equations or mesh equations of the delivery network whenever the load network undergoes some change and solve the circuit in its entirety repeatedly. Thevenin's Theorem and Norton's Theorem help us to avoid this kind of wasted effort and become efficient in solving circuits. They help us to conduct node analysis or mesh analysis of the delivery network and replace it with a simple equivalent circuit for further analysis when different load networks are connected to it. They are two tools indispensable to a circuit analyst.

Consider a memoryless network shown in Fig. 5.5-1 containing linear resistors, linear dependent sources and independent sources with a pair of terminals identified as the output terminals of the

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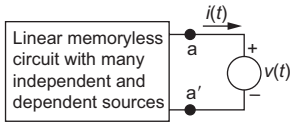


Fig. 5.5-1 A memoryless network terminated in a voltage source at its output terminals

voltage source negative terminal is taken, *i.e.*, a' , as the reference node for writing the node equations of the circuit. We are interested in the behaviour of the current $i(t)$ delivered by the circuit to the terminating voltage source versus the source function $v(t)$.

We remember that any circuit variable in a linear circuit can be expressed as a linear combination of all the *independent source functions* in the circuit. Hence, $i(t)$ in this circuit can be expressed as

$$i(t) = [(a_1 v_{s1}(t) + a_2 v_{s2}(t) + \dots + a_{n_v} v_{sn_v}(t)) + (b_1 i_{s1}(t) + b_2 i_{s2}(t) + \dots + b_{n_i} i_{sn_i}(t))] + a_o v(t)$$

This current has two components – one contributed by all independent current and voltage sources within the circuit and the second contributed by the independent voltage source connected from outside, *i.e.*, $v(t)$. The functions $v_{s1}(t) \dots$ represent the source functions of independent voltage sources within the circuit and the functions $i_{s1}(t) \dots$ represent the source functions of independent current sources within the circuit. n_v and n_i are the number of independent voltage and current sources within the circuit. The contribution coefficients $a_o, a_1, a_2, \dots$ and $b_1, b_2, \dots$ are determined by the circuit parameters. They may be found by node analysis or mesh analysis.

The source functions and contribution coefficients are fixed once and for all by the circuit and the only aspect of the circuit that can change is the network that gets connected at the output terminals. Hence we may represent the terms within the square brackets in the expression for $i(t)$ as a fixed function of time that does not depend on what is connected at the output and term it as $i_{sc}(t)$.

$$\therefore i(t) = i_{sc}(t) + a_o v(t) \quad \text{where} \quad i_{sc}(t) = \sum_{i=1}^{n_v} a_i v_{si}(t) + \sum_{i=1}^{n_i} b_i i_{si}(t) \quad (5.5-1)$$

This equation can be interpreted in an interesting manner if a_o can be written as $-G_o$. The number a_o can be obtained by finding out the current *delivered* to the voltage source when all the independent sources are set to zero. If the circuit contains only resistors, then the current will actually be delivered to the circuit, and hence a_o will be a negative number, making G_o a positive number. The possibility of a_o assuming a positive value does exist if there are dependent sources within the circuit. Therefore, G_o is positive for a purely resistive network, whereas it could be negative for a circuit containing dependent sources.

$$\therefore i(t) = i_{sc}(t) - G_o v(t) \quad (5.5-2)$$

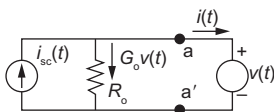


Fig. 5.5-2 A circuit that follows Eqn. 5.5-2

This equation suggests that $i(t)$ behaves as if it is coming from an independent current source of source function $i_{sc}(t)$ that is in parallel with a resistance of $R_o = 1/G_o$ (see Fig. 5.5-2)

How do we get the source function $i_{sc}(t)$? $i(t) = i_{sc}(t)$ when $v(t) = 0$. Therefore, we can find $i_{sc}(t)$ by finding out

the current that flows out into a short-circuit that is put across its output. This is the reason why we used 'sc' as the subscript for this current source function.

Further, how do we find out the value of R_o ? If we can reduce $i_{sc}(t)$ to zero and apply a non-zero $v(t)$, the ratio of current drawn from $v(t)$ to $v(t)$ will be R_o . We can reduce $i_{sc}(t)$ to zero by deactivating all the independent sources within the circuit. Thus, we see that, R_o is nothing but the equivalent resistance of the deactivated network from terminals **a** to **a'**.

We conclude that a linear memoryless circuit containing resistors, dependent sources and independent sources may be replaced by a current source $i_{sc}(t)$ in parallel with a resistance R_o when it is terminated in an independent voltage source where $i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

The circuit was terminated in an independent voltage source $v(t)$ till now. We do a mental flip now and choose to view that voltage source as the result of a *Substitution Operation*. That is, this voltage source came up there because we *substituted* a part of the original network by an independent voltage source by invoking *Substitution Theorem*. We note that *Substitution Theorem does not require the part of the circuit that is being substituted to be linear*. Now we bring that part of the circuit back and dispense with the independent voltage source $v(t)$. We will keep in mind that the circuit must meet all those constraints that Substitution Theorem calls for. Hence, *Norton's Theorem* is stated as follows.

Norton's Theorem

Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent current source of value $i_{sc}(t)$ in parallel with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution.

$i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

This equivalent circuit for N_1 is called its *Norton's Equivalent*.

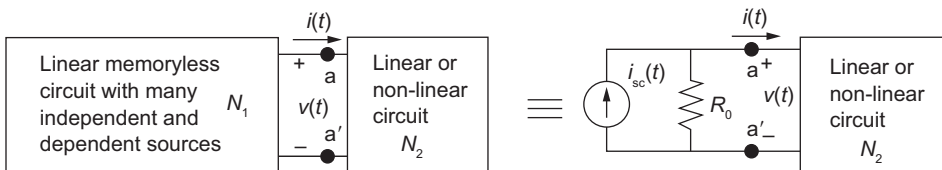


Fig. 5.5-3 Norton's theorem and Norton's equivalent

A similar argument after terminating the network N_1 in an independent current source of source function $i(t)$ will lead us to the conclusion that it may be replaced by an independent voltage source $v_{oc}(t)$ in series with a resistance R_o without affecting the circuit solution in N_2 . $v_{oc}(t)$ in this case will be the voltage generated across **a–a'** by all the independent sources within the network N_1 when the output terminals are kept open. Therefore, it is called the open-circuit voltage. R_o will again be the equivalent resistance of the deactivated network seen from **a–a'**. The resulting equivalent circuit for N_1 is called its *Thevenin's Equivalent*.

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Thevenin's Equivalent may also be derived from Norton's Equivalent by applying Source Transformation Theorem.

Thevenin's Theorem

Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent voltage source of value $v_{oc}(t)$ in series with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution.

$v_{oc}(t)$ is the voltage that will appear across the terminals when they are kept open and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

This equivalent circuit for N_1 is called its *Thevenin's Equivalent*.

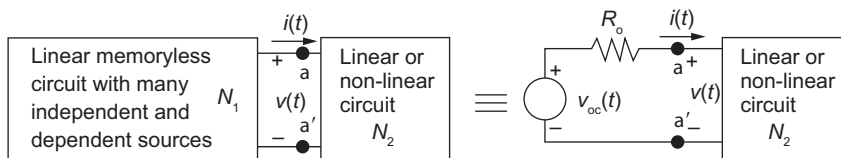


Fig. 5.5-4 Thevenin's theorem and Thevenin's equivalent

Example: 5.5-1

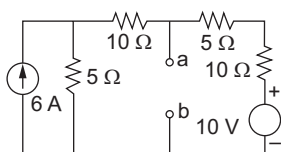


Fig. 5.5-5 Circuit for Example 5.5-1

Find the Thevenin's equivalent and Norton's equivalent of the circuit in Fig. 5.5-5 with respect to terminals a and b .

Solution

Step-1: Find the open-circuit voltage across $a - b$

This step may require nodal analysis or mesh analysis in general. However, in simple resistive circuits like this, one may use superposition principle and solve for the required voltage. The two single-source circuits needed for this are shown in Fig. 5.5-6.

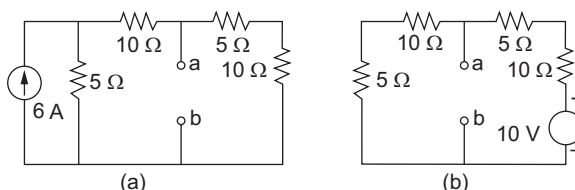


Fig. 5.5-6 Single-source circuits for finding out contributions to open-circuit voltage across $a-b$

The contribution to v_{oc} from the 6A current source = $(10+5) \times \frac{5}{5+(10+5+10)} \times 6 = 15 \text{ V}$

The contribution to v_{oc} from the 10V voltage source = $\frac{10+5}{10+5+10+5} \times 10 = 5 \text{ V}$

Therefore, $v_{oc} = 15 + 5 = 20 \text{ V}$.

Step-2: Find Thevenin's equivalent resistance R_o

The deactivated circuit is shown in Fig. 5.5-7.

The equivalent resistance seen from $a-b = (10+5)/(10+5) = 7.5 \Omega$

Therefore, the Thevenin's equivalent and Norton's equivalent (determined by applying source transformation on Thevenin's equivalent) are as shown in Fig. 5.5-8.

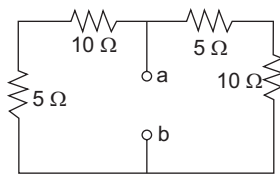


Fig. 5.5-7 The deactivated circuit for determining R_o

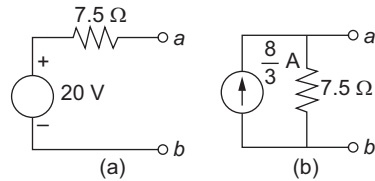


Fig. 5.5-8 (a) Thevenin's equivalent and (b) Norton's equivalent in Example 5.5-1

Thevenin's equivalent and Norton's equivalent are equivalent only as far as the circuit variables in the network that is connected across them. They are not equivalents as far as the circuit variables in the network they replace are concerned. Whenever we use such equivalents, we agree not to seek any information on the circuit variables inside the network that was replaced by equivalent. For instance, *the power dissipated in R_o is not the power actually dissipated in the network that is replaced by Thevenin's equivalent.*

Determining R_o for resistive circuits is easy. Series-parallel equivalents and Star-Delta Transformation will help us at that. However, these are not useful in the case of circuits containing dependent sources since the deactivated source will still contain them. Special procedures are needed in the case of such circuits.

5.6 DETERMINATION OF EQUIVALENTS FOR CIRCUITS WITH DEPENDENT SOURCES

Method-1

- (i) Find v_{oc} by nodal analysis or mesh analysis or superposition principle.
- (ii) Find i_{sc} by nodal analysis or mesh analysis or superposition principle.
- (iii) Obtain R_o by $R_o = \frac{v_{oc}}{i_{sc}}$

Method-2

- (i) Find v_{oc} by nodal analysis or mesh analysis or superposition principle.
- (ii) Assume that a current source of 1 A is applied to the output terminals such that the current flows into the network at the first terminal. Carry out a node or mesh analysis

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and find out the voltage appearing at first terminal with respect to second terminal. The numerical value of this voltage gives the value of R_o .

- (iii) Determine i_{sc} by $i_{sc} = \frac{v_{oc}}{R_o}$

Method-3

- Find i_{sc} by nodal analysis or mesh analysis or superposition principle.
- Assume that a voltage source of 1 V is applied to the output terminals with positive polarity at the first terminal. Carry out a node or mesh analysis and find out the current flowing into the first terminal. The numerical value of this current gives the value of $G_o = 1/R_o$.
- Determine v_{oc} by $v_{oc} = R_o i_{sc}$

Example: 5.6-1

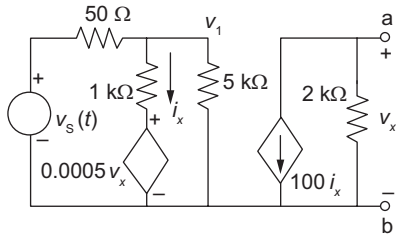


Fig. 5.6-1 Circuit for Example 5.6-1

Find the Thevenin's equivalent of the circuit in Fig. 5.6-1 with respect to the terminals a and b and thereby find the ratio of $v_x(t)$ to $v_s(t)$ when a resistor of $2k\Omega$ is connected across the output. The circuit is the low-frequency signal model for an RC-coupled common emitter amplifier using a bipolar junction transistor.

Solution

The first method is used in this example. Let b be the reference node and let the node voltage at top end of

$5k\Omega$ be v_1 as marked in the figure. Then,

$$\begin{aligned} i_x &= 1 \times 10^{-3} (v_1 - 0.0005 v_x) \quad \text{and} \quad v_x = -2 \times 10^5 i_x = -200(v_1 - 0.0005 v_x) = -200v_1 + 0.1v_x \\ \therefore 0.9v_x &= -200v_1 \Rightarrow v_x = -222.2v_1 \\ \therefore 0.0005v_x &= -0.11v_1 \end{aligned}$$

Now writing KCL at the node where v_1 is assigned,

$$\begin{aligned} 0.2 \times 10^{-3} v_1 + (1 - (-0.11)) \times 1 \times 10^{-3} v_1 + 0.02(v_1 - v_s(t)) &= 0 \\ \text{i.e., } 0.02131v_1 &= 0.02v_s(t) \\ \therefore v_1 &= 0.9385v_s(t) \end{aligned}$$

Since $v_x = -222.2v_1$, $v_x = -208.5v_s(t)$

Therefore $v_{oc} = -208.5v_s(t)$

When the terminals a – a' are shorted, $v_x = 0$, and therefore the independent voltage source at the input side is zero-valued. The value of v_1 under this condition is given by,

$$v_1 = \frac{5k // 1k}{50 + 5k // 1k} v_s(t) = 0.9434v_s(t)$$

Now the current i_x is $0.9434 \times 10^{-3} v_s(t)$, and hence the current in the dependent source at the output side is $0.09434 v_s(t)$. All this current flows out of a' to a through the short-circuit. Hence $i_{sc}(t) = -0.09434 v_s(t)$.

$$\therefore R_o = \frac{v_{oc}}{i_{sc}} = \frac{208.5 v_s(t)}{0.09434 v_s(t)} = 2.21 \text{ k}\Omega$$

The two equivalent circuits are shown in Fig. 5.6-2.



Fig. 5.6-2 (a) Thevenin's equivalent circuit for the amplifier in Example 5.6-1 (b) Norton's equivalent circuit

If a load resistance of $2 \text{ k}\Omega$ is connected at the output, then the output voltage will be $-208.5 v_s(t) \times \frac{2}{2 + 2.21} = -99.05 v_s(t)$. Hence the ratio between output and input (*i.e.*, the gain of the amplifier) is -99.05 .

Example: 5.6-2

The equivalent circuit of DC current source realised using a transistor and few resistors is shown in Fig. 5.6-3. The design is expected to deliver -2 mA at a . Find the Norton's equivalent circuit for this current source design.

Solution

We find the short-circuit current at output first. The circuit for this is shown in Fig. 5.6-4.

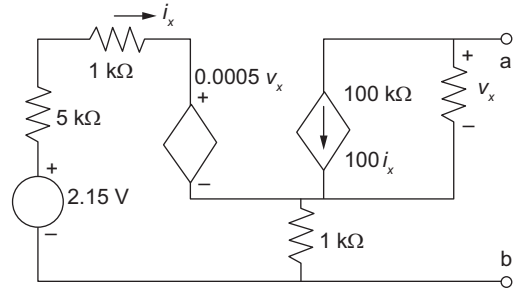


Fig. 5.6-3 Circuit for Example 5.6-2

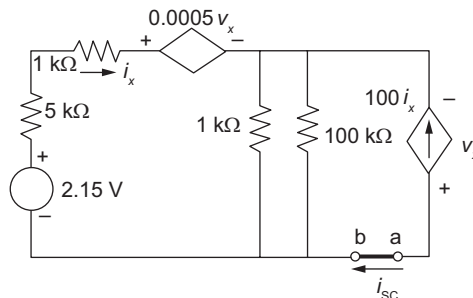


Fig. 5.6-4 Circuit for determining short-circuit current in Example 5.6-2

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We solve this circuit by mesh analysis. The current into $1\text{k}/100\text{k}$ ($= 0.99\text{k}$) is $101i_x$. We will assume that the unit of i_x is in mA. Therefore the KVL in the first mesh is

$$-2.15 + 6 \times i_x + 0.0005 \times -(0.99 \times 101i_x) + 0.99 \times 101i_x = 0$$

$$\text{i.e.,} \quad 105.94i_x = 2.15 \Rightarrow i_x = 0.0203 \text{ mA}$$

Therefore, $i_{sc} = -100i_x = -2.03 \text{ mA}$

To find R_o

We assume that we are injecting 1 mA into the network from terminal a after deactivating the circuit. We determine the voltage v_{ab} . This voltage directly gives R_o in $\text{k}\Omega$ units. Refer to the circuit in Fig. 5.6-5 (a).

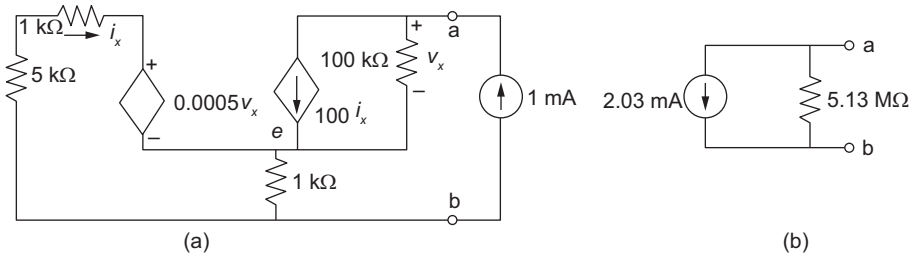


Fig. 5.6-5 (a) Circuit for determining Thevenin's equivalent resistance in Example 5.6-2
(b) The Norton's equivalent circuit required

Let i_x be in mA. The current in 100k resistor is $1 - 100i_x$ (by applying KCL at node a) and therefore $v_x = 100 - 10^4 i_x$ V. The current into 1k resistor at node e is $(1 + i_x)$ mA and hence voltage of node e with respect to node b is $(1 + i_x)$ V. Now applying KVL in the first mesh,

$$5i_x + i_x + 0.0005 \times (100 - 10^4 i_x) + (1 + i_x) = 0, \therefore i_x = -0.5025 \text{ mA}$$

$$\therefore v_x = 100 - 10^4 i_x = 5125 \text{ volts and } \therefore v_{ab} = v_x + v_e = 5125 + (1 - 0.5025) = 5125.5 \text{ volts}$$

$$\Rightarrow R_o = 5125.5 \text{ k}\Omega$$

If we really apply 1 mA into the output of the transistor circuit, either the current source we are applying will fail to function as a current source or the transistor will fail on overvoltage! But then, this is only a 'thought experiment' aimed at evaluating R_o . The Norton's equivalent circuit for this current source design is shown in Fig. 5.6-5 (b).

Example: 5.6-3

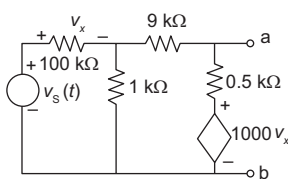


Fig. 5.6-6 Circuit for Example 5.6-3

Find the Thevenin's equivalent of the circuit in Fig. 5.6-6 with respect to a and b .

Solution

We find the open-circuit voltage across $a-b$ first. Assume two mesh currents i_1 (mA) and i_2 (mA) in the clockwise direction in the first and second mesh, respectively. The two mesh equations are

$$100i_1 + (i_1 - i_2) = v_s(t)$$

$$(i_2 - i_1) + 9i_2 + 0.5i_2 + 1000 \times 100i_1 = 0$$

Solving these two equations we get $i_1 = \frac{v_s(t)}{9624.7}$ and $i_2 = -0.9895 v_s(t)$.

$$\therefore v_{oc} = 0.5i_2 + 1000 \times 100i_1 = 9.9 v_s(t)$$

To find R_o

We assume that 1 V is applied across $a-b$ after deactivating the circuit and find out the current drawn by the circuit from this 1 V source. The value of current gives G_o . The circuit required to solve for R_o is shown as circuit in Fig. 5.6-7 (b).

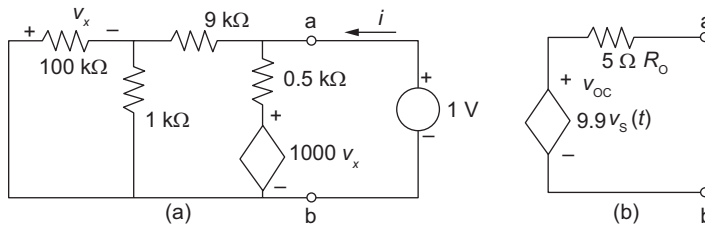


Fig. 5.6-7 (a) Circuit for determining R_o in Example 5.6-3 (b) The Thevenin's equivalent

The current drawn from 1 V has two components – one flowing into the $9\text{ k}\Omega$ resistor and the second flowing into $0.5\text{ k}\Omega$ path. The first component is $1/(9 + 1/100) = 1/9.99 = 0.1\text{ mA}$. The value of v_x is given by applying voltage division principle as $-1 \times (100/1)/(9 + 100/1) = -0.099\text{ V}$. Therefore the voltage across $0.5\text{ k}\Omega$ resistor is $1 - (-0.099 \times 1000) = 100\text{ V}$. Therefore the current flowing into $0.5\text{ k}\Omega$ path is $100/0.5 = 200\text{ mA}$. Then, the total current drawn from 1 V source is 200.1 mA and the value of G_o is 0.2 S . Therefore, the value of R_o is $5\text{ }\Omega$.

The Thevenin's equivalent for the circuit is shown in circuit of Fig. 5.6-7 (b).

5.7 RECIPROCITY THEOREM

The nodal conductance matrix and mesh resistance matrix of a memoryless circuit without any dependent sources in it (*i.e.*, a pure resistive circuit) are symmetric matrices. Reciprocity Theorem for resistive circuits is a restatement of this fact.

Consider a pure resistive circuit with only one independent current source driving it as shown in circuit of Fig. 5.7-1 (a).

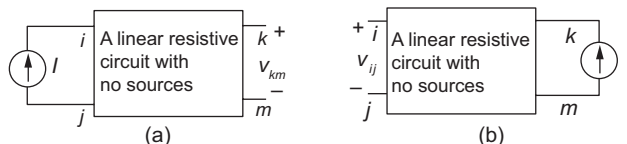


Fig. 5.7-1 Circuits to illustrate Reciprocity Theorem

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A current source of value I is connected across a node-pair i and j . Two other nodes – k and m – form a node-pair across which the voltage can be measured. There is no other independent source or dependent source inside the circuit in the box.

The nodal analysis formulation of this circuit will result in a matrix equation $\mathbf{YV} = \mathbf{I}$, where $\mathbf{Y}$ is a symmetric nodal conductance matrix, $\mathbf{V}$ is a column vector of node voltages and $\mathbf{I}$ is the column vector containing the net current injection at nodes. In this case $\mathbf{I}$ will contain I in the i^{th} row and $-I$ in the j^{th} row. All other entries will be zero. Let $\mathbf{A} = \mathbf{Y}^{-1}$. Then, $\mathbf{A}$ will be a symmetric matrix since $\mathbf{Y}$ is a symmetric matrix. We can write the node voltage vector in terms of $\mathbf{A}$ as $\mathbf{V} = \mathbf{AI}$. But $\mathbf{I}$ contains non-zero entries only in the i^{th} row and in the j^{th} row. Therefore, the node voltages v_k and v_m can be written as

$$\begin{aligned} v_k &= a_{ki}I - a_{kj}I \\ v_m &= a_{mi}I - a_{mj}I \end{aligned}$$

where a_{ki} is the element in $\mathbf{A}$ in k^{th} row and i^{th} column. The other a values also have same interpretation. We can now express the voltage between the two nodes as

$$v_{km} = v_k - v_m = [(a_{ki} + a_{mj}) - (a_{kj} + a_{mi})]I.$$

Therefore the ratio of response measured to excitation applied

$$= [(a_{ki} + a_{mj}) - (a_{kj} + a_{mi})] \quad (5.7-1)$$

Now, consider the circuit of Fig. 5.7-1 (b). The location of excitation and response are interchanged. The current source is applied across the node-pair k and m , and the voltage response is measured between the node-pair i and j .

Now the current injection vector $\mathbf{I}$ will have non-zero entries only in k^{th} row ($= I$) and in m^{th} row ($= -I$). Therefore, we can express the node voltages at node- i and node- j as

$$\begin{aligned} v_i &= a_{ik}I - a_{im}I \\ v_j &= a_{jk}I - a_{jm}I \end{aligned}$$

and the voltage between the two nodes as $v_{ij} = v_i - v_j = [(a_{ik} + a_{jm}) - (a_{jk} + a_{im})]I$

Therefore, the ratio of response measured to excitation applied

$$= [(a_{ik} + a_{jm}) - (a_{jk} + a_{im})] \quad (5.7-2)$$

$\mathbf{A}$ is a symmetric matrix. Therefore, $a_{ik} = a_{ki}$, $a_{mj} = a_{jm}$, $a_{jk} = a_{kj}$ and $a_{im} = a_{mi}$. Therefore, the ratios given by Eqns. 5.7-1 and 5.7-2 are equal.

Does it matter when the two ratios in Eqns. 5.7-1 and 5.7-2 were calculated? For instance, can we calculate the ratio in Eqn. 5.7-1 at t and the other ratio at a different instant t' ? The answer, of course, is yes, provided the entries in $\mathbf{A}$ (i.e., $\mathbf{Y}^{-1}$) matrix are time-invariant quantities. Hence, the circuit has to be a ‘linear time-invariant resistive’ one for Reciprocity Theorem to work.

First form of Reciprocity Theorem

The ratio of voltage measured across a pair of terminals to the excitation current applied at another pair of terminals is invariant to an interchange of excitation and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.

The second form can be obtained by considering a dual situation shown in Fig. 5.7-2.

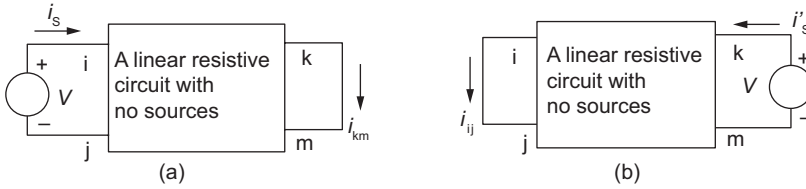


Fig. 5.7-2 Illustrating second form of Reciprocity Theorem

It is possible to show that the ratio $\frac{i_{km}}{V}$ is same as $\frac{i_{ij}}{V}$. It is easy to show this for a planar network using a mesh analysis formulation and exploiting the symmetry properties of mesh resistance matrix. In that case, we view i, j, k and m as mesh identifiers. One has to view the voltage source as participating in i^{th} and j^{th} meshes and the shorting link participating in k^{th} and m^{th} meshes and use an argument similar to the one we used in the case of first form of Reciprocity Theorem. However, $\frac{i_{km}}{V}$ will be equal to $\frac{i_{ij}}{V}$ even for a non-planar resistive network and mesh analysis does not help us with non-planar networks.

It is possible to show that $\frac{i_{km}}{V}$ will be equal to $\frac{i_{ij}}{V}$ using nodal analysis formulation too. In that case, we view i, j, k and m as node identifiers. We view the voltage source as connected between the i^{th} and j^{th} nodes and shorting link between k^{th} and m^{th} nodes in circuit of Fig. 5.7-2 (a). Then we impose the constraints that $v_i - v_j = v$ and $v_k - v_m = 0$ with a current injection of i_s at i^{th} node, $-i_s$ at j^{th} node, $-i_{km}$ at k^{th} node and i_{km} at m^{th} node. This will result in two equations in two unknowns i_{km} and i_s . We solve for i_{km} . The procedure is repeated for circuit of Fig. 5.7-2 (b) and solution for i_{ij} is obtained. Note that i_s will not be the same as i'_s . Comparison of expressions for i_{km} and i_{ij} for the same applied voltage will reveal that they are equal due to symmetry of $\mathbf{Y}^{-1}$ matrix.

We skip the details and state the second form of Reciprocity Theorem.

Second form of Reciprocity Theorem

The ratio of current measured in a short-circuit across a pair of terminals to the excitation voltage applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.

The third and last form of this theorem can be obtained by considering the circuits shown in Fig. 5.7-3.

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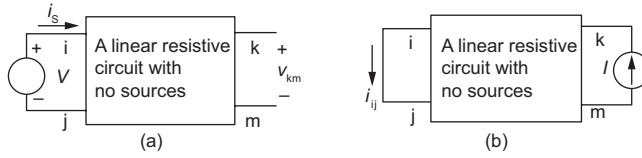


Fig. 5.7-3 Circuit illustrating third form of Reciprocity Theorem

We calculate the ratio $\frac{v_{km}}{V}$ in the circuit of Fig. 5.7-3 (a) first.

$$\text{Node voltage at node-}i = v_i = a_{ii} i_s - a_{ij} i_s$$

$$\text{Node voltage at node-}j = v_j = -a_{jj} i_s + a_{ji} i_s$$

$$\text{Node voltage at node-}k = v_k = a_{ki} i_s - a_{kj} i_s$$

$$\text{Node voltage at node-}m = v_m = a_{mi} i_s - a_{mj} i_s$$

The node voltages at node- i and node- j are constrained to have a difference of V .

$$a_{ii} i_s - a_{ij} i_s - (-a_{jj} i_s + a_{ji} i_s) = V \Rightarrow i_s = \frac{V}{a_{ii} + a_{jj} - a_{ij} - a_{ji}}$$

Substituting this expression for i_s in the equations for v_k and v_m , we get the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals as

$$\frac{v_{km}}{V} = \frac{(a_{ki} - a_{mi}) + (a_{mj} - a_{kj})}{a_{ii} + a_{jj} - a_{ij} - a_{ji}} \quad (5.7-3)$$

Now we calculate $\frac{i_{ij}}{I}$ in the circuit Fig. 5.7-3 (b).

$$v_i = a_{ii} i_{ij} - a_{ij} i_{ij} + a_{ik} I - a_{im} I$$

$$v_j = -a_{jj} i_{ij} + a_{ji} i_{ij} + a_{jk} I - a_{jm} I$$

But $v_i = v_j$

$$\therefore 0 = (a_{ii} i_{ij} - a_{ij} i_{ij} + a_{ik} I - a_{im} I) - (-a_{jj} i_{ij} + a_{ji} i_{ij} + a_{jk} I - a_{jm} I)$$

Solving this we get the ratio of current measured in short-circuit across the first pair of terminals to the current source applied at across the second pair of terminals as

$$\frac{i_{ij}}{I} = \frac{(a_{ik} - a_{im}) + (a_{jm} - a_{jk})}{a_{ii} + a_{jj} - a_{ij} - a_{ji}} \quad (5.7-4)$$

Comparing the two expressions in Eqns. 5.7-3 and 5.7-4 and using the symmetry of A (i.e., the inverse of nodal conductance matrix), we see that the two ratios are equal.

Third form of Reciprocity Theorem

The ratio of current measured in a short-circuit across first pair of terminals to the excitation current applied at the second pair of terminals is same as the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals in the case of a linear time-invariant resistive circuit with no independent sources inside. (Refer Fig. 5.7-3 for polarity of currents and voltages.)

Note that the key to Reciprocity Theorem is that (i) the nodal conductance matrix $\mathbf{Y}$ (and mesh resistance matrix $\mathbf{Z}$) of the circuit must be time-invariant and *symmetric* and (ii) excitation should be applied only at terminals identified, *i.e.*, there should not be independent sources present within the circuit. $\mathbf{Y}$ and $\mathbf{Z}$ matrices of a circuit containing *linear two-terminal time-invariant resistors* will be symmetric. Hence such circuits will obey all the three forms of Reciprocity Theorem unconditionally.

Dependent sources, even if they are linear, bilateral and time-invariant, *can* make these matrices asymmetric. *However, they need not do so always.* There can be dependent sources in the circuit and yet the circuit may have symmetric $\mathbf{Y}$ and $\mathbf{Z}$ matrices. Reciprocity Theorem will hold for such circuits too. See the following example.

Example: 5.7-1

Show that Reciprocity Theorem is valid for the circuit in Fig. 5.7-4.

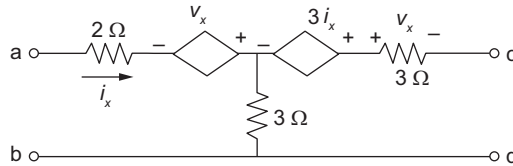


Fig. 5.7-4 Circuit for Example 5.7-1

Solution

We find the mesh resistance matrix of the circuit first and verify whether it is symmetric and time-invariant. We know that the mesh resistance matrix of a circuit can be found from its deactivated version. Since excitation can be applied only across $a-b$ and $c-d$, we short these two ports (since for mesh analysis voltage source is the excitation source) and get the circuit in Fig. 5.7-5. The mesh currents are identified in it.

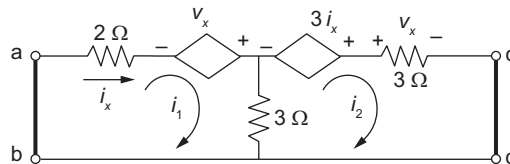


Fig. 5.7-5 Circuit to obtain $\mathbf{Z}$ matrix in Example 5.7-1

The mesh equations are written for the two meshes after observing that $i_x = i_1$ and $v_x = 3i_2$.

$2i_1 - 3i_2 + 3(i_1 - i_2) = 0$ and $3(i_2 - i_1) - 3i_1 + 3i_2 = 0$ are the mesh equations. Therefore, the mesh resistance matrix $\mathbf{Z}$ is

$\mathbf{Z} = \begin{bmatrix} 5 & -6 \\ -6 & 6 \end{bmatrix}$ and it is a symmetric time-invariant matrix. Therefore, Reciprocity Theorem will

be valid in the circuit.

We verify the first form by using the circuit configurations shown in Eqn. 5.7-4. A 1A independent current source is used to drive the $a-b$ terminal pair first and the voltage v_{cd} is noted. Then the same

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current source is used to drive the $c-d$ terminal pair and the voltage v_{ab} is noted. We expect to see that $v_{ab} = v_{cd}$.

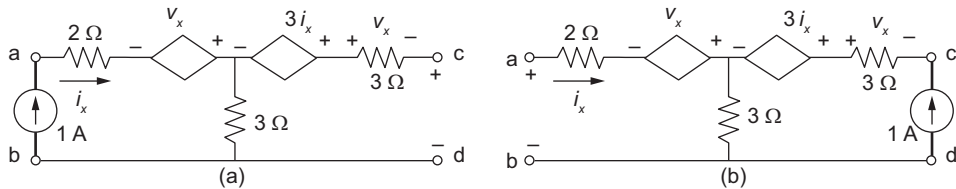


Fig. 5.7-6 Circuits for verifying Reciprocity Theorem in Example 5.7-1

The second mesh current in circuit of Fig. 5.7-6 (a) is zero. First mesh current is 1 A. Applying KVL in the second mesh, $-3 \times 1 - 3 \times 1 + 3 \times 0 + v_{cd} = 0 \Rightarrow v_{cd} = 6$ V.

The first mesh current in circuit (b) is zero. Second mesh current is -1 A. Applying KVL in the first mesh, $-v_{ab} + 2 \times 0 - (3 \times -1) + 3 \times 1 = 0 \Rightarrow v_{ab} = 6$ V.

Thus, we see that first form of the theorem holds in this circuit. It may be verified in a similar manner that the other two forms are also valid in this circuit.

5.8 MAXIMUM POWER TRANSFER THEOREM

All electrical and electronic circuits fall under one of the three broad categories – power generation and delivery circuits, power conditioning circuits and signal generation and conditioning circuits.

In a power delivery context, one part of the circuit acts as a power source and delivers power to the other part of the circuit. In the process of delivering power to load part of the circuit, the source part of the circuit ends up dissipating some of the power within itself. This compromises the efficiency of power delivery as well as the power availability to the load at the same time. Hence the power delivery capability of source part of the circuit for a given load circuit is of crucial practical significance – both in high-power electrical circuits (kW to hundreds of MW) and low-power electronic circuits (pW to 100's of W). We address the issue of power delivery capability of a source circuit in this section.

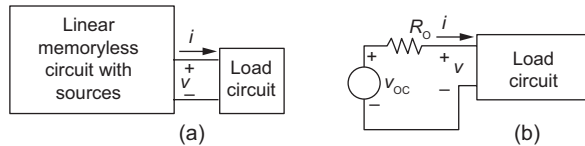


Fig. 5.8-1 (a) The power delivery context (b) Power delivery circuit replaced by its Thevenin's equivalent

Figure 5.8-1 shows a linear time-invariant memoryless circuit containing one or more independent DC sources delivering power to a load circuit that may be linear or non-linear. It is assumed that the constraints required for applying Thevenin's theorem are satisfied by the entire circuit – *i.e.*, the circuits in Fig. 5.8-1 (a) and (b) have unique solution and there is no interaction between the delivery circuit and load circuit other than through the common terminals. Then, we can replace the power delivery circuit by its Thevenin's equivalent comprising an open-circuit voltage in series with the Thevenin's equivalent resistance.

We state the *Maximum Power Transfer Theorem* when the power delivery circuit is a linear time-invariant memoryless circuit containing one or more independent DC sources.

The power delivered by a linear time-invariant memoryless circuit containing independent DC sources is a maximum of $\frac{V_{oc} i_{sc}}{4}$ when it is delivering $\frac{i_{sc}}{2}$ to the load where V_{oc} is the open-circuit voltage in its Thevenin's equivalent and i_{sc} is the short-circuit current in its Norton's equivalent.

We show this as follows:

$$p = vi = [v_{oc} - R_o i]i$$

$$\therefore \frac{dp}{di} = v_{oc} - 2R_o i$$

Equating the derivative of power with respect current to zero, we get the condition for maximum power transfer as $i = \frac{v_{oc}}{2R_o} = \frac{i_{sc}}{2}$ (since $R_o = \frac{v_{co}}{i_{sc}}$). The value of maximum power transferred to load circuit is $\left(v_{oc} - \frac{R_o i_{sc}}{2} \right) \times \frac{i_{sc}}{2} = \frac{v_{oc} i_{sc}}{4}$ w. This transfer will take place with a load voltage of $\frac{v_{oc}}{2}$ V and load current of $\frac{i_{sc}}{2}$ A. The internal dissipation inside the power delivery circuit under this condition will be same as the power transferred and the efficiency of power delivery will be 50%. If the load circuit is a single resistor of value R_L , then the condition for maximum power transfer reduces to $R_L = R_o$ and the maximum power transferred will be $\frac{v_{oc}^2}{4R_L}$ w.

Example: 5.8-1

A DC voltage source of 200 V with an internal resistance of 2 Ω delivers power to another DC source in series with a resistor R . The value of this DC source is 50 V and the two sources oppose each other in the circuit. Find the value of R and the power transferred to load circuit if maximum power transfer is to take place.

Solution

Maximum power transfer takes place when the source is delivering half its short-circuit current, i.e., 100 A in this case. The voltage across the load circuit under this current flow has to be half of the open-circuit voltage of the source. Therefore, the 50 V source and R together should absorb 100 V when 100 A is flowing through them. Then R must be 0.5 Ω .

The maximum power transferred will be $100 \text{ V} \times 100 \text{ A} = 10 \text{ kW}$ of which 5 kW will go into the 50 V DC source and 5 kW will be dissipated in 0.5 Ω resistor. The internal dissipation of the 200 V source will be 10 kW.

It should be obvious that we do not intend to load an electrical power source to the maximum power transfer level since it will result in inefficient operation. But, a comparison between the maximum power that the source can deliver and the power actually required in the load will immediately reveal to us the level of efficiency we can hope for in the delivery system. However, when efficiency is of

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no concern and signal strength is of prime concern, we take care to achieve maximum power transfer condition in the circuit. This is relevant in low-power electronic circuits. The source signal may be weak and it may come through a high R_o . R_L may be very small compared to the internal resistance. Then some kind of matching circuit that makes the low R_L appear high and equal to R_o to the source will be interposed between the source and load.

5.9 MILLMAN'S THEOREM

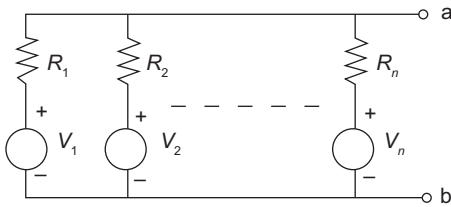


Fig. 5.9-1 Practical voltage sources connected in parallel

Consider a set of n independent voltage sources, each in series with a resistance representing its internal resistance. Assume that these n sources are connected in parallel and are supplying a common load. We wish to replace these n sources by a single independent voltage source in series with a resistance. The relevant circuit is shown in Fig. 5.9-1.

We can achieve our objective by applying Source Transformation Theorem repeatedly. Each voltage source is replaced by a current source in parallel with a resistor across the terminal pair $a-b$. The current

source will have a value equal to the voltage of the voltage source multiplied by the conductance of resistor in series with it. The circuit after this replacement is shown in Fig. 5.9-2.

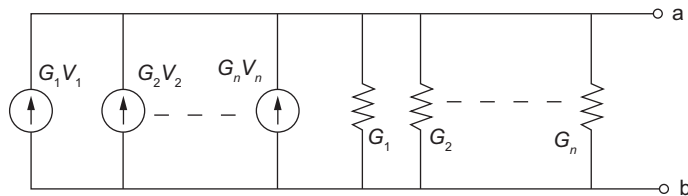


Fig. 5.9-2 Circuit in Fig. 5.9-1 after source transformation

$G_1, G_2, \dots, G_n$ are the conductances of the resistors. Current sources in parallel can be replaced by a single current source with a source function equal to the sum of source functions of each current source. Conductances in parallel can be replaced by a single conductance of value equal to the sum of conductances. This circuit reduction process results in a single current source in parallel with a single resistance. See Fig. 5.9-3 (a). This can be replaced by a single voltage source in series with a single resistance as shown in Fig. 5.9-3 (b).

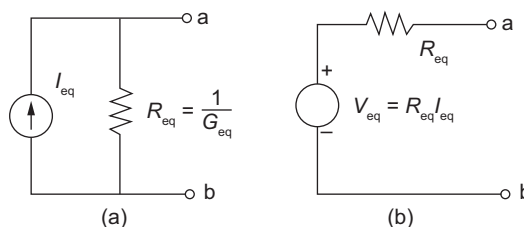


Fig. 5.9-3 (a) Current source equivalent (Norton's equivalent) (b) Voltage source equivalent (Thevenin's equivalent) for the circuit in Fig. 5.9-1

Thus, n ideal independent voltage sources of voltage values $V_1, V_2, \dots, V_n$ each in series with a resistance, delivering power to a common load in parallel, can be replaced by a single ideal independent voltage source in series with a resistance. The value of voltage source

$$V_{eq} = \frac{\sum_{i=1}^n G_i V_i}{\sum_{i=1}^n G_i}; \quad R_{eq} = \frac{1}{\sum_{i=1}^n G_i} \quad \text{where } G_i = R_i \text{ for } i = 1 \text{ to } n.$$

This is known as Millman's Theorem. This theorem is only a restatement of Source Transformation Theorem that is valid under a special context.

5.10 SUMMARY

- *Superposition Theorem* is applicable only to linear circuits. It states that 'the response of any circuit variable in a multi-source *linear memoryless* circuit containing n independent sources can be obtained by adding the responses of the same circuit variable in n single-source circuits with i^{th} single-source circuit formed by keeping only i^{th} independent source active and all the remaining independent sources deactivated'.
- A more general form of Superposition Theorem states that 'the response of any circuit variable in a multi-source *linear memoryless* circuit containing n independent sources can be obtained by adding responses of the same circuit variable in two or more circuits with each circuit keeping a subset of independent sources active in it and remaining sources deactivated such that there is no overlap between the such active source subsets among them'.
- Substitution Theorem is applicable to any circuit satisfying certain stated constraints. Let a circuit with unique solution be represented as interconnection of two networks N_1 and N_2 , and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 and N_2 may be linear or non-linear. Let $v(t)$ be the voltage that appears at the terminals between N_1 and N_2 , and let $i(t)$ be the current flowing into N_2 from N_1 . Then, the network N_2 may be replaced by an independent current source of value $i(t)$ connected across the output of N_1 or an independent voltage source of value $v(t)$ connected across the output of N_1 without affecting any voltage or current variable within N_1 provided the resulting network has unique solution.
- Compensation Theorem is applicable to linear circuits and states that 'in a linear memoryless circuit, the *change* in circuit variables due to *change* in one resistor value from R to $R + \Delta R$ in the circuit can be obtained by solving a single-source circuit analysis problem with an independent voltage source of value $i\Delta R$ in series with $R + \Delta R$ where i is the current flowing through the resistor before its value changed'.
- Thevenin's and Norton's Theorems are applicable to linear circuits. Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear, and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent voltage source of value $v_{oc}(t)$ in series with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution. $v_{oc}(t)$ is the voltage that will appear across the terminals when they are kept open and R_o is the equivalent resistance of the

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deactivated circuit ('dead' circuit) seen from the terminals. This equivalent circuit for N_1 is called its *Thevenin's Equivalent*.

- Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 , and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent current source of value $i_{sc}(t)$ in parallel with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution. $i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.
- Reciprocity Theorem is applicable to linear time-invariant circuits with no dependent sources. First form of Reciprocity Theorem states that 'the ratio of voltage measured across a pair of terminals to the excitation current applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside'.
- Second form of Reciprocity Theorem states that 'the ratio of current measured in a short-circuit across a pair of terminals to the excitation voltage applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside'.
- Third form of Reciprocity Theorem states that 'the ratio of current measured in a short-circuit across first pair of terminals to the excitation current applied at the second pair of terminals is same as the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals in the case of a linear time-invariant resistive circuit with no independent sources inside'.
- Maximum Power Transfer Theorem applicable to linear time-invariant circuit under steady-state conditions and states that 'the power delivered by a linear time-invariant memoryless circuit containing independent DC sources is maximum of $\frac{v_{oc} i_{sc}}{4}$ when it is delivering $\frac{i_{sc}}{2}$ to the load where v_{oc} is the open-circuit voltage in its Thevenin's equivalent and i_{sc} is the short-circuit current in its Norton's equivalent'.

5.11 PROBLEMS

1. Find the change in voltage across 20Ω resistor in Fig. 5.11-1 when the current source value increases by 0.5 A.

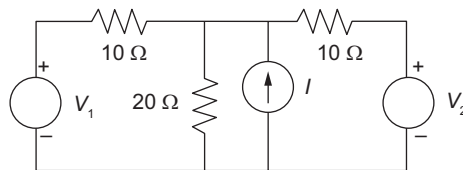
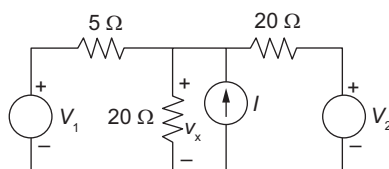
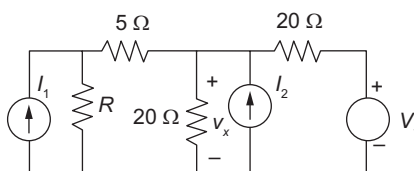


Fig. 5.11-1

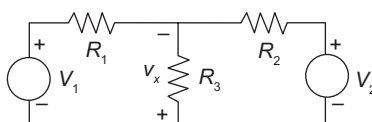
2. $V_1 = V_2 = 6V$ in the circuit in Fig. 5.11-2. (i) If v_x is found to be 8V find the value of current source. (ii) If V_1 increases by 1 V now and I does not change, what should be the change in V_2 such that v_x does not change?

**Fig. 5.11-2**

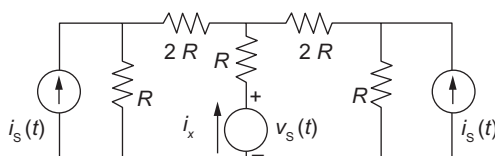
3. (i) Find R in the circuit in Fig. 5.11-3, if $\frac{\partial v_x}{\partial I_1}$ is found to be 7.5 V/A. (ii) With this value of R determine $\frac{\partial v_x}{\partial I_2}$ and $\frac{\partial v_x}{\partial V_1}$. (iii) If I_1 and I_2 change at the rate of 0.5 A/s what should be $\frac{dV_1}{dt}$ such that $\frac{\partial v_x}{dt} = 0$ V/s?

**Fig. 5.11-3**

4. The voltage v_x in the circuit in Fig. 5.11-4 is found to be zero when $V_2 = -2V_1$. dv_x/dt is found to be -0.6 V/s when both source voltages are changing at the rate of 1 V/s. Find at least two sets of values for the three resistors that will explain these observations.

**Fig. 5.11-4**

5. If the two current source are equal at all instants of time, what should be $\frac{dv_s(t)}{di_s(t)}$ such that $\frac{di_x(t)}{dt} = 0$ in the circuit of Fig. 5.11-5? The value of R is 1Ω .

**Fig. 5.11-5**

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6. Find (i) voltage across the 10Ω (ii) power dissipated in all resistors by using superposition principle in the circuit in Fig. 5.11-6.

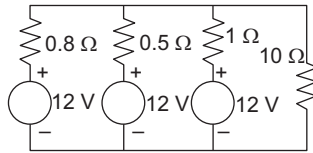


Fig. 5.11-6

7. Find (i) i_x (ii) power dissipated in all resistors by using superposition principle in the circuit in Fig. 5.11-7.

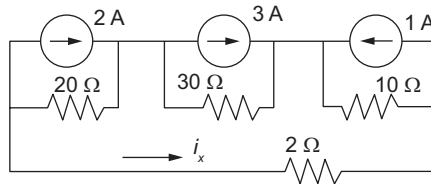


Fig. 5.11-7

8. Find the time instant at which v_x crosses zero first time after $t = 0$ in the circuit in Fig. 5.11-8.

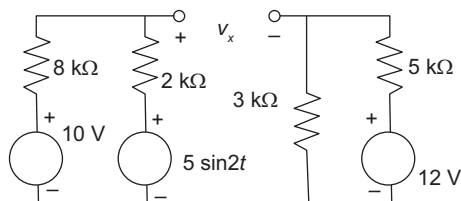


Fig. 5.11-8

9. Show that $v_o = -(v_1 + v_2 + v_3)$ and $v_x \approx 0$ as $k \rightarrow \infty$ in the circuit in Fig. 5.11-9. This is an *inverting summer circuit* that can be realised using an electronic amplifier with high gain.

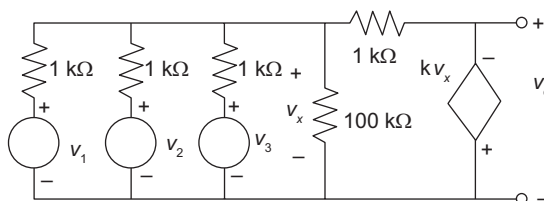


Fig. 5.11-9

10. Show that $v_o = v_1 - v_2$ and $v_x \approx 0$ as $k \rightarrow \infty$ in the circuit in Fig. 5.11-10.

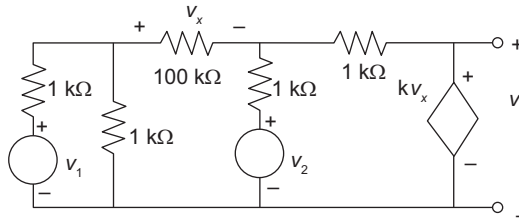


Fig. 5.11-10

11. Find v_x in the circuit in Fig. 5.11-11 by employing Superposition Theorem and Star-Delta Transformation.

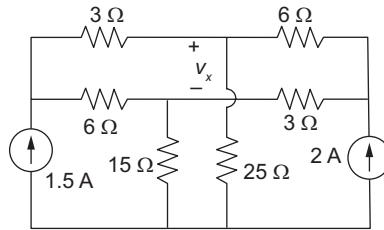


Fig. 5.11-11

12. Find i_x in the circuit in Fig. 5.11-12 by applying Superposition Theorem and Star-Delta Transformation.

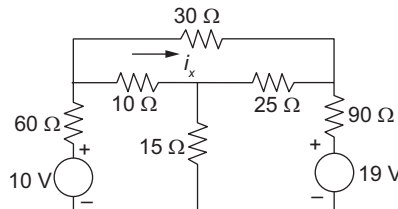


Fig. 5.11-12

13. Refer to Fig. 5.11-10. If all the $1\text{k}\Omega$ resistors can have values between $0.9\text{k}\Omega$ to $1.1\text{k}\Omega$ due to manufacturing tolerances, find an expression for maximum deviation in v_o from the expected $v_o = (v_1 - v_2)$ in terms of v_1 and v_2 by applying Compensation Theorem.
14. (i) Find an expression for v_x in the circuit in Fig. 5.11-13 in terms of v_1 and v_2 by using Superposition Theorem. (ii) If all resistors with nominal value of $1\text{k}\Omega$ can have values between $0.98\text{k}\Omega$ and $1.02\text{k}\Omega$ due to manufacturing tolerances or mismatches between devices, find an expression for deviation in v_x from the expression arrived at in the first step by employing Compensation Theorem. The circuit is an approximate model of a *differential amplifier* using two bipolar junction transistors.

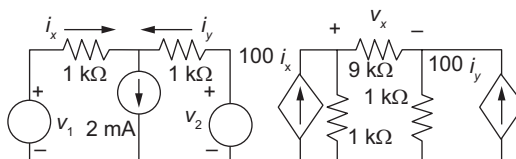


Fig. 5.11-13

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15. (i) Find the value of R in the circuit in Fig. 5.11-14, if i_x is to be zero. (ii) If i_x was seen to be 0.01mA , find the value of R by using Compensation Theorem.

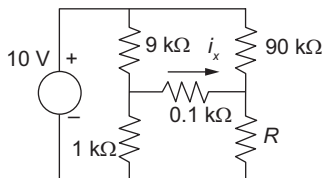


Fig. 5.11-14

16. Find the Thevenin's equivalent of the circuit in Fig. 5.11-15 with respect to terminals a and b .

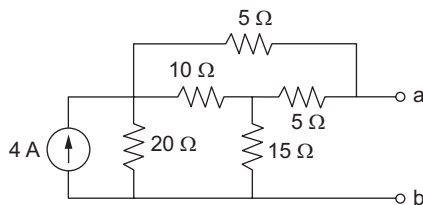


Fig. 5.11-15

17. What should be the value of R if the current through it is to be 1A in the direction shown in the circuit in Fig. 5.11-16?

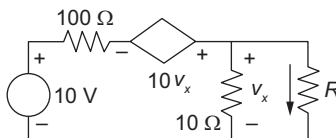


Fig. 5.11-16

18. Find the Thevenin's equivalent and Norton's equivalent of the circuit in Fig. 5.11-17 with respect to terminal pair $a-b$.

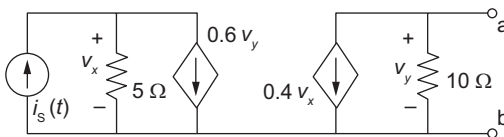


Fig. 5.11-17

19. Find the Thevenin's equivalent and Norton's equivalent of the circuit in Fig. 5.11-18.

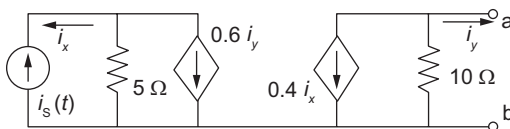


Fig. 5.11-18

20. Find the Thevenin's equivalent and Norton's equivalent for the circuit in Fig. 5.11-19 with respect to terminal pair $a-b$.

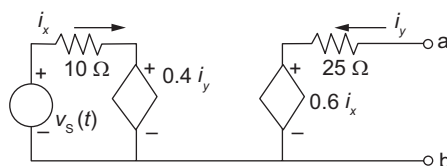


Fig. 5.11-19

21. Find the Thevenin's equivalent and Norton's equivalent for the circuit in Fig. 5.11-20 with respect to terminal pair $a-b$.

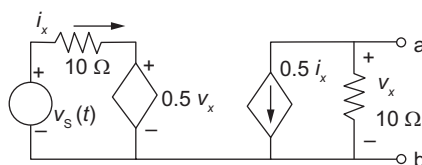


Fig. 5.11-20

22. Approximate equivalent circuit of a *Unity Gain Buffer Amplifier* (also called *Voltage Follower*) using a high gain differential amplifier (an electronic amplifier that amplifies the difference between two voltages) is shown in Fig. 5.11-21. $200\text{k}\Omega$ represents the input resistance of the amplifier and $1\text{k}\Omega$ represents the output resistance of the amplifier. k is the gain of the amplifier. (i) Obtain the Thevenin's equivalent of the circuit in terms of $v_s(t)$ and k . (ii) Show that the Thevenin's equivalent resistance (R_o) approaches zero, $v_{oc}(t)$ approaches $v_s(t)$ and the ratio $v_s(t)/i_s(t)$ (i.e., input resistance of the circuit) approaches infinity as $k \rightarrow \infty$.

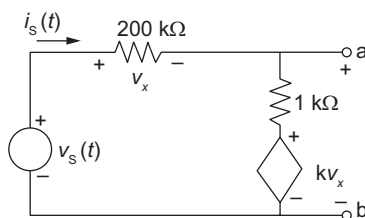


Fig. 5.11-21

23. The circuit in Fig. 5.11-22 shows the approximate equivalent circuit of the so-called *common-base amplifier* using a single bipolar junction transistor. Find the Thevenin's equivalent of the amplifier across the terminals marked a and b .

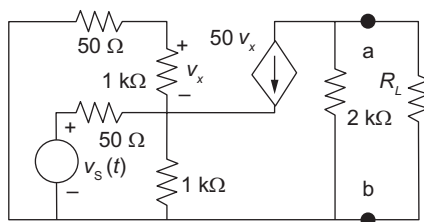


Fig. 5.11-22

5.44 | Circuit Theorems

24. The circuit in Fig. 5.11-23 shows the approximate equivalent circuit of the so-called *common-emitter amplifier with un-bypassed emitter* using a single bipolar junction transistor. Find the Thevenin's equivalent of the amplifier across the terminals marked *a* and *b*.

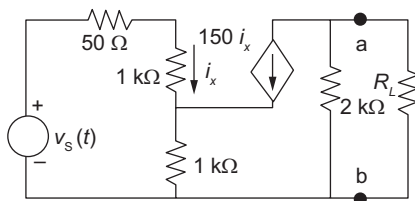


Fig. 5.11-23

25. Refer to the circuit in Fig. 5.11-24. With $V = 10$ V, $I = 1$ A and $R = 10$ Ω , the value of i_x is found to be 0.5 A. The corresponding value with $R = 20$ Ω , $V = 20$ V and $I = -2$ A is -2 A. Find the value of i_x when $R = 5$ Ω , $V = 10$ V and $I = 2$ A. (Hint: Find Thevenin's equivalent using superposition principle.)

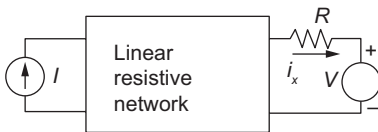


Fig. 5.11-24

26. Refer to the circuit in Fig. 5.11-25. With $V = 10$ V, $I = 1$ A and $R = 10$ Ω , the value of v_x is found to be 10 V. The corresponding value with $R = 20$ Ω , $V = 20$ V and $I = 4$ A is 0 V. Find the value of v_x when $R = 5$ Ω , $V = 10$ V and $I = 2$ A. (Hint: Find Norton's equivalent using superposition principle.)

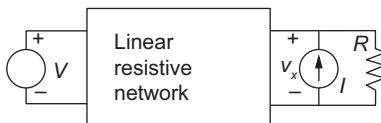


Fig. 5.11-25

27. Using the data shown in the first circuit in Fig. 5.11-26, find I_1 for second circuit. Can I_2 be found using the given data? (Hint: Reciprocity Theorem + Superposition Theorem.)

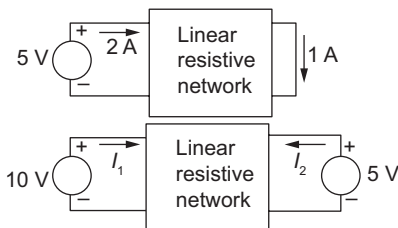


Fig. 5.11-26

28. Using the data shown in the first circuit in Fig. 5.11-27, find v_1 for second circuit. Can v_2 be found using the given data? (Hint: Reciprocity Theorem + Superposition Theorem.)

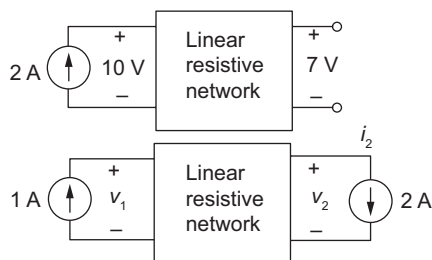


Fig. 5.11-27

29. Using the data shown in the first circuit in Fig. 5.11-28, find v_1 for second circuit. Can i_2 be found using the given data? (Hint: Reciprocity Theorem + Superposition Theorem.)

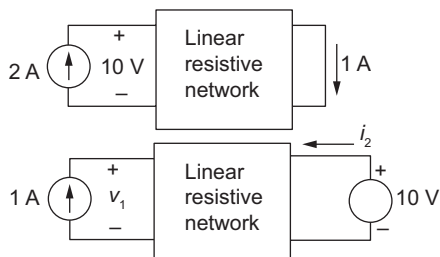


Fig. 5.11-28

30. Show that the circuit in Fig. 5.11-29 is reciprocal with respect to terminal pairs $a-b$ and $c-d$.

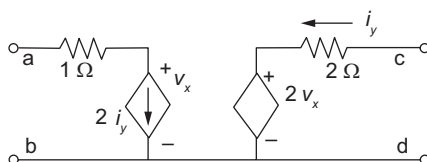


Fig. 5.11-29

31. (i) Find the value of R for maximum power transfer into it and the value of power transferred to it in the circuit in Fig. 5.11-30. (ii) Calculate the power loss in all resistors in the power delivery circuit and find efficiency power transfer with half of the value calculated in the first step for R .

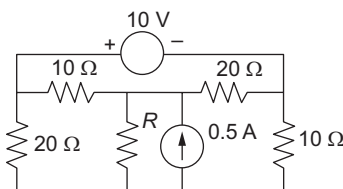


Fig. 5.11-30

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32. A resistor of 20Ω connected across $a-b$ in the circuit of Fig. 5.11-31 draws maximum power from the circuit and the power drawn is 100W . (i) Find the value of R and I_1 . (ii) With 20Ω across $a-b$ find the value of I_1 such that power transferred to it is 0W .

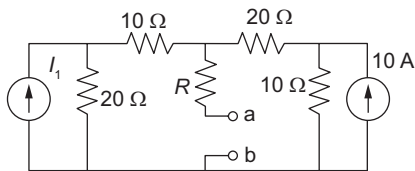


Fig. 5.11-31

33. Find the value of R for maximum power transfer in the circuit of Fig. 5.11-32 and the ratio v_x/v_s with this value of R .

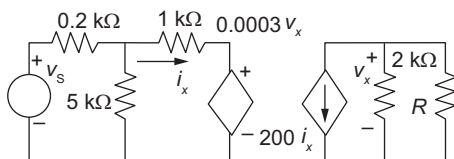


Fig. 5.11-32

34. A composite load consisting of a resistor R in parallel with a 6V DC source in series with 3Ω is connected across terminal pair $a-b$ in the circuit of Fig. 5.11-33. (i) Find the value of R such that maximum power is delivered to the load circuit. (ii) Find the current in the 6V source under this condition.

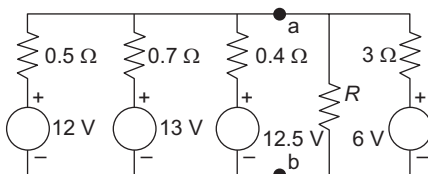


Fig. 5.11-33

35. Find R such that maximum power is transferred to the load connected to the right of $a-b$ in the circuit in Fig. 5.11-34.

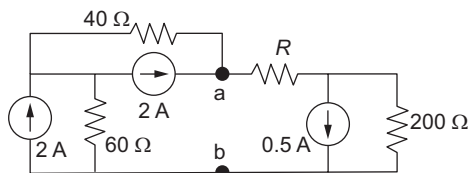


Fig. 5.11-34



Power and Energy in Periodic Waveforms

CHAPTER OBJECTIVES

- To explain the need for sinusoidal waveforms and the importance of sinusoidal analysis.
- To explain the concepts of phase, phase difference, phase lag/lead, phase delay/advance, time delay/advance etc. in the context of sinusoidal waveforms.
- To introduce instantaneous power, cyclic average power and average power in periodic waveforms.
- To define effective value of periodic waveforms and illustrate rms calculations through examples.
- To develop and explain the power superposition principle and to emphasise the pitfalls in applying this principle.
- To develop an expression for effective value of composite periodic waveform and point out pitfalls in applying the result.

INTRODUCTION

Electrical power generation, transmission and distribution employ sinusoidal voltage and current waveforms to carry power. That, in itself, is a sufficient reason for a detailed discussion on steady-state analysis of circuits containing R , L and C .

A sinusoidal waveform is completely specified by three parameters. For example, if $v_s(t) = A \sin(\omega t + \theta)$ V, this waveform is completely specified by three numbers – A , ω and θ . Therefore, sending a pure sine wave from a transmitter to a receiver in a communications context is pointless because such a waveform cannot carry any information other than that contained in just three numbers. And, for that matter, no waveform that is known completely beforehand can carry any information. Certain degree of uncertainty in the waveform to be transmitted is a precondition for information transmission from one location to another. Hence, the correct mathematical description of an information-bearing

6.2 | Power and Energy in Periodic Waveforms

signal can only be a statistical description. However, despite this, the entire area of Electronic and Communication Engineering relies heavily on sinusoidal analysis of circuits and systems.

This raises two questions – (i) Why did the Electrical Power Industry prefer sinusoidal waveform to any other waveform? (ii) Why does Electronics and Communications Engineering concern itself with sinusoidal analysis though a single-frequency sinusoid is hardly ever employed in a communication system?

These two questions are first answered in this chapter. Subsequently, the concepts of instantaneous power, average power, effective value of waveforms etc. are developed for sinusoidal waveforms as well as for other periodic waveforms.

6.1 WHY SINUSOIDS?

An Electrical Power System, like any other electrical circuit, contains passive electrical elements like resistors, inductors, capacitors, mutually coupled coils and active elements in the form of independent voltage sources and current sources. The elements in a power system are modelled by linear time-invariant elements to a first degree of approximation. Resistors produce voltage drops across them that are proportional to the current flowing through them. Inductors demand a voltage that is proportional to the rate of change of current through them. Capacitors demand a voltage across them that are proportional to the integral of current through them.

Electrical power system is usually voltage-driven. That is, independent voltage sources connected at various points in the system serve as sources of power in the system. The voltage sources connected at various source nodes (*i.e.*, the generating stations) drive currents through the series interconnections in the system, get modified by the voltage drops produced across various series path elements and appear as load voltage at load nodes in a modified form. Shunt elements connected from various nodes to the reference node in the system also influence this process of transformation of source voltages into load voltages.

Voltage drops across various elements thus modify the load voltage with respect to source voltage. These voltage drops are decided by a *scaling* of current by resistance value in the case of a resistor. It is decided by *derivative of current* in the case of an inductor and by *integral of current* in the case of a capacitor.

A time-function retains its waveshape when multiplied by a constant. But, in general, it does not maintain its waveshape on *differentiation* and *integration*. Therefore, it follows that, in general, voltages and currents at various locations in an interconnected electrical network will have different waveshapes even if all sources in the network have same waveshape.

That would surely complicate things in an Electrical Power System. In fact, it will not be a viable system at all. That prompts us to raise the question – is there any waveshape that will be *invariant* to time-domain differentiation and integration?

A generalised exponential function, $Ae^{\alpha t}$, has this property, as may be verified easily. The value of α can be complex. If α is real and positive, it represents a growing exponential. Such a waveform is not suited in an electrical system that is expected to operate steadily for extended duration. If α is real and negative, it represents a real decaying waveform that tapers down to zero sometime. An electrical system excited by a set of such sources will settle down ultimately to a state in which all voltages and currents everywhere will be zero. Obviously, such source waveforms cannot help the system to deliver power to loads in a steady manner for extended duration. If $\alpha = \gamma + j\omega$, the exponential function is a complex function of time and is given by $Ae^{\gamma t} \cos \omega t + j Ae^{\gamma t} \sin \omega t$. We

cannot generate an imaginary waveform in a physical system. But that problem can be solved by generating $0.5(Ae^{\alpha t} + Ae^{-\alpha t})$ which is equal to $Ae^{\gamma t} \cos(\omega t)$. Here too, the same objections we raised for a real exponential will hold if γ is non-zero. Therefore, we conclude that $0.5(Ae^{j\omega t} + Ae^{-j\omega t}) = A \cos \omega t$ or $-j0.5(Ae^{j\omega t} - Ae^{-j\omega t}) = A \sin \omega t$ are the possible choices for electrical power system source functions.

Thus, *sinusoidal signals* have gained their pre-eminent position in Electrical Power Engineering since they belong to a special class of time-functions that preserve their waveshape under *differentiation* and *integration*. A linear network excited by *sinusoidal sources* of a particular frequency will have *sinusoidal voltages and currents at the same frequency* everywhere in the system in the long run.

We observe that a *constant time-function* (i.e., DC voltages and currents) is a special case of $Ae^{\alpha t}$ with $\alpha = 0$ and hence must satisfy the waveshape-preservation requirement. Thus, it must be possible to generate, transmit and deliver steady power to loads in an electrical system by means of DC sources. It is indeed possible and that was how Electrical Power Industry started in late 19th century. But the problem associated with DC system was total inflexibility with respect to voltage and current levels used in the system. For instance, if the customer had to be given 220V at his premises, the generators had to generate 220V and all the interconnection system had to work at that voltage level. As the load level increases, the current flow everywhere become excessive and generation/transmission become inefficient due to resistive losses everywhere in the system. It would have been very convenient if generation could be done at a voltage level economical from the point of view of electrical machine design and operation. Similarly, it would have been convenient if the transmission of power through transmission lines could be done at high voltage level so that the current level and consequently losses in lines would decrease. But this calls for generation at low voltage level, transmission at high voltage level and consumption at low voltage level. An efficient ‘voltage level conversion unit’ is needed for this. Such voltage level conversion equipment for DC at high power levels was simply not available at the initial stages of evolution of power systems in late 19th and early 20th centuries. And the same task turned out to be very easy in the case of a sinusoidal voltage system. A two-winding transformer can easily and efficiently change voltage levels in a system in the case of sinusoidal voltages. Thus, economic generation and transmission of huge quantities of electrical power became possible with sinusoidal voltage system and transformers. This is another reason why Electrical Power Industry is firmly committed to sinusoidal waveforms.

Advances in an area called Power Electronics resulted in power system loads becoming non-linear in nature progressively from early 1980. Power Electronic Equipment (AC to DC converters, thyristorised DC motor drives, variable speed AC drives, uninterruptible power supplies, HVDC transmission systems etc.) process the sinusoidal system power using non-linear devices for a variety of purposes. Non-linear loads draw *non-sinusoidal* currents from sinusoidal voltages. Non-sinusoidal current waveforms, subjected to differentiation and integration in various circuit elements, result in non-sinusoidal voltage drops across them. These non-sinusoidal voltage drops, in combination with the sinusoidal voltages produced at various generating stations, result in a non-sinusoidal voltage at load nodes in a power system. This is called the *Power System Harmonics Problem*. Another very important feature of sine waves helps in analysing electrical systems that have non-sinusoidal periodic waveforms.

A broad class of periodic non-sinusoidal waveforms can be expressed as an infinite sum of sinusoidal waveforms with frequencies that are integer multiples of frequency of the periodic wave. For instance, $a \pm 1$ V square wave with 1 cycle per second frequency can be expressed as $(4/\pi)[\sin 2\pi t + (1/3) \sin 6\pi t + (1/5) \sin 10\pi t + \dots + (1/n) \sin 2n\pi t + \dots]$. Infinite terms are needed; but the amplitude

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of sine wave decreases as the order of the term increases. It is possible to truncate this kind of series expansion of a periodic waveform to obtain reasonably accurate results in circuit analysis problems. Thus, series expansion of non-sinusoidal periodic waveforms in terms of sinusoidal waveforms and Superposition Theorem will help us to solve a circuit driven by such periodic waveforms provided we know how to solve it for a sinusoidal waveform. Thus, even a power system under the influence of non-linear loads can be analysed by sinusoidal analysis techniques with the help of this kind of series expansion for non-sinusoidal periodic waveforms.

Further, it turns out that, this series expansion of a periodic waveform leads to an expansion of even *aperiodic waveforms* in terms of sinusoidal waveforms as a limiting case. The series expansion of periodic waveform referred here is called *Fourier Series*, and the expansion of an arbitrary aperiodic waveform in terms of sinusoidal functions is called *Fourier Transforms*.

Fourier Series and Fourier Transforms along with Superposition Theorem help us to solve an electrical circuit driven by sources with arbitrary source functions by solving it for sinusoidal excitation. This is yet another factor that leads to pre-eminence of sinusoidal waveforms in circuit analysis. In fact, this is *the* reason why Electronics and Communication Engineers use sinusoidal analysis at all.

Thus, sinusoidal waveforms and sinusoidal analysis of circuits is of great importance to electrical and electronic circuits due to the following reasons:

- Sinusoidal waveforms preserve their waveshape in linear circuits.
- Sinusoidal waveform render the voltage levels in electrical and electronic systems flexible so that optimisation of sub-system performance in various parts of the system by choosing a suitable voltage level in that part of the system becomes possible. Transformers help us to realise this voltage flexibility.
- Periodic non-sinusoidal waveforms as well as a broad class of aperiodic waveforms can be expressed as a sum of sinusoids by Fourier series and Fourier transforms. Hence, circuit solution with such input waveforms can be obtained with relative ease if solution for a sinusoidal input is known.

6.2 THE SINUSOIDAL SOURCE FUNCTION

The sinusoidal voltage source function is generated in Synchronous Generators in Electrical Power Systems and by low-power electronic oscillator circuits in Electronic Systems, Communication Systems and Instrumentation/Measurement Systems. A source will have to be switched on at some point in time. Switching on a source may be a simple affair of switching on the DC power supply as in the case of an electronic sinusoidal oscillator circuit or switching on the AC mains to a function generator in the laboratory. It may be a complicated affair involving a sequence of steps as in the case of a bringing a generator in a nuclear power station online.

Moreover, the sinusoidal voltage source may not start producing a sinusoidal output as soon as it is powered up. Usually, it goes through a transient period during which its output builds up. During this period, the output will not be a pure sine wave. After an initial period of adjustment, it starts delivering sinusoidal output to whatever that is connected at its output.

Even if it produces a sine wave right from the instant at which it is powered up, it may not start at zero position in the waveform or at peak position in the waveform.

We assume in this section that the sinusoidal sources have been powered up in the past and have become steady. Moreover, we assume that they started at zero position on a sine wave when they were switched on. The concepts we evolve are not really dependent on these assumptions. The assumptions are made only to render clarity to the discussion that follows.

6.2.1 Amplitude, Period, Cyclic Frequency, Angular Frequency

Consider a single sinusoidal voltage source that was powered up in the past. We start observing the waveform of the voltage output in an oscilloscope from a particular point in time. We assign zero value to the time variable at the instant we start our observation of the waveform. Thus, the source was powered up in the past with respect to the instant at which we start observing it. The observed waveform is shown in Fig. 6.2-1. The time variable t is used in the horizontal axis.

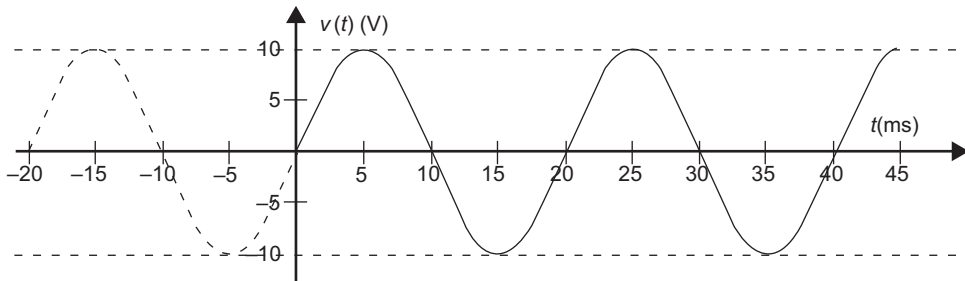


Fig. 6.2-1 Waveform of a sinusoidal voltage

The maximum positive value attained by the waveform is seen to be 10 V and the maximum negative value attained is -10 V. This quantity is called the *amplitude* of the sinusoidal waveform. It so happened that the waveform was crossing zero from negative value to positive value at $t = 0$. This zero crossing is called the *positive-going zero crossing*. The zero crossing that happens when the voltage is crossing over from positive value to negative value is termed as *negative-going zero crossing*. That $t = 0$ happens to be a positive-going zero crossing is a coincidence. But as a result of that coincidence, we are now free to write the voltage waveform that we observe from $t = 0$ onwards as $v(t) = 10 \sin \omega t$. We need to work out the meaning and value of ω .

We observe from Fig. 6.2-1 that the sinusoidal voltage completes one full cycle of variation in 20 ms. That is, if we start at any t and move through the waveform till we reach $t + 20$ ms, we will find that the instantaneous voltage at $t + 20$ ms is the same as the instantaneous voltage at t . Moreover, the shape of voltage variation in any $(t + n \times 20, t + 20 + n \times 20)$ interval is same as in the interval $(t, t + 20)$, where the unit of time is in ms and n is a positive integer. Thus, the waveshape is repetitive with its basic repeating unit decided by any 20ms interval. That is, the waveform is periodic from the instant we start observing it. The *period* of this waveform is 20ms. In general, period of a periodic waveform is the time interval needed to complete one full cycle of the waveform. Symbol ' T ' is used to represent the period of a periodic waveform. In other words, it is the width of the basic repeating unit of the periodic waveform in the time-axis.

The number cycles of variation that the waveform goes through in one second is defined as its *cyclic frequency*. The qualifier '*cyclic*' is often dropped when there is no cause for ambiguity or when the unit employed makes it clear that it is cyclic frequency that is being referred to. The unit of *cyclic frequency* is '*cycles-per-second*' and is given a name *Hertz*. Hertz is written in short form as Hz. The shortened form '*cps*' is also used to designate the unit of cyclic frequency. The cyclic frequency of the waveform in Fig. 6.2-1 is $1/20\text{ms} = 50$ Hz. *Cyclic frequency* is usually indicated by the symbol ' f '.

A sinusoidal function of an angle is periodic with a period of 2π radians. Thus, the argument of the sinusoidal function in a sinusoidal waveform will go through an increment of 2π radians in one period. Therefore, the increment in the argument of the trigonometric function in one second will be $2\pi/T$

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radians, where T is the period of waveform ($= 1/f$). This quantity, which represents the *rate of change of angle argument of the sinusoidal function with respect to time*, is defined as the *angular frequency* or *radian frequency* of the sinusoidal waveform and is usually represented by the symbol ω (lower case omega). The unit of *angular frequency* is radians/seconds, abbreviated as rad/s. Thus,

$$f = \frac{1}{T} \text{ Hz}, \quad \omega = \frac{2\pi}{T} = 2\pi f \text{ rad/s}, \quad T = \frac{2\pi}{\omega} = \frac{1}{f} \text{ s}$$

The sinusoidal waveform of voltage source $v(t) = 10 \sin 100\pi t$ shown against t in Fig. 6.2-1 is redrawn against the angular argument ωt in Fig. 6.2-2.

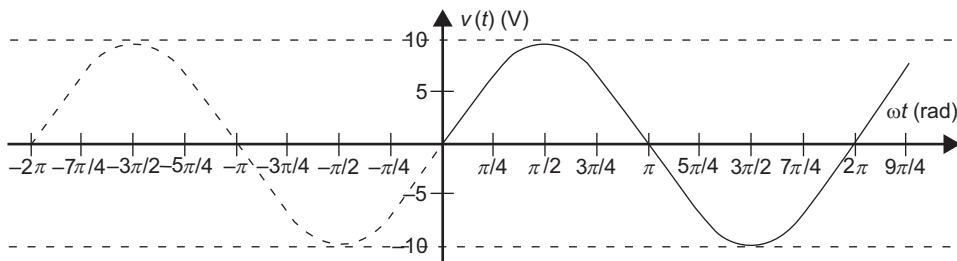


Fig. 6.2-2 Sinusoidal waveform $v(t)$ plotted against ωt

A sinusoidal source voltage waveform that undergoes a positive-going zero crossing at $t = 0$ can be expressed as $v(t) = A \sin \omega t = A \sin (2\pi/T) t = A \sin 2\pi f t$ V, where A is its *amplitude*, T is its *period* in s, f is its *cyclic frequency* in s^{-1} (Hertz, Hz) and ω is its *radian frequency* or *angular frequency* in rad/s.

6.2.2 Phase of a Sinusoidal Waveform

Consider the observed sinusoidal waveform shown in Fig. 6.2-3. This waveform may be interpreted as a sinusoidal source powered up 17.5 ms prior to the start of observation. Assume that the source started delivering the sine wave output from zero value. The instantaneous value observed at the starting instant of observation is not zero, but -7.07 V. The amplitude is observed to be 10 V and the frequency is seen to be 50 Hz. Thus, we can express this waveform as $v(t) = 10 \sin (100\pi t + \theta)$, where θ is an angle to be determined from the observed amplitude and instantaneous value at $t = 0$. The value at $t = 0$ is $10 \sin \theta$ and this is seen to be -7.07 V. Therefore, $\theta = 45^\circ = \pi/4$ rad.

$$\therefore v(t) = 10 \sin(100\pi t - \frac{\pi}{4})$$

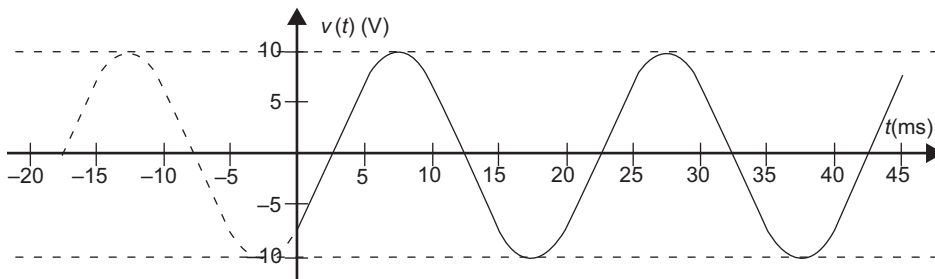


Fig. 6.2-3 Sinusoidal waveform with a non-zero phase

This waveform is plotted against ωt in Fig. 6.2-4. Note that the instantaneous value of $v(t)$ at $\omega t = 100\pi t = \pi/4$ rad is zero as it should be.

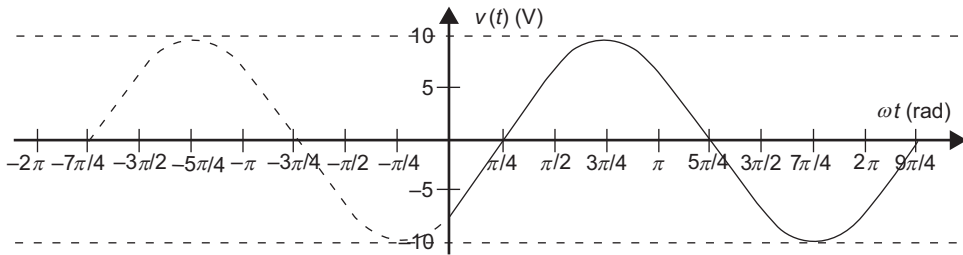


Fig. 6.2-4 A sinusoidal waveform with non-zero phase plotted against ωt

The argument of trigonometric function in the expression for a sinusoidal waveform is always in radians. However, it is permitted to write the angle θ in degrees provided the symbol of degree is clearly shown. That is, $v(t)$ can be written as $10 \sin(100\pi t - 45^\circ)$ V, but not as $10 \sin(100\pi t - 45)$. In the latter case, 45 will be interpreted with radian unit. Moreover, though the form $10 \sin(100\pi t - 45^\circ)$ V is permitted, 45° has to be converted into radians before subtracting from the value of $100\pi t$ for some particular t before evaluating the sine function. In short, the form $10 \sin(100\pi t - 45^\circ)$ is allowed only as a *notation* and not for *function evaluation*. The angular argument has to be in radians when an evaluation of trigonometric function is attempted.

The quantity θ in $v(t) = A \sin(\omega t + \theta)$ is defined as the *phase* of the sinusoidal function.

6.2.3 Phase Difference Between Two Sinusoids

Consider the situation in Fig. 6.2-5. Two observers – A and B – use XY-Recorders A and B, respectively to record the output from two sinusoidal voltage sources $v_1(t)$ and $v_2(t)$ which had been powered up in the past. A closes the two switches onto his recorder at $t = 0$. t represents the time-axis chosen by him. B closes the switches onto his recorder at $t' = 0$. t' represents the time-axis chosen by B. $v_1(t)$ has an amplitude of V_{m1} and $v_2(t)$ has an amplitude of V_{m2} .

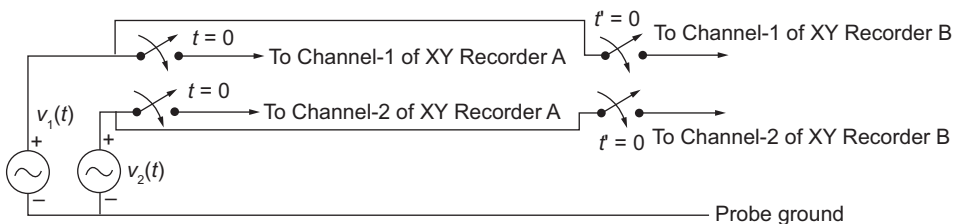


Fig. 6.2-5 Simultaneous observation of two sinusoidal sources by two observers with different starting instants for observation

The waveforms recorded by Observer A are shown in Fig. 6.2-6(a) by the solid curve. The dotted curve shows the sinusoidal variation of sources prior to recording and will not show up in the recorder output. The waveforms in Fig.(a) are normalised with respect to their respective amplitude values to obtain the waveforms in Fig.(b).

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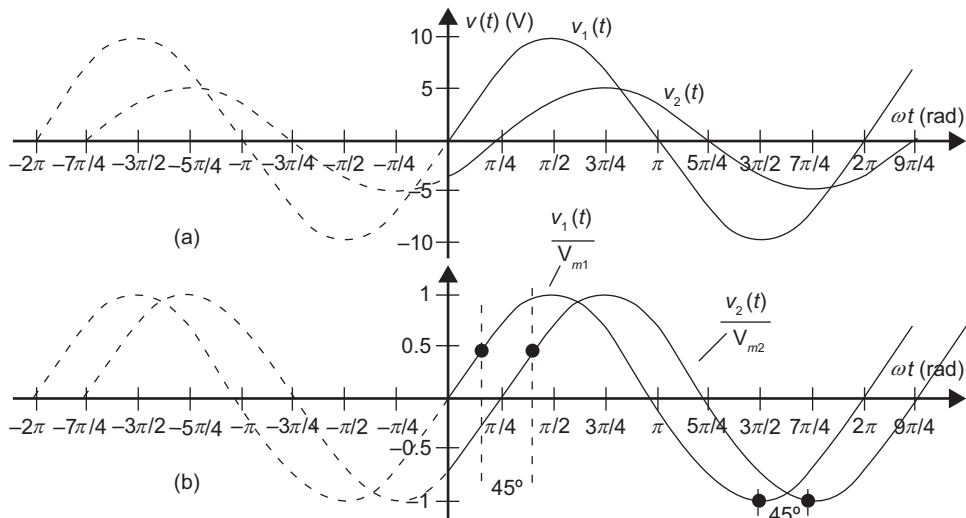


Fig. 6.2-6 Waveform observation by Observer A

Two pairs of *similarly located waveform points within a cycle period* are located in the waveshape of $v_1(t)/V_{m1}$ and $v_2(t)/V_{m2}$ as shown in Fig. 6.2-6(b). Observer A notes that similarly located points in the two waveforms are separated by 45° in angle argument. A also notes that points on $v_2(t)$ come *after* (in a visual sense) similarly located points on $v_1(t)$.

Similar observations recorded by Observer B in $\omega t'$ axis is shown in Fig. 6.2-7.

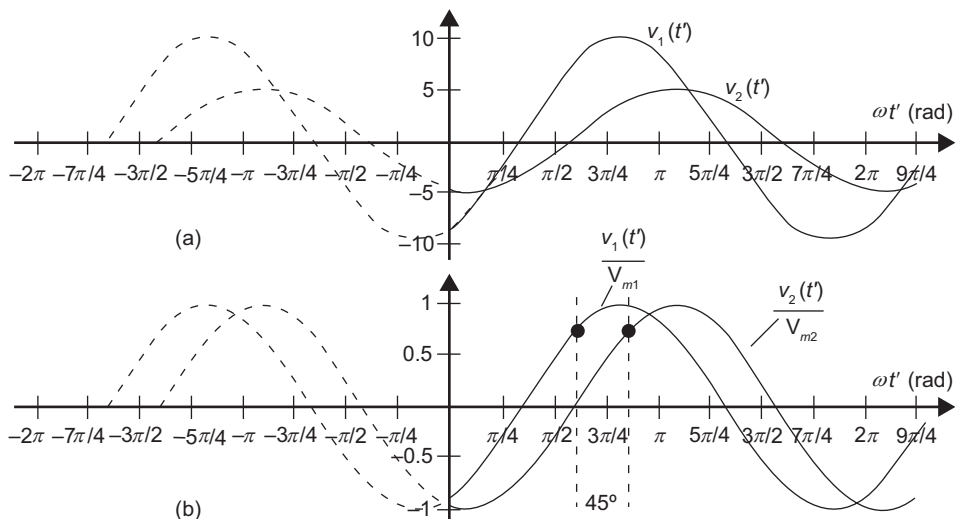


Fig. 6.2-7 Waveform observations by Observer B

B too measures 45° angular separation between similarly located points on normalised $v_1(t)$ and $v_2(t)$. Moreover, B too observes that points on $v_2(t)$ come *after* similarly located points on $v_1(t)$.

The angular difference between similarly located points within a cycle period on two normalised sinusoidal waveforms (normalised with respect to their respective amplitude values) with same frequency is defined as the *phase difference* between them. The *phase difference* between two sinusoids is independent of choice of origin in t or ωt axis. The precedence relationship [i.e., which comes after (in a visual sense) which] between them in t or ωt axis too is independent of choice of origin.

However, based on the observed amplitudes, values at origin and the position of first zero-crossing, A will write the sinusoidal functions as $v_1(t) = 10 \sin 100\pi t$ V and $v_2(t) = 5 \sin(100\pi t - 45^\circ)$ V. Moreover, B will conclude that $v_1(t') = 10 \sin(100\pi t' - 60^\circ)$ V and $v_2(t') = 5 \sin(100\pi t' - 105^\circ)$ V. Thus, the *phase* of $v_1(t)$ is 0° and *phase* of $v_2(t)$ is -45° as far as A is concerned. And they are -60° and -105° , respectively as from B's point of view.

The *phase* of a sinusoidal waveform depends on the choice of origin in t or ωt axis. *Phase difference* between two sinusoidal waveforms at same frequency does not.

When a waveform point on a sinusoidal function $v_2(t)$ appears *after* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ with *same frequency*, $v_2(t)$ is said to *lag* $v_1(t)$ in phase and the corresponding phase difference between them is called a *lag phase angle* under this condition.

Similarly, when a waveform point on a sinusoidal function $v_2(t)$ appears *before* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ with *same frequency*, $v_2(t)$ is said to *lead* $v_1(t)$ in phase and the corresponding phase difference between them is called a *lead phase angle* under this condition.

It must be obvious that if $v_2(t)$ *lags* $v_1(t)$, then $v_1(t)$ must necessarily *lead* $v_2(t)$. Moreover, if $v_2(t)$ *lags* $v_1(t)$, then $v_2(t')$ will also *lag* $v_1(t')$, where t' is a new time variable as result of a different choice of origin.

6.2.4 Lag or Lead?

Similarly located points on two sinusoidal waveforms with same frequency have to be located within a period of the waveforms. But this leads to two choices for locating the point on the second waveform after having chosen a point on the first waveform (refer Fig. 6.2-8).

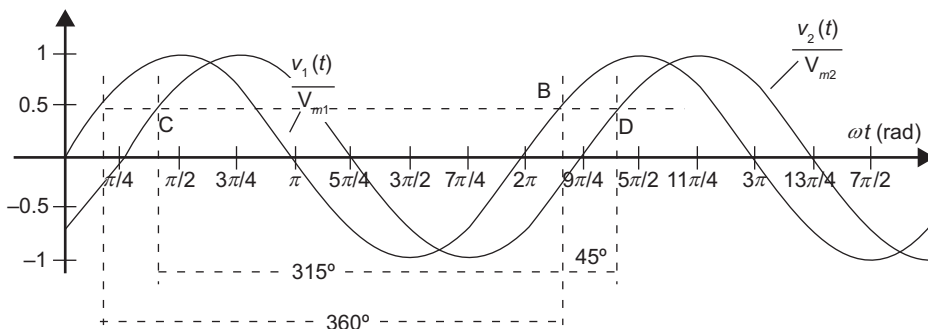


Fig. 6.2-8 Relationship between phase lag and phase lead

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We locate point B on normalised $v_1(t)$ first. We are free to locate the similarly located point on normalised $v_2(t)$ on either side of B within a span of 2π radians or 360° . This gives us two choices – point C and point D on the second waveform. If we choose point C, we can conclude that $v_2(t)$ *leads* $v_1(t)$ by 315° . If we choose point D, we conclude that $v_2(t)$ *lags* $v_1(t)$ by 45° . Therefore, a lag angle of θ radians and a lead angle of $(2\pi - \theta)$ radians mean the same. As a convention, we favour the angle that turns out to be less than 180° (or π radians). Thus, in Fig. 6.2-8, we will term it as a lag angle of 45° .

6.2.5 Phase Lag/Lead Versus Time Delay/Advance

It leaves a certain ambiguity about which waveform is *after* the other. If we accept the point C on $v_2(t)$ in Fig. 6.2-8 as the point corresponding to point B on $v_1(t)$, we conclude that $v_2(t)$ comes *before* $v_1(t)$. Similarly, If we accept point D on $v_2(t)$ as the point corresponding to point B on $v_1(t)$, we conclude that $v_2(t)$ comes *after* $v_1(t)$. However, note carefully that we had been careful to keep the precedence relationships –*before* and *after* – only in relation to our visual perception of the waveform plots. We have not yet ascribed temporal significance to these terms. That is, we have *not* stated till now that if a waveform $v_2(t)$ comes *after* $v_1(t)$ in a *visual sense*, then $v_2(t)$ started later than $v_1(t)$ in time. In other words, we have not correlated the *phase difference* between two waveforms with *time delay* or *time advance* between them. The term ‘*phase lag*’ tends to give us an impression that the waveform that *lags* behind suffered some *time delay* with respect to the other waveform. But this impression can be wrong. Similarly, the waveform that *leads ahead* of another waveform did not necessarily start earlier. The reader is cautioned against equating a ‘*phase lag*’ with a ‘*time delay*’ and a ‘*phase lead*’ with a ‘*time advance*’ indiscriminately. There are situations in which a ‘*phase lag (lead)*’ implies a ‘*time delay (advance)*’ – in that case, we will term the phase lag (lead) as ‘*phase delay (advance)*’. And there are situations in which *lag/lead* cannot be uniquely correlated to *delay/advance* in time-domain.

Consider the two waveforms $v_1(t)$ and $v_2(t)$ in Fig. 6.2-9(a). Additional information in the form of dotted curves is also shown in Fig.(a). The frequency of waveform is 50 Hz. The waveforms in Fig.(a) show that the source $v_1(t)$ started generating a sinusoidal voltage at 20 ms before the observation started and $v_2(t)$ started only 2.5 ms later. It is also clear that $v_2(t)$ lags $v_1(t)$ by 45° . Thus, a *time delay* of 2.5 ms has resulted in a *phase lag* of 45° . Since we know from the additional information provided in the form of dotted curves that the phase difference between two sources resulted from a time delay, we can term this 45° phase lag as a 45° *phase delay* too. Obviously, the following relation between *time delay* and *phase delay* holds:

$$\begin{aligned}\text{Phase delay } \phi \text{ in radians} &= \omega \times \text{time delay } t_d \text{ in seconds} \\ \text{i.e., } \phi &= \omega t_d\end{aligned}$$

Similar statement can be arrived at in the case of *time advance* too, provided we know from information other than we obtained from observing the two waveforms from $t = 0$ that the observed phase difference is due to a *time advance*.

$$\begin{aligned}\text{phase advance } \phi \text{ in radians} &= \omega \times \text{time advance } t_d \text{ in seconds} \\ \text{i.e., } \phi &= \omega t_d\end{aligned}$$

Now, consider the waveforms in Fig. 6.2-9(b). Here, $v_1(t)$ started 17.5 ms earlier than $v_2(t)$. But if we go only by the observation from $t = 0$ onwards, we will conclude that $v_2(t)$ *leads* $v_1(t)$ by 45° or

equivalently $v_2(t)$ lags $v_1(t)$ by 315° , and, as per the agreed convention, we will settle for ' $v_2(t)$ leads $v_1(t)$ by 45° '. But, in the light of the additional information given in the form of dotted curves, a translation to the effect that $v_2(t)$ started 2.5 ms earlier than $v_1(t)$ will be in error. Actually, $v_2(t)$ started 17.5 ms (315°) after $v_1(t)$ and hence *phase delay* of $v_2(t)$ is 315° and *time delay* of $v_2(t)$ is 17.5 ms with respect to $v_1(t)$. The conclusion from observation from $t = 0$ onwards can also be stated as $v_1(t)$ lags $v_2(t)$ by 45° . Again, a translation to the effect that $v_1(t)$ started 2.5 ms after $v_2(t)$ started is wrong in the light of additional information given. Actually, $v_1(t)$ started 17.5 ms before $v_2(t)$ started and hence $v_1(t)$ has a *phase advance* of 315° and a *time advance* of 17.5 ms with respect to $v_2(t)$.

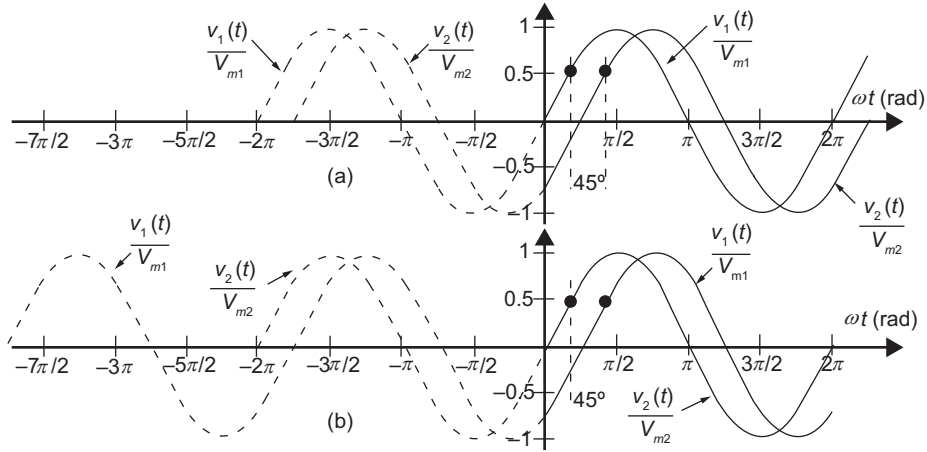


Fig. 6.2-9 Illustrating phase delay and time delay

Thus, additional information is required to translate an observed *phase lag/lead* relationship between two sinusoidal waveforms into a *phase delay/advance* (equivalently, *time delay/advance*) relationship between them. Phase lag is not necessarily a phase delay and phase lead is not necessarily a phase advance. Phase lag does not necessarily imply time delay and phase lead does not necessarily imply time advance.

The additional information needed to decide *time delay/advance* from *phase lag/lead* is not usually available in the case of multiple sinusoidal source waveforms in a complex electrical system. But then, we do not usually need the *time delay/advance* information in Electrical Power Systems.

There is one situation in which this additional information needed is invariably available. Consider a situation in which a sinusoidal voltage source is applied to a linear electrical network at $t = 0$. The circuit variables respond to this excitation and assume pure sinusoidal variation at the same frequency as that of sinusoidal excitation in the long run. There will be a definite phase difference between a response variable (may be a current in some element or a voltage across some element) and the source function. *No physical system can produce a response before the excitation is applied to it.* Response always follows the excitation in a physical system and cannot precede excitation. This intuitively obvious fact is known as the '*law of causality*' for physical systems. Thus, law of causality of physical systems effectively states that the response will be *delayed* with respect to excitation. *Therefore, the response sinusoid in an electrical circuit will always be delayed with respect to the excitation sinusoid quite regardless of whether the phase difference is a lag angle or lead angle. A phase lead that the response variable exhibits with respect to excitation variable in a physical electrical network has to be understood as a phase delay that is more than π radians and a time delay that is more than*

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half-period. The reader is cautioned against the commonly made mistake of assuming that the apparent phase lead exhibited by a response variable implies a phase advance or time advance.

Example: 6.2-1

Two sinusoidal waveforms, $x(t)$ and $y(t)$, recorded from $t = 0$ are shown in Fig. 6.2-10. $x(0) = -2.571$ and $y(0) = 1.25$ (i) Express $x(t)$ and $y(t)$ as sine functions and identify their amplitude, period, cyclic frequency, radian frequency, phase and phase difference. (ii) If no additional information is available, list all possible time delay/advance relations between the two assuming that both started at positive-going zero-crossing position. (iii) If $x(t)$ is voltage waveform that was powered up and applied to a linear electrical circuit long back in the past and $y(t)$ is the current flow in some element in that circuit, find the time delay/advance between the two and phase delay/advance between the two.

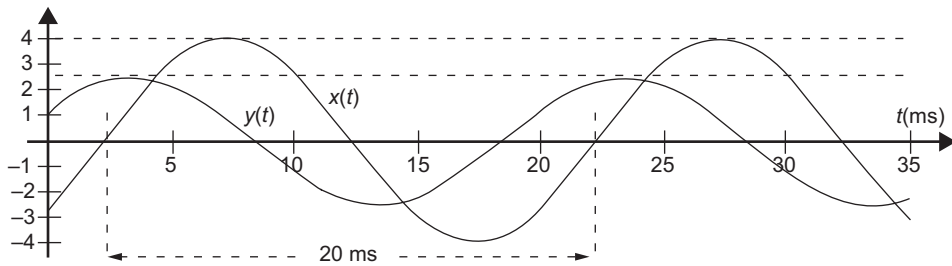


Fig. 6.2-10 Sinusoidal waveforms for Example 6.2-1

Solution

- (i) The period of both waveforms, $T = 20$ ms. Therefore, $f = 50$ Hz and $\omega = 100\pi$ rad/s. Amplitude of $x(t)$ is 4 and amplitude of $y(t)$ is 2.5.

Therefore, $x(t) = 4 \sin(100\pi t + \theta_x)$ and $y(t) = 2.5 \sin(100\pi t + \theta_y)$, where θ_x and θ_y are the phases of $x(t)$ and $y(t)$, respectively. The values of $x(t)$ and $y(t)$ at $t = 0$ are given as -2.571 and 1.25 , respectively.

$$\therefore \sin \theta_x = -2.571/4 = -0.6428 \Rightarrow \theta_x = -40^\circ \text{ or } -140^\circ.$$

The choice between -40° and -140° is made by observing that if it is -40° , the first zero-crossing of the waveform that takes place at $\omega t = -\theta_x$ will take place within the first quarter cycle and if it is -140° , the first zero-crossing after $t = 0$ will take place after first quarter cycle. In the present case, $x(t)$ crosses zero within first 5 ms (first quarter cycle) and hence $\theta_x = -40^\circ$.

$$\therefore x(t) = 4 \sin(100\pi t - 40^\circ)$$

Similarly, $\sin \theta_y = 1.25/2.5 = 0.5 \Rightarrow \theta_y = 30^\circ \text{ or } 150^\circ$. Now the first zero crossing will take place at $\omega t = 150^\circ$ position (*i.e.*, in the second quarter cycle) if $\theta_y = 30^\circ$ and it will take place at $\omega t = 30^\circ$ position (*i.e.*, in the first quarter cycle) if $\theta_y = 150^\circ$. Since it takes place within the second quarter cycle in the present case, $\theta_y = 30^\circ$

$$\therefore y(t) = 2.5 \sin(100\pi t + 30^\circ)$$

The phase difference between the two waveforms has a magnitude of $30^\circ - (-40^\circ) = 70^\circ$ and $x(t)$ lags $y(t)$ by 70° . Equivalently, $y(t)$ leads $x(t)$ by 70° . Equivalently, $x(t)$ leads $y(t)$ by 290° and $y(t)$ lags $x(t)$ by 290° .

- (ii) 70° phase difference translates to $20 \times 70/360 \approx 3.89$ ms time interval. The following are the possibilities:
- (a) $y(t)$ started $(3.89 + 20n)$ ms *before* $x(t)$ and therefore $y(t)$ has a *phase advance* of $(70^\circ + n360^\circ)$ with respect to $x(t)$, where $n = 0, 1, 2, 3, \dots$. This may also be restated as $x(t)$ has a *phase delay* of $(70^\circ + n360^\circ)$ with respect to $y(t)$.
 - (b) $x(t)$ started $(16.11 + 20n)$ ms *before* $y(t)$ and therefore $x(t)$ has a *phase advance* of $(290^\circ + n360^\circ)$ with respect to $y(t)$ where $n = 1, 2, 3, \dots$. This may be restated as $y(t)$ has a *phase delay* of $(290^\circ + n360^\circ)$ with respect to $x(t)$.
- (iii) Now, $x(t)$ is the *cause* and $y(t)$ is the *effect* in a physical electrical circuit. Therefore, by law of causality, $y(t)$ can have only a *phase delay* with respect to $x(t)$. Therefore, $y(t)$ has a *phase delay* of $(290^\circ + n360^\circ)$ with respect to $x(t)$.

However, this does not mean that the circuit waited for $(16.11 + 20n)$ ms after the voltage waveform was applied to it doing nothing in that interval and then started producing a sinusoidal current in the element under consideration! What happens in the electrical circuit is that, as soon as the voltage waveform is applied, the circuit starts a mixed response that includes even non-sinusoidal terms caused by the electrical inertia of the circuit. This period is called the transient period. The non-sinusoidal components in the response die down to zero as time progresses. After a sufficiently long duration decided by circuit parameters, a steady-state comes up in the circuit in which all the response variables become pure sinusoidal waveforms at the same frequency as that of excitation. And, by that time, the current $y(t)$ would have acquired a steady *phase delay* of $(290^\circ + n360^\circ)$ with respect to the applied voltage $x(t)$. The value of n that is applicable can be obtained only if the circuit is known in detail and a full circuit solution is obtained.

Phase lag/lead between various voltages and currents in an Electrical Power System has profound implications in the economic operation of the system. Phase delay and time delay between various sinusoidal waveforms in an electronic system or communication system has great significance in terms of waveform distortion and loss of information contained in a waveform. Therefore, Electrical Power Engineers pay a great deal of attention to *phase lag/lead* between waveforms whereas Electronics and Communication Engineers place even higher emphasis on *phase delay/advance* and *time delay/advance* between waveforms. That is why these terms were discussed in detail in this section.

6.3 INSTANTANEOUS POWER IN PERIODIC WAVEFORMS

A waveform $v(t)$ is said to be periodic with a periodicity of T s if $v(t+nT) = v(t)$ for all integer values of n and for all t . This implies that it must be possible to identify a basic section of the waveform that lasts for T s and that repeats to infinite extent into the past and into the future. Thus, a waveform is strictly periodic only if it is ever-existent. But, in practice, waveforms are switched on at some definite time instant. Such switched waveforms cannot be called periodic waveforms in the strict sense of definition of periodicity. However, we can view them as periodic waveforms for circuit analysis purposes provided we focus our attention to time instants located far away from the instant at which the waveform was switched on.

Consider a two-terminal electrical element with the current and voltage variables marked as per *passive sign convention* in Fig. 6.3-1.

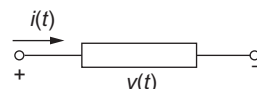


Fig. 6.3-1 A two-terminal element with voltage and current marked as per passive sign convention

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The voltage difference v_{AB} between two points A and B is the work to be done in moving $+1$ C of charge from B to A . Energy has to be spent in carrying charge from a lower potential point to higher potential point. Similarly, energy is released when a charge is allowed to fall through a higher potential point to lower potential point. The amount of charge that went through the element from a higher potential point to lower potential point in one second is given by $i(t)$. Therefore, the product of $v(t)$ and $i(t)$ must be the energy released into element in one second. The rate of change of energy is defined as *instantaneous power* and is denoted by $p(t)$.

Therefore, instantaneous power is *delivered to* a two-terminal element, $p(t) = v(t) i(t)$, where $v(t)$ and $i(t)$ are the element variables defined as per passive sign convention.

Then, energy *delivered to* a two-terminal element is obviously given by

$$E(t) = \int_{-\infty}^t p(t) dt = \int_{-\infty}^t v(t)i(t) dt = E(0) + \int_0^t v(t)i(t) dt$$

where $E(0)$ is the total energy dissipated in the element from infinite past to $t = 0$.

And the relation between the energy function $E(t)$ and the instantaneous power $p(t)$ is given by $p(t) = \frac{dE(t)}{dt}$. Let $E_i(t)$ be the energy dissipation function (*i.e.*, the net energy delivered from $-\infty$ to t) of the i^{th} element in a b element electrical circuit. Then the total energy dissipation function of the circuit is $E_T(t) = \sum_{i=1}^b E_i(t)$. The circuit considered as a whole is an isolated system and the *total energy in an isolated system is a constant* by Law of Conservation of Energy. Therefore,

$$\sum_{i=1}^b E_i(t) = A \text{ Constant.}$$

This implies that if some elements of the circuit are receiving energy, then some other elements must be losing energy.

Since the statement of conservation of energy is true on an instant-to-instant basis, we can differentiate both sides of the above equation with respect to time:

$$\frac{dE_T(t)}{dt} = 0 \Rightarrow \frac{dE_1(t)}{dt} + \frac{dE_2(t)}{dt} + \dots + \frac{dE_b(t)}{dt} = 0$$

But each term of the type $\frac{dE_i(t)}{dt}$ can be interpreted as the *instantaneous power delivered to the i^{th} circuit element*.

$$p_1(t) + p_2(t) + \dots + p_b(t) = 0 \Rightarrow v_1(t)i_1(t) + v_2(t)i_2(t) + \dots + v_b(t)i_b(t) = 0$$

Therefore, *sum of instantaneous power delivered to all elements in a circuit will be zero*. Equivalently, *sum of instantaneous power delivered by all elements in a circuit will be zero*. This implies that in any circuit some elements will be delivering positive power and the remaining elements will be receiving positive power at all instants of time. The sum of positive powers delivered will be equal to the sum of positive powers received (or consumed) at all t . This is the *principle of conservation of instantaneous*

power which is valid for all isolated circuits containing arbitrary kind of elements. *Isolated circuit* is one that has no interaction with environment.

Principle of Conservation of Instantaneous Power – The sum of instantaneous power delivered to all elements in an isolated circuit will be zero, i.e., $p_1(t) + p_2(t) + \dots + p_b(t) = 0$ where b is the total number of elements in the circuit.

Example: 6.3-1

A DC voltage of 10 V is switched on to a 10Ω resistor at $t = 0$. Find and plot $p(t)$ and $E(t)$ for $t \geq 0$.

Solution

$p(t) = \frac{V^2}{R} = 10 \text{ W}$ for $t \geq 0$ and $E(t) = \int_0^t 10 dt + 0 = 10t \text{ J}$. The waveforms of applied instantaneous power and the energy dissipated are shown Fig. 6.3-2.

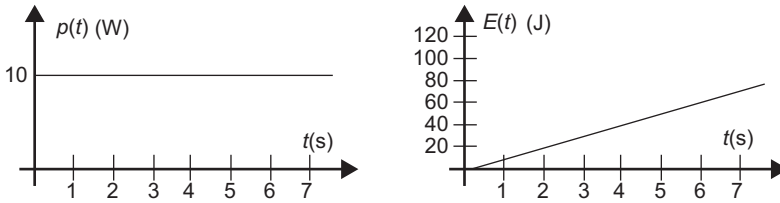


Fig. 6.3-2 Instantaneous power and energy in Example 6.3-1

Example: 6.3-2

A symmetric square wave of 100 Hz frequency and $\pm 10\text{V}$ amplitude is switched on to a resistor of 10Ω at $t = 0$ at the positive-going zero crossing of the square wave. Find and plot the instantaneous power and energy dissipation in the resistor.

Solution

$$p(t) = v(t) i(t) = [v(t)]^2 / R$$

When a symmetric square wave is squared, it becomes a constant quite irrespective of its frequency. Therefore, $[v(t)]^2 = 100 \text{ V}^2$ for $t \geq 0$.

$\therefore p(t) = 10 \text{ W}$ for $t \geq 0$ and $E(t) = 10 t \text{ J}$. The plots are same as in Fig. 6.3-1.

Therefore, we conclude that applying a symmetric square wave voltage of amplitude V and frequency f to a resistive load has the same effect as applying a DC voltage of V to it as far as the instantaneous power and energy dissipation function are concerned.

Example: 6.3-3

A periodic rectangular pulse voltage shown in Fig. 6.3-3 is applied to a resistor of 10Ω . Find and plot instantaneous power and energy dissipation function.

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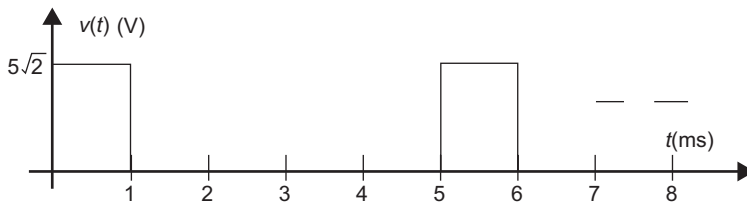


Fig. 6.3-3 The applied voltage waveform for Example 6.3-3

Solution

The waveform of $p(t)$ is obtained by squaring the waveform of $v(t)$ and dividing by 10. $E(t)$ is obtained by integrating (*i.e.*, evaluating the area under $p(t)$ curve) the power waveform. The two plots are shown in Fig. 6.3-4 for the time range [0ms, 30ms].

Energy delivery to the load takes place through power pulses in this example. As a consequence, the energy dissipated in the resistor rises non-uniformly with time. However, $E(t)$ is either increasing or constant at all t consistent with the fact that $p(t)$ never goes negative. The plot of $E(t)$ versus t shows a dotted line. This line is obtained by joining the points representing the total energy dissipation in the resistor at the end of a cycle. It turns out to be straight line with a slope of $0.01 \text{ J/ms} = 10 \text{ W}$. The power pulse located within a cycle of $p(t)$ has an amplitude of 50 W and duration of 1 ms . The cycle duration is 5 ms . Hence, the energy delivered to the resistor in each cycle is $50 \text{ W} \times 0.001 \text{ s} = 0.05 \text{ J}$. The same energy would have been delivered to the resistor in 5 ms if a constant power of 10 W were available for the entire cycle. We note that 10 W is the average of instantaneous power over one cycle period.

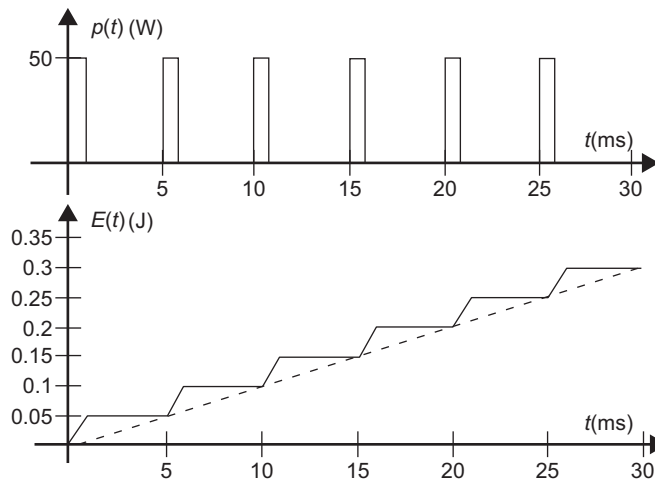


Fig. 6.3-4 Instantaneous power and energy plots for Example 6.3-3

Therefore, a line joining the end-of-cycle energy values in the energy versus time diagram will have a slope that is equal to the average value of instantaneous power over a cycle.

Example: 6.3-4

$v(t) = 10\sqrt{2} \sin t \text{ V}$ is applied to a 10Ω resistor from $t = 0$ onwards. Plot $p(t)$ and $E(t)$ for $t \geq 0$.

Solution

The waveform of $p(t)$ is obtained by squaring the waveform of $v(t)$ and dividing by 10. $E(t)$ is obtained by integrating (*i.e.*, evaluating the area under $p(t)$ curve) the power waveform. Therefore,

$$p(t) = 20 \sin^2 t = 10 - 10 \cos 2t \text{ for } t \geq 0$$

$$E(t) = \int_0^t [10 - 10 \cos 2t] dt + 0 = 10t - 5 \sin 2t$$

Therefore, the instantaneous power has a frequency that is double that of the frequency of voltage. Moreover, $p(t)$ contains a DC component. The plots of $p(t)$ and $E(t)$ are shown in Fig. 6.3-5 for $0 \leq t \leq 4\pi$ s.

The cycle period of power waveform is π s. The values of $E(t)$ at integer multiples of cycle period of power waveform are joined by a dotted line as shown in Fig. 6.3-5. This line turns out to be a straight line with a slope of 10 W. The average value of $p(t)$ over one cycle of π s is also 10 W, as shown in the following derivation:

$$\begin{aligned} \text{Average of } p(t) \text{ over a cycle} &= \frac{\text{Area of } p(t) \text{ over a cycle}}{\text{period of cycle}} \\ &= \frac{\int_0^\pi (10 - 10 \cos 2t) dt}{\pi} = 10 - \frac{5}{\pi} \sin 2t \Big|_0^\pi = 10 \text{ W} \end{aligned}$$

The energy function in this example too is a monotonically increasing function of time. Example 6.3-5 considers a situation in which the energy function is not monotonic on t .

Example: 6.3-5

A sinusoidal voltage source $v(t) = 10\sqrt{2} \sin t$ V is connected across a sinusoidal current source $i(t) = \sqrt{2} \sin(t - 60^\circ)$ A at $t = 0$. $i(t)$ flows out of the positive terminal of $v(t)$. Find the instantaneous power delivered by the voltage source to the current source and the corresponding energy function and plot them.

Solution

Let $p(t)$ be the instantaneous power delivered by the voltage source to the current source. Then,

$$\begin{aligned} p(t) &= v(t)i(t) \\ &= 20 \sin t \sin(t - 60^\circ) \text{ W} \\ &= 10[\cos 60^\circ - \cos(2t - 60^\circ)] \text{ W} = [5 - 10 \cos(2t - 60^\circ)] \text{ W} \end{aligned}$$

Energy is obtained by integrating this function over $t = 0$ to $t = t$.

$$E(t) = \int_0^t [5 - 10 \cos(2t - 60^\circ)] dt = [5t - 5 \sin(2t - 60^\circ) - 4.33] \text{ J}$$

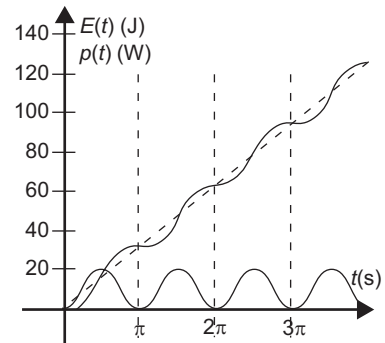


Fig. 6.3-5 Plots of instantaneous power and energy in Example 6.3-4

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The waveforms of voltage, current, power and the plot of energy function are shown in Fig. 6.3-6.

Voltage and current have a frequency of 1 Hz whereas the power waveform has a frequency of 2 Hz. Power waveform is bipolar. The voltage source delivers energy to the current source for some time during a cycle and the current source delivers energy to the voltage source for the remaining time. However, the net energy delivered in one cycle is from voltage source to current source. The energy plot shows that $E(t)$ is non-monotonic on t . This must be so since the power waveform is bipolar.

The line joining the energy values at the end of cycles of power is seen to be a straight line with a slope of 5 W. The average value of $p(t)$ over a cycle must then be 5 W. It is verified to be 5 W in the derivation that follow.

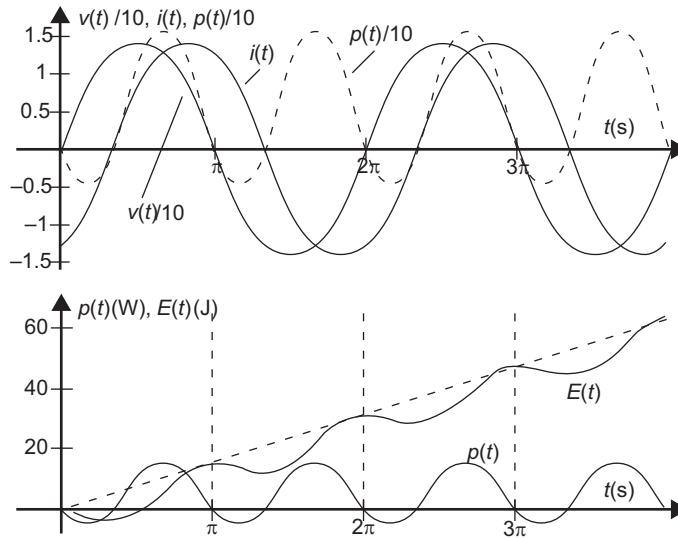


Fig. 6.3-6 Waveforms and plots for Example 6.3-5. Upper traces show voltage, current and power.

$$\begin{aligned} \text{Average of } p(t) \text{ over a cycle} &= \frac{\text{Area of } p(t) \text{ over a cycle}}{\text{period of cycle}} \\ &= \frac{\int_0^\pi [(5 - 10 \cos(2t - 60^\circ))] dt}{\pi} = 5 - \frac{5}{\pi} \sin(2t - 60^\circ) \Big|_0^\pi = 5 \text{ W} \end{aligned}$$

Example: 6.3-6

An electrical element draws a current of $i(t) = -\sqrt{2} \cos 100\pi t$ A from a sinusoidal voltage source of $v(t) = 10\sqrt{2} \sin 100\pi t$ V. Find and plot instantaneous power delivered to the load and energy delivered to it as functions of time.

Solution

$$p(t) = -20 \sin 100\pi t \cos 100\pi t = -10 \sin 200\pi t \text{ W and } E(t) = -0.05(1 - \cos 200\pi t) \text{ J.}$$

Power waveform has 100 Hz frequency whereas the voltage and current waveforms have 50 Hz frequency. The period of the power waveform is 10 ms.

Average value of power over a cycle $= \int_0^{10\text{ms}} -10 \sin 200\pi t \, dt = 0$. Therefore the energy function cannot be a growing function. The net change in $E(t)$ over a cycle of power waveform will be zero. It has to be an alternating function of time with a possible average content, indicating that there is no net energy transfer from source to load over a cycle of power waveform. The plots shown in Fig. 6.3-7 confirm this.

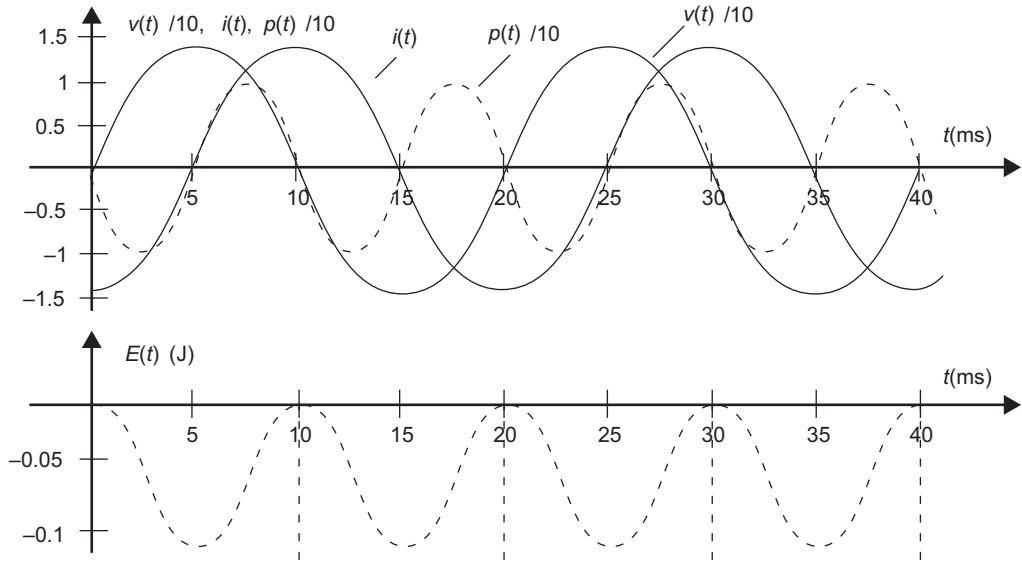


Fig. 6.3-7 Waveforms and plots for Example 6.3-6

6.4 AVERAGE POWER IN PERIODIC WAVEFORMS

The following points emerge from the discussion on instantaneous power in Section 6.3.

- Instantaneous power delivered to an element is a non-constant function of time in general.
- If the voltage across the element and current through the element are periodic waveforms with period T and zero average value over a cycle period, the instantaneous power will be a periodic waveform with period $0.5T$ and may have a non-zero average value over its cycle period of $0.5T$.
- The energy delivered to the element will also be a function of time. The value of total energy delivered to the element at end-of-cycle points will fall on a straight line with a slope equal to the average value of instantaneous power waveform over its cycle period.
- If the instantaneous power waveform is unipolar, the energy function will be monotonic on t . If instantaneous power waveform is bipolar, the energy function will be non-monotonic.

A 100 W incandescent lamp draws about 0.615 A peak sinusoidal current when a 325 V peak sinusoidal voltage at 50 Hz is applied across it. This results in a $p(t) = 200 \sin^2 200\pi t$ W of instantaneous

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power in it. $p(t)$ can also be expressed as $p(t) = (100 - 100 \cos 200\pi t)$ W. Thus, the instantaneous power varies from 0 to 200 W and goes back to zero in 10 ms. Does the lamp filament respond to this power input variation?

It does respond to the power variation. However, 10ms is too small a time interval for significant variations in the temperature of the filament to take place. This is due to the thermal capacity of the lamp system that tends to behave like thermal inertia when it comes to changes in lamp temperature. Thus, though the lamp responds to both the 100 W constant component and the 100 W cosine term in instantaneous power, the response of temperature variable to the 100 W cosine term is very small compared to the response to the 100 W constant term. Thus, the lamp temperature and hence its light output is more or less constant in the long run (after about 50 to 100ms of switching on in this context) corresponding to the 100 W constant term in $p(t)$, with the 100 W rippling term contributing only a negligible amplitude oscillation in them. And, persistence of human vision virtually blots out even this small rippling component in light output. Hence, we do not get to see the small amount light flicker that takes place invariably due to the power changing periodically with a period of 10 ms. If we apply a 5 Hz voltage instead of 50 Hz voltage to a lamp, the oscillating component in lamp light output will be much more in amplitude and the frequency of this oscillating component will be 10 Hz. We will experience a prominent flicker in the lamp light output.

Thus, we conclude that, if the frequency of pulsation in instantaneous power delivered to an electrical load (lamps, heating element, motors etc.) is sufficiently high, the output from the load (light, temperature, torque, speed etc.) will be constant in the long run. The magnitude of this constant output is decided by the average of instantaneous power over its cycle period. The load will ignore the pure alternating component/s in instantaneous power. Therefore, the average value of $p(t)$, averaged over its cycle period, is a much more relevant quantity in practice.

The Cycle Average Power in the context of periodic waveforms is defined as the cycle average of instantaneous power over a cycle of instantaneous power and is denoted by P ,

i.e., $P = \frac{1}{\text{Period}} \int_{\text{over one period of } p(t)} p(t) dt = \frac{2}{T} \int_t^{t+0.5T} v(t)i(t) dt$, where T is the period of the periodic voltage and current waveforms and t is any arbitrary time instant after $t = 0$. $p(t)$ is assumed to be zero for $t < 0$. In practice, the integration is carried out from the beginning of a power cycle to the end of that cycle.

The *Average Power* contained in an instantaneous power waveform and the *Cycle Average Power* are two different concepts altogether. First of all, there can be a cycle average power only if $p(t)$ is a periodic waveform. But average power can be defined and calculated for any $p(t)$. Let $v(t)$ be the voltage across an element and $i(t)$ be the current through it over an interval of time denoted by $[t_1, t_2]$. Then the energy delivered to the element during this interval is given by the area under $p(t) = v(t) i(t)$ from t_1 to t_2 . Then, the *Average Power* (P_{av}) delivered during this interval is the value of constant power that would have delivered the same amount of energy to the element in the interval between t_1 to t_2 .

$$\therefore P_{av} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} v(t)i(t) dt$$

The instant t_1 is usually chosen to be the instant at which the voltage was applied to the element and the instant t_2 is the instant at which the supply to the electrical element was switched off. If the

two instants are chosen this way, the P value gives the average rate at which energy was delivered to the load element during the entire period of connection.

A simple relation exists between *Average Power* (P_{av}) and *Cyclic Average Power* (P) in the context of periodic voltages and currents. Let $v(t) = V_m \sin \omega t$ V and $i(t) = I_m \sin(\omega t + \theta)$ A. Let the interval duration $t_2 - t_1$ extend over a large number of cycles of $p(t)$. The instantaneous power in this case will be periodic with a period of $0.5T$. But $t_2 - t_1$ may not be an integer multiple of $0.5T$. Hence, we express $t_2 - t_1$ as $0.5nT + 0.5kT$ where k is a real number between -1 and 1 and n is an integer.

$$\begin{aligned}
 p(t) &= V_m I_m \sin \omega t \sin(\omega t + \theta) = \frac{V_m I_m}{2} [\cos \theta - \cos(2\omega t + \theta)] \\
 \therefore P &= \frac{2}{T} \int_0^{0.5T} p(t) dt = \frac{\pi}{\omega} \int_0^{\frac{\pi}{\omega}} \frac{V_m I_m}{2} [\cos \theta - \cos(2\omega t + \theta)] dt \\
 &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2 \times 2\omega} \sin(2\omega t + \theta) \Big|_0^{\frac{\pi}{\omega}} = \frac{V_m I_m}{2} \cos \theta \\
 P_{av} &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t) dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{V_m I_m}{2} [\cos \theta - \cos(2\omega t + \theta)] dt \\
 &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2 \times 2\omega(t_2 - t_1)} \sin(2\omega t + \theta) \Big|_{t_1}^{t_2} \\
 &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2\pi(n+k)} \sin[\pi n + \pi k] \cos[2\omega(t_1 + t_2) + \theta]
 \end{aligned}$$

We used $t_2 - t_1 = 0.5nT + 0.5kT$ in the last step. $\sin(\pi n + \pi k)$ is zero if there are integer periods of length $0.5T$ within $t_2 - t_1$. If not, the magnitude of second term in the expression for P_{av} is upper bounded by $V_m I_m / 2\pi(n+k)$. For a sufficiently large n , i.e., if the length of the interval over which the average power is calculated is very large compared to the period of instantaneous power waveform, then, the second term becomes negligible. The first term is same as cycle average power. Therefore, *Average Power*, P_{av} = *Cycle Average Power*, P , if the waveforms last for sufficient duration compared to their period. If $v(t)$ and $i(t)$ persist for more than 20 or more cycles, the error in taking P as P_{av} will be $< 1\%$.

Example: 6.4-1

The periodic 50Hz voltage waveform applied across a 5Ω resistor is shown in Fig. 6.4-1. (i) Find the average power delivered to the load. (ii) Find the DC voltage that will deliver the same average power. (iii) Find the amplitude of a 50 Hz sinusoidal voltage that will deliver the same average power.

Solution

- (i) The power waveform will be described by the following equation for the first period:

$$p(t) = \begin{cases} 2000(200t)^2 & \text{for } 0 \leq t \leq 5\text{ms} \\ 2000(2 - 200t)^2 & \text{for } 5\text{ms} < t \leq 10\text{ms} \end{cases}$$

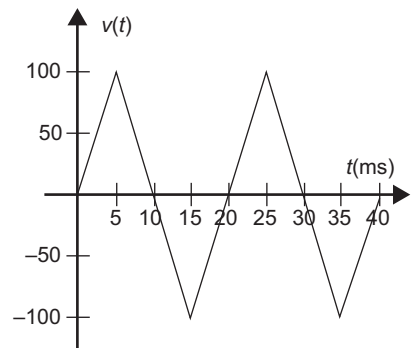


Fig. 6.4-1 Applied voltage waveform in Example 6.4-1

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Energy delivered in one cycle of $p(t)$ = The area under $p(t)$ over one cycle (10ms) of $p(t)$ =
 $2 \times \text{area under } p(t) \text{ for } 0 \text{ to } 5\text{ms interval} = 2 \int_0^{0.005} 8 \times 10^7 t^2 dt = 2 \times 8 \times 10^7 \times 0.005^3 \div 3 = 6.67 \text{ J}$

Therefore, Cycle Average Power = $6.67 \text{ J} / 0.01 \text{ s} = 667 \text{ W}$

Therefore, $P_{av} = 667 \text{ W}$

(ii) The required DC voltage V is such that $V^2/5 = 667$. Therefore, $V = 57.74 \text{ V}$

(iii) Let $v(t) = V_m \sin 100\pi t$. Then, $i(t) = 0.2 V_m \sin 100\pi t$ and $p(t) = 0.2 V_m^2 \sin^2 \omega t \text{ W}$. Therefore, cycle average power = $0.2 V_m^2 \div 2 = 0.1 V_m^2$. This has to be 667 W . Therefore, the required amplitude of sinusoidal voltage is $V_m = 81.67 \text{ V}$.

Example: 6.4-2

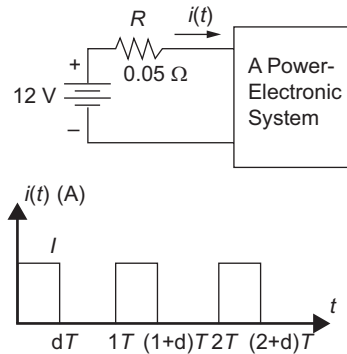


Fig. 6.4-2 Circuit and waveform for Example 6.4-2

A 12V battery delivers power to a power electronic system through its internal resistance of 0.05Ω as shown in Fig. 6.4-2. The figure also shows that the current drawn by the load is a pulsed current with period of $T = 1 \text{ ms}$. The load draws the current in such a way that the average value of current is kept constant at 10 A while d and I vary under different operating conditions. Calculate the average power delivered by the source, average power delivered to load and efficiency for (i) $I = 100 \text{ A}$ (ii) $I = 40 \text{ A}$, (iii) $I = 12 \text{ A}$, and (iv) $I = 10 \text{ A}$.

Solution

Since the average value of $i(t)$ is kept constant, Id must be constant.

Average power delivered by the source is

$$P_s = \frac{1}{0.001} \int_0^{0.001} 12 \times i(t) dt = 1000 \int_0^{0.001d} 12I dt = 12Id \text{ watts} = 120 \text{ W. This value is independent of } I \text{ and } d$$

as long as the average value of current is kept at 10 A . The power delivered by a *constant voltage source* is equal to product of *source voltage* and the *cycle average value of current*.

Average power dissipated in the internal resistance, P_R , is

$$\begin{aligned} P_R &= \frac{1}{0.001} \int_0^{0.001} 0.05 \times [i(t)]^2 dt = \frac{1}{0.001} \int_0^{0.001d} 0.05 \times I^2 dt \\ &= 0.05 I^2 d = 0.05 \times 10 \times I = 0.5I \text{ W} \end{aligned}$$

We have used the fact that $Id = 10 \text{ A}$ in the last step.

(i) $I = 100 \text{ A}$. Therefore, $d = 10/100 = 0.1$.

$P_R = 0.5 \times 100 = 50 \text{ W}$, $P_s = 100 \text{ W}$, $\therefore$ Load power, $P_L = 100 - 50 = 50 \text{ W}$
 Efficiency = 50%

(ii) $I = 40 \text{ A}$. Therefore, $d = 10/40 = 0.25$

$P_R = 0.5 \times 40 = 20 \text{ W}$, $P_s = 100 \text{ W}$, $\therefore$ Load power, $P_L = 100 - 20 = 80 \text{ W}$
 Efficiency = 80%

- (iii) $I = 12$ A. Therefore, $d = 10/12 = 0.833$
 $P_R = 0.5 \times 12 = 6$ W. $P_S = 100$ W, $\therefore$ Load power, $P_L = 100 - 6 = 94$ W
 Efficiency = 94%
- (iv) $I = 10$ A. Therefore, $d = 10/10 = 1$, $i(t)$ becomes a constant current of 10 A.
 $P_R = 0.5 \times 10 = 5$ W. $P_S = 100$ W,
 $\therefore$ Load power, $P_L = 100 - 5 = 95$ W
 Efficiency = 95%

Example: 6.4-3

A DC voltage source of V V delivers a current $i(t)$ to an external load through its internal resistance of $R \Omega$. Show that, among the infinite possible periodic waveforms for $i(t)$ with the same value of cycle average, $i(t) = \text{constant}$ is the waveform that results in minimum loss and maximum efficiency.

Solution

Let I be the cycle average value of a periodic $i(t)$. Then $i(t)$ can be expressed as a constant plus a pure alternating component as $i(t) = I + i_{ac}(t)$, where $i_{ac}(t)$ is bipolar and has equal areas under positive half-cycle and negative half-cycle. Let P_S be the average power delivered by the source, P_L be the average power delivered to the load and P_R be the average power dissipated in R . Average powers are equal to corresponding cycle average powers if $i(t)$ lasts for a long time compared to its period. Then,

$P_S = \frac{1}{T} \int_0^T V[I + i_{ac}(t)] dt$, where T is the period of $i_{ac}(t)$. The cycle average value of a pure alternating waveform with equal magnitude areas under its positive and negative half-cycles will be zero. Therefore, $P_S = VI$ W.

$$P_R = \frac{1}{T} \int_0^T R(I + i_{ac}(t))^2 dt = \frac{1}{T} \int_0^T RI^2 dt + \frac{1}{T} \int_0^T 2RI i_{ac}(t) dt + \frac{1}{T} \int_0^T R(i_{ac}(t))^2 dt$$

Since the instantaneous power in R contains a term proportional to $i_{ac}(t)$, its basic period is T and not $0.5T$. This is why the integration in the above step is from 0 to T . The second integral goes to zero since cycle average of a pure alternating component is zero.

$$\therefore P_R = \frac{1}{T} \int_0^T RI^2 dt + \frac{1}{T} \int_0^T R(i_{ac}(t))^2 dt$$

The quantity $[i_{ac}(t)]^2$ is always positive and hence the second integral in the previous equation will be positive-valued. Therefore,

$P_R \geq \frac{1}{T} \int_0^T RI^2 dt$ with the equality sign applicable only when $i_{ac}(t) = 0$ for all t – i.e., only when $i(t) = I$, a constant.

Therefore, for a given amount of average current drawn from a DC source, the power loss in the internal resistance and connecting link resistance will be a minimum when the energy is drawn at a constant rate – i.e., power delivered is kept constant by keeping current constant.

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The optimum way to draw power from a DC source is by drawing a DC current. Similarly, the optimum way to charge a battery is by delivering a constant current into it. No other current waveform is as energy efficient as the DC current waveform when drawing power from a DC voltage source or delivering power to it.

Example: 6.4-4

An AC voltage source (*i.e.*, a sinusoidal source) $v(t) = 325 \sin 100\pi t$ V delivers 1 kW of average power to a load through a resistance of 5Ω . This resistance is the sum of the internal resistance of the source and the resistance of cable connecting the load to the source. Assume that the load draws a current $i(t) = I_m \sin(100\pi t + \theta)$ and that I_m and θ can be varied keeping the power delivered by the source at 1 kW itself. Find the amplitude of current, power dissipated in the series resistance and the efficiency of power transfer when (i) $\theta = -80^\circ$ (ii) $\theta = 45^\circ$ (iii) $\theta = -30^\circ$ and (iv) $\theta = 0^\circ$.

Solution

Let $v(t) = V_m \sin \omega t$ and $i(t) = I_m \sin(\omega t + \theta)$. Then the average power delivered by the source = $\frac{V_m I_m}{2} \cos \theta$ W. (We have derived this many times in earlier sections.). The average power delivered depends on magnitude of phase difference between voltage and current waveforms. But it does not depend on the sign of phase difference.

The power delivered by the source is kept constant at 1000 W in this example. Therefore, $I_m \cos \theta = 1000/325 = 3.077$ A.

Let P_R be the average power dissipated in the resistor. Then,

$$\begin{aligned} P_R &= \frac{2}{T} \int_0^{0.5T} R[i(t)]^2 dt = \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} R I_m^2 \sin^2(\omega t + \theta) dt \\ &= \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} R I_m^2 \left[\frac{1}{2} - \frac{1}{2} \cos(2\omega t + 2\theta) \right] dt = \frac{R I_m^2}{2} \text{ watts} \end{aligned}$$

But $I_m = 3.077/\cos \theta$. Therefore, $P_R = 4.734R/\cos^2 \theta$. Now we evaluate the numbers for the various cases:

- (i) $\theta = -80^\circ$. Then, $\cos \theta = 0.1736$, $\therefore I_m = 3.077/0.1736 = 17.72$ A, $\therefore P_R = 4.734 \times 5/0.1736^2 = 785.4$ W, $P_S = 1000$ W, $\therefore P_L = 214.6$ W and efficiency = 21.46 %.
- (ii) $\theta = 45^\circ$. Then, $\cos \theta = 0.707$, $\therefore I_m = 3.077/0.707 = 4.35$ A, $\therefore P_R = 4.734 \times 5/0.707^2 = 47.3$ W, $P_S = 1000$ W, $\therefore P_L = 952.7$ W and efficiency = 95.27 %.
- (iii) $\theta = -30^\circ$. Then, $\cos \theta = 0.866$, $\therefore I_m = 3.077/0.866 = 3.55$ A, $\therefore P_R = 4.734 \times 5/0.866^2 = 31.6$ W, $P_S = 1000$ W, $\therefore P_L = 968.4$ W and efficiency = 96.84 %.
- (iv) $\theta = 0^\circ$. Then, $\cos \theta = 1$, $\therefore I_m = 3.077$ A, $\therefore P_R = 4.734 \times 5/1^2 = 23.7$ W, $P_S = 1000$ W, $\therefore P_L = 976.3$ W and efficiency = 97.63 %.

Increasing magnitude of phase difference between voltage and current makes power transfer to the load highly inefficient.

Example: 6.4-5

Show that the fixed amount of average power delivered by a sinusoidal voltage source through its internal resistance and connecting link resistance to a load will reach the load with minimum loss and maximum efficiency when the current drawn by the load has zero phase difference with respect to the source voltage.

Solution

Let $v(t) = V_m \sin \omega t$ and $i(t) = I_m \sin (\omega t + \theta)$. Let the total series resistance in the path be R . Then the average power delivered by the source $= \frac{V_m I_m}{2} \cos \theta$ W. The average power dissipated in R is $0.5 R I_m^2$ W (see Example 6.4-4).

But the power delivered by the source is stated to be fixed. Let this fixed value be P W. Then $0.5 V_m I_m \cos \theta = P$. Therefore, $I_m = 2P / (V_m \cos \theta)$. Therefore, the average power loss in R is

$$P_R = 0.5 R I_m^2 = \frac{2RP^2}{V_m^2 \cos^2 \theta}.$$

The minimum of this loss takes place when $\cos^2 \theta$ is a maximum – i.e., $\theta = 0^\circ$ or 180° . $\theta = 180^\circ$ is relevant when the load is delivering power to source.

Therefore, when a sinusoidal voltage source is delivering power to load, a given amount of source power is transferred to load with minimum losses and maximum efficiency when the load draws current from source at zero phase difference at the source terminal.

This fact has great significance in Electrical Power Distribution systems and Electricity tariff structure.

6.5 EFFECTIVE VALUE (RMS VALUE) OF PERIODIC WAVEFORMS

The cycle average power delivered to the same resistance R by different voltage or current waveforms forms the basis for comparison of their effectiveness in delivering useful power to a load. We can define a measure of effectiveness of a given voltage or current waveform by calculating the cyclic average power that will be delivered to a resistance R and by answering the question – what is the value of a DC voltage or current that will result in same average power in R ? The measure defined this way is called the *Root-Mean-Square Value (rms value)* of the waveform. It is also called the *Effective Value* of the waveform.

Effective value or rms value of a waveform is the value of DC quantity that will produce the same heating effect as that produced by the waveform when it is applied as a voltage across a resistance of 1Ω or as a current through a resistance of 1Ω .

The assumption of 1Ω is a matter of convenience. It could have been $R\Omega$ and that will not make any difference since R will get cancelled out when the heating effects are equated.

We can develop an expression for rms value of a periodic waveform as shown in the following. Let $x(t)$ be a periodic waveform with period of T s. We find out the cyclic average power as if $x(t)$ is a voltage signal applied across 1Ω .

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$$P = \frac{1}{T} \int_0^T \frac{[x(t)]^2}{1} dt$$

Let X_{rms} be the value of DC quantity that will produce same power in 1Ω . Then,

$$\begin{aligned} \frac{1}{T} \int_0^T [x(t)]^2 dt &= (X_{\text{rms}})^2 \\ \therefore X_{\text{rms}} &= \sqrt{\frac{1}{T} \int_0^T [x(t)]^2 dt} \end{aligned} \quad (6.5-1)$$

Therefore, finding rms value of a waveform involves *squaring* the waveform, finding the *mean* (i.e., average over a cycle) of the *squared waveform* over a cycle and then finding the *root* of the *mean*. Hence the name – *root mean square*.

Now we can express the average power delivered to a resistance of $R \Omega$ by a periodic voltage waveform $v(t)$ applied across it as $P = (V_{\text{rms}})^2 / R$ W. The average power delivered to a resistance of $R \Omega$ by a periodic current $i(t)$ flowing through it is $P = R(I_{\text{rms}})^2$ W.

6.5.1 RMS Value of Sinusoidal Waveforms

Let $v(t) = V_m \sin \omega t$ V. We find its rms value as,

$$V_{\text{rms}} = \sqrt{\frac{2\pi}{\omega} \int_0^{\omega} V_m^2 \sin^2 \omega t dt} = \sqrt{\frac{2\pi}{\omega} \int_0^{\omega} \frac{1}{2} V_m^2 [1 - \cos 2\omega t] dt} = \sqrt{\frac{1}{2} V_m^2} = \frac{V_m}{\sqrt{2}}.$$

Thus, a sinusoidal voltage is only as effective as a DC voltage that is 70.7% of its peak value as far as its heating capability is concerned.

Let $v(t) = V_m \sin \omega t$ V and $i(t) = I_m \sin(\omega t + \theta)$ A be the voltage across and current through an electrical element as per passive sign convention, respectively. Then we know that the average power delivered to the element is given by $0.5 V_m I_m \cos \theta$ W. Now we can express this power equation in terms of rms values of voltage and current.

$$P = V_{\text{rms}} I_{\text{rms}} \cos \theta \quad \text{W} \quad (6.5-2)$$

Two other measures are useful in the context of periodic waveforms. They are *cycle average* and *cycle average of absolute value of the waveform*. We denote them by V_{cav} and V_{caav} , respectively.

$$V_{\text{cav}} = \frac{1}{T} \int_0^T v(t) dt \quad \text{and} \quad V_{\text{caav}} = \frac{1}{T} \int_0^T |v(t)| dt \quad (6.5-3)$$

Note that both in the definitions of rms value in Eqn. 6.5-1 and average values in Eqn. 6.5-3, we are free to carry out the integration over any interval of width equal to the period of the waveform. It does not have to be between 0 to T .

A *pure alternating waveform* will have a cycle average of zero.

The cycle average of absolute value of a pure alternating waveform that has identical positive half-cycle width and negative half-cycle width will be same as the average of positive half-cycle over $0.5T$. Hence, the cycle average of absolute value is also called *half-cycle average* in the case of such a waveform. A sinusoidal waveform is one such waveform. The V_{caav} value for a sinusoidal waveform $v(t) = V_m \sin \omega t$ is obtained as follows:

$$V_{\text{caav}} = \frac{2}{T} \int_0^{0.5T} v(t) dt = \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} V_m \sin \omega t dt = \frac{V_m}{\pi} \cos \omega t \Big|_{\frac{\pi}{\omega}}^0 = \frac{2V_m}{\pi}$$

The ratio between the rms value and the half-cycle average value of a periodic waveform with equal half-cycle widths and zero cycle average (*i.e.*, a pure alternating waveform) is defined as its *form factor*. Therefore, the form factor of a pure sinusoidal waveform $= \frac{\pi}{2\sqrt{2}} \approx 1.11$.

The ratio between peak value and rms value of an alternating waveform with equal half-cycle widths and zero cycle average is defined as its *crest factor*. The crest factor of a pure sinusoidal waveform is $\sqrt{2}$.

Example: 6.5-1

If $v(t)$ is a pure alternating periodic waveform with period T and rms value V_{rms} find the rms value of a new periodic waveform $v_1(t) = V + v(t)$ where V is a constant quantity. Thereby, find the rms value of $10+10\sin 100\pi t$ V.

Solution

$$V_{\text{1rms}} = \sqrt{\frac{1}{T} \int_0^T [V + v(t)]^2 dt} = \sqrt{\frac{1}{T} \int_0^T [V^2 + (v(t))^2 + 2Vv(t)] dt}$$

The component $2Vv(t)$ represents a pure alternating component with zero cycle average since $v(t)$ is pure alternating waveform and V is a constant. Hence, integration of this term over one cycle period yields zero integral.

$$\therefore V_{\text{1rms}} = \sqrt{\frac{1}{T} \int_0^T [V^2 + (v(t))^2] dt} = \sqrt{V^2 + V_{\text{rms}}^2}$$

$$\text{Therefore rms value of } 10+10\sin 100\pi t = \sqrt{10^2 + \left(\frac{10}{\sqrt{2}}\right)^2} = \sqrt{150} = 12.25 \text{ V}$$

Example: 6.5-2

If $v(t)$ is a periodic waveform with period T , rms value V_{rms} and cycle average value V_{cav} , find the rms value and cycle average value of a new periodic waveform $v_1(t) = V + v(t)$, where V is a constant quantity.

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Solution

$v(t)$ has a non-zero cycle average value. Therefore it can be written as $V_{\text{cav}} + v_{\text{ac}}(t)$, where $v_{\text{ac}}(t)$ is a pure alternating waveform with zero cycle average. Using the result we arrived at in Example 6.5-1,

$$V_{\text{rms}}^2 = V_{\text{cav}}^2 + V_{\text{ac rms}}^2$$

$$\therefore V_{\text{ac rms}}^2 = V_{\text{rms}}^2 - V_{\text{cav}}^2$$

Now, $v_1(t) = V + v(t) = V + V_{\text{cav}} + v_{\text{ac}}(t)$.

Using the result from Example 6.5-1 again,

$$V_{1\text{rms}}^2 = (V + V_{\text{cav}})^2 + V_{\text{ac rms}}^2$$

Substituting for $V_{\text{ac rms}}^2$,

$$V_{1\text{rms}}^2 = (V + V_{\text{cav}})^2 + (V_{\text{rms}}^2 - V_{\text{cav}}^2)$$

$$= V^2 + 2VV_{\text{cav}} + V_{\text{rms}}^2$$

$$\therefore V_{1\text{rms}} = \sqrt{V^2 + 2VV_{\text{cav}} + V_{\text{rms}}^2}$$

The cycle average of $v_1(t) = V_{\text{cav}} + V$

Example: 6.5-3

Find the rms value of a symmetric square wave of ± 10 V and 50 Hz frequency.

Solution

When a symmetric square wave of amplitude V is squared, the resulting waveform is the same as squaring a DC voltage of V V. Therefore, rms value of a symmetric square wave is same as the amplitude of the wave. Hence, the rms value of square wave voltage in this example is 10 V.

Example: 6.5-4

Find the rms value, half-cycle value, crest factor and form factor for a symmetric triangular waveform shown in Fig. 6.5-1.

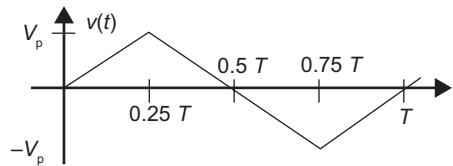


Fig. 6.5-1 Symmetric triangle waveform for Example 6.5-4

Solution

Due to symmetry, the squared waveform needs to be integrated only from 0 to $0.25T$. The integral for 0 to T range is four times this value.

$$V_{\text{rms}} = \sqrt{\frac{4}{T} \int_0^{0.25T} \left(\frac{4V_p t}{T} \right)^2 dt} = \sqrt{\frac{64V_p^2}{3T^3} \times (0.25T)^3} = \frac{V_p}{\sqrt{3}} \text{ V}$$

$$V_{\text{caav}} = \frac{2}{T} \int_0^{0.5T} v(t) dt = 2 \times \frac{2}{T} \int_0^{0.25T} \frac{4V_p t}{T} dt = \frac{V_p}{2} \text{ V}$$

$$\therefore \text{Form factor} = \frac{V_p / \sqrt{3}}{V_p / 2} = \frac{2}{\sqrt{3}} = 1.1547$$

$$\text{Crest factor} = \frac{V_p}{V_p / \sqrt{3}} = \sqrt{3} = 1.732$$

Example: 6.5-5

Find an expression for rms value of the periodic pulse train shown in Fig. 6.5-2.

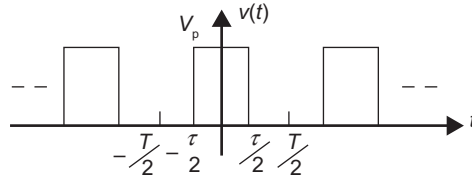


Fig. 6.5-2 Periodic waveform for Example 6.5-5

Solution

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt} = \sqrt{\frac{1}{T} \int_{-0.5\tau}^{0.5\tau} [V_p]^2 dt} = V_p \sqrt{\frac{\tau}{T}} \text{ V}$$

Note the limits of integration.

Example: 6.5-6

Let $v(t) = V_m \sin \omega t$ V. A new voltage waveform is generated from this waveform by a process called *half-wave rectification*, which is represented mathematically as

$$v_r(t) = \begin{cases} 0 & \text{if } v(t) < 0 \\ v(t) & \text{if } v(t) \geq 0 \end{cases}$$

Find the rms value and cycle average value of $v_r(t)$.

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Solution

The area under squared $v_r(t)$ over one T will be half the area under squared $v(t)$ since one half-cycle is missing in $v_r(t)$. Therefore, rms value of $v_r(t)$ will be $\frac{1}{\sqrt{2}}$ times the rms value of $v(t)$. Therefore, rms value of $v_r(t) = 0.5V_m$ V.

Half-cycle average of $v(t)$ is $2V_m/\pi$. Therefore, half-cycle area $= \frac{2V_m}{\pi} \times \frac{T}{2} = V_m T/\pi$. This area becomes the full cycle area in $v_r(t)$ since second half cycle is zero-valued. Therefore, cycle average of $v_r(t) = V_m T/\pi \div T = V_m/\pi$ V.

6.6 THE POWER SUPERPOSITION PRINCIPLE

Current variables and voltage variables in a *linear* electric circuit obey the Superposition Theorem. However, instantaneous power in a linear circuit does not follow superposition principle. Consider a circuit driven by two sources $v_{s1}(t)$ and $v_{s2}(t)$ and let the corresponding response components for the voltage across a particular element be $v_{11}(t)$ and $v_{12}(t)$. The corresponding response components of current through that element are $i_{11}(t)$ and $i_{12}(t)$ as per the passive sign convention. Then,

The instantaneous power delivered to the element when $v_{s1}(t)$ is acting alone,

$$p_1(t) = v_{11}(t) i_{11}(t)$$

The instantaneous power delivered to the element when $v_{s2}(t)$ is acting alone,

$$p_2(t) = v_{12}(t) i_{12}(t)$$

The instantaneous power delivered to the element when $v_{s1}(t)$ and $v_{s2}(t)$ are acting together,

$$\begin{aligned} p(t) &= [v_{11}(t) + v_{12}(t)][i_{11}(t) + i_{12}(t)] = v_{11}(t) i_{11}(t) + v_{12}(t) i_{12}(t) + [v_{11}(t) i_{12}(t) + v_{12}(t) i_{11}(t)] \\ &= p_1(t) + p_2(t) + \text{Cross Power Terms} \end{aligned}$$

$p(t)$ should have been equal to $p_1(t) + p_2(t)$ if instantaneous power obeys superposition principle. Therefore, instantaneous power does not follow superposition principle in a linear circuit. This conclusion is general in nature and there are no exceptions.

However, there are exceptions to this rule when it comes to *average power*. Let us confine to the case of periodic waveforms that are applied to circuits for sufficient duration such that average power can be taken as cyclic average power. Let the average power delivered to the element in question when $v_{s1}(t)$ is acting alone be P_1 and the corresponding value when $v_{s2}(t)$ is acting alone be P_2 . We assume that $v_{s1}(t)$ and $v_{s2}(t)$ have a least common period of T s. This common period need not be the period of any one of them. For instance, let the period of $v_{s1}(t)$ be 1 sec and that of $v_{s2}(t)$ be 1.25 s. Then the value of T will be 5 s since there will be integer number of cycles of both in a 5 s period. Five seconds is the lowest interval that will contain integer number of cycles of both. Cyclic average power can be found by integrating over multiple cycles provided the averaging is done over the same number of cycle periods. Therefore, taking T as the integration interval is permitted in the evaluation of P_1 and P_2 .

$$P_1 = \frac{1}{T} \int_0^T v_{11}(t) i_{11}(t) dt$$

$$P_2 = \frac{1}{T} \int_0^T v_{12}(t) i_{12}(t) dt$$

The average power delivered to the element when both sources are acting together, P , is given by

$$\begin{aligned}
P &= \frac{1}{T} \int_0^T [v_{11}(t) + v_{12}(t)][i_{11}(t) + i_{12}(t)] dt \\
&= \frac{1}{T} \int_0^T [v_{11}(t)i_{11}(t) + v_{12}(t)i_{12}(t) + v_{12}(t)i_{11}(t) + v_{11}(t)i_{12}(t)] dt \\
&= P_1 + P_2 + \frac{1}{T} \int_0^T v_{12}(t)i_{11}(t) dt + \frac{1}{T} \int_0^T v_{11}(t)i_{12}(t) dt
\end{aligned} \tag{6.6-1}$$

Hence, in general, average power does not follow superposition principle. However, unlike the case of instantaneous power, the cross power terms in the average power can go to zero under special conditions.

Consider a case where $v_{s1}(t)$ and $v_{s2}(t)$ are two sinusoidal waveforms at different angular frequencies of ω_1 and ω_2 . Note that one of them can be zero as a special case indicating a DC or steady input. Assume that $\omega_1 < \omega_2$. In this case, T will be an integer multiple of $2\pi/\omega_1$ and another integer multiple of $2\pi/\omega_2$, i.e., $T = k_1 \frac{2\pi}{\omega_1} = k_2 \frac{2\pi}{\omega_2}$ and $\frac{k_1}{k_2} = \frac{\omega_1}{\omega_2}$, where k_1 and k_2 are two integers. It is possible to find a value for T satisfying this condition only if the ratio ω_1/ω_2 is a rational number.

The circuit is linear. Therefore $v_{11}(t)$ and $i_{11}(t)$ will be sinusoids at ω_1 rad/s and $v_{12}(t)$ and $i_{12}(t)$ will be sinusoids at ω_2 rad/s. Therefore, both cross power terms $-v_{11}(t)i_{12}(t)$ and $v_{12}(t)i_{11}(t)$ will involve products of two sinusoidal waveforms of different frequencies ω_1 and ω_2 . By applying trigonometric identities they may be expressed as sinusoidal waveforms of sum frequency and difference frequency. Therefore, $v_{11}(t)i_{12}(t) + v_{12}(t)i_{11}(t)$ will yield two sinusoids – one with a frequency of $\omega_1 + \omega_2$ rad/s and the second with a frequency of $\omega_2 - \omega_1$ rad/s.

Both these sinusoidal waveforms will have integer number of cycles within an interval of length T as shown next.

$$\begin{aligned}
\text{Period of } \omega_1 + \omega_2 \text{ rad/s component } T_s &= \frac{2\pi}{\omega_1 + \omega_2} \\
\frac{T}{T_s} &= \frac{k_1 \frac{2\pi}{\omega_1}}{\frac{2\pi}{\omega_1 + \omega_2}} = k_1 + k_1 \frac{\omega_2}{\omega_1} = k_1 + k_2 = \text{an integer} \left(\because \frac{k_1}{k_2} = \frac{\omega_1}{\omega_2} \right) \\
\text{Period of } \omega_2 - \omega_1 \text{ rad/s component } T_d &= \frac{2\pi}{\omega_2 - \omega_1} \\
\frac{T}{T_d} &= \frac{k_1 \frac{2\pi}{\omega_1}}{\frac{2\pi}{\omega_2 - \omega_1}} = -k_1 + k_1 \frac{\omega_2}{\omega_1} = k_2 - k_1 = \text{an integer} \left(\because \frac{k_1}{k_2} = \frac{\omega_1}{\omega_2} \right)
\end{aligned}$$

Therefore, the area under these two sinusoidal waveforms over an interval of T s long will be zero. Thus, the cross power terms in the instantaneous power contribute only zero to the average power in this case. Therefore, $P = P_1 + P_2$. We can extend this analysis to a case involving many sinusoidal sources of different frequencies easily and arrive at the *Power Superposition Principle* stated in the following:

The average power delivered to an element in a linear circuit excited by sinusoidal sources of different frequencies (including DC, i.e., zero frequency) obeys superposition principle.

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6.6.1 RMS Value of a Composite Waveform

We will employ the superposition principle for average power to arrive at an expression for the rms value (*i.e.*, the effective value) of a periodic waveform that is the sum of many sinusoidal periodic waveforms of different frequencies.

Let $v(t) = v_1(t) + v_2(t) + \dots + v_n(t)$ be a composite waveform comprising n *distinct frequency sinusoidal waveforms*. Notice the word *distinct* used in the last sentence. No two components in the sum can have same frequency value. If there are two sources of same frequency, they have to be combined and treated as a single term.

Now imagine that we apply this signal as a voltage across 1Ω resistor. The average power delivered to the resistor is found by applying power superposition principle as the sum of average power delivered to the same resistor when each source is acting alone. But that will be nothing but the square of rms value of each component waveform. The rms value of $v(t)$ is obtained by finding the square root of average power delivered to the 1Ω resistor. Therefore,

$$\begin{aligned} V_{\text{rms}}^2 &= V_{1\text{rms}}^2 + V_{2\text{rms}}^2 + \dots + V_{n\text{rms}}^2 \\ \therefore V_{\text{rms}} &= \sqrt{V_{1\text{rms}}^2 + V_{2\text{rms}}^2 + \dots + V_{n\text{rms}}^2}. \end{aligned} \quad (6.6-2)$$

The effective value of a waveform that is the sum of many sinusoidal waveforms with distinct frequencies (including zero frequency) is obtained by taking the square root of sum of squares of rms values of individual components.

Example: 6.6-1

The voltage applied across an electrical load circuit is $v(t) = 10 + 50 \sin 100\pi t + 30 \sin (150\pi t - 30^\circ)$ V and the current delivered to the load circuit is seen to be $i(t) = 2.5 + 5 \sin (100\pi t - 45^\circ) + 1.5 \sin (150\pi t - 80^\circ)$ A. (i) Find the average power delivered to the load circuit, rms value of applied voltage and load current. (ii) What is the period over which averaging was carried out in the last step?

Solution

- (i) All the three components in voltage waveform have distinct frequencies. Similarly, all the three components in current waveform too have distinct frequencies.

Therefore, average power delivered by applying power superposition principle is obtained as

$$\begin{aligned} P &= 10 \times 2.5 + \frac{50}{\sqrt{2}} \times \frac{5}{\sqrt{2}} \cos(0 - (-45^\circ)) + \frac{30}{\sqrt{2}} \times \frac{1.5}{\sqrt{2}} \cos(-30^\circ - (-80^\circ)) \\ &= 25 + 125 \cos 45^\circ + 22.5 \cos 50^\circ = 127.84 \text{ W} \end{aligned}$$

The rms values of voltage and current waveforms are obtained by applying Eqn. 6.6-2.

$$V_{\text{rms}} = \sqrt{10^2 + \left(\frac{50}{\sqrt{2}}\right)^2 + \left(\frac{30}{\sqrt{2}}\right)^2} = 42.43 \text{ V}$$

$$I_{\text{rms}} = \sqrt{2.5^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{1.5}{\sqrt{2}}\right)^2} = 4.46 \text{ A}$$

$$\omega_1 = 100\pi \text{ and } \omega_2 = 150\pi$$

$$(ii) \frac{\omega_1}{\omega_2} = \frac{100\pi}{150\pi} = \frac{2}{3} = \frac{k_1}{k_2}$$

The lowest values possible for k_1 and k_2 are 2 and 3 respectively.

$$\text{Therefore, } T = \frac{k_1 2\pi}{\omega_1} = \frac{2 \times 2\pi}{100\pi} = 0.04 \text{ s}$$

Therefore, the period over which averaging was carried out in calculating average power and rms values is 40 ms.

Example: 6.6-2

Find the rms value of a voltage waveform given by $v(t) = 10 \sin(100\pi t - 30^\circ) + 5 \cos(100\pi t + \pi/6) + 5 \cos(150\pi t - 65^\circ) + 7 \sin(300\pi t - \pi/3) - 7 \cos(300\pi t - 45^\circ)$ V.

Solution

The terms in the expression for $v(t)$ contain non-distinct frequencies. Terms with same frequency will have to be combined into a single term first.

$$\begin{aligned} & \underline{10 \sin(100\pi t - 30^\circ) + 5 \cos(100\pi t + \pi/6)} \\ &= \sin 100\pi t [10\cos 30^\circ - 5 \sin \pi/6] + \cos 100\pi t [-10\sin 30^\circ + 5\cos \pi/6] \\ &= 6.16 \sin 100\pi t - 0.67 \cos 100\pi t \\ &= (6.16^2 + 0.67^2)^{0.5} \sin[100\pi t - \tan^{-1}(0.67/6.16)] \\ &= 6.196 \sin(100\pi t - 6.2^\circ) \end{aligned}$$

$$\begin{aligned} & \underline{7 \sin(300\pi t - \pi/3) - 7 \cos(300\pi t - 45^\circ)} \\ &= \sin 300\pi t [7\cos \pi/3 - 7\sin 45^\circ] + \cos 300\pi t [-7\sin \pi/3 - 7\cos 45^\circ] \\ &= -1.45 \sin 300\pi t - 11.01 \cos 300\pi t \\ &= -(1.45^2 + 11.01^2)^{0.5} \sin[300\pi t + \tan^{-1}(11.01/1.45)] \\ &= -11.105 \sin(300\pi t + 82.5^\circ) \\ &= 11.105 \sin(300\pi t - 97.5^\circ) \\ \therefore v(t) &= 6.196 \sin(100\pi t - 6.2^\circ) + 5 \cos(150\pi t - 65^\circ) + 11.105 \sin(300\pi t - 97.5^\circ) \text{ V.} \end{aligned}$$

$$V_{\text{rms}} = \sqrt{\left(\frac{6.196}{\sqrt{2}}\right)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{11.105}{\sqrt{2}}\right)^2} = 9.66 \text{ V}$$

Example: 6.6-3

The current through a 10Ω resistor is seen to be $i(t) = 2 + 3 \sin 500\pi t + 1.5 \cos 501\pi t$ A. (i) Find the amplitude of a symmetric triangle wave current with 50 Hz frequency that will produce same heating in the resistor. (ii) Will the answer arrived at in (i) be correct if the current lasted only for 0.5 s?

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Solution

- (i) The rms value of a triangular current with a peak value of I_p A will be $\frac{I_p}{\sqrt{3}}$ (see Example 6.5-3)
This must be equal to the rms value of $i(t)$.

$$I_{\text{rms}} = 2^2 + \left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{1.5}{\sqrt{2}}\right)^2 = 3.1 \text{ A}$$

$$\therefore I_p = 3.1 \times \sqrt{3} = 5.37 \text{ A}$$

- (ii) No. The period over which averaging is done in the calculation of rms value is 500 times the period of $\sin 500\pi t$. That is, $T = 500 \times 4\text{ms} = 2 \text{ s}$.

Approximating average power by cyclic average power is valid only if the waveform lasts for duration much longer than the period over which averaging is carried out in cyclic average calculation. This is the reason why the frequency information does not figure in the expressions for P and rms values. The effective value of a waveform does not depend on frequency provided the waveform remains applied to the circuit for much longer than its period.

6.7 SUMMARY

- Sinusoidal waveforms are of great importance to electrical and electronic circuits due to the following reasons: (i) They preserve their waveshape in linear circuits. (ii) They render the voltage levels in electrical and electronic systems flexible so that optimisation of sub-system performance in various parts of the system by choosing a suitable voltage level in that part of the system becomes possible. Transformers help us to realise this voltage flexibility. (iii) Periodic non-sinusoidal waveforms as well as a broad class of aperiodic waveforms can be expressed as a sum of sinusoids by Fourier series and Fourier transforms. Hence, circuit solution with such input waveforms can be obtained with relative ease if solution for a sinusoidal input is known.
- A sinusoidal waveform of period T can be expressed as $v(t) = A \sin(\omega t + \theta) = A \sin(2\pi t/T + \theta) = A \sin(2\pi f t + \theta)$, where A is its *amplitude*, T is its period in s, $f = 1/T$ is its *cyclic frequency* in s^{-1} (Hertz, Hz) and $\omega = 2\pi f = 2\pi/T$ is its *radian frequency* or *angular frequency* in rad/s. The quantity θ in $v(t) = A \sin(\omega t + \theta)$ is defined as the *phase* of the sinusoidal function.
- The angular difference between *similarly located points within a cycle period* on two normalised sinusoidal waveforms with same ω is defined as the *phase difference* between them. The *phase difference* between two sinusoids is independent of choice of origin in t or ωt axis. The precedence relationship [*i.e.*, which comes after (in a visual sense) which] between them in t or ωt axis too is independent of choice of origin. However, the *phase* of a sinusoidal waveform depends on the choice of origin in t or ωt axis.
- When a waveform point on a sinusoidal function $v_2(t)$ appears *after* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ *with same frequency*, $v_2(t)$ is said to *lag* $v_1(t)$ in phase and $\theta_1 - \theta_2$ is called a *lag phase angle* under this condition.
- Similarly, when a waveform point on a sinusoidal function $v_2(t)$ appears *before* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ *with same frequency*, $v_2(t)$ is said to *lead* $v_1(t)$ in phase and $\theta_2 - \theta_1$ is called a *lead phase angle* under this condition.

- Phase lag is not necessarily a phase delay and phase lead is not necessarily a phase advance. Phase lag does not necessarily imply time delay and phase lead does not necessarily imply time advance.
- Instantaneous power *delivered to* a two-terminal element, $p(t) = v(t) i(t)$, where $v(t)$ and $i(t)$ are the element variables defined as per passive sign convention. The sum of $p(t)$ delivered to all elements in an isolated circuit will be zero.
- $p(t)$ delivered to an element is a non-constant function of time in general. If $v(t)$ and $i(t)$ are periodic waveforms with period T and zero average value over a cycle period, $p(t)$ will be a periodic waveform with period $0.5T$ and may have a non-zero average value over its cycle period of $0.5T$.
- The energy delivered to the element will also be a function of time. The value of total energy delivered to the element at end-of-cycle points will fall on a straight line with a slope equal to the average value of $p(t)$ waveform over its cycle period.
- The Cycle Average Power in the context of periodic waveforms is defined as the average of $p(t)$ over one cycle period and is denoted by P . Average power, P_{av} over a time interval $(t_2 - t_1)$ is defined

as $P_{av} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} v(t)i(t) dt$. Average Power, $P_{av} = \text{Cycle Average Power, } P$, if the waveforms

remain applied to the circuit for sufficient duration compared to their period.

- Effective value or rms value of a waveform is the value of DC quantity that will produce the *same heating effect* as that produced by the waveform when it is applied as a voltage across a resistance of 1Ω or as a current through a resistance of 1Ω . If $x(t)$ is a periodic waveform with period T , its

rms value is given by $X_{rms} = \sqrt{\frac{1}{T} \int_0^T [x(t)]^2 dt}$

- Let $v(t) = V_m \sin \omega t$ V and $i(t) = I_m \sin(\omega t + \theta)$ A be the voltage across and current through an electrical element as per passive sign convention. Then rms value of $v(t)$ is $V_m/\sqrt{2}$, rms value of current is $I_m/\sqrt{2}$ and the average power delivered to the element is $V_{rms} I_{rms} \cos \theta$ W.
- The average power delivered to an element in a linear circuit excited by sinusoidal sources of different frequencies (including DC, i.e., zero frequency) obeys superposition principle.
- Let $v(t) = v_1(t) + v_2(t) + \dots + v_n(t)$ be a composite waveform comprising n distinct frequency sinusoidal waveforms. Then, $V_{rms} = \sqrt{V_{1rms}^2 + V_{2rms}^2 + \dots + V_{n rms}^2}$.

6.8 PROBLEMS

1. A periodic waveform is the sum of three sinusoidal waveforms of period T , $0.5T$ and $0.2T$ respectively. Will this periodic waveform preserve its waveshape when differentiated and integrated?
2. What is the amplitude of $v(t) = 5 \sin \omega t - 3 \cos \omega t$ and what is its phase in degrees with respect to $v_1(t) = 4 \sin(\omega t - \pi/5)$?
3. Is the composite waveform $v(t) = 2 \sin \omega t + 3 \cos \sqrt{2} \omega t$ periodic? If yes, what is its period?
4. Is the composite waveform $v(t) = 2 \sin 200\pi t + 4 \cos(200.0001\pi t - 45^\circ)$ periodic? If yes, what is its period and cyclic frequency?
5. The voltage across a linear load is $v(t) = 100 \sin(100\pi t - 25^\circ)$ V. If the load current $i(t)$ is found to lag $v(t)$ by 36° and $i(0)$ is 2 A find $i(t)$ as a function of time.
6. What is the phase of the first derivative of a sinusoidal waveform with respect to the original waveform?

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7. What is the phase of the indefinite integral of a sinusoidal waveform with respect to the original waveform?
8. If $v_1(t) = V_1 \sin \omega t$ and $v_2(t) = V_2 \sin (\omega t + \theta)$, what is the value of θ such that (i) $v_1(t) + v_2(t)$ has a minimum amplitude, (ii) $v_1(t) + v_2(t)$ has a maximum amplitude and (iii) $v_1(t) + v_2(t)$ has an amplitude of $(V_1 + V_2)/2$.
9. If $v_1(t) = V_1 \sin \omega t$, $v_2(t) = V_2 \sin (\omega t + \theta)$ and $v_3(t) = v_1(t) + v_2(t) = V_3 \sin (\omega t + \phi)$, find θ such that (V_1, V_2, V_3) is a Pythagorean triplet and find ϕ under this condition in terms of V_1 and V_2 .
10. Derive an expression for phase angle of $v_2(t)$ with respect to $v_1(t)$ if $v_1(t) = a \sin \omega t + b \cos \omega t$ and $v_2(t) = c \sin \omega t + d \cos \omega t$.
11. Find the amplitude, frequency, angular frequency and phase of the following sinusoidal functions of time.

$$x(t) = 10 \sin 500\pi t + 5 \cos(500\pi t - \pi/5)$$

$$x(t) = 7 \sin(120\pi t - 50^\circ) + 5 \cos(120\pi t + 30^\circ)$$

$$x(t) = 7 \cos(50\pi t) + 2 \cos(50\pi t + 30^\circ) - 3 \sin(50\pi t + \pi/4)$$

12. Find the phase of $x(t)$ and $y(t)$ with respect to a $\sin \omega t$ function, phase difference between $x(t)$ and $y(t)$ and the lead/lag relationship between them for the following cases:

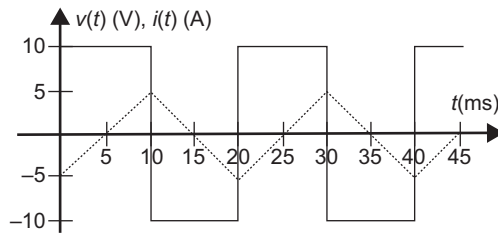
- (i) $x(t) = 3 \cos 57\pi t$
 $y(t) = 3 \sin(57\pi t - 15^\circ)$

- (ii) $x(t) = 3 \cos(7\pi t - \pi/7)$
 $y(t) = 3 \sin(7\pi t + 15^\circ)$

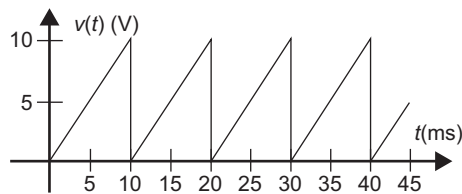
- (iii) $x(t) = 3 \cos(7\pi t - \pi/7)$
 $y(t) = 3 \sin(7\pi t + 15^\circ)$

- (iv) $x(t) = \sin \pi t + 2 \cos(\pi t - \pi/3)$
 $y(t) = \cos \pi t - 2 \sin(7\pi t + 15^\circ)$

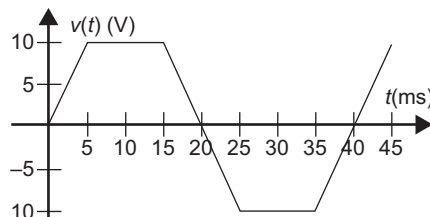
13. If $x(t) = 2 \sin 50\pi t + 3 \cos 55\pi t + 3 \sin 60\pi t$, find the period and cyclic frequency of $x(t)$.
14. A linear circuit contains 6 two-terminal elements. The sum of instantaneous power in five of them at a particular instant is found to be 17 W. What is the nature of the remaining element and what is the instantaneous power delivered to it at that instant?
15. A linear element has $v(t) = 10 \sin 10\pi t$ V and $i(t) = 3 \cos (10\pi t - 45^\circ)$ A as per passive sign convention. (i) Is the element a resistor? If not, why? (ii) Can the element be a simple element like an inductor or a capacitor or must it be a composite element? (iii) Find the average rate of growth of energy dissipated in the element. (iv) Is this average rate accurate enough if the element remained powered up only for 0.43 s? If not, calculate the average rate at which energy was dissipated in it.
16. The square wave in Fig. 6.8-1 is the voltage across an element and the triangle wave is the current through it. (i) Find and plot the instantaneous power delivered to the element and the energy delivered to it as functions of time for $t > 0$. (ii) Calculate the cycle average power.

**Fig. 6.8-1**

17. Calculate the average power an element with $v(t)$ same as the square wave in Fig. 6.8-1 and $i(t) = (5 + \text{the triangle wave in Fig. 6.8-1})$ for $t \geq 0$.
18. Show that average power in an isolated circuit is a conserved quantity.
19. Show that cyclic average power in an isolated circuit is a conserved quantity.
20. An active circuit delivers $i(t) = 10 \sin 300\pi t$ A into the positive terminal of a 12 V battery. The battery has an internal resistance of 0.2Ω . (i) What is the average power delivered to the 12 V source inside the battery? (ii) What is the average power loss in the internal resistance of the battery?
21. The voltage across a linear load is $v(t) = 325 \sin(100\pi t - 25^\circ)$ V. If the load current $i(t)$ is found to lead $v(t)$ by 36° and the average power delivered to the load is 250 W, find $i(t)$ as a function of time.
22. Derive an expression for average power if $v(t) = a \sin \omega t + b \cos \omega t$ and $i(t) = c \sin \omega t + d \cos \omega t$ in terms of a , b , c and d .
23. A sinusoidal voltage source of 230 V rms delivers 1 kW of average power to a load. What is the minimum possible amplitude of load current?
24. The periodic ramp voltage waveform in Fig. 6.8-2 is applied across a resistor of 10Ω . Find the average power dissipated in it.

**Fig. 6.8-2**

25. The absolute value of $v(t)$ shown in Fig. 6.8-3 is applied across a 5Ω resistor. Find the average power delivered to it.

**Fig. 6.8-3**

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26. A pure sinusoidal voltage $v(t) = 100 \sin 100\pi t$ V is applied across a 20Ω resistor. (i) Find the amount of energy that is moving back and forth between the voltage source and the resistor. (ii) Find the time required such that this amount is less than 1 % of the energy dissipated in the resistor.
27. A linear load draws $i(t) = 5 \sin 200\pi t$ A from a voltage source $v(t) = 200 \cos 200\pi t$ V. (i) Find the average power delivered to the load. (ii) Find the amount of energy that is moving back and forth between the voltage source and the load.
28. A linear composite load draws $i(t) = 5 \sin(100\pi t - 30^\circ)$ A from a voltage source $v(t) = 200 \sin 100\pi t$ V. (i) Find the average power delivered to the load. (ii) Find the amount of energy that is moving back and forth between the voltage source and the load. (iii) Find the time required such that this amount is less than 1% of the energy dissipated in the load.
29. A 48-V battery with a series resistance of 0.1Ω delivers the current shown in Fig. 6.8-4 to a power electronic load through a connecting wire resistance of 0.05Ω . (i) Find the cycle average and rms value of $i(t)$. (ii) Find the average power delivered by the source and received by the load. (iii) What is the waveshape and value of $i(t)$ that would have resulted in minimum loss in the system with source delivering same average power?

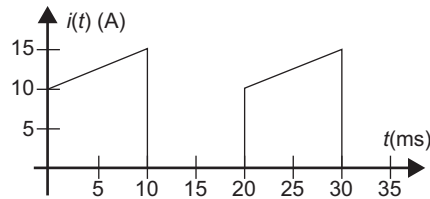


Fig. 6.8-4

30. What is the ratio between rms values of a symmetric square wave and a sinusoidal waveform with same amplitude?
31. What are the ratios between rms values and half-cycle average values of a symmetric triangle waveform and a sinusoidal waveform with same amplitude?
32. The rms value of a periodic waveform $v(t)$ with a frequency of 1 kHz is found to be 10 V. What is the rms value of $v_1(t)$ which has same amplitude and waveshape as that of $v(t)$, but has a frequency of 10 Hz?
33. The rms value of $V_m \sin \omega t$ is $V_m/\sqrt{2}$ for any value of ω . Explain why ω value does not affect the rms value.
34. If the average power delivered by a 50 Hz sinusoidal voltage to a resistor of 10Ω is 2 W, what is the average power delivered to the same resistor by another sinusoidal voltage source with same amplitude and 5 kHz frequency?
35. Show that if V_{rms} is the rms value of a periodic waveform $v(t)$, the rms value of $av(t)$ is aV_{rms} where a is a real constant.
36. A sinusoidal voltage source $v(t) = 100 \sin 20\pi t$ is applied across a 10Ω resistor for 0.42 s from $t = 0$. (i) Find the cyclic average power. (ii) Find the average power delivered during the connection duration. (iii) Why are the two values different?
37. A non-sinusoidal periodic voltage waveform with rms value of 100 V is applied to a 10Ω resistor. What is the average power delivered to the resistor?
38. If $v(t) = V_m \sin \omega t + 0.5V_m \cos 3\omega t$ V and $i(t) = I_m \sin(\omega t - 45^\circ) - 2I_m \cos(3\omega t - 45^\circ)$ A are the terminal variables of a load, find the average power delivered to the load.
39. (a) If $v_1(t) = 5 \sin 100\pi t$ and $v_2(t) = 5 \sin 100\pi t$, find the rms value of $v_1(t) + v_2(t)$, average power that will be delivered to a 10Ω resistor by $v_1(t) + v_2(t)$, and the energy that will be delivered to

- 10Ω resistor in 0.5 s. (b) If $v_1(t) = 5 \sin 100\pi t$ and $v_2(t) = 5 \sin 100.000001\pi t$, repeat part (a). Can the energy delivered in 0.5 s be found from average power in this case? Why are the rms values so different in (a) and (b) for such a small change in frequency of one component?
40. A current $i(t) = 10 \sin 200\pi t$ flows through a resistor of 5Ω from $t = 0$ to $t = 32$ ms. Explain why energy delivered to the resistor is *not equal* to average power multiplied by duration of connection. Calculate the actual value of energy dissipated in the resistor.
41. A sinusoidal voltage source $v(t) = V_m \sin \omega t$ V delivers a fixed amount of average power P through its internal resistance of R to a load. The load current has a general form of

$$i(t) = \sum_{n=1}^{\infty} a_n \sin n\omega t + b_n \cos n\omega t \text{ A.}$$

- Show that the fixed amount of average power delivered by the source reaches the load with maximum efficiency and minimum loss, when $i(t)$ is a pure sinusoidal waveform at ω rad/s with zero phase difference from $v(t)$, i.e., all a 's and b 's are zero-valued except a_1 .
42. Show that the rms value of a periodic waveform $x(t)$ and its absolute value $|x(t)|$ are equal.
43. The voltage appearing across a power electronic load and the current drawn by it are shown in Fig. 6.8-5. (i) Find the average power delivered to the load. (ii) Find the cycle average value and rms value of $v(t)$ and $i(t)$.

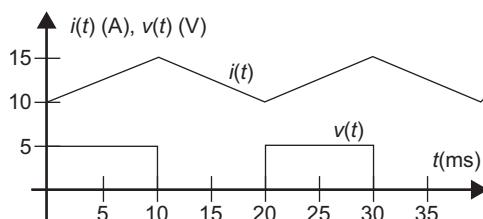


Fig. 6.8-5

44. Find the value of δ such that the periodic waveform shown in Fig. 6.8-6 will have the same rms value as that of a sinusoidal waveform with same peak voltage of V_p .

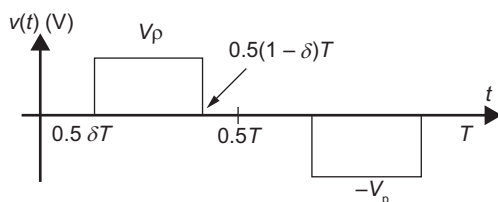


Fig. 6.8-6

45. Find the cycle average and rms value of the periodic waveform in Fig. 6.8-7.

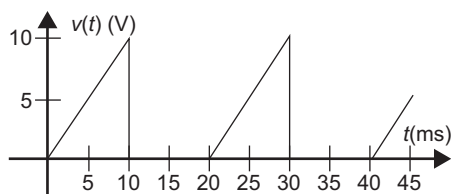


Fig. 6.8-7

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46. If $v(t)$ across an element is $10 \sin 100\pi t$ V and $i(t)$ in that element is $2 + 3\sin 200\pi t + 2 \cos 300\pi t$ A, find the average power delivered?
47. If $v(t)$ across an element is $5 + 10 \sin 100\pi t$ V and $i(t)$ in that element is $2 + 3\sin 100\pi t + 2 \cos 100\pi t + 2 \sin 300\pi t$ A, find the average power delivered?
48. Calculate the form factor and crest factor for the waveform shown in Fig. 6.8-8.

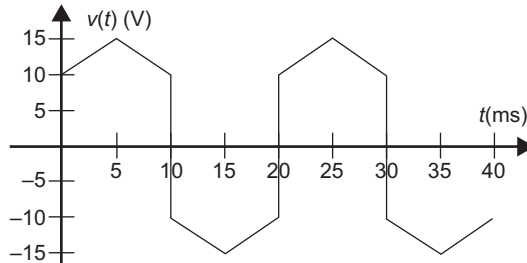


Fig. 6.8-8

49. A rectifier-type voltmeter reads the rms value of a sine wave by measuring the half-cycle average of the waveform and graduating the meter scale after accounting for the form factor of the waveform. What will be the meter reading if (i) a 10 V peak sinusoidal waveform is applied to it, (ii) a 10 V peak symmetrical square waveform is applied to it and (iii) a 10 V peak symmetric triangular waveform is applied to it?
50. Find the form factor and crest factor of the current waveform shown in Fig. 6.8-9.

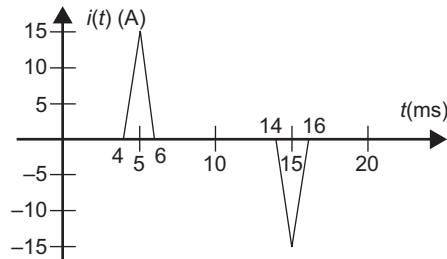


Fig. 6.8-9

51. The half-cycle average of a periodic voltage waveform is 25 V. Form factor is 1.2. Find the power delivered to a 10Ω resistor when this waveform is applied across it?
52. The cycle average of a periodic voltage waveform is 25 V. Its rms value is 35 V. Find the power delivered to a 5Ω resistor when this waveform is applied to it *after removing its DC component*?
53. Find the rms value of $v(t) = 20 + 20 \sin 100\pi t + 35 \cos (100\pi t - 35^\circ) + 23 \sin (250\pi t - 0.2\pi) + 21 \cos (250\pi t + 45^\circ)$ V.
54. If the current in an element as per passive sign convention is $i(t) = 3 + 2 \sin (100\pi t - 36^\circ) + 3.5 \cos (100\pi t - 71^\circ) + 1.15 \sin (250\pi t - 0.45\pi) + 1.05 \cos (250\pi t)$ A when the voltage waveform in Problem 24 is applied across it, find the average power delivered to it.

55. One period of two periodic voltage waveforms $v_1(t)$ and $v_2(t)$ is shown in Fig. 6.8-10. Find the power delivered by $(v_1(t) + v_2(t))$ to a $10\ \Omega$ resistor. Explain why average power does not satisfy superposition principle in this case.

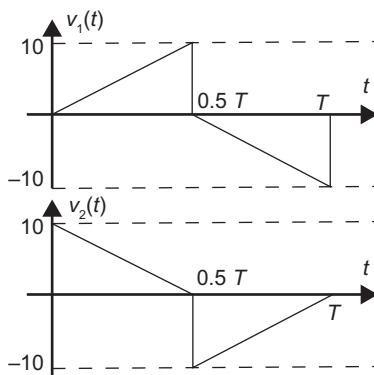


Fig. 6.8-10

56. $v_1(t)$ is a composite periodic waveform containing many sinusoidal waveforms of *distinct* frequencies. $v_2(t)$ is another composite periodic waveform containing many sinusoidal waveforms of *distinct* frequencies. $v_1(t)$ and $v_2(t)$ do not share a sinusoidal waveform component of same frequency. Show that the rms values of $[v_1(t) + v_2(t)]$ and $[v_1(t) - v_2(t)]$ will be the same.

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Chapter 7

The Sinusoidal Steady-State Response

CHAPTER OBJECTIVES

- To define and explain the concept of sinusoidal steady-state.
- To develop a systematic procedure to analyse sinusoidal steady-state in circuits in terms of steady-state solution for a complex exponential input function.
- To show how to use phasors, phasor equivalent circuits and phasor diagrams for solving circuits under sinusoidal steady-state condition.
- To illustrate the application of circuit theorems in phasor equivalent circuits.
- To introduce complex power and its components and provide detailed interpretation of the power components.
- To introduce magnetically coupled circuits and explain the application contexts for linear perfectly coupled transformer and ideal transformer.
- To familiarise the reader with analysis strategies for sinusoidal steady-state response through a large number of solved examples.

INTRODUCTION

A memoryless circuit responds to *present value* of input and its response variables are quite independent of what the source function values were at prior instants. It does not *remember* the past inputs in any manner. Hence, response variable in a memoryless circuit will have a waveshape same as that of the input source function.

We had dealt with the memory elements – inductor and capacitor – in single-element circuits in Chapter 3. Now we bring them back into circuits and take up the time-domain analysis of circuits containing R , L , C , linear dependent sources and independent sources. Such circuits are called *dynamic circuits*. The response at a particular time instant t in such circuits depends not only on the value of inputs at that instant but also on the value of inputs from infinite past to the current instant.

7.2 | The Sinusoidal Steady-State Response

The energy storage elements – L and C – *remember* what was done to them from infinite past to the current instant. Their ability to *remember* can be understood in terms of an inductor's ability to store flux linkage and a capacitor's ability to store charge. Alternatively, their memory-capability can be understood in terms of the inductor's capability to store magnetic energy and capacitor's capability to store electrostatic energy.

A circuit that contains *memory-elements* can produce a response that can have a different kind of time-variation compared to the time-variation of input source functions. For instance, if there is only one source and that varies as $\sin \omega t$, the response variables can be of $\sin(\omega t + \theta)$ type – that is, response can contain $\cos \omega t$. If there is only one source and that is a square wave of a particular frequency, the response variables need not be square waves at all. In fact, they will assume quite complicated shapes in practice – but they will be periodic waveforms with same period as that of input square wave in the case of *linear* dynamic circuits.

In general, the response of a dynamic circuit to application of input at $t = 0$ will contain two components – the natural response component and the forced response component. The total response at any instant after $t = 0$ is a mixture of the two. The natural response component represents the reaction of *inertia* in the circuit against the compelling input source function. There is inertia in the circuit since there is *memory* in the circuit. *Memory* brings about *resistance* to *change*. The circuit adjusts its natural response component in such a way that no inductors and no capacitors are required to change their initial energy storage suddenly as a result of application of input. The natural response terms in a stable circuit usually die down with time. They are called transient response terms due to this. Only the forced response remains after the transient response terms die down to zero. Under this circumstance, the forced response is called the steady-state response *if the notion of steady-state* is applicable in relation to the input. The notion of *steady-state* is applicable only if input function possesses some aspects that remain steady in time.

Sinusoidal steady-state in a dynamic circuit is that state when all the response variables (*i.e.*, all element currents and element voltages) contain just one component with a *sinusoidal waveshape* with the frequency same as that of the sinusoidal forcing function applied.

Sinusoidal steady-state, like any other steady-state, can come up in a circuit only after the circuit goes through the transient period that follows the application of sources. The transient period is deemed to have completed when all the transient response terms, which usually have waveshape different from the applied sinusoidal function, have died down to negligible levels.

The next section attempts to provide an overview of transient response of circuits in order to set a background for taking up the study of sinusoidal steady-state. All the topics briefly touched upon in that section will be taken up in great detail in later chapters on time-domain analysis of dynamic circuits. The aim of the next section is only to place the sinusoidal steady-state in the correct perspective.

7.1 TRANSIENT STATE AND STEADY-STATE IN CIRCUITS

The mesh equations (node equations) needed to solve for the response of any circuit – memoryless or dynamic – are obtained by applying Kirchhoff's Voltage Law in meshes (Kirchhoff's Current Law at nodes) along with the element equations. The element equation of an inductor is $v_L = L \frac{di_L}{dt}$ and that of

a capacitor is $v_C = \frac{1}{C} \int_{-\infty}^t i_C dt$. Therefore, the mesh equations (node equations) describing a *dynamic circuit* will be *integro-differential equations*. These equations are true for all t since they are obtained from KVL and KCL that are true on an instant to instant basis. Therefore, both sides of such integro-differential equations can be differentiated or integrated with respect to time, if needed. We will be able to eliminate the integral terms in the equations by this technique. The resulting equations will be differential equations and the coefficients of differential equation will be decided by the values of R , L , C and M . Since they are constants in the circuits we consider, the resulting differential equations will be a set of simultaneous linear differential equations with constant coefficients. We illustrate this in two examples in the next sub-section.

7.1.1 Governing Differential Equation of Circuits – Examples

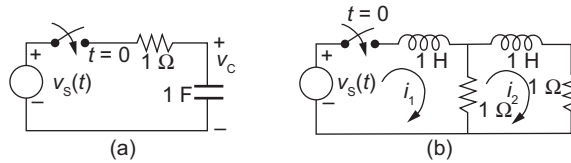


Fig. 7.1-1 Example circuits illustrating circuit differential equations

Consider the simple Series RC Circuit in Fig. 7.1-1(a). Assume that capacitor was initially uncharged. The voltage across the capacitor is taken as the response variable. Then, the current through the resistor from left to right can be expressed as $(v_s - v_C)/R$. Writing KCL at the positive terminal of capacitor,

$$v_s - v_C = 1 \times \frac{dv_C}{dt}$$

$$\therefore \frac{dv_C}{dt} + v_C = v_s$$

Now, we take up the second example circuit in Fig. 7.1-1 and write the two mesh equations as,

$$\frac{di_1}{dt} + i_1 - i_2 = v_s \quad (7.1-1)$$

$$\frac{di_2}{dt} + 2i_2 - i_1 = 0 \quad (7.1-2)$$

This is a set of two first-order simultaneous ordinary differential equations with constant coefficients. The first mesh current variable i_2 is selected as the circuit response variable. We have to eliminate i_1 from these two equations and get a single differential equation for i_2 in order to find the differential equation describing the circuit. This may be done by substituting Eqn. 7.1-2 in Eqn. 7.1-1 as shown below:

$i_1 = \frac{di_2}{dt} + 2i_2$ - from Eqn. 7.1-2. Substituting this in Eqn. 7.1-1, we get,

$$\frac{d}{dt} \left(\frac{di_2}{dt} + 2i_2 \right) + \left(\frac{di_2}{dt} + 2i_2 \right) - i_2 = v_s$$

7.4 | The Sinusoidal Steady-State Response

Therefore, the describing differential equation in Fig. 7.1-1 (b) is

$$\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = v_s$$

These two examples help us to understand the following generalisation.

A dynamic circuit is described by a linear, constant-coefficient, ordinary differential equation for one chosen response variable. The coefficients will be decided by circuit parameters. The right-hand side will contain the applied forcing function terms (including their derivatives in general).

7.1.2 Solution of the Circuit Differential Equation

We accept the following from the basic courses in Mathematics. More detailed exposition on these matters will be provided in later chapters on time-domain analysis of circuits.

- The total solution of a linear constant-coefficient differential equation contains two kinds of terms – the complementary solution terms and the particular integral terms.
- Complementary solution terms are obtained by solving the differential equation with right-hand side set to zero.
- Particular integral terms are obtained by solving the differential equation with the forcing function in the right-hand side.
- Complementary solution terms are of the type $Ae^{\alpha t}$, where A and α are to be solved for. An n^{th} order linear differential equation with constant coefficients will have n such solution terms. α 's can be obtained by substituting this solution in the differential equation with right-hand side set to zero. A 's can be obtained by applying initial conditions for all derivatives of the dependent variable from zero to $(n - 1)^{\text{th}}$ order on the *total solution*.

The differential equation governing the circuit (a) was. $\frac{dv_C}{dt} + v_C = v_s$.

We try $Ae^{\alpha t}$ as the solution of $\frac{dv_C}{dt} + v_C = 0$. On substituting the trial solution, we get, $A\alpha e^{\alpha t} + Ae^{\alpha t} = 0$. Since this has to be true for all t , we conclude that $\alpha + 1 = 0$ leading to $\alpha = -1$.

Now, we attempt to solve for the particular integral. We must specify v_s for that. Let us assume that $v_s = V$, a constant source – that is, a DC source. Then the equation we have to solve is given by $\frac{dv_C}{dt} + v_C = V$. The only possibility that a time-function and its first derivative can combine to yield a constant for all t occurs when the time-function is a *constant*. Therefore, we try a constant function as trial solution. Since the first derivative of a constant is zero, the solution must be $v_C = V$.

Therefore, the total solution for v_C is $v_C = Ae^{-t} + V$ for $t > 0$.

Since the voltage across a capacitor cannot change instantaneously unless there is an impulse current flow through it, we expect the above expression to approach zero as $t \rightarrow 0$ from right side.

$$\therefore 0 = Ae^{-0} + V \Rightarrow A = -V$$

Therefore, the complete solution for DC switching problem in Fig. 7.1-1 (a) is. $v_C = V(1 - e^{-t})$ V.

The solution contains two terms; $-Ve^{-t}$ is the transient response term and V is the steady-state response term. The transient response term vanishes in about 5s or so leaving only the steady-state term. DC steady-state, thus, comes up in this circuit in about 5s after application of DC voltage. The period during which the transient response term is active and non-negligible is called the *transient period*. The circuit reaches steady-state only after the transient period is over.

The differential equation describing circuit in Fig. 7.1-1 (b) was $\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = v_s$. Consider a DC switching problem with zero initial currents in inductors in this case too.

The complementary solution terms are obtained by trying out $Ae^{\alpha t}$ in the homogeneous differential equation leading to $\alpha^2 + 3\alpha + 1 = 0$. Therefore, α has two values and they are -0.382 and -2.618 . Therefore the complementary solution is $A_1 e^{-0.382t} + A_2 e^{-2.618t}$.

The particular integral for DC switching problem is a constant in this case too. The value of that constant has to be V in order to satisfy the differential equation with V on right-hand side. Therefore, the total solution is $i_2 = 1 + A_1 e^{-0.382t} + A_2 e^{-2.618t}$. We can find out A_1 and A_2 by using the initial current values for the inductors, if we desire so. But those values are not important to us here. What is more important is the observation that the total solution again contains two sets of terms – one set which vanishes with time and hence transient in nature and the other which lasts even after transients vanish. The transient response terms come from complementary solution and the lasting component (*i.e.*, the steady-state component) comes from the particular integral. The transient response terms vanish in duration decided by the index in the exponential terms, which are decided by circuit parameters in turn.

7.1.3 Complete Response with Sinusoidal Excitation

Consider the circuit in Fig. 7.1-1 (a) with $v_s = V_m \cos \omega t$ V now. This sinusoidal waveform is switched on to the circuit at $t = 0$.

The complementary solution is the same as before – it is Ae^{-t} . The particular integral has to be obtained from $\frac{dv_C}{dt} + v_C = V_m \cos \omega t$. This equation has to be true for all t . That can happen only if both sides of the equation are time-functions with same waveshape. Therefore, $\frac{dv_C}{dt} + v_C$ must have same waveshape as that of $\cos \omega t$. That will imply that both v_C and $\frac{dv_C}{dt}$ must have sinusoidal waveshape. Therefore, we can try $v_C = a \sin \omega t + b \cos \omega t$ as a trial solution. Substituting this trial solution in the differential equation and collecting terms, we get,

$$(a\omega + b) \cos \omega t + (a - b\omega) \sin \omega t = V_m \cos \omega t$$

This equation can be true for all t only if the coefficients of $\sin \omega t$ on both sides of the equation are equal and the coefficients of $\cos \omega t$ on both sides of the equation are equal.

$$\therefore a\omega + b = V_m \text{ and } a - b\omega = 0 \Rightarrow a = \frac{\omega V_m}{\omega^2 + 1} \text{ and } b = \frac{V_m}{\omega^2 + 1}.$$

$$\therefore \text{particular solution} = \frac{\omega V_m}{\omega^2 + 1} \sin \omega t + \frac{V_m}{\omega^2 + 1} \cos \omega t = \frac{1}{\sqrt{\omega^2 + 1}} \cos(\omega t - \tan^{-1} \omega)$$

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The total solution $= v_c = Ae^{-t} + \frac{1}{\sqrt{\omega^2 + 1}} \cos(\omega t - \tan^{-1} \omega)$ for $t > 0$. This solution must approach zero value as $t \rightarrow 0$ from right side since the initial value of voltage across capacitor is zero. Therefore,

$$0 = Ae^{-0} + \frac{V_m}{\sqrt{\omega^2 + 1}} \cos(-\tan^{-1} \omega) \Rightarrow A = -\frac{V_m}{\sqrt{\omega^2 + 1}} \cos(-\tan^{-1} \omega)$$

$$\therefore v_c = \frac{V_m}{\sqrt{\omega^2 + 1}} \cos(\omega t - \tan^{-1} \omega) - \frac{V_m}{\sqrt{\omega^2 + 1}} \cos(-\tan^{-1} \omega) e^{-t} \text{ V}$$

Once again we see that the total response contains a transient term that vanishes as $t \rightarrow \infty$ and a steady-state term that persists. *The steady-state term is a sinusoidal waveform of same frequency as that of input sinusoid. But it has a phase difference with respect to input sinusoid.* It is a phase lag. Both the amplitude and phase of steady-state response component depend on the angular frequency ω of input sinusoid. The steady-state response is given by particular integral and the transient response is contributed by complementary function.

We can proceed the same way in the case of Fig. 7.1-1 (b) too. We solve only for the particular integral of differential equation this time since we know that the transient response will vanish in this circuit. We try out the solution $i_2 = a \sin \omega t + b \cos \omega t$ in $\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = V_m \cos \omega t$ to find the sinusoidal steady-state solution for second mesh current in the circuit when the circuit is driven by $V_m \cos \omega t$ V. Substituting the trial solution in the differential equation and collecting terms, we get,

$$(-\omega^2 b + 3\omega a + b) \cos \omega t + (-\omega^2 a - 3\omega b + a) \sin \omega t = V_m \cos \omega t$$

Equating coefficients of $\sin \omega t$ and $\cos \omega t$ on both sides of the equation, we get

$$(-\omega^2 b + 3\omega a + b) = V_m \text{ and } (-\omega^2 a - 3\omega b + a) = 0$$

Solving for a and b we get $a = \frac{3\omega V_m}{(1 - \omega^2)^2 + 9\omega^2}$ and $b = \frac{(1 - \omega^2)V_m}{(1 - \omega^2)^2 + 9\omega^2}$

Therefore, the sinusoidal steady-state response for i_2 in Fig. 7.1-1 is

$$i_2(t) = \frac{3\omega V_m}{(1 - \omega^2)^2 + 9\omega^2} \sin \omega t + \frac{(1 - \omega^2)V_m}{(1 - \omega^2)^2 + 9\omega^2} \cos \omega t$$

$$= \frac{V_m}{\sqrt{(1 - \omega^2)^2 + 9\omega^2}} \cos \left(\omega t - \tan^{-1} \frac{3\omega}{1 - \omega^2} \right)$$

We observe once again that the response is a pure sinusoidal waveform at same frequency as that of input sinusoid. But the response has a phase lag with respect to the input. Both the amplitude and phase of response depend on the angular frequency ω of the input sinusoid.

The steady-state response of a circuit variable in a linear dynamic circuit under sinusoidal excitation is a sinusoidal waveform of same frequency as that of input. The response will, in general, have a phase difference with respect to input. The amplitude and phase of response under steady-state condition will depend on the amplitude of input and the angular frequency of input sinusoid.

7.2 THE COMPLEX EXPONENTIAL FORCING FUNCTION

The method outlined in the previous section to determine the sinusoidal steady-state response of a dynamic circuit can be summarized as follows:

1. Use mesh or nodal analysis to obtain integro-differential equations of the circuit.
2. Differentiate equations again to eliminate integrals, if needed.
3. Choose one of the mesh currents (or node voltages) as the describing variable for the circuit. Eliminate all the other variables and obtain an n^{th} order linear constant-coefficient differential equation describing the chosen circuit variable.
4. Assume solution in the form $(a \sin \omega t + b \cos \omega t)$. Substitute the assumed solution in the differential equation. Equate coefficients of cosine and sine on both sides of the equation. Solve for a and b using the resulting equations.
5. Express the solution in the form of a single sinusoid with phase shift.
6. Find the other mesh currents (or node voltages) which were eliminated earlier by using the elimination equations in the reverse. Once all the mesh currents (node voltages) are available, any element variable can be obtained from them.

There is nothing wrong with this method – except that it is going to be very tedious if the circuit contains more than two energy storage elements.

Hence, we look for another simpler and more elegant method to obtain sinusoidal steady-state. Euler's identity, which relates a complex exponential time-function to trigonometric time-functions, is the key to this new method.

$$\text{Euler's Identity } e^{j\theta} = \cos \theta + j \sin \theta. \quad (7.2-1)$$

By letting $\theta = \omega t$ and using Euler's Identity, we can express $e^{j\omega t}$ as $e^{j\omega t} = \cos \omega t + j \sin \omega t$ and by letting $\theta = -\omega t$ and using Euler's Identity, we can express $e^{-j\omega t}$ as $e^{-j\omega t} = \cos \omega t - j \sin \omega t$. Therefore,

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \text{ and } \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \quad (7.2-2)$$

$e^{j\omega t}$ is the *complex exponential function of unit amplitude*. Equation. 7.2-2 expresses unit amplitude cosine and sine functions of time with an angular frequency of ω in terms of two complex exponential functions of time with ω and $-\omega$ in the indices of exponential functions. We can also express sine and cosine functions in terms of complex exponential function in another way too as in Eqn. 7.2-3.

$$\cos \omega t = \text{Re}[e^{j\omega t}] \text{ and } \sin \omega t = \text{Im}[e^{j\omega t}] \quad (7.2-3)$$

7.2.1 Sinusoidal Steady-State Response from Response to $e^{j\omega t}$

These two ways of expressing trigonometric functions in terms of complex exponential functions suggest two methods to obtain the sinusoidal steady-state response in dynamic circuits. The first method is to obtain steady-state response to complex exponential inputs $e^{j\omega t}$ and $e^{-j\omega t}$ and obtain the steady-state response for $\cos \omega t$ as the sum of responses for $e^{j\omega t}$ and $e^{-j\omega t}$. But this will be correct if and only if the particular integral of a linear constant-coefficient differential equation obeys superposition principle. The mathematical theory of such differential equations assures us that it is indeed so.

7.8 | The Sinusoidal Steady-State Response

The steady-state response component in linear time-invariant circuits obeys superposition principle.

The second method will be to obtain the steady-state response for a $\cos\omega t$ input as the *real part* of steady-state response to a complex exponential function $e^{j\omega t}$ and the steady-state response for a $\sin\omega t$ input as its *imaginary part*. The underlying reasoning is that since $\cos\omega t$ is the *real part* of $e^{j\omega t}$, the response for $\cos\omega t$ must be the *real part* of response for $e^{j\omega t}$. It looks intuitively evident. But this turns out to be true only for linear circuits. This will be true only if the *real part* of the input function does not affect the *imaginary part* of response and the *imaginary part* of input function does not affect the *real part* of response. We employ the superposition principle for particular integral of a constant-coefficient linear differential equation to verify that it is so in the case of such differential equations.

$e^{j\omega t}$ can be viewed as a linear combination of two input functions – $\cos\omega t$ multiplied by 1 added to $\sin\omega t$ multiplied by j . We can view the steady-state response of a linear circuit to $e^{j\omega t}$ input as the particular solution of the describing differential equation of the circuit to a composite input – $\cos\omega t$ multiplied by 1 added to $\sin\omega t$ multiplied by j . But, particular solution obeys superposition principle. Therefore, steady-state response to $e^{j\omega t}$ = steady-state response to $\cos\omega t$ + j times steady-state response to $\sin\omega t$. Therefore,

The *real part* of steady-state response to $e^{j\omega t}$ = steady-state response to $\cos\omega t$ and the *imaginary part* of steady-state response to $e^{j\omega t}$ = steady-state response to $\sin\omega t$.

Therefore, the second method for determining the sinusoidal steady-state response in terms of steady-state response to complex exponential function will yield correct result for linear circuits.

7.2.2 Steady-State Solution to $e^{j\omega t}$ and the $j\omega$ Operator

Both the methods based on complex exponential function will be meaningful only if finding the steady-state response to complex exponential function is simpler and more elegant than finding the steady-state response to sinusoidal input straightaway. We show in the following paragraphs that this is true.

Consider the describing differential equation for the second mesh current in Fig. 7. 1-1 (b) with a unit amplitude complex exponential function driving the circuit. The equation is $\frac{d^2 i_2}{dt^2} + 3 \frac{di_2}{dt} + i_2 = e^{j\omega t}$.

This equation has to be true for all t and that will be possible only if both the sides of the equation have same waveshape in time. Therefore, the particular integral has to be chosen in such a way that, on substituting the trial solution in the differential equation, the left-hand side yields $(.)e^{j\omega t}$. But this

will imply that $\frac{d^2 i_2}{dt^2}$, $\frac{di_2}{dt}$ and i_2 must have $e^{j\omega t}$ in them. [In fact, there are special situations in circuits

under which this is not strictly true. But we leave special cases to later chapters that deal extensively with time-domain analysis of circuits.] Therefore, we look for some function that has $e^{j\omega t}$ in its zeroth-order, first-order and second-order derivatives. The only function that comes to our mind is $e^{j\omega t}$ itself. Therefore, the trial solution for particular integral is $A e^{j\omega t}$, where A can now be a complex number.

Substituting this solution in the differential equation, we get,

$$(j\omega)^2 [Ae^{j\omega t}] + 3(j\omega)[Ae^{j\omega t}] + [Ae^{j\omega t}] = e^{j\omega t}$$

Since this has to be true for all t , we cancel out $e^{j\omega t}$ and get $[(j\omega)^2 + 3(j\omega) + 1]A = 1$. The solution is completed by solving for A .

$$i_2 = \frac{1}{(1 - \omega^2) + 3j\omega} e^{j\omega t} = \left[\frac{1}{\sqrt{(1 - \omega^2)^2 + 9j\omega^2}} e^{j\phi} \right] e^{j\omega t}, \text{ where } \phi = \tan^{-1} \frac{3\omega}{1 - \omega^2}$$

We note that each differentiation in time has got replaced by a multiplication by $j\omega$ in the equation determining the complex amplitude in the assumed particular solution. Once we appreciate that point, we can straightaway obtain the equation governing the complex amplitude A of the steady-state response to a complex exponential input by replacing every differentiation in the differential equation by a multiplication by $j\omega$ and solve for A easily.

Let us generalise this. We consider a linear circuit with one sinusoidal source at angular frequency of ω driving it. If there are more sources, we employ superposition principle and solve many single-source circuits. Hence, the basic problem is to solve the circuit for a single source. The most general differential equation governing a chosen response variable for a linear circuit is

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x \quad (7.2-4)$$

where y is the chosen response variable, x is the input function and a 's and b 's are real positive constants decided by circuit parameters. The order of differential equation, n , is equal to the number of independent energy storage elements in the circuit.

If $x = e^{j\omega t}$, then $y = A e^{j\omega t}$, where A is a complex amplitude decided by the equation

$$A[(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_1(j\omega) + a_0] = b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \dots + b_1(j\omega) + b_0.$$

Therefore, the complex amplitude of steady-state response to a unit amplitude complex exponential input comes out as a ratio of rational polynomials in the variable $j\omega$. The desired steady-state response

is obtained as $y = \frac{\sum_{k=0}^m b_k (j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i (j\omega)^i} e^{j\omega t}.$

Hence, solving for steady-state response to complex exponential function is much simpler and more elegant than solving for steady-state response to cosine or sine input functions. Therefore, trying to obtain sinusoidal steady-state response indirectly from steady-state response to complex exponential input function is worthwhile.

The method based on Eqn. 7.2-2 and superposition principle – we called it the first method – is the Electronics and Communication Engineer's favourite. Electrical Power Engineers usually prefer the second method that is based on Eqn. 7.2-3 and superposition principle. Both lead to the same result of course. But, the analysis of sinusoidal steady-state response from the so-called *phasor concept* evolves from the second method. Hence, we take up the second method for detailed discussion. In this method, we obtain sinusoidal steady-state response for $\cos\omega t$ by obtaining the *real part* of

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steady-state response to $e^{j\omega t}$ input. The sinusoidal steady-state response for $\sin\omega t$ input is similarly the *imaginary part* of steady-state response to $e^{j\omega t}$ input. A separate determination of response for cosine and sine input is unnecessary. $\sin\omega t$ is 90° phase delayed version of $\cos\omega t$ and therefore it must be possible to obtain the steady-state response for $\sin\omega t$ input by delaying the steady-state response for $\cos\omega t$ input by 90° . We will verify this in later sections.

7.3 SINUSOIDAL STEADY-STATE RESPONSE USING COMPLEX EXPONENTIAL INPUT

The method developed in the previous section for determining the sinusoidal steady-state response of a linear circuit for sinusoidal input is summarized as follows:

1. Obtain the differential equation of the circuit in terms of a chosen describing variable y by nodal analysis or mesh analysis and subsequent elimination of all variables other than the chosen one. The differential equation is put in the form shown in Eqn. 7.2-4.
2. Express the sinusoidal input as $x = X_m \cos(\omega t + \theta)$. This form is needed to accommodate even sine functions. It could be a current source or voltage source and X_m represents the amplitude of the function. Then this function can be seen as the *real part* of $X_m e^{j(\omega t + \theta)}$.
3. Assume that the input is $x = X_m e^{j(\omega t + \theta)}$ instead of $x = X_m \cos(\omega t + \theta)$.
4. Let the steady-state solution for y for this input be $y_{ss} = Y e^{j(\omega t + \theta)}$ where Y is the *complex amplitude* of the response. **We use bold face italics for complex amplitudes.**

5. Obtain Y as $Y = X_m \times \frac{\sum_{k=0}^m b_k (j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i (j\omega)^i}$ where a 's and b 's are as in Eqn. 7.2-6. This complex

number (which is a function of real variable ω) can be expressed in exponential form as $Y_m e^{j\phi}$ where Y_m is its magnitude and ϕ is its argument (*i.e.*, angle). It can also be expressed in polar form as $Y_m \angle \phi$.

6. Now, $y_{ss} = Y_m e^{j\phi} e^{j(\omega t + \theta)} = Y_m e^{j(\omega t + \theta + \phi)}$.
7. The desired sinusoidal steady-state response is the *real part* of this. Hence the sinusoidal steady-state response to $x = X_m \cos(\omega t + \theta)$ is $y = Y_m \cos(\omega t + \theta + \phi)$.

This procedure is applied on the circuit shown in Fig. 7.3-1.

Applied input is $v_s(t) = V_m \cos\omega t$ V. The second mesh current is the chosen circuit variable. Two mesh equations are written first.

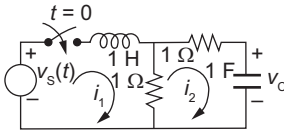


Fig. 7.3-1 A two-mesh circuit for illustrating sinusoidal steady-state solution

$$\frac{di_1}{dt} + i_1 - i_2 = v_s \quad (7.3-1)$$

$$\frac{1}{1} \int_{-\infty}^t i_2 dt + 2i_2 - i_1 = 0 \quad (7.3-2)$$

Differentiating both sides of second mesh equation with respect to time, we get,

$$2 \frac{di_2}{dt} + i_2 - \frac{di_1}{dt} = 0 \quad (7.3-3)$$

We need to eliminate i_1 from Eqns. 7.3-1 and 7.3-3 to get a differential equation for i_2 .

$$\frac{di_1}{dt} = 2 \frac{di_2}{dt} + i_2 \text{ (from Eqn. 7.3-3)}$$

$$\therefore \frac{d^2 i_1}{dt^2} = 2 \frac{d^2 i_2}{dt^2} + \frac{di_2}{dt}$$

Differentiating the Eqn. 7.3-1 with respect to time gives us,

$$\frac{d^2 i_1}{dt^2} + \frac{di_1}{dt} - \frac{di_2}{dt} = \frac{dv_s}{dt}$$

Substituting for $\frac{di_1}{dt}$ and $\frac{d^2 i_1}{dt^2}$, we get,

$$\frac{d^2 i_2}{dt^2} + \frac{di_2}{dt} + 0.5i_2 = 0.5 \frac{dv_s}{dt}$$

Now, we solve the steady-state response for $V_m e^{j\omega t}$. Let the response be $Y = Y_m e^{j\omega t}$. Then, substituting the solution in the above differential equation,

$$Y[(j\omega)^2 + (j\omega) + 0.5] e^{j\omega t} = 0.5(j\omega) V_m e^{j\omega t}$$

$$\therefore Y = \frac{0.5(j\omega)V_m}{(0.5 - \omega^2) + (j\omega)} = \frac{0.5\omega V_m}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} e^{j\phi}, \text{ where } \phi = \frac{\pi}{2} - \tan^{-1} \frac{\omega}{(0.5 - \omega^2)}.$$

Therefore, the steady-state response for $V_m e^{j\omega t} = V_m e^{j\omega t} = \frac{0.5\omega V_m}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} e^{j(\omega t + \phi)}$

The desired sinusoidal steady-state response is obtained by taking the real part of this solution and is = $\frac{0.5\omega V_m}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} \cos \left[\omega t + \frac{\pi}{2} - \tan^{-1} \frac{\omega}{(0.5 - \omega^2)} \right]$.

7.4 THE PHASOR CONCEPT

Solving for particular integral of a differential equation for sinusoidal input has been rendered easy by the use of complex exponential function as shown in the previous sections. But deriving the differential equation remains a tedious affair even now. Arriving at the proper elimination steps require considerable ingenuity in the case of circuits containing many inductors and capacitors. But we can avoid all that. We proceed to see how.

The steady-state response of second mesh current i_2 in the circuit in Fig. 7.3-1 for a complex exponential input of $V_m e^{j\omega t}$ was seen to be $\frac{0.5\omega V_m e^{j\phi}}{\sqrt{(0.5 - \omega^2)^2 + \omega^2}} e^{j\omega t}$. We evaluate this for $\omega = 1$ rad/sec.

Then $i_2 = 0.447 V_m e^{-j26.6^\circ} e^{jt}$

The equation used for eliminating the first mesh current was $\frac{di_1}{dt} = 2 \frac{di_2}{dt} + i_2$. Substituting for i_2 , we get,

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$$\frac{di_1}{dt} = j0.894V_m e^{-j26.6^\circ} e^{jt} + 0.447V_m e^{-j26.6^\circ} e^{jt} = (0.447 + j0.894)V_m e^{-j26.6^\circ} e^{jt}$$

Integrating this equation gives us,

$$\begin{aligned} i_1 &= \frac{1}{j1} (0.447 + j0.894)V_m e^{-j26.6^\circ} e^{jt} \\ &= (0.894 - j0.447)V_m e^{-j26.6^\circ} e^{jt} \\ &= 1e^{-j26.6^\circ} V_m e^{-j26.6^\circ} e^{jt} \\ &= V_m e^{-j53.2^\circ} e^{jt} \end{aligned}$$

We have got the two mesh currents now. Therefore, we can obtain all the circuit variables now. For instance, the current in the common resistor is given by

$$\begin{aligned} i_1 - i_2 &= V_m (e^{-j53.2^\circ} - 0.447e^{-j26.6^\circ}) e^{jt} = V_m (0.6 - j0.8 - 0.4 + j0.2) e^{jt} \\ &= V_m (0.2 - j0.6) e^{jt} = 0.632V_m e^{-j71.6^\circ} e^{jt} \end{aligned}$$

The voltage across 1H inductor is given by

$$\frac{di_1}{dt} = \frac{d}{dt} (V_m e^{-j53.2^\circ} e^{jt}) = V_m j e^{-j53.2^\circ} e^{jt} = V_m e^{j90^\circ} e^{-j53.2^\circ} e^{jt} = V_m e^{j26.6^\circ} e^{jt}.$$

Based on the above example, we arrive at the following conclusion.

All element voltage variables and all element current variables in a linear dynamic circuit driven by a complex exponential function $X_m e^{j\omega t}$ will assume the form $(Y_m e^{j\phi}) e^{j\omega t}$ under steady-state condition where $(Y_m e^{j\phi})$ represents the relevant *complex amplitude* for the variables. Y_m will be proportional to X_m .

7.4.1 Kirchhoff's Laws in Terms of Complex Amplitudes

Consider a mesh containing n elements in a general linear dynamic circuit. Let it be in steady-state condition under complex exponential drive. Let the angular frequency of the complex exponential function be ω . Then we can represent each element voltage variable, v_i for $i = 1$ to n , as $v_i = (V_{im} e^{j\phi_i}) e^{j\omega t}$ under steady-state condition. We assume for simplicity that we encounter the positive polarity of voltage variable first when we traverse the mesh in clock-wise direction. Then, applying KVL in the mesh results in the following KVL equation.

$$\begin{aligned} (V_{1m} e^{j\phi_1}) e^{j\omega t} + (V_{2m} e^{j\phi_2}) e^{j\omega t} + \dots + (V_{nm} e^{j\phi_n}) e^{j\omega t} &= 0 \text{ for all } t \\ \text{i.e., } [(V_{1m} e^{j\phi_1}) + (V_{2m} e^{j\phi_2}) + \dots + (V_{nm} e^{j\phi_n})] e^{j\omega t} &= 0 \text{ for all } t \\ \text{i.e., } [(V_{1m} e^{j\phi_1}) + (V_{2m} e^{j\phi_2}) + \dots + (V_{nm} e^{j\phi_n})] &= 0. \end{aligned}$$

Thus, the KVL equation under complex exponential steady-state response condition can be written entirely in terms of the *complex amplitudes* of the steady-state element voltages. The common

complex exponential function format $e^{j\omega t}$ that appears in *all* element voltages may be suppressed in KVL equations.

Similar conclusion can be stated for KCL equations at nodes of a dynamic circuit under complex exponential steady-state response condition.

KVL equation in a mesh (KCL equation at a node) becomes a *mesh equation* (*node equation*) only when *mesh currents* (*node voltages*) are used to replace element voltages (currents). We need the *element equations* for that. Hence we need to see how the *complex amplitudes* of element voltage and current are related to each other in the case of R, L, C etc.

7.4.2 Element Relations in Terms of Complex Amplitudes

Consider a resistor. Passive sign convention is assumed everywhere. The element equation of a resistor is $v_R = Ri_R$. Substituting $v_R = (V_{Rm}e^{j\phi_v})e^{j\omega t}$ and $i_R = (I_{Rm}e^{j\phi_i})e^{j\omega t}$ we get $(V_{Rm}e^{j\phi_v})e^{j\omega t} = R(I_{Rm}e^{j\phi_i})e^{j\omega t}$ and hence $V_{Rm} = RI_{Rm}$ and $\phi_v = \phi_i$. This implies that a resistor can not bring about a phase difference between the current and voltage variables. Using the bold face italic notation for the complex amplitude, we write the element relation of a resistor for complex amplitudes as follows:

$$\mathbf{V}_R = R\mathbf{I}_R \quad (7.4-1)$$

The element equation of an inductor is $v_L = L \frac{di_L}{dt}$. Substituting $v_L = (V_{Lm}e^{j\phi_v})e^{j\omega t}$ and $i_L = (I_{Lm}e^{j\phi_i})e^{j\omega t}$ we get $(V_{Lm}e^{j\phi_v})e^{j\omega t} = L(j\omega)(I_{Lm}e^{j\phi_i})e^{j\omega t}$. This implies that $V_{Lm} = \omega LI_{Lm}$ and $\phi_v = \phi_i + 90^\circ$. Thus an inductor scales the current amplitude by ωL and adds a 90° phase advance to the current to generate the voltage across it. Using the bold face italic notation for the complex amplitude, we write the element relation of an inductor for *complex amplitudes* as below.

$$\begin{aligned} \mathbf{V}_L &= (j\omega L)\mathbf{I}_L = (\omega L e^{j90^\circ})\mathbf{I}_L \\ \mathbf{I}_L &= \frac{1}{j\omega L}\mathbf{V}_L = \left[\frac{1}{\omega L} e^{-j90^\circ} \right] \mathbf{V}_L \end{aligned} \quad (7.4-2)$$

Thus, the inductor *operates* upon the *complex amplitude* of current by the operator $j\omega L$ to generate the voltage *complex amplitude* across itself.

The amplitude of current in an inductor is $1/\omega L$ times the amplitude of voltage and the current *lags* the voltage by 90° under sinusoidal steady-state condition.

The element equation of a capacitor is $i_C = C \frac{dv_C}{dt}$. Substituting $v_C = (V_{Cm}e^{j\phi_v})e^{j\omega t}$ and $i_C = (I_{Cm}e^{j\phi_i})e^{j\omega t}$ we get $(I_{Cm}e^{j\phi_i})e^{j\omega t} = C(j\omega)(V_{Cm}e^{j\phi_v})e^{j\omega t}$. This implies that $I_{Cm} = \omega CV_{Cm}$ and $\phi_i = \phi_v + 90^\circ$. Thus a capacitor scales the voltage amplitude by ωC and adds a 90° phase advance to the voltage to generate the current through itself. Using the bold face italic notation for the complex amplitude, we write the element relation of a capacitor for *complex amplitudes* as below.

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$$\begin{aligned} I_C &= (j\omega C)V_C = (\omega C e^{j90^\circ})V_C \\ V_C &= \frac{1}{j\omega C} I_C = \left[\frac{1}{\omega C} e^{-j90^\circ} \right] I_C \end{aligned} \quad (7.4-3)$$

Thus, the capacitor *operates* upon the *complex amplitude* of voltage by the *operator* $j\omega C$ to generate the complex amplitude of current through it.

The amplitude of current in a capacitor is ωC times the amplitude of voltage and the current *leads* the voltage by 90° under sinusoidal steady-state condition.

7.4.3 The Phasor

Electrical Engineers decided long back that a new name is required for what we have been calling the *complex amplitude*. Complex amplitude is the number that gives the amplitude of complex exponential function and the phase of the complex exponential function in the form of a single complex number. Its magnitude gives the amplitude of the signal and its angle gives the phase of the signal. The phase is referred to the standard $e^{j\omega t}$ reference function. Electrical Engineers call complex amplitude a *phasor*.

Phasor is a complex number that gives the amplitude of complex exponential function and the phase of the complex exponential function with the time-variation of the function understood as $e^{j\omega t}$. It can be used as a representation for a sinusoidal function.

Thus *phasor* is just a new name for what we have understood till now as *complex amplitude*.

The process of starting with a sinusoidal function and ending up with its *phasor representation* is called *Phasor Transformation*. We summarize the steps involved in this transformation.

- Express the given sinusoidal function in the form $x(t) = X_m \cos(\omega t + \theta)$.
- Write $x(t)$ as the *real part* of $X_m e^{j(\omega t + \theta)}$.
- Suppress the qualifier *real part*.
- Suppress $e^{j\omega t}$ after noting the value of ω for later use.
- The resulting complex number $\mathbf{X} = X_m e^{j\theta}$ is the *phasor representation* for $x(t)$. The bold face italic notation stands for the magnitude and angle together. The symbol $\bar{X}$ may be used in hand-written text.

Once we get the answer for a circuit analysis problem in *phasor representation*, we need to go back to time-domain to get the time-domain output that we wanted really. We start in the time-domain, we want to end up in the time-domain and the netherworld of *phasors* is only a temporary sojourn. The steps involved in *inverse phasor transformation* are listed below.

- Obtain the magnitude X_m and angle θ of the *phasor* and put it in $X_m e^{j\theta}$ form.
- Multiply $X_m e^{j\theta}$ by $e^{j\omega t}$ and express it in $X_m e^{j(\omega t + \theta)}$ form.
- Get the *real part* as $X_m \cos(\omega t + \theta)$ by using Euler's Identity.

We are free to express the time-function as $x(t) = X_m \sin(\omega t + \theta)$. The *phasor* representation will remain the same. But *imaginary part* will be implied everywhere.

The steps in forward and inverse *phasor* transformation are easy and can be done by inspection with a little practice. For instance, if $X = 1 + j1$, then the magnitude is $\sqrt{2}$ and angle is 45° and the time-domain waveform is $\sqrt{2}\cos(\omega t + 45^\circ)$ if we started with a cosine and it is $\sqrt{2}\sin(\omega t + 45^\circ)$ if we started with a sine. Of course we need to know the value of ω in addition to X . The phasor representation of a sinusoidal waveform *will not* contain the frequency information. Frequency has to be known separately.

7.5 TRANSFORMING A CIRCUIT INTO PHASOR EQUIVALENT CIRCUIT

We have already seen that we can write the KVL and KCL equations directly in terms of *complex amplitudes* (i.e., phasors) and that there are well-defined relations between *complex voltage amplitude* (i.e., voltage phasor) and *complex current amplitude* (i.e., current phasor) for all two-terminal elements.

The ratio of *voltage phasor* to *current phasor* is equal to R in the case of a resistor. It is $j\omega L$ in the case of an inductor and it is $1/j\omega C$ in the case of a capacitor.

These facts suggest that we need not write down the mesh and node equations in time-domain at all. We can write them in terms of mesh current phasors or node voltage phasors using the element relation that ties up voltage phasor of the element to current phasor of the element. The resulting equations will be algebraic equations involving phasors. Thus *phasor transformation* of all circuit variables results in a set of simultaneous algebraic equations rather than simultaneous differential equations. Eliminating variables and solving for the desired circuit variable is far easier when we deal with algebraic equations than when we deal with differential equations.

The circuit that will help us to deal with steady-state response to complex exponential input as if it is a memoryless circuit is called the *phasor equivalent circuit* of the dynamic circuit.

7.5.1 Phasor Impedance, Phasor Admittance and Phasor Equivalent Circuit

First we generalise the concept of resistance to accommodate a more general relationship between voltage and current than a simple scaling. The ratio between the voltage phasor and current phasor at a pair of terminals will, in general, be a complex number indicating that the circuit connected between the pair of terminals is capable of scaling the amplitude and imparting phase shift to one quantity with respect to the other. We define this ratio as the *driving-point phasor impedance* at the pair of terminals and represent it as $Z(j\omega)$. Its unit is Ohms. The reciprocal of the ratio – i.e., the ratio of current phasor to voltage phasor at a pair of terminals – is defined as the *driving-point phasor admittance* at the terminals and represent it as $Y(j\omega)$. Its unit is siemens or mhos. See Fig. 7.5-1.

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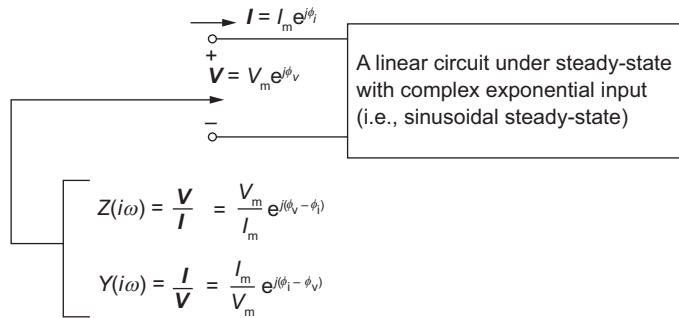


Fig. 7.5-1 Driving-point impedance and admittance under sinusoidal steady-state

Thus, if $V = V_m e^{j\phi_V}$ and $I = I_m e^{j\phi_I}$ are the voltage and current phasors as per passive sign convention

at a pair of terminals, then, the phasor impedance function $Z(j\omega) = \frac{V}{I} = \left[\frac{V_m}{I_m} \right] e^{j(\phi_V - \phi_I)}$ and the phasor

admittance function $Y(j\omega) = \frac{I}{V} = \left[\frac{I_m}{V_m} \right] e^{j(\phi_I - \phi_V)}$. Note that both impedance function and admittance

function is represented as functions of $j\omega$. They are in fact complex functions of a real variable ω and not complex functions of an imaginary variable $j\omega$ as indicated by the notation. The j in $j\omega$ serves to remind us they are complex functions.

The magnitude of the complex Z gives the ratio of amplitudes of voltage phasor and current phasor. The angle of Z gives the angle by which the current lags the voltage phasor.

The real part of Z is the *resistance* part of Z and the imaginary part of Z is defined as its *reactance* part. Similarly, the real part of Y is the *conductance* part of Y and the imaginary part of Y is defined as its *susceptance* part. Thus $Z = R + jX$ and $Y = G + jB$ where X is the reactance and B is the susceptance. Reactance has ohms as its unit and susceptance has siemens as its unit.

We have already derived the relation between voltage and current phasors for R , L and C earlier. Thus we conclude the following with the help of Eqns. 7.4-1 to 7.4-3.

$Z(j\omega) = R$ and $Y(j\omega) = 1/R$ for a resistor of $R \Omega$.

$Z(j\omega) = j\omega L$ and $Y(j\omega) = 1/j\omega L$ for an inductor of L henries.

$Z(j\omega) = 1/j\omega C$ and $Y(j\omega) = j\omega C$ for a capacitor of C farads.

The *phasor equivalent circuit* is formed by carrying out the following steps:

1. Convert all sinusoidal sources at a single frequency ω into their phasor representations and mark them near the source symbols. There is no change in the graphic symbols used. Cosine function is assumed in time-domain by default.
2. Replace all passive elements by their phasor impedance/admittance and linear dependent sources by their phasor relations. The graphic symbols used for all elements will be the same in the original circuit and in its phasor equivalent circuit.

The procedure is illustrated for the circuit in Fig. 7.5-2.

The first source function $200 \sin 314t$ is expressed as $200 \cos(314t - 90^\circ)$. Then the first source voltage phasor is $200e^{j90^\circ}$ in exponential form, $200 \angle -90^\circ$ in polar form and $0 - j200$ in rectangular form.

The second source function $250 \sin(314t - 45^\circ)$ is expressed as $250 \cos(314t - 135^\circ)$. Then the second source voltage phasor is $250e^{-j135^\circ}$ in exponential form, $200 \angle -135^\circ$ in polar form and $-176.77 - j176.77$ in rectangular form.

The value of $\omega = 314$ rad/s. Therefore the 4mH inductor will have an impedance of $j1.256 \Omega$, the 5mH inductor will have $j1.57 \Omega$ and the 10mH inductor will have $j3.14 \Omega$. The impedance of $100 \mu\text{F}$ capacitor will be $1/j0.0314 = -j31.85 \Omega$.

We see that impedances of inductor and capacitor are purely reactive. The reactance of an inductor is a positive quantity, whereas the reactance of a capacitor is a negative quantity. Similarly, the susceptance of an inductor is a negative quantity, whereas the susceptance of a capacitor is a positive quantity.

The phasor equivalent circuit of the circuit in Fig. 7.5-2 is now completed as in Fig. 7.5-3.

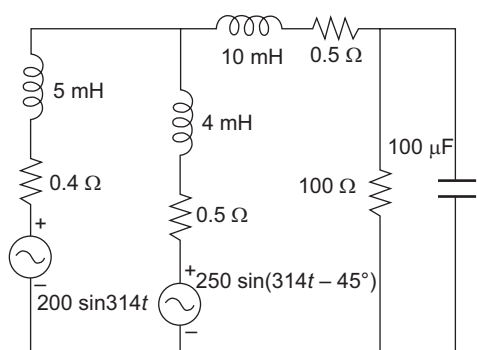


Fig. 7.5-2 Circuit for illustrating phasor equivalent circuit

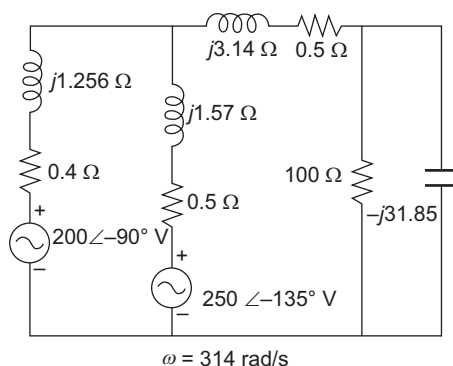


Fig. 7.5-3 Phasor equivalent circuit of circuit in Fig. 7.5-2

7.6 SINUSOIDAL STEADY-STATE RESPONSE FROM PHASOR EQUIVALENT CIRCUIT

We now know how to convert the sinusoidal source functions to their phasor representations and how to construct the phasor equivalent circuit employing phasor impedances and phasor admittances for steady-state analysis of dynamic circuits for complex exponential inputs. Steady-state analysis for complex exponential input and steady-state analysis for sinusoidal input are effectively the same. Only phasor transformation comes in between them.

Systematic application of KVL and KCL, along with element relationships that tie up the phasor voltage of an element to the phasor current through that element, will lead to sinusoidal steady-state solution in circuits in principle. However, systematic and routine analysis procedures will be quite welcome in this case too, as they were in the case of time-domain analysis of memoryless circuits. This prompts us to compare the memoryless circuit time-domain analysis problem with the sinusoidal steady-state analysis problem in dynamic circuits. The aim of such a comparison is to determine

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whether we can employ the analysis methods we developed in the context of memoryless circuits to the sinusoidal steady-state analysis problem. Moreover, we would like to verify whether the circuit theorems developed in the context of memoryless circuits will hold in the case of dynamic circuits in sinusoidal steady-state.

7.6.1 Comparison between Memoryless Circuits and Phasor Equivalent Circuits

1. The sources and circuit variables in a memoryless circuit are, in general, functions of time. DC resistive circuits form a sub-class of memoryless circuits in which all sources are constant-valued. The source functions and circuit variables in a phasor equivalent circuit are *phasors* which are *complex amplitudes* of complex exponential functions of a common format $e^{j\omega t}$, where ω is the angular frequency of all sinusoidal sources active in the circuit. *A phasor equivalent circuit can be drawn only if all sources are of the same frequency in that circuit.* If a dynamic circuit contains sinusoidal sources with different frequencies, phasor equivalent circuits for different frequencies have to be prepared separately and superposition principle has to be used to combine the solution from various phasor equivalent circuits. Thus a phasor equivalent circuit prepared for a particular value of ω is similar to a memoryless circuit driven by constant sources. The difference is that, in phasor equivalent circuit the constant sources are complex-valued whereas in memoryless circuit the constant sources are real-valued. Similarly, all circuit variables in a phasor equivalent circuit are complex-valued constant quantities in time whereas they are real-valued constant quantities in time in the case of a DC memoryless circuit. Indeed, all the circuit variables in a dynamic circuit under sinusoidal steady-state are varying in time; but this time-variation has been absorbed in the term $e^{j\omega t}$ which is a common factor in all circuit variables and which is suppressed in the phasor equivalent circuit.
2. The only passive element permitted in the memoryless circuit is the resistor. The voltage across a resistor is proportional to current through it with proportionality constant that is real-valued. Dependent sources with source functions of the form $y = kx$, where x is the controlling variable, y is the controlled variable and k is a real-valued proportionality constant, are allowed in a memoryless circuit. Thus, we see that, all the elements permitted to be present in such a circuit (except independent sources) can only *scale* (i.e., result in a multiplication by a constant) circuit variables. The resulting circuit equations will be *simultaneous algebraic equations* tying up all the instantaneous voltage and current variables (real-valued constant quantities in the case of DC excitation) in the circuit.
3. All kinds of linear passive elements, including dependent sources of the type $y = k_1 x + k_2 x' + k_3 x'' + \dots$, where primed variable indicates derivatives, are permitted in a dynamic circuit. However, once a phasor equivalent circuit using phasor impedances or phasor admittances for the elements is constructed, only *scaling* of phasors through impedances and admittances can take place in the circuit. The resulting circuit equations will be *simultaneous algebraic equations* tying up the various phasor voltage and phasor current values (complex-valued constant quantities for a particular ω) in the phasor equivalent circuit.
4. Therefore, a memoryless circuit driven by DC sources and a dynamic circuit driven by sinusoidal sources of same frequency under sinusoidal steady-state will have *similarly structured equations* of analysis. The only difference is that the variables in a memoryless circuit are real-valued, whereas variables in a dynamic circuit under sinusoidal steady-state are complex-valued.

5. Two circuit analysis techniques called *nodal analysis* and *mesh analysis* were developed for memoryless circuits in Chapter 4. We did not depend explicitly or implicitly in any way on the ‘real-valued’ nature of circuit variables to arrive at these analysis procedures in that chapter. Hence those analysis techniques should remain valid even when the circuit variables and element impedances turn out to be complex numbers.
6. Similarly, all the circuit theorems which depend on properties of Nodal Conductance Matrix and Mesh Resistance Matrix and linearity for their proof should remain applicable to phasor equivalent circuits too, provided, the criterion of ‘sinusoidal steady-state’ does not conflict with the assumptions underlying the theorems in any manner.
7. A straightforward application of KVL and KCL will show that the series equivalent impedance of n phasor impedances connected in series will be given by $\mathbf{Z}_{\text{eq}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n$ and that the parallel equivalent admittance of n phasor admittances connected in parallel will be equal to $\mathbf{Y}_{\text{eq}} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n$

The nodal analysis and mesh analysis techniques developed for memoryless circuits apply to phasor equivalent circuits with no change except that *impedance* $\mathbf{Z}$ takes the place of *resistance* R and *admittance* $\mathbf{Y}$ takes the place of *conductance* G . Nodal Conductance Matrix will get called Nodal Admittance Matrix $\mathbf{Y}_m$ and Mesh Resistance Matrix will get called Mesh Impedance Matrix $\mathbf{Z}_m$ in the sinusoidal steady-state analysis using phasor equivalent circuits. They will be symmetric complex matrices if the phasor equivalent circuit contains no dependent sources.

7.6.2 Nodal Analysis and Mesh Analysis of Phasor Equivalent Circuits – Examples

The nodal analysis and mesh analysis techniques for obtaining sinusoidal steady-state response quantities using phasor equivalent circuit is illustrated through some examples in this sub-section.

Example: 7.6-1

Find the steady-state current and average power dissipated in the resistor in an R – L series circuit with $R = 100 \, \Omega$ and $L = 1 \, \text{H}$ when driven by a switched sinusoidal source $v_s(t) = 325 \sin 100\pi t \, u(t) \, \text{V}$.

Solution

That $v_s(t) = 325 \sin 100\pi t \, u(t) \, \text{V}$ makes it clear that the sinusoidal source was switched on to the circuit only at $t = 0$. Hence, the steady-state situation will come up in the circuit only after some time and we should not expect the solution that we work out based on phasor equivalent circuit to hold during the initial period after switching on the source.

The angular frequency of the source is $\omega = 100\pi \, \text{rad/sec}$. The value of reactance of the 1H inductor at this angular frequency $= 100\pi \times 1 = 314.15 \, \Omega$ and hence the impedance of this inductor is $j314.15 \, \Omega$. Note that reactance is a *real number*, whereas impedance is a *complex number*.

We need to represent the source function in cosine form first. $v_s(t) = 325 \sin 100\pi t = 325 \cos(100\pi t - 90^\circ)$. Therefore, the phasor representation of the source is $\mathbf{V}_s = 325 \angle -90^\circ \, \text{V}$. The circuit in time-domain and circuit in phasor domain are shown in Fig. 7.6-1.

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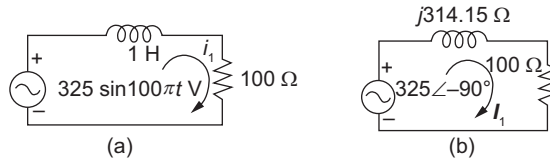


Fig. 7.6-1 (a) The circuit in time-domain and (b) The phasor equivalent circuit for Example 7.6-1

This is a single mesh circuit and the mesh current I_1 is identified in the phasor equivalent circuit in Fig. 7.6-1 (b). The mesh equation is obtained as

$$(100 + j314.15)I_1 = 325 \angle -90^\circ$$

Solving for I_1 , we get,

$$I_1 = \frac{325 \angle -90^\circ}{(100 + j314.15)} = \frac{325 \angle -90^\circ}{329.7 \angle 72.34^\circ} = 0.986 \angle -162.34^\circ \text{ A}$$

Going back to time-domain by inverse phasor transformation, we get,

$$\begin{aligned} i_1(t) &= 0.986 \cos(100\pi t - 162.34^\circ) \text{ A} \\ &= 0.986 \cos(100\pi t - 90^\circ - 72.34^\circ) \text{ A} \\ &= 0.986 \sin(100\pi t - 72.34^\circ) \text{ A} \end{aligned}$$

The source voltage and circuit current waveforms are shown in Fig. 7.6-2 (a) and (b).

The current waveform as drawn in (a) is wrong. Remember that we have obtained the sinusoidal steady-state solution only and not the complete circuit solution for all $t > 0$. Sinusoidal steady-state gets established only in the long run. The time taken for that will depend on circuit parameters. We will learn how to estimate the time required for a given circuit to reach sinusoidal steady-state in later chapters. We may accept the fact that it takes about $5L/R$ seconds (*i.e.*, about 50 ms in this circuit) for an R – L circuit to reach steady-state. Therefore, strictly speaking, the sinusoidal steady-state waveforms should be marked in time-axis as shown in Fig. 7.6-2 (b). It is understood that the t used in the axis marking in (b) can have any value greater than 50 ms or so.

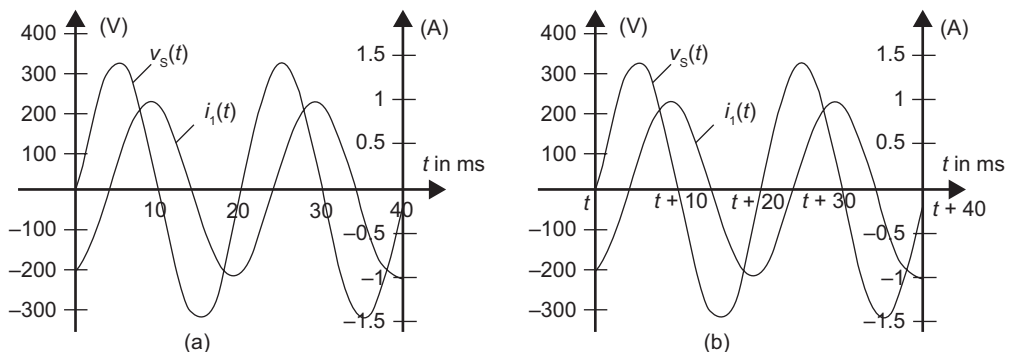


Fig. 7.6-2 Source voltage and circuit current waveforms in Example 7.6-1 with (a) Misleading time-axis marking (b) Correct time-axis marking

The waveform as shown in (a) is wrong from another point of view too. We remember that the voltage applied to the circuit was zero prior to $t = 0$. According to (a), the current suddenly changed from zero to a -ve value at $t = 0$. It is true that this value of current will exist in the circuit whenever voltage goes through a positive-going zero-crossing once the circuit has reached steady-state. But the current cannot do that at the first zero-crossing of voltage itself since it will be the violation of law of causality then. How did the circuit know while it was at $t = 0$ that the zero voltage that it is being subjected to at that instant is somehow different from the zero voltage that it was subjected to at the prior instants? Could it have anticipated that the voltage is going to rise and could it have raised its current instantaneously as per its anticipation about what the voltage waveform is going to do in future after $t = 0$ while it was at $t = 0$? No physical system can do that sort of a thing. *All physical systems are non-anticipatory*. The last sentence is yet another form of law of causality. Hence the current waveform as shown in (a) violates law of causality.

We note from this example that (i) the impedance of an R - L circuit has positive angle which is $\tan^{-1}(\omega L/R)$ in general (ii) the current in an R - L circuit lags the voltage waveform under steady-state conditions by $\tan^{-1}(\omega L/R)$ in general.

$$\text{Average power delivered to resistor} = (I_{\text{rms}})^2 R = (0.986/\sqrt{2})^2 \times 100 = 48.6 \text{ W}$$

Average power delivered to the resistor can also be calculated by calculating the power delivered by the voltage source minus the average power delivered to the inductor. The first quantity is given by $0.5 V_m I_m \cos \theta$ where θ is the phase angle by which the voltage phasor *leads* the current phasor. The angle in this case is $+72.34^\circ$. Therefore average power delivered by the source is $0.5 \times 325 \times 0.986 \times \cos(72.34^\circ) = 48.6 \text{ W}$.

$$\text{The voltage phasor across the inductor} = j314.15 \times 0.986 \angle -162.34^\circ = 309.75 \angle -72.34^\circ \text{ V.}$$

$$\therefore \text{Voltage across inductor} = 309.75 \cos(100\pi t - 72.34^\circ) = 310.34 \sin(100\pi t + 17.66^\circ) \text{ V.}$$

$$\therefore \text{The phase angle between inductor voltage and current} = +17.66^\circ - (-72.34^\circ) = +90^\circ$$

This is the expected value since the voltage across an inductor is expected to *lead* ahead of its current under sinusoidal steady-state. Since cosine of 90° is zero, the average power delivered to the inductor under sinusoidal steady-state condition is zero. Therefore, the average power delivered to the resistor is the same as the average power delivered by the voltage source and is equal to 48.6 W.

Example: 7.6-2

Find the steady-state current and average power dissipated in the resistor in an R - C series circuit with $R = 100 \Omega$ and $C = 1 \mu\text{F}$ when driven by a switched sinusoidal source $v_s(t) = 325 \sin 100\pi t u(t) \text{ V}$.

Solution

The impedance of the capacitor at an angular frequency of $\omega \text{ rad/s}$ is a purely reactive one of the form $0 + jX_C$ and is equal to $1/j\omega C$. Its reactance value X_C is then $-1/\omega C$. The time-domain circuit and phasor equivalent circuit are shown in Fig. 7.6-3.

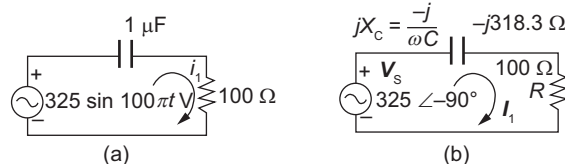


Fig. 7.6-3 (a) The RC circuit in time-domain and (b) Its phasor equivalent circuit in Example 7.6-2

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This is a single mesh circuit and may be solved by using its mesh equation. But we do not need even the mesh equation. It is a series connection of two impedances across a source voltage. The equivalent impedance is $R + 1/j\omega C = R - j/\omega C$. Hence, the impedance of an RC series circuit has a negative angle.

The current phasor in the circuit is given by voltage phasor divided by phasor impedance. Hence

$$I_1 = \frac{V_s}{Z} = \frac{V_s}{R + \frac{1}{j\omega C}} = \frac{j\omega C V_s}{1 + j\omega RC} = \frac{V_s \omega C}{\sqrt{1 + (\omega RC)^2}} \angle (90^\circ - \tan^{-1} \omega RC) V_s$$

Thus, a Series RC Circuit adds a positive phase angle to the voltage phasor in transforming it into current phasor. The current in an RC circuit *leads* the voltage waveform by $90^\circ - \tan^{-1}(\omega RC)$.

We calculate the impedance in the present instance by substituting the relevant numbers.

$$Z = 100 - j318.3 = 333.65 \angle -72.54^\circ \Omega.$$

$$\therefore I_1 = 325 \angle -90^\circ + 333.65 \angle -72.54^\circ = 0.975 \angle -17.46^\circ$$

$$\begin{aligned} \therefore i_1(t) &= 0.975 \cos(100\pi t - 17.46^\circ) = 0.975 \cos(100\pi t - 90^\circ + 72.54^\circ) \\ &= 0.975 \sin(100\pi t + 72.54^\circ) \text{ A.} \end{aligned}$$

The circuit current *leads* the applied voltage by 72.54° .

Average power delivered to resistor $= (I_{\text{rms}})^2 R = (0.975/\sqrt{2})^2 \times 100 = 47.53 \text{ W}$.

Average power delivered to the resistor can also be calculated by calculating the power delivered by the voltage source minus the average power delivered to the capacitor. The first quantity is given by $0.5 V_m I_{\text{rms}} \cos \theta$, where θ is the phase angle by which the voltage phasor *leads* the current phasor. The angle in this case is -72.54° . Therefore average power delivered by the source is $0.5 \times 325 \times 0.975 \times \cos(-72.54^\circ) = 47.53 \text{ W}$.

The voltage phasor across the capacitor $= -j318.3 \times 0.975 \angle -17.46^\circ = 310.34 \angle -107.46^\circ$

$\therefore$ Voltage across capacitor $= 310.34 \cos(100\pi t - 107.46^\circ) = 310.34 \sin(100\pi t - 17.46^\circ) \text{ V}$.

$\therefore$ The phase angle between capacitor voltage and current through it $= -17.46^\circ - 72.54^\circ = -90^\circ$

This is the expected value since the voltage across a capacitor is expected to *lag* behind its current under sinusoidal steady-state. Since cosine of -90° is zero, the average power delivered to the capacitor under sinusoidal steady-state condition is zero. Therefore, the average power delivered to the resistor is the same as the average power delivered by the voltage source and is equal to 47.53 W .

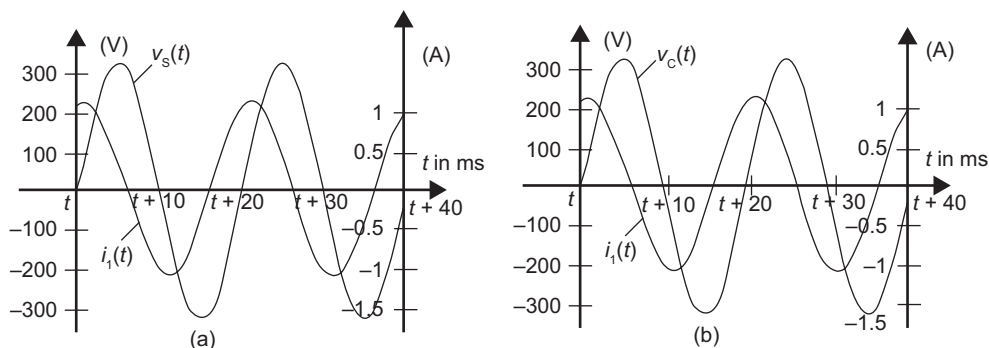


Fig. 7.6-4 (a) Applied voltage and current (b) Capacitor voltage and current in Example 7.6-2

Figure 7.6-4 shows the applied voltage waveform *lagging* behind the circuit current in (a) and the capacitor voltage *lagging* behind the current by 90° in (b). Of course, a physical system can only delay the response with respect to input. Hence the phase lead that the current in a capacitive circuit exhibits under sinusoidal steady-state condition should not be understood as a time-advance. The apparent time-advance comes up only after the response undergoes a delay during the transient period.

Example: 7.6-3

Find the (i) source current, (ii) source power, (iii) output voltage and (iv) power delivered to 25Ω resistor in the circuit in Fig. 7.6-5.

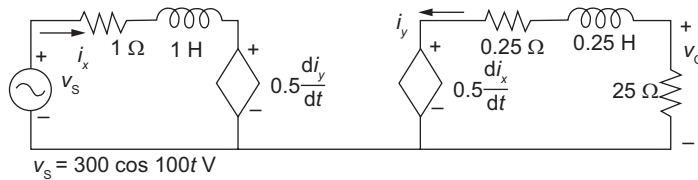


Fig. 7.6-5 Circuit for Example 7.6-3

Solution

The source voltage phasor is $300\angle 0^\circ$ V.

The dependent sources involve first derivative of controlling currents. But differentiation in time-domain is to be replaced by multiplication by $j\omega$ in the phasor equivalent circuit. The value of ω in this example is 100 rad/s. The phasor equivalent circuit of the circuit is given in Fig. 7.6-6. The mesh equations are written as follows:

$$\begin{aligned} -300\angle 0^\circ + I_1 + j100I_1 + j50(-I_2) &= 0 \\ -j50I_1 + 0.25I_2 + j25I_2 + 25I_2 &= 0 \end{aligned}$$

Recasting these equations in matrix form,

$$\begin{bmatrix} 1 + j100 & -j50 \\ -j50 & 25.25 + j25 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 300\angle 0^\circ \\ 0 \end{bmatrix}$$

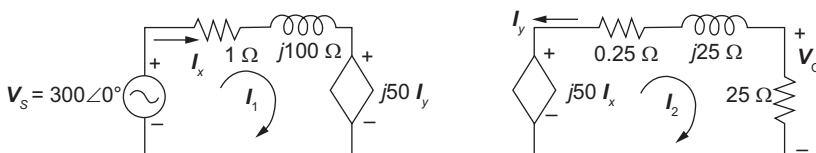


Fig. 7.6-6 The phasor equivalent circuit for the circuit in Fig. 7.6-5

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Solving for I_1 and I_2 , we get

$$I_1 = \frac{300\angle 0^\circ \times (25.25 + j25)}{(1 + j100)(25.25 + j25) - (j50)(j50)} = \frac{300\angle 0^\circ \times (25.25 + j25)}{(25.25 + j2550)} \text{ A}$$

$$= \frac{300\angle 0^\circ \times 35.53\angle 44.71^\circ}{2550.13\angle 89.43^\circ} = 4.18\angle -44.72^\circ \text{ A}$$

$$I_2 = \frac{300\angle 0^\circ \times j50}{2550.13\angle 89.43^\circ} = 5.88\angle 0.57^\circ \text{ A}$$

$$\therefore V_o = 25 \times 5.88\angle 0.57^\circ = 147\angle 0.57^\circ \text{ V}$$

$$\therefore i_1(t) = 4.18 \cos(100t - 44.72^\circ) \text{ A}$$

$$i_2(t) = 5.88 \cos(100t + 0.57^\circ) \text{ A}$$

$$v_o(t) = 147 \cos(100t + 0.57^\circ) \text{ V}$$

Source power = $0.5 \times 300 \times 4.18 \times \cos(0 - (-44.72^\circ)) = 445.5 \text{ W}$.

Average power delivered to $25\Omega = 0.5 \times 5.88 \times 5.88 \times 25 = 432.2 \text{ W}$.

[This is an equivalent circuit for a 2:1 two-winding transformer using dependent sources to model the mutual coupling between the windings. Note that the output voltage is almost equal to half of the input voltage and almost in phase with it. The presence of the 1Ω and 0.25Ω resistors make the output amplitude slightly less than 150 V and phase of output slightly different from zero. These resistors model the inevitably present winding resistances.]

Example: 7.6-4

A pair of AC voltage sources with same frequency connected through an inductance is called a *synchronous link* in Electrical Power System Engineering. The sources are generating stations and the inductance is that of the high voltage transmission line that links up the stations. When two DC sources are linked together by means of a resistance, the higher voltage source sends power to the lower voltage source. But when two AC sources at same frequency are linked together, it is not the magnitude of voltage that decides the magnitude and direction of power flow.

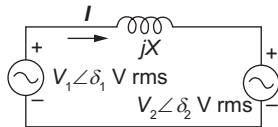


Fig. 7.6-7 A synchronous link

Show that in the synchronous link in Fig. 7.6-7, the *leading voltage source* sends average power to the *lagging voltage source* and that for small phase difference between the two sources the power exchanged is proportional to phase difference in radians.

Solution

The frequency is not given specifically, however, the reactance value of the inductor is directly specified as X and the word *synchronous* implies that both sources are at the same frequency. The source voltages are specified as rms values and this is a common practice in Electrical Power Engineering.

Electronics and Communication Engineers prefer to specify amplitude rather than rms value. In this book, if rms value is specified, it will be explicitly mentioned after the unit of the quantity.

The current phasor I from first source to second source is found in the following:

$$\begin{aligned}
 I &= \frac{V_1 \angle \delta_1 - V_2 \angle \delta_2}{jX} \\
 &= \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2) + j(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{jX} \\
 &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} - j \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \\
 &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} \angle 0^\circ - \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \angle 90^\circ \\
 \therefore i(t) &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} \cos \omega t - \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \cos(\omega t + 90^\circ) \\
 &= \frac{(V_1 \sin \delta_1 - V_2 \sin \delta_2)}{X} \cos \omega t + \frac{(V_1 \cos \delta_1 - V_2 \cos \delta_2)}{X} \sin \omega t
 \end{aligned}$$

The source voltage time-function for first source is

$$v_1(t) = \sqrt{2}V_1 \cos(\omega t + \delta_1)$$

We find the average power delivered by the first source by taking the two terms in current one by one. The first component is a $\cos \omega t$ component and the phase angle by which the voltage leads this component is δ_1 . Therefore, average power delivered through this current, P_1 , is

$$\begin{aligned}
 P_1 &= 0.5 \times 2 \frac{V_1^2 \sin \delta_1 \cos \delta_1 - V_1 V_2 \sin \delta_2 \cos \delta_1}{X} \\
 &= \frac{V_1^2 \sin \delta_1 \cos \delta_1 - V_1 V_2 \sin \delta_2 \cos \delta_1}{X}
 \end{aligned}$$

The second current component is a $\sin \omega t$ i.e., a $\cos(\omega t - 90^\circ)$ component. The phase angle by which the voltage leads this component is $90^\circ + \delta_1$. Therefore, the average power delivered through this current, P_2 , is

$$\begin{aligned}
 P_1 &= 0.5 \times 2 \frac{V_1^2 \cos \delta_1 \cos(90^\circ + \delta_1) - V_1 V_2 \cos \delta_2 \cos(90^\circ + \delta_1)}{X} \\
 &= \frac{-V_1^2 \cos \delta_1 \sin \delta_1 - V_1 V_2 \cos \delta_2 \sin \delta_1}{X}
 \end{aligned}$$

Adding P_1 and P_2 to get the total average power delivered by the first source,

$$P = \frac{V_1^2 \sin \delta_1 \cos \delta_1 - V_1 V_2 \sin \delta_2 \cos \delta_1}{X} + \frac{-V_1^2 \cos \delta_1 \sin \delta_1 + V_1 V_2 \cos \delta_2 \sin \delta_1}{X}$$

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$$\therefore P = \frac{V_1 V_2 [\sin \delta_1 \cos \delta_2 - \cos \delta_1 \sin \delta_2]}{X} = \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2)$$

$$P \approx \frac{V_1 V_2}{X} (\delta_1 - \delta_2) \text{ if } (\delta_1 - \delta_2) \text{ in radians } \ll 1$$

Therefore, in the synchronous link shown in Fig. 7.6-7, average power flows *from leading* voltage source *to lagging* voltage source quite independent of the voltage magnitude relationship between them. Moreover, for small phase difference between the two sources, the power flow is proportional to the phase difference in radians.

Example: 7.6-5

The source current in the circuit in Fig. 7.6-8 is $i_s(t) = I_m \cos \omega t$ A. Find ω and k such that the current i_y is in phase with $i_s(t)$ and has the same amplitude as that of $i_s(t)$.

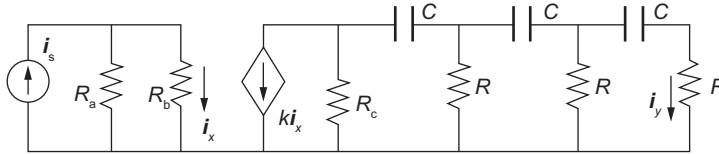


Fig. 7.6-8 Circuit for Example 7.6-5

Solution

The phasor equivalent circuit is shown in Fig. 7.6-9.

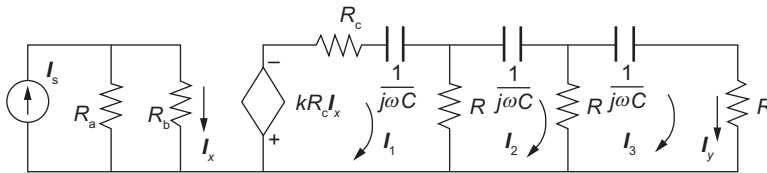


Fig. 7.6-9 The phasor equivalent circuit for circuit in Fig. 7.6-8

The dependent source current value is $= \frac{kR_a I_s}{R_a + R_b}$. This source is transformed into a dependent

voltage source of value $= \frac{kR_a R_c I_s}{R_a + R_b} = \alpha I_s$ in series with R_c . Source transformation theorem is

applicable under sinusoidal steady-state condition. Three meshes and the mesh current phasors are as shown in Fig. 7.6-9. The mesh impedance matrix can be written by inspection. Hence, the mesh equations in matrix form will be as below.

$$\begin{bmatrix} R_c + R + \frac{1}{j\omega C} & -R & 0 \\ -R & 2R + \frac{1}{j\omega C} & -R \\ 0 & -R & 2R + \frac{1}{j\omega C} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -\alpha I_s \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the mesh impedance matrix after some simplification is

$$\Delta_Z = \left[R^3 + 3R_c R^2 - \frac{R_c + 5R}{(\omega C)^2} \right] - j \left[\frac{R(6R + 4R_c)}{(\omega C)} - \frac{1}{(\omega C)^3} \right]$$

We need to solve for I_3 ,

$$I_3 = \frac{-R^2 \alpha I_s}{\Delta_Z}$$

We want the angle of I_3 to be the same as that of I_s . This is possible only if Δ_Z is a negative real number because the numerator is a real negative number. Therefore imaginary part of Δ_Z should go to zero at the ω we are looking for.

$$\text{Therefore, } \left[\frac{R(6R + 4R_c)}{(\omega C)} - \frac{1}{(\omega C)^3} \right] = 0 \Rightarrow \omega = \frac{1}{RC} \frac{1}{\sqrt{6 + 4\beta}}, \text{ where } \beta = \frac{R_c}{R}$$

We also want the magnitude of the two currents to be the same. Therefore,

$$\left[R^3 + 3R_c R^2 - \frac{R_c + 5R}{(\omega C)^2} \right] = -R^2 \alpha$$

Substituting $\omega = \frac{1}{RC} \frac{1}{\sqrt{6 + 4\beta}}$, where $\beta = \frac{R_c}{R}$ and $\alpha = \frac{kR_a R_c}{R_a + R_b}$ and simplifying, we get the

$$\text{required value of } k \text{ as, } k = \frac{(R_a + R_b)}{R_b} \left[4 \frac{R_c}{R} + 23 + 29 \frac{R}{R_c} \right]$$

[This is the small-signal equivalent circuit used to analyse the conditions for sinusoidal oscillation to take place in a *Transistor Phase Shift Oscillator* circuit that is widely used to generate low power sinusoidal signals up to about 20 kHz frequency.]

Example: 7.6-6

The circuit in Fig. 7.6-10 is the phasor equivalent circuit of a small power system running at 50 Hz containing two generating stations and one load sub-station. The transmission lines connecting the generating stations to the load station and between them are modelled by a series R - L impedance.

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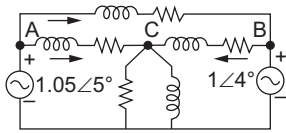


Fig. 7.6-10 Phasor equivalent circuit of a simple power system for Example 7.6-6

A modern power system will have voltage values ranging from 230 Vrms at customer premises to hundreds of kV at transmission level. Similarly, the currents at various locations may have a spread of 1:1000 or even more. Thus, the range of numbers involved in a power system analysis problem is numerically inconvenient in the Solution process.

Power System Engineers have solved this problem by evolving a special kind of *normalization* scheme for the quantities in the phasor equivalent circuit of a power system. This scheme is called *per unit* representation. We do not intend to go into the details of *per unit* system. However, we note the fact that all nominal voltage values everywhere become values close to unity in this scheme. The values marked in the circuit in Fig. 7.6-10 is as per this scheme. But for our purposes here, we may choose to view them as actual rms values themselves.

All the three lines have $0.02 + j0.1 \Omega$ impedance at 50 Hz. The load connected at C is 2Ω in parallel with $j2\Omega$ inductive reactance. Solve the given circuit for (i) all line phasor currents, (ii) phasor voltage at load station C, (iii) power delivered by sources at A and B and (iv) power delivered to the load at node C.

Solution

This is a nodal analysis problem. The fact that two nodes are constrained by voltage source does not prevent us from writing node equations at those nodes. We simply assume that the currents injected into node A and node B by the respective voltage sources are I_A and I_B and make use of these in nodal equations.

The admittance of the R - L impedance of lines is $1/(0.02 + j0.1) = 9.81\angle-78.7^\circ = 1.92 - j9.62$ S. The admittance of the parallel R - L connection at node C is $0.5 - j0.5$. We write the node equations by inspection.

$$\begin{bmatrix} 3.84 - j19.24 & -1.92 + j9.62 & -1.92 + j9.62 \\ -1.92 + j9.62 & 3.84 - j19.24 & -1.92 + j9.62 \\ -1.92 + j9.62 & -1.92 + j9.62 & 4.34 - j19.74 \end{bmatrix} \begin{bmatrix} 1.05\angle 5^\circ \\ 1\angle 4^\circ \\ V_C \end{bmatrix} = \begin{bmatrix} I_A \\ I_B \\ 0 \end{bmatrix}$$

We have used the known voltage phasors at node A and node B. There is no source connected at node C and hence there is no current injection at that node from reference node. V_C can be obtained by using the third equation,

$$(-1.92 + j9.62)(1.05\angle 5^\circ) + (-1.92 + j9.62)(1\angle 4^\circ) + (4.34 - j19.74)V_C = 0.$$

Solving for V_C , we get,

$$V_C = 0.969 + j0.0574 = 0.971\angle 3.39^\circ \text{ V rms}$$

Next we find the currents injected by the voltage sources by making use of this value of V_C in the first two equations from the matrix nodal equation.

$$I_A = (3.84 - j19.24)(1.05\angle 5^\circ) + (-1.92 + j9.62)(1\angle 4^\circ) + (-1.92 + j9.62)(0.971\angle 3.39^\circ)$$

$$I_B = (-1.92 + j9.62)(1.05\angle 5^\circ) + (3.84 - j19.24)(1\angle 4^\circ) + (-1.92 + j9.62)(0.971\angle 3.39^\circ)$$

Therefore,

$$I_A = 0.5037 - j0.1580 = 0.528\angle -17.42^\circ \text{ A rms}$$

$$I_B = 0.0095 - j0.2978 = 0.298\angle -88.2^\circ \text{ A rms}$$

$V_A = 1.05\angle 5^\circ$ and $I_A = 0.528\angle -17.42^\circ$. Therefore, the phase angle by which the voltage leads current is $5^\circ - (-17.42^\circ) = 22.42^\circ$. Therefore, average power delivered by source at node A = $1.05 \times 0.528 \times \cos 22.42^\circ = 0.513$ W.

$V_B = 1\angle 4^\circ$ and $I_B = 0.298\angle -88.2^\circ$. Therefore, the phase angle by which the voltage leads current is $4^\circ - (-88.2^\circ) = 92.2^\circ$. Therefore average power delivered by source at node A = $1 \times 0.298 \times \cos 92.2^\circ = -0.0114$ W. We used $P = V_{\text{rms}} I_{\text{rms}} \cos \theta$ for these calculations, where θ is the phase angle by which voltage phasor leads current phasor.

Power delivered to the load at node C = power delivered to 2Ω resistor + average power delivered to 2Ω reactance = $0.971 \times 0.971 / 2 + 0 = 0.471$ W. We used the expression $P = V_{\text{rms}}^2 / R$ for this calculation.

The line currents are found out by dividing the phasor difference between voltage phasors at two ends of the line by the line impedance.

Line AC – from A to C = $[1.05\angle 5^\circ - 0.971\angle 3.39^\circ] \div (0.02 + j0.1) = 0.339 - j0.204$ A rms

Line BC – from B to C = $[1\angle 4^\circ - 0.971\angle 3.39^\circ] \div (0.02 + j0.1) = 0.174 - j0.251$ A rms

Line AB – from A to B = $[1.05\angle 5^\circ - 1\angle 4^\circ] \div (0.02 + j0.1) = 0.165 + j0.047$ A rms

Verification

The line currents from A to C and from A to B must add up to I_A . The line current from B to C minus current from A to B must be I_B . These are verified within numerical rounding errors.

The reader is encouraged to verify the power flow principle in synchronous links brought out in Example: 7.6-4 with the results obtained in this example.

Example: 7.6-7

The circuit in Fig. 7.6-11 shows a 50 Hz, 230 V rms AC voltage source delivering power to a $10\Omega // j20\Omega$ inductive load through a feeder line of series impedance $0.4 + j1 \Omega$. (i) Find the current phasor delivered by the source, load voltage phasor, average power delivered to load and efficiency of power delivery if the capacitor C is disconnected. (ii) Repeat part (i) with $X_C = -20 \Omega$.

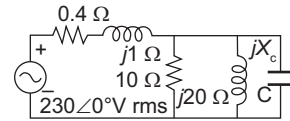


Fig. 7.6-11 Phasor equivalent circuit for Example 7.6-7

Solution

(i) We use voltage division principle to determine load voltage first. The parallel combination of 10Ω and $j20\Omega$ is in series with $0.4 + j1 \Omega$. Therefore, voltage across load, V_L is obtained as

$$\begin{aligned} V_L &= \frac{10 // j20}{0.4 + j1 + 10 // j20} \times 230\angle 0^\circ = \frac{j200}{-16 + j218} \times 230\angle 0^\circ \\ &= 0.915\angle -4.2^\circ \times 230\angle 0^\circ \\ &= 210.2\angle -4.2^\circ \text{ V rms} \end{aligned}$$

Now the current delivered by the source is obtained as

$$I = \frac{230\angle 0^\circ - 210.4\angle -4.2^\circ}{0.4 + j1} = 20.2 - j12.0 = 23.5\angle -30.8^\circ \text{ A rms.}$$

7.30 | The Sinusoidal Steady-State Response

Power delivered to load can be found in two ways. First method is to apply $V_{\text{rms}} I_{\text{rms}} \cos \theta$ to the load voltage and load current. Load voltage is $210.2 \angle -4.2^\circ$ and load current is $23.5 \angle -30.8^\circ$. Therefore, $\theta = -4.2^\circ - (-30.8^\circ) = 26.6^\circ$ and $\cos \theta = 0.8942$. Therefore, average power delivered to load $= 210.2 \times 23.5 \times 0.8942 = 4418 \text{ W}$.

The second method to find load power is to find the power delivered to the 10Ω resistor by applying $P = V_{\text{rms}}^2 / R$. The average power delivered to an inductance is zero. Therefore load power is the same as average power in resistor. Therefore, load power is $210.2^2 / 10 = 4418 \text{ W}$.

The average power delivered by source is calculated as $230 \times 23.5 \times \cos 30.8^\circ = 4643 \text{ W}$.

Therefore efficiency of power delivery $= 100 \times 4418 / 4643 = 95.15\%$

(ii) With X_C at -20Ω , the reactance of inductor and capacitor in parallel cancel each other as can be seen from $j20 // -j20 = \frac{-400}{j20 - j20} = \text{open-circuit}$. Therefore, the load circuit is effectively only a resistor of 10Ω .

$$\therefore V_L = \frac{10}{0.4 + j1 + 10} \times 230 \angle 0^\circ = 220.14 \angle -5.5^\circ \text{ V rms.}$$

Now the current delivered by the source is obtained as

$$I = \frac{230 \angle 0^\circ - 220.14 \angle -5.5^\circ}{0.4 + j1} = 21.9 - j2.1 = 22.0 \angle -5.5^\circ \text{ A rms.}$$

The average power delivered to load $= 220.14^2 / 10 = 4846 \text{ W}$

The average power delivered by source $= 230 \times 22 \times \cos(-5.5^\circ) = 5037 \text{ W}$

Efficiency of power delivery $= 100 \times 4846 / 5037 = 96.21\%$.

We observe that connecting a capacitor across an inductive load results in (i) better load voltage, (ii) lower source current, (iii) lower phase difference between source voltage and source current and (iv) lower losses in feeder line. Thus there is overall improvement in system performance when the capacitor is connected across the inductive load. We had shown in Example: 6.4-5 in Section 6.4 of Chapter 6 that when a sinusoidal voltage source is delivering power to load, a given amount of source power is transferred to load with minimum losses and maximum efficiency when the load draws current from source at zero phase difference at the source terminal. An inductive load draws a lagging current. Connecting a capacitor across such a load makes the total current less lagging and closer to zero phase condition.

This method of improving the phase between source voltage and source current is called *capacitive compensation* of an inductive load.

Example: 7.6-8

The circuit shown in Fig. 7.6-12 is the small-signal equivalent circuit of a transistor amplifier for analysis of operation at high frequency. Find the gain, i.e., ratio of output voltage phasor to input voltage phasor at 1 MHz.

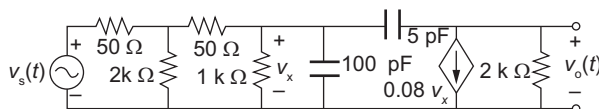


Fig. 7.6-12 Equivalent circuit of a transistor amplifier at high frequency

Solution

As a first step, we convert the circuit to the left of 100pF capacitor to current source in parallel with a resistor by applying Norton's Theorem in time-domain itself. The resulting Norton's equivalent and the complete circuit with the Norton's equivalent in place are shown in Fig. 7.6-13.

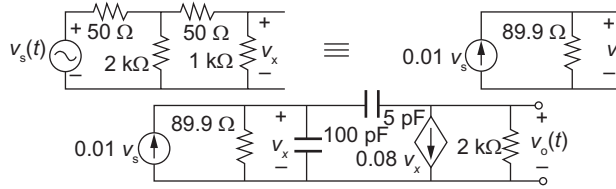


Fig. 7.6-13 Norton's equivalent of input side of the circuit in Fig. 7.6-12

Let $v_s(t) = V_m \cos \omega t$ V with $\omega = 2\pi \times 10^6$ rad/s. The admittance of 5pF and 100pF capacitors are calculated as $j3.1416 \times 10^{-5}$ S and $j6.283 \times 10^{-4}$ S, respectively, at 1 MHz. The conductance of 89.9Ω and 2kΩ are 0.01117S and 0.0005 S, respectively. The phasor equivalent circuit constructed using these values is shown in Fig. 7.6-14. Two nodes and corresponding node voltage phasors and the ground node are also marked.

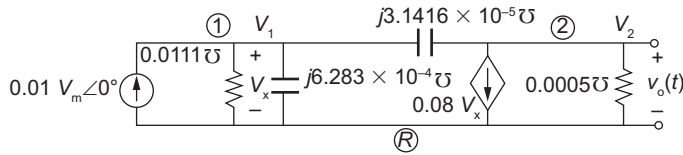


Fig. 7.6-14 The phasor equivalent circuit at 1 mhz for the circuit in Fig. 7.6-12

The node equation at first node is:

$$(0.0111 + j6.6 \times 10^{-4})V_1 - j3.1416 \times 10^{-5}V_2 = 0.01V_m \angle 0^\circ \quad (7.6-1)$$

We note that the controlling voltage phasor V_x is same as the node voltage phasor V_1 . Therefore, the node equation at second node is:

$$(0.08 - j3.1416 \times 10^{-5})V_1 + (0.0005 + j3.1416 \times 10^{-5})V_2 = 0 \quad (7.6-2)$$

We need to solve for only V_2 . Substituting for V_1 in terms of V_2 as determined from Eqn. 7.6-2 in Eqn. 7.6-1 yields the solution for V_2 .

$$\left[-(0.0111 + j6.6 \times 10^{-4}) \frac{(0.0005 + j3.1416 \times 10^{-5})}{(0.08 - j3.1416 \times 10^{-5})} - j3.1416 \times 10^{-5} \right] V_2 = 0.01V_m \angle 0^\circ$$

Solving for V_2 , we get, $V_2 = (-108.5 + j62.7) V_m$ V.

Therefore, gain at 1 MHz = $(-108.5 + j62.7) = -(108.5 - j62.7) = -125.3 \angle -30^\circ$. We express the gain this way since this is an inverting amplifier and the overall negative sign accounts for that. Thus, the output sine wave of amplifier lags by 30° at 1 MHz compared to the output when input is at low frequencies of few 100's of Hz.

7.7 CIRCUIT THEOREMS IN SINUSOIDAL STEADY-STATE ANALYSIS

Sinusoidal steady-state, as we have defined, is a concept relevant only for a linear dynamic circuit. A nonlinear circuit excited by a sinusoidal waveform may reach a steady-state; but that steady-state will not be a sinusoidal steady-state since the response of a nonlinear circuit to a sinusoidal waveform may contain distorted and non-sinusoidal content. Phasors and phasor equivalent circuits are applicable only if all response variables are sinusoidal waveforms with frequency same as that of input, *i.e.*, only if the dynamic circuit is linear. *Thus, when a phasor equivalent circuit is mentioned, linearity of the time-domain circuit is implied.*

All the circuit theorems we developed in Chapter 5, except the maximum power transfer theorem, are applicable to phasor equivalent circuits used to solve for sinusoidal steady-state variables in a linear dynamic circuit excited by sinusoidal sources with a common frequency. Thus we may use Source Transformation Theorem, Superposition Theorem, Compensation Theorem, Thevenin's Theorem, Norton's Theorem and Reciprocity Theorem in phasor equivalent circuits. The maximum power transfer theorem needs some modification.

7.7.1 Maximum Power Transfer Theorem for Sinusoidal Steady-State Condition

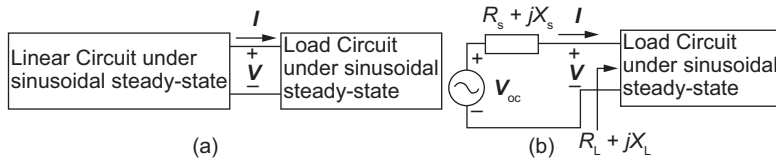


Fig. 7.7-1 (a) The power transfer context (b) The power delivery circuit replaced by its thevenin's equivalent

Figure 7.7-1 shows a linear dynamic circuit containing one or more independent sinusoidal sources with a common frequency delivering power to a load circuit *which is also under sinusoidal steady-state*. It is assumed that the constraints required for applying Thevenin's theorem are satisfied by the entire circuit – that is the circuit in Fig 7.7-1 (a) and (b) have unique solution and there is no interaction between the delivery circuit and load circuit other than through the common terminals. Then we can replace the power delivery circuit by its Thevenin's equivalent comprising an open-circuit voltage in series with the Thevenin's equivalent impedance $R_s + jX_s$. The load voltage and current phasors are identified.

We can assume with no loss of generality that the angle of V_{oc} phasor is zero, *i.e.*, $V_{oc} = V_{ocm} \angle 0^\circ$. Let $I = I_m \angle \phi$ and $Z = Z \angle \theta = R_s + jX_s$. Therefore, $R_s = Z \cos \theta$ and $X_s = Z \sin \theta$. The load voltage phasor $V = V_{oc} - ZI = V_{ocm} - ZI_m \angle (\phi + \theta)$

Then, we may write the following time-domain expressions under sinusoidal steady-state condition.

$$\begin{aligned} v_{oc}(t) &= V_{ocm} \cos \omega t \\ i(t) &= I_m \cos(\omega t + \phi) \\ v(t) &= V_{ocm} \cos \omega t - ZI_m \cos(\omega t + \phi + \theta) \end{aligned}$$

The average power delivered to the load circuit is the cycle average of $v(t) \times i(t)$.

$$v(t)i(t) = V_{ocm} I_m \cos \omega t \cos(\omega t + \phi) - ZI_m^2 \cos(\omega t + \phi) \cos(\omega t + \phi + \theta)$$

Hence the average power delivered to the load $P_L = 0.5V_{ocm}I_m \cos \phi - 0.5ZI_m^2 \cos \theta$.

We want to maximize this. But P_L is a function of I_m and ϕ since both of them will be influenced by the load circuit. Therefore, we set the partial derivatives of P_L with respect to I_m and ϕ to zero.

$$\frac{\partial P_L}{\partial I_m} = 0 \Rightarrow 0.5V_{ocm} \cos \phi - ZI_m \cos \theta = 0$$

$$\frac{\partial P_L}{\partial \phi} = 0 \Rightarrow 0.5V_{ocm} \sin \phi = 0$$

Second equation leads to $\phi = 0$ and with this value of ϕ , the first equation leads to $I_m = \frac{V_{ocm}}{2Z \cos \theta}$. But $Z \cos \theta = R_S$. Therefore, $I_m = V_{ocm}/2R_S$.

The condition $\phi = 0$ implies that the circuit has to be resistive for maximum power transfer to take place. Therefore, the effective reactance of the load at its terminals must be $-X_S$ such that it will cancel the Thevenin's reactance of the power delivery circuit and make the entire circuit resistive. Moreover, the condition that $I_m = V_{ocm}/2R_S$ implies that the effective resistance at the load terminals must be R_S such that the entire circuit becomes a resistor of $2R_S$ in series with V_{oc} . We now state the *Maximum power transfer theorem for circuits in sinusoidal steady-state*.

Maximum average power is transferred to a load circuit from a power delivery circuit under sinusoidal steady-state when the driving-point impedance $\mathbf{Z}_L = R_L + jX_L$ of the load is the *conjugate* of Thevenin's impedance $\mathbf{Z}_S = R_S + jX_S$ of the power delivery circuit. Therefore $R_L = R_S$ and $X_L = -X_S$ are the required conditions. The maximum average power transferred under this condition will be $P_{L,max} = \frac{(V_{ocm})^2}{4R_S} = \frac{(V_{rms})^2}{2R_S} \text{ W}$.

Example: 7.7-1

(i) Find the Thevenin's equivalent across **a–b** in the circuit in Fig. 7.7-2. (ii) Find the voltage across a 500Ω connected across **a–b**. (iii) Find the value of capacitive reactance to be connected across **a–b** if the magnitude of voltage across the 500Ω load is to be raised to 132 kV rms . (iv) If **a–b** gets shorted what is the current that flows through the short?

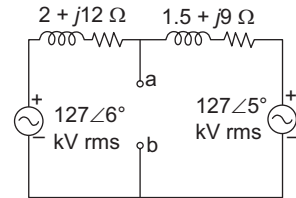


Fig. 7.7-2 Circuit for Example 7.7-1

Solution

(i) We use superposition principle to obtain the V_{oc} phasor.

$$V_{oc} = 127\angle 6^\circ \times \frac{1.5 + j9}{3.5 + j21} + 127\angle 5^\circ \times \frac{2 + j12}{3.5 + j21} = 127\angle 5.43^\circ \text{ kV rms}$$

$$Z_{th} = (2 + j12) \parallel (1.5 + j9) = \frac{(2 + j12)(1.5 + j9)}{3.5 + j21} = 0.857 + j5.143 \Omega.$$

7.34 | The Sinusoidal Steady-State Response

The Thevenin's equivalent circuit for sinusoidal steady-state is shown as the circuit in Fig. 7.7-3 (a)

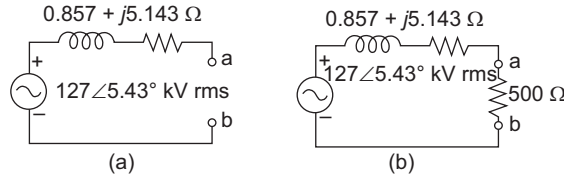


Fig. 7.7-3 (a) Thevenin's equivalent circuit for the circuit in Fig. 7.7-2. (b) With 500Ω load

(ii) The voltage phasor across the load of 500Ω

$$= 127\angle 5.43^\circ \times \frac{500}{500.857 + j5.143} = 126.77\angle 4.84^\circ \text{ kV rms}$$

(iii) We first determine a new Thevenin's equivalent of the circuit in Fig. 7.7-3 (b) for further load connection across **a–b**. This equivalent circuit will have an open-circuit voltage phasor = $126.77\angle 4.84^\circ \text{ kV rms}$ and Thevenin's equivalent impedance = $(0.857 + j5.143)/500 = 0.908 + j5.125 \Omega$. Let $-jX_C$ be the capacitive reactance connected now across the output of this new equivalent circuit. The magnitude of voltage phasor across the capacitor has to be 132 kV rms .

$$\therefore \left| \frac{-jX_C}{0.908 + j(5.125 - X_C)} \right| = \frac{132}{126.77} = 1.0413$$

$$\text{i.e., } \frac{X_C^2}{0.908^2 + (5.125 - X_C)^2} = 1.0413^2 = 1.0843$$

There are two solutions for X_C . They are 2.7Ω and 129.1Ω . The higher value is accepted. (Why?).

Therefore, the required reactance is -129.1Ω and the required *capacitive reactance* is 129.1Ω . Once we qualify a reactance by using *capacitive* we do not have to include the negative sign.

(iv) The short-circuit current at **a–b** is $126.77\angle 4.84^\circ \div (0.908 + j5.125) = 126.77\angle 4.84^\circ \div 5.205\angle -80^\circ = 6.26 \angle -75.16^\circ \text{ kA rms} = 24.36\angle -75.16^\circ \text{ kA rms}$.

Example: 7.7-2

Two terminals **a** and **b** are identified as output terminals of a linear circuit containing sinusoidal sources at a common frequency. The open-circuit voltage measured across **a–b** is seen to be 100 V rms . The voltage phasor across **a–b** goes down to 63.25 V rms when a 20Ω resistor is connected at the output and it goes down to 44.71 V rms when a 10Ω resistor is connected at the output. (i) Find the resistive load that will draw maximum average power from this circuit if a reactance can be introduced in series with the resistive load and can be varied to maximize the power. (ii) Find the resistive load that will draw maximum average power for this circuit if no reactance can be added in series with it.

Solution

We have to find the Thevenin's equivalent of the circuit with respect to **a–b** first. Let the Thevenin's equivalent impedance be $R + jX \Omega$.

Then the magnitude of voltage phasor across **a–b** when a resistor R_L is connected across the terminals = $\left| \frac{R_L}{R + R_L + jX} \right| |V_{oc}| = \frac{R_L}{\sqrt{(R + R_L)^2 + X^2}} |V_{oc}|$

$|V_{oc}|$ is given as 100 V rms. Also known are the rms output voltage values when $R_L = 10 \Omega$ and $R_L = 20 \Omega$. Substituting the numerical values, we get the following two equations in two unknowns, R and X .

$$\frac{20}{\sqrt{(20 + R)^2 + X^2}} = \frac{63.25}{100} = 0.6325 \text{ and } \frac{10}{\sqrt{(10 + R)^2 + X^2}} = \frac{44.71}{100} = 0.4471$$

Squaring both sides of equations and carrying out the required algebraic manipulation, we get, $(20 + R)^2 + X^2 = 999.86$ and $(10 + R)^2 + X^2 = 500.25$. Subtracting the second equation from the first yields

$(20 + R)^2 - (10 + R)^2 + X^2 = 499.61$, i.e., $(30 + 2R) \times 10 = 499.61 \Rightarrow R = 9.98 \Omega \approx 10 \Omega$. Using the value of $R = 10 \Omega$ in $(10 + R)^2 + X^2 = 500.25$ we get $X = 10.01 \Omega \approx 10 \Omega$.

Thus the Thevenin's equivalent of the circuit is $100 \angle 0^\circ$ V rms in series with $(10 + j10) \Omega$.

- (i) If a reactance can be put in series with the load resistance and the value of reactance and resistance can be independently adjusted, then, maximum power transfer will take place under *conjugate impedance matching* condition. Therefore, R_L must be 10Ω and X_L must be $-j10 \Omega$ for maximum average power transfer to load. The maximum power transferred under this condition will be $= R(V_{oc}/2R)^2 = V_{oc}^2/4R$ W where V_{oc} is the rms open-circuit voltage. Therefore, the power transferred to $(10 - j10) \Omega$ is 250 W.

- (ii) The condition for maximum power transfer has to be derived for this case. Let R_L be the load resistance. Then the current phasor = $\frac{V_{oc}}{(R + R_L) + jX}$ and rms value (i.e., magnitude of the

phasor, assuming V_{oc} is the rms open-circuit voltage) of current is $= \frac{V_{oc}}{\sqrt{(R + R_L)^2 + X^2}}$. Power

in $R_L = R_L \times I_{rms}^2 = \frac{R_L V_{oc}^2}{(R + R_L)^2 + X^2}$. Value of R_L for maximizing this quantity is found by equating its derivative with respect to R_L to zero.

$$\frac{dP_L}{dR_L} = 0 \Rightarrow (R + R_L)^2 + X^2 - 2R_L(R + R_L) = 0 \Rightarrow R_L = \sqrt{R^2 + X^2}.$$

Thus, maximum power transfer takes place in a pure resistive load when load resistance is equal to magnitude of Thevenin's equivalent impedance of the power delivery circuit. The required load resistance in this example is $\sqrt{10^2 + 10^2} = 14.14 \Omega$ and the power transferred is =

$$\frac{100^2}{(10 + 14.14)^2 + 10^2} \times 14.14 = 207 \text{ W}.$$

7.8 PHASOR DIAGRAMS

We have understood a *phasor* as a *complex amplitude* of a complex exponential function that varies in time as per $e^{j\omega t}$ till now. We lend a little more color to *phasors* in this section. We are motivated by uniform circular motion that is a part of school Physics. We take up a time-domain signal $v_s(t) = V_m \cos \omega t$ represented by a phasor $V_s = V_m \angle 0^\circ$ and arrive at a *geometric interpretation* for the phasor.

Concept No. 1 – Consider a line of length V_m with an arrow at the end (instead of a stone at the end of a taut string) rotating at a constant angular velocity of ω rad/sec in the *counter-clockwise* direction. Let the coordinates of arrow-tip be represented as $x(t)$ and $y(t)$ in the horizontal and vertical directions in a right-handed Cartesian coordinate system as shown in (a) of Fig. 7.8-1.

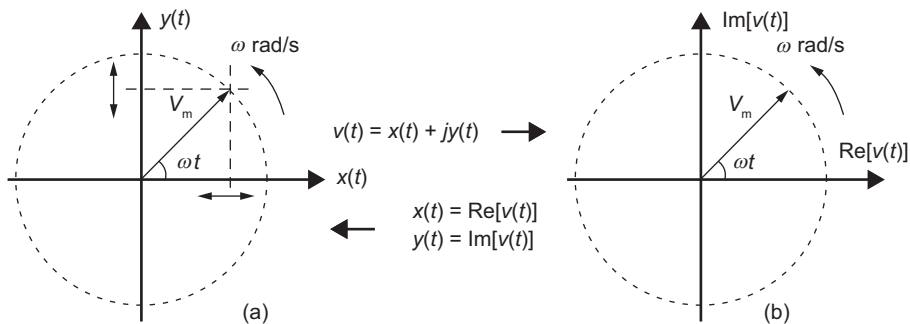


Fig. 7.8-1 (a) A rotating line of length V_m in x - y coordinate system (b) A time-varying complex number in complex plane representing a complex signal constructed using coordinates of arrow-tip in (a)

Assume that the line was collinear with x -axis at $t = 0$ and then started rotating at ω rad/sec in the direction shown. Then, the angular position of the line in space is given by ωt radians measured in counter-clockwise direction from positive x -axis. The projection of the arrow-tip on the x -axis will then be $V_m \cos \omega t$ and the projection of the arrow-tip on the y -axis will be $V_m \sin \omega t$. Therefore, the signal we started with can be given a *geometric interpretation* of *horizontal projection of arrow-tip of a line of length V_m rotating in counter-clockwise direction with a constant angular velocity of ω rad/s, starting from x -axis position at $t = 0$.*

Concept No. 2 – Projections on both axes are functions of time. We define a composite function by using these two projection functions. We define a *complex function of time* $v(t) = [x(t) + jy(t)] u(t)$ by treating the horizontal projection as the *real part* of a complex number and the vertical projection as the *imaginary part* of the same complex number. This complex number can be represented as a point in a *complex plane*. As the line progresses in its rotation, the value of complex number, constructed as explained, too will change. Therefore, the point representing this number in the complex plane also will change with time. A complex number can be geometrically represented by a line with one end at origin and with an arrow at the other end in the complex plane. When the complex number changes with time, the arrow-tip of line representing the number in complex plane will trace out a path in that plane. It must be evident in this case that when the rotating line moves in (a), the corresponding path traced out by the complex number in the complex plane in (b) will also be a circle of radius V_m and the arrow-tip will traverse this path with a constant angular velocity of ω rad/sec.

Thus, a complex signal $v(t) = V_m (\cos \omega t + j \sin \omega t) u(t) = V_m e^{j\omega t} u(t)$ constructed from coordinates of arrow-tip of a uniformly rotating line in space will be represented geometrically in the ‘complex signal plane’ by a directed line of length V_m rotating uniformly, starting from real axis, in the plane.

The values read on the axes of 'complex signal plane' at any instant t are the real and imaginary components of the complex number representing the signal value at that instant.

Concept No. 3 – Let $v_s(t) = V_m \cos(\omega t + \theta) u(t)$. Now the arrow-tip of rotating line in (a) of Fig. 7.8-1 (a) will start at $(V_m \cos \theta, V_m \sin \theta)$ i.e., at an angular position of θ at $t = 0$ and will start rotating at ω rad/sec from there. Therefore, its angular position at $t = t$ will be $(\omega t + \theta)$. The corresponding complex signal $V_m e^{j(\omega t + \theta)} u(t)$ positions in complex signal plane are marked in Fig. 7.8-2 for $t = 0$ and $t = t$.

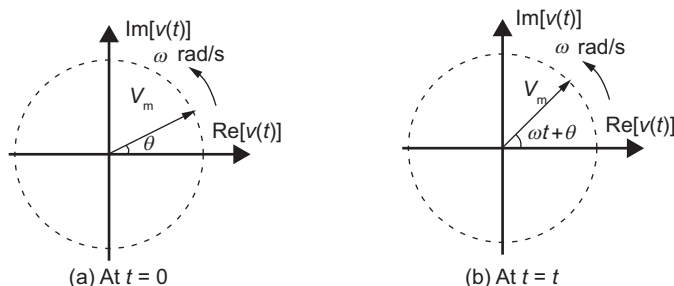


Fig. 7.8-2 Signal positions for $V_m e^{j(\omega t + \theta)}$ (a) At $t = 0$ (b) At $t = t$

Concept No. 4 – Consider two signals $v_s(t) = V_m \cos(\omega t + \theta_v)$ and $i_s(t) = I_m \cos(\omega t + \theta_i)$ with same angular frequency. Their representations in complex signal plane at $t = 0$ and at $t = t$ are shown in Fig. 7.8-3 (a) and (b).

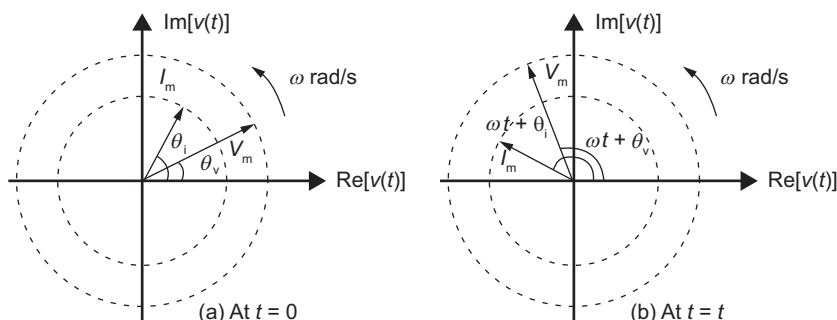


Fig. 7.8-3 Signal positions of two complex exponential functions at (a) $t = 0$ (b) $t = t$

The directed lines of different lengths do change their angular positions with time; but they maintain a constant angular difference at all t . This constant angular difference is the value of angular difference they had at $t = 0$.

Therefore, a set of complex exponential signals, all with same angular frequency but with different initial angular positions, will maintain their relative positions with respect to each other as they rotate in counter-clockwise direction with a constant angular velocity of ω rad/sec. Such a set of signals with same angular frequency form a coherent group and always stays together with their relative positions unchanged.

Concept No. 5 – Thus rotation aspect is common to all signals at the same angular frequency. That is a piece of information that we can supply at any time and does not have to be carried always in the diagram. What we are usually interested in is the relative phase angles between members of a coherent group of complex exponential signals. Therefore, we may suppress the rotation of lines representing complex exponential signals in the complex signal plane – i.e., we may freeze the lines at their position

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at $t = 0$. This ‘freezing’ the signal lines at their initial position converts complex time-functions into complex numbers – i.e., *constant-valued* signals in *complex signal plane*. These *constant-valued* signals in *complex signal plane* are our *phasors*.

Thus going from *phasor* to time-function involves ‘unfreezing’ the directed lines in complex signal plane, allowing them to rotate in counter-clockwise direction at a constant angular velocity of ω rad/sec and extracting the horizontal projection of line end-points, i.e., extracting their real parts.

Concept No. 6 – A diagram depicting a group of coherent (i.e., of same angular frequency) complex exponential signals frozen at their initial position is called a phasor diagram. Angles measured in counter-clockwise direction in a phasor diagram are lead angles and angles measured in clockwise direction in a phasor diagram are lag angles

Phasor diagram shows the magnitude of phasors as the length of directed arrows to some scale and angle of phasors as angles measured from a reference phasor in counter-clockwise direction. The reference phasor is aligned along the horizontal direction. Since only the relative positions of various phasors in a circuit really matter, any one phasor may be taken as reference phasor and the directions of all other phasors may be marked with respect to this reference phasor provided the absolute phase of reference phasor preserved for later use. This amounts to stating that a group of directed lines originating from origin may be rotated as a whole to a new position without affecting the relative positions among the members of the group.

Concept No. 7 – Multiplying a phasor by j or e^{j90° or $1\angle 90^\circ$ amounts to rotating it by 90° in the counter-clockwise direction in the phasor diagram. This amounts to converting a $\cos\omega t$ to $\cos(\omega t + 90^\circ) = -\sin\omega t$ in time-domain and hence is equivalent to introducing a phase lead of 90° . Multiplying a phasor by $-j$ or e^{-j90° or $1\angle -90^\circ$ amounts to rotating it by 90° in the clockwise direction in the phasor diagram. This amounts to converting a $\cos\omega t$ to $\cos(\omega t - 90^\circ) = \sin\omega t$ in time-domain and hence is equivalent to introducing a phase lag of 90° .

Concept No. 8 – Phasors in a phasor diagram can be added and subtracted by employing parallelogram law of addition of complex numbers in complex plane. This law is same as the law of addition of vectors in space coordinates.

Phasors are complex amplitudes used to represent sinusoidal quantities under sinusoidal steady-state condition. There should be only one value of angular frequency in the circuit. All sinusoidal sources must be at the same frequency. Therefore, a phasor diagram can be drawn only for a circuit that is in sinusoidal steady-state under the influence of one or more sinusoidal sources at same angular frequency. If there are different frequency sinusoids present in the same circuit, different phasor diagrams – one each for each frequency – should be drawn. The phasor solution arrived at from different phasor diagrams will have to be transformed into time-domain quantities using relevant angular frequency values before combining the solutions by invoking Superposition Theorem.

Example: 7.8-1

Draw the phasor diagram showing all voltage phasors and current phasors for a series RL circuit with $R = 6\ \Omega$, $X = 8\ \Omega$ at 50 Hz and $v_s(t) = 20 \cos 100\pi t$ V.

Solution

The impedance of the circuit, $Z = (6 + j8)\Omega = 10\angle 53.13^\circ\ \Omega$. Applied voltage phasor $V_s = 20\angle 0^\circ$ V. The circuit current phasor $I = 20\angle 0^\circ \text{ V} \div 10\angle 53.13^\circ\ \Omega = 2\angle -53.13^\circ$ A.

Voltage phasor across resistor $V_R = 6\ \Omega \times 2\angle -53.13^\circ \text{ A} = 12\angle -53.13^\circ$ V. Voltage phasor across inductor $V_L = j8\ \Omega \times 2\angle -53.13^\circ \text{ A} = 16\angle 36.87^\circ$ V.

We choose the applied voltage phasor as the reference phasor and align it along horizontal direction. Different scaling for voltage phasor magnitudes and current phasor magnitudes will have to be employed when a phasor diagram shows voltage phasors and current phasors together.

The circuit and phasor diagrams are shown in Fig. 7.8-4.

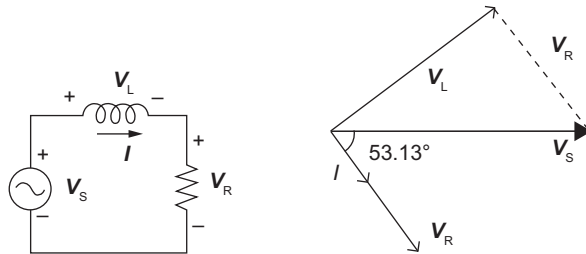


Fig. 7.8-4 An RL circuit and its phasor diagram

Note that V_L and V_R add to form V_S by parallelogram law of addition. The inductor voltage is seen to lead the circuit current by 90° and current lags the applied voltage by the impedance angle equal to 53.13° .

Example: 7.8-2

Draw the phasor diagram showing all voltage phasors and current phasors for a series RC circuit with $R = 6\ \Omega$, $X = -8\ \Omega$ at 50 Hz and $v_s(t) = 20 \cos 100\pi t$ V.

Solution

The impedance of the circuit, $Z = (6 - j8)\ \Omega = 10 \angle -53.13^\circ\ \Omega$. Applied voltage phasor $V_s = 20 \angle 0^\circ$. The circuit current phasor $I = 20 \angle 0^\circ\text{ V} \div 10 \angle -53.13^\circ\ \Omega = 2 \angle 53.13^\circ\text{ A}$. Voltage phasor across resistor $V_R = 6\ \Omega \times 2 \angle 53.13^\circ\text{ A} = 12 \angle 53.13^\circ\text{ V}$. Voltage phasor across inductor $V_L = -j8\ \Omega \times 2 \angle 53.13^\circ\text{ A} = 16 \angle -36.87^\circ\text{ V}$. We choose the applied voltage phasor as the reference phasor and align it along horizontal direction. The circuit and phasor diagrams are shown in Fig. 7.8-5.

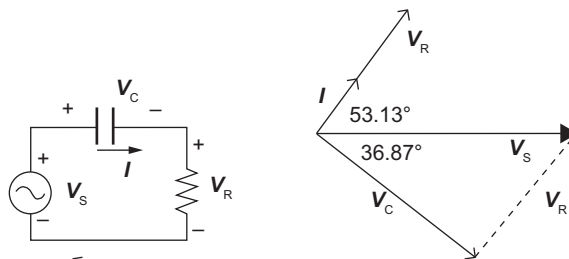


Fig. 7.8-5 The RC circuit and its phasor diagram

Example: 7.8-3

A series RLC circuit is excited by a voltage source $v_s(t) = 100 \cos \omega t$ V. The inductive reactance at ω is $10\ \Omega$ and capacitive reactance at ω is $10\ \Omega$. The resistor has $1\ \Omega$ value. Draw the phasor diagram of the circuit under sinusoidal steady-state condition.

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Solution

The impedance of the circuit at ω rad/s $Z = 10 + j10 - j10 = 10 \angle 0^\circ \Omega$

Applied voltage phasor, $V_S = 100 \angle 0^\circ$ V

$\therefore$ Circuit current phasor, $I = 100 \angle 0^\circ$ A

$\therefore$ Voltage phasor across R , $V_R = 100 \angle 0^\circ$ V

$\therefore$ Voltage phasor across L , $V_L = j10 \Omega \times 100 \angle 0^\circ$ A = $1000 \angle 90^\circ$ V

$\therefore$ Voltage phasor across C , $V_C = -j10 \Omega \times 100 \angle 0^\circ$ A = $1000 \angle -90^\circ$ V

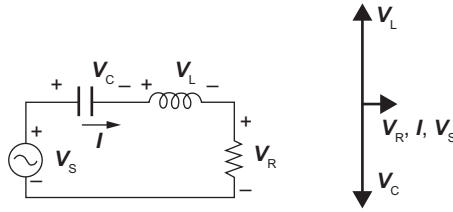


Fig. 7.8-6 A RLC circuit and its phasor diagram

Note that the voltage across capacitor and inductor are in phase opposition. Therefore, they cancel each other completely, thereby leaving the entire supply voltage to the resistor. Hence, the current under this condition is the maximum current that the circuit can have with given amplitude of applied voltage. This happens because the impedance of capacitor and inductor are equal in magnitude and opposite in sign. They cancel out, making the impedance of the circuit a minimum of R at this frequency. This is the *resonance* condition in a series RLC circuit.

Example: 7.8-4

A sinusoidal current source $i_s(t) = 5 \cos(100\pi t - 45^\circ)$ A is applied across a parallel combination of an inductor, a resistor and a capacitor. Find the steady-state currents in elements and voltage across the combination using phasor diagram. The resistance value is 6Ω and the values of reactance of inductor and capacitor at $\omega = 100\pi$ rad/sec are 8Ω and -4Ω respectively.

Solution

This example calls for use of phasor diagram to solve the circuit under sinusoidal steady-state. Hence the phasor quantities are unknown when we draw the phasor diagram. In this kind of a situation, the phasor that we choose as reference phasor has to have the property that all the other phasors in the circuit can be worked out from this phasor employing KCL, KVL and element relationship. Applied voltage or current will not be suitable for this purpose. It has to be one of the response variables. But, if it is one of the response variables, its magnitude will be unknown and hence we cannot fix the scale in phasor diagram. Thus, the phasor diagram is drawn by assigning an arbitrary, but known, length to the phasor that is chosen as the reference phasor. The scale in the diagram will emerge from the known applied voltage or current phasor once the diagram is completed according to KCL, KVL and element relationships.

We choose the current in the resistor as the reference phasor in this example. The circuit and phasor diagrams are shown in Fig. 7.8-7.

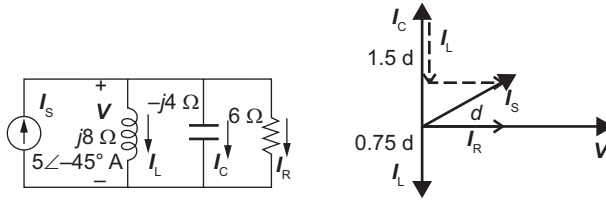


Fig. 7.8-7 Circuit and phasor diagram in Example 7.8-4

We have set I_R in the horizontal direction. This does not mean that $i_R(t)$ will be a $\cos 100\pi t$ wave. Once we solve the phasor diagram completely, the source current phasor will come out with some angle other than its actual angle of -45° . Then we will rotate the entire phasor diagram such that I_S takes up -45° position in the diagram. The other phasors will then take up suitably shifted positions. The new angular positions can be calculated from the known angular position of I_S and apparent angular position of I_S in the diagram shown in Fig. 7.8-7.

Let d be the length that we used to represent I_R . Then the current through the capacitor $I_C = RI_R / jX_C$ will have a magnitude of $6/4 = 1.5$ times that of I_R and hence needs a line of length $1.5d$ in 90° position in the diagram. The current through inductor $I_L = RI_R / jX_L$ will have a magnitude of $6/8 = 0.75$ times that of I_R and hence needs a line of length $0.75d$ in -90° position in the diagram. Moving a copy of I_L to the tip of I_C takes us to $(I_C + I_L)$ and placing a copy of I_R at the tip of $(I_C + I_L)$ takes us to the tip of I_S phasor. Thus we complete the diagram. Now, we either measure the length of I_S phasor or calculate it as $\sqrt{(0.75d)^2 + d^2} = 1.25d$ from the geometry of the figure. But this must be equal to 5 A since the amplitude of $i_S(t)$ is stated to be 5 A. Therefore, a length of d stands for $5/1.25 = 4$ A. Therefore magnitude of I_R is 4 A, of I_C is 6 A and of I_L is 3 A.

The angle of I_S phasor in the phasor diagram is $\tan^{-1}(0.75) = 36.9^\circ$. But we know that actual angle of I_S phasor is -45° . Therefore, angle of $-(45^\circ + 36.9^\circ) = -81.9^\circ$ will have to be added to all phasors in the diagram.

$\therefore I_S = 5\angle -45^\circ$ A, $I_R = 4\angle -81.9^\circ$ A, $I_C = 6\angle 8.1^\circ$ A and $I_L = 3\angle -171.9^\circ$ A and $V = 24\angle -81.9^\circ$ V is the phasor solution of the circuit.

Corresponding time-domain functions are:

$i_S(t) = 5 \cos(100\pi t - 45^\circ)$ A, $i_R(t) = 4 \cos(100\pi t - 81.9^\circ)$ A, $i_C(t) = 6 \cos(100\pi t + 8.1^\circ)$ A, $i_L(t) = 3 \cos(100\pi t - 171.9^\circ)$ A and $v(t) = 24 \cos(100\pi t - 81.9^\circ)$ V.

We could have used I_C or I_L or V as the reference phasor and developed the phasor diagram to arrive at the same solution. However, we could not have used I_S as the reference phasor since we would not have been able to proceed any further with that choice.

Example: 7.8-5

Two impedances $Z_1 = 6 + j8 \Omega$ and $Z_2 = 8 - j6 \Omega$ are in parallel and the whole combination is in series with a third impedance $Z_3 = 5 + j5 \Omega$. The circuit is driven by a sinusoidal voltage source $v_S(t) = 50 \sin 100\pi t$ V. Solve the circuit by phasor diagram method.

Solution

The circuit and phasor diagrams are shown in Fig. 7.8-8.

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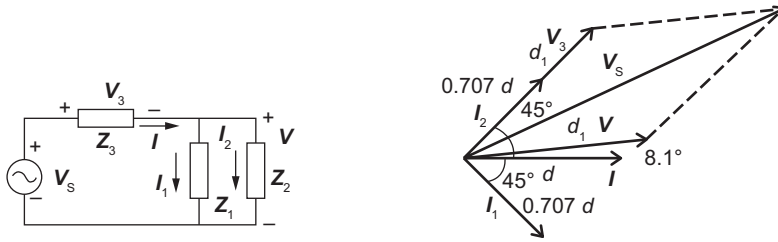


Fig. 7.8-8 Circuit and phasor diagram for Example 7.8-5

We choose the current phasor I as the reference phasor.

The impedance values are converted to polar form as $Z_1 = 10 \angle 5.31^\circ \Omega$, $Z_2 = 10 \angle -36.9^\circ \Omega$ and $Z_3 = 7.07 \angle 45^\circ \Omega$. The parallel combination, $Z_1 // Z_2$ is $7 + j1 \Omega = 7.07 \angle 8.1^\circ \Omega$.

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I = (0.5 - j0.5) I = 0.707 \angle -45^\circ I$$

$$I_2 = \frac{Z_1}{Z_1 + Z_2} I = (0.5 + j0.5) I = 0.707 \angle 45^\circ I$$

We use a length d for I . Then, the length to be used for I_1 and I_2 are $0.707d$ and they are oriented at -45° and 45° respectively. Next we draw the $V = 7.07 \angle 8.1^\circ I$ phasor at 8.1° with respect to horizontal and use a convenient length d_1 if the length $7.07d$ is not suitable. The V_3 phasor is also of the same length since $V_3 = Z_3 I = 7.07 \angle 45^\circ I$. But it is to be drawn at 45° position.

The phasors V and V_3 on addition as per parallelogram law should result in V_s . The length of V_s must be $2 \times d_1 \times \cos [(45^\circ - 8.1^\circ)/2] = 1.9 d_1$. But this length must stand for 50 V and hence d_1 must stand for 26.3 V. Therefore, magnitudes of V and V_3 are 26.3 V. Since $I = V_3 / Z_3$, magnitude of I will be 3.72 A. Now, magnitudes of I_1 and I_2 are 0.707 times the magnitude of I . Hence they are of 2.63 A magnitude.

The angle of V_s as per the phasor diagram is $8.1^\circ + (45^\circ - 8.1^\circ)/2 = 26.55^\circ$. But since $v_s(t) = 50 \sin 100\pi t$, the actual phase angle of V_s is -90° with respect standard cosine wave. Therefore, an angle of -116.55° has to be added to the angle of all phasors in the phasor diagram shown in Fig. 7.8-8.

Therefore, the sinusoidal steady-state solution of the circuit is obtained as:

$$V_s = 50 \angle -90^\circ, \text{ and}$$

$$V = 26.3 \angle -108.45^\circ \text{ V and } V_3 = 26.3 \angle -71.55^\circ \text{ V.}$$

$$I = 3.72 \angle -116.55^\circ \text{ A, } I_1 = 2.63 \angle -151.55^\circ \text{ A, } I_2 = 2.63 \angle -71.55^\circ \text{ A}$$

The time-domain functions are:

$$v_s(t) = 50 \cos(100\pi t - 90^\circ) = 50 \sin 100\pi t \text{ V}$$

$$v(t) = 26.3 \cos(100\pi t - 108.45^\circ) = 26.3 \sin(100\pi t - 18.45^\circ) \text{ V}$$

$$v_3(t) = 26.3 \cos(100\pi t - 71.55^\circ) = 26.3 \sin(100\pi t + 18.45^\circ) \text{ V}$$

$$i(t) = 3.72 \cos(100\pi t - 116.55^\circ) = 3.72 \sin(100\pi t - 26.55^\circ) \text{ A}$$

$$i_1(t) = 2.63 \cos(100\pi t - 151.55^\circ) = 2.63 \sin(100\pi t - 61.55^\circ) \text{ A}$$

$$i_2(t) = 2.63 \cos(100\pi t - 71.55^\circ) = 2.63 \sin(100\pi t + 18.45^\circ) \text{ A}$$

Example: 7.8-6

Three sinusoidal voltage sources $-v_1(t)$, $v_2(t)$ and $v_3(t)$ – with angular frequency of 100π rad/sec and amplitudes of 63 V, 52 V and 25 V, respectively, are connected in series along with a 10Ω resistor to

form a closed loop. The voltage sources are connected in such a way that they aid each other in the loop. The current in $10\ \Omega$ resistor is found to be zero. Find $v_1(t)$, $v_2(t)$ and $v_3(t)$.

Solution

The statement of the problem makes it clear that $v_1(t) + v_2(t) + v_3(t) = 0$. Therefore, the phasor diagram of the three voltage phasors will form a closed triangle. The phasor diagram is shown in Fig. 7.8-9.

The phasor diagram is drawn as follows. Choose a suitable scale and draw the line OP to represent magnitude of V_1 . With O as centre, draw a circle of radius 52 to scale. Draw another circle of radius 25 to scale with P as its centre. Let the two circles intersect at Q. They will intersect; otherwise the three voltages would not have added up to zero. Join QO and PQ. Create a copy of QO and move it to form V_2 . Similarly, create a copy of PQ and move it in parallel such that the non-arrow end comes to O to form V_3 .

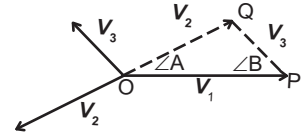


Fig. 7.8-9 Phasor diagram in Example 7.8-6

Now $\angle A$ and $\angle B$ can be measured from the diagram. Then $V_1 = 63\angle 0^\circ$, $V_2 = 52\angle -(180-A)^\circ$ and $V_3 = 25\angle (180-B)^\circ$.

The angles $\angle A$ and $\angle B$ can also be calculated by *Law of Cosines*.

$$25^2 = 63^2 + 52^2 - 2 \times 63 \times 52 \times \cos A \Rightarrow A = 22.62^\circ$$

$$52^2 = 63^2 + 25^2 - 2 \times 63 \times 25 \times \cos B \Rightarrow B = 53.13^\circ$$

$$\therefore V_1 = 63\angle 0^\circ \text{ V}, V_2 = 52\angle -157.38^\circ \text{ V and } V_3 = 25\angle 126.87^\circ \text{ V}.$$

$$\therefore v_1(t) = 63 \cos 100\pi t \text{ V}, v_2(t) = 52 \cos(100\pi t - 157.4^\circ) \text{ V and } v_3(t) = 25 \cos(100\pi t + 126.9^\circ) \text{ V}.$$

7.9 APPARENT POWER, ACTIVE POWER, REACTIVE POWER AND POWER FACTOR

Consider a sinusoidal voltage source $v(t) = V_m \cos \omega t$ delivering power to a resistive load R . The current in the resistor is $i(t) = I_m \cos \omega t$ where $I_m = V_m/R$.

The instantaneous power is $p(t) = V_m I_m \cos^2 \omega t = 0.5 V_m I_m + 0.5 V_m I_m \cos 2\omega t$ W. The first term is a constant and the second term produces an average of zero over a cycle. Therefore, the average power delivered to resistor is $0.5 V_m I_m = 0.5 V_m^2/R = 0.5 I_m^2 R$. The average power can be expressed as $V_{\text{rms}} I_{\text{rms}}$ in terms of rms values of voltage and current. Thus, a sinusoidal voltage/current is only as effective as a DC voltage /current of magnitude that is only 70.7% of the amplitude of the sinusoid. The presence of the second term – the term that has as much strength as the average power; but is oscillating at twice the supply frequency – indicates this relative inefficiency of sinusoids compared to DC quantities in carrying power to a load. This is the inevitable price that we have to pay for having opted for sinusoidal waveforms. Hence, we do not complain about the inevitable double-frequency power pulsation that has as much amplitude as the average power that is being delivered to the load.

Now, consider the same voltage source delivering power to the same resistor, but the resistor is in *parallel* with an inductor of reactance X at ω rad/s as shown in Fig. 7.9-1.

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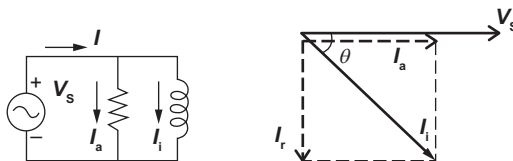


Fig. 7.9-1 A parallel RL load and its phasor diagram

Such a load is called a *reactive load*. The load impedance now is given by

$$\mathbf{Z} = R // jX = \frac{jRX}{R + jX} = \left[\frac{X^2}{R^2 + X^2} \right] R + j \left[\frac{R^2}{R^2 + X^2} \right] X = \frac{RX}{\sqrt{R^2 + X^2}} \angle \tan^{-1} \frac{R}{X}$$

The current delivered by the voltage source will be

$$i(t) = \frac{V_m}{R} \cos \omega t + \frac{V_m}{X} \sin \omega t = I_m \cos(\omega t - \theta), \text{ where } I_m = \frac{V_m \sqrt{R^2 + X^2}}{RX} \text{ and } \theta = \tan^{-1} \frac{R}{X}$$

The instantaneous power is

$$\begin{aligned} p(t) &= V_m I_m \cos \omega t \cos(\omega t - \theta) \\ &= V_m (I_m \cos \theta) \cos^2 \omega t + V_m (I_m \sin \theta) \sin \omega t \cos \omega t \\ &= \{ [0.5 V_m (I_m \cos \theta)] + [0.5 V_m (I_m \cos \theta)] \cos 2\omega t \} + [0.5 V_m (I_m \sin \theta)] \sin 2\omega t \end{aligned}$$

The average power is $0.5 V_m I_m \cos \theta = 0.5 V_m \times \frac{V_m \sqrt{R^2 + X^2}}{RX} \times \cos(\tan^{-1}(R/X)) = 0.5 V_m^2/R$.

Thus, $I_m \cos \theta = V_m/R$ is the same as the current drawn by the resistor alone.

Thus, average power is due to the current drawn by resistor and is the same as before. However, the source has to deliver a higher current to deliver the same amount of power now. The first double-frequency power pulsation (*i.e.*, the $\cos 2\omega t$ term in $p(t)$) is the expected double-frequency pulsation when an average power is being delivered. The second double-frequency pulsating power (*i.e.*, the $\sin 2\omega t$ term) is solely due to the inductor in parallel with resistor – *i.e.*, due to the *reactive* nature of load) and has an amplitude of $0.5 V_m I_m \sin \theta = 0.5 V_m^2/X$. (since $\sin \theta = \sin(\tan^{-1}(R/X)) = \frac{R}{\sqrt{R^2 + X^2}}$).

The presence of the second pulsating power term with non-zero amplitude is an indicator to the fact that the magnitude of current is more than the minimum magnitude of current required to pass on the average power to the load. The minimum current that is required in the circuit to deliver the average power it is delivering now is only $\cos \theta$ times the present current.

If the voltage in a DC circuit is same as V_{rms} of this sinusoidal voltage source and the current in the DC circuit is same as the I_{rms} in this AC circuit, then, the DC source would have delivered $V_{\text{rms}} I_{\text{rms}}$ watts of average power to the load. Compared to that, the AC circuit delivers only $\cos \theta$ times this power. Thus, effectiveness of utilisation of voltage and current in a reactive circuit under sinusoidal steady-state is compromised by the factor $\cos \theta$ compared to a DC circuit carrying similar voltage and current. This observation leads to a definition of *apparent power* in an AC circuit.

Apparent power carried by a sinusoidal voltage of rms value V_{rms} and a sinusoidal current of rms value I_{rms} is defined as the actual power that will be carried by a DC voltage of same effective value and a DC current of same effective value –i.e., Apparent Power = $V_{rms} I_{rms}$

Since the average power in an AC circuit can be different by a factor $\cos\theta$, where θ is the angle between voltage phasor and current phasor, the unit of *watts* is reserved for average power and a unit of *Volt-Ampere (VA)* is assigned to apparent power. Since only the average power contained in the apparent power is *active* in generating useful output from the circuit, average power is called *active power*. The ratio between the active power and apparent power is called the *power factor* of the circuit.

Apparent Power = $V_{rms} I_{rms}$ VA

Active Power, P = Average Power = $V_{rms} I_{rms} \cos\theta$ W, where θ is the angle by which the voltage phasor leads the current phasor:

$$\text{Power Factor} = \frac{\text{Active Power}}{\text{Apparent Power}} = \cos\theta$$

Note that the definitions of apparent power, active power and power factor are applicable for any general periodic waveform context. But the expressions, $V_{rms} I_{rms} \cos\theta$ for active power and $\cos\theta$ for power factor, are applicable only under sinusoidal steady-state condition.

7.9.1 Active and Reactive Components of Current Phasor

$I_m \cos\theta$ is the amplitude of $\cos\omega t$ term in current and $I_m \sin\theta$ is the amplitude of $\sin\omega t$ term in current. $\cos\omega t$ and $\sin\omega t$ terms are represented by phasors that have 90° between them. They are called *quadrature components* for this reason. Thus, $I_m \cos\theta$ is the *in-phase* component in current phasor and $I_m \sin\theta$ is the *quadrature* component in current phasor with respect to the voltage phasor. $I_m \cos\theta$, the in-phase component, carries the average power (along with an unavoidable double-frequency pulsating power of equal amplitude), and, $I_m \sin\theta$, the quadrature component, produces a pure double-frequency pulsating power term with zero average content. This pulsating power term is avoidable by making $\theta = 0$ – i.e., by making the load purely resistive.

Any current phasor can be resolved into two components – one in the direction of voltage phasor and one in a direction perpendicular to the voltage phasor. The component in the direction of voltage phasor is the *in-phase* component and this component will carry *active power*. Therefore, this component is called *active component of current* and is denoted by a phasor I_a . The component in the perpendicular direction to voltage phasor is the *quadrature* component and this component *will not* carry *any average power*. This component is decided by the reactance in the circuit and goes to zero when the circuit is purely resistive. Hence this component is called the *reactive component of current* and is denoted by a phasor I_r . The current phasor I is the phasor sum of I_a and I_r . This is shown in the phasor diagram in Fig. 7.9-1.

Note that the definition of *active and reactive components of a current phasor* is based on projecting the current phasor along and perpendicular to voltage phasor and is applicable to any current phasor. In the circuit we considered in Fig. 7.9-1, the active component of I could be identified as the current in

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R and the reactive component could be identified as the current in jX . However, such an identification of active and reactive current components of a given current phasor is not a precondition for their definition. Consider a series RL load and its phasor diagram in Fig. 7.9-2.

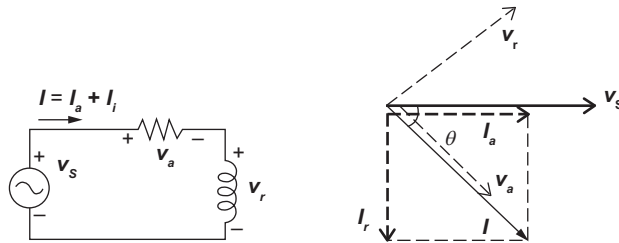


Fig. 7.9-2 A series RL load and its phasor diagram

The current phasor I can be resolved into in-phase component I_a and quadrature component I_r , with respect to voltage phasor as shown in the phasor diagram in Fig. 7.9-2. However, we cannot identify these components as real currents flowing in any element since there is only one current in a series circuit and that is I . However, we observe that the voltage phasor also can be resolved into two components – one along the current phasor direction and one in a perpendicular direction. These are the *active component* and *reactive component* of voltage phasor.

We can always think of a parallel-connected resistance and reactance that has same impedance as that of a series connected resistance and reactance at a particular frequency. Then, we can identify the active component of I in a series circuit as the current that will flow in the resistor of a parallel circuit that has same impedance as that of the series circuit. Similarly, we can identify the reactive component of I in a series circuit as the current that will flow in the reactance of a parallel circuit that has same impedance as that of the series circuit. Hence, though resolving voltage phasor along current phasor and resolving current phasor along voltage phasor have the same effect in power equations, we choose to use the quadrature components of current phasor rather than voltage phasor in subsequent discussion.

I_a is in phase (or with 180° phase) with voltage phasor by definition and hence its phase is known. Therefore, it is a common practice to refer to I_a as if it is a real number with positive or negative sign. In the case of I_r , it can be specified as a real number with positive or negative sign. Or else, one may add the qualifiers ‘capacitive’ or ‘inductive’ and then skip the sign. The j in I_r is taken to be implied by the word ‘reactive’ in ‘reactive component’. Thus, if $V_s = V_m \angle 0^\circ$ and $I = I_m \angle -\theta$ then, *active component of current* is $I_m \cos \theta$ ($I_{\text{rms}} \cos \theta$ A rms) and *reactive component of current* is $-I_m \sin \theta$ ($-I_{\text{rms}} \sin \theta$ A rms). If the negative sign is skipped, then we have to state that *inductive component of current* is $I_m \sin \theta$ ($I_{\text{rms}} \sin \theta$ A rms). For instance, a 5 A *reactive current* implies 5 A of a capacitive current, a –5 A *reactive current* implies 5 A of an inductive current, a 5 A *inductive current* implies 5 A of an inductive current and a –5 A *inductive current* implies 5 A of capacitive current. Similarly a 5 A *capacitive current* implies 5 A of a capacitive current and –5 A *capacitive current* implies 5 A of an inductive current. Inductive current component *lags* voltage phasor by 90° and capacitive current component *leads* voltage phasor by 90° .

7.9.2 Reactive Power and the Power Triangle

We restate the concepts we developed in Section 7.9-1 before we continue with the concept of *reactive power* that tends to be a confusing one to beginners in Electrical Engineering.

- Let $v(t) = V_m \cos \omega t$ V and $i(t) = I_m \cos(\omega t - \theta)$ A be the steady-state voltage and current at a pair of load terminals as per passive sign convention. Then apparent power which is the power that a DC circuit with same effective values of voltage and current would have delivered is $V_{rms} I_{rms} = 0.5 V_m I_m$ VA. The average power which is also called *active power* is $P = V_{rms} I_{rms} \cos \theta$ W = $0.5 V_m I_m \cos \theta$ W. The power factor of the circuit which is defined as the ratio of active power to apparent power is $\cos \theta$.
- The minimum magnitude of current required to deliver a given amount of power P is given by $I_{rms} = P/V_{rms}$ with $\cos \theta = 1$. This happens when the driving-point impedance of the load at ω rad/sec is effectively a resistance. The circuit has $\theta = 0$ then, and, draws power at unity power factor with minimum magnitude of current.
- Reactive component in the driving-point impedance of the load circuit makes θ non-zero and increases the current magnitude for a given amount of active power. Thus current is underutilized as far as active power delivery is considered. The power factor of the circuit will be less than unity.
- The current phasor can be resolved into active component ($= I_{rms} \cos \theta$ A rms) and reactive component ($= -I_{rms} \sin \theta$ A rms) by finding its projection along voltage phasor and along a perpendicular to voltage phasor respectively. The active component may be thought of as the current drawn by the resistor in a parallel connected resistance-reactance combination that has same impedance as that of the load circuit. The reactive component of current is the current drawn by the reactance in that equivalent parallel circuit. The active current component carries the entire active power.

The utilisation of load current in its role as a vehicle to carry active power can be judged from the relative proportion of its active and reactive current components. It can be shown easily that $I_{rms} = \sqrt{I_{a,rms}^2 + I_{r,rms}^2}$ and $I_m = \sqrt{I_{am}^2 + I_{rm}^2}$ where $I_{a,rms}$ and $I_{r,rms}$ indicate the rms values of the components, whereas I_{am} and I_{rm} indicate their amplitudes. The following relations also hold between various quantities.

$$\cos \theta = \text{power factor} = \frac{I_{a,rms}}{I_{rms}} = \frac{I_{am}}{I_m}$$

$$\sin \theta = -\frac{I_{r,rms}}{I_{rms}} = -\frac{I_{rm}}{I_m}$$

$$\tan \theta = -\frac{I_{r,rms}}{I_{a,rms}} = -\frac{I_{rm}}{I_{am}}$$

$I_{r,rms}$ and I_{rm} contain the sign of reactive component. Therefore, while power factor is always positive, $\sin \theta$ and $\tan \theta$ are positive for a lagging load and negative for a leading load. *Note that θ is the angle by which voltage phasor leads the current phasor.* Power factor of a load is independent of sign of θ and hence a qualifier *lag* or *lead* is to be appended to the number representing power factor to distinguish between positive and negative values of θ . Thus, if $\theta = 45^\circ$, the power factor is ‘0.7 lag’ and if $\theta = -45^\circ$ the power factor is ‘0.7 lead’.

Another method to describe the utilisation of load current in carrying active power will be to compare the active and reactive rms components of current after scaling the active rms current component by $+V_{rms}$ and reactive rms current component by $-V_{rms}$. But then, if the *active component of current* is multiplied by $+V_{rms}$, the result is *active power*. Then, it will be tempting to call the

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product of the *reactive component of current multiplied by* $-V_{\text{rms}}$ a power – but not a real average power, since this component of current produces only $V_{\text{rms}} I_{r,\text{rms}} \sin 2\omega t$ term in instantaneous power. Electrical Engineers yielded to this temptation long back and they called it *reactive power* in contrast to *active power*. **Thus reactive power is not a power at all; it is only a power-like measure of reactive component of current.**

To state that there is some reactive power flow into a load is a disguised way of stating that (i) the load impedance has a reactive component (ii) the load current has a reactive component which reduces the efficacy of current in carrying active power (iii) therefore, the current magnitude is more than the minimum magnitude needed that is commensurate with actual power transfer taking place (iv) therefore, circuit is operating at a power factor less than unity.

This ‘fictitious power’ that is not a power at all in the normal sense of that word, is, in essence, a stand-in for the reactive component of current. It is usually denoted by Q and its unit is *Volt-Ampere-reactive*, shortened as *VAR*. Thus, $Q = V_{\text{rms}} I_{\text{rms}} \sin \theta$ VAR where θ is the phase angle by which the voltage phasor leads the current phasor. Therefore, the reactive power consumed by an inductive load is positive in sign and the reactive power consumed by a capacitive load is negative in sign by definition.

Notice that the Q value is the same as the amplitude of double-frequency power pulsation caused by reactive component of current.

Note carefully that the sign of *reactive component of current* and *reactive power* carried by that current are opposite. Thus, an inductive load *draws* a ‘negative reactive current’ and *consumes* ‘positive reactive power’. A capacitive load *draws* ‘positive reactive current’ and *consumes* ‘negative reactive power’. This is matter of convention and convenience rather than of necessity.

If a circuit element is *consuming* a certain amount of reactive power, it may equivalently be thought of as *delivering* negative of that amount of reactive power. Thus an element that *draws* positive reactive power (i.e., inductive Q) can be said to *deliver* negative reactive power (i.e., capacitive Q). Similarly, an element that *draws* negative reactive power (i.e., capacitive Q) can be said to *deliver* positive reactive power (i.e., inductive Q). Thus, a capacitor is a *source* of inductive reactive power and an inductor is a *source* of capacitive reactive power.

One may easily show that $(\text{Apparent Power})^2 = P^2 + Q^2$. Thus, a closed triangle can be constructed by treating apparent power, active power and magnitude of reactive power as its sides – the triangle will be called, obviously, the *power triangle*. This fact is also expressed in alternative forms as $(\text{VA})^2 = (W)^2 + (\text{VAR})^2$ or $(\text{kVA})^2 = (\text{kW})^2 + (\text{kVAR})^2$.

It may also be noted that active power is alternatively called *real power* and *in-phase power*. Similarly, reactive power is also called *quadrature power*.

Many expressions are commonly employed to calculate reactive power. The first expression is used when the load circuit is a composite circuit containing many resistive and reactive elements. If $V = V_{\text{rms}} \angle \phi_v$ and $I = I_{\text{rms}} \angle \phi_i$ are the voltage phasor and current phasor at load terminals as per passive sign convention, then the Q delivered to the load circuit is $V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i)$ VAR. The other expressions are relevant when the voltage phasor and/or current phasor across a pure reactive element is known. In this case $(\phi_v - \phi_i)$ is assured to be 90° if the element is an inductor and -90° if the element is a capacitor. Then, the reactive power delivered to that element, i.e., Q is given by

$Q = V_{\text{rms}} I_{\text{rms}} = I_{\text{rms}}^2 X = \frac{V_{\text{rms}}^2}{X}$, where V_{rms} is the rms value of voltage across the element, I_{rms} is the rms value of current through the element and X is the reactance of that element. *Note that reactance of an inductor is a positive value and that of a capacitor is a negative value.*

7.10 COMPLEX POWER UNDER SINUSOIDAL STEADY-STATE CONDITION

Can we get these P and Q values from the voltage phasor and current phasor straightaway by multiplying them together? We will try. Let $\mathbf{V} = V_{\text{rms}} \angle \phi_v$ V rms be the voltage phasor and $\mathbf{I} = I_{\text{rms}} \angle \phi_i$ A rms be the current phasor; both specified as rms quantities. Then, $\mathbf{VI} = V_{\text{rms}} I_{\text{rms}} \angle (\phi_v + \phi_i) = V_{\text{rms}} I_{\text{rms}} \cos(\phi_v + \phi_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\phi_v + \phi_i)$.

We are not able to identify P or Q in the real and imaginary parts of this quantity since we know that $P = V_{\text{rms}} I_{\text{rms}} \cos(\phi_v - \phi_i)$ and $Q = V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i)$. But this observation prompts us to try $\mathbf{VI}^*$ instead of $\mathbf{VI}$.

$$\mathbf{VI}^* = V_{\text{rms}} I_{\text{rms}} \angle (\phi_v - \phi_i) = V_{\text{rms}} I_{\text{rms}} \cos(\phi_v - \phi_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i) = P + jQ$$

Thus the quantity $\mathbf{VI}^*$ contains the active power as its real part and the reactive power as its imaginary part.

This quantity, $\mathbf{VI}^*$, is defined as *Complex Power* and is denoted by $\mathbf{S}$ with unit of VA.

$$\mathbf{S}(\text{VA}) = \mathbf{VI}^* = P(\text{W}) + jQ(\text{VAr}) = \sqrt{P^2 + Q^2} \angle (\phi_v - \phi_i) \text{VA}.$$

Therefore the magnitude of *complex power* is the apparent power and the angle of *complex power* is the angle by which the voltage phasor *leads* the current phasor. This angle is the same as the angle of driving-point impedance of the load circuit. $\mathbf{S}$, $P + j0$ and $0 + jQ$ form a closed triangle in complex plane.

If the reader feels uncomfortable with the way the complex power was arrived at, let him be consoled by the fact that this was precisely how the expression for complex power was arrived at in the history of Electrical Engineering.

We had shown that both instantaneous power and average power are conserved in any circuit. Thus, active power under sinusoidal steady-state condition is a conserved quantity. That is, algebraic sum of active power delivered to all the elements of an isolated circuit under sinusoidal steady-state is zero. Does a similar conservation law hold for reactive power? It is possible to show that it does, by using the expression for instantaneous power under sinusoidal steady-state conditions. Since both real part and imaginary parts of $\mathbf{S}$ are conserved, $\mathbf{S}$ itself is a conserved quantity. That is, algebraic sum of complex powers in all elements of an isolated circuit will be zero.

$$\sum_{\text{over all elements}} P = 0; \sum_{\text{over all elements}} Q = 0 \text{ and } \sum_{\text{over all elements}} \mathbf{S} = 0 \text{ for an isolated circuit under sinusoidal steady-state.}$$

Example: 7.10-1

Refer to Example: 7.6-4. Derive expressions for complex power delivered by the first source and complex power absorbed by the second source in a synchronous link and obtain approximate expressions for a situation when the phase difference between the sources is small and the difference in magnitude of voltages is small.

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Solution

The synchronous link under consideration is shown in Fig. 7.10-1.

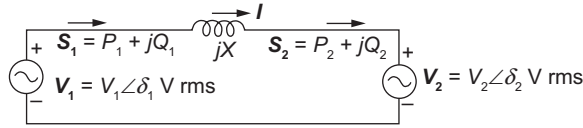


Fig. 7.10-1 A synchronous link

Let the current phasor from left to right be I .

$$I = \frac{V_1 \angle \delta_1 - V_2 \angle \delta_2}{jX} = \frac{V_1 \angle (\delta_1 - \pi/2)}{X} - \frac{V_2 \angle (\delta_2 - \pi/2)}{X}$$

$$\begin{aligned} S_1 = V_1 I^* &= V_1 \angle \delta_1 \left[\frac{V_1 \angle -\delta_1 - V_2 \angle -\delta_2}{-jX} \right] = \frac{V_1^2}{X} \angle \pi/2 - \frac{V_1 V_2}{X} \angle (\pi/2 + \delta_1 - \delta_2) \\ &= \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) + j \frac{V_1 [V_1 - V_2 \cos(\delta_1 - \delta_2)]}{X} \\ \therefore P_1 &= \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) \text{ W and } Q_1 = \frac{V_1 [V_1 - V_2 \cos(\delta_1 - \delta_2)]}{X} \text{ VAr} \end{aligned}$$

Similarly,

$$\begin{aligned} S_2 = V_2 I^* &= V_2 \angle \delta_2 \left[\frac{V_1 \angle -\delta_1 - V_2 \angle -\delta_2}{-jX} \right] = \frac{V_2^2}{X} \angle \pi/2 - \frac{V_1 V_2}{X} \angle (\pi/2 + \delta_1 - \delta_2) \\ &= \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) + j \frac{V_2 [V_1 \cos(\delta_1 - \delta_2) - V_2]}{X} \\ \therefore P_2 &= \frac{V_1 V_2}{X} \sin(\delta_1 - \delta_2) \text{ W and } Q_2 = \frac{V_2 [V_1 \cos(\delta_1 - \delta_2) - V_2]}{X} \text{ VAr} \end{aligned}$$

The link is purely inductive and we do not expect any loss of active power in the link. This is borne out by the fact that $P_1 = P_2$. The link inductor has voltage across it and current through it. Therefore this inductor will consume positive reactive power. Hence, we expect Q_2 to be less than Q_1 . Let Q_L be the reactive power absorbed by the link inductor. Then,

$$Q_L = Q_1 - Q_2 = \frac{V_1^2 + V_2^2 - 2V_1 V_2 \cos(\delta_1 - \delta_2)}{X} = \frac{(V_1 - V_2)^2 + 2V_1 V_2 [1 - \cos(\delta_1 - \delta_2)]}{X}$$

Obviously, Q_L is a positive quantity and hence $Q_2 < Q_1$.

Special Case – $\delta = \delta_1 - \delta_2 \ll \pi/2$ and $V_1 = V + \Delta V$, $V_2 = V$

$$P_1 \approx \frac{V^2}{X} \delta \text{ W}; Q_1 \approx \frac{V \times \Delta V + (\Delta V)^2}{X} \text{ VAr}$$

$$P_2 \approx \frac{V^2}{X} \delta \text{ W}; Q_2 \approx \frac{V \times \Delta V}{X} \text{ VAr}$$

$$Q_L \approx \frac{(\Delta V)^2}{X} \text{ VAr}$$

Thus, in a synchronous link operating with small phase difference and voltage magnitude difference between sources, the active power flows *from the leading source to the lagging source*. The active power will be proportional to phase difference (in radians) and will be relatively independent of voltage magnitude difference. Positive reactive power flow will take place *from the source with higher voltage magnitude to the source with lower voltage magnitude*. Reactive power flow in the link is proportional to voltage magnitude difference and is relatively insensitive to phase difference.

Example: 7.10-2

A 50 Hz, 63.5 kV rms sub-station provides power to a large industrial consumer through a 20 km long high voltage line that can be modelled by a resistance of $5\ \Omega$ and inductive reactance of $20\ \Omega$. The voltage magnitude at receiving end is to be maintained at 63.5 kV. This is done by adjusting the sending end voltage magnitude by tap-changing transformers or otherwise. If the consumer draws 20 MW of power at 0.707 lag power factor, find (i) the magnitude of sending end voltage and power factor, (ii) line current (iii) sending end active and reactive power, (iv) active and reactive power absorbed by the line impedance and (v) line power efficiency.

Solution

The magnitude of receiving end current = $20 \times 10^6 \div (0.707 \times 63.5 \times 10^3) = 445.5\ \text{A rms}$

The angle of current with respect to receiving end voltage phasor = $-\cos^{-1}0.707 = -45^\circ$

We take the receiving end voltage as the reference phasor. Then, $V_R = 63.5 \angle 0^\circ\ \text{kV rms}$ and $I = 445.5 \angle -45^\circ\ \text{A rms}$.

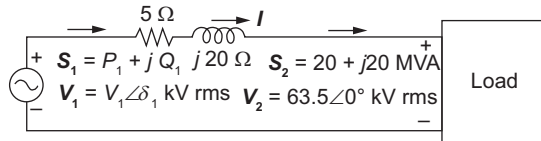


Fig. 7.10-2 Circuit for Example 7.10-2

Active power delivered at receiving end = 20 MW

Reactive power delivered at receiving end = $63.5\ \text{kV} \times 0.4455\ \text{kA} \times \sin 45^\circ = 20\ \text{MVAr}$

Active power consumed by the line impedance = $5\ \Omega \times (0.4455\ \text{kA})^2 = 1\ \text{MW}$

Reactive power consumed by the line impedance = $20\ \Omega \times (0.4455\ \text{kA})^2 = 4\ \text{MVAr}$

$\therefore$ Active power at sending end = $20 + 1 = 21\ \text{MW}$

$\therefore$ Reactive power at sending end = $20 + 4 = 24\ \text{MVAr}$

$\therefore$ Complex power at sending end, $S_1 = 21 + j24\ \text{MVA}$

Since $S_1 = V_1 I^*$, $V_1 = S_1 \div I^* = (21 + j24) \times 10^6 \div (445.5 \angle -45^\circ)^* = 71.6 \angle 3.8^\circ\ \text{kV rms}$.

The angle between sending end voltage phasor and current phasor = $3.8^\circ - (-45^\circ) = 48.8^\circ$. Therefore, sending end power factor = $\cos 48.8^\circ = 0.66\ \text{lag}$.

(i) Sending end voltage magnitude = 71.6 kV rms (12.75% above nominal value)

(ii) Sending end power factor = 0.66 lag

(iii) Sending end active power = 21 MW

(iv) Sending end reactive power = 24 MVAr

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- (v) Active power loss in line = 1 MW
- (vi) Reactive power loss in line = 4 MVar
- (vii) Line power efficiency = 95.2%

Example: 7.10-3

If a capacitor is connected directly across the load at customer side in the problem stated in Example: 7.10-2 such that the receiving end current is at unity power factor with respect to receiving end voltage, find the reactive power drawn by the capacitor and the capacitance value. Also calculate all the quantities calculated under Example: 7.10-2 and comment on the differences.

Solution

Refer to the circuit in Fig. 7.10-3. The capacitor should supply all the reactive power requirement of load, i.e., all of 20 MVar if the current in the line at receiving end is to be at unity power factor. Therefore, the current taken by capacitor will be $20 \times 10^6 / 63.5 \times 10^3 = 315$ A. Therefore, capacitive reactance = $63.5 \times 10^3 / 315 = 201.6 \Omega$. This value is $1/\omega C$ and $\omega = 2\pi \times 50 = 100\pi$. Therefore, $C = 15.8 \mu\text{F}$. With this capacitor in place, the current at receiving end of line will have a magnitude of $20 \times 10^6 / 63.5 \times 10^3 = 315$ A and its angle with respect to voltage will be 0° . Then, $V_R = 63.5 \angle 0^\circ$ kV rms and $I = 315 \angle 0^\circ$ A rms.

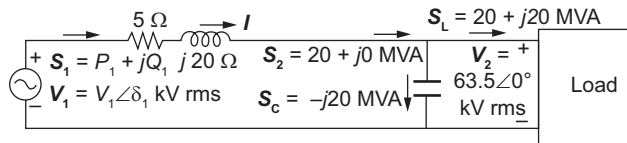


Fig. 7.10-3 Circuit for Example 7.10-3

Active power delivered at receiving end = 20 MW

Reactive Power delivered at receiving end = $63.5 \text{ kV} \times 0.315 \text{ kA} \times \sin 0^\circ = 20 \text{ MVar}$

Active power consumed by the line impedance = $5\Omega \times (0.315 \text{ kA})^2 = 0.5 \text{ MW}$

Reactive power consumed by the line impedance = $20\Omega \times (0.315 \text{ kA})^2 = 2 \text{ MVar}$

$\therefore$ Active power at sending end = $20 + 0.5 = 20.5 \text{ MW}$

$\therefore$ Reactive power at sending end = $0 + 2 = 2 \text{ MVar}$

$\therefore$ Complex power at sending end, $S_1 = 20.5 + j 2 \text{ MVA}$

Since $S_1 = V_1 I^*$, $V_1 = S_1 \div I^* = (20.5 + j 2) \times 10^6 \div (315 \angle -45^\circ)^* = 65.4 \angle 5.6^\circ \text{ kV rms}$.

The angle between sending end voltage phasor and current phasor = $5.6^\circ - (0^\circ) = 5.6^\circ$. Therefore sending end power factor = $\cos 5.6^\circ = 0.995$ lag.

- (i) Sending end voltage magnitude = 65.4 kV rms (3% above nominal value)
- (ii) Sending end power factor = 0.995 lag
- (iii) Sending end active power = 20.5 MW
- (iv) Sending end reactive power = 2 MVar
- (v) Active power loss in line = 0.5 MW
- (vi) Reactive power loss in line = 2 MVar
- (vii) Line power efficiency = 97.6%

Comments on the results

A load that draws power at a lag power factor causes higher magnitude current in the line and source. Higher current magnitude results in higher voltage drop in the line, thereby requiring higher

value of sending end voltage to maintain a specified receiving end voltage. Higher current magnitude results in higher active power loss in the line and thereby reduces power efficiency of the line.

The reactive component of load current undergoes a rotation by 90° to form the voltage drop in the line inductive reactance and results in an in-line voltage drop. An in-line voltage drop affects sending end voltage much more than a quadrature voltage drop. See the phasor diagram in Fig. 7.10-4.

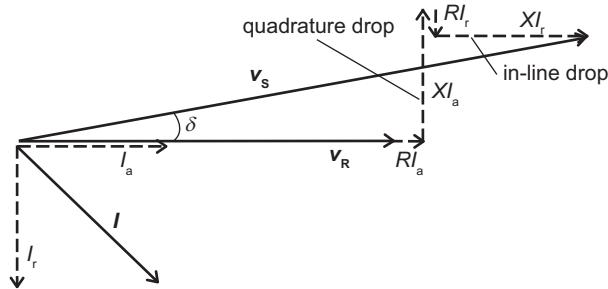


Fig. 7.10-4 Phasor diagram of a line delivering power to a lagging load (Not to scale)

Therefore, reducing the reactive component of current drawn by a lagging load results in (i) lower current magnitude in the line and source (ii) lower line power loss and improved transmission efficiency (iii) lower voltage drop in the line. This reduction is effected by making a local capacitor act as a source of lagging reactive power required by the load. The line is thereby relieved from the task of supplying this reactive power. This is called *capacitive compensation of lagging loads*. Capacitive compensation is routinely employed in Power Systems and Industrial Electrical Systems.

Example: 7.10-4

A 230 V rms source supplies two loads in parallel. The first one draws 10 kVA at 0.8 lag power factor. The second one draws 10 kW at 0.8 lead power factor. Find the source current rms value, complex power delivered by source and source power factor.

Solution

Complex power of first load = $10 \times 0.8 + j 10 \times \sin(\cos^{-1}0.8) = 8 + j6$ kVA.

Complex power of second load = $10 - j (10/0.8) \times \sin(\cos^{-1}0.8) = 10 - j7.5$ kVA.

Complex power is a conserved quantity. Therefore, complex power delivered by source = total complex power delivered to loads = $18 - j1.5$ kVA. Source voltage is $230 \angle 0^\circ$ V rms. Therefore, source current = $[(18 - j1.5) \times 10^3 / 230]^* = 78.53 \angle 4.76^\circ$ A rms.

Therefore, source current rms value is 78.53 A and source power factor is $\cos 4.76^\circ = 0.997$ lead.

7.11 SUMMARY

- *Sinusoidal steady-state* in a dynamic circuit is that state when all the response variables contain just one component with a *sinusoidal waveshape* with the frequency same as that of the sinusoidal

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forcing function applied. The response will, in general, have a phase difference with respect to input. This state will be established only after a transient period that follows the application of sources. Sinusoidal steady-state response in linear circuits obeys superposition principle.

- The *real part* of steady-state response to a complex exponential function $e^{j\omega t}$ = steady-state response to $\cos\omega t$ and *imaginary part* of steady-state response to $e^{j\omega t}$ = steady-state response to $\sin\omega t$.
- All element voltage variables and all element current variables in a linear dynamic circuit driven by a complex exponential function $X_m e^{j\omega t}$ will assume the form $(Y_m e^{j\phi}) e^{j\omega t}$ under steady-state condition, where $(Y_m e^{j\phi})$ represents the relevant *complex amplitude* for the variables. Y_m will be proportional to X_m .
- The amplitude of current in an inductor is $1/\omega L$ times the amplitude of voltage and the current *lags* the voltage by 90° under sinusoidal steady-state condition.
- The amplitude of current in a capacitor is ωC times the amplitude of voltage and the current *leads* the voltage by 90° under sinusoidal steady-state condition.
- *Phasor* is a complex number that gives the amplitude of complex exponential function and the phase of the complex exponential function with the time-variation of the function understood as $e^{j\omega t}$. It can be used as a representation for a sinusoidal function.
- The ratio of voltage phasor to current phasor of an element is called its phasor impedance. It is $R \Omega$ for a resistor, $j\omega L \Omega$ for an inductor and $1/j\omega C$ for a capacitor. Phasor equivalent circuit is constructed for sinusoidal steady-state analysis by replacing all sources by their phasor values and all elements by their phasor impedances.
- The nodal analysis and mesh analysis techniques developed for memoryless circuits apply to phasor equivalent circuits with no change except that *impedance* Z takes the place of *resistance* R and *admittance* Y takes the place of *conductance* G . All circuit theorems, except maximum power transfer theorem, apply to phasor equivalent circuits without modification.
- Maximum average power is transferred to a load circuit from a power delivery circuit under sinusoidal steady-state when the driving-point impedance $Z_L = R_L + jX_L$ of the load is the *conjugate* of Thevenin's impedance $Z_S = R_S + jX_S$ of the power delivery circuit.
- A diagram depicting a group of coherent (*i.e.*, of same angular frequency) complex exponential signals, frozen at their initial position, is called a phasor diagram. Angles measured in counter-clockwise direction in a phasor diagram are lead angles and angles measured in clockwise direction in a phasor diagram are lag angles.
- Apparent power carried by a sinusoidal voltage of rms value V_{rms} and a sinusoidal current of rms value I_{rms} is defined as the actual power that will be carried by a DC voltage of same effective value and a DC current of same effective value – *i.e.*, Apparent Power = $V_{rms} I_{rms}$.
- Active Power, $P = V_{rms} I_{rms} \cos\theta$, where θ is the angle by which the voltage phasor leads the current phasor and Power Factor = $\cos\theta$ under sinusoidal steady-state.
- The current phasor can be resolved into active component ($= I_{rms} \cos\theta$ A rms) and reactive component ($= -I_{rms} \sin\theta$ A rms) by finding its projection along voltage phasor and along a perpendicular to voltage phasor respectively. The active current component carries the entire active power.
- Reactive Power Q under sinusoidal steady-state condition is a quantity that stands for the reactive component of current. It is a scaled version of reactive component of current, the scaling factor being negative of rms value of voltage.
- Complex power S in circuits under sinusoidal steady-state condition is defined as $S(VA) = VI^* = P(W) + jQ(VAr) = \sqrt{P^2 + Q^2} \angle(\phi_v - \phi_i) VA$. Complex power and its components – P and Q – are conserved quantities.

- Let $v(t) = V_m \cos(\omega t + \phi_v) = \sqrt{2} V_{\text{rms}} \cos(\omega t + \phi_v)$ and $i(t) = I_m \cos(\omega t + \phi_i) = \sqrt{2} I_{\text{rms}} \cos(\omega t + \phi_i)$ be the voltage and current as per passive sign convention in a load circuit. Let the series equivalent of the load circuit be $R_s + jX_s$ and the parallel equivalent of the same circuit be $R_p + jX_p$. Then, the following table summarizes the various phasor quantities and their interrelations.

Quantity	Relationship using peak values	Relationship using rms values	Unit
Voltage Phasor V	$V_m \angle \phi_v$	$V_{\text{rms}} \angle \phi_v$	V
Current Phasor I	$I_m \angle \phi_i$	$I_{\text{rms}} \angle \phi_i$	A
Complex Power S	$0.5 V_m I_m \cos(\phi_v - \phi_i) + j0.5 V_m I_m \sin(\phi_v - \phi_i)$	$V_{\text{rms}} I_{\text{rms}} \cos(\phi_v - \phi_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i)$	VA
Apparent Power $ S $	$0.5 V_m I_m$	$V_{\text{rms}} I_{\text{rms}}$	VA
Active Power P	$0.5 V_m I_m \cos(\phi_v - \phi_i)$	$V_{\text{rms}} I_{\text{rms}} \cos(\phi_v - \phi_i)$	W
Reactive Power Q	$0.5 V_m I_m \sin(\phi_v - \phi_i)$	$V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i)$	Var
Power Factor PF	$\cos(\phi_v - \phi_i)$	$\cos(\phi_v - \phi_i)$	—
Input Impedance Z	$(V_m/I_m) \angle (\phi_v - \phi_i)$	$(V_{\text{rms}}/I_{\text{rms}}) \angle (\phi_v - \phi_i)$	Ω
R_s	$(V_m/I_m) \cos(\phi_v - \phi_i)$; or $2P/I_m^2$	$(V_{\text{rms}}/I_{\text{rms}}) \cos(\phi_v - \phi_i)$ or P/I_{rms}^2	Ω
X_s	$(V_m/I_m) \sin(\phi_v - \phi_i)$ or $2Q/I_m^2$	$(V_{\text{rms}}/I_{\text{rms}}) \sin(\phi_v - \phi_i)$ or Q/I_{rms}^2	Ω
R_p	$V_m/[I_m \cos(\phi_v - \phi_i)]$ or $2P/[I_m \cos(\phi_v - \phi_i)]^2$	$V_{\text{rms}}/[I_{\text{rms}} \cos(\phi_v - \phi_i)]$ or $P/[I_{\text{rms}} \cos(\phi_v - \phi_i)]^2$	Ω
X_p	$V_m/[I_m \sin(\phi_v - \phi_i)]$ or $2Q/[I_m \sin(\phi_v - \phi_i)]^2$	$V_{\text{rms}}/[I_{\text{rms}} \sin(\phi_v - \phi_i)]$ or $Q/[I_{\text{rms}} \sin(\phi_v - \phi_i)]^2$	Ω

7.12 PROBLEMS

- Express $v(t) = 5 \cos(\omega t - 45^\circ) - 4 \sin(\omega t + 30^\circ)$ as a single complex exponential function.
- What should be the values of A , ω and ϕ if the signal $v(t) = (2 + j3)e^{j(10t + 0.5\pi)} + Ae^{j(\omega t + \phi)}$ is a real-valued signal?
- The current in a particular element in a linear circuit excited by a sinusoidal voltage source $v_s(t) = 10 \cos(100t - 60^\circ)$ V is found to be $i(t) = 0.75 \sin(100t - 15^\circ)$ A. (i) What is the complex amplitude of current if $v_s(t) = 5e^{j100t}$ V? (ii) Explain why you can not determine the complex amplitude of current when $v_s(t) = 5e^{j200t}$ V using this data?
- Express the following signals as phasors by carrying out the addition in phasor domain. Draw the phasor diagrams.
 - $x(t) = 10 \sin 500 \pi t + 5 \cos(500 \pi t - \pi/5)$
 - $x(t) = 7 \sin(120 \pi t - 50^\circ) + \cos(120 \pi t + 30^\circ)$
 - $x(t) = 7 \cos(50\pi t) + 2 \cos(50\pi t + 30^\circ) - 3 \sin(50\pi t) + 2 \cos(50\pi t + \pi/4)$

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5. Express the following phasors as time-domain signals after carrying out the required additions in phasor domain.
 - (i) $15\angle -60^\circ + 5e^{j\pi/3}$
 - (ii) $20-j10 + 10\angle -20^\circ$
 - (iii) $4e^{-j\pi/12} - 3e^{j\pi/8} + 3\angle -50^\circ$
6. Let $v(t) = V_m \cos \omega t$ V and $i(t) = I_m \cos(\omega t - \phi)$ A be the voltage and current of an element in a linear circuit under sinusoidal steady-state. If the peak value of power pulsation is found to be twice the value of average power what must be the value of ϕ ?
7. Express the angular frequency at which a series RLC circuit excited by a sinusoidal voltage source will have the peak of double-frequency power pulsation equal to the average power in terms of circuit parameters.
8. The current drawn by a linear load circuit from a sinusoidal voltage source $v_s(t) = V_m \cos \omega t$ V is seen to be $I_{m1} \cos(\omega_1 t - 30^\circ)$ A when the angular frequency of voltage source is ω_1 and $I_{m2} \cos(\omega_2 t + 45^\circ)$ A when it is ω_2 . Explain why there should exist a value of ω between ω_1 and ω_2 such that the circuit will draw power at unity power factor at that angular frequency.
9. A resistor R_1 in series with an inductor L is connected in parallel to another resistor R_2 that is in series with a capacitor C . (i) What will be the nature of power factor – lag or lead – for very low frequency sinusoidal input voltage? (ii) What will be the nature of power factor – lag or lead – for very high frequency sinusoidal input voltage? (iii) Will there exist a value of angular frequency at which this load will have unity power factor?
10. The current phasor drawn by a load circuit containing two circuit elements from a voltage of $100\angle 0^\circ$ V rms is $10\angle -36^\circ$ A. Explain why the load circuit has to be an RL circuit?
11. The power factor of a load circuit containing three circuit elements is found to be unity at $\omega = 100\pi$ rad/sec. Is it necessary that all the three elements are resistors?
12. The power factor of a load circuit is found to be 0.7 lag at $\omega = 100\pi$ rad/sec. Is it necessary that the power factor is a lag power factor for all values of ω ? If your answer is no, give an example circuit to justify the answer.
13. The ratio of magnitudes of two load impedances is 1:2. If the first one has 0.7 lag power factor, what must be the power factor of the second one such that a parallel combination of these two loads draws power at unity power factor?
14. The phasor impedance of a multi-element RLC circuit is $3 + j17 \Omega$ at 100π rad/s. Show that there must be at least one resistor and one inductor in the circuit.
15. The phasor impedance of a multi-element passive circuit is $-j20 \Omega$ at 100π rad/s. Show that the circuit contains at least one capacitor and that it contains no resistor.
16. The current drawn by a multi-element passive circuit is found to lag the voltage phasor by 90° . Show that the circuit contains at least one inductor and no resistor.
17. The power factor of a multi-element passive circuit is found to be 0.3 lead at 200 rad/s. It is found to be 0.4 lag at 400 rad/s. Show that the circuit contains at least one resistor, capacitor and inductor.
18. The sum of reactive power delivered to 5 elements in a 6-element circuit is -20 kVAR. Show that the sixth element is either an independent source or an inductor.
19. The sum of reactive power delivered to 4 elements in a 5-element circuit is $+10$ kVAR and the sum of active power delivered to the same 4 elements is 20 kW. Show that the fifth element has to be an independent source and find the power factor at which it is *delivering* power.
20. The sum of reactive power delivered to 4 elements in a 5-element circuit is -10 kVAR and the sum of active power delivered to the same 4 elements is 0 kW. Show that the fifth element has to be an inductor.

21. The sum of complex power delivered to 7 elements in an 8-element circuit is $20 - j10$ VA. Identify the nature of the 8th element.
22. A voltage source $v_s(t) = 325\cos 100\pi t$ V delivers power to a load through an impedance $0.5 + j4 \Omega$. The load voltage peak value is seen to be 350 V. Argue with the help of a phasor diagram that the load has to be a capacitive one.
23. Find the equivalent impedance and admittance at $\omega = 100\pi$ rad/sec for the circuits in Fig. 7.12-1.

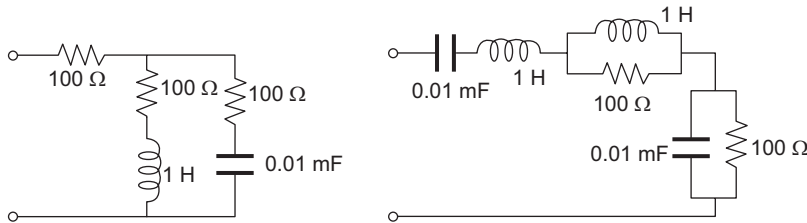


Fig. 7.12-1

24. A resistor of 200Ω and an inductor of 0.5 H are in series. (i) Find the impedance of the combination at 60 Hz in rectangular form and polar form. (ii) Find a parallel RL circuit that has same impedance at same frequency. (iii) Find the impedance of the parallel equivalent in step (ii) for 50 Hz. Are the impedance values equal for the two circuits at 50 Hz?
25. A resistor of 200Ω and a capacitor of 0.05 mF are in parallel. (i) Find the impedance of the combination at 60 Hz in rectangular form and polar form. (ii) Find a series RC circuit that has same impedance at same frequency. (iii) Find the impedance of the series equivalent in step (ii) for 50 Hz. Are the impedance values equal for the two circuits at 50 Hz?
26. The voltage across an impedance is $v(t) = 100 \sin 120\pi t$ V and the current through it is $i(t) = 5 \sin(120\pi t - 60^\circ)$ A. Find the impedance and obtain its series equivalent and parallel equivalent. Can these equivalents be used at another frequency?
27. Find the (i) current delivered by each source (ii) the complex power delivered by each source and power factor of each source (iii) the voltage of LN with respect to SN (iv) total active and reactive power delivered to the three load branches (v) the sum of instantaneous power delivered to the three load branches in the circuit shown in Fig. 7.12-2. $V_1 = 110 \angle 0^\circ$ V rms, $V_2 = 110 \angle -120^\circ$ V rms and $V_3 = 110 \angle 120^\circ$ V rms. The frequency of operation is 60 Hz. Use mesh analysis procedure. Show phasor diagram

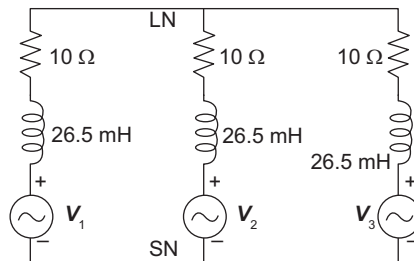


Fig. 7.12-2

28. Repeat Problem 27 by using nodal analysis procedure. The node voltages at nodes between the resistor and inductor are not required. Take SN as the reference node.

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29. Find the phasor currents delivered by sources and the current I_n in the circuit in Fig. 7.12-3 by using Superposition Theorem. $V_1 = 110\angle 0^\circ$ V rms, $V_2 = 110\angle -120^\circ$ V rms and $V_3 = 110\angle 120^\circ$ V rms. The frequency of operation is 60 Hz. Draw the phasor diagram.

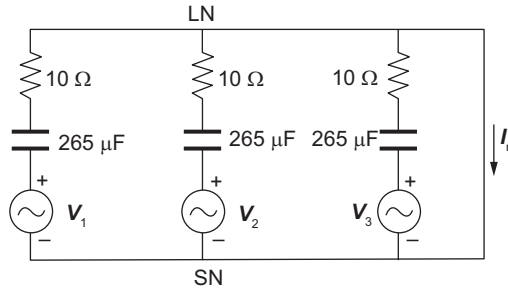


Fig. 7.12-3

30. (a) Find the phasor currents delivered by sources and the current I_n in the circuit in Fig. 7.12-4 by using Superposition Theorem. $V_1 = 110\angle 0^\circ$ V rms, $V_2 = 110\angle -120^\circ$ V rms and $V_3 = 110\angle 120^\circ$ V rms. (b) draw the phasor diagram showing all voltages and currents. The frequency of operation is 60 Hz. Also find (i) the complex power delivered by each source and its power factor and (ii) the sum of instantaneous power delivered to the three load branches in the circuit.

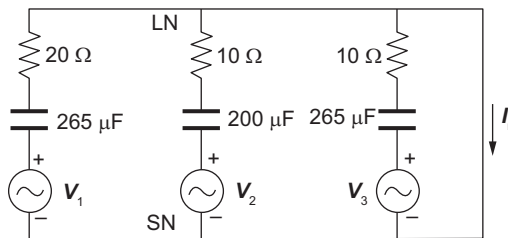


Fig. 7.12-4

31. Find the (i) current delivered by each source (ii) S delivered by each source and power factor of each source (iii) the voltage of LN with respect to SN (iv) total P and Q delivered to the three loads (v) the sum of instantaneous power delivered to the three load branches in the circuit shown in Fig. 7.12-5. $V_1 = 230\angle 0^\circ$ V rms, $V_2 = 230\angle -120^\circ$ V rms and $V_3 = 230\angle 120^\circ$ V rms. The frequency of operation is 50 Hz. Use node analysis procedure. Show phasor diagram.

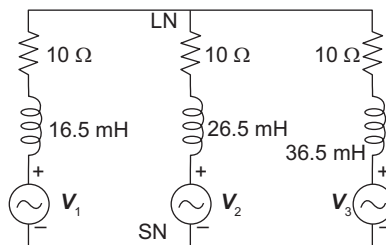


Fig. 7.12-5

32. Find the (i) current delivered by each source (ii) S delivered by each source and power factor of each source (iii) the voltage of LN with respect to SN (iv) the current I_n in the circuit shown in Fig. 7.12-6. $V_1 = 100\angle 0^\circ$ V rms, $V_2 = 120\angle -140^\circ$ V rms and $V_3 = 110\angle 110^\circ$ V rms. The frequency of operation is 60 Hz. Use node analysis procedure. Show phasor diagram.

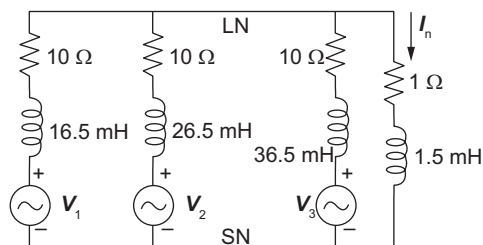


Fig. 7.12-6

33. Find the Thevenin's equivalent of the circuit in Fig. 7.12-6 with respect to terminal pair LN-SN. Thereby find the current that would flow in a short-circuit put across SN-LN node pair.
34. (i) What must be the value of R_L in the circuit in Fig. 7.12-7 if maximum power is to be transferred to it? What is the maximum power transferred? (ii) How will the answer differ if an adjustable reactance X_L can be put in series with R_L and independent adjustment of R_L and X_L for maximum power transfer is possible?

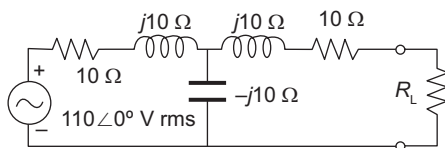


Fig. 7.12-7

35. Find the current in 1Ω as a function of time by using source transformation theorem and current division principle in the circuit shown in Fig. 7.12-8. The frequency of the source is 60 Hz.

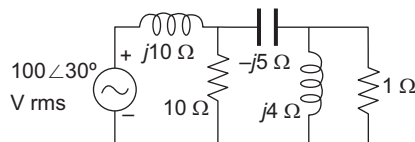


Fig. 7.12-8

36. Find the Norton's Equivalent for the circuit in Fig. 7.12-9 for $\omega = 1000$ rad/sec.

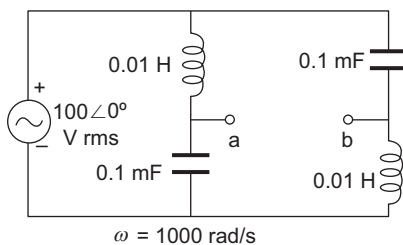


Fig. 7.12-9

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37. Two impedances in parallel are supplied by $V = 100\angle-0^\circ$ V rms. The source is seen to deliver a power of 10 kW at 0.8 lag power factor. The power consumed by the first impedance is 4 kW at 0.9 lead power factor. (i) Find the currents in the impedances and impedance values (ii) power factor and reactive power consumption of second impedance (iii) source current, source power factor and complex power delivered to the impedances if these two impedances are connected in series across the same voltage source.
38. Find the Thevenin's equivalent of the circuit in Fig. 7.12-10 across **a–b** if $v_{s1}(t) = 100 \sin 120\pi t$ V and $v_{s2}(t) = 100 \sin(120\pi t - 60^\circ)$ V.

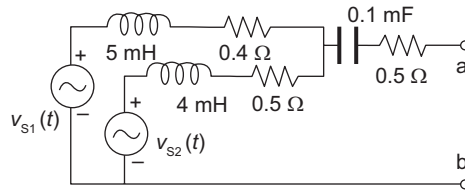


Fig. 7.12-10

39. Two impedances $Z_1 = 6 + j8 \Omega$ and $Z_2 = 8 - j6 \Omega$ are in series and the whole combination is in parallel with a third impedance $Z_3 = 5 + j5 \Omega$. The circuit is driven by a sinusoidal current source $i_s(t) = 10 \cos(120\pi t - 30^\circ)$ A. (i) Solve the circuit by phasor diagram method (ii) Find the complex power delivered to the three impedances. (iii) Verify the conservation of active power and reactive power in the circuit.
40. The star–delta transformation equation developed for resistive circuits is applicable to impedances in phasor equivalent circuits. Apply star–delta transformation and obtain the complex power delivered to the Z in the circuit in Fig. 7.12-11

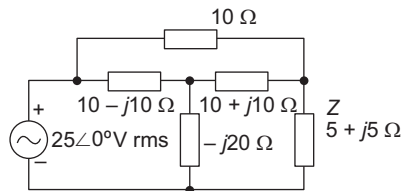


Fig. 7.12-11

41. A 230 V rms, 50 Hz distribution line feeds power to a load located at the far end of the line. The line can be modelled by a series combination of 0.07Ω resistance and 0.3Ω inductive reactance. The load is represented by impedance of $15 + j15 \Omega$. (i) Obtain the load voltage and line current by phasor diagram method. (ii) Calculate the source complex power, source power factor and load complex power and transmission efficiency.
42. A 110 V rms, 60 Hz distribution line feeds power to a load located at the far end of the line. The line can be modelled by a series combination of 0.05Ω resistance and 0.2Ω inductive reactance. The load is represented by impedance of $5 + j10 \Omega$. (i) Obtain the load voltage and line current. (ii) Find the capacitance value of a capacitor that should be connected across the load if the sending end and receiving end voltage magnitudes are to be 230 V rms. (ii) Calculate the source complex power, source power factor and load complex power and transmission efficiency with and without the capacitor.

43. If the source delivers 1 kW at unity power factor in the circuit in Fig. 7.12-12 determine the impedance Z .

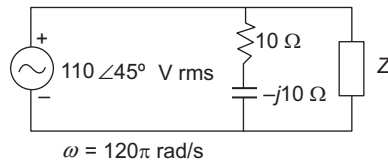


Fig. 7.12-12

44. The current delivered by the source in the circuit in Fig. 7.12-13 is $10 \cos(120\pi t - 15^\circ)$ A under steady-state condition. The load impedance Z_1 is seen to consume 300 W of active power and the load Z_2 is found to deliver 300 VAR of reactive power. (i) Find the currents delivered to the impedances as functions of time. (ii) the impedance values.

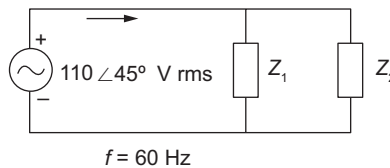


Fig. 7.12-13

45. A motor connected to a 50 Hz, 230 V rms supply line draws 9 A rms and consumes 1.6 kW. (i) Find the apparent power, reactive power and power factor. (ii) Find the capacitance of a parallel capacitor that will make the power factor at the source unity. (iii) Find the source current with this capacitor installed.
46. Find the complex power delivered by the current source in the circuit in Fig. 7.12-14.

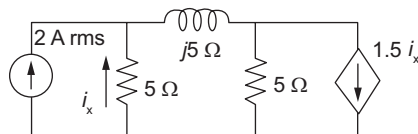


Fig. 7.12-14

47. Find the value of angular frequency such that $v_o(t) = v_s(t)$ in the circuit in Fig. 7.12-15 if $v_s(t) = V_m \cos \omega t$ V. Assume ideal opamp model for the Opamp..

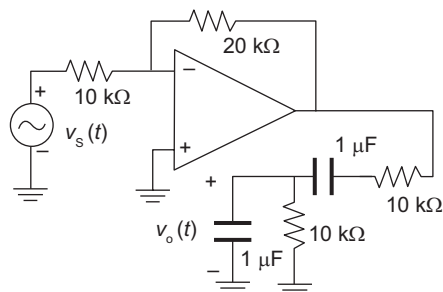


Fig. 7.12-15

7.62 | The Sinusoidal Steady-State Response

48. Using the approximate power flow equations for a synchronous link, find out the complex power delivered by the two sources, currents delivered by the sources and power factor at which they are operating in the circuit shown in Fig. 7.12-16. Also find the voltage phasor at node C. (Hint – Let the voltage at C be $V\angle\delta$. Then AC and BC are two synchronous links.)

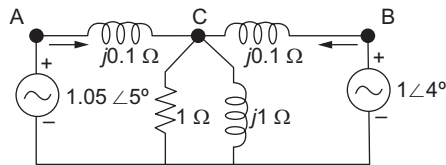


Fig. 7.12-16

49. The current source at C in Fig. 7.12-17 can deliver only Q . Q delivered by it is adjusted such that the voltage at C is 1.0 V rms. (i) Find the Q that it has to deliver. (ii) Find the S delivered by the two voltage sources. (iii) Find the angle of voltage phasor at node C. Use synchronous link equations.

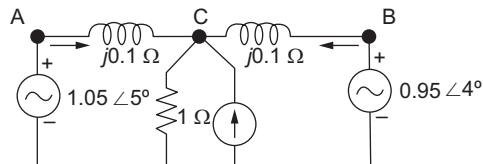


Fig. 7.12-17



Sinusoidal Steady-State in Three-Phase Circuits

CHAPTER OBJECTIVES

- To define and explain balanced and unbalanced three-phase systems and to extend the concepts of active power, reactive power and complex power to such systems.
- To develop and employ relations between line and phase quantities in Y-connected and Δ -connected three-phase circuits.
- To emphasize constraints on line quantities in three-phase three-wire and four-wire systems.
- To develop a systematic procedure to analyse sinusoidal steady-state in three-phase balanced circuits using Y-equivalent and single-phase equivalent circuits.
- To show how to use phasors, phasor equivalent circuits and phasor diagrams for solving three-phase circuits under sinusoidal steady-state condition.
- To introduce and illustrate the problem of neutral-shift voltage in four-wire systems.
- To illustrate circuit analysis procedures for unbalanced three-phase circuit analysis through solved examples.
- To introduce and explain symmetrical components in detail and use it in analysis of unbalanced circuits.
- To develop power relations using sequence components.

INTRODUCTION

Electrical power generation, transmission and distribution employ a particular system of three sinusoidal voltages/currents with certain symmetry in voltage/current magnitude and relative phase. This system of three sinusoidal voltage/current waveforms is called a three-phase voltage/current and

8.2 Sinusoidal Steady-State in Three-Phase Circuits

a circuit employing a three-phase voltage/current system is called a three-phase circuit. A three-phase circuit, like any other circuit, will have to go through a transient period subsequent to the application of source functions before it can settle down to a sinusoidal steady-state condition. We focus on analysis of three-phase circuits under sinusoidal steady-state condition in this chapter.

No new concepts are required for analysis of three-phase circuits under sinusoidal steady-state conditions. It requires only the application of all the concepts we developed in the last two chapters. But before we take up the analysis of three-phase circuits, let us see why electrical engineers thought up three-phase system in the first place.

8.1 THREE-PHASE SYSTEM VERSUS SINGLE-PHASE SYSTEM

Industrial heaters are used in various material processing industries for a variety of heating applications. Typically, heater boxes that contain heating elements powered by AC supply, transfer the heat developed to a gaseous (usually air) or liquid medium which conducts the heat to the material that is to be heated up. Resistor elements made of some specially constructed wire are laid out uniformly inside the heater box to develop uniform heating of the contact surface. Many such heating elements are finally connected in parallel to form the heater resistive load. Consider such a heater which, after all parallel connection of sub-elements, is finally represented by three equal resistors of R each. These three resistors may be connected in parallel and supplied from a single-phase AC source or they may be supplied individually from different AC sources – for instance, for redundancy and reliability.

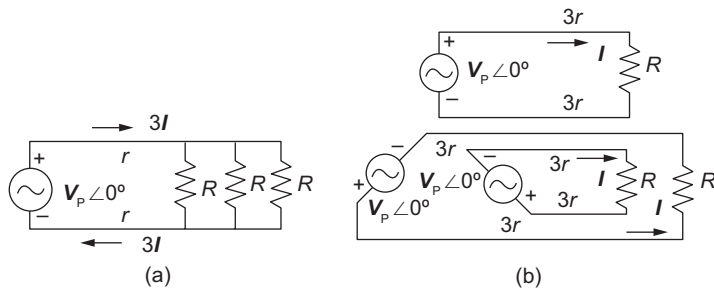


Fig. 8.1-1 (a) Three equal resistors powered from a single-phase supply (b) Three equal resistors supplied from three identical sources

The circuit in Fig. 8.1-1(a) shows the three heater elements tied in parallel and supplied from a single-phase AC supply. The rms value of source is specified as V_p . Let r be the resistance of the connecting cable and $3I$ be the phasor current in the cable. r is usually small compared to R and hence we can take the total circuit resistance as $R/3$ ignoring the contribution $2r$. Therefore, circuit current magnitude $\approx 3V_p/R$ A rms. Therefore, magnitude of I is $\approx V_p/R$ A rms. The total power loss in the cable is $\approx 18r (V_p/R)^2$ W.

The circuit in Fig. 8.1-1(b) shows the same three heater elements supplied from same single-phase supply – but with sources shown as three identical sources. Since the current in the connecting cable is only one-third of the earlier value we will choose a cable that has one-third the area of cross section that the cable in the circuit in Fig. 8.1-1(a) had. This implies that the cables in the circuit in Fig. 8.1-1(b) have a resistance of $3r$ each. Thus, the total power loss in the circuit in Fig. 8.1-1(b) is $\approx 6 \times 3r \times (V_p/R)^2 = 18r (V_p/R)^2$ W again.

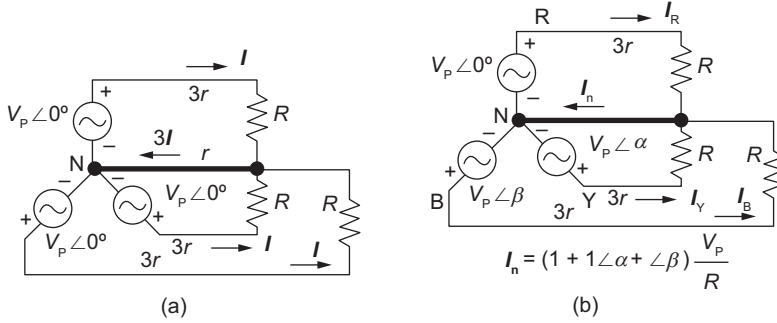


Fig. 8.1-2 From single-phase circuit to three-phase circuit

Next, we imagine that the negative polarity terminals of all the three sources are joined together to form a common node and that a single cable connects this node to a similarly formed node on the load side as shown in the circuit of Fig. 8.1-2(a). The return currents from the three load branches flow through this cable. Then, this cable must have three times the cross-sectional area of the other cables in circuit in Fig. 8.1-2(a). Therefore, its resistance will be r . The power loss in four cables put together will again be $18r (V_p/R)^2 \text{ W}$.

Non-uniform heating will result if the three branches get different powers delivered to them. Therefore, we wish to keep the power dissipated in the three resistors equal. Therefore, the three sources must have the same rms voltage. But that does not mean that they should be in-phase too. We are not particularly worried about the relative phase of currents flowing in the three branches of a heater resistance. However, is there any advantage in making the phase angles non zero?

We assume in circuit of Fig. 8.1-2(b) that the phase of source connected at the terminal identified Y is at an angle α with respect to the source connected at terminal marked as R. Similarly, the phase of source connected at the terminal identified as B is at an angle β with respect to the source connected at terminal marked as R. Moreover, we assume that now we are using an extra-thick cable between node N to a similar node on the load side such that it is virtually a resistance-free connection. Now, the currents flowing out from the sources have different phase; but same magnitude of $V_p/(R + 3r) \text{ A rms}$. Note that we use 'R' as a terminal marking and 'R' as symbol and value for a resistance. The cable connecting the node N to a similar node on the load side will have a current I_n in it.

$$I_n = (1 + 1\angle\alpha + 1\angle\beta) \frac{V_p}{R + 3r} \text{ A rms} = [(1 + \cos\alpha + \cos\beta) + j(\sin\alpha + \sin\beta)] \frac{V_p}{R + 3r} \text{ A rms.}$$

Why do not we make the magnitude of this current zero so that the cable power loss will become zero in the corresponding cable even if we reduce the size of that cable? We try to find a (α, β) pair such that magnitude of I_n is zero. α and β should be such that $(\sin\alpha + \sin\beta) = 0$ and $(\cos\alpha + \cos\beta) = -1$ simultaneously. First equation implies that $\beta = -\alpha$ and with this constraint the second equation results in $\alpha = \pm 120^\circ$. Therefore, $(\alpha, \beta) = (-120^\circ, 120^\circ)$ or $(120^\circ, -120^\circ)$ are the two choices for (α, β) pair such that the current $I_n = 0$.

If we make one of these two choices, the total power delivered to the resistors remains the same; however, the total cable loss becomes $\approx 9r (V_p/R)^2 \text{ W}$, i.e., only 50% of the cable loss in the case of single-phase supply.

8.4 Sinusoidal Steady-State in Three-Phase Circuits

If a cable (or any resistance for that matter) carries only zero current in a circuit, then, that cable can be removed without affecting the behaviour of any circuit variable in the circuit. This implies that we can remove the cable connecting the common point N of the sources to the common point in the load without affecting the load currents or load power. This is possible since the instantaneous currents in the three lines add up to zero at all t leaving only zero current in the common point link. Let us verify this with $\alpha = -120^\circ$ $\beta = 120^\circ$.

$$i_R(t) = \frac{\sqrt{2}V_P}{R+3r} \cos \omega t \text{ A}, i_Y(t) = \frac{\sqrt{2}V_P}{R+3r} \cos(\omega t - 120^\circ) \text{ A}, i_B(t) = \frac{\sqrt{2}V_P}{R+3r} \cos(\omega t + 120^\circ) \text{ A}$$

$$\therefore i_R(t) + i_Y(t) + i_B(t) = \frac{\sqrt{2}V_P}{R+3r} [\cos \omega t + \cos(\omega t - 120^\circ) + \cos(\omega t + 120^\circ)]$$

$$= \frac{\sqrt{2}V_P}{R+3r} [\cos \omega t + \{-0.5 \cos \omega t + 0.866 \sin \omega t\} + \{-0.5 \cos \omega t - 0.866 \sin \omega t\}] = 0 \text{ for all } t$$

Hence, no wire is needed between the common point of sources and common point of load. Therefore, we need only three wires of $3r \Omega$ each in circuit of Fig. 8.1-2(b) instead of two wires of $r \Omega$ each as in circuit in Fig. 8.1-1(a). Cross-sectional area of wires in circuit of Fig. 8.1-1(a) will be three times that of the wires in circuit of Fig. 8.1-2(b). Therefore, the amount of copper (volume or weight) we use in wiring up circuit in Fig. 8.1-2(b) is only 50% of the copper that we use in wiring up circuit in Fig. 8.1-1(a).

A set of three sinusoidal quantities, all at the *same frequency*, with equal peak (and hence rms) values and shifted successively by 120° in phase is defined as a *Balanced Three-Phase Quantity*. Therefore, if $x_1(t)$, $x_2(t)$ and $x_3(t)$ is a three-phase set, then,

$$x_1(t) = X_m \cos \omega t, x_2(t) = X_m \cos(\omega t - 120^\circ) \text{ and } x_3(t) = X_m \cos(\omega t + 120^\circ) \text{ or}$$

$$x_1(t) = X_m \cos \omega t, x_2(t) = X_m \cos(\omega t + 120^\circ) \text{ and } x_3(t) = X_m \cos(\omega t - 120^\circ)$$

and $x_1(t) + x_2(t) + x_3(t) = 0$ for all t .

Each limb or branch of a three-phase system (source or load) is termed as a *phase*. R, Y and B are used to designate the line terminals of a three-phase source or load.

If the peak values are unequal and/or successive phase shifts are different from 120° , the set will be called an *Unbalanced Three-Phase Quantity*.

A balanced three-phase system requires only 50% of the copper and incurs only 50% of the power loss to transmit a given amount of power to load compared to a single-phase system.

The sub elements of an industrial heater designed for three-phase operation are sometimes arranged in such a way that adjacent sub-elements belong to different phases. That is, if three sub-elements are adjacent, the first one will be a part of R that gets connected to R-line, the second one will be part of Y that gets connected to Y-line and the third one will be part of B that gets connected to B-line. Then, the cycle repeats. This is, obviously, done on purpose. We discuss the sum of instantaneous power delivered by the three phases of a three-phase system under balanced operation to appreciate the purpose involved.

Let us assume that the impedance per branch (*i.e.*, phase) is $Z = Z \angle \theta$ and that Y-phase voltage lags R-phase by 120° and B-phase voltage lags Y-phase voltage by 120° . Then, the instantaneous currents in three lines can be expressed as

$$i_R(t) = I_m \cos(\omega t - \theta), i_Y(t) = I_m \cos(\omega t - 120^\circ - \theta), \text{ and } i_B(t) = I_m \cos(\omega t + 120^\circ - \theta)$$

where $I_m = V_m/Z$.

Also, $v_{RN}(t) = V_m \cos(\omega t)$, $v_{YN}(t) = V_m \cos(\omega t - 120^\circ)$, and $v_{BN}(t) = V_m \cos(\omega t + 120^\circ)$.

where $V_m = \sqrt{2}V_p$. Now, the instantaneous powers delivered by the sources in each phase are expressed as

$$\begin{aligned} p_R(t) &= V_m I_m \cos(\omega t) \cos(\omega t - \theta) \\ &= 0.5 V_m I_m [\cos \theta + \cos \theta \cos 2\omega t] + 0.5 V_m I_m [\sin \theta \sin 2\omega t] \text{ W.} \\ p_Y(t) &= V_m I_m \cos(\omega t - 120^\circ) \cos(\omega t - 120^\circ - \theta) \\ &= 0.5 V_m I_m \cos \theta + \cos \theta \cos(2\omega t - 240^\circ) + 0.5 V_m I_m [\sin \theta \sin(2\omega t - 240^\circ)] \text{ W.} \\ p_B(t) &= V_m I_m \cos(\omega t + 120^\circ) \cos(\omega t + 120^\circ - \theta) \\ &= 0.5 V_m I_m [\cos \theta + \cos \theta \cos(2\omega t + 240^\circ)] + 0.5 V_m I_m [\sin \theta \sin(2\omega t + 240^\circ)] \text{ W.} \end{aligned}$$

The sum of instantaneous powers can now be expressed as

$$\begin{aligned} p_R(t) + p_Y(t) + p_B(t) &= 1.5 V_m I_m \cos \theta \\ &\quad + 0.5 V_m I_m \cos \theta \{ \cos 2\omega t + \cos(2\omega t - 240^\circ) + \cos(2\omega t + 240^\circ) \} \\ &\quad + 0.5 V_m I_m \sin \theta \{ \sin 2\omega t + \sin(2\omega t - 240^\circ) + \sin(2\omega t + 240^\circ) \} \text{ W.} \end{aligned}$$

A phase angle of -240° is same as $+120^\circ$ and a phase angle of $+240^\circ$ phase is same as -120° . Therefore, the three terms in the sums enclosed by curly brackets in the preceding equation form balanced three-phase sets. Sum of a three-phase balanced set is zero for all time instants. Therefore,

$$p_R(t) + p_Y(t) + p_B(t) = 1.5 V_m I_m \cos \theta = 3 \frac{V_p}{Z} \cos \theta \text{ W.}$$

If the power dissipated in the three branches of a balanced three-phase load acts on the *same physical system*, then, a balanced three-phase source delivers active power to the load with zero power pulsation. The double-frequency power pulsation in the case of a single-phase AC supply can never be reduced to zero.

Some instances in which the total active power from three branches of a three-phase load acts on the same physical system are industrial heaters and three-phase motors. In a three-phase heater, power dissipated in three branches will heat up the same body. In a three-phase motor, the torque developed by the branch windings act on the same mechanical shaft. This leads to steady heating and steady temperature in a heater and speed without fluctuations in the case of a motor.

The next point of superiority of three-phase system over single-phase system has to be accepted as a statement of fact. It concerns matters beyond the scope of a book on circuits. It is the fact that Electrical Machines and Power Systems Components are more efficiently designed in three-phase

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version rather than in single-phase version in all senses of the word ‘efficiency’ – efficiency in material utilisation, efficiency in power utilisation etc. Hence, we conclude:

A balanced three-phase system comprising balanced three-phase sources and balanced three-phase loads is superior to a single-phase system thanks to the following facts:

- (i) Power loss in transmission system is lower in three-phase system.
- (ii) Copper utilisation is superior in three-phase system.
- (iii) Power delivered to a balanced three-phase load by a balanced three-phase supply is free of pulsation.
- (iv) Electrical equipment designed for three-phase operation is more efficient than their single-phase counter parts.

8.2 THREE-PHASE SOURCES AND THREE-PHASE POWER

Three-phase sources and loads can take two forms – star connected or delta connected. The star connection is also referred to as Y-connection. The delta connection is referred to as Δ -connection and mesh connection too.

8.2.1 The Y-connected Source

Two options to construct a three-phase Y-connected source are shown in Fig. 8.2-1.

In Fig. 8.2-1(a) the terminal identified by Y has a voltage phasor that lags the voltage phasor of terminal identified as R by 120° . Further, the terminal identified as B has a voltage phasor that leads the R-line by 120° – i.e., B-line voltage lags R-line voltage by 240° . Thus, similarly located points on the waveform will appear first in R-line, then in Y-line after one-third of a cycle period and finally in B-line after another one-third of a cycle period. See the waveforms in Fig. 8.2-1(a). Thus, the sequence of appearance of peaks will be ...-RYB-RYB-... A three-phase source that has this sequence of appearance of similarly located waveform points is called a *positive sequence three-phase source*. Note that the sequence can also be written as ...-YBR-YBR-... or BRY-BRY-...

The source in Fig. 8.2-1(b) illustrates a *negative phase sequence source*. In this case, the voltage of B-line lags that of R-line by 120° and voltage of Y-line lags that of R-line by 240° . Thus, the order of appearance of similarly located waveform points in time will be ...-RBY-RBY-... (equivalently, ...-BYR-BYR-... or ...-YRB-YRB-... too) in this case.

There are loads that *cannot* distinguish between the two sequences – a three-phase industrial heater is one. There are loads that *can* distinguish between the sequences – a three-phase induction motor is one. If it rotates in clockwise direction for positive sequence applied voltage, it will rotate in counter-clockwise direction for negative sequence applied voltage. So will a three-phase synchronous motor. The three-phase sources we consider from this point onwards will be positive sequence sources by default.

The voltage V_p used in Fig. 8.2-1 is an rms value. We will specify three-phase source in rms values in this chapter.

The Y-connected source has a common point at which the three phase-sources join. This point is called the ‘*neutral*’ point of the source. *Phases* in the context of three-phase source or load refer to the three branches that constitute the three-phase system. *Lines* refer to the three terminals – R, Y and B –

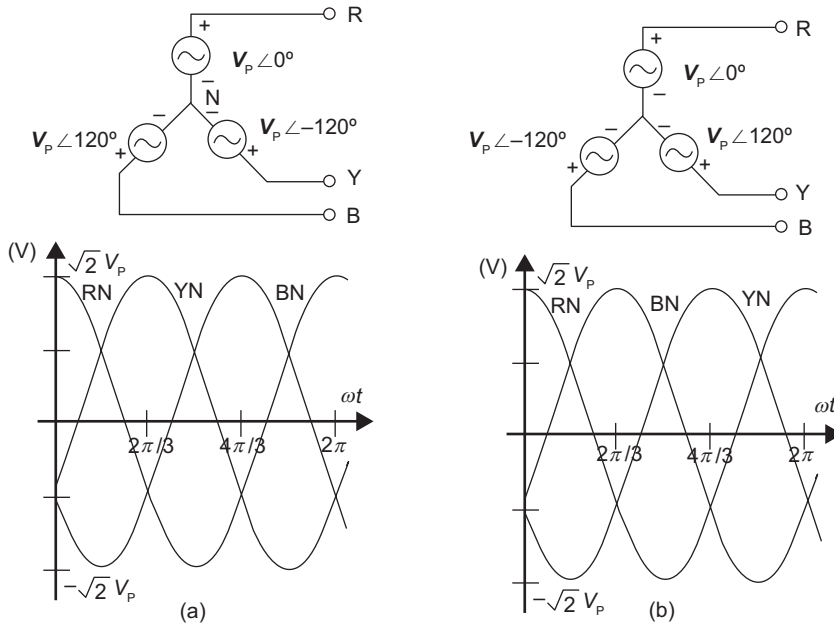


Fig. 8.2-1 (a) A positive sequence Y-connected three-phase source (b) A negative sequence Y-connected three-phase source

that come out of the three-phase source or load and are available for external connections. Thus, in the case of Y-connected source, the *phase-source* is connected between a *Line* and the *Neutral*. Therefore, the *phase-source voltage* is same as *Line-to-Neutral voltage* in the case of Y-connected source. As a matter of common practice, *phase-source voltage* is shortened to *phase voltage*. Beginners often get confused between *phase voltage* (i.e., phase-source voltage or phase-load voltage) and *line-to-neutral voltage* and conclude that they are the same. They are not the same always. *Phase voltage* is the voltage across phase source or phase load. Phase voltages exist for any three-phase source or load. Line-to-neutral voltage will exist only if there is a neutral point – i.e., only for Y-connected source or load.

Though line-to-neutral voltage does not exist unless it is a Y-connected system, *line-to-line voltages* exist for all three-phase systems. V_{RY} is the phasor voltage of the R-line with respect to the Y-line. V_{YB} and V_{BR} are similarly defined. *Line-to-line voltage* is usually shortened to *line voltage*.

Similarly, *phase current* is the *phase-source current* or *phase-load current* flowing in one of the branches that constitute the three-phase source or load as per passive sign convention. *Line current* is the current that flows out of or into one of the three terminals R, Y or B. Line current is usually taken to be *flowing out of a source line terminal* and *flowing into a load line terminal*. Refer to Fig. 8.2-2 that shows all the phase and line quantities in a Y-connected source. V_{RN} , V_{YN} and V_{BN} are the *phase voltages* and I_{RN} , I_{YN} and I_{BN} are the *phase currents*. There is a neutral point in an Y-connected source. Hence, *line-to-neutral voltages* exist and they are the same as *phase voltages*. *Line currents* are I_R , I_Y and I_B and are assumed to flow out of the source towards the load. In a Y-connected source, the *line currents* defined in this manner are equal to negative of *phase currents* defined as per passive sign convention.

Since the R-line is same as positive terminal of the source that feeds the R-line in a Y-connected source, this source may be referred to as the R-phase source. Similarly for other two sources too.

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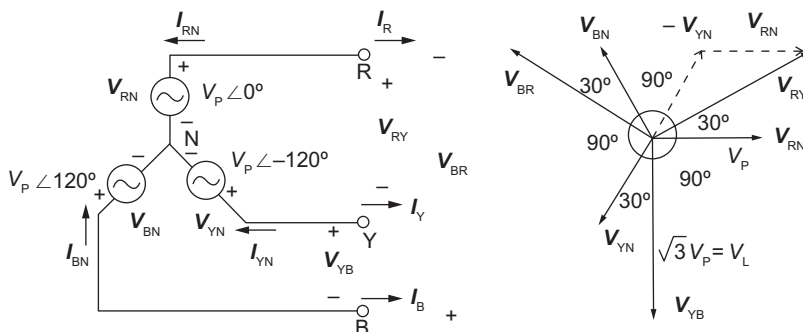


Fig. 8.2-2 A Y-connected source with all phase and line quantities identified

We start at the Y-line terminal and traverse through the two sources, reach R-line terminal and move to Y-terminal by falling through V_{RY} . The KVL equation we get is $V_{RY} = V_{RN} - V_{YN}$. Refer to the phasor diagram shown in Fig. 8.2-2. V_{RN} is taken as reference and a horizontal line of length V_P (the rms value of phase sources) to suitable scale is drawn to represent it. The remaining two phase voltages $-V_{YN}$ and V_{BN} are at -120° and -240° positions as shown. Now, the equation $V_{RY} = V_{RN} - V_{YN}$ is implemented by rotating V_{YN} by 180° , moving a copy of V_{RN} to the tip of $-V_{YN}$ and completing the triangle. The result is the first line voltage V_{RY} .

We observe by projecting the lengths of V_{RN} and $-V_{YN}$ on to V_{RY} that magnitude of V_{RY} is $2 \cos 30^\circ V_P = \sqrt{3} V_P$. We represent the magnitude of line voltage by V_L . Hence, $V_L = \sqrt{3} V_P$. We observe further that the first line voltage, V_{RY} , leads the first phase voltage V_{RN} by 30° .

The remaining two line voltages are also obtained by using equations $V_{YB} = V_{YN} - V_{BN}$ and $V_{BR} = V_{BN} - V_{RN}$. Obviously, the three line voltages form a three-phase balanced set of voltages.

The line voltages in a balanced Y-connected source form a balanced three-phase set of voltages that are $\sqrt{3}$ times in magnitude and 30° ahead in phase with respect to phase voltages.

The line currents in Y-connected source need not be cophasal with the phase voltages. That depends on the load. Assuming that the load is a balanced one, the line currents themselves will form a three-phase balanced set of phasors that are displaced in phase by a phase-lag angle θ with respect to respective phase voltages.

Thus, $I_R = I_L \angle -\theta$, $I_Y = I_L \angle -\theta - 120^\circ$ and $I_B = I_L \angle -\theta + 120^\circ$, where I_L is the rms value of line current. These current phasors are shown in relation to phase voltage phasors in Fig. 8.2-3(b). Also shown are the phase positions of line voltages with respect to phase voltages in Fig. 8.2-3(a). We make the following observations from these phasor diagrams.

1. The line current lags the phase-source voltage by θ and it lags behind the corresponding line voltage by $(30^\circ + \theta)$.
2. Each source contributes an active power of $V_P I_L \cos \theta$ W. Hence, total three-phase active power delivered by the three-phase source $= 3 V_P I_L \cos \theta = \sqrt{3} V_L I_L \cos \theta$ W where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RN}$ in the case of R-line) lags behind the phase voltage.

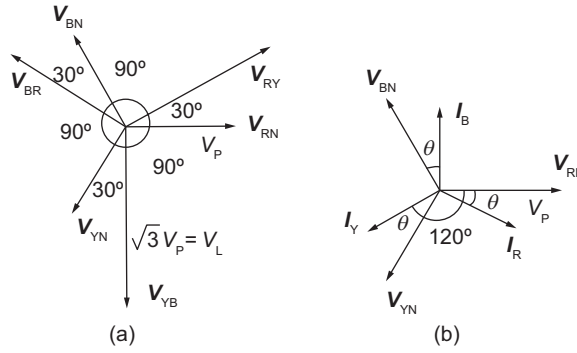


Fig. 8.2-3 (a) Phasor diagram of phase and line voltages of a Y-connected source (b) Phasor diagram of line currents and phase voltages in Y-connected source

- Each source contributes a reactive power of $V_P I_L \sin \theta$ W. Hence, total three-phase reactive power delivered by the three-phase source $= 3 V_P I_L \sin \theta = \sqrt{3} V_L I_L \sin \theta$ VAr, where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RN}$ in the case of R-line) lags behind the phase voltage. Equivalently, θ is the angle by which the current delivered by a phase source lags behind the phase voltage.
- Therefore, the complex power delivered by the three-phase source $S = \sqrt{3} V_L I_L \angle \theta$ VA.

8.2.2 The Δ -connected Source

The generators in a power system are almost invariably Y-connected. However, power systems use Y- Δ transformers and a Δ -connected three-phase source can be used to model the secondary side of such a transformer. A Δ -connected three-phase source and the phasor diagram of its voltages and currents are shown in Fig. 8.2-4.

We assume that the load on the source is balanced. V_{RY} is taken as the reference phasor for phasor diagrams.

Note that *phase voltage* and *line voltage* are the same for this source and that there is no neutral point and hence no *line-to-neutral voltage* can be defined for this source. Applying KCL at the three corners of delta, we get the following equations.

$$\begin{aligned} I_{BR} - I_{RY} &= I_R \\ -I_{YB} + I_{RY} &= I_Y \\ I_{YB} - I_{BR} &= I_B \end{aligned}$$

This system of equations has no unique solution since we can add a constant to I_{RY} , I_{YB} and I_{BR} without affecting these equations. Such a constant added to the three currents will represent a circulating current within the delta-loop. Such a circulating current is possible only since three ideal sources are connected in a loop. However, in practice, all the sources in the delta will have some small impedance or other which will force the condition that $(I_{RY} + I_{YB} + I_{BR})Z = 0$ where Z is the small source impedance of the sources. Assuming that the circulating current component in delta is zero, let the current $-I_{RY}$ be $I_p \angle -\theta$. Then, the remaining two phase currents will be $-I_{YB} = I_p \angle -\theta - 120^\circ$ and $-I_{BR} = I_p \angle -\theta + 120^\circ$, where I_p is the magnitude of phase currents.

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Now, I_R can be constructed by $I_R = -I_{RY} - (-I_{BR})$. This is shown in the phasor diagram in Fig. 8.2-4. The resulting $I_R = \sqrt{3} I_p \angle -(30^\circ + \theta)$ A rms.

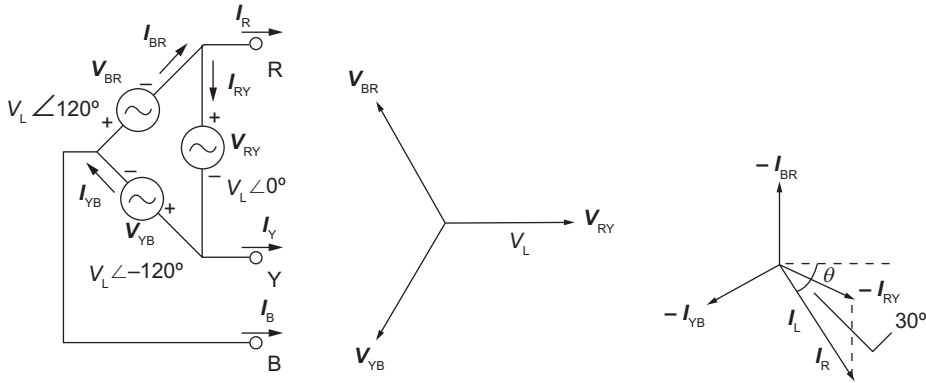


Fig. 8.2-4 A delta-connected source and its phasor diagrams

Hence, the line current magnitude is $I_L = \sqrt{3} I_p$ in a delta-connected three-phase source.

The line currents delivered by a balanced Δ -connected source form a balanced three-phase set of currents that are $\sqrt{3}$ times in magnitude and 30° behind in phase with respect to phase currents delivered by the phase-sources.

We make the following observations from these phasor diagrams.

1. The delivered phase current lags the phase-source voltage by θ and line current lags behind the corresponding line voltage by $(30^\circ + \theta)$.
2. Each source contributes an active power of $V_L I_p \cos \theta$ W. Hence, total three-phase active power delivered by the three-phase source $= 3 V_L I_p \cos \theta = \sqrt{3} V_L I_L \cos \theta$ W, where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RY}$ in the case of RY-phase source) lags behind the phase voltage.
3. Each source contributes a reactive power of $V_L I_p \sin \theta$ W. Hence, total three-phase reactive power delivered by the three-phase source $= 3 V_L I_p \sin \theta = \sqrt{3} V_L I_L \sin \theta$ VAr where θ is the angle by which the 'negative of phase current defined as per passive sign convention' (i.e., $-I_{RY}$ in the case of RY-phase source) lags behind the phase voltage. Equivalently, θ is the angle by which the current delivered by a phase-source lags behind the phase voltage.
4. Therefore, the complex power delivered by the three-phase source $S = \sqrt{3} V_L I_L \angle \theta$ VA.

Therefore, we state the following conclusion:

The three-phase complex power delivered by a three-phase source is given by

$$S = \sqrt{3} V_L I_L \angle \theta = [\sqrt{3} V_L I_L \cos \theta + j \sqrt{3} V_L I_L \sin \theta] \text{ VA}$$

where θ is the angle by which the current phasor delivered by a phase source lags behind its voltage phasor. This relationship is independent of whether the source is Y-connected or Δ -connected. It should be noted that θ is *not* the angle between line voltage phasor and line current phasor.

Electrical Power Engineering adopts a convention to specify a three-phase source by specifying its **line-to-line voltage in rms value**. We follow this convention unless stated otherwise in the remaining portions of this chapter. Thus a 400 V, 50 Hz source means a source that has a line-to-line voltage magnitude of 400 V rms at 50 Hz. If it is Y-connected its phase voltage will be $400/\sqrt{3}$ V rms and peak of its phase voltage will be $400\sqrt{2}/\sqrt{3}$ V.

8.3 ANALYSIS OF BALANCED THREE-PHASE CIRCUITS

Balanced three-phase circuits comprising a balanced three-phase source and a balanced three-phase load can be of four varieties, viz., $-Y-Y$, $Y-\Delta$, $\Delta-Y$ or $\Delta-\Delta$. We develop a common analysis procedure for all the possible configurations.

8.3.1 Equivalence Between a Y-connected Source and a Δ -connected Source

Let $V_{RY} = V_L \angle 0^\circ$, $V_{YB} = V_L \angle -120^\circ$ and $V_{BR} = V_L \angle 120^\circ$ be the three line voltages observed in a three-phase circuit and let $I_R = I_L \angle -(\theta + 30^\circ)$, $I_Y = I_L \angle -(\theta + 150^\circ)$ and $I_B = I_L \angle -(\theta - 90^\circ)$ be the observed line currents where the observed phase lag of first line current is expressed as 30° plus some angle designated by θ . If the source is within a black box and we are to guess whether it is a Y-connected source or a Δ -connected source, we will find that we will not be able to resolve the matter using the observed line voltage phasors and line current phasors. This is so because both configurations shown in Fig. 8.3-1 will result in these observed line voltages and line currents.

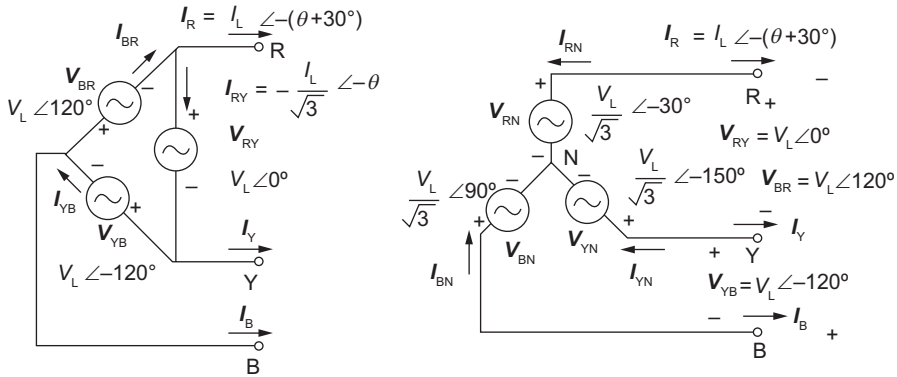


Fig. 8.3-1 Illustrating equivalence between a Y-connected source and a Δ -connected source

Hence, for every set of line voltages and line currents observed in a three-phase circuit, there exist a delta-connected source and a star-connected source which will justify the observed line quantities and which are indistinguishable from outside.

Therefore, if a balanced three-phase source is really Δ -connected, it may be replaced by an equivalent Y-connected source for purposes of analysis. The details of phase-source currents in the real delta-connected source can be obtained after the circuit problem is solved using the star equivalent source.

8.3.2 Equivalence Between a Y-connected Load and a Δ -connected Load

But there is nothing new in this equivalence – we already know that any Y-connected set of impedances can be transformed into a Δ -connected impedance and vice versa. Therefore, any given three-phase balanced Δ -connected impedance can be converted into a three-phase balanced Y-connected impedance such that the terminal voltage and current behaviour remain unaffected.

The individual branch currents in the branches of delta in the real delta-connected load can be obtained after line quantities have been obtained by using its equivalent Y-connected model. Three-phase symmetry and $1/\sqrt{3}$ factor connecting line currents and phase currents in a delta-connected system can be used for this purpose. The impedance to be used in Y-connected load is $1/3$ times the impedance present in Δ -connected load. See Fig. 8.3-2.

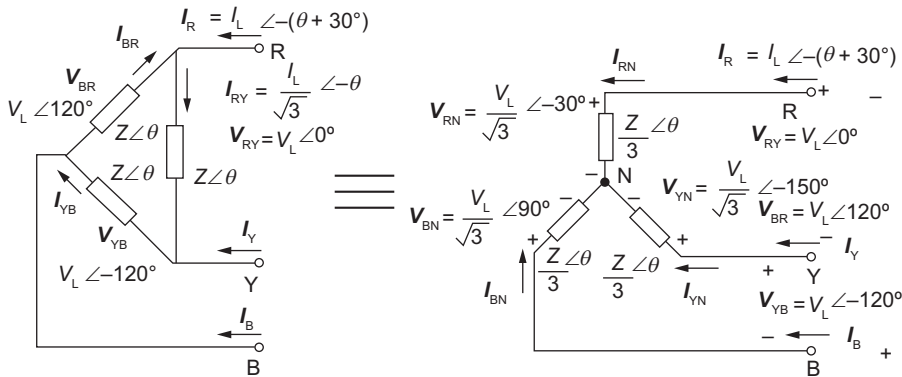


Fig. 8.3-2 Equivalent Y-connected load for a Δ -connected load

8.3.3 The Single-Phase Equivalent Circuit for a Balanced Three-Phase Circuit

Any balanced three-phase circuit can be equivalenced to a Y–Y balanced three-phase circuit by replacing Δ -connected sources and loads (if any) by their Y-connected equivalents developed in the last two subsections. The equivalence is valid as far as line voltages, line currents and complex power flow are concerned.

Thus, analysis of balanced three-phase circuits reduces to a three-step procedure. The Δ -connected sources and loads are replaced by their Y-connected equivalents in the first step. The resulting balanced Y-connected circuit is solved for line voltages and line currents everywhere in the second step. The relevant line voltage values and line current values are used to determine the phase-source voltages and currents in the case of those Δ -connected sources that were replaced by star equivalents earlier and phase-load voltages and currents in the case of those Δ -connected loads that were replaced by star equivalents earlier. This will constitute the third step.

The difference between a real Y-connected source/load and Y-connected equivalent of Δ -connected source/load is that the neutral point of the first is accessible for connection, whereas the neutral point of the second is an inaccessible node. However, the neutral connection in a balanced three-phase circuit does not carry any current (even if the neutral link has non-zero impedance) and hence the voltage difference between source neutral and load neutral will remain zero whether we connect the neutral points together or not. Therefore, the circuit solution in a balanced Y-connected circuit is independent of neutral connection.

If all the neutral points in a balanced three-phase circuit are at the same potential, then, each phase-source appears directly across the net impedance connected from the respective source line terminal (R or Y or B) to the load neutral point. Thus, the circuit may be solved phase by phase independent of each other. But then, why should we solve all the three phases at all? We expect the circuit solution to be balanced set of three-phase phasors in any case. Hence, we can determine the solution for Y-line and B-line if we know the solution for phasors in R-line by using three-phase symmetry. Therefore, we need to determine only the R-line phasors. The equivalent circuit that we employ to solve for R-line phasors is called the *single-phase equivalent circuit for a balanced three-phase circuit*. This equivalent circuit will contain the R-line phase-source in series with all impedances that come between R-line terminal of the source and neutral point of load. The neutral points of source and load are shorted to form the reference node in a single-phase equivalent circuit even if the neutral points are connected through impedance in the actual circuit. This is so since the neutral connection in a balanced circuit will not carry any current.

Note that if a three-phase circuit is star connected everywhere, the use of ‘R-phase’ instead of ‘R-line’ does not create any confusion. Such confusion can arise in the case of delta connection. A delta-connected source has only R-line and no R-phase. However, we have now decided that all our circuits will be converted to star equivalents. Thus, use of ‘R-phase’ instead of R-line is permitted.

Example: 8.3-1

A balanced Y-connected load with phase impedance of $7.9 + j5.7 \, \Omega$ at 50 Hz is supplied from a 400V, 50 Hz balanced Y-connected source through connection impedance of $0.1 + j0.3 \, \Omega$ in each line. (i) Find the line current and load voltage. (ii) Find the active and reactive power delivered by the source and delivered to the load.

Solution

It is a Y–Y connection already. The single-phase equivalent circuit required for solving the problem is shown in Fig. 8.3-3.

A three-phase source is specified by specifying line-to-line voltage rms value. Thus, the source voltage in single-phase equivalent must be $400/\sqrt{3} = 230.9 \, \text{V rms}$ in magnitude. The R-phase source voltage is taken as the reference phasor with an angle of 0° . Solving the circuit in Fig. 8.3-3, we get,

$$I_R = \frac{230.9 \angle 0^\circ}{(0.1 + j0.3) + (7.9 + j5.7)} = \frac{230.9 \angle 0^\circ}{8 + j6} = \frac{230.9 \angle 0^\circ}{10 \angle 36.9^\circ} = 23.09 \angle -36.9^\circ \, \text{A rms.}$$

$$V_L = 23.09 \angle -36.9^\circ \times (7.9 + j5.7) = 23.09 \angle -36.9^\circ \times 9.74 \angle 35.8^\circ = 224.9 \angle -1.1^\circ \, \text{V rms.}$$

Active power delivered by source = $3 \times 230 \times 23.09 \times \cos(0 - (-36.9^\circ)) = 12.75 \, \text{kW}$

Reactive power delivered by source = $3 \times 230 \times 23.09 \times \sin(36.9^\circ) = 9.56 \, \text{kVAr}$

Active power delivered to load = $3 \times 224.9 \times 23.09 \times \cos(-1.1^\circ - (-36.9^\circ)) = 12.64 \, \text{kW}$

Reactive power delivered to load = $3 \times 224.9 \times 23.09 \times \sin(-1.1^\circ - (-36.9^\circ)) = 9.11 \, \text{kVAr}$

Magnitude of line voltage across load = $\sqrt{3} \times 224.9 = 389.56 \, \text{V rms}$

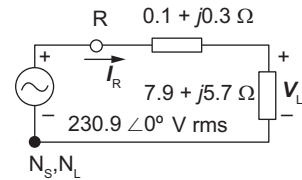


Fig. 8.3-3 Single-phase equivalent for Example 8.3-1

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Complete Phasor Solution

Phase voltages at source: $230\angle 0^\circ$, $230\angle -120^\circ$, $230\angle 120^\circ$ (V rms)

The first line voltage, i.e., V_{RY} , is known to lead the first phase voltage, i.e., V_{RN} , by 30° . Therefore, line voltages at source terminals: $400\angle 30^\circ$, $400\angle -90^\circ$, $400\angle 150^\circ$ (V rms)

Phase voltages at load: $224.9\angle -1.1^\circ$, $230\angle -121.1^\circ$, $230\angle 118.9^\circ$ (V rms)

Line voltages at load terminals: $389.56\angle 28.9^\circ$, $389.56\angle -91.1^\circ$, $389.56\angle 148.9^\circ$ (V rms)

Line currents: $23.09\angle -36.9^\circ$, $23.09\angle -156.9^\circ$, $23.09\angle 83.1^\circ$ (A rms)

Complex power delivered by source = $\sqrt{3} \times 400 \times 23.09 \angle (0 - (36.9^\circ)) = 16\angle 36.9^\circ$ kVA

Complex power in load = $\sqrt{3} \times 389.56 \times 23.09 \angle (-1.1 - (36.9^\circ)) = 15.54\angle 35.8^\circ$ kVA

Example: 8.3-2

A balanced Δ -connected load with phase impedance of $23.7 + j17.1 \Omega$ at 50 Hz is supplied from a 400 V, 50 Hz balanced Y-connected source through connection impedance of $0.1 + j0.3 \Omega$ in each line.

(i) Find the line current and load voltage. (ii) Find the load-branch currents. (iii) Find the active and reactive power delivered by the source and delivered to the load.

Solution

The circuit with all the relevant phasors identified is shown in Fig. 8.3-4.

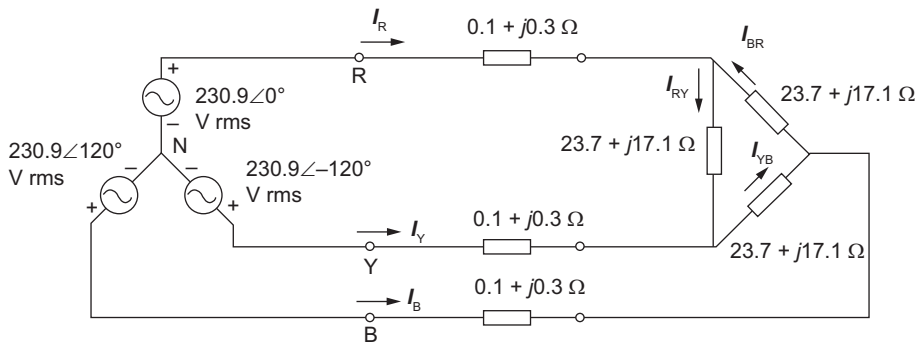


Fig. 8.3-4 A Y-connected source delivering power to a Δ -connected load through connection impedance

The delta-connected impedance is transformed into an Y-connected one by star-delta transformation equations. Since the delta impedances are equal, the impedance in star connection will be one-third of delta branch impedance, i.e., $7.9 + j5.7 \Omega$. The circuit after this transformation is shown in Fig. 8.3-5.

Now, the single-phase equivalent of this circuit can be identified as the same as in Example: 8.3-1 and hence the solution is same as in that example.

$\therefore$ Line current (R) = $23.09\angle -36.9^\circ$ A rms, Load phase voltage (R) = $224.9\angle -1.10$ V rms and Load line voltage (RY) = $389.56\angle 28.9^\circ$ V rms.

Load active power can be found as $3 \times 224.9 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$ or as $\sqrt{3} \times 389.56 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$. The value is 12.64 kW. Similarly, the load reactive power is $3 \times 224.9 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$ or $\sqrt{3} \times 389.56 \times 23.09 \times \cos(36.9^\circ - 1.1^\circ)$. The value is 9.11 kVAR.

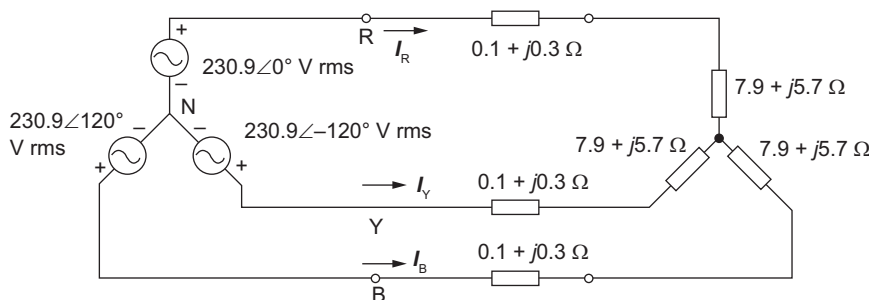


Fig. 8.3-5 The Y-connected equivalent of circuit in Fig. 8.3-4

Branch Currents in Delta-connected Load

Applying KCL at the corners of delta in R-line and Y-line in Fig. 8.3-4, we get

$$I_{RY} - I_{BR} = I_R \text{ and } I_{YB} - I_{RY} = I_Y$$

Multiplying the second equation by -1 and adding the result to the first equation, we get

$$2I_{RY} - (I_{BR} + I_{YB}) = I_R - I_Y$$

But $(I_{RY} + I_{BR} + I_{YB}) = 0$ since we expect these three currents to be a three-phase balanced set. Therefore, $-(I_{BR} + I_{YB}) = I_{RY}$.

Therefore, $2I_{RY} + I_{RY} = I_R - I_Y \Rightarrow I_{RY} = (I_R - I_Y)/3$.

We know that $I_R = 23.09\angle-36.9^\circ$ A rms. I_R , I_Y and I_B form a balanced three-phase set of phasors. Therefore, $I_Y = 23.09\angle-156.9^\circ$ A rms.

$$\therefore 3I_{RY} = (23.09\angle-36.9^\circ - 23.09\angle-156.9^\circ) \text{ A rms.}$$

This is the difference between two phasors with 120° between them. We have determined the result of this kind of difference operation before. The result will have a magnitude that is $\sqrt{3} \times 23.09$ and will lead the first quantity in the difference by 30° .

$$\therefore 3I_{RY} = 40\angle-6.9^\circ \text{ and } I_{RY} = 13.33\angle-6.9^\circ \text{ A rms.}$$

This can also be obtained by observing that RY line voltage at load is $389.56\angle28.9^\circ$ V rms and I_{RY} is the current drawn by $23.7 + j17.1\Omega$ ($29.225\angle35.8^\circ \Omega$) from this voltage.

Therefore, $I_{RY} = 389.56\angle28.9^\circ \div 29.225\angle35.8^\circ = 13.33\angle-6.9^\circ$ A rms as before.

Now, the remaining two branch currents can be obtained by using three-phase symmetry as $I_{YB} = 13.33\angle-126.9^\circ$ A rms and $I_{BR} = 13.33\angle-113.1^\circ$ A rms

Example: 8.3-3

A delta-connected source with an open-circuit voltage of 200 V rms at 50 Hz is used to deliver power to a star-connected load with phase impedance of $5.9 + j7.7 \Omega$. Each phase source in the Δ -connected source has internal impedance of $0.3 + j0.9 \Omega$ in series with it. (i) Find the phase voltage and line voltage across the load. (ii) Find the line currents and delta branch currents. (iii) Find the active power and reactive power delivered to the load and consumed by the internal impedance of the source.

Solution

The three-phase circuit to be analysed is shown in Fig. 8.3-6.

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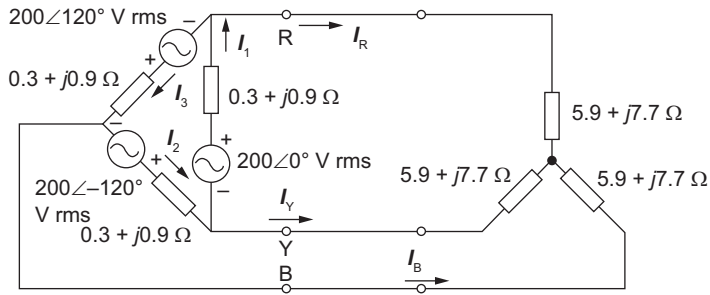


Fig. 8.3-6 A Delta-connected source with source impedance supplying a star-connected load

The open-circuit line voltage across source terminals is given to be 200 V rms. The balanced delta connection shown in Fig. 8.3-6 will not have any current in delta when the terminals are left open since the loop voltage inside delta is zero. Therefore, the source voltages must themselves be 200 V rms. A straightforward determination of Y-equivalent of delta connected source is not possible in this case. We transform the ‘voltage source in series with impedance combinations’ into ‘current source in parallel with impedance combinations’ by using source transformation theorem. This is shown in Fig. 8.3-7.

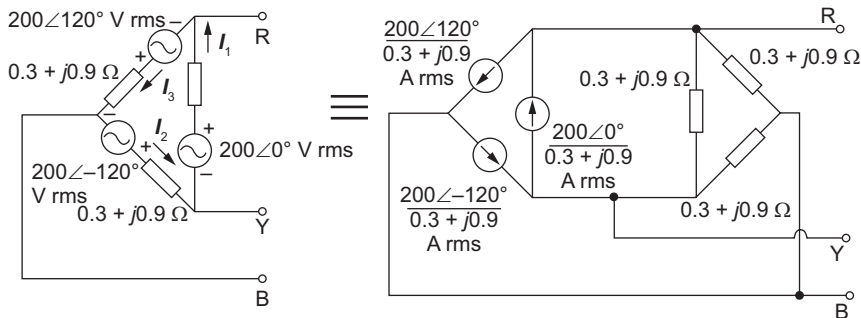


Fig. 8.3-7 Applying source transformation theorem on the delta-connected source in Fig. 8.3-6

We take the R_Y line quantity as the reference phasor in this example. Now, the delta-connected current source has a star equivalent and the delta-connected source impedance also has a star equivalent. Combining the current source in star equivalent and the impedance in star equivalent into a voltage source in series with impedance, we get $(200/\sqrt{3})\angle-30^\circ = 115.5\angle-30^\circ$ V rms in series with $(0.3 + j0.9)/3 = 0.1 + j0.3 \Omega$ in R-phase of the final star equivalent. The complete star equivalent of the source along with the load is shown in Fig. 8.3-8.

The single-phase equivalent of this circuit is shown in Fig. 8.3-9.

- (i) The line current (R) is $115.5\angle-30^\circ \text{ V rms} \div 6 + j8 \Omega = 11.55 \angle -83.13^\circ \text{ A rms}$ and the phase voltage (RN) across load is $11.55\angle-83.13^\circ \text{ A rms} \times (5.9 + j7.7) \Omega = 11.55\angle-83.13^\circ \times 9.7\angle52.54^\circ = 112\angle-30.6^\circ \text{ V rms}$.

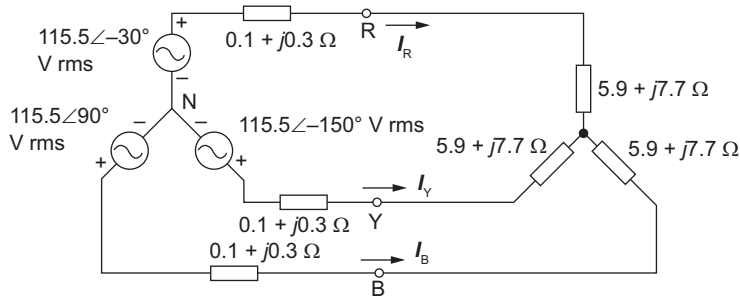


Fig. 8.3-8 The star equivalent of the circuit in Fig. 8.3-6

Hence, line voltage (RY) across the load = $112\sqrt{3}\angle(-30.6^\circ + 30^\circ) = 194\angle-0.6^\circ$ V rms.

- (ii) The line current (R) is $115.5\angle-30^\circ \div 6 + j8 = 11.55\angle-83.13^\circ$ A rms. Refer to Fig. 8.3-6. The current marked as $I_1 = [200\angle 0^\circ - \text{line voltage (RY) across load}] \div (0.3 + j0.9) = [200\angle 0^\circ - 194\angle-0.6^\circ] \div 0.9487\angle 71.57^\circ = (6 + j2) \div 0.9487\angle 71.57^\circ = 6.68\angle-51.13^\circ$ A rms. Then, I_2 will be $6.68\angle-171.13^\circ$ A rms and I_3 will be $6.68\angle 68.87^\circ$ A rms

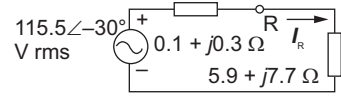


Fig. 8.3-9 Single-phase equivalent of circuit in Fig. 8.3-6

- (iii) The active power delivered to the load = $\sqrt{3} \times 194 \times 11.55 \times \cos(-30^\circ - (-83.13^\circ)) = 2.33$ kW.
The reactive power delivered to load = $\sqrt{3} \times 194 \times 11.55 \times \sin(-30^\circ - (-83.13^\circ)) = 3.11$ kVar.

The internal impedance of the source is $0.3 + j0.9 \Omega$ and it carries an rms current of 6.68 A. Therefore, the active power consumed by the internal impedance of the delta-connected source = $3 \times 6.68^2 \times 0.3 = 40.2$ W. The reactive power consumed by this impedance = $3 \times 6.68^2 \times 0.9 = 120.6$ VAR.

8.4 ANALYSIS OF UNBALANCED THREE-PHASE CIRCUITS

An unbalanced three-phase circuit is one that contains at least one source or load that does not possess three-phase symmetry. A source with the three source-function magnitudes unequal and/or the phase displacements different from 120° can make a circuit unbalanced. Similarly, a three-phase load with unequal phase impedances can make a circuit unbalanced.

The single-phase equivalent circuit technique of analysis *does not* work for unbalanced three-phase circuits. General circuit analysis techniques like mesh analysis or nodal analysis will have to be employed for analysing such circuits.

8.4.1 Unbalanced Y–Y Circuit

Y–Y connection is typically used in the last mile in a power system, *i.e.*, in the low-tension (LT) distribution system.

The primary distribution system usually runs at 11 kV line voltage level. Distribution transformers that are rated for 11 kV/400 V and are Δ -connected in the primary side and Y-connected in the secondary side are used to step down the primary distribution voltage to the LT distribution level. Such transformers are located at the load centre location of the area that a particular transformer is expected to serve. *The neutral point of secondary side of a distribution transformer is usually earthed solidly.*

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4-wire LT lines emanating from the secondary of transformer distribute power to various subareas of the service area. Individual consumers are provided service by means of service drops tapped from these lines. Both single-phase loads and three-phase loads are provided power from these LT lines at various points. Thus, the load on the LT line will be unbalanced in general. Even if the single phase loads are to get distributed equally among three lines (R, Y and B), only the transformer secondary will perceive a balanced load and various sections of the LT line will see varying degree of unbalance in load. This is due to the spatially dispersed nature of loads. Thus, the neutral wire in an LT distribution line will invariably carry current – that too, different currents in different locations – even if the aggregate load is balanced.

The system in which the neutral wire is available for connecting single-phase loads and three-phase loads that require a neutral tie-up is called a *four-wire system*. The system in which neutral wire is *not available* for load connection is called a *three-wire system*. This could be Y-Y system with neutrals isolated, Y- Δ , Δ -Y or Δ - Δ system.

The exposed metal parts of all electrical equipment at the consumer location are tied together and earthed locally. This earth point is made available as the third pin of all three-pin power points installed in the consumer electrical system. If the earthing resistances at the transformer neutral and at the consumer location are negligibly small, then, the earth-pin at all consumer locations will be at the same potential as the transformer secondary neutral. However, the neutral-pin at the consumer locations will differ in potential from the transformer neutral potential due to the neutral wire drop. This difference will be more if there is unbalance in the load and neutral is carrying heavy currents. Moreover, this potential difference between the earth-pin and neutral-pin will be different for different consumers due to spatially distributed nature of line load. This problem of neutral – earth voltage difference is called the *neutral-shift* problem. Electronic equipment in general, and, computing equipment in particular, are sensitive to this neutral-shift voltage and often malfunction when it exceeds certain pre-specified levels. Damage to components can also take place due *differences* in neutral-shift potential that exist among various power points when different components of an interconnected computing system are powered from different power points.

Example: 8.4-1

A balanced three-phase source of 400 V, 50 Hz supplies an unbalanced load through an LT 4-wire line as in Fig. 8.4-1. Each wire in the line has impedance of $0.1 + j0.3 \Omega$. (a) Find (i) phase and line voltages at load location and neutral-shift voltage, (ii) wire currents, (iii) power and reactive power delivered to the load, (iv) three-phase apparent power and power factor. (b) Repeat (a) assuming that the neutral wire is a thick conductor with zero impedance. (c) Repeat (a) assuming that the neutral wire is open.

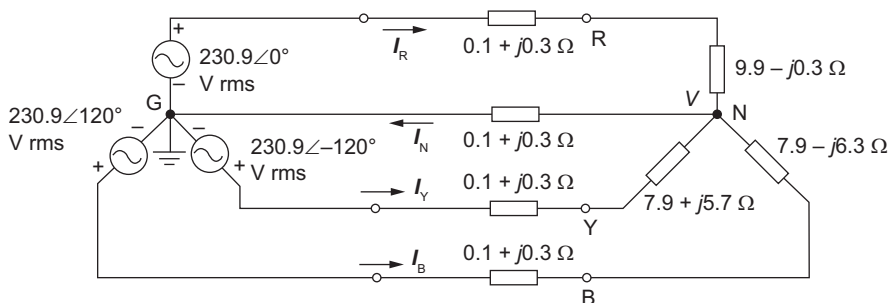


Fig. 8.4-1 Unbalanced three-phase circuit in Example 8.4-1

Solution

(a) We solve the problem by finding out the phasor voltage at load neutral with respect to the ground (earth) node G by using the KCL equation at load neutral. The sum of currents flowing away from that node is set to zero.

$$\frac{V - 230.9\angle 0^\circ}{10} + \frac{V}{0.1 + j0.3} + \frac{V - 230.9\angle -120^\circ}{8 + j6} + \frac{V - 230.9\angle 120^\circ}{8 - j6} = 0,$$

$$\text{i.e., } V \left[\frac{1}{10} + \frac{1}{0.1 + j0.3} + \frac{1}{8 + j6} + \frac{1}{8 - j6} \right] = \frac{230.9\angle 0^\circ}{10} + \frac{230.9\angle -120^\circ}{8 + j6} + \frac{230.9\angle 120^\circ}{8 - j6}.$$

Solving for V , we get, $V = -2.31 - j5.49 = 5.96\angle -112.8^\circ$ V rms.

Now, we can find wire currents as

$$I_R = \frac{230.9\angle 0^\circ - 5.96\angle -112.8^\circ}{10} = 23.33\angle 1.35^\circ \text{ A rms}$$

$$I_Y = \frac{230.9\angle -120^\circ - 5.96\angle -112.8^\circ}{8 + j6} = 22.5\angle -157.06^\circ \text{ A rms}$$

$$I_B = \frac{230.9\angle 120^\circ - 5.96\angle -112.8^\circ}{8 - j6} = 23.45\angle 155.71^\circ \text{ A rms}$$

$$I_N = I_R + I_Y + I_B = 18.83\angle 175.65^\circ \text{ A rms}$$

The load phase voltages can be determined by multiplying line current phasors by phasor impedance of each phase.

$$V_{RN} = 23.33\angle 1.35^\circ \times (9.9 - j0.3) = 231.04\angle -0.39^\circ \text{ V rms}$$

$$V_{YN} = 22.5\angle -157.06^\circ \times (7.9 + j5.7) = 219.18\angle -121.25^\circ \text{ V rms}$$

$$V_{BN} = 23.45\angle -155.71^\circ \times (7.9 + j6.3) = 237\angle -123.65^\circ \text{ V rms}$$

The line voltages across the load can be determined in terms of the phase voltage phasors now.

$$V_{RY} = V_{RN} - V_{YN} = 231.04\angle -0.39^\circ - 219.18\angle -121.25^\circ = 391.63\angle 28.33^\circ \text{ V rms}$$

$$V_{YB} = V_{YN} - V_{BN} = 219.18\angle -121.25^\circ - 237\angle -123.65^\circ = 398.32\angle -90.8^\circ \text{ V rms}$$

$$V_{BR} = V_{BN} - V_{RN} = 237\angle -123.65^\circ - 231.04\angle -0.39^\circ = 400.2\angle 147.94^\circ \text{ V rms}$$

The active power and reactive power delivered to the load is determined by adding up the corresponding power delivered to each phase of the load.

$$P = P_R + P_Y + P_B = 231.04 \times 23.33 \times \cos(-0.39^\circ - 1.35^\circ) \\ + 231.04 \times 22.5 \times \cos(121.25^\circ - (-157.06^\circ)) \\ + 237 \times 23.45 \times \cos(123.65^\circ - (-155.71^\circ)) \\ = 5387.1\text{W} + 3999\text{W} + 4345.9\text{kW}$$

$$Q = Q_R + Q_Y + Q_B = 231.04 \times 23.33 \times \sin(-0.39^\circ - 1.35^\circ) \\ + 231.04 \times 22.5 \times \sin(121.25^\circ - (-157.06^\circ)) \\ + 237 \times 23.45 \times \sin(123.65^\circ - (-155.71^\circ)) \\ = 163.25\text{VAr} + 2885.3\text{VAr} - 3465.7\text{VAr} = -0.744\text{kVAr}$$

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Two different definitions of apparent power and power factor are possible in the case of unbalanced three-phase circuits. The first one is using the total complex power, $\mathcal{S}$, delivered to the load. In this definition, the magnitude of $\mathcal{S}$ is taken as apparent power and ratio of active power to the magnitude of $\mathcal{S}$ is taken as average power factor of the circuit. The apparent power in the load in the circuit in this example then is $(13.73^2 + 0.744^2)^{0.5} = 13.75$ kVA and power factor is 0.9985 lead.

The second definition accepts the sum of apparent powers of load phases as the three-phase apparent power. Power factor is taken as the ratio of active power to apparent power. The apparent power in the load in the circuit in this example then is 15.88 kVA and power factor is 0.865. We cannot specify it as a lead or lag power factor in this case.

The second definition is more appropriate in the sense that power factor and apparent power are expected to give us an indication as to the utilisation efficiency of current in carrying active power. The first method of calculating apparent power and power factor will hide the fact that underutilization of current is taking place when one phase takes leading reactive power and another phase takes lagging reactive power. The phases may consume large reactive power and yet the complex power may come out with only real part or with small imaginary part. Therefore, the definition of apparent power and power factor based on apparent power of individual phases of load is a more meaningful one.

(b) The neutral-shift voltage in this case is zero since the source neutral and load neutral are tied together by a zero impedance link. The circuit becomes a collection of three single-phase circuits sharing a common point at neutral. Therefore, the line currents can be obtained as

$$\begin{aligned} I_R &= \frac{230.9 \angle 0^\circ}{10} = 23.09 \angle 0^\circ \text{ A rms} \\ I_Y &= \frac{230.9 \angle -120^\circ}{8 + j6} = 23.09 \angle -156.87^\circ \text{ A rms} \\ I_B &= \frac{230.9 \angle 120^\circ}{8 - j6} = 23.09 \angle 156.87^\circ \text{ A rms} \\ I_N &= I_R + I_Y + I_B = -19.38 \text{ A rms} \end{aligned}$$

The load phase voltages can be determined by multiplying line current phasors by phasor impedance of each phase.

$$\begin{aligned} V_{RN} &= 23.09 \angle 0^\circ \times (9.9 - j0.3) = 228.7 \angle -1.74^\circ \text{ V rms} \\ V_{YN} &= 23.09 \angle -156.87^\circ \times (7.9 + j5.7) = 224.93 \angle -121.06^\circ \text{ V rms} \\ V_{BN} &= 23.09 \angle -156.87^\circ \times (7.9 - j6.3) = 233.31 \angle 118.3^\circ \text{ V rms} \end{aligned}$$

The line voltages across the load can be determined in terms of the phase voltage phasors now.

$$\begin{aligned} V_{RY} &= V_{RN} - V_{YN} = 228.7 \angle -1.74^\circ - 224.93 \angle -121.06^\circ = 391.51 \angle 28.32^\circ \text{ V rms} \\ V_{YB} &= V_{YN} - V_{BN} = 224.93 \angle -121.06^\circ - 233.31 \angle 118.3^\circ = 398.15 \angle -90.8^\circ \text{ V rms} \\ V_{BR} &= V_{BN} - V_{RN} = 233.31 \angle 118.3^\circ - 228.7 \angle -1.74^\circ = 400.2 \angle 147.95^\circ \text{ V rms} \\ V_{RY} &= V_{YB} - V_{BR} = 391.51 \angle 28.32^\circ + 398.15 \angle -90.8^\circ + 400.2 \angle 147.95^\circ \\ &= 344.65 + j185.73 - 5.56 - j398.11 - 339.204 + j212.37 \\ &\approx 0 \text{ within round-off error} \end{aligned}$$

The active power and reactive power delivered to the load are determined by adding up the corresponding power delivered to each phase of the load.

$$\begin{aligned}
 P &= P_R + P_Y + P_B = 231.04 \times 23.33 \times \cos(0^\circ - 1.74^\circ) \\
 &\quad + 231.04 \times 22.5 \times \cos(-121.06^\circ - (-156.87^\circ)) \\
 &\quad + 237 \times 23.45 \times \cos(118.3^\circ - 156.87^\circ) \\
 &= 5287.2\text{W} + 4211.9\text{W} + 4211.9\text{W} = 13.7\text{ kW} \\
 Q &= Q_R + Q_Y + Q_B = 231.04 \times 23.33 \times \sin(0^\circ - 1.74^\circ) \\
 &\quad + 231.04 \times 22.5 \times \sin(-121.06^\circ - (-156.87^\circ)) \\
 &\quad + 237 \times 23.45 \times \sin(118.3^\circ - 156.87^\circ) \\
 &= 159.94\text{VAr} + 3038.9\text{VAr} - 3358.8\text{VAr} = -0.48\text{kVAr}
 \end{aligned}$$

The three-phase apparent power is 15.86 kVA and average power factor is 0.864.

(c) The neutral points are unconnected and hence the system is a three-wire system in this case. The neutral-shift voltage V is found out from the node equation at the load neutral.

$$\begin{aligned}
 \frac{V - 230.9\angle 0^\circ}{10} + \frac{V - 230.9\angle -120^\circ}{8 + j6} + \frac{V - 230.9\angle 120^\circ}{8 - j6} &= 0 \\
 \text{i.e., } V \left[\frac{1}{10} + \frac{1}{8 + j6} + \frac{1}{8 - j6} \right] &= \frac{230.9\angle 0^\circ}{10} + \frac{230.9\angle -120^\circ}{8 + j6} + \frac{230.9\angle 120^\circ}{8 - j6}
 \end{aligned}$$

Solving for V , we get, $V = -74.55\text{ V rms}$.

Observe the unacceptably large neutral-shift voltage when the source neutral and load neutral are isolated.

Now, we can find wire currents as

$$\begin{aligned}
 I_R &= \frac{230.9\angle 0^\circ + 74.55}{10} = 30.545\angle 0^\circ\text{ A rms} \\
 I_Y &= \frac{230.9\angle -120^\circ + 74.55}{10} = 20.414\angle -138.43^\circ\text{ A rms} \\
 I_B &= \frac{230.9\angle 120^\circ + 74.55}{8 - j6} = 20.414\angle 138.43^\circ\text{ A rms.} \\
 I_N &= I_R + I_Y + I_B = 0\text{ A rms.}
 \end{aligned}$$

The load phase voltages can be determined by multiplying line current phasors by phasor impedance of each phase.

$$\begin{aligned}
 V_{RN} &= 30.545\angle 0^\circ \times (9.9 - j0.3) = 302.53\angle -1.74^\circ\text{ V rms} \\
 V_{YN} &= 20.414\angle -138.43^\circ \times (7.9 + j5.7) = 198.87\angle -102.62^\circ\text{ V rms} \\
 V_{BN} &= 20.414\angle 138.43^\circ \times (7.9 - j6.3) = 206.27\angle 99.86^\circ\text{ V rms}
 \end{aligned}$$

Observe the over-voltage in R-phase. The line voltages across the load can be determined in terms of the phase voltage phasors now.

$$\begin{aligned}
 V_{RY} &= V_{RN} - V_{YN} = 302.53\angle -1.74^\circ - 198.87\angle -102.62^\circ = 392.16\angle 28.13^\circ\text{ V rms} \\
 V_{YB} &= V_{YN} - V_{BN} = 198.87\angle -102.62^\circ - 206.27\angle 99.86^\circ = 397.37\angle -91.2^\circ\text{ V rms} \\
 V_{BR} &= V_{BN} - V_{RN} = 206.27\angle 99.86^\circ - 302.53\angle -1.74^\circ = 398.94\angle 147.83^\circ\text{ V rms}
 \end{aligned}$$

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The active power and reactive power delivered to the load are determined by adding up the corresponding power delivered to each phase of the load.

$$P = P_R + P_Y + P_B = 15.82 \text{ kW}$$
$$Q = Q_R + Q_Y + Q_B = -0.53 \text{ kVAR}$$

Apparent power is 17.11 kVA and average power factor is 0.925.

Note that, in all the cases, the three line voltages add up to zero. The sum of line voltages in any three-phase system is zero since these three voltages are voltages in a loop formed by R to Y, Y to B and B to R.

This example demonstrated the need for a strong neutral tie in unbalanced four-wire circuits to maintain the phase voltages at load terminal at near-nominal values.

8.4.2 Circulating Current in Unbalanced Delta-connected Sources

A delta-connected source comprises three sinusoidal sources in a closed loop. If the sources are *ideal independent voltage sources* they *have to* add up to zero at all instants. Otherwise, there will be infinitely large circulating current in the closed loop. The sources will have some non-zero impedance in practice and this source impedance will limit the circulating current arising out of residual voltage in the loop. However, if the source impedance is small (and, it should be small for a good source), the circulating current will consume the current carrying capacity of the source and leave only a portion of its capacity to deliver useful power to the external load. Moreover, large circulating current in a delta loop produces large dissipation within the sources.

Therefore, the phase-source voltage phasors in a delta-connected source should add up to zero preferably. This happens naturally in the case of a balanced source. But it need not happen always in the case of unbalanced source though it can for certain combinations of magnitudes and phase angles. Essentially, the magnitudes and phase angles of the three phase-sources must be such that the phasors form a closed triangle in complex plane. This is difficult to arrange in a synchronous generator. That is why generators are always Y-connected.

It is rarely that we encounter a delta-connected source. One situation where it can appear is when a transformer secondary is delta-connected. We can use a delta-connected voltage source, with each arm of delta containing a source in series with the transformer series impedance, as a circuit model for the secondary of the transformer. Exact symmetry in a three-phase equipment is hard to achieve in practice. For instance, there can be small differences in the number of turns employed in the three phase-windings of the transformer. This may produce unbalanced voltages in the three secondary windings. Further, those voltage phasors may not ‘close’ into a triangle. Circulating current in the secondary winding of the transformer is the result.

The next example illustrates this problem as well as the application of mesh analysis and node analysis in solving unbalanced three-phase circuits.

Example: 8.4-2

A delta-connected unbalanced source delivers power to a balanced Y-connected resistive load as shown in Fig. 8.4-2. Solve the circuit completely by using mesh analysis.

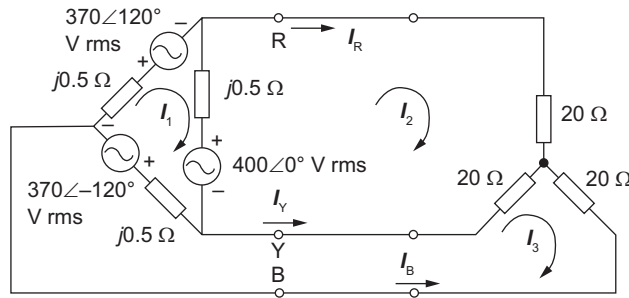


Fig. 8.4-2 Unbalanced delta-connected source supplying a Y-connected load in Example 8.4-2

Solution

The mesh currents are assigned as shown in Fig. 8.4-2. The mesh equations are written in matrix form by inspection.

$$\begin{bmatrix} j1.5 & -j0.5 & -j0.5 \\ -j0.5 & 40 + j0.5 & -20 \\ -j0.5 & -20 & 40 + j0.5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -(400\angle 0^\circ + 370\angle -120^\circ + 370\angle 120^\circ) \\ 400\angle 0^\circ \\ 370\angle -120^\circ \end{bmatrix} = \begin{bmatrix} -30 \\ 400\angle 0^\circ \\ 370\angle -120^\circ \end{bmatrix}$$

Solving for I_1 , I_2 and I_3 ,

$I_1 = 14.98\angle 77.64^\circ$ A rms, $I_2 = 11.12\angle -29.2^\circ$ A rms and $I_3 = 10.68\angle -90.48^\circ$ A rms.

Now, the line currents are $I_R = I_2 = 11.12\angle -29.2^\circ$ A rms, $I_Y = I_3 - I_2 = 11.116\angle -151.77^\circ$ A rms and $I_B = -I_3 = 10.68\angle 89.52^\circ$ A rms.

Current delivered by $400\angle 0^\circ$ V rms source $= I_2 - I_1 = 21.08\angle -72.04^\circ$ A rms

Current delivered by $370\angle -120^\circ$ V rms source $= I_3 - I_1 = 25.53\angle -97.42^\circ$ A rms

Current delivered by $370\angle 120^\circ$ V rms source $= -I_1 = 14.98\angle -102.36^\circ$ A rms

Power delivered to load $= [11.12^2 + 11.12^2 + 10.68^2] \times 20 = 7227.4$ W

Power delivered by $400\angle 0^\circ$ V rms source $= 400 \times 21.08 \times \cos(72.04^\circ) = 2600.3$ W

Power delivered by $370\angle -120^\circ$ V rms source $= 370 \times 25.53 \times \cos(-120^\circ + 97.42^\circ) = 8772$ W

Power delivered by $370\angle 120^\circ$ V rms source $= 370 \times 14.98 \times \cos(120^\circ + 102.36^\circ) = -4095.57$ W

Notice the marked deviation in phase-source currents from the levels we expect from the observed line currents. We would have expected about 6–7 A rms in the sources considering the fact that the line currents are around 11 A rms. The circulating current due to the residual delta-loop voltage of 30 V rms makes the delta currents much larger than what they need to be in view of load power. This results in excess power being delivered by some branches, forcing the remaining branch to absorb power rather than deliver it. This is why the $370\angle 120^\circ$ V rms source had to absorb a large quantity of power.

Of course, the delta-connected source used in this example can be of no practical use. A delta-connected source is of practical utility only if the residual voltage within the delta-loop can be kept at zero or a value quite low compared to source voltage.

The line voltages appearing across the load can be calculated by subtracting the voltage drop in the source impedance of each phase-source from its source voltage value. The result of this calculation is $V_{RY} = 390\angle -0.48^\circ$ V rms, $V_{YB} = 375.3\angle -121.8^\circ$ V rms and $V_{BR} = 375.3\angle -120.83^\circ$ V rms.

8.5 SYMMETRICAL COMPONENTS

Analysis of balanced three-phase circuits is considerably simpler than analysis of unbalanced three-phase circuits. Three-phase symmetry exhibited by a balanced circuit makes it possible to solve the circuit employing a single-phase equivalent circuit. This prompts us to ask the question – is it possible to bring back the symmetry enjoyed by a balanced circuit into an unbalanced three-phase circuit by some means?

We realise that an unbalanced three-phase circuit can result from unbalanced three-phase sources acting on balanced loads or balanced sources acting on unbalanced loads or unbalanced three-phase sources acting on unbalanced loads. Let us tackle the first case – unbalanced three-phase sources acting on balanced loads.

8.5.1 Three-Phase Circuits with Unbalanced Sources and Balanced Loads

In this case, maybe we can bring three-phase symmetry back into the circuit if we can somehow express the unbalanced three-phase source voltages/currents as a *superposition of balanced sets of three-phase voltages/currents*. Is that possible?

Specialised version of a general theorem called *Fortesque's Theorem* assures us that it is possible. If a *set* of unbalanced three-phase source functions can be expressed as a sum of balanced *sets* of three-phase source functions, then, the solution of unbalanced three-phase circuit can be obtained as a *superposition* of solution of the circuit for various balanced sets of three-phase source functions. If all the loads are balanced, each of the circuit problems that needs to be solved for applying superposition principle, will be a balanced circuit problem. Thus, symmetry can be restored to unbalanced three-phase circuits this way, *provided the unbalance is only due to sources and not due to loads*. The sets of three-phase balanced source components and possible single-phase components of an unbalanced source are called its *Symmetrical Components*.

There are three symmetrical components for an unbalanced three-phase source function. *Each symmetrical component is a set of three source functions*. The first set – called the *positive sequence component* – is a balanced three-phase set of source functions that has positive phase sequence. The second set – called the *negative sequence component* – is a balanced three-phase set of source functions that has negative phase sequence. The third set – called the *zero sequence component* – is a set of three cophasal (*i.e.*, of same phase) single-phase source functions. *It is not a three-phase set at all*.

Symmetrical components are denoted by the first phasor element in each set. Thus, positive sequence component is denoted by R-phase quantity or R-line quantity of the balanced three-phase source function of positive phase sequence in phasor form. Similarly, negative sequence component is denoted by R-phase quantity or R-line quantity of the balanced three-phase source function of negative phase sequence in phasor form. Further, zero sequence component is denoted by one of the three cophasal single-phase source functions.

For instance, let $200\angle-10^\circ$, $100\angle-50^\circ$ and $25\angle-30^\circ$ be the rms values of positive, negative and zero sequence components of some unbalanced phase voltage set. Then, $200\angle-10^\circ$ stands for a three-phase balanced voltage set ($200\angle-10^\circ$, $200\angle-130^\circ$, $200\angle110^\circ$) in *RN*, *YN* and *BN* phase voltages, respectively. The $100\angle-50^\circ$ negative sequence component stands for ($100\angle-50^\circ$, $100\angle70^\circ$, $100\angle-170^\circ$) in *RN*, *YN* and *BN* phase voltages, respectively. Note the phase sequence of the bracketed quantity. The zero sequence component of $25\angle-30^\circ$ stands for ($25\angle-30^\circ$, $25\angle-30^\circ$, $25\angle-30^\circ$) in *RN*, *YN* and *BN* phase

voltages, respectively. Then, by *Fortesque's Symmetrical Components Theorem*, the phase voltages are given by

$$V_{RN} = 200\angle -10^\circ + 100\angle -50^\circ + 25\angle -30^\circ = 308.8\angle -23.64^\circ \text{ V rms}$$

$$V_{YN} = 200\angle -130^\circ + 100\angle 70^\circ + 25\angle -30^\circ = 109.4\angle -131.7^\circ \text{ V rms}$$

$$V_{BN} = 200\angle 110^\circ + 100\angle -170^\circ + 25\angle -30^\circ = 214.66\angle 132.6^\circ \text{ V rms}$$

The phase voltage set is an unbalanced one and we have expressed it as a sum of three components – one balanced positive sequence three-phase voltage, one balanced negative sequence three-phase voltage and one single-phase voltage.

We could have expressed these equations in another format too.

$$V_{RN} = 1 \times 200\angle -10^\circ + 1 \times 100\angle -50^\circ + 1 \times 25\angle -30^\circ$$

$$V_{YN} = 1\angle -120^\circ \times 200\angle -10^\circ + 1\angle 120^\circ \times 100\angle -50^\circ + 1 \times 25\angle -30^\circ$$

$$V_{BN} = 1\angle 120^\circ \times 200\angle -10^\circ + 1\angle -120^\circ \times 100\angle -50^\circ + 1 \times 25\angle -30^\circ$$

Let us define a constant complex number a as $a = 1\angle 120^\circ$. When a phasor is multiplied by this operator, its magnitude stays unaffected, but it gets rotated in counter clockwise direction by 120° . Then, $a^2 = 1\angle 240^\circ = 1\angle -120^\circ$ is the operator that can rotate a phasor by 120° in clockwise direction. Note that $1 + a + a^2 = 0$, $a^2 = a^*$, $a^3 = 1\angle 0^\circ$ and $a^4 = a$. Now, the above equations can be expressed in terms of this operator a as

$$V_{RN} = 1 \times 200\angle -10^\circ + 1 \times 100\angle -50^\circ + 1 \times 25\angle -30^\circ$$

$$V_{YN} = a^2 \times 200\angle -10^\circ + a \times 100\angle -50^\circ + 1 \times 25\angle -30^\circ$$

$$V_{BN} = a \times 200\angle -10^\circ + a^2 \times 100\angle -50^\circ + 1 \times 25\angle -30^\circ$$

Now, let us introduce X_+ , X_- and X_0 as the values of positive, negative and zero sequence components (note that they are phasors) of some voltage or current instead of the three numbers – $200\angle -10^\circ$, $100\angle -50^\circ$ and $25\angle -30^\circ$ – we used till now. Then, the unbalanced quantities are given in terms of symmetrical components (*i.e.*, X_+ , X_- and X_0) by

$$X_R = 1 \times X_+ + 1 \times X_- + 1 \times X_0$$

$$X_Y = a^2 \times X_+ + a \times X_- + 1 \times X_0$$

$$X_B = a \times X_+ + a^2 \times X_- + 1 \times X_0$$

We can express this equation set in matrix form as below.

$$\begin{bmatrix} X_R \\ X_Y \\ X_B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_+ \\ X_- \end{bmatrix} \quad (8.5-1)$$

X_R , X_Y and X_B can represent any phase voltage, phase current, line voltage or line current phasors in an unbalanced system. The 3×3 matrix in Eqn. 8.5-1 is called the *Symmetric Transformation Matrix*. The inverse equation required for determining sequence components from phase quantities is as below.

$$\begin{bmatrix} X_0 \\ X_+ \\ X_- \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} X_R \\ X_Y \\ X_B \end{bmatrix} \quad (8.5-2)$$

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The reader is urged to verify the matrix inversion involved.

The circuit interpretation of symmetrical components is given in Fig. 8.5-1. Here, an unbalanced star-connected voltage source is resolved into its symmetrical components. Symmetrical Components theorem assures us that the source in Fig. 8.5-1(a) can be viewed as the composite source in Fig. 8.5-1(b) in which each source limb contains three sources in series – one each from each sequence component set. Further, the superposition principle assures us that applying the source in Fig. 8.5-1(b) is the same as applying the three source sets shown in Fig. 8.5-1(c) individually.

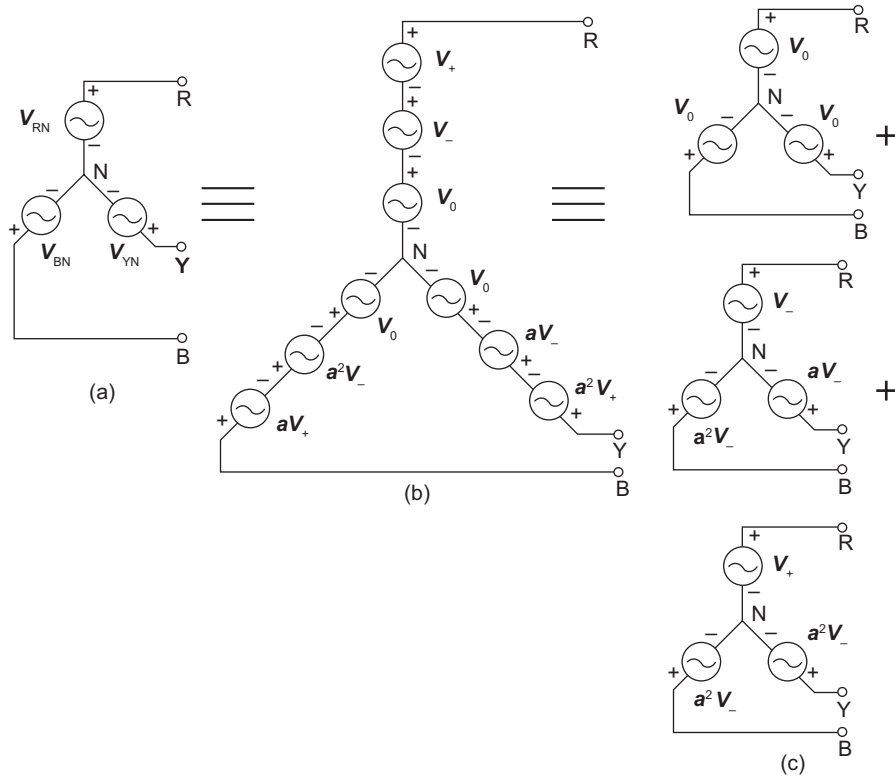


Fig. 8.5-1 Interpretation of symmetrical components

8.5.2 The Zero Sequence Component

This component needs special attention. We get an expression for zero sequence component from Eqn. 8.5-2 as $X_0 = \frac{1}{3}(X_R + X_Y + X_B)$.

Zero sequence component is the average of the three three-phase quantities.

Line voltages in any three-phase circuit, balanced or unbalanced, will add up to zero by KVL. Therefore, line voltages in a three-phase system cannot have a zero sequence content anywhere in the system.

Line currents in a three-phase three-wire system will have to add up to zero by KCL. *Therefore, line currents in a three-phase three-wire system cannot have a zero sequence content anywhere in the system.*

Phase voltages in a three-phase system *need not* add up to zero. *Hence, phase voltages can have zero sequence content.*

Line currents in a four-wire system *need not* add up to zero. The sum $I_R + I_Y + I_B$ can flow through the fourth wire (neutral wire) in the return direction. *Therefore, three-phase four-wire systems can have zero sequence content in their line currents.*

8.5.3 Active Power in Sequence Components

Let the phase voltages across a *balanced three-phase load circuit* be V_{RN} , V_{YN} and V_{BN} where N is the neutral point in the load itself or in its Y-equivalent. Let I_R , I_Y and I_B be the line currents flowing into the load. Further, let V_0 , V_+ and V_- be the sequence components of phase voltages and I_0 , I_+ and I_- be the sequence components of line currents.

Then, the total active power that flows into the load,

$$\begin{aligned} P &= \text{Re}[V_{RN} I_R^*] + \text{Re}[V_{YN} I_Y^*] + \text{Re}[V_{BN} I_B^*] \\ &= \text{Re}[V_{RN} I_R^* + V_{YN} I_Y^* + V_{BN} I_B^*] \end{aligned}$$

Let us define four column vectors as below.

$$V_{\text{ryb}} = \begin{bmatrix} V_{RN} \\ V_{YN} \\ V_{BN} \end{bmatrix}; \quad I_{\text{ryb}} = \begin{bmatrix} I_R \\ I_Y \\ I_B \end{bmatrix}; \quad V_{0+-} = \begin{bmatrix} V_0 \\ V_+ \\ V_- \end{bmatrix}; \quad I_{0+-} = \begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix}$$

Then, the power equation can be expressed as $P = \text{Re}[V_{\text{ryb}}^t I_{\text{ryb}}^*]$ where the superscripts t indicates matrix transpose operation and $*$ indicates complex conjugation operation.

Eqn. 8.5-1 expresses the three-phase quantities in terms of its sequence components. It is reproduced below.

$$\begin{bmatrix} X_R \\ X_Y \\ X_B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_+ \\ X_- \end{bmatrix}$$

Using this equation, we write the column vector V_{ryb} in terms of the column vector V_{0+-} and write the column vector I_{ryb} in terms of the column vector I_{0+-} as below.

$$V_{\text{ryb}} = A V_{0+-} \quad \text{and} \quad I_{\text{ryb}} = A I_{0+-}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

Then,

$$\begin{aligned} P &= \text{Re}[V_{\text{ryb}}^t I_{\text{ryb}}^*] = \text{Re}[(A V_{0+-})^t (A I_{0+-})^*] = \text{Re}[V_{0+-}^t A^t A^* I_{0+-}^*] \\ A^t &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & (a^2)^* & a^* \\ 1 & a^* & (a^2)^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \end{aligned}$$

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Therefore, $\mathbf{A}\mathbf{A}^*$ is three times identity matrix of 3×3 order. Hence,

$$\begin{aligned} P &= 3 \operatorname{Re}[\mathbf{V}_{0+-} \mathbf{I}_{0+-}^*] = 3 \operatorname{Re}[\mathbf{V}_0 \mathbf{I}_0^* + \mathbf{V}_+ \mathbf{I}_+^* + \mathbf{V}_- \mathbf{I}_-^*] \\ &= 3 \operatorname{Re}[\mathbf{V}_0 \mathbf{I}_0^*] + 3 \operatorname{Re}[\mathbf{V}_+ \mathbf{I}_+^*] + 3 \operatorname{Re}[\mathbf{V}_- \mathbf{I}_-^*] \end{aligned} \quad (8.5-3)$$

Thus, we observe that (i) active power carried by sequence components obey superposition principle and that (ii) positive sequence component of voltage can send active power *only through* positive sequence component of current; negative sequence component of voltage can send active power *only through* negative sequence component of current and zero sequence component of voltage can send active power *only through* zero sequence component of current.

Balanced impedance will have negative sequence current in it only if there is negative sequence voltage applied to it. This is so since balanced impedance can produce only negative sequence voltage drop when a negative sequence current flows through it.

That is, balanced impedance is incapable of bringing about a sequence conversion involving translation of positive sequence current into negative sequence voltage and vice versa. This is called sequence-decoupling property of balanced three-phase impedances.

8.5.4 Three-Phase Circuits with Balanced Sources and Unbalanced Loads

Three-phase sources, at least at generation and transmission level, are more or less balanced. They may show considerable unbalance at LT distribution level. However, three-phase circuit unbalance is usually the result of unbalanced loads rather than unbalanced sources. Symmetrical components can be used for analysing such circuits too. However, we do not take up such analysis since it is somewhat beyond the scope of an introductory text like this one. Hence, we close our discussion on symmetrical components after covering two salient points in this subsection.

The first point is that an unbalanced load does not possess *sequence-decoupling property*. Rather, it *creates sequence coupling*. Consider a balanced three-phase voltage source supplying an unbalanced three-phase load. Since the voltage source is balanced, it has no negative and zero sequence components. It is a purely positive sequence voltage source. However, the load will draw unbalanced currents since it is unbalanced. We know that unbalanced current set will contain negative sequence component. Thus, a positive sequence voltage source has produced a negative sequence current in the circuit. Moreover, the voltage drop produced across the load impedance due to the negative sequence current must be a positive sequence voltage since there is only positive sequence component in the source voltage to meet the KVL requirement. Therefore, negative sequence current succeeds in producing a positive sequence voltage drop in the case of an unbalanced load. Therefore, *an unbalanced load produces sequence coupling between positive sequence components and negative sequence components*.

The second observation we want to make is regarding active power. We observed in the last subsection that a negative sequence current flowing through a positive sequence voltage could not carry active power. Hence, the negative sequence current drawn by an unbalanced three-phase load from a balanced three-phase source is a wasteful current that ties up equipment capacity and increases system losses without carrying productive power. We had met with another such current before – the reactive current component. Thus, both the reactive component of positive sequence current and the entire negative sequence current drawn by an unbalanced three-phase load from a balanced supply are unproductive in effect and detrimental to the system.

The most energy-efficient way to draw active power from a balanced three-phase source is to draw it through balanced three-phase currents at unity power factor.

Symmetrical components afford us the required information to evaluate the effectiveness of current in carrying power in unbalanced loading of the system equipment. We can solve the unbalanced circuit problem by applying mesh or nodal analysis and obtain the symmetrical components of currents to evaluate how bad the current utilisation is.

Thus, with this introductory coverage on symmetrical components, we can now apply sequence components (i) to solve circuits with *unbalanced sources* and *balanced loads* and (ii) to evaluate the effectiveness of current flow in carrying active power in circuits with *balanced sources* and *unbalanced loads*. There is much more to symmetrical components. Power System Engineers go deep into symmetrical components in analysing power systems under fault condition.

Example: 8.5-1

An unbalanced 4-wire system is shown in Fig. 8.5-2. (i) Find the symmetrical components of the source phase voltages. (ii) Determine the line voltages at the source and verify that line voltage does not contain zero sequence component. (iii) Determine the line currents and neutral current by using symmetrical components. (iv) Find the load neutral voltage with respect to earth. (v) Find the phase voltages and line voltages across the load by symmetrical components. (vi) Find the total power delivered by source and delivered to load using symmetrical components.

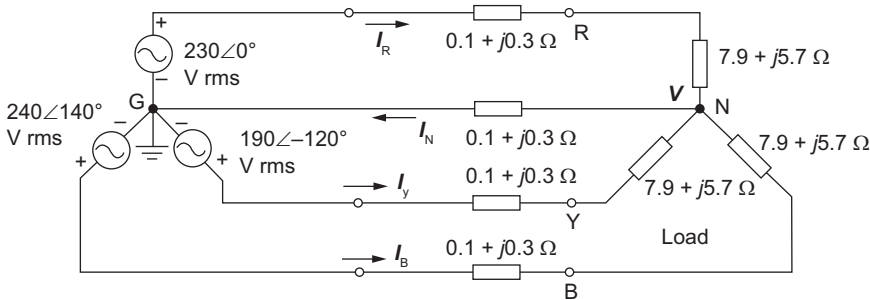


Fig. 8.5-2 Unbalanced three-phase circuit for Example 8.5-1

Solution

$$(i) \begin{bmatrix} V_{s0} \\ V_{s+} \\ V_{s-} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} V_{sRG} \\ V_{sYG} \\ V_{sBG} \end{bmatrix}. \text{ The subscript 's' stands for source quantities. Substituting } a =$$

$$1\angle120^\circ, V_{sRN} = 230\angle0^\circ, V_{sYN} = 190\angle-120^\circ \text{ and } V_{sBN} = 240\angle140^\circ,$$

$$\begin{bmatrix} V_{s0} \\ V_{s+} \\ V_{s-} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1\angle120^\circ & 1\angle-120^\circ \\ 1 & 1\angle-120^\circ & 1\angle120^\circ \end{bmatrix} \begin{bmatrix} 230\angle0^\circ \\ 190\angle-120^\circ \\ 240\angle140^\circ \end{bmatrix} = \begin{bmatrix} 16.64\angle-168.12^\circ \\ 216.91\angle7.25^\circ \\ 39.25\angle-37.58^\circ \end{bmatrix} \text{ V rms.}$$

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(ii) Line voltages are obtained as below.

$$V_{sRY} = V_{sRG} - V_{sYG} = 230\angle 0^\circ - 190\angle -120^\circ = 325 + j164.54 = 364.28\angle 26.85^\circ \text{ Vrms}$$

$$V_{sYB} = V_{sYG} - V_{sBG} = 190\angle -120^\circ - 240\angle 140^\circ = 88.85 - j318.81 = 330.96\angle -74.43^\circ \text{ Vrms}$$

$$V_{sBR} = V_{sBG} - V_{sRG} = 240\angle 140^\circ - 230\angle 0^\circ = -413.85 + j154.27 = 441.67\angle 159.56^\circ \text{ Vrms}$$

$$V_{sRY} + V_{sYB} + V_{sBR} = (325 + j164.54) + (88.85 - j318.81) + (-413.85 + j154.27) = 0 + j0$$

Therefore, zero sequence component in line voltage is zero.

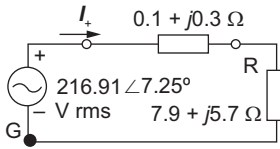


Fig. 8.5-3 Single-phase equivalent circuit of the circuit in Fig. 8.5-2 for positive sequence component

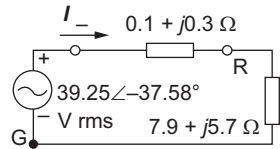


Fig. 8.5-4 Single-phase equivalent circuit of the circuit in Fig. 8.5-2 for negative sequence component

(iii) The circuit is solved for the three symmetrical components by applying superposition principle. We apply the positive sequence voltage source component first and obtain the positive sequence component of line currents and neutral current. We note that the connection impedances and load impedances are balanced. Therefore, we can use single-phase equivalent circuit for obtaining currents. The single-phase equivalent circuit is shown in Fig. 8.5-3. The source neutral and load neutral are at the same potential and hence the neutral impedance of $0.1 + j0.3 \Omega$ will not appear in the single-phase equivalent circuit for positive sequence component.

Therefore, $I_+ = 216.91\angle 7.25^\circ \div (8 + j6) = 21.69\angle -29.62^\circ \text{ A rms}$.

The single-phase equivalent circuit for negative sequence input is shown in Fig. 8.5-4. Passive balanced impedance cannot differentiate between positive phase sequence and negative phase sequence. However, this is not true in the case of all electrical equipment. AC motors can distinguish

between the two and hence the equivalent circuits of motors will be different for the two phase sequences.

The load circuit is balanced. Negative sequence component is a balanced component. Therefore, neutral potential at source and load will be the same and the neutral impedance will not appear in the single-phase equivalent circuit for negative sequence input too.

Therefore, $I_- = 39.25\angle -37.58^\circ \div (8 + j6) = 3.925\angle -74.45^\circ \text{ A rms}$.

The circuit to be solved with zero sequence input is shown in Fig. 8.5-5.

Note that all the three voltage sources have the same phase and hence all the three lines R, Y and B will carry cophasal currents in the same direction. Therefore, the neutral return current will be $3I_0$. Applying KVL in any one mesh, we get,

$$[(8 + j6) + 3 \times (0.1 + j0.3)] \times I_0 = 16.64\angle -168.12^\circ \Rightarrow I_0 = 1.542\angle 152.14^\circ \text{ A rms}$$

Note that the effective value of neutral line impedance in limiting the zero sequence current is three times its actual value due to $3I_0$ flowing in it.

Now, the line currents can be obtained by applying symmetrical component transformation equation.

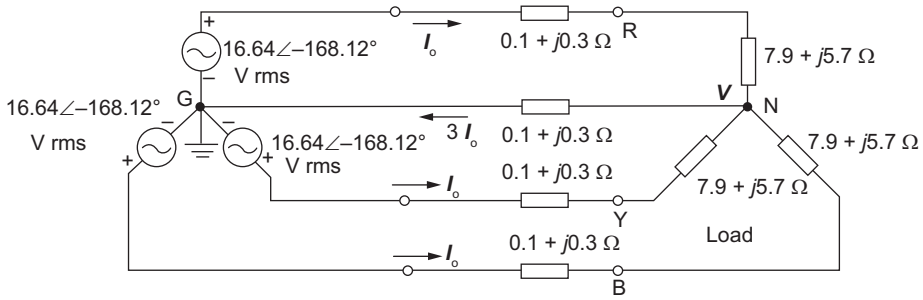


Fig. 8.5-5 Circuit with only zero sequence component of input voltage acting in the circuit in Fig. 8.5-2

$$\begin{bmatrix} I_R \\ I_Y \\ I_B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix}$$

The result will be, $I_R = 23.106\angle-36.62^\circ$ A rms, $I_Y = 18.86\angle-156.75^\circ$ A rms and $I_B = 23.984\angle-102.78^\circ$ A rms.

Neutral current can arise only from zero sequence component of source voltage since the remaining two components are balanced three-phase components and the load circuit is balanced. Hence, neutral current $= 3I_0 = 4.63\angle152.14^\circ$ A rms.

- (iv) Load neutral voltage with respect to earth = neutral current $\times$ neutral impedance, since the source neutral is earthed. The value is $= 4.63\angle152.14^\circ \times (0.1 + j0.3) = 1.463\angle-136.3^\circ$ V rms.
- (v) The symmetrical components of load phase voltages can be obtained as

$$\begin{bmatrix} V_{I0} \\ V_{I+} \\ V_{I-} \end{bmatrix} = \begin{bmatrix} 7.9 + j5.7 & 0 & 0 \\ 0 & 7.9 + j5.7 & 0 \\ 0 & 0 & 7.9 + j5.7 \end{bmatrix} \begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix}$$

The values are $V_{I0} = 15.018\angle-172.05^\circ$ V rms, $V_{I+} = 211.305\angle6.2^\circ$ V rms and $V_{I-} = 38.237\angle-38.64^\circ$ V rms. Now, the load phase voltages can be determined by symmetrical components transformation.

$$\begin{bmatrix} V_{IRN} \\ V_{IYN} \\ V_{IBN} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} V_{I0} \\ V_{I+} \\ V_{I-} \end{bmatrix}$$

The values obtained on substitution of sequence component values are,
 $V_{IRN} = 225.09\angle-0.81^\circ$ V rms, $V_{IYN} = 183.73\angle-120.93^\circ$ V rms and $V_{IBN} = 233.65\angle138.54^\circ$ V rms.

The load line voltages are found as

$$V_{IRY} = V_{IRN} - V_{IYN} = 225.09\angle-0.81^\circ - 183.73\angle-120.93^\circ = 354.87\angle25.8^\circ \text{ V rms}$$

$$V_{IYB} = V_{IYN} - V_{IBN} = 183.73\angle-120.93^\circ - 233.65\angle138.54^\circ = 322.41\angle-75.5^\circ \text{ V rms}$$

$$V_{IBR} = V_{IBN} - V_{IRN} = 233.65\angle138.54^\circ - 225.09\angle-0.81^\circ = 430.26\angle158.5^\circ \text{ V rms}$$

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(vi) The values of sequence components at source end are

$V_{s0} = 16.64 \angle -168.12^\circ$ V rms; $V_{s+} = 216.91 \angle 7.25^\circ$ V rms; $V_{s-} = 39.25 \angle -37.58^\circ$ V rms
and $I_0 = 1.542 \angle 152.14^\circ$ A rms; $I_+ = 21.69 \angle -29.62^\circ$ A rms; $I_- = 3.925 \angle -74.45^\circ$ A rms.

$\therefore$ Power delivered through zero sequence components
 $= 16.64 \times 1.542 \times \cos(-168.12^\circ - 152.14^\circ) = 19.73$ W per phase

$\therefore$ Power delivered through positive sequence components
 $= 216.91 \times 21.69 \times \cos(7.25^\circ + 29.62^\circ) = 3763.9$ W per phase

$\therefore$ Power delivered through negative sequence components
 $= 39.25 \times 3.925 \times \cos(-37.58^\circ + 74.45^\circ) = 123.3$ W per phase

$\therefore$ Total power delivered by the source $= 3 \times (19.73 + 3763.9 + 123.3)$ W $= 11.72$ kW.

The values of sequence components at load end are,

$V_{l0} = 15.018 \angle -172.05^\circ$ V rms, $V_{l+} = 211.305 \angle 6.2^\circ$ V rms and $V_{l-} = 38.237 \angle -38.64^\circ$ V rms
and $I_0 = 1.542 \angle 152.14^\circ$ A rms; $I_+ = 21.69 \angle -29.62^\circ$ A rms; $I_- = 3.925 \angle -74.45^\circ$ A rms.

$\therefore$ Power delivered to the load through zero sequence components
 $= 15.018 \times 1.542 \times \cos(-172.05^\circ - 152.14^\circ) = 18.8$ W per phase

$\therefore$ Power delivered to the load through positive sequence components
 $= 211.305 \times 21.69 \times \cos(6.2^\circ + 29.62^\circ) = 3716.9$ W per phase

$\therefore$ Power delivered to the load through negative sequence components
 $= 38.237 \times 3.925 \times \cos(-38.64^\circ + 74.45^\circ) = 121.7$ W per phase

$\therefore$ Total power delivered to the load $= 3 \times (18.8 + 3716.9 + 121.7)$ W $= 11.572$ kW

Example: 8.5-2

Single-phasing of a three-phase load leads to an extreme case of unbalanced operation in practice. An undetected open in the wiring path in one phase or a fuse blowing in a phase that leads to this kind of operation of three-phase loads in industries. It is so common that specific measures in the form of a single-phasing prevention relay is incorporated in the control gear of high-power induction motor drives in the Industry.

The circuit in Fig. 8.5-6 shows a balanced three-phase induction motor running from a balanced supply. The balanced operation of this circuit was studied in Example: 8.3-2. Assume that the line goes open in B-phase leading to single-phased operation of the motor. Find (i) line currents and sequence components of line currents, (ii) currents in the three motor windings, (iii) active and reactive power delivered to the motor and (iv) apparent power and power factor of the source.

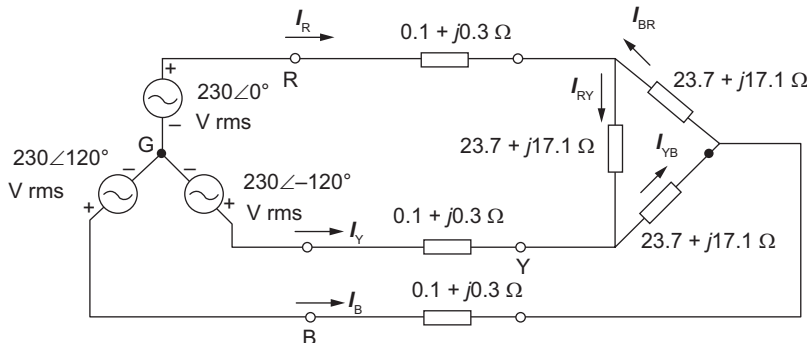


Fig. 8.5-6 Circuit for Example 8.5-2

Solution

This circuit was analysed in Example: 8.3-2 for balanced operation. The following results were obtained.

Line current (R) = $23.09\angle-36.9^\circ$ A rms, load phase voltage (R) = $224.9\angle-1.10$ V rms and load line voltage (RY) = $389.56\angle28.9^\circ$ V rms. The delta branch currents had an rms value of 13.33 A.

Load active power = 12.64 kW, load reactive power = 9.11 kVar and source apparent power = 16 kVA.

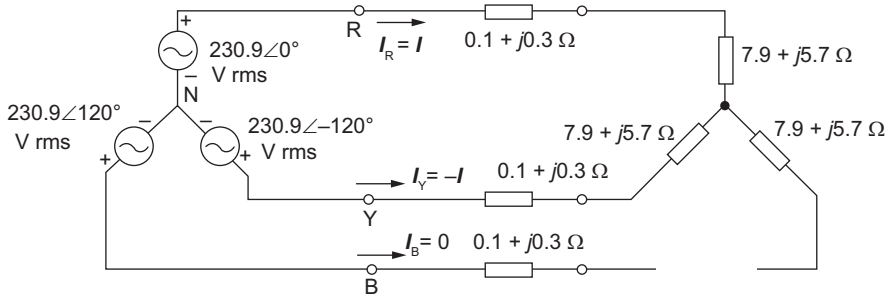


Fig. 8.5-7 The Y-connected equivalent circuit of circuit in Fig. 8.5-6 with B-line open

We replace the balanced delta-connected load impedance by a balanced star-connected impedance first to get a Y-connected equivalent circuit for the circuit in Fig. 8.5-6. This Y-connected equivalent circuit with the break in B-line is shown in Fig. 8.5-7. Note that there is only one current variable in the circuit now and that current phasor is marked as I .

- (i) The current phasor I can be obtained by writing the KVL equation in the only mesh that is present in the circuit.

$$I = [230.9\angle0^\circ - 230.9\angle-120^\circ] \div [16 + j12] = 400\angle30^\circ \div 20\angle36.87^\circ = 20\angle-6.87^\circ \text{ A rms}$$

Therefore, the line currents are $I_R = 20\angle-6.87^\circ$ A rms, $I_Y = -20\angle-6.87^\circ$ A rms and $I_B = 0$.

The symmetrical components of line current is found by using the equation

$$\begin{bmatrix} I_0 \\ I_+ \\ I_- \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} I_R \\ I_Y \\ I_B \end{bmatrix}.$$

The sequence components are $I_0 = 0$, $I_+ = 11.545\angle-36.87^\circ$ A rms and $I_- = 11.545\angle23.13^\circ$ A rms. Observe that the positive sequence and negative sequence components of current have equal magnitude.

- (ii) The current I divide in two paths in the delta-connected windings. All the windings have equal impedances. Therefore, the current is shared in 2:1 ratio in RY winding and other two windings in series. The winding currents are 13.33 A rms, 6.67 A rms and 6.67 A rms.
- (iii) Active power delivered to motor = Active power delivered by source – power dissipated in connection impedance = $400 \times 20 \times \cos(30^\circ - (-6.87^\circ)) - 20^2 \times 0.1 \times 2 \text{ W} = 6.32 \text{ kW}$
- Reactive power delivered to motor = Q delivered by source – Q absorbed by connection impedance = $400 \times 20 \times \sin(30^\circ - (-6.87^\circ)) - 20^2 \times 0.3 \times 2 \text{ Var} = 4.56 \text{ kVar}$.

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- (iv) Apparent power of source = $230.9 \times 20 + 230.9 \times 20 + 230.9 \times 0 \text{ VA} = 9.236 \text{ kVA}$
 Power factor = $6.4/9.236 = 0.693$

Example: 8.5-3

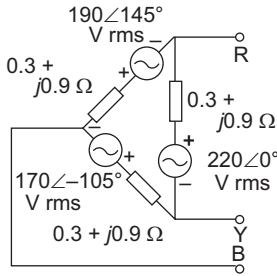


Fig. 8.5-8 Delta-connected unbalanced source in Example 8.5-3

Convert the delta-connected unbalanced voltage source with source impedance in Fig. 8.5-8 into its Y-connected equivalent.

Solution

Let the three source voltage phasors be identified as V_1 , V_2 and V_3 . Let the source impedance be identified as Z . We convert the delta branches into current sources in parallel with impedance form as shown in Fig. 8.5-9.

The delta-connected unbalanced current source is now resolved in terms of its sequence components as shown in Fig. 8.5-10 where $I_+ = V_+ / Z$, $I_- = V_- / Z$ and $I_0 = V_0 / Z$.

The first two delta-connected current sources are three-phase balanced sources and have Y-equivalents. Their Y-equivalent is obtained by finding out the first line current delivered by each.

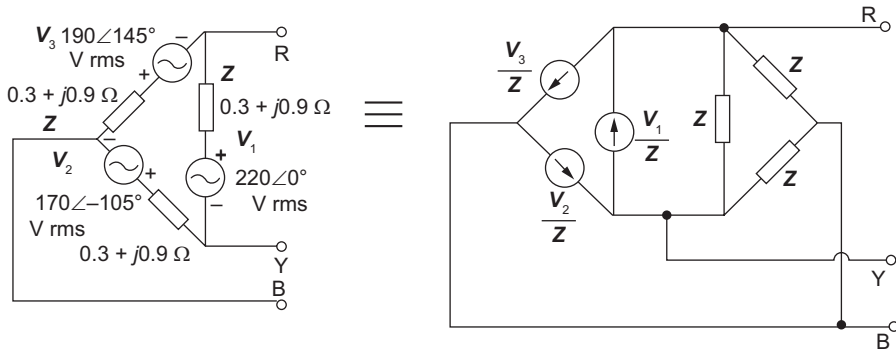


Fig. 8.5-9 Voltage source to current source transformation applied to the source in Fig. 8.5-8

The positive sequence sources deliver $I_+ - aI_+$ into R-line. The star equivalent must have this as the source function in R-phase. $(1 - a) = \sqrt{3} \angle 30^\circ$. Therefore, the star equivalent will have a positive sequence current source of value of $\sqrt{3} \angle 30^\circ I_+$.

The negative sequence sources deliver $I_- - a^2 I_-$ into R-line. The star equivalent must have this as the source function in R-phase. $(1 - a^2) = \sqrt{3} \angle -30^\circ$. Therefore, the star equivalent will have a positive sequence current source of value of $\sqrt{3} \angle -30^\circ I_-$.

The zero sequence current inside the delta circulates within it and cannot come out. Therefore, the Y-equivalent has no zero sequence current source. This is consistent with the fact that the line currents in a three-wire system cannot have zero sequence content.

Therefore, the Y-equivalent of the circuit in Fig. 8.5-10 is as shown in circuit Fig. 8.5-11(a). Note that the delta-connected impedance is also converted into equivalent star.

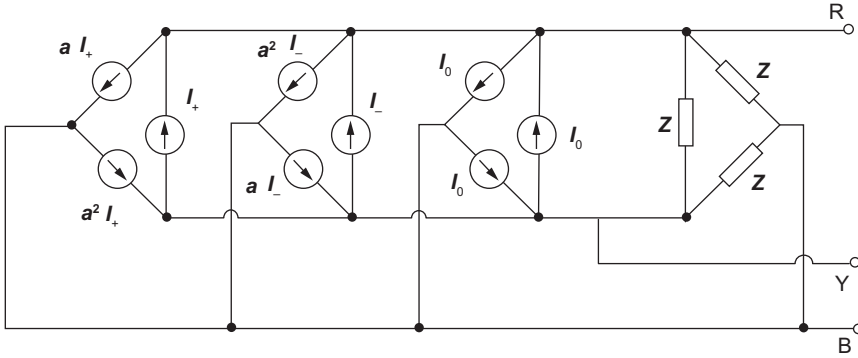


Fig. 8.5-10 Resolution of three-phase current source in Fig. 8.5-9 in terms of its sequence components

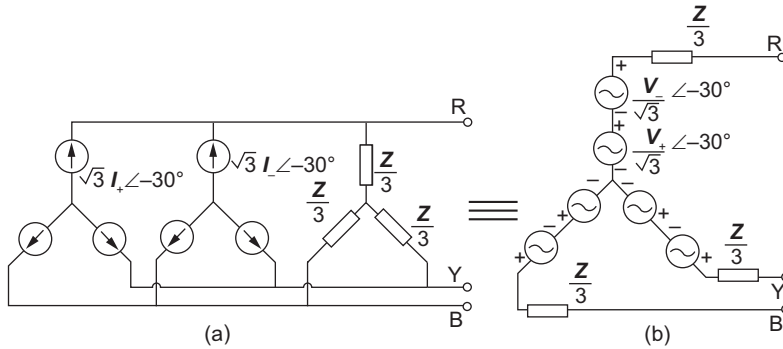


Fig. 8.5-11 Y-equivalent of circuit in Fig. 8.5-10

Now, all the three three-phase Y-connected subcircuits in circuit Fig. 8.5-11(a) are balanced circuits and hence the neutrals will be at the same potential. Therefore, they can be joined together. Then, the resulting parallel connection of two current sources and impedance in each phase can be converted into two voltage sources in series with the same impedance. The last step is to substitute for I_+ and I_- in terms of sequence components of delta phase-source voltages. The resulting Y-connected equivalent circuit is shown in circuit Fig. 8.5-11(b).

Now, we have determined the positive sequence and negative sequence components of phase voltages of the Y-connected equivalent. They are $\frac{V}{\sqrt{3}} \angle -\theta^\circ$ and $\frac{V}{\sqrt{3}} \angle \theta^\circ$, respectively. The zero sequence component is zero. Note that there is non-zero zero sequence content in delta-connected sources.

$$\begin{bmatrix} V_0 \\ V_+ \\ V_- \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Substituting for V_1 , V_2 , V_3 and a , we get $V_0 = 19.62 \angle -69.76^\circ$ V rms, $V_+ = 190.04 \angle 12.6^\circ$ V rms, and $V_- = 36.051 \angle -39.69^\circ$ V rms.

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The star equivalent phase voltage are obtained by

$$\begin{bmatrix} V_{RN} \\ V_{YN} \\ V_{BN} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} \frac{1\angle -30^\circ}{\sqrt{3}} V_+ \\ \frac{1\angle 30^\circ}{\sqrt{3}} V_- \\ 0 \end{bmatrix} = \begin{bmatrix} 130.38\angle -16.17^\circ \\ 103.63\angle -148.11^\circ \\ 98.38\angle 112.23^\circ \end{bmatrix} \text{ V rms.}$$

8.6 SUMMARY

- A set of three sinusoidal quantities, all at the *same frequency*, with equal peak (and hence rms) values and shifted successively by 120° in phase is defined as a *Balanced Three-Phase Quantity*. Therefore, if $x_1(t)$, $x_2(t)$ and $x_3(t)$ is a three-phase set, then,

$$x_1(t) = X_m \cos \omega t, x_2(t) = X_m \cos(\omega t - 120^\circ) \text{ and } x_3(t) = X_m \cos(\omega t + 120^\circ) \text{ or}$$

$$x_1(t) = X_m \cos \omega t, x_2(t) = X_m \cos(\omega t + 120^\circ) \text{ and } x_3(t) = X_m \cos(\omega t - 120^\circ)$$

and $x_1(t) + x_2(t) + x_3(t) = 0$ for all t .

If the peak values are unequal and/or successive phase shifts are different from 120° , the set will be called an *Unbalanced Three-Phase Quantity*. A three-phase source is specified by specifying line-to-line voltage rms values.

- Each limb or branch of a three-phase system (source or load) is termed as a *phase*. R, Y and B are used to designate the line terminals of a three-phase source or load.
- A balanced three-phase system comprising balanced three-phase sources and balanced three-phase loads is superior to a single-phase system due to the following facts: (i) Power loss in transmission system is lower in three-phase system (ii) Copper utilisation is superior in three-phase system. (iii) Power delivered to a balanced three-phase load by a balanced three-phase supply is free of pulsation. (iv) Electrical equipment designed for three-phase operation is more efficient than their single-phase counter parts.
- The sum of line voltages in any three-phase system is zero since these three voltages are voltages in a loop formed by R to Y, Y to B and B to R. The sum of line currents in any three-phase three-wire system is zero since these three currents have to satisfy KCL.
- The line voltages in a balanced Y-connected source form a balanced three-phase set of voltages that are $\sqrt{3}$ times in magnitude and 30° ahead in phase with respect to phase voltages.
- The line currents delivered by a balanced Δ -connected source form a balanced three-phase set of currents that are $\sqrt{3}$ times in magnitude and 30° behind in phase with respect to phase currents delivered by the phase sources.
- The three-phase complex power delivered by a three-phase source is given by

$$S = \sqrt{3} V_L I_L \angle \theta = [\sqrt{3} V_L I_L \cos \theta + j \sqrt{3} V_L I_L \sin \theta] \text{ VA}$$
 where θ is the angle by which the current phasor *delivered by a phase-source* lags behind its voltage phasor. This relationship is independent of whether the source is Y-connected or Δ -connected. θ is not the angle between line voltage phasor and line current phasor.
- Any balanced three-phase circuit can be equivalenced to a Y-Y balanced three-phase circuit by replacing Δ -connected sources and loads (if any) by their Y-connected equivalents. The equivalence is valid as far as line voltages, line currents and complex power flow are concerned.

- A balanced Y-Y circuit can be solved by solving its R-phase and employing three-phase symmetry to arrive at the solution for Y-phase and B-phase. The R-phase circuit is solved by using the single-phase equivalent circuit.
- The sets of three-phase balanced source components and possible single-phase components of an unbalanced source are called its *Symmetrical Components*.
- There are three symmetrical components for an unbalanced three-phase source function. *Each symmetrical component is a set of three source functions*. The first set – called the *positive sequence component* – is a balanced three-phase set of source functions that has positive phase sequence. The second set – called the *negative sequence component* – is a balanced three-phase set of source functions that has negative phase sequence. The third set – called the *zero sequence component* – is a set of three cophasal (*i.e.*, of same phase) single-phase source functions. *It is not a three-phase set at all*.
- The zero sequence component is the *average* of the three three-phase quantities. Line voltages in a three-phase system cannot have a zero sequence content anywhere in the system. Line currents in a three-phase three-wire system cannot have a zero sequence content anywhere in the system.
- Active power carried by sequence components obeys superposition principle.
- Positive sequence component of voltage can send active power *only through* positive sequence component of current; negative sequence component of voltage can send active power *only through* negative sequence component of current and zero sequence component of voltage can send active power *only through* zero sequence component of current.
- Balanced impedance is incapable of bringing about a sequence conversion involving translation of positive sequence current into negative sequence voltage and vice versa. An unbalanced load produces sequence coupling between positive sequence components and negative sequence components
- The most energy-efficient way to draw active power from a balanced three-phase source is to draw it through balanced three-phase currents at unity power factor.

8.7 PROBLEMS

1. Three voltage phasors $100\angle 0^\circ$ V, $100\angle -120^\circ$ V and $110\angle 120^\circ$ V cannot be the line voltages of a three-phase source. Explain why.
2. The line currents flowing into a load are found to be $10\angle -90^\circ$ A, $10\angle -200^\circ$ A and $10\angle 30^\circ$ A. Explain why the load cannot be a delta-connected one.
3. The line currents flowing into a load are found to be $10\angle -30^\circ$ A, $10\angle -150^\circ$ A and $9\angle 90^\circ$ A. Show that the three-phase system has to be a 4-wire system and that the load has to be star-connected.
4. The line currents flowing into a load from a balanced supply are found to be $10\angle 0^\circ$ A, $5\sqrt{2}\angle -135^\circ$ A and $5\sqrt{2}\angle 135^\circ$ A. Is it possible to decide whether the load is delta-connected or star connected from this data?
5. The phase-voltage phasors of a Y-connected three-phase voltage source are $100\angle 0^\circ$ V, $100\angle -90^\circ$ V and $100\angle 90^\circ$ V. Calculate the line voltages and explain why line voltage magnitude is not equal to $100\sqrt{3}$ V.
6. Three sinusoidal current sources $10\angle 0^\circ$ A, $10\angle -120^\circ$ A and $8\angle 120^\circ$ A are used to form a three-phase 3-wire current source. Explain why this current source has to be a delta-connected one.

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7. A balanced three-phase voltage source of 400 V, 50 Hz delivers 10 A at unity power factor to a balanced three-phase load through a connection impedance of $1 + j0 \Omega$. Is the line voltage magnitude across the load equal to $400 \text{ V} - 10 \text{ V} = 390 \text{ V}$? If not, why and what is the correct value?
8. Find the symmetrical components of a set of three currents $10\angle-35^\circ \text{ A}$, $10\angle-35^\circ \text{ A}$ and $10\angle-35^\circ \text{ A}$.
9. An electro-dynamometer type wattmeter is a measuring instrument that measures average power. It has two coils called current coil and pressure coil. The current coil terminals are usually marked M and L and the pressure coil terminals are usually marked V and COM. The meter measures the average of the product $v(t) \times i(t)$ where $v(t)$ is the voltage of COM with respect to V and $i(t)$ is the current flowing from M to L in the meter. In the case of sinusoidal steady-state, the meter measures $V_{\text{rms}} I_{\text{rms}} \cos \phi$ where V_{rms} is the rms value of voltage applied across pressure coil, I_{rms} is the rms value of current flowing through current coil and ϕ is the angle between these two phasors. Two such wattmeters can be used to measure active power in a *three-wire system* irrespective of the nature of source and load connections (*i.e.*, star or delta). The wattmeter connection to do this is shown in Fig. 8.7-1.

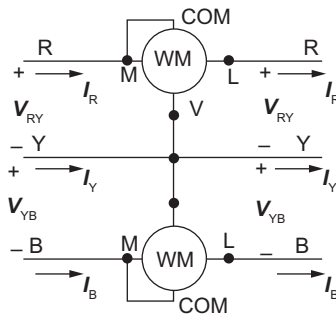


Fig. 8.7-1 Two-wattmeter connection

- (i) Let the reference phasor be the R-phase phase voltage in the equivalent Y-connected system and let the current I_R be $I_L \angle -\theta$. Then, show that, under balanced operation, (i) the wattmeter in R-line reads $W_1 = \sqrt{3} V_L I_L \cos(30^\circ + \theta) \text{ W}$, (ii) the wattmeter in B-line reads $W_2 = \sqrt{3} V_L I_L \cos(30^\circ - \theta) \text{ W}$ and (iii) the total active power in the three-phase system is $= W_1 + W_2 \text{ W}$. Draw the required phasor diagrams.
 - (ii) What is the range of load power factor for which the first wattmeter in two-wattmeter method of power measurement in Fig. 8.7-1 will try to read negative value? What is the corrective action needed to take readings? How is the correction incorporated in the expression for total three-phase power?
 - (iii) Wattmeters used in two-wattmeter method are often constructed with reversing switch that will reverse the voltage applied to pressure coil. What is the need for this reversing switch?
10. A 400 V rms Δ -connected source shown in Fig. 8.7-2 has $I_1 = 10\angle-30^\circ \text{ A rms}$, $I_2 = 10\angle-120^\circ$ and $I_3 = 10\sqrt{2}\angle 105^\circ \text{ A rms}$. (i) Find the line currents delivered, (ii) Find the three-phase active power and reactive power delivered (iii) Find a delta-connected load that will result in this loading.

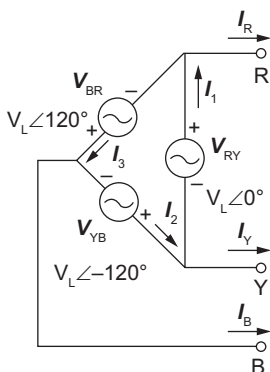


Fig. 8.7-2

11. A star-connected source with zero source impedance delivers $V_{RY} = 200\angle -20^\circ$ V rms, $V_{YB} = 210\angle -130^\circ$ V rms. The R-phase phase voltage $V_{RN} = 115\angle 0^\circ$ V rms. Find the third line voltage and the remaining two phase voltages. Draw the phasor diagram showing phase voltages and line voltages.
12. A delta-connected voltage source with zero source impedance delivers $I_R = 20\angle -30^\circ$ A rms, $I_B = 15\angle 130^\circ$ A rms. The source that is directly between R-line and Y-line is found to deliver $11\angle 0^\circ$ A rms. Find Y-line current and the remaining two phase-source currents. Draw the phasor diagram showing all the currents.
13. A star-connected source with zero source impedance and isolated neutral delivers $V_{RY} = 400\angle -25^\circ$ V rms, $V_{YB} = 390\angle -135^\circ$ V rms. The Y-phase phase voltage $V_{RN} = 230\angle 0^\circ$ V rms. The line currents delivered by the source in R-line and B-line are $I_R = 10\angle -30^\circ$ A rms, $I_B = 12\angle 130^\circ$ A rms. Find the three-phase active power and reactive power delivered by the source. Draw the phasor diagram showing phase voltages, line voltages and line currents.
14. A delta-connected voltage source with zero source impedance delivers $I_R = 5\angle -40^\circ$ A rms, $I_Y = 15\angle -100^\circ$ A rms. The source that is directly between R-line and Y-line is found to deliver $11\angle 0^\circ$ A rms. V_{RY} and V_{YB} are found to be $200\angle -0^\circ$ V rms and $200\angle -110^\circ$ V rms, respectively. (i) Find the three-phase active power and reactive power delivered by the source. (ii) Find a delta-connected impedance that will cause this loading in the source. (iii) Draw the phasor diagram showing phase currents, line currents and line voltages.
15. A star-connected balanced voltage source of 400 V rms, 50 Hz rating delivers $6-j3$ kVA of complex power to a passive balanced Y-connected load. Use the R-phase phase voltage as reference phasor. (i) Find the line currents. (ii) Determine the load impedance. (iii) Draw the phasor diagrams showing all phase voltages, line voltages and line currents. (iii) What will be the readings in two wattmeters in two-wattmeter method of power measurement as in Fig. 8.7-1?
16. A star-connected balanced voltage source of 400 V rms, 50 Hz rating with a source impedance of $1 + j1 \Omega$ in series with the source in each phase delivers power to a delta-connected balanced load impedance with a branch impedance of $20 + j20 \Omega$ at 50 Hz. Find the three-phase complex power delivered to the load.
17. A delta-connected balanced voltage source of 200 V rms, 60 Hz rating with zero source impedance delivers power to a delta-connected balanced load impedance with a branch impedance of $8-j6 \Omega$ through an impedance-less connection. Find the line currents, load-branch currents

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and three-phase complex power delivered to the load without transforming the circuit into its Y-Y equivalent. Draw the phasor diagram for the circuit taking V_{RY} as the reference phasor.

18. A delta-connected balanced voltage source of 200 V rms, 60 Hz rating with zero source impedance delivers power to a delta-connected balanced load impedance with a branch impedance of $8 + j6 \Omega$ through an connection impedance of $1 + j1 \Omega$ in each line. Find the line currents, load-branch currents and three-phase complex power delivered to the load by employing single-phase equivalent circuit method. Draw the phasor diagram for the circuit taking V_{RY} as the reference phasor.
19. A delta-connected balanced voltage source of 400 V rms, 50 Hz rating with zero source impedance delivers power to a star-connected balanced load impedance with a branch impedance of $11 - j17 \Omega$ through an connection impedance of $1 + j1 \Omega$ in each line. Find the line currents, load-branch currents and three-phase complex power delivered to the load by using star-delta transformation of impedances.
20. A star-connected balanced voltage source of 400 V rms, 50 Hz rating with zero source impedance delivers power to a star-connected balanced load impedance with a branch impedance of $10 - j10 \Omega$ in parallel with a balanced delta-connected impedance with a branch impedance of $30 + j60 \Omega$ through an connection impedance of $1 + j1 \Omega$ in each line. Find the line currents, load-branch currents in both loads, three-phase complex power delivered to each load, the complex power delivered by source and source power factor by using single-phase equivalent circuit method.
21. A star-connected balanced voltage source of 200 V rms, 60 Hz rating has an internal resistance of 1Ω in series with an internal inductance of 3 mH in each phase. It supplies power to an unbalanced Y-connected load with 10Ω , 15Ω and 8Ω in R, Y and B phases, respectively through a 3-wire connection with a connection impedance of $0.8 \Omega + 4 \text{ mH}$ inductance in each wire. The neutral of the source is firmly earthed. (i) Find the neutral potential with respect to earth at the load, (ii) Find line currents, line and phase voltages across the load and (iii) The active power delivered to each phase of the load and the total active power.
22. Repeat Problem 21 if it is a 4-wire connection with same connection impedance in the neutral wire. Find the neutral wire current too.
23. A star-connected unbalanced voltage source with $V_{RN} = 115\angle 0^\circ$ V rms, $V_{YN} = 115\angle -110^\circ$ V rms and $V_{BN} = 132\angle 125^\circ$ V rms delivers power to a balanced delta-connected load with a branch-impedance of $23.4 + j17.1 \Omega$ through a 3-wire line with a line impedance of $0.2 + j0.3 \Omega$. Determine (i) line currents, (ii) line voltages across the load, (iii) active power and reactive power delivered by each phase of the source and the three-phase values and (iv) three-phase active power and reactive power delivered to the load by using star-delta transformation on load impedance and mesh analysis.
24. A delta-connected unbalanced voltage source uses $115\angle 0^\circ$ V rms, $110\angle -110^\circ$ V rms and $122\angle 125^\circ$ V in the three branches R–Y, Y–B and B–R, respectively. Each branch source has $1 \Omega + 3 \text{ mH}$ inductance in series with it. The frequency of the system is 60 Hz. This source is delivering power to an unbalanced delta-connected load that has 10Ω resistor, 10Ω resistor in series with 30 mH inductor and 10Ω resistor in series with $1000 \mu\text{F}$ capacitor between R–Y, Y–B and B–R, respectively. Determine (i) line currents, (ii) source and load branch currents, (iii) line voltages across the load, (iv) active power and reactive power delivered by each phase of the source and the three-phase values and (v) three-phase active power and reactive power delivered to the load by using star-delta transformation on load impedance and mesh analysis.
25. The positive sequence component of line current drawn by a balanced delta-connected load of branch impedance $8 + j6 \Omega$ is $20\angle -30^\circ$ A rms. The negative sequence component of line voltage

- across the load is $15\angle-90^\circ$ V rms. (i) Find the line voltages across the load and line currents. (ii) Find the three-phase active power and reactive power delivered to the load from three-phase quantities. (iii) Find the three-phase active power and reactive power delivered to the load from sequence components.
26. An unbalanced star-connected voltage source delivers power to a balanced star-connected load with $8 - j6\ \Omega$ in each phase over a 3-wire impedance-less connection. The load draws a line current that has $10\angle 0^\circ$ A rms positive sequence component and $2\angle-90^\circ$ A negative sequence component. The source neutral is earthed solidly and the load neutral is found to be at $10\angle-150^\circ$ V rms with respect to earth. (i) Find the phase voltages and line voltages across the source and load (ii) Find the active power and reactive power delivered by the source and delivered to the load.
 27. An unbalanced star-connected voltage source delivers power to a balanced star-connected load with $7.9 + j5.7\ \Omega$ in each phase over a 3-wire connection with a wire-impedance of $0.1 + j0.3\ \Omega$ in each wire. The load draws a line current that has $10\angle 0^\circ$ A rms positive sequence component and $2\angle-90^\circ$ A rms negative sequence component. The source neutral is earthed solidly and the load neutral is found to be at $10\angle 50^\circ$ V rms with respect to earth. (i) Find the phase voltages and line voltages across the source and load (ii) Find the active power and reactive power delivered by the source and delivered to the load.
 28. An unbalanced star-connected voltage source delivers power to a balanced star-connected load with $7.9 + j5.7\ \Omega$ in each phase over a 4-wire connection with a wire-impedance of $0.1 + j0.3\ \Omega$ in each wire. The load draws a line current that has $10\angle 0^\circ$ A rms positive sequence component and $2\angle-90^\circ$ A rms negative sequence component. The neutral return current is found to be at $1.5\angle-100^\circ$ A rms. (i) Find the phase voltages and line voltages across the source and load (ii) Find the active power and reactive power delivered by the source and delivered to the load.
 29. The magnitudes of line voltages at a certain point in a three-phase system are measured as 400 V rms, 380 V rms and 410 V rms between R–Y, Y–B and B–R, respectively. Find the symmetrical components of the line voltage. [Hint: The equation $V_{RY} + V_{YB} + V_{BR} = 0$ implies that line voltage phasors in a three-phase system will have to form a closed triangle.]
 30. The line voltages of an ideal three-phase unbalanced source are measured to be 420 V rms, 390 V rms and 380 V rms between R–Y, Y–B and B–R, respectively. This source drives a balanced *RLC* load through a 3-wire connection with wire-impedance of $1 + j3\ \Omega$ in each wire. If the magnitude of positive sequence component of line voltage across load is found to be 380 V rms find the magnitude of negative sequence component of line voltage across the load.

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Dynamic Circuits with Periodic Inputs – Analysis by Fourier Series

CHAPTER OBJECTIVES

- (i) To explain how a periodic waveform can be expanded in terms of sinusoids and why such an expansion is necessary, (ii) to show how such an expansion may be obtained for a given periodic waveform; and (iii) to show how the expansion can be used to solve for the forced response of a circuit.
- To discuss the difference between steady-state response and forced response of a circuit to set the background for application of sinusoidal expansion of periodic waveforms in Circuit Analysis.
- To bring out the important properties of Fourier Series expansion through solved examples.

INTRODUCTION

This chapter goes into the determination of steady-state response component in the output of dynamic circuits when they are excited by a periodic input waveform. The problem of expressing a periodic waveform as a sum of pure sinusoidal components is addressed first. Subsequently, the use of frequency response data to solve for forced response when the forcing function is such a sum of sinusoidal components will be taken up.

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A waveform $v(t)$ is periodic on t with a period of T if and only if the following condition is satisfied by it for all t and n where n is a positive integer.

$$v(t) = v(t \pm nT), \text{ for any } t \text{ and for any } n; n = 1, 2, 3, \dots$$

This implies that the values of $v(t)$ at similarly positioned time points, at an interval of T seconds between adjacent points, will be same.

Obviously, if $v(t)$ is periodic it must extend from $-\infty$ to $+\infty$ in the time-axis, since the value of integer n in the definition of periodicity is not limited to any finite number. Therefore, it is equally obvious that, there is no strictly periodic waveform in nature. All waveforms in electrical circuits start at some time instant and stop at some other time instant. *Hence, all practical circuit waveforms are necessarily aperiodic.*

Fourier series expansion deals with periodic waveforms. It *resolves* a periodic waveform *into* pure sinusoidal components. Equivalently, it expands the periodic waveform as a linear combination of infinitely many *harmonically* related sinusoidal waveforms. Two sinusoids are *harmonically* related if their frequencies are integer multiples of some common frequency value. Fourier series expresses the periodic waveform as a sum of infinitely many sinusoids with frequencies which are integer multiples of the frequency of that waveform. This frequency is called the *fundamental frequency*. The sinusoidal component that is at the same frequency as that of the periodic waveform is termed as *the fundamental component* and all the other sinusoidal components with frequencies which are integral multiples of this frequency are called the *harmonic components*.

Granted that a periodic waveform can be expanded in this manner and that frequency response information for a circuit can then be employed to find the steady-state response of the circuit for such a periodic input, two questions come up at this point.

- Why the obsession with sinusoids? Is it because a periodic waveform can be expanded only in terms of sinusoids?
- If a periodic waveform is only a mathematical entity with no corresponding physical counterpart, why bother to study the steady-state response of circuits to such a hypothetical input?

9.1 PERIODIC WAVEFORMS IN CIRCUIT ANALYSIS

We consider the first question.

Expansion in terms of sinusoids is not the only expansion possible for a periodic waveform. However, this expansion turns out to be of great utility in Circuit Analysis.

Complex exponential functions of the form e^{st} , where s is a complex number, are eigen functions of linear time-invariant circuits. This means that the *total response* of the circuit when $v_s(t) = e^{st}$, for all t , is applied is a scaled copy of e^{st} itself. The scaling factor will be a function of s and circuit parameters. It will be a complex number (this complex number is called an *eigen value*). Note that e^{st} has to be applied from $t = -\infty$ for this to be true.

Forced response components in a linear time-invariant circuit due to simultaneously acting forcing functions obey superposition principle.

Therefore, if an input function with an arbitrary waveshape can be expressed as a sum of many (finite or infinite) functions of a type which has a simple waveshape, we will be able to arrive at the forced response of the circuit for this arbitrary waveshape by superposing forced response components for the simple waveshape.

The advantage involved will be further enhanced if the forced response component for the simple waveshape is easy to determine. Therefore, the simple function that should be used as a *basis* function for expanding the complex waveshape must be e^{st} since that is the input function for which a simple scaling will result in the forced response component.

s in e^{st} is, in general, a complex number and e^{st} is a complex signal. A real physical waveform that we find in a physical circuit cannot have an imaginary part. Therefore, we expect that if we find e^{st} with a particular complex value for $s = \alpha + j\omega$ necessary in the expansion for a real waveform, we will find that (i) the conjugate signal of e^{st} will also be needed and that (ii) the scaling factors for e^{st} and its conjugate will turn out to be conjugate numbers. Thus, these two conjugate contributions will combine to yield a component of $e^{\alpha t} \sin(\omega t + \theta)$ type. Depending on the values of α and ω , this component may be a decaying exponential, a growing exponential, an exponentially damped sinusoid, an exponentially growing sinusoid or a sinusoid of fixed amplitude.

But, is it possible to expand any arbitrary function of time, say $v(t)$, as a sum of scaled complex exponential functions even if we allow all possible values of s to contribute to the sum? That is a question for mathematicians. The short answer for the student of circuit analysis is that, yes, it is possible, except for some very peculiar and pathological functions which exist only in the pages of books on mathematics and never in a physical system.

An exponentially damped sinusoid is a complex waveshape. So is an exponentially growing sinusoid. True, they are eigen functions of linear circuits and do not pose any difficulty in solving for forced response in circuits. But let us try to take the simpler ones first. The real exponential function does not suit our purpose since there is nothing *steady* about it. Therefore we choose the simplest – the steady amplitude sinusoid which is a special case of e^{st} with the real part of s set to zero. That is, we choose all those signals of type $e^{j\omega t}$ along the $j\omega$ axis in the signal plane as our *basis* functions. Limiting ourselves to sinusoidal signals drawn from the entire imaginary axis of the signal plane, we raise the question – can any arbitrary waveform $v(t)$ be expressed as a sum of scaled functions of $e^{j\omega t}$ type?

The answer is no. But all waveforms that can really appear in a physical circuit can be expanded this way. Some of the commonly used idealised waveforms like step, impulse ramp etc. do pose some problems in this regard – but the problems are not insurmountable.

We constrain $v(t)$ further and state that it is periodic with period T . Now, do we need all the signals on the entire $j\omega$ axis for its expansion? We note that two waveforms with *same period* will result in a periodic waveform with same period when they are combined in a linear combination. Therefore, it is reasonable to expect that a periodic $v(t)$ can be expressed as sum of many sinusoidal waveforms with same period. But how many sinusoids can be there with a particular period T ? There are infinite of them – because a sinusoid with a period of T/n , where n is an integer, is periodic with a period of T too; there will be n full cycles of it in T s. Therefore, the signal represented by the finite series,

$$\sum_{n=0}^N a_n \cos(n\omega_0 t) + \sum_{n=1}^N b_n \sin(n\omega_0 t),$$
 will be a periodic wave with frequency of ω_0 rad/s for any finite value of N , a_n and b_n .

For a particular value of ω_0 and for each choice of N , we can vary the $2N + 1$ numbers – a_n and b_n – to synthesise infinite number of distinct periodic waveforms of periodicity $2\pi/\omega_0$ s. Each combination of these $2N + 1$ numbers over real number field will result in a unique periodic waveform. And we will get yet another set of infinite unique periodic waveforms for another choice of N .

That prompts a question – given a non-sinusoidal periodic waveform $v(t)$ with angular frequency ω_0 rad/s, can we find some N and a set of $2N + 1$ real numbers for a_n and b_n such that

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$v(t) = \sum_{n=0}^N a_n \cos(n\omega_0 t) + \sum_{n=1}^N b_n \sin(n\omega_0 t)$ at all instants? Yes, provided N is allowed to become infinite if necessary and $v(t)$ satisfies certain criteria. Almost all periodic functions appearing in Circuit Analysis satisfy the required criteria.

We have answered the first question raised in the introductory section. We take up the second one now.

We recapitulate that the total response of a circuit contains two components – the *natural response* and *forced response components*. Forced response in a circuit is the component of total response contributed by the input source function, *i.e.*, the particular integral of the differential equation describing the circuit. Memory elements like inductor and capacitor store energy in them. The stored energy in them cannot be changed instantaneously unless infinite voltage/current are allowed in the circuit. There may arise a mismatch between the value of stored energy in various components immediately before and immediately after applying the input if forced response was the only response component. Natural response terms arise due to the difference between energy storage in memory elements before applying input and the energy storage predicted by the particular integral part of the circuit response. Natural response terms are the complementary solution terms of the circuit differential equation and are of e^{st} type, where s values are the natural frequencies (*i.e.* roots of characteristic equation of the circuit differential equation) of the circuit.

Though the terms, *forced response* and *steady-state response*, are used interchangeably, they are not the same. If a forcing function is applied to a circuit, there will be a forced response. But there need not be a steady-state response always. There can be a steady-state in the circuit only if three conditions are satisfied.

First, the circuit natural response must be damped and should approach zero as time increases without limit.

Second, there must exist features in the input source function waveform that can be used to define what is meant by *steady-state* at the output (*i.e.*, the meaning of ‘*steady-state*’ is decided by input waveform. If there is nothing about the waveform that can be used to define a meaning for ‘*steady-state*’, then there is no meaning for ‘*steady-state*’ at the output too). For example, if input is a periodic waveform with period of T , switched on to a linear time-invariant circuit at $t = 0$, then *steady-state output* is a waveform that is periodic with same period – such a *steady-state* is called *periodic steady-state*. Sinusoidal steady-state is a special case of periodic steady-state. The difference between the two is that the output waveform need not have the same waveshape as that of input in the case of a general periodic steady-state, whereas the output waveform will be of same waveshape as that of input in the case of sinusoidal steady-state.

The third condition is that the input source function must remain applied to the circuit for enough time for the circuit to reach steady-state. For example, we may switch on a periodic wave such as a 50 Hz sine wave to an RC circuit with 0.2 s time constant and switch off after 0.3 s. The circuit transient response does not get enough time to decay down to zero and there is no question of the circuit reaching sinusoidal steady-state.

Steady-state response, if steady-state gets established in the circuit, is the same as forced response.

9.1.1 The Sinusoidal Steady-State Frequency Response Function

In Chapter 7 we noted that the most general differential equation governing a chosen response variable for a linear circuit is

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x$$

where y is the chosen response variable, x is the input function and a 's and b 's are real positive constants decided by circuit parameters. The order of differential equation, n , is equal to the number of independent energy storage elements in the circuit.

If $x = e^{j\omega t}$, then $y = A e^{j\omega t}$, where A is a complex amplitude decided by the equation

$$A[(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_1(j\omega) + a_0] = b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \dots + b_1(j\omega) + b_0$$

Therefore, the complex amplitude of steady-state response to a unit amplitude complex exponential input comes out as a ratio of rational polynomials in the variable $j\omega$.

The desired steady-state response is obtained as
$$y = \frac{\sum_{k=0}^m b_k(j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i(j\omega)^i} e^{j\omega t}.$$

The Sinusoidal Steady-State Frequency Response Function of a linear time-invariant circuit is defined as the complex valued gain offered by the circuit to a signal $e^{j\omega t}$. It is usually represented by $H(j\omega)$.

$$\therefore H(j\omega) = \frac{\sum_{k=0}^m b_k(j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i(j\omega)^i}$$

Let the input to a linear time-invariant circuit be $x(t) = A \cos(\omega t + \theta)u(t)$. Then the output after the circuit attains steady-state (if it can reach a steady-state) is $y(t) = |H(j\omega)| A \cos(\omega t + \theta + \phi)$ where $|H(j\omega)|$ is the *gain magnitude of frequency response function* and $\phi = \angle H(j\omega)$ is the *phase of frequency response function*. Similarly if the input is $x(t) = A \sin(\omega t + \theta)u(t)$, the output after the circuit achieves steady-state is $y(t) = |H(j\omega)| A \sin(\omega t + \theta + \phi)$.

The magnitude part of frequency response function $H(j\omega)$ gives the ratio of 'output sinusoid amplitude' to 'input sinusoid amplitude'. The angle part of frequency response function gives the phase angle by which the output sinusoid leads the input sinusoid.

Note that we did not specify the nature of x and y in the definition of frequency response function. If both x and y are voltage (or current) variables, the frequency response function is a *transfer gain function*. If x is a voltage variable and y is the current variable at the same terminals where the voltage x is present, the frequency response function is a *driving point admittance function*. If x is a current variable and y is the voltage variable at the same terminals, the frequency response function is a *driving point impedance function*.

It is not necessary to derive the differential equation describing the circuit to obtain its frequency response function. $H(j\omega)$ can be found from the phasor equivalent circuit by circuit reduction or by nodal/mesh analysis. For instance, the frequency response function connecting the circuit

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current to applied voltage in a series RL circuit can be obtained from its phasor equivalent circuit as

$$= \frac{1}{R + j\omega L} = \frac{1/R}{1 + j\omega L/R} = \frac{1/R}{\sqrt{1 + (\omega L/R)^2}} \angle -\tan^{-1} \omega L/R.$$

Similarly, the frequency response function describing the transfer gain between the voltage across capacitor and input voltage in a series RC circuit can be determined from its phasor equivalent circuit by applying voltage division rule in a series circuit.

$$H(j\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} = \frac{1}{\sqrt{1 + (\omega RC)^2}} \angle -\tan^{-1} \omega RC$$

Many circuits do reach periodic steady-state after the switch-on transient and operate for long periods before they are switched off. Electrical power system and the loads that draw power from it constitute an important example of this kind of operation. Many features of such steady-state operation are of considerable practical interest to the electrical engineer. Therefore, one reason why we study the forced response of circuits to periodic waveforms is that we are interested in steady-state response of circuits when such waveforms are switched on to them at $t = 0$. Expanding the periodic waveform in terms of sinusoidal components and using frequency response function and superposition principle will help us to get the required steady-state solution with relative ease. However, if we want to get the complete response we have to bring in the natural response terms too and arrange for compliance with initial conditions.

Consider an example. An RL circuit with no initial energy storage, $R = 1 \Omega$ and $L = 1 \text{ H}$ is connected to a source $v(t) = \sin t + 0.3 \sin 3t + 0.2 \sin 5t \text{ V}$, at $t = 0$. Subsequently it is removed from the supply at $t = 100 \text{ s}$ and the circuit is kept shorted after that. We wish to find the *total response* of the circuit current.

Let $v_s(t)$ stand for the voltage applied to the circuit. What is $v_s(t)$?

$$v_s(t) = (\sin t + 0.3 \sin 3t + 0.2 \sin 5t)[u(t) - u(t - 100)]$$

The forcing function is a sum of three sinusoids. Forced response component for each sinusoid may be obtained using frequency response data. The relevant frequency response function is

$$\frac{1/R}{\sqrt{1 + (\omega L/R)^2}} \angle -\tan^{-1} \omega L/R = \frac{1}{\sqrt{1 + \omega^2}} \angle -\tan^{-1} \omega.$$

Forced response components for the three sinusoids may be obtained by using corresponding values of ω in this function. For instance, when $\omega = 1 \text{ rad/s}$, the gain magnitude is $0.707 \Omega^{-1}$ and lead phase angle is -45° . The individual steady-state components may be combined to obtain the total steady-state response.

The result will be $= 0.707 \sin(t - 45^\circ) + 0.0945 \sin(3t - 71.6^\circ) + 0.04 \sin(5t - 78.7^\circ) \text{ A}$.

This solution predicts that the current at $t = 0^+$ is -0.63 A . But the initial condition for current at $t = 0^+$ is zero. Therefore the natural response component has to be $-0.63 e^{-t} \text{ A}$. Therefore the solution for current for $t \geq 0^+$ is $i(t) = -0.63 e^{-t} + 0.707 \sin(t - 45^\circ) + 0.0945 \sin(3t - 71.6^\circ) + 0.04 \sin(5t - 78.7^\circ) \text{ A}$.

The circuit reaches periodic steady-state after about 5 time constants, *i.e.* at about 5 s. It operates under periodic steady-state till 100 s. At that point, the applied voltage in the circuit goes to zero suddenly. Therefore, the solution we determined above is valid only up to 100 s. The value of current

at $t = 100^-$ s can be obtained by evaluating $i(t)$ with $t = 100$ s. The value is 0.66 A. This current decreases exponentially with a time constant of 1 s since the circuit is kept shorted after 100 s. Therefore the complete solution for current in the circuit is

$$i(t) = \begin{cases} -0.6e^{-t} + 0.7\sin(t - 45^\circ) + 0.095\sin(3t - 71.6^\circ) + 0.04\sin(5t - 78.7^\circ) \text{ A}; & 0^+ \leq t \leq 100^- \\ 0.66e^{-(t-100)} \text{ A}, & \text{for } t \geq 100^+ \end{cases}$$

We could arrive at the solution only because we could solve for the forced response component using frequency response analysis. We could do that since the statement of the problem already expressed the applied source as a sum of sinusoids. Fourier series helps us to get this sum for a wide class of periodic waveforms.

If a waveform that is applied to a circuit can be expressed as a *gated periodic waveform* – i.e. if $v_s(t) = v(t) \times [u(t) - u(t - t_0)]$, where $v_s(t)$ is the applied waveform, $v(t)$ is the underlying periodic waveform and t_0 is the duration for which the underlying periodic waveform is passed on to the circuit – then, the forced response component in the output can be obtained by using sinusoidal expansion of the underlying waveform along with frequency response data for the circuit. Fourier series is needed and will be of great help in this kind of circuit problems.

9.2 THE EXPONENTIAL FOURIER SERIES

Let $v(t)$ be a periodic function of time and let it satisfy the conditions required to be satisfied for its expansion in terms of sinusoids to exist. Let the period of $v(t)$ be T s and let its angular frequency be ω_0 rad/s ($\omega_0 = 2\pi/T$). Then Fourier theorem, in effect, states that $v(t)$ may be represented by the infinite series

$$v(t) = \dots + \tilde{v}_{-3}e^{-j3\omega_0 t} + \tilde{v}_{-2}e^{-j2\omega_0 t} + \tilde{v}_{-1}e^{-j\omega_0 t} + \tilde{v}_0 + \tilde{v}_1e^{j\omega_0 t} + \tilde{v}_2e^{j2\omega_0 t} + \tilde{v}_3e^{j3\omega_0 t} + \dots$$

Using summation notation we write this series as

$$v(t) = \sum_{n=-\infty}^{n=\infty} \tilde{v}_n e^{jn\omega_0 t}; \quad n \text{ integer} \quad (9.2-1)$$

This equation states that the periodic function $v(t)$ can be *constructed* or *synthesised* from infinitely many complex exponential functions of time drawn from $j\omega$ axis in the signal plane. $\tilde{v}_n$ in Eqn. 9.2-1 are called the *coefficients* of exponential Fourier series. ω_0 is the *fundamental frequency*.

$v(t)$ is usually a real function of time since it represents some voltage or current waveform in a circuit. Therefore we expect that $\tilde{v}_{-n}$ will turn out to be $\tilde{v}_n^*$ for any non-zero value of n .

$$\text{Let } \tilde{v}_n = \frac{a_n}{2} - j\frac{b_n}{2}. \text{ Then } \tilde{v}_n^* = \frac{a_n}{2} + j\frac{b_n}{2}.$$

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Then the contribution of n^{th} harmonic to $v(t)$ can be expressed as

$$\begin{aligned}
 &= \tilde{v}_{-n} e^{-jn\omega_0 t} + \tilde{v}_n e^{jn\omega_0 t} \\
 &= \tilde{v}_n^* e^{-jn\omega_0 t} + \tilde{v}_n e^{jn\omega_0 t} = \left[\frac{a_n}{2} + j \frac{b_n}{2} \right] e^{-jn\omega_0 t} + \left[\frac{a_n}{2} - j \frac{b_n}{2} \right] e^{jn\omega_0 t} \\
 &= \left(\frac{a_n}{2} \cos n\omega_0 t + \frac{b_n}{2} \sin n\omega_0 t \right) - j \left(\frac{a_n}{2} \sin n\omega_0 t - \frac{b_n}{2} \cos n\omega_0 t \right) \\
 &\quad + \left(\frac{a_n}{2} \cos n\omega_0 t + \frac{b_n}{2} \sin n\omega_0 t \right) + j \left(\frac{a_n}{2} \sin n\omega_0 t - \frac{b_n}{2} \cos n\omega_0 t \right) \\
 &= a_n \cos n\omega_0 t + b_n \sin n\omega_0 t
 \end{aligned}$$

Hence $\tilde{v}_{-n} = \tilde{v}_n^*$ will result in the two complex exponential contributions adding up to yield a real function of time.

Equation 9.2-1 tells us how to construct the periodic waveform $v(t)$ from its harmonic components. But how do we get the exponential Fourier series coefficients $\tilde{v}_n$ given the function $v(t)$?

We proceed as follows.

First we introduce a new index variable k in the place of n in Eqn. 9.2-1 and restate that equation as follows:

$$v(t) = \sum_{k=-\infty}^{k=\infty} \tilde{v}_k e^{jk\omega_0 t}$$

Then we multiply both sides by $e^{-jn\omega_0 t}$ where n is a particular value of k .

$$\begin{aligned}
 v(t) e^{-jn\omega_0 t} &= e^{-jn\omega_0 t} \sum_{k=-\infty}^{k=\infty} \tilde{v}_k e^{jk\omega_0 t} \\
 &= \tilde{v}_n \sum_{\substack{k=-\infty \\ k \neq n}}^{k=\infty} \tilde{v}_k e^{j(k-n)\omega_0 t} \\
 &= \tilde{v}_n \sum_{\substack{k=-\infty \\ k \neq n}}^{k=\infty} \tilde{v}_k [\cos(k-n)\omega_0 t + j \sin(k-n)\omega_0 t]
 \end{aligned}$$

We wish to extract $\tilde{v}_n$. We remember an interesting property of sinusoids – the area under a sinusoidal curve over one period is zero, since the area accumulated under the positive half-cycle is cancelled exactly by the area accumulated under the negative half-cycle. More generally, the area under a sinusoid over any time interval equal to its period or integer multiples of its period will be zero.

$k - n$ is an integer. Thus, a sinusoid with angular frequency of $(k - n)\omega_0$ will have integral number of cycles in T seconds since $T = 2\pi/\omega_0$. Therefore,

$$\int_t^{t+T} \cos(k-n)\omega_0 t dt = 0 \text{ and } \int_t^{t+T} \sin(k-n)\omega_0 t dt = 0 \text{ for } k \neq n.$$

We make use of this fact to extract $\tilde{v}_n$ as,

$$\int_t^{t+T} v(t) e^{-jn\omega_0 t} dt = \int_t^{t+T} \tilde{v}_n dt + \sum_{\substack{k=-\infty \\ k \neq n}}^{k=\infty} \int_t^{t+T} \tilde{v}_k [\cos(k-n)\omega_0 t + j \sin(k-n)\omega_0 t] dt = \tilde{v}_n T + 0$$

$$\therefore \tilde{v}_n = \frac{1}{T} \int_t^{t+T} v(t) e^{-jn\omega_0 t} dt$$

The required integration can be carried out over any interval of width T . However, this interval is usually chosen to be $[-T/2, +T/2]$ in order to exploit certain symmetries that the waveform $v(t)$ may possess. Therefore,

$$\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt \text{ -- analysis equation} \quad (9.2-2)$$

Equations 9.2-1 and 9.2-2 are called the *synthesis equation* and the *analysis equation*, respectively and the two together form the *Fourier series pair*.

We expect that $\tilde{v}_{-n}$ will turn out to be $\tilde{v}_n^*$ for any non-zero value of n . We show that it is indeed so.

$$\begin{aligned} \tilde{v}_{-n} &= \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-j(-n)\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{jn\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} v(t) (e^{-jn\omega_0 t})^* dt \\ &= \left(\frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt \right)^* \quad (\text{since } v(t) \text{ is a real function } t) \\ &= \tilde{v}_n^* \end{aligned}$$

The value $n = 0$ is a special one. The harmonic coefficient at $n = 0$ appears alone without a conjugate companion. We examine this coefficient further.

$$\tilde{v}_0 = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-j \cdot 0 \cdot \omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} v(t) dt$$

Thus, $\tilde{v}_0$ is a real value representing the *cycle average* value of $v(t)$. The area of $v(t)$ in one cycle is divided by the period to arrive at $\tilde{v}_0$. It represents the *DC content* in the waveform. If this DC content is removed from $v(t)$, it becomes a pure AC signal that has zero area under one cycle.

9.3 TRIGONOMETRIC FOURIER SERIES

The trigonometric form of Fourier series affords better insight into how sinusoids combine to produce the periodic waveform $v(t)$. This form is derived from the exponential form below as follows.

$$\text{Let } \tilde{v}_n = \frac{a_n}{2} - j \frac{b_n}{2}.$$

$$\text{Then } \tilde{v}_n^* = \frac{a_n}{2} + j \frac{b_n}{2}.$$

Then,

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$$\begin{aligned}
 v(t) &= \tilde{v}_0 + \sum_{n=1}^{\infty} [\tilde{v}_{-n} e^{-jn\omega_o t} + \tilde{v}_n e^{jn\omega_o t}] \\
 &= \tilde{v}_0 + \sum_{n=1}^{\infty} [\tilde{v}_n^* e^{-jn\omega_o t} + \tilde{v}_n e^{jn\omega_o t}] = \tilde{v}_0 + \sum_{n=1}^{\infty} \left[\left[\frac{a_n}{2} + j \frac{b_n}{2} \right] e^{-jn\omega_o t} + \left[\frac{a_n}{2} - j \frac{b_n}{2} \right] e^{jn\omega_o t} \right] \\
 &= \tilde{v}_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t \\
 \therefore v(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t \text{ where} \\
 a_0 &= \tilde{v}_0 = \frac{1}{T} \int_{-T/2}^{T/2} v(t) dt, \\
 a_n &= \tilde{v}_n + \tilde{v}_{-n} = \tilde{v}_n + \tilde{v}_n^* = 2 \operatorname{Re}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \cos n\omega_o t dt \text{ for } n = 1, 2, 3, \dots \\
 b_n &= -\tilde{v}_n + \tilde{v}_{-n} = -\tilde{v}_n + \tilde{v}_n^* = -2 \operatorname{Im}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \sin n\omega_o t dt \text{ for } n = 1, 2, 3, \dots
 \end{aligned} \tag{9.3-1}$$

This can be written in the following form by combining the cosine and sine contributions for a particular harmonic order n using trigonometric identities.

$$\begin{aligned}
 \therefore v(t) &= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_o t - \phi_n) \text{ where} \\
 c_0 &= \tilde{v}_0, \\
 c_n &= \sqrt{a_n^2 + b_n^2} = 2\tilde{v}_n \tilde{v}_n^* = 2|\tilde{v}_n|, \\
 \phi_n &= \tan^{-1} \frac{b_n}{a_n} = -\angle \tilde{v}_n \text{ for } n = 1, 2, 3, \dots
 \end{aligned} \tag{9.3-2}$$

9.4 CONDITIONS FOR EXISTENCE OF FOURIER SERIES

The exponential and trigonometric Fourier series exists for all $v(t)$ which satisfy a set of conditions known as Dirichlet's conditions.

- (i) The function $v(t)$ must be single-valued everywhere.
- (ii) The integral $\int_{t_0}^{t_0+T} |v(t)| dt$ is *finite* for any t_0 . This condition ensures that all Fourier series coefficients are finite-valued.
- (iii) $v(t)$ has only a finite number of maxima and minima in any one period.
- (iv) $v(t)$ has only a finite number of discontinuities in any one period. Moreover, each discontinuity is a finite discontinuity.

There are functions that violate one or more of Dirichlet's conditions. But they do not usually come up in electrical circuits. Hence we can safely assert that all waveforms we encounter in physical circuits will satisfy these conditions.

If $v(t)$ satisfies all the Dirichlet's conditions, its Fourier series is guaranteed to exist and converge to the function value *except at the points of discontinuity*. At the points of discontinuity, the Fourier series will converge to the average value – *i.e.* to half the sum of value of $v(t)$ at the left and right of the discontinuity.

This implies that, for a $v(t)$ that satisfies all the four conditions, the partial sum of its Fourier series tends to approach the value of $v(t)$ as the number of series terms included in the partial sum approaches infinity.

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^{n=N} \tilde{v}_n e^{jn\omega_o t_0} = v(t_0) \text{ if } v(t) \text{ is continuous at } t_0 \text{ and}$$

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^{n=N} \tilde{v}_n e^{jn\omega_o t_0} = \frac{v(t_0^-) + v(t_0^+)}{2} \text{ if } v(t) \text{ is discontinuous at } t_0 \quad (9.4-1)$$

All the terms in a Fourier series are continuous functions of time. Eqn. 9.4-1 seems to suggest that the Fourier series converges to a discontinuity – *i.e.* the series converges to $v(t_0^-)$ at $t = t_0^-$, to $0.5v(t_0^-) + 0.5v(t_0^+)$ at $t = t_0$ and to $v(t_0^+)$ at $t = t_0^+$. How can a sum of continuous functions produce jump discontinuity? In fact it doesn't. The partial sum of Fourier series in Eqn. 9.4-1 oscillates with time around the point of discontinuity. As N is increased, these oscillations in partial sum value get crowded more and more towards a small neighborhood of t_0 . The oscillations never really disappear.

9.5 WAVEFORM SYMMETRY AND FOURIER SERIES COEFFICIENTS

Waveforms can exhibit symmetry about the vertical axis. Figure 9.5-1 shows two waveforms that exhibit the so-called *even symmetry*.

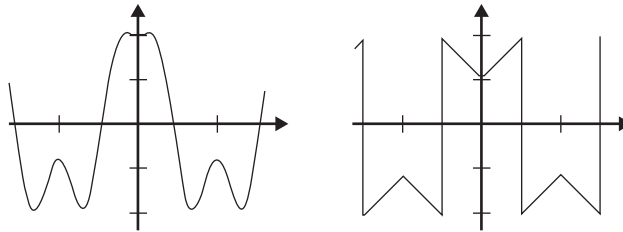


Fig. 9.5-1 Waveforms exhibiting even symmetry

If the right-side portion of the waveform, – *i.e.* the portion of waveform for $t > 0$, is mirror reflected about vertical axis, the reflection coincides with the portion of waveform located in the left side of time-axis. This can be expressed as $v(-t) = v(t)$ for any t . A function $v(t)$ that exhibits this kind of symmetry is said to possess the property of *even symmetry* and is called an *even function*.

Examine the two waveforms shown in Fig. 9.5-2. If the right-side portion of the waveform is mirror reflected about vertical axis, the reflection is equal and opposite to the waveform located in the left side of time-axis. This can be expressed as $v(-t) = -v(t)$ for any t . A function $v(t)$ that exhibits this kind of symmetry is said to possess the property of *odd symmetry* and is called an *odd function*.

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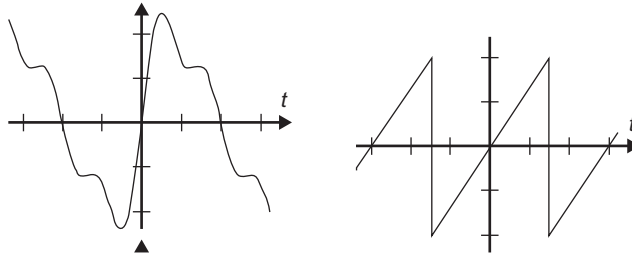


Fig. 9.5-2 Waveforms exhibiting odd symmetry

Even and *odd* symmetry in waveforms is dependent on choice of $t = 0$ point in the horizontal axis. If the vertical axis is moved to the right or left of its current position all the waveforms in Figs. 9.5-1 and 9.5-2 will lose their symmetry properties.

Consider a $v(t)$ which does not exhibit *even* or *odd* symmetry. Let us express $v(t)$ in the following equivalent form.

$$v(t) = \frac{v(t) + v(-t)}{2} + \frac{v(t) - v(-t)}{2}$$
 $v(t)$ is expressed here as sum of two functions. The first function is an even function. So we call it $v_e(t)$.

$$v_e(t) = \frac{v(t) + v(-t)}{2}$$

$$\text{Then, } v_e(-t) = \frac{v(-t) + v(t)}{2} = v_e(t)$$

$\therefore v_e(t)$ is an *even* function on t for any $v(t)$

The second function is an odd function. So we call it $v_o(t)$.

$$v_o(t) = \frac{v(t) - v(-t)}{2}$$

Then,

$$v_o(-t) = \frac{v(-t) - v(t)}{2} = -v_o(t)$$

$\therefore v_o(t)$ is an *odd* function on t for any $v(t)$

Any time-function $v(t)$ can be expressed as the sum of an even function and an odd function. $v_e(t) = \frac{1}{2}[v(t) + v(-t)]$ is termed as the *even part* of $v(t)$ and $v_o(t) = \frac{1}{2}[v(t) - v(-t)]$ is called the *odd part* of $v(t)$.

How will *even* and *odd* symmetry affect the exponential Fourier series coefficients? If $v(t)$ is *even* on t ,

$$\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_o t} dt = \frac{1}{T} \int_0^{T/2} v(t) [e^{-jn\omega_o t} + e^{-jn\omega_o (-t)}] dt = \frac{2}{T} \int_0^{T/2} v(t) \cos n\omega_o t dt.$$

Thus, its Fourier series coefficients $\tilde{v}_n$ will be real numbers. This indicates that its trigonometric Fourier series will contain only cosine terms.

If $v(t)$ is *even* on t , then $\tilde{v}_n$ is real and $b_n = 0$, for all n

$$\therefore v(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\omega_o t$$

$$a_o = \frac{2}{T} \int_0^{T/2} v(t) dt \quad \text{and} \quad a_n = \frac{4}{T} \int_0^{T/2} v(t) \cos n\omega_o t dt$$

The coefficients of exponential Fourier series of an *even* periodic waveform on t will be *real* and its trigonometric Fourier series will contain only *cosine* terms.

This is quite reasonable since sum of odd functions will produce only an odd function, and hence an *even* $v(t)$ cannot contain even a single sine term.

It can be shown similarly that,

If $v(t)$ is *odd* on t , then

$$\tilde{v}_n \text{ is pure imaginary and } = -j \frac{2}{T} \int_0^{T/2} v(t) \sin n\omega_o t dt, a_n = 0, \text{ for all } n$$

and $v(t)$ will contain only sine terms in its trigonometric Fourier series. The DC content in the waveform has to be zero since DC value is an *even* function of time.

If $v(t)$ is *odd* on t , then $\tilde{v}_n$ is imaginary and $a_n = 0$, for all n

$$\therefore v(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_o t \text{ and } b_n = \frac{4}{T} \int_0^{T/2} v(t) \sin n\omega_o t dt$$

The coefficients of exponential Fourier series of an *odd* periodic waveform on t will be *imaginary* and its trigonometric Fourier series will contain only *sine* terms.

Another kind of symmetry exhibited by waveforms is called *half-wave symmetry*. A periodic waveform $v(t)$ is *half-wave symmetric* if one half-cycle of the waveform has the same appearance of the other half cycle inverted about time-axis. This is expressed as

$$\text{Half-wave symmetry} \Rightarrow v(t) = -v\left(t \pm \frac{1}{2}T\right), \text{ for all } t.$$

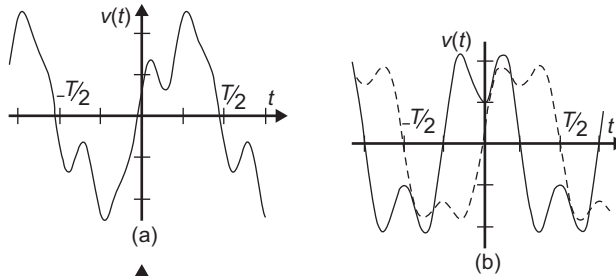


Fig. 9.5-3 Waveforms showing half-wave symmetry

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Figure 9.5-3 shows three waveforms exhibiting *half-wave* symmetry. The waveform in Fig. 9.5-3 (a) shows *half-wave* symmetry, but it does not possess *even* or *odd* symmetry. The waveform represented by solid curve in Fig. 9.5-3 (b) shows *half-wave* symmetry along with *even* symmetry and the one represented by dotted curve shows *half-wave* symmetry along with *odd* symmetry. When a periodic waveform possesses both *half-wave* symmetry and *even* or *odd* symmetry it is said to have *quarter-wave* symmetry. The effect of *half-wave* symmetry on Fourier series coefficients is derived below in the following:

$$\tilde{v}_n = \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} v(t) e^{-jn\omega_o t} dt = \frac{1}{T} \int_0^{\frac{1}{2}T} v(t) e^{-jn\omega_o t} dt + \frac{1}{T} \int_{-\frac{1}{2}T}^0 v(t) e^{-jn\omega_o t} dt = I_1 + I_2$$

Substitute $t' = t + \frac{T}{2}$ in the second integral

$$\begin{aligned} I_2 &= \frac{1}{T} \int_{-\frac{1}{2}T}^0 v(t) e^{-jn\omega_o t} dt = \frac{1}{T} \int_0^{\frac{1}{2}T} v(t' - T/2) e^{-jn\omega_o t'} e^{-jn\omega_o T/2} dt' \\ &= \frac{1}{T} \int_0^{\frac{1}{2}T} -v(t') e^{-jn\omega_o t'} e^{jn\omega_o T/2} dt' \quad (\because v(t) \text{ is half-wave symmetric}) \\ &= \frac{1}{T} \int_0^{\frac{1}{2}T} -v(t') e^{-jn\omega_o t'} e^{jn\pi} dt' \quad (\because \omega_o = \frac{2\pi}{T}) \\ &= (-1)^n \frac{1}{T} \int_0^{\frac{1}{2}T} -v(t') e^{-jn\omega_o t'} dt' \quad (\because e^{jn\pi} \text{ is } 1, \text{ for even } n \text{ and } -1 \text{ for odd } n) \\ &= (-1)^{n+1} \frac{1}{T} \int_0^{\frac{1}{2}T} v(t') e^{-jn\omega_o t'} dt' = -I_1 \text{ for even } n \text{ and } +I_1 \text{ for odd } n \\ \therefore \tilde{v}_n &= \begin{cases} 0 & \text{for even } n \\ 2 \frac{1}{T} \int_0^{\frac{1}{2}T} v(t) e^{-jn\omega_o t} dt & \text{for odd } n \end{cases} \\ a_o &= 0 \\ a_n &= \begin{cases} 0 & \text{for even } n \\ 4 \frac{1}{T} \int_0^{\frac{1}{2}T} v(t) \cos n\omega_o t dt & \text{for odd } n \end{cases} \text{ and } b_n = \begin{cases} 0 & \text{for even } n \\ 4 \frac{1}{T} \int_0^{\frac{1}{2}T} v(t) \sin n\omega_o t dt & \text{for odd } n \end{cases} \end{aligned}$$

The coefficients of exponential Fourier series of an *even* periodic waveform on t will be *real* and its trigonometric Fourier series will contain only *cosine* terms.

The coefficients of exponential Fourier series of an *odd* periodic waveform on t will be *imaginary* and its trigonometric Fourier series will contain only *sine* terms.

A periodic waveform with *half-wave symmetry* does not contain any average value (or DC content) and does not contain any even harmonics.

If a periodic waveform is *even* on t and is *half-wave symmetric*, its Fourier series expansion will contain only *cosine functions* at *odd harmonic frequencies*.

If a periodic waveform is *odd* on t and is *half-wave symmetric*, its Fourier series expansion will contain only *sine functions* at *odd harmonic frequencies*.

9.6 PROPERTIES OF FOURIER SERIES AND SOME EXAMPLES

We take up some examples of Fourier series and develop the important properties of Fourier series through them. The waveforms used in the examples that follow are very important signal waveforms that appear frequently in electrical and electronic engineering applications and are not just some functions used to illustrate Fourier series. The waveforms appearing in the examples are important in their own right.

The first property of Fourier series is almost self-evident. It is the property of *linearity*. It states that if $v_1(t)$ and $v_2(t)$ are two periodic waveforms with same period T and $v_3(t) = a_1 v_1(t) + a_2 v_2(t)$, then, $\tilde{v}_{3n} = a_1 \tilde{v}_{1n} + a_2 \tilde{v}_{2n}$ for all n , where $\tilde{v}_{1n}$, $\tilde{v}_{2n}$ and $\tilde{v}_{3n}$ are the coefficients of exponential Fourier series of $v_1(t)$, $v_2(t)$ and $v_3(t)$, respectively. This may be proved easily.

Example: 9.6-1

Find the exponential Fourier series of $v(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$ and derive the trigonometric Fourier series from it.

Solution

This waveform is a periodic sequence of unit impulses with a period of 1 s. It is shown in Fig. 9.6-1.

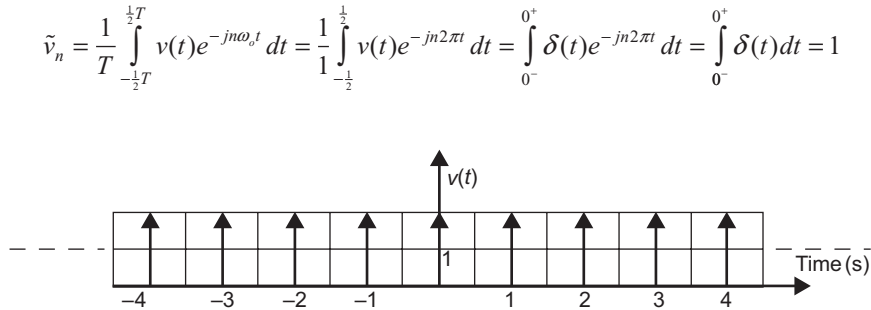


Fig. 9.6-1 Waveform $v(t)$ in Example: 9.6-1

The waveform has zero value at all points in the first period $[-0.5, 0.5]$ except between 0^- and 0^+ . In that interval it is an impulse of unit magnitude. And the value of exponential in that interval is 1. Therefore all coefficients in the exponential Fourier series are equal to 1.

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{n=-\infty}^{\infty} e^{j2\pi n t} \text{ is the exponential Fourier series of } v(t).$$

We obtain the trigonometric Fourier series from exponential Fourier series by taking two components with harmonic order $-n$ and $+n$ together.

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{n=-\infty}^{\infty} e^{j2\pi n t} = e^{j2\pi \cdot 0 \cdot t} + \sum_{n=1}^{\infty} e^{j2\pi n t} + e^{-j2\pi n t} = 1 + \sum_{n=1}^{\infty} 2 \cos 2\pi n t \quad (9.6-1)$$

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The waveform contains a DC component of 1 unit. This should be so since the total area content of the waveform under one period is the area content of unit impulse, *i.e.* unity. This area divided by T must be the DC content in the waveform. T in the example is 1 s.

The waveform contains only cosine terms. This too is expected since $v(t)$ is an *even* function of t .

But most important aspect to be noted is that periodic impulse train contains *all harmonics with equal strength*, the amplitude of harmonic components does not show any let up as the frequency of harmonic component goes up. Sinusoids at all harmonic frequencies with uniform amplitude are required to synthesise the periodic impulse train.

But didn't we stretch the concept of Fourier series a bit too far? Impulse is a highly discontinuous waveform. In fact, the $v(t)$ in this example violates the Dirichlet's condition that, discontinuity, if present, must be of finite value. Hence, though we have found a Fourier series for $v(t)$, will it really converge to $v(t)$ at all t ?

The answer, strictly speaking, is that, it will not. As usual, faced with such mathematical difficulties, we just change our viewpoint and make the Fourier series we derived in this example a useful one! We simply view the impulse as rectangular pulse of large height and small width and unit area. Then we argue that over the width of rectangular pulse in the first period (which is centered around $t = 0$) the exponential factor $e^{-jk\omega_0 t}$ is close to 1 and may be approximated as such. But won't the approximation fail if k becomes very large though t is small? It may, but we will not let it fail; we will state that we will compress the pulse a little more while keeping its area at unity! Hence the Fourier series in Eqn. 9.6-1 represents the Fourier series of a periodic rectangular pulse train with each pulse containing unit area as the width of the pulse is made infinitesimal and height of the pulse is made infinitely large. We keep Dirichlet happy that way!

Example: 9.6-2

Obtain the Fourier series of $v(t) = \sum_{n=-\infty}^{\infty} \delta(t - n - 0.5)$ shown in Fig. 9.6-2.

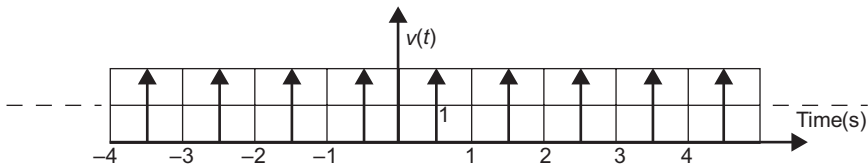


Fig. 9.6-2 Waveform $v(t)$ for Example: 9.6-2

Solution

We observe by comparing Figs. 9.6-1 and 9.6-2 that this waveform is only a *delayed* version of waveform in Fig. 9.6-1. The delay involved is 0.5 s. Therefore if we delay all the sinusoidal components in the Fourier series of waveform in Fig. 9.6-1, we should get the Fourier series of waveform in Fig. 9.6-2. Delaying a sinusoid by t_d s amounts to adding a phase delay of ωt_d rad to its argument, where ω is its radian frequency. That is, ωt has to be replaced by $(\omega t - \phi)$ where $\phi = \omega t_d$.

n^{th} harmonic component in the exponential Fourier series of any waveform is denoted by $\tilde{v}_n e^{jn\omega_0 t}$, where ω_0 is the fundamental radian frequency. Delaying this component by t_d s results in

$\tilde{v}_n e^{j(n\omega_o t - n\omega_o t_d)} = [\tilde{v}_n e^{-jn\omega_o t_d}] e^{jn\omega_o t}$. Therefore, the Fourier series of a time-shifted waveform can be expressed in terms of Fourier series of the original version as follows.

If $\tilde{v}_n$ are the coefficients of exponential Fourier series of a periodic waveform $v(t)$, then the coefficients of exponential Fourier series of the time-shifted periodic waveform $v(t - t_d)$ are given by $\tilde{v}_n e^{-jn\omega_o t_d}$. This is called the ‘*Time-shifting property of Fourier series expansion*’.

In this example, the time delay is 0.5 s and that value is half the period. Therefore, $e^{-jn\omega_o t_d} = e^{-j\pi n} = 1$ for even n and -1 for odd n .

Therefore, Fourier series of $v(t)$ is as shown below as follows:

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t - k - 0.5) = \sum_{k=-\infty}^{\infty} e^{-j\pi n} e^{j2\pi nt} = 1 + \sum_{n=1}^{\infty} (-1)^n 2 \cos 2\pi nt \quad (9.6-2)$$

There is no change in the amplitude of cosine waves, but the phase of waves alternate between 0 and 180° with harmonic order n .

Example: 9.6-3

Find the exponential Fourier series and trigonometric Fourier series of waveforms in Examples 9.6-1 and 9.6-2 if the magnitude of each impulse in both cases is V units and the period is T s.

Solution

We remember that there is a $1/T$ factor in the Fourier series analysis equation. It happened to be 1 when T was set to 1 s. We have to bring that factor back. Similarly, since $\omega_o = \frac{2\pi}{T}$ we have to bring in $1/T$ in the index of exponential in exponential Fourier series and in the argument of trigonometric functions in trigonometric Fourier series.

Hence the required Fourier series for waveform in Example 9.6-1 is

$$\therefore \sum_{k=-\infty}^{\infty} V \delta(t - kT) = \sum_{n=-\infty}^{\infty} \frac{V}{T} e^{j\frac{2\pi}{T} nt} = \frac{V}{T} + \sum_{n=1}^{\infty} \frac{2V}{T} \cos \frac{2\pi}{T} nt$$

and the required Fourier series for waveform in Example 9.6-2 is

$$\therefore \sum_{k=-\infty}^{\infty} V \delta(t - kT - \frac{T}{2}) = \sum_{n=-\infty}^{\infty} \left(\frac{V}{T} e^{-j\pi n} \right) e^{j\frac{2\pi}{T} nt} = \frac{V}{T} + \sum_{n=1}^{\infty} \frac{2V}{T} (-1)^n \cos \frac{2\pi}{T} nt$$

Example: 9.6-4

Let $v_1(t)$ be a periodic waveform same as the one used in Example 9.6-1 and $v_2(t)$ be a periodic waveform same as the waveform used in Example 9.6-2. Then, find $v(t) = v_1(t) - v_2(t)$ and its Fourier series.

Solution

The waveform $v(t)$ is constructed and shown in Fig. 9.6-3.

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We construct the Fourier series of this waveform by using Eqns. 9.6-1 and 9.6-2 and property of linearity of Fourier series.

$$\therefore v(t) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt} - \sum_{n=-\infty}^{\infty} e^{-jn\pi} e^{j2\pi nt} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} 2e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} 4\cos 2\pi nt$$

It contains only odd harmonics of cosine format. Its average value is zero. It should be so since this waveform has *even* symmetry and *half-wave* symmetry.

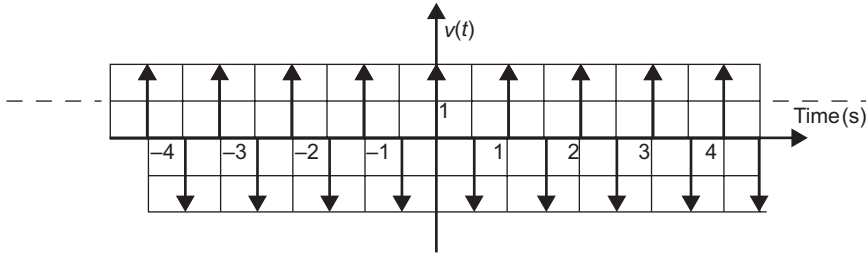


Fig. 9.6-3 Waveform of $v(t)$ in Example: 9.6-4

Example: 9.6-5

$v(t)$ is a periodic waveform with same waveform as that of Fig. 9.6-3, but with 2 units of magnitude in each impulse. Find and plot $v_i(t) = \int_{-\infty}^t v(t)dt$ and obtain Fourier series of $v_i(t)$.

Solution

Integration usually results in an arbitrary constant. This constant will become a DC component in $v_i(t)$. Since there is no way to find out this arbitrary constant we choose to ignore it and we qualify the Fourier series of $v_i(t)$ by adding a clause that a DC term can always come in the series without upsetting other coefficients. The waveform of $v_i(t)$, ignoring a possible DC term, is shown in Fig. 9.6-4.

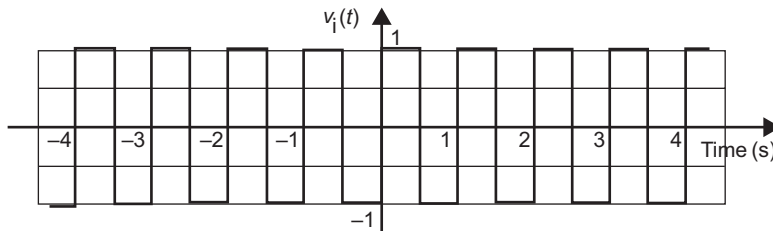


Fig. 9.6-4 Waveform of $v_i(t)$ in Example: 9.6-5

The DC content was ignored. Therefore $v_i(t)$ must be pure AC. At $t = 0$ its value will change by the area content of impulse at that point. This area content is 2 units. Hence the value of $v_i(t)$ must change by 2 units at $t = 0$. Then, it must remain constant. At $t = 0.5$ s the negative impulse of 2 unit magnitude will again change $v_i(t)$ by -2 units. With the assumption of AC output, $v_i(t)$ must then be alternating between $+1$ and -1 with a period of 1 s. Hence $v_i(t)$ is a symmetric square wave of amplitude 1 unit and period 1 s.

We wish to find the Fourier series of $v_i(t)$. There are two ways to do it. The first method is to employ the analysis equation of Fourier series, Eqn. 9.2-2, and carry out the required integration to determine the Fourier series coefficients. The second method relies upon the fact the waveform $v_i(t)$ is the integral of $v(t)$ and that we already know the Fourier series of $v(t)$. That leads us to the question – what is the relationship between Fourier series coefficients for a periodic waveform and Fourier series coefficients for its integral?

$$v_i(t) = \int_{-\infty}^t v(t) dt = \int_{-\infty}^t \sum_{n=-\infty}^{\infty} \tilde{v}_n e^{jn\omega_o t} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^t \tilde{v}_n e^{jn\omega_o t} dt = \sum_{n=-\infty}^{\infty} \frac{\tilde{v}_n}{jn\omega_o} e^{jn\omega_o t}$$

$$\therefore \tilde{v}_{in} = \frac{\tilde{v}_n}{jn\omega_o} \quad (9.6-3)$$

Note that $v(t)$ should not have an average value, i.e. $\tilde{v}_0 = a_0 = 0$. If it has a non-zero average value, then its integral will have a linearly increasing term which makes it aperiodic and unbounded. There is no Fourier series for such a waveform.

We have arrived at the *Integration in Time* property of Fourier series.

If $v(t)$ is a zero-average periodic waveform with $\tilde{v}_n$ as its exponential Fourier series coefficients, then the exponential Fourier series coefficients of $\int_{-\infty}^t v(t) dt$ is $\frac{\tilde{v}_n}{jn\omega_o}$.

The Fourier series coefficients of half of $v(t)$ in this example was derived in Example 9.6-4.

$$v(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} 4e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} 8 \cos 2\pi nt$$

$$\therefore v_i(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{4}{j2\pi n} e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi nt} + \frac{2}{j\pi(-n)} e^{j2\pi(-n)t} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{4}{\pi n} \sin 2\pi nt$$

A unit amplitude symmetric square wave of period T will also have same coefficients, i.e. $\frac{4}{\pi n}$. The factor of $\frac{1}{T}$ gets cancelled by a similar factor that is included in ω_o in Eqn. 9.6-3.

A symmetric square wave contains only odd harmonics and the harmonic amplitude $\left(= \frac{4}{\pi n} \text{ in trigonometric Fourier series} \right)$ decreases in inverse proportion to harmonic order n .

Example: 9.6-6

$v(t)$ is a ± 4 unit symmetric square wave with $T = 1$ s. Find and plot $v_i(t) = \int_{-\infty}^t v(t) dt$ and find its Fourier series.

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$v(t)$ and $v_i(t)$ are shown in Fig. 9.6-5.

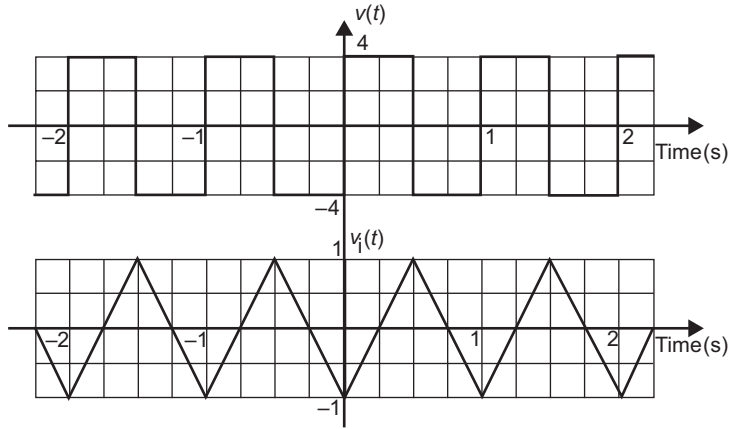


Fig. 9.6-5 Waveforms of $v(t)$ and its integral, $v_i(t)$, in Example: 9.6-6

Using the result from the previous example, we get, $v(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{8}{j\pi n} e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{16}{\pi n} \sin 2\pi nt$.

The exponential Fourier series coefficients get divided by $jn\omega_0$ when the waveform gets integrated in time.

$$\therefore v_i(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{16}{(j2\pi n)^2} e^{j2\pi nt} = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} -\frac{4}{\pi^2 n^2} (e^{j2\pi nt} + e^{-j2\pi nt}) = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{8}{\pi^2 n^2} \cos 2\pi nt$$

The triangle wave in Fig. 9.6-5 has ± 1 unit amplitude and has *even half-wave* symmetry. Hence it contains odd cosine harmonics.

A symmetric triangle wave contains only odd harmonics and the harmonic amplitude $\left(= \frac{8}{\pi^2 n^2} \right.$ in trigonometric Fourier series) decreases in inverse proportion to square of harmonic order n .

Example: 9.6-7

Let $v(t)$ be a periodic waveform with period T and let $v_c(t)$ be defined as $v_c(t) = v(\alpha t)$, where $\alpha > 0$. Show that $v(t)$ and $v_c(t)$ have same coefficients in their Fourier series.

Solution

$v_c(t)$ will be a time-compressed version of $v(t)$ for $\alpha > 1$ and time-expanded version of $v(t)$ for $\alpha < 1$.

Therefore period of $v_c(t)$ will be $\frac{T}{\alpha}$. Let $\tilde{v}_{cn}$ and $\tilde{v}_n$ be the exponential Fourier series coefficients of $v_c(t)$ and $v(t)$ respectively.

$$\begin{aligned}\tilde{v}_{cn} &= \frac{\alpha}{T} \int_{-\frac{1}{2\alpha}}^{\frac{1}{2\alpha}} v_c(t) e^{j\frac{2\alpha\pi}{T}nt} dt \\ &= \frac{\alpha}{T} \int_{-\frac{1}{2\alpha}}^{\frac{1}{2\alpha}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} dt \\ &= \frac{\alpha}{T} \int_{-\frac{1}{2\alpha}}^{\frac{1}{2\alpha}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} \frac{1}{\alpha} d(\alpha t) \\ &= \frac{1}{T} \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} d(\alpha t) \\ &= \tilde{v}_n\end{aligned}$$

Compression or expansion of a periodic waveform in time does not change its Fourier series coefficients. But fundamental frequency and harmonic frequency values will change. This property is called *time-scaling property of Fourier series*.

Note that *time-scaling* is not the same as a simple change in T alone. Let $v(t)$ a periodic train of rectangular pulses of unit amplitude and 0.1 s width repeating with a period of 1 s. Then $v(0.5t)$ is a periodic train of rectangular pulses of unit amplitude and 0.2 s repeating with a period of 2 s. Its Fourier series coefficients will be same as that of $v(t)$, but its fundamental frequency will be π rad/s whereas that of $v(t)$ will be 2π rad/s.

However, consider another periodic train of rectangular pulses of unit height and 0.1 s width repeating with a period of 2 s. This is not the same as $v(0.5t)$. Its Fourier series coefficients will be different from that of $v(t)$. In fact, all Fourier series coefficients will get multiplied by $\frac{1}{2}$.

Example: 9.6-8

Find the Fourier series of the periodic rectangular pulse train shown in Fig. 9.6-6.

Solution

$$\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_o t} dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} e^{-jn\omega_o t} dt = \frac{-1}{jn\omega_o T} [e^{-jn\omega_o \tau/2} - e^{jn\omega_o \tau/2}]$$

$$e^{-jn\omega_o \tau/2} - e^{jn\omega_o \tau/2} = -2j \sin \frac{n\omega_o \tau}{2}, \text{ by Euler's Formula}$$

$$\therefore \tilde{v}_n = \frac{\tau}{T} \frac{2 \sin \frac{n\omega_o \tau}{2}}{n\omega_o T} = \frac{\tau}{T} \frac{\sin \frac{n\omega_o \tau}{2}}{n\omega_o \tau/2} = \left(\frac{\tau}{T}\right) \left(\frac{\sin x}{x}\right), \text{ where } x = \frac{n\omega_o \tau}{2}$$

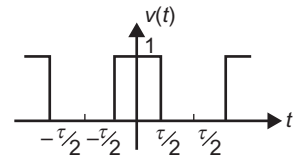


Fig. 9.6-6 Waveform for Example: 9.6-8

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The DC content of the waveform is $\frac{\tau}{T}$. The Fourier series contains only cosine terms since exponential Fourier series coefficients are real.

Example: 9.6-9

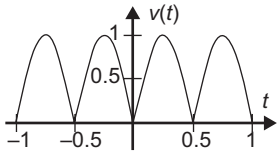


Fig. 9.6-7 Full-wave rectified waveform in Example: 9.6-9

Figure 9.6-7 shows the output of an absolute value circuit when input is a sinusoidal wave. This is the shape of output voltage of a full-bridge diode rectifier routinely used in AC–DC conversion applications. We wish to obtain the Fourier series for this waveshape. We bring out an important property of Fourier series first and use that property to obtain the required Fourier series in this example.

Solution

The waveform $v(t)$ is visualised first as the product of two waveforms as in Fig. 9.6-8.

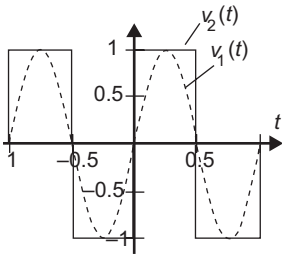


Fig. 9.6-8 Two waveforms in a product results in $v(t)$ of Example: 9.6-9

The product of $v_2(t)$ which is a unit amplitude square wave with $v_1(t)$ which is a pure sine wave results in $v(t)$, the full-wave rectified waveform.

That raises the question – what are the exponential Fourier series coefficients of a product of two waveforms with same period in terms of exponential Fourier series coefficients of the constituent waveforms?

$$v_1(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} e^{jn\omega_o t} \quad \text{and} \quad v_2(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_{2n} e^{jn\omega_o t}$$

$$v(t) = v_1(t) v_2(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_k e^{jk\omega_o t}$$

Consider k th component in the exponential Fourier series of $v(t)$. The waveform contributed to $v(t)$ by this component is $\tilde{v}_k e^{jk\omega_o t}$. Since $v(t)$ is the product of $v_1(t)$ and $v_2(t)$, this contribution can come up in $v(t)$ due to products of n th contribution $\tilde{v}_{1n} e^{jn\omega_o t}$ in $v_1(t)$ and $(k-n)$ th contribution $\tilde{v}_{2(k-n)} e^{j(k-n)\omega_o t}$ with n varying from $-\infty$ to $+\infty$.

$$\therefore \tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}$$

This is the so-called *Multiplication in Time* property of Fourier series.

If $v_1(t)$ and $v_2(t)$ are two periodic waveforms with same period and $v(t) = v_1(t) \times v_2(t)$, then the exponential Fourier series coefficients of $v(t)$ is given by $\tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}$ for $-\infty < k < \infty$, where $\tilde{v}_{1n}$ and $\tilde{v}_{2n}$ are the exponential Fourier series coefficients of $v_1(t)$ and $v_2(t)$, respectively.

$v_1(t)$ is a sine wave in this example.

$$\begin{aligned} v_1(t) &= \sin 2\pi t \\ &= \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} \quad (\text{By Euler's Formula}) \\ \therefore \tilde{v}_1 &= \frac{1}{2j}, \tilde{v}_{-1} = \frac{-1}{2j} \text{ and } \tilde{v}_n = 0, \text{ for all other values of } n \end{aligned}$$

$v_2(t)$ is a unit square wave. Its Fourier series was obtained in Example 9.6-5 as

$$\begin{aligned} \tilde{v}_{2n} &= \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi n t} \\ \therefore \tilde{v}_k &= \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)} = -\frac{1}{2j} \frac{2}{j\pi(k+1)} + \frac{1}{2j} \frac{2}{j\pi(k-1)}, \text{ for even } k \\ &= \frac{-2}{\pi(k^2 - 1)}, \text{ for even } k \end{aligned}$$

Since n takes only two values -1 and 1 , and square wave has only odd harmonics, $(k+1)$ and $(k-1)$ have to be odd for $\tilde{v}_k$ to be non-zero.

$$\therefore v(t) = \sum_{\substack{k=-\infty \\ \text{even } k}}^{\infty} \frac{-2}{\pi(k^2 - 1)} e^{j2\pi kt} = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\substack{k=1 \\ \text{even } k}}^{\infty} \frac{1}{(k^2 - 1)} \cos 2\pi kt \text{ is the Fourier series of a full-wave}$$

rectified sinusoid of unit amplitude.

The average value of rectified waveform is $\frac{2}{\pi}$ or $\frac{2V_m}{\pi}$ in general, where V_m is the peak value of sine wave undergoing rectification. It has only even cosine harmonics in it. What kind of symmetry is responsible for this? It is *even* function and hence only cosines are expected. Notice that the fundamental period of the waveform is 0.5 s and not 1 s. This 1 s period comes up only in the square waveform and sine waveform we used to form the product. Hence, there is no component at odd harmonic order in the rectified output. If we treat it as a waveform with 0.5-s period, all odd and even harmonic components of its fundamental frequency are present in it. Its fundamental frequency will be double the fundamental frequency of the sinusoid that underwent rectification.

9.7 DISCRETE MAGNITUDE AND PHASE SPECTRUM

Spectral plot for a time-domain waveform displays the Fourier series coefficients graphically against frequency. Since the frequencies involved in a Fourier series are discrete values (fundamental frequency and its multiples), a plot of Fourier series coefficients cannot be a continuous curve. Therefore, the spectral plot is called a *discrete spectrum*. The exponential Fourier series coefficients are complex in general and two plots – one for magnitude of coefficients and the other for phase angle of the coefficients – will be needed.

The information on coefficients is portrayed as a series of vertical lines located at harmonic frequencies. These lines will be equidistant and the length of the lines will be proportional to

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magnitude of the coefficient in the case of magnitude spectrum and to phase in the phase spectrum. The harmonic order n is also used in the abscissa instead of ω or f . The discrete spectral plots of the unit amplitude square wave we covered in Example 9.6-5 is shown in Fig. 9.7-1 for illustration. Its exponential Fourier series is $\tilde{v}_n = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi n t}$.

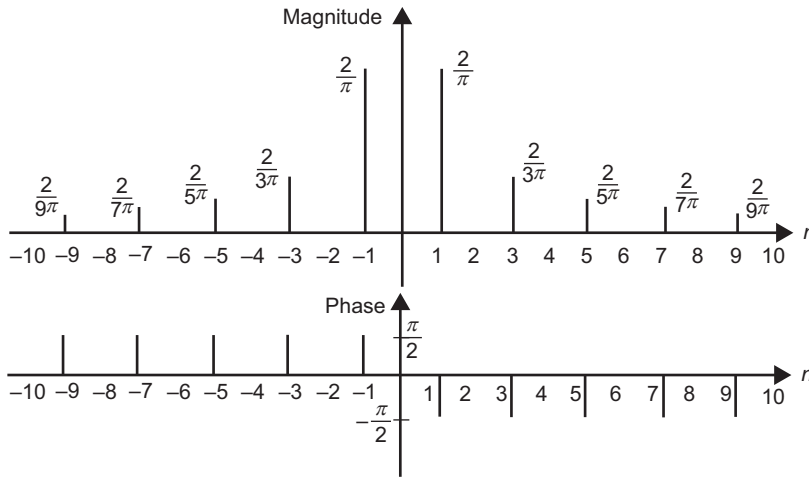


Fig. 9.7-1 Discrete magnitude spectrum and phase spectrum for $a \pm 1$ square wave against harmonic order

The Fourier series coefficients of exponential Fourier series were plotted in the spectrum and that results in the so-called *two-sided spectrum*. It has been pointed out in earlier discussion that, in exponential Fourier series, the two companion components at n and $-n$ always go together. Two such components will add up to yield a *real* sinusoid always. They cannot be split.

That the two components similarly placed on the left-hand and right-hand side of origin in a two-sided spectrum should be viewed as an integral unit rather than as two separate components is to be kept in mind, especially when interpreting two-sided spectral plots drawn against ω . If we forget that, we will be tempted to ask that often repeated question – *what is the meaning of negative frequency?*

There is no negative *cyclic frequency*. There is no negative *angular frequency*. There are only two complex exponential functions $-e^{j\omega t}$ and $e^{-j\omega t}$. These two always get scaled by complex conjugate numbers and enter into a sum. They never appear individually once the circuit problem has been solved. They always go together and produce either a $\sin \omega t$ or a $\cos \omega t$ or a mixture of the two. Whatever they produce at the end will have an angular frequency of ω rad/s and a cyclic frequency of $\omega/2\pi$ Hz. *No electrical linear circuit can ever do any processing on $e^{j\omega t}$ without carrying out the same processing on $e^{-j\omega t}$.*

Both of them are complex exponential functions of time. Hence they have real and imaginary parts. Both, real and imaginary parts, are sinusoids. Those sinusoids have angular frequency of ω rad/s and cyclic frequency of $\omega/2\pi$ Hz, whether they come from $e^{j\omega t}$ or from $e^{-j\omega t}$. Therefore, *there is no negative radian frequency or cyclic frequency.*

However, we want to represent the magnitudes of scaling factors of $e^{j\omega t}$ and $e^{-j\omega t}$ and phases of scaling factors separately in a spectral plot. Therefore, as a part of notation for presenting information efficiently, we decide to extend the ω -axis to the left and put the data on scaling factor of $e^{-j\omega t}$ there. That does not make a value on the left-hand side of ω -axis a *negative frequency*.

Note that the magnitude spectrum of a real $v(t)$ has to be necessarily *even* on ω and its phase spectrum has to be necessarily *odd* on ω . (Why?)

Example: 9.7-1

Some desktop off-line uninterruptible power supply (UPS) units used for supplying single PC units deliver the waveform as shown in Fig. 9.7-2 instead of a sine wave. (a) Find α if the third harmonic content is to become zero. (b) With this value of α , find V such that the rms voltage is 220 V. (c) Plot the magnitude and phase spectra with this value of α and V . (d) The purity of a sine wave is measured in terms of a quantity called ‘Total Harmonic Distortion (THD)’. It is usually quoted in percentage and is defined as follows:

$$\text{THD} = \frac{\sqrt{\sum_{n=2}^{\infty} |\tilde{v}_n|^2}}{|\tilde{v}_1|} \times 100\%, \text{ where } \tilde{v}_n \text{ is the}$$

exponential Fourier series coefficient. The rms value of all harmonic components together is expressed as a percentage of rms value of fundamental component in the THD measure. Amplitudes may be used instead of rms values since it is a ratio. Calculate the THD of the waveform in this example.

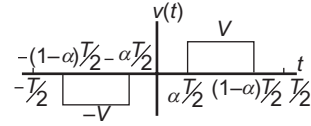


Fig. 9.7-2 Waveform for Example: 9.7-1

Solution

The trigonometric Fourier series of the waveform with $V = 1$ and $T = 1$ is determined first. It is an *odd half-wave symmetric* waveform. Its trigonometric Fourier series will contain only odd sine harmonics. With $V = 1$ and $T = 1$,

$$\begin{aligned} \therefore v(t) &= \sum_{n=1}^{\infty} b_n \sin n\omega_o t \text{ and } b_n = \frac{4}{T} \int_0^{T/2} v(t) \sin n\omega_o t dt \\ b_n &= 4 \int_{\alpha/2}^{(1-\alpha)/2} \sin 2\pi n t dt \\ &= \frac{4}{2\pi n} [\cos n\pi\alpha - \cos n\pi(1-\alpha)] \\ &= \frac{4}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi(1-2\alpha)}{2} \end{aligned}$$

The trigonometric Fourier series coefficients tends to zero for even n as expected. Further, the Fourier series of this waveform approaches that of a unit amplitude square wave as $\alpha \rightarrow 0$ as expected.

- (a) If the third harmonic content is to be zero, then $\sin \frac{3\pi(1-2\alpha)}{2}$ must become zero. Therefore, $\alpha = \frac{1}{6}$. With this value of α , the waveform will be zero for one-third of a half-cycle.

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- (b) Therefore rms value = $\sqrt{\frac{2}{3}V^2} = 0.8165V$. If this is to be 220 V, V must be ≈ 270 V.
- (c) The exponential Fourier series of $v(t)$ can be constructed from trigonometric Fourier series by noting that coefficients of sine terms will be twice the negative of imaginary part of exponential Fourier series coefficients. Therefore, with $V = 1$, $T = 1$ and $\alpha = 1/6$,

$$\therefore v(t) = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} -\frac{j2}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi}{3} e^{j2\pi nt} \quad (9.7-1)$$

The two-sided spectrum is plotted in Fig. 9.7-3 with the scaling factor of 270 V incorporated.

- (d) Refer to Eqn. 9.7-1. $\sin \frac{n\pi}{2}$ has a magnitude of 1 for all odd n . $\sin \frac{n\pi}{3}$ has a magnitude of 0 for all odd multiples of 3 and 0.866 for all other odd n including $n = 1$. $\frac{8}{\pi}$ is a common factor.

$$\text{Therefore, THD} = \sqrt{\frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \dots} \times 100 \approx 28.5\%$$

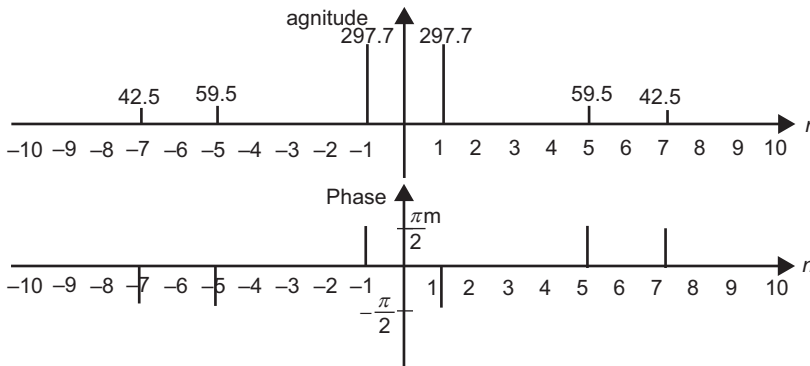


Fig. 9.7-3 Spectral plots for $v(t)$ in Example: 9.7-1

9.8 RATE OF DECAY OF HARMONIC AMPLITUDE

Fourier series is an infinite series. It requires infinite number of sinusoids with frequencies ranging from fundamental frequency to infinitely large frequency to synthesise a non-sinusoidal periodic waveform in general. There may be special cases where the Fourier series terminates at some finite harmonic order. But they are only special cases.

This indicates that we have to find out the AC steady-state response of the circuit to each and every component in Fourier series of input and sum them up, to get the periodic steady-state response of the circuit. That calls for infinite computation – we will not get done with it. Hence, the issue of rate of decay of harmonic amplitudes is of practical significance in deciding how many terms from the Fourier series should we carry in any analysis problem.

Circuits carrying high frequency voltages and currents cause electromagnetic interference (EMI) in themselves as well as in neighbouring circuits. This EMI takes the form of induced voltages and currents due to electromagnetic coupling and electrostatic coupling between circuits as well as due to electromagnetic radiation. Every circuit carrying time-varying voltage and current acts as a transmitting antenna and receiving antenna simultaneously. The induced voltages and currents can lead to malfunction in circuits if not actual damage.

EMI happens at all frequencies. However, the induced voltages are usually of negligible magnitude at low frequencies. Therefore, a designer will often be forced to take out high frequency content from circuit waveforms for reducing destructive electromagnetic interference. An appreciation of how the Fourier series coefficients vary with harmonic order and the factors governing such variation helps him in such a task.

We noted in Example 9.6-1 and subsequent examples in Section 9.6 that periodic impulse train waveforms of different type will have Fourier series with coefficients which *do not vary* with harmonic order n . The amplitude of harmonics is independent of harmonic order in such waveforms.

Starting from such impulse train waveforms, the *integration in time* property of Fourier series helps us to see that periodic square waveforms and periodic rectangular pulse waveforms should possess Fourier series with their coefficients decreasing in inverse proportion to the harmonic order. Integrating an impulse train results in a waveform that will have step discontinuities in one period. Thus, we conclude that periodic waveforms which contain step discontinuities will have Fourier series coefficients that are proportional to $\frac{1}{n}$, where n is the harmonic order.

Integration brings a factor of $\frac{1}{n}$ in the Fourier series coefficients. Integrating square or rectangular pulse waveforms result in waveforms which contain sections in which it varies linearly with time and sections in which it remains constant. Such waveforms are continuous, but their first derivative will have step discontinuities. Their second derivative will contain impulse train along with other possible components. Thus we conclude that periodic waveforms that have impulses in their second derivative will have Fourier series with coefficients decreasing with $\frac{1}{n^2}$ at the least. There may be terms involving $\frac{1}{n^3}, \frac{1}{n^4}$ etc. in the Fourier series of such a waveform, but the terms involving $\frac{1}{n^2}$ are the ones which decide how many terms in the Fourier series are to be included in a circuit analysis context or EMI context.

By extending this reasoning we may state qualitatively that if a periodic waveform $v(t)$ requires m successive differentiation operations before impulses make their appearance, then the harmonic amplitude in its Fourier series will decrease with $\frac{1}{n^m}$ at the least.

Example: 9.8-1

Some desktop UPS units supplying single PC units deliver the waveform $v(t)$ shown in Fig. 9.8-1. The trapezoidal shape is expected to reduce the THD of the wave compared to the square wave in Example 9.7-1. (a) Obtain an expression for rms value of this voltage in terms of V and α . (b) Find the Fourier series coefficients for $v(t)$. (c) Find V and α such that the third harmonic content is zero and the rms value is 220 V. (d) With these values for V and α , find the THD of $v(t)$.

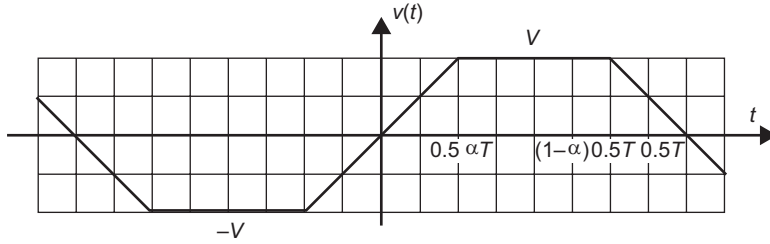


Fig. 9.8-1 Waveform for Example: 9.8-1

Solution

The waveform $v(t)$ exhibits *odd* symmetry and *half-wave* symmetry. Its Fourier series will contain only odd harmonics of sine format.

$$(a) \text{ rms value of } v(t) = \sqrt{\frac{4}{T} \int_0^{0.5\alpha T} \left(\frac{2Vt}{\alpha T} \right)^2 dt + V^2(0.25T - 0.5\alpha T)} = V \sqrt{1 - \frac{4}{3}\alpha} \quad V$$

- (b) Fourier series coefficients can be found by a straight application of the analysis equation of Fourier series (Eqn. 9.2-2). However, a more elegant method is to consider a waveform which will produce $v(t)$ on integration. Find its Fourier series coefficients and obtain the Fourier series coefficients of $v(t)$ by dividing those coefficients by $j n \omega_0$. The waveform shown in Fig. 9.8-2 is the required one.
- (c) Exponential Fourier series coefficients for this waveform is found by splitting the waveform into two – the upper half-cycles alone will constitute a periodic rectangular pulse train with period T and the lower half-cycles alone will constitute the same rectangular pulse train delayed by $0.5T$ and negated. Refer Example 9.6-8 for Fourier series of such a periodic rectangular pulse train.

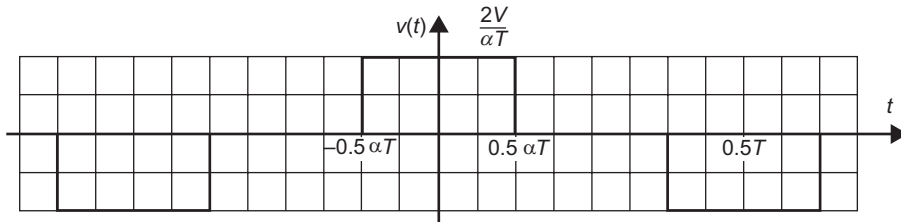


Fig. 9.8-2 Waveform of derivative of $v(t)$ in Example: 9.8-1

$$v_n = \frac{2V}{\alpha T} \frac{1}{jn\omega_0 T} \left(e^{j0.5n\omega_0 \alpha T} e^{-j0.5n\omega_0 \alpha T} \right) - \frac{2V}{\alpha T} \frac{e^{-j0.5n\omega_0 T}}{jn\omega_0 T} \left(e^{j0.5n\omega_0 \alpha T} - e^{-j0.5n\omega_0 \alpha T} \right)$$

$$e^{-j0.5n\omega_0 T} = e^{-jn\pi} = 1 \text{ for even } n \text{ and } -1 \text{ for odd } n$$

$$\begin{aligned}\therefore \tilde{v}'_n &= \begin{cases} 0, & \text{for even } n \\ \frac{2V}{\alpha T} \frac{2}{jn\omega_o T} (e^{j0.5n\omega_o \alpha T} - e^{-j0.5n\omega_o \alpha T}), & \text{for odd } n \end{cases} \\ &= \begin{cases} 0, & \text{for even } n \\ \frac{4V}{T} \frac{\sin \pi n \alpha}{\pi n \alpha}, & \text{for odd } n \end{cases}\end{aligned}$$

Coefficients of exponential Fourier series of $v(t)$ is obtained by dividing these coefficients by $jn\omega_o$.

$$\therefore \tilde{v}_n = \begin{cases} 0, & \text{for even } n \\ -j \frac{2V}{\pi n} \frac{\sin \pi n \alpha}{\pi n \alpha}, & \text{for odd } n \end{cases} \quad 9.8-1)$$

The rate of decay of harmonic amplitude must be $\propto \frac{1}{n^2}$ and it is seen to be so. Notice that the waveform becomes a square waveform when $\alpha = 0$. The Fourier series coefficients given in Eqn. 9.8-1 become the same as that of a square wave, i.e. $\frac{2V}{\pi n}$ for even n and when $\alpha = 0.5$, $v(t)$ becomes a triangle waveform with $\frac{4V}{\pi^2 n^2}$ for odd n as magnitude of exponential Fourier series coefficient magnitude.

- (a) Third harmonic content will go to zero when $\sin 3\pi\alpha = 0$. Therefore, $\alpha = 1/3$ is the required value. The rms value with this value of α is $0.7453V$ and for 220 V rms value V must be ≈ 295 V.
- (b) The THD is evaluated as

$$\text{THD} = 100 \times \sqrt{\sum_{\substack{n=3 \\ \text{odd } n}}^{\infty} \frac{1}{n^2} \left(\frac{\sin n\pi/3}{n\pi/3} \right)^2} \div \frac{\sin \pi/3}{\pi/3} \approx 4.6\%.$$

The faster decay of harmonic amplitude with harmonic order due to the slanting portion of this trapezoidal waveform has yielded a much better approximation to a pure sine wave than the waveform in Example 9.7-1. The next step to improve THD further will be to replace the flat portion of the waveform by another slanting portion with lesser slope than in the first section.

9.9 ANALYSIS OF PERIODIC STEADY-STATE USING FOURIER SERIES

The first step of analysis is the determination of Fourier series of the periodic input waveform in exponential format or trigonometric format. The second step is the determination of frequency response function connecting the required output variable to the input variable. Third step is the determination of steady-state response to each term in the Fourier series of input waveform. Fourth step is to combine all these steady-state response components and to obtain the instantaneous waveform of output after deciding the number of terms to be retained in the truncated Fourier series of output waveform.

This procedure is illustrated through examples to follow.

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Example: 9.9-1

A periodic ramp voltage waveform shown in Fig. 9.9-1 is applied to an RC low-pass circuit with time constant of 0.1 s. The period of the input waveform is 1 s. Find and plot the steady-state output voltage across the capacitor.

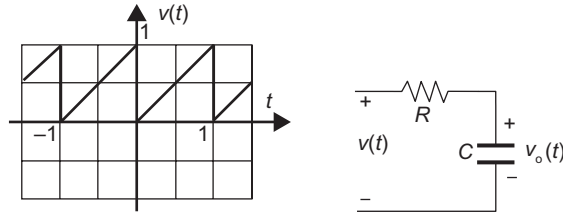


Fig. 9.9-1 The input waveform and the circuit for Example: 9.9-1

Solution

Step 1: Find the Fourier series of $v(t)$

$$\tilde{v}_n = \frac{1}{T} \int_0^T v(t) e^{-jn\omega_o t} dt \quad (\because \text{integration can be performed over any interval that is one period wide})$$

$$\begin{aligned} \therefore \tilde{v}_n &= \int_0^1 t e^{-j2\pi n t} dt \quad (\because T = 1, \omega_o = 2\pi) \\ &= -\frac{1}{(-j2\pi n)^2} e^{-j2\pi n t} \Big|_0^1 + \frac{1}{-j2\pi n} t e^{-j2\pi n t} \Big|_0^1 \\ &= 0 + \frac{1}{-j2\pi n} = \frac{j}{2\pi n} \end{aligned}$$

This expression is valid only for $n \neq 0$

$$\begin{aligned} \tilde{v}_0 &= \frac{1}{T} \int_0^T v(t) dt = \int_0^1 t dt = \frac{1}{2} \\ \therefore a_0 &= 0.5, a_n = 0 \text{ for } n \neq 0, b_n = -\frac{1}{\pi n} \\ v(t) &= 0.5 - \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin 2n\pi t \end{aligned}$$

Step 2: Find the frequency response function

$$\frac{V_o(j\omega)}{V(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega\tau} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \angle -\tan^{-1} \omega\tau$$

Substituting $\tau = 0.1$ s,

$$\frac{V_o(j\omega)}{V(j\omega)} = \frac{1}{\sqrt{1 + 0.01\omega^2}} \angle -\tan^{-1} 0.1\omega$$

Step 3: Find the steady-state response to input components

Output corresponding to DC content in input = 0.5 (since DC gain = 1)

Output phasor for n^{th} harmonic component

$$= -\frac{1}{\pi n} \frac{1}{\sqrt{1+0.01 \times (2\pi n)^2}} \sin(2n\pi t - \tan^{-1} 0.2\pi n)$$

$$= -\frac{1}{\pi n} \frac{1}{\sqrt{1+0.395n^2}} \sin(2n\pi t - \tan^{-1} 0.63n)$$

Step 4: Form the total output and decide the number of terms needed in the truncated series.

$$v_o(t) = 0.5 - \sum_{n=1}^{\infty} \frac{1}{\pi n} \frac{1}{\sqrt{1+0.395n^2}} \sin(2n\pi t - \tan^{-1} 0.63n)$$

The DC content in the output is 0.5. We can choose to ignore all those harmonic terms that have amplitude less than 1% of this value.

$$\frac{1}{\pi n} \frac{1}{\sqrt{1+0.395n^2}} = 0.005$$

Ignore 1 under square root for an approximate solution.

$$\Rightarrow n = 10$$

$$\text{Then, } v_o(t) \approx 0.5 - \sum_{n=1}^{10} \frac{1}{\pi n} \frac{1}{\sqrt{1+0.395n^2}} \sin(2n\pi t - \tan^{-1} 0.63n)$$

This output is plotted in Fig. 9.9-2. More terms will have to be included to remove the fine oscillations that appear in the waveform. Note that the circuit is not able to follow the sharp fall that takes place in the input at the end of every period. *Any non-zero time constant is too slow to follow an instantaneous change.* $v_o(t)$ can almost reach 1 if time constant is reduced further, but it will not reach zero for any time constant.

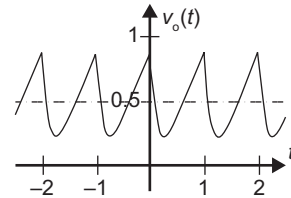


Fig. 9.9-2 Output waveform $v_o(t)$ in Example: 9.9-1

Example: 9.9-2

The current source in the circuit in Fig. 9.9-3 represents a power electronic load (called a DC–DC chopper, used to step down DC voltages) that is drawing a pulsed current at 20 kHz. The current drawn by the load under a particular operating condition is shown as $i_s(t)$ in Fig. 9.9-3. The LC filter is expected to hold the voltage presented to the load at a constant level and to smooth the current in the battery. Pulsed current has adverse impact on battery life. Solve for $i_L(t)$, $i_C(t)$ and $v_o(t)$ under steady-state.

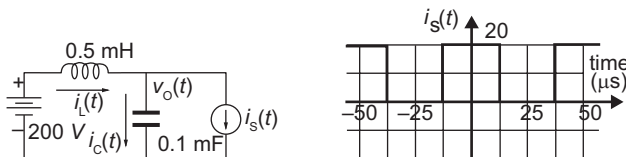


Fig. 9.9-3 Circuit and waveform for Example: 9.9-2

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Solution

The Fourier series coefficients of a symmetric square wave has been worked out in Example 9.6-5. They are $\frac{4}{\pi n}$ in the trigonometric Fourier series. The waveform of $i_s(t)$ here contains a DC component of 10 A. Hence $i_s(t) - 10$ will be a symmetric square wave that has *even* symmetry. The square wave in Example 9.6-5 had odd symmetry and had sine terms in its Fourier series. Here it will be cosine terms.

$$\therefore i_s(t) = 10 + \frac{40}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \cos 40 \times 10^3 n \pi t \text{ A}$$

Inductor behaves as an open-circuit and capacitor behaves as a short-circuit under DC steady-state conditions. $\therefore$ DC component in $v_o(t) = 200 \text{ V}$, in $i_L(t) = 10 \text{ A}$ and in $i_C(t) = 0 \text{ A}$.

The 200V battery is replaced by a short-circuit when the circuit solution for the AC components of $i_s(t)$ is attempted. This results in a parallel LC circuit with current excitation. Current division principle in parallel impedances gives the frequency response of inductor current as

$$\frac{I_L(j\omega)}{I_s(j\omega)} = \frac{\frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}} = \frac{1}{1 - \omega^2 LC} = \frac{1}{1 - 789.6n^2} \approx -\frac{1.267 \times 10^{-3}}{n^2}$$

Thus the amplitude of inductor current at 20 kHz will be $1.267 \times 10^{-3} \times 40 / \pi = 0.016 \text{ A}$. This is 0.16% of the DC current in it. We do not have to calculate the contribution due to other current harmonics. They will be very negligible due to the n^2 factor in the denominator of current division ratio and the $1/n$ factor in the Fourier series of $i_s(t)$. Therefore, the inductor current is practically DC. Similarly, it can be shown that $v_o(t)$ is practically DC.

However, in practice, the 100 μF aluminium electrolytic capacitor used will have a series resistance along with its capacitance. It is called effective series resistance (ESR). It comes up due to the lead resistance and foil resistance. In this case, a 100 μF capacitor is likely to have an ESR of 0.3 Ω to 0.8 Ω depending on the grade and quality of the capacitor chosen. Note that the impedance of 100 μF capacitor at 20 kHz is $-j0.08 \Omega$ and that of 0.5 mH inductor is $j62.8 \Omega$. Obviously, the ESR of capacitor rules the situation. Current division will take place between the impedance of inductor and ESR of capacitor and the filter performance is not going to be as good as we calculated.

Increasing the capacitance value decreases the ESR. Thus the size of capacitor chosen in this kind of applications is based on ESR considerations rather than the kind of calculations we carried out without taking the ESR into account.

Example: 9.9-3

A linear electrical element draws only sinusoidal current from a pure sinusoidal source. However, a non-linear load can draw a non-sinusoidal current that is rich in harmonics from a voltage source of sinusoidal nature. Various industrial electronic equipment such as AC–DC converters, inverter fed induction motor drives, thyristor controlled industrial heaters, converter fed DC motor drives, uninterruptible power supplies, fluorescent lamps etc. belong to this category.

The voltage generated in synchronous generators located at various generating stations in a power system is generally a pure single frequency (50 Hz in India) sinusoid. However this voltage reaches individual customer's electrical installation through transmission and distribution network. Hence, the voltage received by a customer will be a superposition of generated voltage and the voltage drops in various system impedances like transformer series reactance, line resistances and reactance and so on. Harmonic currents drawn by non-linear loads flow through these system impedances producing harmonic voltage drops in them. These harmonic voltage drops produce distorted voltage waveform everywhere in the power system despite the pure sinusoidal voltage generated at the generating stations.

Distorted voltage results in mal-functioning of electronic equipment, inefficient operation of motors, increased losses everywhere in the system etc. It can also cause damage to the power capacitors used in a power system for various purposes.

Harmonic current filters are used to prevent the harmonic currents generated by a load from getting into the power system. Both passive filters and filters using power semi conductor devices are in use. A passive harmonic filter is essentially a tuned LC series combination connected in parallel to the offending load. The LC circuit is tuned to the harmonic frequency to be eliminated. The tuned combination will have zero impedance at that frequency and the currents at that frequency will flow into the filter instead of getting into the power system. Obviously more than one such combination will have to be used in parallel if elimination of multiple harmonics is desired.

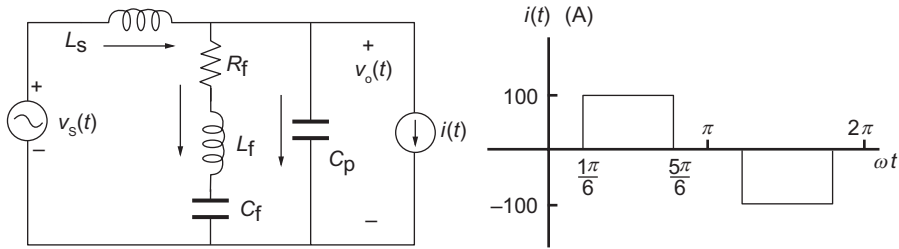


Fig. 9.9-4 Circuit diagram for the harmonic filtering context in Example: 9.9-3

L_s in the circuit in Fig. 9.9-4 represents the Thevenin's impedance of the power system at a point where a non-linear load is connected. The current drawn by the load is modelled as a current source $i(t)$. The waveform of load current is also given in the same figure. L_f and C_f are the harmonic filter components. They are tuned to 250 Hz, i.e. it is a fifth harmonic current filter. R_f is the series resistance of the inductor. The component values are $L_s = 0.3$ mH, $L_f = 1.84$ mH, $C_f = 0.22$ mF and $R_f = 0.072$ Ω . C_p is a power factor correction capacitor that is employed to draw leading current in order to cancel the lagging current drawn by another load which is not active in the current context. Its value is 0.47 mF. The source voltage is a 320V amplitude, zero phase sinusoid at 50 Hz.

Solve for the three currents and the output voltage under steady-state conditions and plot them.

Solution

Step 1: Fourier series of $i(t)$.

Refer Example 9.7-1 for Fourier series of this waveform. It is

$$i(t) \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} -\frac{j200}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi}{3} e^{j\omega_o t} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \left(\frac{400}{\pi n} \sin \frac{n\pi}{2} \sin \frac{n\pi}{3} \right) \sin n\omega_o t$$

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All harmonics of 3 and its multiples will have amplitude of zero.

Step 2: Solve for steady-state response for each harmonic.

The phasor equivalent circuit for fundamental component is shown in circuit of Fig. 9.9-5 (a). The value of ω_0 is ≈ 314 rad/s.

Peak values of voltage and current sources are marked instead of the rms value. It is a two-source problem and the solution is obtained by applying superposition principle.

Solution for (c)

$$\text{Total effective impedance} = j0.0942 + (-j13.9)/(-j6.78) = j0.0942 - j4.56 = -j4.463 \Omega$$

$$\therefore I_S = 320 \angle 0^\circ \div -j4.463 = 71.7 \angle 90^\circ \text{ A}$$

$$V_O = 320 \angle 0^\circ \times (-j4.56 \div -j4.463) = 327 \angle 0^\circ \text{ V}$$

$$I_f = 71.7 \angle 90^\circ \times (-j6.78 \div -j20.68) = 23.5 \angle 90^\circ \text{ A}$$

$$I_c = 71.7 \angle 90^\circ \times (-j13.9 \div -j20.68) = 48.2 \angle 90^\circ \text{ A}$$

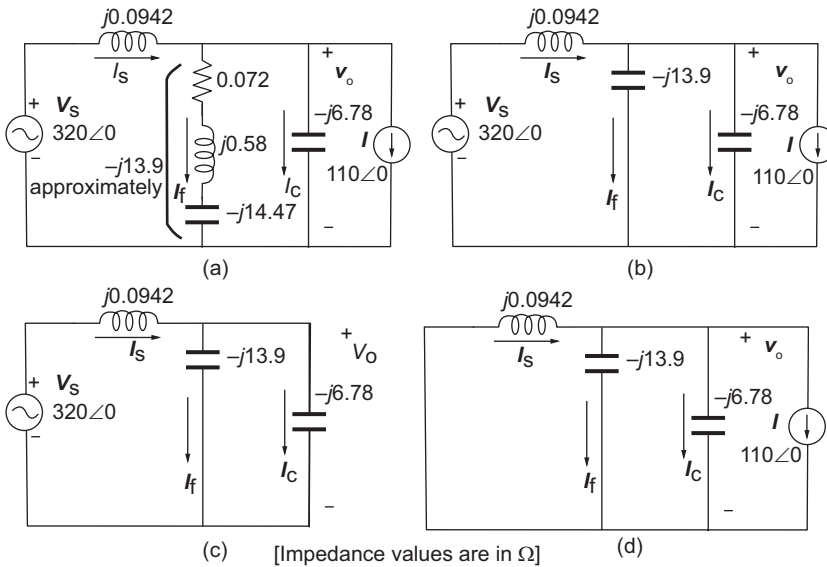


Fig. 9.9-5 (a) and (b) Phasor equivalent circuit (c) and (d) Circuits for applying superposition principle

Solution for (d)

$$\text{Total effective admittance} = -j10.4 \text{ S}$$

$$\text{Total effective impedance} = j0.096$$

$$\therefore V_O = -110 \angle 0^\circ \times j0.096 = -10.6 \angle 90^\circ \text{ V}$$

$$I_S = 10.6 \angle 90^\circ \div j0.094 = 112.8 \angle 0^\circ \text{ A}$$

$$I_C = 10.6 \angle 90^\circ \div (-j6.78) = 1.54 \angle 180^\circ \text{ A}$$

$$I_f = 10.6 \angle 90^\circ \div (-j13.9) = 0.74 \angle 180^\circ \text{ A}$$

Solution for (c) + (d)

$$V_O = 327 \angle 0 - 10.6 \angle 90^\circ \text{ V} = 327.2 \angle -1.9^\circ \text{ V}$$

$$I_S = 71.7 \angle 90^\circ + 112.8 \angle 0^\circ \text{ A} = 133.66 \angle 32.4^\circ \text{ A}$$

$$I_f = 23.5 \angle 90^\circ + 0.74 \angle 180^\circ \text{ A} = 23.5 \angle 91.8^\circ \text{ A}$$

$$I_c = 48.2 \angle 90^\circ + 1.54 \angle 180^\circ \text{ A} = 48.2 \angle 91.8^\circ \text{ A}$$

Time-domain solution

$$v_o(t) = 327.2 \sin(314t - 1.9^\circ) \text{ V}$$

$$i_s(t) = 133.7 \sin(314t + 32.4^\circ) \text{ A}$$

$$i_f(t) = 23.5 \sin(314t + 91.8^\circ) \text{ A}$$

$$i_c(t) = 48.2 \sin(314t + 91.8^\circ) \text{ A}$$

The equivalent circuits for solving the harmonic currents are shown in Fig. 9.9-6. Only the first four non-zero harmonics in load current are considered.

The circuits can be solved by current division principle – current divides among parallel elements in proportion to their admittance values. Solution is illustrated in the case of $n = 11$.

$$V_O = 10 \angle 0 \div (j1.63 - j0.2 - j0.96) = 21.3 \angle -90^\circ \text{ V}$$

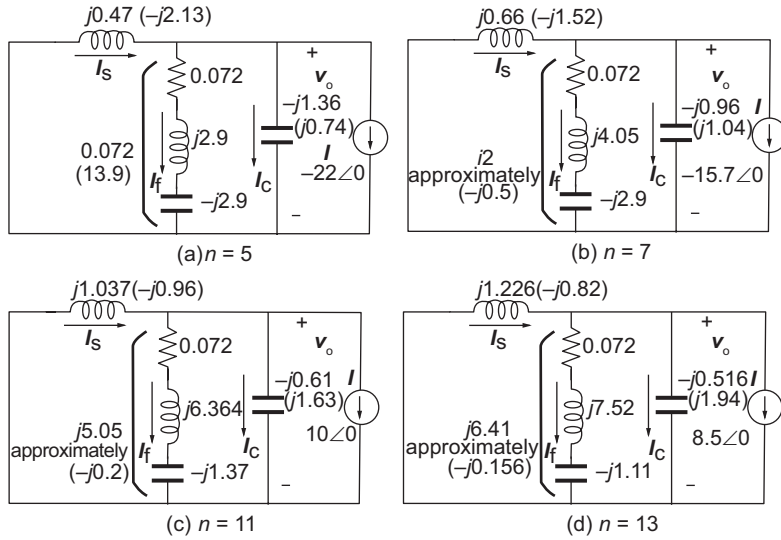
$$I_S = 10 \angle 0 \times -j0.96 \div (j1.63 - j0.2 - j0.96) = -20.4 \angle 0^\circ \text{ A}$$

$$I_f = 10 \angle 0 \times -j0.2 \div (j1.63 - j0.2 - j0.96) = -4.26 \angle 0^\circ \text{ A}$$

$$I_C = 10 \angle 0 \times j1.63 \div (j1.63 - j0.2 - j0.96) = 34.7 \angle 0^\circ \text{ A}$$

Observe that the load current had only 10 A at 550 Hz (11th harmonic). However, the source current has 21.3 A and the power factor correction capacitor C_p has 34.7 A in it. If C_p were not there, the source current would have been close to 10 A only and the 11th harmonic component in voltage would have been close to 10 V instead of its present value of 21.3 V. This amplification of 11th harmonic illustrates a potential problem that can exist in harmonic context in power systems. The admittance of source inductance and the power factor capacitor cancel each other partially at 11th harmonic and this results in an increase in the impedance level presented to 11th harmonic. Higher impedance results in higher voltage and higher currents in individual parallel paths. There can be dangerous harmonic resonance if the parallel resonant frequency of the circuit coincides or comes near one of the harmonics of supply frequency. In this case, the resonant frequency is around 424 Hz (decided practically by the 0.3 mH inductor and 0.47 mF capacitor) and there can be large harmonic amplification if the load current contains 8th or 9th harmonics.

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[Impedance values are in Ω . Admittance values (in brackets) are in siemens.]

Fig. 9.9-6 Circuit equivalents for current harmonics

The time-domain solution is obtained by using $\omega = 11 \times 314 \text{ rad/s}$.

$$v_o(t) = 21.3 \sin(11 \times 314t - 90^\circ) \text{ V}$$

$$i_s(t) = -20.4 \sin(11 \times 314t) \text{ A}$$

$$i_f(t) = -4.26 \sin(11 \times 314t) \text{ A}$$

$$i_c(t) = 34.7 \sin(11 \times 314t) \text{ A}$$

The equivalent circuits for other harmonics also can be solved similarly.

The time-domain solution for $n = 5$ is given in the following. Note that the tuned circuit diverts most of the 22 A fifth harmonic into it. The source current still contains fifth harmonic and this is due to the fact that the filter path has a resistance of 0.072Ω .

$$v_o(t) = -1.58 \sin(5 \times 314t + 5.7^\circ) \text{ V}$$

$$i_s(t) = -3.35 \sin(5 \times 314t - 84.3^\circ) \text{ A}$$

$$i_f(t) = -21.9 \sin(5 \times 314t + 5.7^\circ) \text{ A}$$

$$i_c(t) = -1.17 \sin(5 \times 314t + 95.7^\circ) \text{ A}$$

The time-domain solution for $n = 7$ is given in the following:

$$v_o(t) = -16.1 \sin(7 \times 314t + 90^\circ) \text{ V}$$

$$i_s(t) = -24.4 \sin(7 \times 314t) \text{ A}$$

$$i_f(t) = -8.05 \sin(7 \times 314t) \text{ A}$$

$$i_c(t) = 16.7 \sin(7 \times 314t) \text{ A}$$

The time-domain solution for $n = 13$ is given in the following:

$$v_o(t) = 8.8 \sin(13 \times 314t - 90^\circ) \text{ V}$$

$$i_s(t) = -7.2 \sin(13 \times 314t) \text{ A}$$

$$i_f(t) = -1.38 \sin(13 \times 314t) \text{ A}$$

$$i_c(t) = 17 \sin(13 \times 314t) \text{ A}$$

The total solution is obtained by combining all these solutions. We neglect the harmonics above 13th in this example.

$$v_o(t) = 327 \sin(\omega_0 t - 1.9^\circ) - 1.6 \sin(5\omega_0 t + 5.7^\circ) - 16.1 \sin(7\omega_0 t + 90^\circ) \\ + 21.3 \sin(11\omega_0 t - 90^\circ) + 8.8 \sin(13\omega_0 t - 90^\circ) \text{ V}$$

$$i_s(t) = 133.7 \sin(\omega_0 t + 32.4^\circ) - 3.35 \sin(5\omega_0 t - 84.3^\circ) - 24.4 \sin(7\omega_0 t) \\ - 20.4 \sin(11\omega_0 t) - 7.2 \sin(13\omega_0 t) \text{ A}$$

$$i_f(t) = 23.5 \sin(\omega_0 t + 91.8^\circ) - 21.9 \sin(5\omega_0 t + 5.7^\circ) - 8.05 \sin(7\omega_0 t) \\ - 4.26 \sin(11\omega_0 t) - 1.38 \sin(13\omega_0 t) \text{ A}$$

$$i_c(t) = 48.2 \sin(\omega_0 t + 91.8^\circ) - 1.17 \sin(5\omega_0 t + 95.7^\circ) + 16.7 \sin(7\omega_0 t) \\ + 34.7 \sin(11\omega_0 t) + 17 \sin(13\omega_0 t) \text{ A}$$

These waveforms are plotted in Fig. 9.9-7. The waveforms reveal that the presence of power factor correction capacitor has caused deterioration in the harmonic performance of the system and that the harmonic filter is not very effective in this context. Observe the high frequency currents in the capacitor.

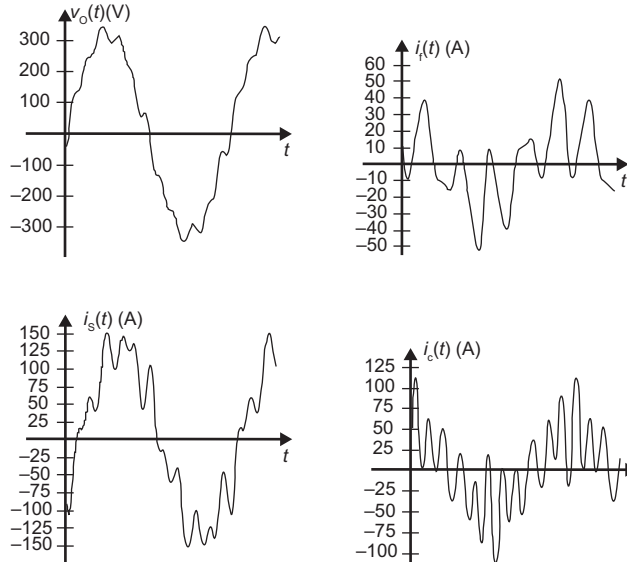


Fig. 9.9-7 Waveforms of $v_o(t)$, $i_s(t)$, $i_c(t)$ and $i_f(t)$ in Example: 9.9-3

9.10 NORMALISED POWER IN A PERIODIC WAVEFORM AND PARSEVAL'S THEOREM

The concept of normalised power in a periodic waveform is often employed in communication engineering and allied areas as a measure of signal strength. It is defined as the *average power* that will be delivered to 1Ω resistance if the periodic waveform is thought of as a voltage waveform applied to that resistor. The averaging is done over any interval equal to the period of the waveform.

$$P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt \quad (9.10-1)$$

If we consider $v_3(t) = [v(t)]^2$ as a new time-function, the term on the right-hand side of Eqn. 9.10-1 can be identified as the DC component of $v_3(t)$. Hence P_n must be equal to the exponential Fourier series coefficient of $v_3(t)$, $\tilde{v}_{3k}$, for $k = 0$.

We developed the *multiplication-in-time* property of Fourier series in Example 9.6-9. This property states that if $v_1(t)$ and $v_2(t)$ are two periodic waveforms with same period and $v_3(t) = v_1(t) \times v_2(t)$, then

the exponential Fourier series coefficients of $v_3(t)$ is given by $\tilde{v}_{3k} = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}$, for $-\infty < k < \infty$,

where $\tilde{v}_{1n}$ and $\tilde{v}_{2n}$ are the exponential Fourier series coefficients of $v_1(t)$, and $v_2(t)$ respectively. We use this property with $v_1(t) = v_2(t) = v(t)$ and evaluate the exponential Fourier series coefficient of $[v(t)]^2$, for $k = 0$ as

$$\tilde{v}_{30} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n}. \text{ Therefore,}$$

$$P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_n^* = \sum_{n=-\infty}^{\infty} |\tilde{v}_n|^2 = |\tilde{v}_0|^2 + 2 \sum_{n=1}^{\infty} |\tilde{v}_n|^2$$

This is Parseval's Theorem on normalised power of periodic waveforms.

The trigonometric Fourier series for $v(t)$ is

$$v(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t$$

$$\text{where } a_o = \tilde{v}_o = \frac{1}{T} \int_{-T/2}^{T/2} v(t) dt,$$

$$a_n = \tilde{v}_n + \tilde{v}_{-n} = \tilde{v}_n + \tilde{v}_n^* = 2 \operatorname{Re}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \cos n\omega_o t dt, \text{ for } n = 1, 2, 3 \dots$$

$$a_n = -\tilde{v}_n + \tilde{v}_{-n} = -\tilde{v}_n + \tilde{v}_n^* = 2 \operatorname{Im}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \sin n\omega_o t dt, \text{ for } n = 1, 2, 3 \dots$$

Therefore, Parseval's Theorem can be expressed in terms of trigonometric Fourier series coefficients

$$\text{as } P_n = a_o^2 + \sum_{n=1}^{\infty} \left(\frac{a_n^2 + b_n^2}{2} \right).$$

The second form of trigonometric Fourier series is shown in the following:

$$\therefore v(t) = c_o + \sum_{n=1}^{\infty} c_n \cos(n\omega_o t - \phi_n)$$

where

$$c_o = \tilde{v}_o, c_n = \sqrt{a_n^2 + b_n^2} = 2\tilde{v}_n \tilde{v}_n^* = 2|\tilde{v}_n|$$

and

$$\phi_n = \tan^{-1} \frac{b_n}{a_n} = -\angle \text{ of } \tilde{v}_n, \text{ for } n = 1, 2, 3 \dots$$

$$\therefore P_n = c_o^2 + \sum_{n=1}^{\infty} \frac{c_n^2}{2}$$

The normalised power of a periodic waveform $v(t)$, P_n , is given by

$$P_n = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_n^* = \sum_{n=-\infty}^{\infty} |\tilde{v}_n|^2 = |\tilde{v}_o|^2 + 2 \sum_{n=1}^{\infty} |\tilde{v}_n|^2 = a_o^2 + \sum_{n=1}^{\infty} \frac{(a_n^2 + b_n^2)}{2} = c_o^2 + \sum_{n=1}^{\infty} \frac{c_n^2}{2}$$

Though the *multiplication-in-time* property easily led us to Parseval's theorem, it does not help us to see the significance of this theorem. Neither does it tell us how this total normalised power is distributed among various frequency components. Hence, we use the trigonometric Fourier series

$v(t) = c_o + \sum_{n=1}^{\infty} c_n \cos(n\omega_o t - \phi_n)$ for further appreciation of P_n .

Consider a simpler situation in which $v(t)$ contains just three components.

$$v(t) = c_o + c_m \cos(m\omega_o t - \phi_m) + c_k \cos(k\omega_o t - \phi_k), k \text{ and } m \text{ are integers}$$

$$\begin{aligned} \therefore [v(t)]^2 &= c_o^2 + c_m^2 \cos^2(m\omega_o t - \phi_m) + c_k^2 \cos^2(k\omega_o t - \phi_k) \\ &\quad + 2c_o c_m \cos(m\omega_o t - \phi_m) + 2c_o c_k \cos(k\omega_o t - \phi_k) \\ &\quad + 2c_m c_k \cos(m\omega_o t - \phi_m) \cos(k\omega_o t - \phi_k) \end{aligned}$$

$$\begin{aligned} \therefore [v(t)]^2 &= c_o^2 + \frac{1}{2} c_m^2 + \frac{1}{2} c_k^2 + \frac{1}{2} c_m^2 \cos 2(m\omega_o t - \phi_m) + \frac{1}{2} c_k^2 \cos 2(k\omega_o t - \phi_k) \\ &\quad + 2c_o c_m \cos(m\omega_o t - \phi_m) + 2c_o c_k \cos(k\omega_o t - \phi_k) \\ &\quad + c_m c_k \cos[(m+k)\omega_o t - (\phi_m + \phi_k)] + c_m c_k \cos[(m-k)\omega_o t - (\phi_m - \phi_k)] \end{aligned}$$

k and m are integers. Thus, if $k \neq m$, the frequencies $m\omega_o$, $k\omega_o$, $2m\omega_o$, $2k\omega_o$, $(m-k)\omega_o$ and $(m+k)\omega_o$ are integer multiples of ω_o . Hence, all the cosine waves in $[v(t)]^2$ will have integer number of cycles in T where T is the period of $v(t)$. Therefore, their average over one T will be zero.

$$\begin{aligned} \therefore P_n &= \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = \frac{1}{T} \left[c_o^2 + \frac{1}{2} c_m^2 + \frac{1}{2} c_k^2 \right] \times T \\ &= c_o^2 + \frac{1}{2} c_m^2 + \frac{1}{2} c_k^2 = c_o^2 + \left(\frac{c_m}{\sqrt{2}} \right)^2 + \left(\frac{c_k}{\sqrt{2}} \right)^2 \end{aligned}$$

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Generalising the result for infinite term Fourier series,

$$\therefore P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = c_o^2 + \sum_{n=1}^{\infty} \left(\frac{c_n}{\sqrt{2}} \right)^2 \quad (9.10-2)$$

$$i.e., P_n = (\text{DC component})^2 + \sum_{n=1}^{\infty} (\text{rms value of } n^{th} \text{ harmonic component})^2$$

Since the rms value of a DC component is same as its value, we can express this as

$$P_n = \sum_{n=0}^{\infty} (\text{rms value } n^{th} \text{ harmonic component})^2 \quad (9.10-3)$$

Square root of this quantity will give the rms value of $v(t)$ itself.

$$\text{rms value of } v(t) = \sqrt{\sum_{n=0}^{\infty} (\text{rms value of } n^{th} \text{ harmonic component})^2} \quad (9.10-4)$$

The normalised power of a particular harmonic component with amplitude c_n when acting alone will be $0.5 c_n^2$. Equations 9.10-2 shows that it contributes the same amount to the total power even when it is acting along with other harmonics.

Consider two *arbitrary waveforms* $v_1(t)$ and $v_2(t)$. Let average of $[v_1(t)]^2$ and $[v_2(t)]^2$ over some interval be a_1 and a_2 , respectively. Will the average of $[v_1(t) + v_2(t)]^2$ over the same interval be $a_1 + a_2$? The answer depends on whether the average of $2v_1(t)v_2(t)$ in that interval is zero or not. In general, it is not zero, and average of $[v_1(t) + v_2(t)]^2$ is not the same as the sum of averages of $[v_1(t)]^2$ and $[v_2(t)]^2$. However, if $v_1(t)$ and $v_2(t)$ are two sinusoids with different frequencies and if their frequencies are integer multiples of some basic frequency, then the average of $2v_1(t)v_2(t)$ in an interval that is equal to the period corresponding to the basic frequency is zero. Therefore, if $v(t)$ is a mixture of harmonically related sinusoids and DC, the normalised power contributions from each component is unaffected by the presence of other components. Hence, the normalised power of the waveform is the sum of normalised power of individual components.

Now we understand that each harmonic component in the trigonometric Fourier series of a waveform contributes to normalised power. We can ascribe the power contributed by a particular component to its frequency and plot the information against as a line spectrum. This spectral plot is called *discrete power spectrum*. However, it will be a single-sided spectrum since we derived it from trigonometric Fourier series. Spectral lines will be located at $0, \omega_0, 2\omega_0, 3\omega_0$ etc. and the length of the spectral line will be proportional to $0.5 c_n^2$. By Parseval's theorem,

$$P_n = \frac{1}{T} \int_{-0.5T}^{0.5T} [v(t)]^2 dt = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_n^* = \sum_{n=-\infty}^{\infty} |\tilde{v}_n|^2.$$

Therefore, we can draw the two-sided discrete power spectrum by plotting two lines of height proportional to $|\tilde{v}_n|^2$ at $n\omega_0$ and $-n\omega_0$. We had noted earlier that two spectral components located at $\pm n\omega_0$ in the two-sided magnitude and phase spectra based on exponential Fourier series have to be thought of as an integral unit rather as individual components. Those two components always go together and form a *real* sinusoid. Similarly, it is understood that the power spectral components located at $\pm n\omega_0$ in the two-sided power spectrum always go together to make a total contribution of $2|\tilde{v}_n|^2$ to P_n .

Example: 9.10-1

The output of a fully controlled AC–DC converter operating from a sinusoidal voltage of 320 V peak and 50 Hz frequency is shown in Fig. 9.10-1. (i) Find and plot its discrete power spectrum. (ii) This waveform is applied to RL circuit with $L = 150$ mH and $R = 10 \Omega$. Find and plot the discrete power spectrum of voltage appearing across the resistor and find the power dissipation in it.

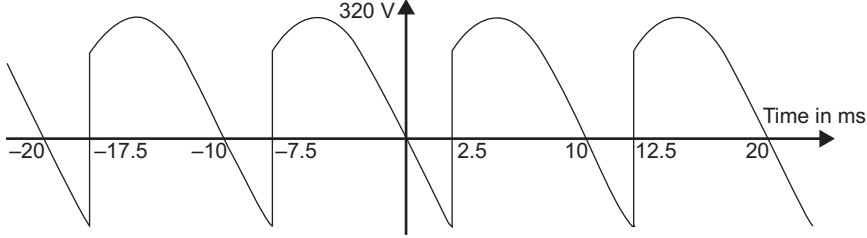


Fig. 9.10-1 Waveform for Example: 9.10-1

Solution-(i)

This waveform, $v(t)$, can be expressed as the product of two waveforms – $v_1(t) = 320\sin 100\pi t$ and $v_2(t)$ which is a symmetric ± 1 , 50 Hz square wave which is delayed by 2.5 ms.

$$\begin{aligned} v_1(t) &= 320 \sin 100\pi t \\ &= 320 \frac{e^{j100\pi t} - e^{-j100\pi t}}{2j} \quad (\text{By Euler's Formula}) \\ \therefore \tilde{v}_{11} &= \frac{160}{j}, \tilde{v}_{1-1} = \frac{-160}{j} \text{ and } \tilde{v}_{1n} = 0, \text{ for all other values of } n \end{aligned}$$

$v_2(t)$ is a unit amplitude square wave. Fourier series of a unit amplitude square wave of 1 s period that crosses zero at origin was obtained in Example 9.6-5 as $\sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi nt}$. Using *time-shift property* of exponential Fourier series we write the exponential Fourier series of the square wave in

this example as $\tilde{v}_{2n} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{-j\frac{\pi}{4}n} e^{j100\pi nt}$. Using the *multiplication in time* property of exponential

Fourier series we write the exponential Fourier series coefficients of the waveform $v(t)$ as

$$\begin{aligned} \therefore \tilde{v}_k &= \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)} = -\frac{160}{j} \frac{2e^{-j\frac{k+1}{4}\pi}}{j\pi(k+1)} + \frac{160}{j} \frac{2e^{-j\frac{k-1}{4}\pi}}{j\pi(k-1)}, \text{ for even } k \\ &= \frac{-320}{\pi} \left(\frac{e^{-j\frac{k+1}{4}\pi}}{k+1} + \frac{e^{-j\frac{k-1}{4}\pi}}{k-1} \right), \text{ for even } k \end{aligned}$$

The magnitude part of the quantity inside the brackets can be shown as $\frac{\sqrt{2(1+k^2)}}{k^2-1}$.

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$$\therefore |\tilde{v}_n|^2 = 2 \times \left(\frac{320}{\pi} \right)^2 \frac{k^2 + 1}{(k^2 - 1)^2}, \text{ for even } k$$

The discrete power spectrum of $|\tilde{v}_n|^2$ is plotted against harmonic order n in Fig. 9.10-2. The common factor $2 \times \left(\frac{320}{\pi} \right)^2$ is normalised to unity. 88% of P_n of this waveform is contributed by the DC component and the 100 Hz component ($n = 2$). The next harmonic component at 200 Hz ($n = 4$) adds another 6.3%. Thus all the harmonics above 200 Hz contribute only 5.7% of power to P_n . The contribution from a spectral component to normalised power depends on the square of amplitude of spectral component. The sharp edge in the waveform in this example is synthesised by small amplitude high frequency components. But they contribute almost nothing to the normalised power.

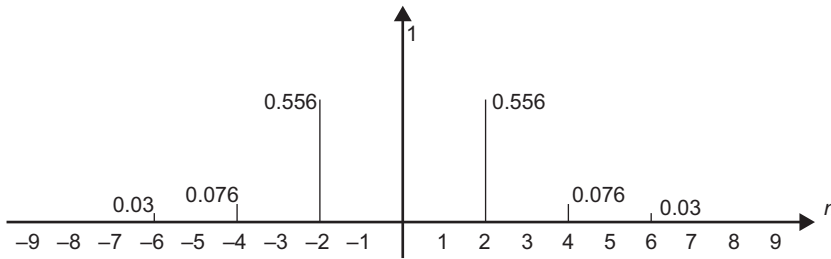


Fig. 9.10-2 Two-sided discrete power spectrum of waveform $v(t)$ in Example: 9.10-1

Solution (ii)

The frequency response of resistor voltage in a series RL circuit is obtained as $\frac{V_R(j\omega)}{V(j\omega)} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\tau}$. $\tau = 150 \text{ mH}/10\Omega = 15 \text{ ms}$ and $\omega = 100\pi$ in this case.

$$\frac{V_R(j\omega)}{V(j\omega)} = \frac{1}{1 + j1.5\pi n} = \frac{1}{\sqrt{1 + 22.21n^2}} \angle \tan^{-1} 4.71n$$

Exponential Fourier series coefficients of output = exponential Fourier series coefficients of input $\times$ value of frequency response function at the corresponding frequency.

$\therefore$ Power spectral component in output = power spectral component at input $\times$ square of magnitude of frequency response function value at the corresponding frequency.

The values of square of magnitude of frequency response function is 1 at $n = 0$, 0.1055 at $n = 2$, 0.053 at $n = 4$ and 0.035 at $n = 6$. Hence, the power spectral component (normalised with respect to maximum value) in the output at $n = 0$ is 1, at $n = \pm 2$ is $0.1055 \times 0.556 = 0.059$, at $n = \pm 4$ is $0.053 \times 0.076 = 0.004$ and at $n = \pm 6$ is $0.035 \times 0.03 = 0.001$. Thus almost the entire normalised power in the output waveform comes from its DC component. This spectrum is plotted in Fig. 9.10-3.

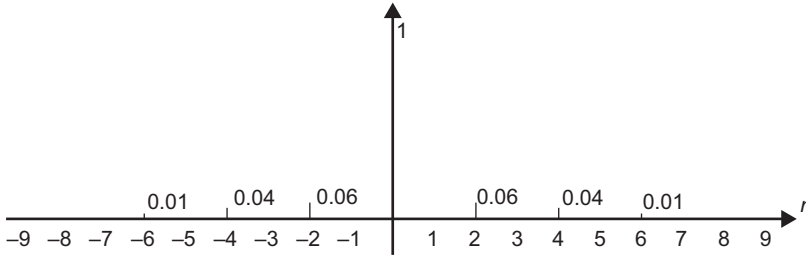


Fig. 9.10-3 Discrete power spectrum of output waveform in Example: 9.10-1

Power dissipated in the $10\ \Omega$ resistor $= P_n / 10 \approx 2 \times \left(\frac{320}{\pi} \right)^2 \times (1 + 0.06 + 0.06 + 0.04 + 0.04 + 0.01 + 0.01 + \dots) / 10 = 2.53\ \text{kW}$.

9.11 POWER AND POWER FACTOR IN AC SYSTEM WITH DISTORTED WAVEFORMS

Let $v(t)$ and $i(t)$ be the voltage across an electrical element and current through that element under periodic steady-state conditions in an AC system working under distorted waveform conditions. Both $v(t)$ and $i(t)$ are expressed in the form of trigonometric Fourier series below where ω_o is the fundamental radian frequency of the system. AC system voltages and currents usually do not contain any DC component under steady-state and hence DC components are not included in these Fourier series.

$$v(t) = \sum_{n=1}^{\infty} \sqrt{2} V_n \cos(n\omega_o t - \phi_{vn})$$

$$i(t) = \sum_{n=1}^{\infty} \sqrt{2} I_n \cos(n\omega_o t - \phi_{in})$$

V_n and ϕ_{vn} represent the rms value of n^{th} harmonic component of voltage and its phase. Similarly I_n and ϕ_{in} represent the rms value of n^{th} harmonic component of current and its phase.

The power delivered to the element, assuming passive sign convention, is given by $v(t) \times i(t)$. The average power delivered, P , is obtained by averaging this quantity over one period of the AC system.

The product $v(t)i(t)$ contains two kinds of terms – product of cosine functions of same harmonic order which have the general form of $2V_k I_k \cos(k\omega_o t - \phi_{vk}) \cos(k\omega_o t - \phi_{ik})$ is the first kind. The second kind of terms will be of the form $2V_k I_r \cos(k\omega_o t - \phi_{vk}) \cos(r\omega_o t - \phi_{ir})$; $k \neq r$. The first kind can be rewritten as $V_k I_k [\cos(2k\omega_o t - \phi_{vk} - \phi_{ik}) + \cos(\phi_{ik} - \phi_{vk})]$. When integrated over a period and divided by the period, this term will result in a contribution of $V_k I_k \cos(\phi_{ik} - \phi_{vk})$ to the average delivered power. The second kind of terms can be expressed as $V_k I_r [\cos((k+r)\omega_o t - \phi_{vk} - \phi_{ir}) + \cos((k-r)\omega_o t + \phi_{ir} - \phi_{vk})]$; $k \neq r$. Since k and r are integers, $(k+r)$ and $(k-r)$ are also integers. Therefore, the two cosine components at the sum and difference frequencies will have zero average value over one period of the system. Hence, only the first kind of terms contribute to the average power.

$$\therefore P = \sum_{n=1}^{\infty} V_n I_n \cos(\phi_{in} - \phi_{vn})\ \text{W}$$

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Thus, the active power (*i.e.* the average power) delivered by each voltage and current harmonic pair with same harmonic order is independent of whether they are acting alone or acting along with other harmonic pairs. Each harmonic pair delivers an active power given by the product of rms voltage, rms current and power factor of that pair.

However, the volt–ampere product (VA) is given by the product of rms values of $v(t)$ and $i(t)$. We express the rms value of $v(t)$ in terms of rms values of its harmonic components by using

Eqn. 9.10-4 as $V_{\text{rms}} = \sqrt{\sum_{n=1}^{\infty} (V_n)^2}$ and the rms value of $i(t)$ as $I_{\text{rms}} = \sqrt{\sum_{n=1}^{\infty} (I_n)^2}$. Then V-A product

$$S = V_{\text{rms}} I_{\text{rms}} = \sqrt{\sum_{n=1}^{\infty} (V_n)^2} \sqrt{\sum_{n=1}^{\infty} (I_n)^2} \text{ VA}$$

The power factor (*apparent* power factor) is the ratio between P and S . It includes the effect of reactive power of each harmonic voltage–current pair as well as the cross-power terms resulting from product of voltage and current of different harmonic order. Cross-power terms do not contribute to active power, but they contribute to VA.

The *fundamental* power factor is $\cos(\phi_{i1} - \phi_{v1})$ and is lagging if $(\phi_{i1} - \phi_{v1})$ is negative and is leading if $(\phi_{i1} - \phi_{v1})$ is positive.

Example: 9.11-1

Fourier analysis of the AC voltage received at the substation of an industry that used uses a large number of variable-speed drives shows that the received phase voltage can be represented by the following truncated Fourier series approximately.

$$v(t) = 6.35\sqrt{2} \sin(100\pi t) + 0.2\sqrt{2} \sin(500\pi t - 0.5\text{rad}) + 0.15\sqrt{2} \sin(700\pi t - 1\text{rad}) \text{ kV}$$

The industry draws a distorted current from this supply due to the power electronics involved in variable speed drives. Fourier analysis of current drawn shows that the following Fourier series can represent it approximately.

$$i(t) = 110\sqrt{2} \sin(100\pi t - 0.6\text{rad}) + 22\sqrt{2} \sin(500\pi t - 1.5\text{rad}) + 15\sqrt{2} \sin(700\pi t - 1\text{rad}) \\ + 10\sqrt{2} \sin(1100\pi t - 0.5\text{rad}) + 7\sqrt{2} \sin(1300\pi t - 0.5\text{rad}) \text{ A}$$

These waveforms are shown in Fig. 9.11-1.

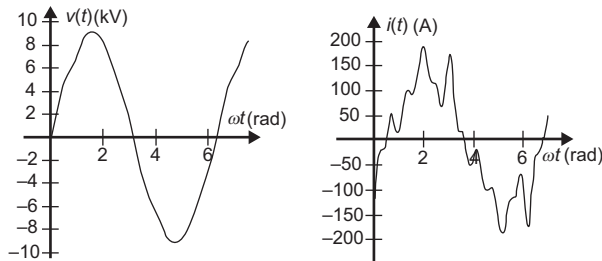


Fig. 9.11-1 Phase voltage and line current waveforms in Example: 9.11-1

Find the active power, VA, power factor, fundamental power factor and THD of voltage and current assuming balanced three-phase operation.

Solution

$$\begin{aligned}
 v(t) &= 6.35\sqrt{2} \sin(100\pi t) + 0.2\sqrt{2} \sin(500\pi t - 0.5\text{rad}) + 0.15\sqrt{2} \sin(700\pi t - 1\text{rad}) \text{ kV} \\
 i(t) &= 100\sqrt{2} \sin(100\pi t - 0.6\text{rad}) + 22\sqrt{2} \sin(500\pi t - 1.5\text{rad}) + 15\sqrt{2} \sin(700\pi t - 1\text{rad}) \\
 &\quad + 10\sqrt{2} \sin(1100\pi t - 0.5\text{rad}) + 7\sqrt{2} \sin(1300\pi t - 0.5\text{rad}) \text{ A} \\
 \therefore P &= 6.35 \times 110 \times \cos(0.6\text{rad}) + 0.2 \times 22 \times \cos(1.5 - 0.5\text{rad}) + 0.15 \times 15 \times \cos(1 - 1\text{rad}) \\
 &= 581.1 \text{ kW per phase} \\
 &= 1.743 \text{ MW (three phase)} \\
 V_{\text{rms}} &= \sqrt{6.35^2 + 0.2^2 + 0.15^2} = 6.355 \text{ kV (phase)} \\
 &= 11 \text{ kV (line)} \\
 I_{\text{rms}} &= \sqrt{110^2 + 22^2 + 15^2 + 10^2 + 7^2} = 113.8 \text{ A} \\
 \therefore S &= V_{\text{rms}} I_{\text{rms}} = 723 \text{ kVA (per phase)} \\
 &= 2.17 \text{ MW (three-phase)}
 \end{aligned}$$

$$\text{Power factor} = \frac{P}{S} = \frac{1.74}{2.17} = 0.8$$

$$\text{Fundamental power factor} = \cos(0.6\text{rad}) = 0.825 \text{ lag}$$

$$\text{Total harmonic distortion in voltage} = \frac{\sqrt{0.2^2 + 0.15^2}}{6.35} \times 100 = 3.94\%$$

$$\text{Total harmonic distortion in current} = \frac{\sqrt{22^2 + 15^2 + 10^2 + 7^2}}{110} \times 100 = 26.63\%$$

9.12 SUMMARY

- Almost all periodic signals employed in circuits can be expressed as the sum of infinite number of sinusoids. Given a periodic waveform $v(t)$ with period T , its exponential Fourier series is given by

$$v(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_n e^{-jn\omega_0 t}, \text{ where } \omega_0 = \frac{2\pi}{T} \text{ and } \tilde{v}_n \text{ are the exponential Fourier series coefficients of } v(t).$$

- The exponential Fourier series coefficients $\tilde{v}_n$ are found by $\tilde{v}_n = \frac{1}{T} \int_{-T/2}^{T/2} v(t) e^{-jn\omega_0 t} dt$.
- The Fourier series may also be expressed in trigonometric form as follows:

$$\therefore v(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\text{where } a_0 = \tilde{v}_0 = \frac{1}{T} \int_{-T/2}^{T/2} v(t) dt,$$

$$a_n = 2 \operatorname{Re}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \cos n\omega_0 t dt \text{ and } b_n = -2 \operatorname{Im}(\tilde{v}_n) = \frac{2}{T} \int_{-T/2}^{T/2} v(t) \sin n\omega_0 t dt$$

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- Another form of trigonometric Fourier series is given in the following:

$$\therefore v(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \phi_n)$$

where $c_0 = \tilde{v}_0$,

$$c_n = \sqrt{a_n^2 + b_n^2} = 2\tilde{v}_n \tilde{v}_n^* = 2|\tilde{v}_n| \text{ and } \phi_n = \tan^{-1} \frac{b_n}{a_n} = -\angle \text{ of } \tilde{v}_n, \text{ for } n = 1, 2, 3 \dots$$

- Following properties are exhibited by the Fourier series for a *real* $v(t)$.

Table 9.12-1 Properties of Exponential Fourier Series for a Real $v(t)$

Aperiodic Signal Property	Fourier Series
$v(t)$	$\tilde{v}_n, a_n$ and b_n, c_n and ϕ_n – Definition
$a v_1(t) + b v_2(t)$	$a\tilde{v}_{1n} + b\tilde{v}_{2n}$ – Linearity
$v(t-t_d)$	$e^{-j\omega_d t} \tilde{v}_n$ – Time-shifting
$e^{jk\omega_0 t} v(t)$	$\tilde{v}_{n-k}$ – Frequency-shifting
$v(-t)$	$\tilde{v}_{-n}$ – Time reversal
$v(\alpha t), \alpha > 0$	$\tilde{v}_n$ – Time-scaling
$v_1(t) v_2(t)$	$\sum_{k=-\infty}^{\infty} \tilde{v}_{1k} \tilde{v}_{2(n-k)}$
$\frac{dv(t)}{dt}$	$jn\omega_0 \tilde{v}_n$ – Time-domain differentiation
$\int_{-\infty}^t v(t) dt$ ($v(t)$ has zero DC content)	$\frac{\tilde{v}_n}{jn\omega_0}$ – Time-domain integration
$v(t)$ real	$\tilde{v}_{-n} = \tilde{v}_n^*$ – Conjugate symmetry $\text{Re}(\tilde{v}_n) = \text{Re}(\tilde{v}_{-n})$ and $\text{Im}(\tilde{v}_n) = -\text{Im}(\tilde{v}_{-n})$ $ \tilde{v}_n = \tilde{v}_{-n} $ and $\angle \tilde{v}_n = -\angle \tilde{v}_{-n}$
$v(t) = -v(t \pm \frac{1}{2}T)$, for all t – half-wave symmetry	$\tilde{v}_n = 0$ for even n
$v(-t) = v(t)$, even symmetry	$\text{Im}(\tilde{v}_n) = 0$, for all n . Only cosines in trigonometric Fourier series.
$v(-t) = -v(t)$, odd symmetry	$\text{Re}(\tilde{v}_n) = 0$, Only sine terms in trigonometric Fourier series.
$v_e(t) = 0.5[v(t) + v(-t)]$	$\text{Re}(\tilde{v}_n)$
$v_o(t) = 0.5[v(t) - v(-t)]$	$\text{Im}(\tilde{v}_n)$
Parseval's power relation	$P_n = \int_{-r/2}^{r/2} [v(t)]^2 dt = \sum_{n=-\infty}^{\infty} \tilde{v}_n ^2$

- The magnitude of exponential Fourier series coefficients plotted against frequency in a two-sided line plot is called the discrete magnitude spectrum of the waveform. A similar plot of phase of exponential Fourier series coefficients is the discrete phase spectrum.
- The two components at $\pm n\omega_0$ in the discrete spectrum form a pair that cannot be separated in any analysis. They always go together to form a *real* sinusoidal component.
- The steady-state response of a circuit to a periodic input can be found in four steps. (i) Find the Fourier series of the input. (ii) Find out the frequency response function between the desired output variable and the input variable by using phasor equivalent circuit. (iii) Obtain the steady-state response in time-domain for each sinusoidal component in the input. (iv) Decide how many terms in the output Fourier series are needed to represent the output waveform with the desired degree of accuracy.
- Normalised power P_n of $v(t)$ is defined as the average power that will be dissipated in a 1Ω resistance if this waveform is applied to it as a voltage. Parseval's theorem states that the normalised power of a periodic waveform $v(t)$, P_n , is given by

$$P_n = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}_n \tilde{v}_n^* = \sum_{n=-\infty}^{\infty} |\tilde{v}_n|^2 = |\tilde{v}_0|^2 + 2 \sum_{n=1}^{\infty} |\tilde{v}_n|^2 = a_0^2 + \sum_{n=1}^{\infty} \frac{(a_n^2 + b_n^2)}{2} = c_0^2 + \sum_{n=1}^{\infty} \frac{c_n^2}{2}$$

- This is equivalent to $P_n = \sum_{n=0}^{\infty} (\text{r.m.s value of } n^{\text{th}} \text{ harmonic component})^2$
- The rms value of a waveform $v(t)$ containing harmonics can be found by

$$\text{rms value of } v(t) = \sqrt{\sum_{n=0}^{\infty} (\text{r.m.s value of } n^{\text{th}} \text{ harmonic component})^2}$$

9.13 PROBLEMS

1. Is the signal $v(t) = \sin(21t) + \sin(15t - 75^\circ)$ periodic? If it is, what is the period?
2. Is the signal $v(t) = \sin(\sqrt{2}t) + \sin(t - 75^\circ)$ periodic? If it is, what is the period?
3. Show that (i) the product of two odd functions of t will be an even function of t , (ii) the product of two even functions of t will be an even function of t and (iii) the product of an odd function of t and an even function of t will be an odd function of t .
4. $v(t)$ in Fig. 9.13-1 has $\omega = 1$ rad/s. It is shifted by t_0 to produce $v_1(t) = v(t - t_0)$. (i) What is t_0 if $v_1(t)$ has odd symmetry? (ii) Is there any t_0 such that $v_1(t)$ is even on t ?

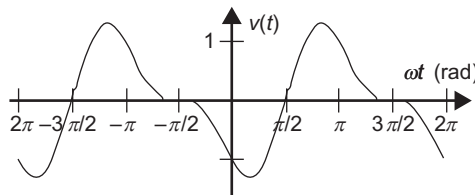


Fig. 9.13-1

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5. The waveform in Fig. 9.13-2 has $\omega = 1$ rad/s.

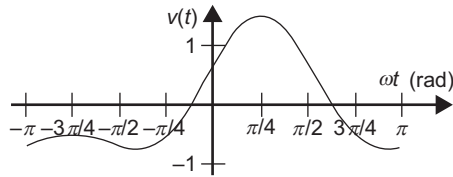


Fig. 9.13-2

- It is time-shifted by t_0 to produce $v_1(t) = v(t - t_0)$. (i) Can a value for t_0 be found such that $v_1(t)$ has odd symmetry? (ii) Is it possible to find value for t_0 such that $v_1(t)$ is even on t ?
6. The waveform in Fig. 9.13-3 has $T = 1$ s. (i) Does it possess odd or even symmetry? (ii) If not, will its time-shifted version, $v_1(t) = v(t - t_0)$ have odd or even symmetry for some value of t_0 ? (iii) Find and plot the even part and odd part of this waveform.

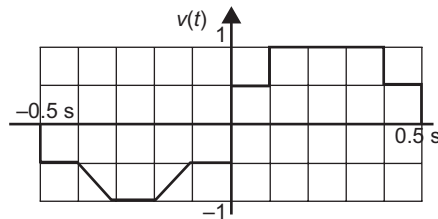


Fig. 9.13-3

7. A waveform $v(t) = 2\sin(\omega t + \pi/7) + 0.5\sin(5\omega t + \phi)$ is known to possess odd symmetry when it is delayed by $1/14$ s. Find ω and one possible value of ϕ .
8. Show that if $v(t)$ is a periodic waveform with ω_0 as its fundamental frequency, the exponential Fourier series coefficients of $\frac{dv(t)}{dt}$ are given by $j n \omega_0 \tilde{v}_n$ where $\tilde{v}_n$ are its exponential Fourier series coefficients.
9. Show that the power spectral components of output voltage of a circuit is given by $|H(j\omega)|^2 \times$ power spectral component of input where $H(j\omega)$ is the frequency response function of the circuit.
10. If $v(t)$ is a periodic waveform with a period of T s and $v_1(t) = 2v(t - 0.5T) + 7$, find the relationship between the trigonometric Fourier series coefficients of $v(t)$ and $v_1(t)$?
11. $v(t)$ is a distorted sinusoidal waveform with a fundamental frequency of ω_0 rad/s and zero DC content. $v_1(t) = \frac{dv(t)}{dt}$ and $v_2(t) = \int_{-\infty}^t v(t)dt$. Will the THD value of $v_1(t)$ and $v_2(t)$ be less than, equal or greater than that of $v(t)$? How does the answer depend on ω_0 ?
12. A battery of open circuit voltage V and internal resistance R is delivering a load current $i(t) = I \sin^2 \omega t$ A. Power is measured by connecting a voltmeter across the battery terminals and an ammeter in series with the battery and multiplying the readings. Calculate the percentage error (ignore meter errors) in measured power if meters are of (i) moving coil type (ii) moving iron type?
13. One cycle of a waveform $v(t)$ is shown in Fig. 9.13-4. It is a symmetrically clipped sinusoid. (i) Obtain a time-shifted version of this waveform such that the resulting waveform has odd symmetry. (ii) Find the trigonometric Fourier series of the shifted version and thereby obtain the discrete Fourier spectrum for $v(t)$.

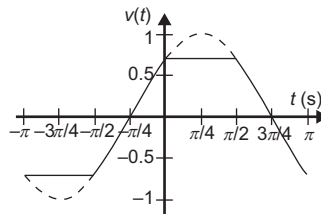


Fig. 9.13-4 Clipped sinusoidal waveform

14. One cycle of a periodic impulse train is shown in Fig. 9.13-5. Find its exponential Fourier series and plot the two-sided spectra.

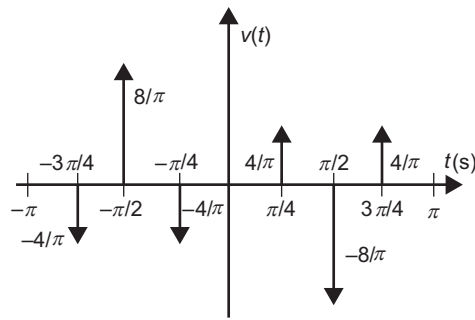


Fig. 9.13-5

15. One cycle of $v(t)$ is shown in Fig. 9.13-6. (i) What is the relationship between this waveform and the one in Fig. 9.13-5? (ii) Obtain the exponential Fourier series of this waveform by using this relationship. (iii) Obtain the trigonometric Fourier series of this waveform and plot the one-sided spectrum of this waveform.

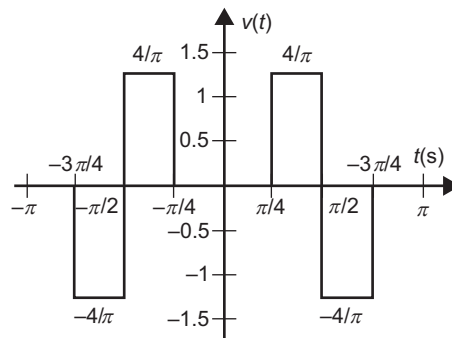


Fig. 9.13-6

16. One cycle of a periodic impulse train is shown in Fig. 9.13-7. (i) What is the relationship between this waveform and the one in Fig. 9.13-6? (ii) Obtain the exponential Fourier series of this waveform by using this relationship. (iii) Obtain the trigonometric Fourier series of this waveform and plot the one-sided spectrum of this waveform.

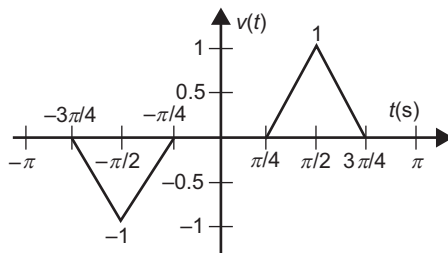


Fig. 9.13-7

17. Find the trigonometric Fourier series of the waveform $v(t)$ in Fig. 9.13-8 and plot its spectrum.

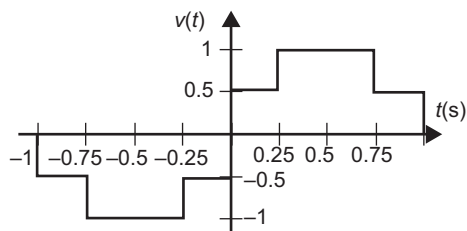


Fig. 9.13-8

18. $v(t)$ is a cosine wave and $v_1(t)$ is a square wave in Fig. 9.13-9. (i) Find $v_2(t) = v(t)v_1(t)$ and plot it. (ii) Find the trigonometric Fourier series of $v_3(t)$ from Fourier series of $v(t)$ and $v_1(t)$ and plot its spectrum.

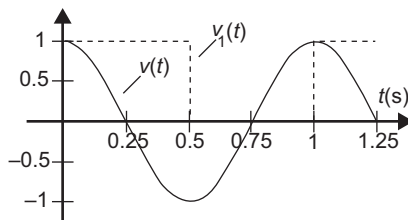


Fig. 9.13-9

19. $v(t)$ is a sine wave and $v_1(t)$ is a square wave in Fig. 9.13-10. (i) Find $v_2(t) = v(t)v_1(t)$ and plot it. (ii) Find the trigonometric Fourier series of $v_3(t)$ from Fourier series of $v(t)$ and $v_1(t)$ in terms of α . (iii) Plot its spectrum for $\alpha = \pi/6$.

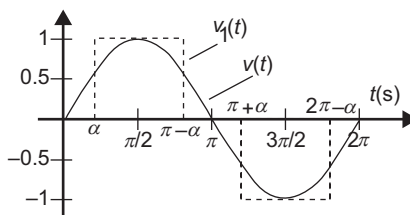


Fig. 9.13-10

20. Positive half-cycle of $v(t)$ with a period of 2 s is shown in Fig. 9.13-11. The waveform has odd symmetry. Find the exponential and trigonometric Fourier series of this waveform and plot its one-sided spectrum. If this waveform is used as an approximation to a sine wave find its THD.

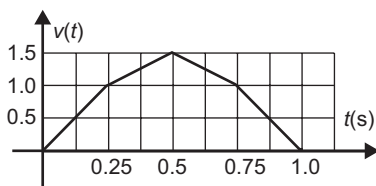


Fig. 9.13-11

21. $v_s(t) = 5|\sin \omega_0 t|$ V with $\omega_0 = 100\pi$ rad/s in Fig. 9.13-12. Assume ideal Opamp and find the output voltage $v_o(t)$. Draw its one-sided spectrum. What function does this circuit perform?

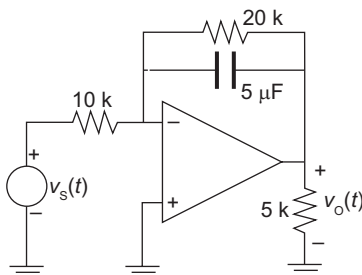


Fig. 9.13-12

22. The input voltage applied to the Opamp circuit in Fig. 9.13-13 is a symmetric triangle periodic waveform moving between +5 V and −5 V with a period of 1 ms. Find the plot the output voltage as a function of time. What function does this circuit perform?

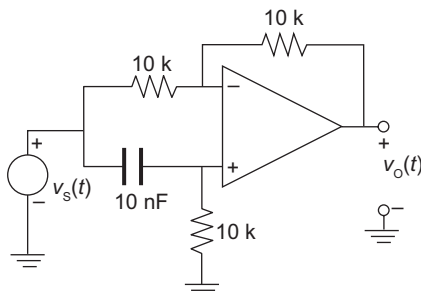


Fig. 9.13-13

23. The circuit in Fig. 9.13-14 is a *practical differentiator circuit* using an Opamp. The components C and R are sufficient to carry out differentiation. However, the non-ideal frequency response of the Opamp makes the circuit highly under-damped usually and the additional component, R_d , imparts damping to the circuit. But, with R_d present, the circuit is no more a differentiator at high frequencies. The input voltage applied to the practical differentiator circuit using Opamp in Fig. 9.13-14 is a ± 1 V symmetric triangular periodic waveform at 2.5 kHz. Obtain and plot the output voltage waveform. What is the expected output from a good differentiator for this input waveform? How does the calculated output compare with it?

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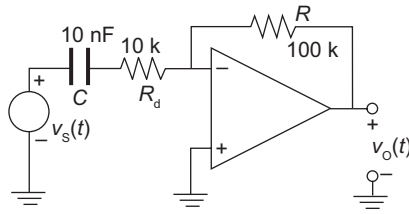


Fig. 9.13-14

24. The circuit in Fig. 9.13-15 is a *practical integrator* using an Opamp. The resistor R_{off} is needed to control the DC offset at output terminals. However, R_{off} makes the circuit an imperfect integrator. The input to this integrator is the waveform shown in Fig. 9.13-6. Find and plot the output taking the first five non-zero harmonics of input into account.

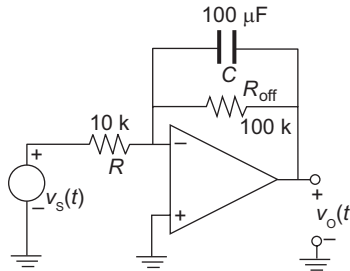


Fig. 9.13-15

25. (i) Predict the DC content in current through $6\ \Omega$ and in voltage across the parallel combination without finding out Fourier series coefficients in the circuit in Fig. 9.13-16. (ii) Find the output voltage $v_o(t)$ and plot its one-sided Fourier spectrum. (iii) Find the rms value of current through $0.3\ \Omega$ and the power dissipated in it.

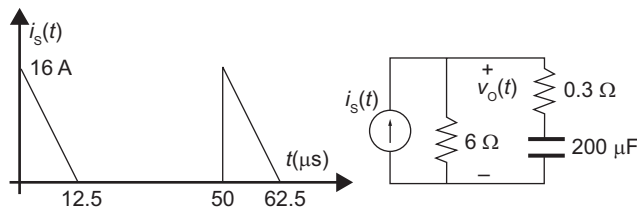


Fig. 9.13-16

26. Find the output voltage $v_o(t)$ in the circuit in Fig. 9.13-17 considering the DC component and first two non-zero harmonics in the input current source.

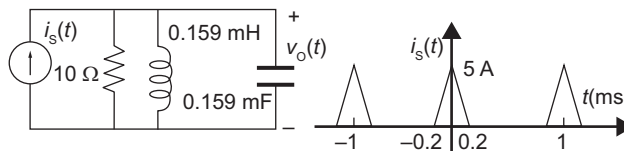


Fig. 9.13-17

27. $R = 1\text{ k}\Omega$ and $C = 1\text{ }\mu\text{F}$ in the circuit in Fig. 9.13-18. The source voltage is a periodic impulse train given by $v_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n \times 10^{-3})\text{ V}$. Find and plot the two-sided discrete power spectrum of $v_o(t)$.

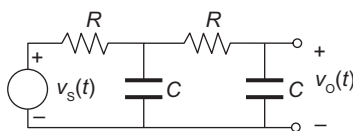


Fig. 9.13-18

28. The output voltage of a Power Electronic Inverter Circuit is related to the DC voltage used in the inverter by the equation $v_o(t) = V_{dc} \times m \sin 100\pi t$, where m is the so-called modulation index. Assume that V_{dc} is not a pure DC source and it contains AC components. Let $V_{dc} = 400 + 20\cos 200\pi t - 10\cos 400\pi t\text{ V}$ and $m = 0.8$. (i) Find and plot the output $v_o(t)$ of the Inverter. (ii) Find the THD and rms value of Inverter output. (iii) Plot the two-sided power spectrum of output voltage.
29. The switch S in the circuit in Fig. 9.13-19 operates periodically with a frequency of 10 kHz, spending $27\text{ }\mu\text{s}$ in position-1 and $73\text{ }\mu\text{s}$ in position-2. (i) Find the average charging current in the 12 V battery under periodic steady-state operation. (ii) Find the exponential Fourier series of $i(t)$ under steady-state operation and plot its power spectrum. (iii) Find the rms value of $i(t)$ and the power dissipated in the resistor.

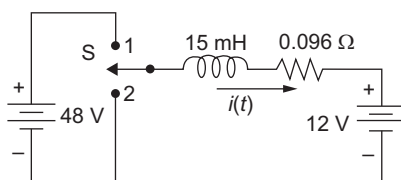


Fig. 9.13-19

30. The switch S in the circuit in Fig. 9.13-20 operates periodically with a frequency of 10 kHz, spending $77\text{ }\mu\text{s}$ in position-1 and $23\text{ }\mu\text{s}$ in position-2. (i) Find the average current delivered by the 12V battery under periodic steady-state operation. (ii) Find the exponential Fourier series of $i(t)$ and plot its power spectrum. (iii) Find the rms value of $i(t)$, the power dissipated in the resistor and the power delivered by the 12V battery. (iv) Find the average charging current in 48V battery.

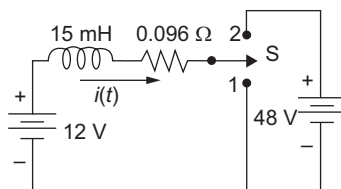


Fig. 9.13-20

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31. The applied voltage $v_s(t)$ in the circuit in Fig. 9.13-21 is $(320 \sin 100\pi t - 40 \sin 300\pi t - 20 \sin 500\pi t)$ V. (i) Find the rms value of applied voltage. (ii) Find the current delivered by the source as a function of time. (iii) Find the power delivered by the source and the VA delivered by it.

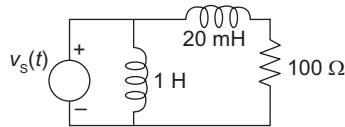


Fig. 9.13-21

32. The exponential Fourier series coefficients of $i(t)$ in the circuit in Fig. 9.13-22 are $\tilde{i}_0 = 1 \text{ A}$, $\tilde{i}_1 = 1 - j1 \text{ A}$, $\tilde{i}_{-1} = 1 + j1$, $\tilde{i}_3 = 0.3 + j0.2$ and $\tilde{i}_{-3} = 0.3 - j0.2$. The value of L is 10 mH and value of R is 100Ω . The period of $v_s(t)$ is 50 ms . Find the Fourier series of $v_s(t)$.

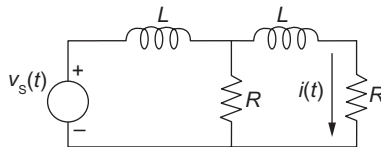


Fig. 9.13-22

Chapter 10

First-Order *RL* Circuits

CHAPTER OBJECTIVES

- *To develop the differential equations for series and parallel RL circuits*
- *Initial condition, its need and interpretation*
- *Complementary function, particular integral and their interpretation*
- *Natural response, transient response and forced response in an RL circuit*
- *Interpretation of various response components*
- *Nature and details of step response of RL circuit and time-domain specifications based on it*
- *Time constant and various interpretations for it*
- *Steady-state response versus forced response and various kinds of steady-state response*
- *Zero-input response and zero-state response and their interpretation*
- *Linearity and superposition principle as applied to various response components*
- *Impulse response and its importance*
- *Equivalence between impulse forcing function and non-zero initial condition*
- *Zero-state response for other inputs from impulse response*
- *Relations between $\delta(t)$, $u(t)$ and $r(t)$ and corresponding responses*
- *Zero-state response of RL circuit for exponential and sinusoidal inputs*
- *Frequency response of RL circuit*
- *General analysis procedure for single time constant RL circuits*

INTRODUCTION

Energy storage elements – inductance and capacitance – lend memory to the electrical circuits and make interesting dynamic behaviour in them possible. The simplest of such dynamic circuits will contain one such energy storage element along with passive resistors and linear dependent sources. Further simplification is possible by ruling out dependent sources and limiting the resistors to one. We consider one such two-element circuit containing one energy storage element in detail in this chapter.

10.2 | First-Order RL Circuits

Specifically, we study series and parallel RL circuits with voltage source and current source excitations of different waveshape with and without initial energy storage.

These circuits are simple in structure. However, that does not imply that their dynamic behaviour is trivial. The simple nature of these circuits does not imply that they are unimportant. In fact, simple RL and RC circuits find applications in almost all areas of Electrical and Electronics Engineering. Moreover, they display almost all the important features of dynamics displayed by the general class of ‘Linear Time-Invariant (LTI) Circuits’. They form very simple and effective circuit examples to convey important concepts in the analysis of LTI circuits.

Circuits containing a large number of inductors, capacitors and dependent sources will require simulation tools for quantitative analysis. However, a thorough understanding of the behaviour of simple circuits like the ones covered in this chapter will help the engineer develop qualitative insight into such circuits and arrive at approximate quantitative results. No complex and costly circuit simulation package is a satisfactory substitute for this kind of analytic capability in an engineer.

We use simple RL circuits to bring out many basic concepts in linear dynamic circuits in this chapter.

10.1 THE SERIES RL CIRCUIT

The series RL circuit with all circuit variables identified as per passive sign convention is shown in Fig. 10.1-1. Unit step voltage applied to the circuit appears in the same figure. The circuit contains three two-terminal elements. Each two-terminal element is associated with two variables – a voltage variable and a current variable. We use passive sign convention and in passive sign convention the positive current enters the element at the positive of voltage variable assigned to the element. Thus, v_L and i_L are the voltage and current variables of the inductor as per passive sign convention. The corresponding variables for the remaining two elements are shown in Fig. 10.1-1. All these six variables are functions of time though the dependence on time is not explicitly marked.

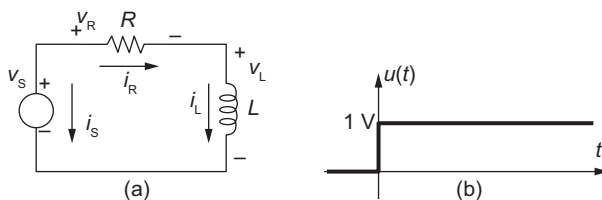


Fig. 10.1-1 (a) The series RL circuit (b) Unit step voltage waveform

10.1.1 The Series RL Circuit Equations

The analysis of this circuit aims at solving for all these six variables as functions of time. The circuit has three nodes. The three elements are in series and they will have a common current through them and hence the following equation is valid at all t :

$$i_L(t) = i_R(t) = -i_s(t)$$

The circuit has one mesh and it has to satisfy KVL at all t in this mesh. Applying KVL in this mesh will give us an equation tying up the three voltage variables – v_s , v_R and v_L – and thereby reducing the number of independent variables among the three to two. The relevant KVL equation is

$$-v_S(t) + v_R(t) + v_L(t) = 0 \text{ for all } t$$

$$\therefore v_R(t) = v_S(t) - v_L(t) \text{ for all } t$$

Thus, we have eliminated v_R by expressing it as the difference between the other two voltage variables v_S and v_L . v_S is a source function – it is the time function representing the voltage waveform of an independent source and hence it is a known function. Therefore, there is only one voltage variable and we choose that to be v_L .

Now, we have two variables – v_L and i_L . We also have the element equations of inductor and resistor to tie them up and get a single equation in i_L as follows.

$$v_R = Ri_R = Ri_L \text{ and } v_L = L \frac{di_L}{dt}$$

$$v_L + v_R = v_S \text{ by KVL in the mesh}$$

$$\therefore L \frac{di_L}{dt} + Ri_L = v_S \text{ for all } t$$

$$\text{i.e. } \frac{di_L}{dt} + \frac{R}{L}i_L = \frac{1}{L}v_S \text{ for all } t$$

Thus, a *first-order linear differential equation with constant coefficients* describes the behaviour of current in a series RL circuit for all time t . The *order* of a differential equation is the degree of highest derivative appearing in the equation and it is one in this equation. The differential equation is linear since all derivatives of the unknown function i_L and the function i_L itself appear in the equation with an exponent of unity, the three coefficients in the three terms in the equation are not functions of i_L and the forcing function on the right-hand side of the equation is a function of time only and is not dependent on the unknown function i_L in any way. R and L are circuit parameters that are assumed to be constants. Therefore, the coefficients of differential equation turn out to be constants.

10.1.2 Need for Initial Condition Specification

The differential equation derived above describes the variation of the inductor current for all t in the time-domain. It is so because we have used KCL and KVL in deriving it and these two laws hold at all time instants. Does it mean that we will be able to solve for $i_L(t)$ for all t – from infinite past to infinite future? Yes, provided we know the function $v_S(t)$ for the entire time range $-\infty < t < \infty$. That is going to be a problem!

Do we really have to get back as far as $-\infty$ in the time axis? Not really. We need to get back to the time instant at which our inductor was manufactured. It is indeed difficult to manufacture an inductor such that it takes birth with some initial energy trapped in it! Therefore, we may safely assume that the inductor had zero initial energy and hence zero initial current (energy storage in an inductor is proportional to square of current flowing through it) when it was manufactured. In fact, when we talk of infinite past, we have this time instant in mind. Therefore, we can be sure that the inductor had zero current in it in the infinite past *i.e.*, at $t = -\infty$. Now, if we know $v_S(t)$ from that instant onwards up to the present instant, we will be able to solve for $i_L(t)$ from $t = -\infty$ to the present instant by integrating the governing differential equation. But didn't we assume that the inductor was connected along with the resistor and source right at the instant it was manufactured and it remained connected so from then on? We cannot assume any such thing. Therefore, we must know whatever that happened to the inductor from the instant it was manufactured to the instant it was connected in this circuit we are trying to

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solve, if we are to solve this circuit at all. This is due to the fact that the function $v_s(t)$ can at best be known only from the instant at which this circuit came into existence and the inductor may have been subjected to various voltages in various other circuits before this circuit was wired up. In that case, the inductor will be carrying a current

$$I_0 = \frac{1}{L} \int_{-\infty}^{t_{\text{create}}} v_L dt$$

where $-\infty$ refers to the instant of manufacturing of the inductor and t_{create} refers to the instant at which the RL circuit under discussion came into being. The voltage $v_L(t)$ in the above integral refers to all the voltage that was applied to the inductor during this time interval. Thus, the inductor carries its accumulated past in the form of an initial current I_0 given by the above integral when it enters the RL circuit we are trying to analyse. Notice that the voltage appearing in the integrand has no relation with the source that is applied subsequently to the RL circuit.

Briefly, *we need to know the past of inductor*. However, fortunately we need not know everything about its past – we need only the value of the above integral. We will be able to solve for $i_L(t)$ for all instants after t_{create} if we know the value of I_0 along with $v_s(t)$ from the instant it was applied, *i.e.*, from t_{create} . Obviously, we need to know everything about its past if we want to solve for $i_L(t)$ from $t = -\infty$ onwards. That is too much of a past to carry. Therefore, we would want to solve for $i_L(t)$ only for $t \geq t_{\text{create}}$ usually. We need the value of I_0 , the initial current in the inductor at an instant just prior to t_{create} for that. This single number condenses all the past history of inductor as far as the effect of voltages applied to it in the past on the evolution of its current in future is concerned. This number is called the *initial condition* for the inductor.

The time instant t_{create} is to be understood as the instant from which we know the data required for solving the differential equation. That is the time instant at which the initial condition is specified for the inductor and it is the time instant from which we have complete knowledge about the input source voltage. Moreover, it is the time instant from which the inductor should act as an element in this circuit and only in this circuit. It is customary in Circuit Theory to set this instant as the time-zero instant, *i.e.*, $t_{\text{create}} = 0$ unless there is some specific reason for making it different.

Usually, some kind of switching action takes place in the circuit at this instant. It could be a switching that applies a specific voltage waveform at its input. Or, it could be a switching operation which changes the structure of the circuit – for example, one element in the circuit may have been kept shorted by closing a switch across it and now at $t = 0$ that switch is opened. Such switching action usually brings in jump discontinuities in circuit variables. Jump discontinuities in variables involved in differential equations are difficult to handle mathematically unless singularity functions are brought in. We do not want to do that in this book. Our way of handling such discontinuities in circuit variables will be circuit-theoretic and we need to define two more time instants to facilitate our circuit-theoretic reasoning in such situations. These two time instants are $t = 0^-$ and $t = 0^+$. $t = 0^-$ is a time instant which is to the left of $t = 0$ in the time axis. However, the time interval $[0^-, 0]$ is of infinitesimal width, *i.e.*, 0^- is arbitrarily close to 0, but always less than 0. Similarly, 0^+ is on the right of 0 and is arbitrarily close to 0. Thus, $0^- < 0 < 0^+$ while $0 - 0^- \approx 0$ and $0^+ - 0 \approx 0$.

10.1.3 Sufficiency of Initial Condition

We defined three important time instants – $t = 0$, 0^- and 0^+ . We also concluded that, with the initial condition for i_L specified and with the knowledge of v_s from the instant at which the initial condition

is specified, we will be able to solve the differential equation for i_L . But, which instant among the three instants above should be used for specifying initial condition?

The three time instants have been defined only to make it easy to handle possible discontinuities in the input function v_s . If the source function v_s is a continuous function of t we do not need $t = 0^-$ and $t = 0^+$. In that case, initial condition is specified at $t = 0$. However, if v_s has a discontinuity or singularity (for example, a jump discontinuity as in step function or an impulse located at $t = 0$) at $t = 0$, it is possible that either i_L or its derivative will undergo step changes at $t = 0$. Therefore, the time instant at which the initial condition is applicable must be clearly specified. That instant has to be $t = 0^-$. Since it will not make any difference in the case where v_s is a continuous function, we will stipulate that initial condition be always be specified at $t = 0^-$.

The initial current in the inductor at $t = 0^+$ can be different from its value specified at $t = 0^-$ for certain kinds of input functions. We will illustrate how the initial condition value at 0^+ can be calculated from initial condition value at 0^- and nature of v_s later in this chapter. We have to address another issue at present.

We now assert that $i_L(t)$ for $t \geq 0^+$ can be obtained if initial condition I_0 at $t = 0^-$ is known and $v_s(t)$ for $t \geq 0^+$ is known. This is equivalent to asserting that *the net effect of all the voltage applied across the inductor in the time range $-\infty < t \leq 0^-$ on the evolution of its current in the time range $t \geq 0^+$ is encoded in a single number I_0 , the initial condition at $t = 0^-$. This in effect says that the past which the inductor remembers is contained in I_0 .*

They are strong assertions and they need to be proved. We offer a plausibility reasoning to convince ourselves that these assertions are true. We assume that the function $v_s(t)$ is continuous at $t = 0$ in the reasoning that follows.

We recast the circuit equations of the series RL circuit in Fig. 10.1-1 in the following manner:

$$\begin{aligned} v_L(t) &= v_s(t) - v_R(t) \text{ for all } t \text{ (from KVL)} \\ \text{i.e. } v_L(t) &= v_s(t) - Ri_R(t) \text{ for all } t \\ \text{i.e. } v_L(t) &= v_s(t) - Ri_L(t) \quad (\text{Since } i_L(t) = i_R(t)) \\ \text{i.e. } v_L(t) &= v_s(t) - \frac{R}{L} \int_{-\infty}^t v_L(t) dt \text{ for all } t \text{ (from element equation of inductor)} \end{aligned}$$

Further, we rewrite the last equation by splitting the range of integration into two sub-ranges as follows. Notice the change in time-range of applicability.

$$v_L(t) = v_s(t) - \frac{R}{L} \int_{-\infty}^0 v_L(t) dt - \frac{R}{L} \int_0^t v_L(t) dt \text{ for } t \geq 0$$

We recognise the second term on the right-hand side as $(R/L)I_0$ where I_0 is the initial condition for the inductor. Therefore, we write

$$v_L(t) = v_s(t) - \frac{R}{L} \int_0^t v_L(t) dt - \frac{R}{L} I_0 \text{ for } t \geq 0 \quad (10.1-1)$$

It is obvious from Eqn. 10.1-1 that the solution for $v_L(t)$ for $t \geq 0$ will depend only on I_0 and the values of v_s for $t \geq 0$. A simple method to integrate the equation numerically is outlined below.

Divide the time interval $[0, t]$ into many small intervals, each of width Δt . Let N be the number of such intervals. Then

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$$v_L(0) = v_S(0) - \frac{R}{L} I_0$$

$$v_L(n\Delta t) = v_S(n\Delta t) - \frac{R}{L} v_L((n-1)\Delta t) - \frac{R}{L} I_0, \quad n=1, \dots, N$$

gives $N + 1$ values of $v_L(t)$ in the interval $[0, t]$ at equally spaced sub-intervals. The accuracy of calculation can be improved by increasing the number of sub intervals (*i.e.*, by decreasing Δt). $i_L(t)$ can be found by evaluating the first derivative of $v_L(t)$ and multiplying it by L once $v_L(t)$ is calculated with sufficient accuracy.

Thus, I_0 specified at some time point and $v_S(t)$ from that time point will be sufficient data needed to solve for inductor current in an RL circuit from that time point onwards.

10.2 SERIES RL CIRCUIT WITH UNIT STEP INPUT – QUALITATIVE ANALYSIS

We study the behaviour of inductor current in a series RL circuit excited by a unit step voltage source qualitatively in this section. The initial current in the inductor at $t = 0^-$ is assumed to be zero for the purpose of this analysis. The relevant circuit with the manner in which the unit step voltage is realised in practice shown clearly appears in Fig. 10.2-1.

Unit step voltage is defined the following manner. It has a step discontinuity at $t = 0$ and strictly speaking the time instant $t = 0$ is excluded from the domain of the function. However, it is customary in Circuit Theory to represent this function with a solid line jumping from 0 to 1 at $t = 0$ in its plot as shown in Fig. 10.1-1

$$u(t) = \begin{cases} 0 & \text{for } t \leq 0^- \\ 1 \text{ Volt} & \text{for } t \geq 0^+ \end{cases}$$

The switch S_2 was closed at $t = -\infty$ and is opened at $t = 0$ and remains open up to $t = +\infty$. The switch S_1 is open from $t = -\infty$, is closed at $t = 0$ and remains closed up to $t = +\infty$. We assume that the switches open and close in zero time intervals. The switch S_2 applies zero V to the circuit for all $t \leq 0^-$ and the switch S_1 applies the 1V available from the DC battery for all $t \geq 0^+$, thereby effecting unit step voltage application to the circuit.

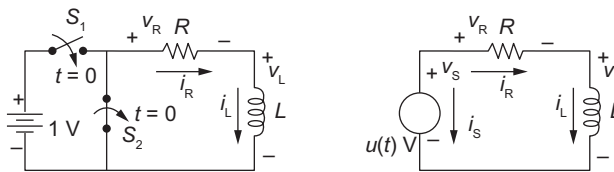


Fig. 10.2-1 Series RL circuit with unit step voltage input

10.2.1 From $t = 0^-$ to $t = 0^+$

The voltage applied to inductor in the interval $t \in (-\infty, 0^-]$ is zero. Hence, the current in the inductor at $t = 0^-$ is 0 A. The voltage applied by the unit step input source has a jump discontinuity from 0V to 1V at $t = 0$. Can this jump discontinuity result in a change in inductor current at $t = 0^+$? We relate

the possible change in inductor current over the time interval $[0^-, 0^+]$ to the voltage across inductor during that interval by employing the element equation of inductor.

$$i_{L(0^+)} - i_{L(0^-)} = \frac{1}{L} \int_{0^-}^{0^+} v_L dt$$

The resistor and the inductor always share the input voltage. Thus, the resistor too may absorb part of the input voltage during the interval $[0^-, 0^+]$ if there is a current in the circuit. Therefore, the voltage available to the inductor during this interval can only be less than what is available in the source. More importantly, this voltage can assume only finite (though not determinate in view of the discontinuity in the input function) values in this interval. Further, the interval is infinitesimal in width. The area under a finite-valued function over an infinitesimal interval is zero. Therefore, $i_{L(0^-)} = i_{L(0^+)}$ i.e., the inductor current is continuous across the interval $[0^-, 0^+]$. It takes an impulse function at $t = 0$ to generate finite area over infinitesimal time interval. Thus, we conclude that, *the current in an inductor at $t = 0^-$ and $t = 0^+$ in any circuit will be same unless the circuit can support impulse voltage across that inductor at $t = 0$.*

There is a difference between *applying impulse voltage* and *supporting impulse voltage*. If we connect an independent voltage source with its source function $= \delta(t)$ to a circuit, then, we are *applying* an impulse voltage to the circuit. However, suppose we are connecting an independent current source which suddenly changes its current from 0A to 1A at $t = 0$ to an inductor. Now, the inductor has to change its current from 0A to 1A instantaneously at $t = 0$ and it will produce a back e.m.f of $L\delta(t)$ V across it in that process. The current source has to absorb that voltage in order to satisfy Kirchhoff's Voltage Law. But ideal current sources do not complain even if they are called upon to support infinite voltage. Hence, in this instance, the circuit *supports* impulse voltage, but does not *apply* it.

10.2.2 Inductor Current Growth Process

The initial current in the inductor at $t = 0^-$ is zero in the circuit under consideration. This implies that the initial energy storage in the inductor is zero. This state of zero initial energy is also indicated often by phrases '*initially at rest*' and '*initially relaxed*'. The input voltage is a unit step function and it remains within finite limits at all instants. Thus, the current in the inductor cannot change over the small time interval between $t = 0^-$ and $t = 0^+$. Therefore, current in the circuit at $t = 0^+$ is zero.

However, the input voltage which was at 0V at $t = 0^-$ has become 1V at $t = 0^+$ while the current remained at zero value. The resistor can absorb only 0V with a zero current flowing through it. This implies that the voltage across inductor will undergo a sudden jump at $t = 0$ from 0V to 1V. In fact, all sudden changes in the input voltage of a series RL circuit will have to appear across the inductor i.e., the resistor will refuse to have anything to do with the sudden jumps in input voltage and will dump all such jumps onto the inductor. This is so because if the resistor absorbs even a small portion of the jump in input voltage, the circuit current too will have to have a jump discontinuity. But the inductor does not allow that unless the circuit can apply or support impulse voltage. Therefore, the inductor insists that the current through it remain continuous (unless impulse voltage can be supported) and it is willing to pay the price for that by absorbing the discontinuities in the input source voltage. Since the inductor will maintain the current continuous, the voltage across the resistor too will remain continuous. In fact, this is one of the applications of RL circuit – to make the voltage across a load resistance smooth when the input voltage is choppy.

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The current in an inductor is related to voltage across it through the relation

$$v_L = L \frac{di_L}{dt}$$

This shows that, with a voltage of 1V at $t = 0^+$ across it, the current in the inductor cannot remain at its current value of zero forever. The current has to grow since its derivative is +ve and hence it starts growing at the rate of $1/L$ A/s initially. This in turn implies that the slope of current Vs. time curve at $t = 0^+$ will be $1/L$ A/s.

However, as the current in the inductor grows under the compulsion of voltage appearing across it, the resistor starts absorbing voltage. This results in a decrease in the inductor voltage since v_R and v_L will add up to 1V by KVL at all $t \geq 0^+$. Decrease in v_L results in a decreasing rate of change of i_L since rate of change of i_L is directly proportional to v_L . This means that the tangent drawn on the i_L vs. t curve will start with a slope of $1/L$ A/s at $t = 0^+$ and will bend progressively towards the time axis as t increases. The current in the inductor which starts out with a certain initial momentum loses its momentum with time and grows at slower and slower rates as time increases due to the resistor robbing an increasing portion of source voltage from the inductor. Therefore, we expect the current function to be of convex shape.

Will the growth process ever end and will the inductor current reach a steady value *i.e.*, a constant value? Let us assume that it does so. Then, what is the value of current that can remain constant in the circuit? If the current is constant, the inductor will demand only zero voltage for allowing that current. That will mean that all the source voltage, *i.e.*, 1V will have to be absorbed by the resistor. The resistor will do that only if the current through it is $1/R$ A. Therefore, if the circuit can reach the end of growth process and become steady, it will do so only at this unique value of current – $1/R$ A. Thus, $i_L = 1/R$ A is a possible steady end to the growth process we described above. However, detailed mathematical solution will tell us that inductor current never attains this value; but approaches this value asymptotically with time.

10.3 STEP RESPONSE OF RL CIRCUIT BY SOLVING DIFFERENTIAL EQUATION

We have already derived the first-order linear differential equation with constant coefficients that describe the behaviour of inductor current in a series RL circuit for all t . This governing differential equation is

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} v_s(t) \quad \text{for all } t \quad (10.3-1)$$

where $\tau = L/R$, $i_L(t)$ is the current in the inductor (as well as the circuit current) and $v_s(t)$ is the input voltage source function (also called excitation function, forcing function, input function etc.) which must be defined for all t if Eqn. 10.3-1 is to be used.

Notice that the domain of the functions appearing in the governing differential equation is the entire time axis from $-\infty$ to $+\infty$ since the differential equation is obtained by a systematic application of KCL and KVL. They are basic conservation laws in essence and hence must remain true at all instants of time. Therefore, we must know the input function for all t if we are to solve for $i_L(t)$ for all t . But, neither can we know input source function for all t in a practical circuit problem nor do we want to solve for current for all t . Also, the input function may be ill-defined at certain instants of time. This makes a detailed discussion of the way input functions are specified in circuit problems necessary.

10.3.1 Interpreting the Input Forcing Functions in Circuit Differential Equations

We can identify a sequence of time instants at which certain ‘switching events’ will take place in any circuit problem in general. Quite often, this sequence of time points may contain only one entry – as in the case of an RL circuit with battery connected to it by a switch that closes at some specified time instant. However, it is quite possible that a sequence of switching events will take place in our circuit in a more complex setting. Among these various switching instants there will be one that is the earliest instant after which we have complete information about all source functions applied to the circuit – this instant is customarily (though not necessarily) marked as $t = 0$ in circuit problems. The circuit problem involves solving for all the circuit variables as functions of time from the first switching instant onwards.

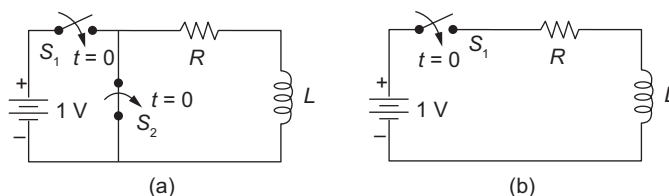


Fig. 10.3-1 (a) Applying unit step voltage to RL circuit (b) 1V DC switching in RL circuit

The switch S_1 in the circuit in Fig. 10.3-1 (a) is closed at $t = 0$ and the switch S_2 is open at $t = 0$ to bring about an abrupt change in the applied voltage from 0 to 1 V. If S_2 was closed from $t = -\infty$, the current in inductor at $t = 0^-$ can only be zero. Whatever energy was possibly trapped in the inductor at $t = -\infty$ would have got dissipated in the resistor by current flow through closed circuit. Hence, there is no need for specifying initial current in the inductor since the applied voltage is known for all t . The differential equation describing the circuit in Fig. 10.3-1 (a) is

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{V}{L} u(t) \text{ for all } t. \quad (10.3.1)$$

The solution of the equation will give the so-called *unit step response* of the RL circuit. *Unit Step function* is defined from $-\infty$. Hence, the initial instant in this circuit is $t = -\infty$ and not $t = 0$. All circuits are assumed to be relaxed at $t = -\infty$ by default.

We just do not know anything about applied voltage for $t < 0^-$ in the circuit in Fig. 10.3-1 (b). This is so because voltage across an open-circuit is not decided by the open-circuit; but it is decided by the elements connected on the right side of open-circuit. The initial condition at $t = 0^-$ for inductor current in this circuit can be zero or non-zero. Zero initial condition does not imply that no voltage was ever applied to inductor in the past. Rather, it means that whatever be the voltage waveform applied to the inductor in the past, the area under that waveform from $-\infty$ to 0^- in the time axis was zero. Similarly, a non-zero initial condition will imply that, some extra circuit arrangement that is not shown in the circuit diagram was employed to apply suitable voltage to inductor in the past in order to take its current to the specified value at $t = 0^-$. In any case, it does not matter if we are interested in the circuit solution only for $t \geq 0^+$. The differential equation describing the circuit is as in the following equation:

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} \text{ for } t \geq 0^+ \text{ with } i_L(0^-) = I_0 \quad (10.3-2)$$

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Note that Eqn. 10.3.1 cannot describe the circuit in Fig. 10.3-1 (b) even if the initial condition is specified as 0 A. This is so since solution of Eqn. 10.3.1 will show that the circuit current is = 0 for all $t \leq 0^-$, whereas zero current specified at $t = 0^-$ in circuit (b) does not imply that the current was zero for all $t \leq 0^-$. However, the circuit solution for $t \geq 0^+$ will be the same in both circuits if $I_0 = 0$ A is specified for the circuit in Fig. 10.3-1 (b).

10.3.2 Complementary Function and Particular Integral

The governing equation for inductor current in a series RL circuit with unit step voltage input is

$$\frac{di_L}{dt} + \alpha i_L = \beta u(t) \quad (10.3-3)$$

where $\alpha = R/L$ and $\beta = 1/L$. There is a jump discontinuity from 0 to 1V at $t = 0$ in the applied voltage. This jump in applied voltage travels straight to the inductor and appears as a jump in voltage across it at $t = 0$.

Particular Integral of a differential equation is the solution term due to the forcing function. The *domain* of particular integral is same as the domain of the input function. In this case, the input is 1V in $t \in (0, \infty)$. Hence, the particular integral will be valid for $t \in (0, \infty)$. The particular integral in the present case is obtained by solving the following equation in the interval $t \in (0, \infty)$:

$$\frac{di_L}{dt} + \alpha i_L = \beta \text{ for all } t \in (0, \infty) \quad (10.3-4)$$

The differential equation in Eqn. 10.3-4 is to be true for all $t \in (0, \infty)$. This is possible only if i_L is such a function that its first derivative along with its own copy becomes a constant for all t . The only such function is $i_L = c$, a constant. Substituting this trial solution in Eqn. 10.3-4, we get that the particular integral is β/α A for $t \in (0, \infty)$.

What about particular integral for $t \in (-\infty, 0)$? The voltage applied in this interval was 0 V. Therefore, the particular integral is 0 for $t \in (-\infty, 0)$.

Therefore, the particular integral of the circuit changes from 0 to β/α A at $t = 0$. That is, there is a step discontinuity in the particular solution at $t = 0$. However, such an instantaneous change in inductor current is not permitted in an RL circuit unless impulse voltage is applied across the inductor. The inductor current has to be continuous at $t = 0$.

Hence, we need another term in the circuit solution which will force the total circuit solution to satisfy the inductor current continuity requirement at $t = 0$.

This additional function has to satisfy a constraint. The solution $i_L(t) = \begin{cases} \frac{\beta}{\alpha} = \frac{1}{R} \text{ A} ; t \in [0^+, \infty) \\ 0 \text{ A} ; t \in (-\infty, 0^-] \end{cases}$

satisfies the differential equation in Eqn. 10.3-4 at all $t \neq 0$. Therefore, the additional function we are to going to add to this solution to enforce compliance with continuity requirement at $t = 0$ should not add anything to the right side of differential equation. This implies that it has to be a function that will satisfy the following differential equation for all t excluding $t = 0$.

$$\frac{di_L}{dt} + \alpha i_L = 0; t \neq 0 \quad (10.3-5)$$

A trivial solution to this equation is $i_L = 0$. We are not interested in that. We recast the above equation as

$$\frac{di_L}{dt} = -\alpha i_L; t \neq 0$$

and note the fact that if i_L is a function of time, then both sides of this equation will be functions of time. *Two functions of time can be equal to each other over their entire domain if and only if they are same kind of functions – they must have same shape when plotted.* Hence, we look for functions which produce a copy (probably scaled versions) of themselves on differentiation.

Sinusoidal function comes to our mind first. But then, we remember that sinusoidal function can be covered by exponential function with imaginary exponent by Euler's formula. Hence, $e^{\gamma t}$ where γ can be a complex number is a function with the desired property. So we try out $Ae^{\gamma t}$, where A is an arbitrary constant, as a possible solution in Eqn. 10.3-5. We get

$$A\gamma e^{\gamma t} + A\alpha e^{\gamma t} = 0 \text{ for all } t \text{ in any interval}$$

$$Ae^{\gamma t}(\gamma + \alpha) = 0 \text{ for all } t \text{ in any interval}$$

$A = 0$ is a trivial solution. $e^{\gamma t} = 0$ cannot be a solution since the equation has to be true for all t . Therefore, γ has to be equal to $-\alpha$. Thus, $Ae^{-\alpha t}$ is the solution for the Eqn. 10.3-5 in any given interval. The differential equation cannot help us in deciding the value of A . It decides only the value of exponent in the exponential function. This solution we have arrived at i.e., $Ae^{-\alpha t}$ is called the *complementary function* of the differential equation in Eqn. 10.3-5 and the differential equation with zero forcing function is called the *homogeneous differential equation*.

Now that we have got the function required to enforce compliance with the current continuity requirement in the circuit, let us proceed to form the *total solution* for i_L in series RL circuit with step input. There are two intervals over which the particular integral is known. They are $(-\infty, 0^-]$ and $[0^+, \infty)$. Let the complementary solution that we accept for $t \in (-\infty, 0^-]$ be $A_- e^{-\alpha t}$ and the complementary solution that we accept for $t \in (0^+, \infty)$ be $A_+ e^{-\alpha t}$.

$$i_L(t) = A_+ e^{-\alpha t} + \frac{1}{R} \text{ for } t \geq 0^+ \text{ and } i_L(0^+) = 0 \quad (10.3-6)$$

The voltage applied to the circuit was zero right from $t = -\infty$ up to $t = 0^-$. Hence, the circuit current has to remain zero over $(-\infty, 0^-]$. Therefore, the amplitude of complementary solution for $t \in (-\infty, 0^-]$ has to be chosen as zero, i.e., $A_- = 0$. However, this will mean that there will be a step discontinuity at $t = 0$ in the complementary solution for any non-zero A_+ . That is precisely what we want. We want the discontinuity in complementary solution to cancel the discontinuity in particular integral at $t = 0$ and thereby make inductor current continuous at $t = 0$. Obviously, A_+ must be chosen as negative of the size of discontinuous jump in a particular integral. This gives us $A_+ = -1/R$. Hence, the final solution is

$$i_L(t) = \begin{cases} \frac{1}{R}(1 - e^{-t/\tau}); & t \geq 0^+ \\ 0; & t \leq 0^- \end{cases} \quad \text{where } \tau = \frac{1}{\alpha} = \frac{L}{R} \quad (10.3-7)$$

Equation 10.3-7 gives the unit step response (abbreviated as *step response*) of current in a series RL circuit.

10.3.3 Series *RL* Circuit Response in DC Voltage Switching Problem

We have seen that switching on a 1V DC source to a series *RL* circuit is not the same as applying a *unit step voltage source* across it. Let us see how the difference affects the circuit solution. The relevant circuit is given as circuit in Fig. 10.3-1 (b). We do not know what was done to the inductor before $t = 0^-$. All we know is that the voltage applied across the circuit is 1V for all $t \geq 0^+$ with switching taking place at $t = 0$. Hence, we need to know the inductor current at $t = 0^-$ if we are to solve for inductor current for $t \geq 0^+$. Let this current $i_L(0^-) = I_0$ be given as data.

We divide the time-axis into two non-overlapping semi-infinite intervals $(-\infty, 0^-]$ and $[0^+, \infty)$. The time-point $t = 0$ is excluded from both.

We do not know the particular integral for $t \in (-\infty, 0^-]$ since we do not know what was the voltage function applied to the circuit during that interval. We know that the complementary solution part for $t \in (-\infty, 0^-]$ can be $A_- e^{-\alpha t}$; but we cannot fix the value of A_- since the nature of input during the relevant interval is unknown. Therefore, we cannot solve for inductor current in this circuit for $t \in (-\infty, 0^-]$.

We know the particular solution $t \in [0^+, \infty)$. It is $1/R$ A as in the case of unit step response. We also know that the complementary solution for this interval can be $A_+ e^{-\alpha t}$. We are given that $i_L(0^-) = I_0$. No impulse voltage is applied across the circuit. Hence, the inductor current has to remain continuous in the interval $[0^-, 0^+]$ and $i_L(0^+)$ has to be equal to $i_L(0^-)$. Therefore, the total solution for $t \in [0^+, \infty)$ has to start at I_0 .

$$\therefore i_L(t) = A_+ e^{-\alpha t} + \frac{1}{R} \quad \text{for } t \geq 0^+ \quad \text{and} \quad i_L(0^+) = i_L(0^-) = I_0$$

$$\text{Substituting initial condition, } I_0 = A_+ + \frac{1}{R} \Rightarrow A_+ = I_0 - \frac{1}{R}$$

$$\begin{aligned} \therefore i_L(t) &= \left(I_0 - \frac{1}{R}\right) e^{-\alpha t} + \frac{1}{R} \quad \text{for } t \geq 0^+ \\ &= I_0 e^{-t/\tau} + \frac{1}{R} (1 - e^{-t/\tau}) \quad \text{for } t \geq 0^+ \end{aligned}$$

The complete solution for the current can be written as

$$i_L(t) = \begin{cases} I_0 e^{-t/\tau} + \frac{1}{R} (1 - e^{-t/\tau}) \text{ A for } t \geq 0^+ \\ \text{Unknown for } t < 0^- \end{cases}$$

For the particular case with $I_0 = 0$ A, the circuit solution will be

$$i_L(t) = \begin{cases} \frac{1}{R} (1 - e^{-t/\tau}) \text{ A for } t \geq 0^+ \\ 0 \text{ A at } t = 0^- \\ \text{Unknown for } t < 0^- \end{cases} \quad (10.3-8)$$

This is not the same as unit step response given in Eqn. 10.3-7.

10.4 FEATURES OF *RL* CIRCUIT STEP RESPONSE

Step response in electrical circuit analysis context implies application of the *unit* step function, $u(t)$, as the input. The response to this unit step application can be described in terms of a chosen circuit variable, which may be a voltage variable, a current variable or a linear combination of voltage and current variables. We had chosen the current through inductor as the response variable in the case of series *RL* circuit. The current waveform was shown to be

$$i_L = \frac{1}{R}(1 - e^{-t/\tau}); \quad t \geq 0^+ \quad (10.4-1)$$

The primary objective in applying an input function to a circuit is to make some chosen output variable in the circuit behave in a desired manner. This is why the input function is called a *forcing function*. Input function is a command to the circuit to vary its response variable in a manner similar to its own time variation. Application of unit step input is equivalent to a command to the circuit to change its response variable in a step-wise manner in this sense. Similarly, when we switch on a voltage $v_S(t) = V_m \sin \omega t$ V at $t = 0$ to any circuit, we are, in effect, commanding the circuit to make the chosen response variable follow this function in shape. A purely memory less circuit will follow the input command with no delay. However, circuits with memory elements will not.

Inductor constitutes electrical inertia. It does not like to change its current and resists any such current change by producing a back e.m.f across it – the magnitude of this e.m.f is directly proportional to the rate at which the inductor current changes. Other elements in the circuit (usually voltage sources, switches etc.) will have to supply the voltage demanded by the inductor if the desired current change is to take place. This is the price the other elements in the circuit have to pay for demanding the lazy inductor to change its current. The price is heavier if the required change in current is to be accomplished faster.

It is still more instructive to look at the ‘inertia’ aspect of inductor from its energy storage capability. An inductor stores energy in its magnetic field. The energy stored in the field is proportional to square of current through inductor. Thus, if we want to change the current through inductor, we have to supply energy to the inductor or absorb energy from the inductor. Note that we do not have to do any such thing if the current through inductor is at a constant level. Let us assume that we want to change the inductor current from I_1 to I_2 ($I_2 > I_1$). By the time we have done it, we would have given the inductor $0.5L(I_2^2 - I_1^2)$ Joules of energy. We can pump energy into the inductor only by pumping power into it. Therefore, a voltage has to appear across the inductor whenever its current tries to change. Further, energy has to be pumped into inductor at a fast rate if the current in inductor is to change fast. That means that power flow into inductor has to be increased if the inductor current is to change fast. And that is why voltage across inductor becomes higher and higher when a given amount of current change is sought to be attained in shorter and shorter time intervals.

Consider a similar situation in translational mechanics. A mass M is forced to move against friction. Assume that the frictional force is proportional to velocity of the mass and that there is no sticking friction. Now, if we apply a constant force to the mass we know that (i) the mass reaches a final speed at which the applied force is met exactly by the frictional force acting against motion and (ii) it takes some time to reach this situation. Mass M does not like to move due to its inertia – it is in the nature of objects in this world to stay put. They prefer it that way. Similarly, it is in the nature of inductor to stay put as far as its current is concerned. However, objects in this world do yield to forces eventually. In the above case, since the mass M shows a tendency to stay put even after the force has come into action, it has to absorb the entire force initially. In that process, it gets accelerated. Hence, for a brief

period initially, a major portion of the applied force is used to accelerate the mass and only a minor portion is used for meeting friction. This proportion will change with time and finally no force will be spent on accelerating the mass and entire force will be spent on countering friction. Hence, initially the ‘inertial nature’ of mass dominates the situation and puts up a stiff fight with the force that is a command to the mass to move at a constant speed. Slowly the resistance from the mass weakens and inexorably the force subjugates the inertial nature of mass. And after sufficient time has elapsed, the applied force wins the situation; the mass yields almost completely to the force command and moves at an almost constant speed commensurate with the level of friction present in the system.

This tussle between the inherent inertial nature of systems and the compelling nature of forcing functions is a common feature in dynamic systems involving memory elements and is present in electrical circuits too. Thus, the response immediately after the application of a forcing function in a circuit will be a compromise between the inherent natural laziness of the system and the commanding nature of forcing function. The circuit expresses its dislike to *change* by spewing out a time function, which quantitatively describes its unwillingness to *change*. The forcing function wears down this natural cry from the circuit gradually and establishes its supremacy in the circuit in the long run – by forcing all circuit variables to vary as per its dictate in the long run.

The total response in the circuit is always a mixture of these two with the component from forcing function dominating almost entirely in the long run and the natural component from the circuit’s inherent inertia ruling in the beginning. It should be noted at this point that it is quite possible that neither component will succeed in overpowering the other in some circuits. Such circuits are called *marginally stable circuits*. Further, there are circuits in which the natural component will not only refuse to yield but grow without limit as time increases; thereby overpowering the forcing function with time. Such circuits are called *unstable circuits*. We will take up such circuits in later chapters. However, at present, we deal with circuits that yield to the forcing function in the long run – called *stable circuits*.

The time function that the circuit employs to protest against *change* is called the *natural response* of the circuit and the time function that the forcing function establishes in the response variable is called the *forced response*. The *natural response* means precisely that – it encodes the basic nature of the circuit and has nothing to do with the nature of forcing function. Its shape and other features (except amplitude) are decided by the nature and number of energy storage elements in the circuit, the way these energy storage elements are connected along with resistive elements to form the circuit, etc. Thus, its shape depends only on the nature of elements and the topology of the circuit and does not depend on the particular shape and value of forcing function – it is *natural* to the circuit. But its magnitude will depend on initial condition and forcing function too.

The series *RL* circuit with voltage source excitation howls ‘*exponentially*’ when forcing function commands its current to change. In fact, all stable dynamic systems described by a ‘linear first-order ordinary differential equation with constant coefficients’ will cry out exponentially when they are asked to change. They all have a natural response of the type $Ae^{-\alpha t}$ where α , which decides the shape of response, is decided by system parameters (R and L in the present instance) and A is decided by initial condition and the initial value of forced response. The forcing function along with initial condition will decide the magnitude of natural response, but not the shape.

The shape of natural response does not depend on forcing function and hence must be the same for a zero forcing function and a non-zero forcing function. A non-zero response with a zero forcing function can exist if the circuit starts out with initial energy at $t = 0^-$. This is similar to a mass, which has been accelerated to some velocity before $t = 0$, slowing down to zero speed after $t = 0$ under the effect of friction with no other force applied to it. Thus, it follows that we can find out the shape of natural response by solving the differential equation describing the circuit response with forcing function set

to zero. But that will be the homogeneous differential equation and we know that its solution is the complementary function of the equation. The complementary solution of the differential equation describing the current in the inductor in our RL circuit was shown to be an exponential function with negative real index earlier. Thus, we conclude that *the complementary solution of the describing differential equation of a circuit yields the natural response of the circuit, whereas the particular integral corresponding to the applied forcing function yields the forced response.*

10.4.1 Step Response Waveforms in Series RL Circuit

There are only three circuit variables in a series RL circuit and they are $i_L(t)$, $v_R(t)$ and $v_L(t)$ as marked in Fig. 10.1-1. The expressions for $v_R(t)$ and $v_L(t)$ may be worked out from the solution for $i_L(t)$.

$$\begin{aligned} i_L(t) &= \frac{1}{R}(1 - e^{-t/\tau}) \text{ A}; t \geq 0^+ \\ v_R(t) &= (1 - e^{-t/\tau}) \text{ V}; t \geq 0^+ \\ v_L(t) &= L \frac{di_L(t)}{dt} = \frac{L}{R} \times \frac{e^{-t/\tau}}{\tau} = e^{-t/\tau} \text{ V}; t \geq 0^+ \end{aligned} \quad (10.4-2)$$

We introduce normalisation in these expressions before we plot them by dividing the expressions by the final steady value or the maximum value as applicable. Similarly, we define normalised time variable t_n as t/τ . This results in

$$\begin{aligned} i_{Ln}(t) &= \frac{i_L(t)}{1/R} = (1 - e^{-t_n}); t_n \geq 0^+ \\ v_{Rn}(t) &= \frac{v_R(t)}{1} = (1 - e^{-t_n}); t_n \geq 0^+ \\ v_{Ln}(t) &= \frac{v_L(t)}{1} = e^{-t_n}; t_n \geq 0^+ \end{aligned} \quad (10.4-3)$$

where the second subscript ‘n’ indicates normalised variables.

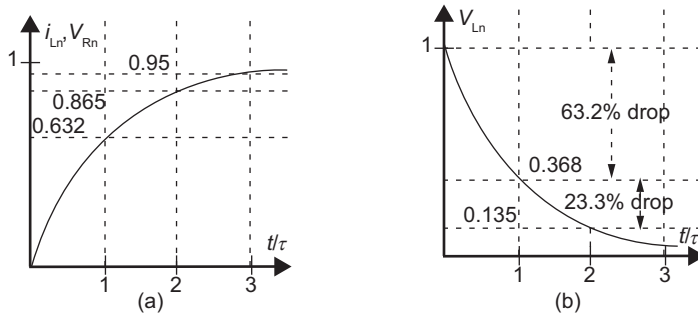


Fig. 10.4-1 Series RL circuit step response – normalised waveforms

These waveforms appear in Fig. 10.4-1. The inductor current rises from zero level at $t = 0^+$ and approaches the normalised final value of 1 as t_n approaches ∞ . It never touches the final value of 1 since the exponential function never becomes zero. The growth of inductor current gradually loses

momentum resulting in the convex shape of current waveform. Simultaneously, the voltage across inductor decays exponentially and tends to go to zero as t_n approaches ∞ .

Most of the rise in i_L and fall in v_L take place within first three units of normalised time, *i.e.*, within 3τ s of actual time. The values of e^{-1} , e^{-2} and e^{-3} are 0.368, 0.135 and 0.05, respectively. Hence, the step response current in a *RL* series circuit rises to 63.2% of its final value in the first τ seconds. The voltage across inductor in the same circuit falls by 63.2% from its value at $t = 0^+$ to reach 36.8% of its initial value in the first τ seconds. During the second τ s period, inductor current rises by another 23.3% to reach 86.5% of its final value. Correspondingly, the voltage across inductor falls to 13.5% of its initial value at the end of 2τ s. And at the end of 3τ s, 95% of the transient is over and the inductor current is only 5% away from its final value.

The value of inductor current is 99% and 99.33% of final value at 4.6τ s and 5τ s respectively. Hence, we can consider the natural response of a series *RL* circuit to be practically over within 5τ s where $\tau = L/R$. During the first five τ periods, the response in the circuit is undergoing a *transient* phase before reaching a practically *steady* situation. This period – *i.e.*, the period during which the natural response component is not negligible – is termed as the *transient period* and a value of 5τ is usually assigned to it in the case of first-order circuits. This leads to another name for natural response – it is also called the *transient response*. However, we have to be careful about this name since it gives us an impression that this response component is only transient and hence it will vanish with time invariably. This is not always so. There are circuits in which natural response either persists indefinitely or increases with time. Therefore, do not expect transients to vanish with time in all cases.

In the case of series *RL* circuit with unit step voltage input, the natural response (or transient response) term in inductor current is $-e^{-t/\tau}/R$ A and the forced response is $1/R$ A.

10.4.2 The Time Constant ' τ ' of a Series *RL* Circuit

The quantity L/R symbolically represented by τ has turned out to be an important one for *RL* circuit by now. The unit of L is V-s/A and the unit of R is V/A resulting in a dimension of time for this quantity with Seconds as its unit. Hence, this quantity is defined as the *Time Constant* of series *RL* circuit.

One of the interpretations assigned to the time constant is that it is the time taken by an initially relaxed *RL* circuit to reach 63.2% of its final current value. That is not very satisfactory. After all, 63.2% is not a neat round number or a particularly significant one. However, 50% would have been a neat measure. In fact, the time taken by a first-order system step response to reach 50% of its final value is termed its *half-life* and this turns out to be $\approx 0.693\tau$ in the case of an *RL* circuit.

Speaking qualitatively, we can appreciate the fact that τ is a measure of the duration taken by the circuit to reach the final current value. *RL* circuit causes delay in step response current due to the memory capability of inductor. The inductor remembers its initial state and its memory prevents it from allowing sudden changes in current through it. But how deep is its memory in time? How persistent is its memory? Time constant provides answers to these questions.

A large τ implies deeper memory and therefore increased duration will be required for the forcing function to compel the inductor to go to final current value. A larger τ implies a more persistent memory and a heightened tendency on the part of the circuit to keep its current smooth in time-domain.

Time constant has another interesting interpretation. We had noted earlier that the current in the initially relaxed *RL* circuit starts rising at the rate of $1/L$ A/s at $t = 0^+$. Refer to Fig. 10.4-2. Tangents to the normalised inductor current plot and the normalised inductor voltage plot are drawn at normalised time instants of 1, 2 and 3. It is clear from this figure that if the inductor current had continued to rise

at the same rate of rise it had at $t = 0^+$, it would have reached the final value at $t_n = 1$ unit *i.e.*, at $t = \tau$. The slope of i_{L_n} at $t = 0^+$ is 1 normalised unit of current per unit of normalised time. Therefore, the normalised current would have reached 1 unit at $t_n = 1$ if the rate of rise had remained unchanged. One unit of normalised time amounts to one τ of real time. Thus, time constant is the time the current in an initially relaxed RL circuit would have taken to reach the final steady value had the initial rate of rise been maintained throughout. Equivalently, it is the time the voltage across inductor would have taken to reach zero if the initial rate of fall could be maintained throughout.

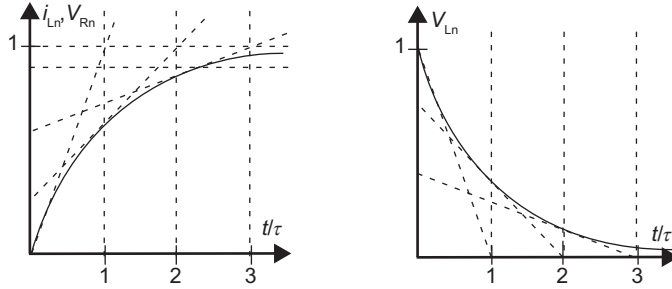


Fig. 10.4-2 Current slope based interpretation of time constant

However, time constant is even more than that. Consider the remaining two tangents at $t_n = 2$ and $t_n = 3$ in Fig. 10.4-2. Moving along either tangent line will take us to the final value of inductor current in one unit of normalised time away from the instant at which the tangent is drawn. If this is true about these two time instants, then, it must be true for all time instants since there is nothing special about these two. Let us examine the slope of inductor current in detail.

$$i_{L_n}(t) = (1 - e^{-t_n}); t_n \geq 0^+$$

$$\therefore \frac{di_{L_n}(t)}{dt_n} = e^{-t_n}; t_n \geq 0^+$$

Let Δt_n be the time required from t_n to reach the final value of 1 with rate of change of $i_{L_n}(t)$ held at this value. Then,

$$\Delta t_n = \frac{1 - (1 - e^{-t_n})}{e^{-t_n}} = 1$$

$$\therefore \Delta t = \tau \text{ s}$$

Thus, time constant of RL circuit is the additional time required from the current instant for the step response in the circuit to reach the final steady value assuming that the rate of rise of response is held constant at its current value from that instant onwards.

10.4.3 Rise Time and Fall Time in First-Order Circuits

'Rise time (t_r)' and 'Fall time (t_f)' are two measures of time delay defined in the context of unit step response for linear dynamic systems. These two measures are quite general in definition in order to

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accommodate wide variety of systems having many terms in their transient response. However, in the case of simple first-order systems there is a direct relationship between the time constant and rise and fall times.

Rise time is defined as the time interval between the first 10% point and first 90% point in the rising step response of a system where the percentages are to the base of the final step response value. Similarly, fall time is defined as the time interval between the first 90% point and the first 10% point in the step response of a system where the response variable is such that it starts at a non-zero initial value and decays to zero value in the long run. The percentages are to the base of the initial response value in this case.

These two definitions are illustrated in the case of series *RL* circuit step response in Fig. 10.4-3. Normalised variables are used in this figure and the corresponding normalised time points at which the 10% and 90% crossover takes place are also marked in the figure. From the figure, it is clear that rise time and fall time of this circuit are equal to $\approx 2.2\tau$ s, where τ is the time constant of the circuit. This result is valid for any circuit described by a first-order linear differential equation with constant coefficients and has nothing to do with the inductive nature of the circuit under consideration.

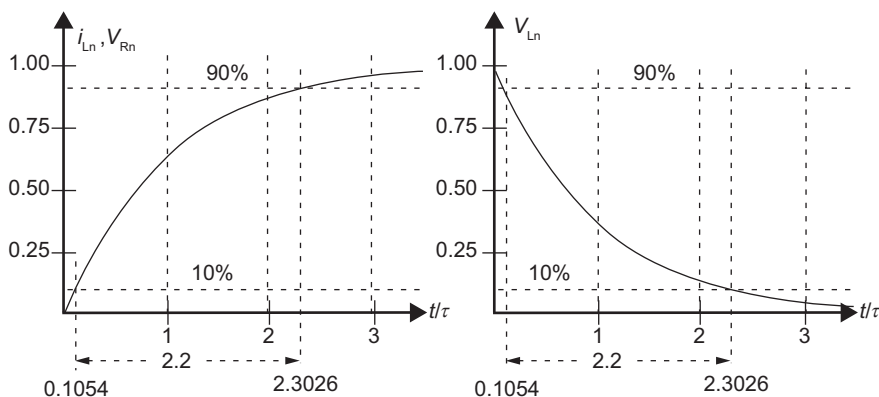


Fig. 10.4-3 Rise time and fall time in series *RL* circuit

These two measures are defined for a general circuit of any order and therefore serve as measures of delay in response and depth of memory in the circuit in situations where a single time constant cannot be identified as the major delaying factor in the circuit.

10.4.4 Effect of Non-Zero Initial Condition on DC Switching Response of *RL* Circuit

We have been dealing with the step response of an *RL* circuit till now. We will deepen our understanding in this sub section by taking up the DC switching response of a series *RL* circuit with non-zero initial current in the inductor at $t = 0^-$.

We have already derived the expression for inductor current in this case as

$$i_L(t) = I_0 e^{-t/\tau} + \frac{1}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+ \quad (10.4-4)$$

Normalising this equation by using $1/R$ A as the current base and τ s as the time base, we get the normalised form with I_{0n} as the normalised initial current at $t = 0^-$.

$$i_{Ln}(t) = I_{0n} e^{-t_n} + (1 - e^{-t_n}) \text{ for } t_n \geq 0^+ \quad (10.4-5)$$

where $i_{Ln}(t) = \frac{i_L(t)}{1/R}$, $I_{0n} = \frac{I_0}{1/R}$ and $t_n = \frac{t}{\tau}$

This expression can also be written in a form that shows the natural response (transient response) and forced response components clearly separated out as below.

$$i_{Ln}(t) = \underbrace{1}_{\text{forced response}} - \underbrace{(1 - I_{0n}) e^{-t_n}}_{\text{natural response}} ; \text{ for } t_n \geq 0^+ \quad (10.4-6)$$

This equation is plotted in Fig. 10.4-4 with solid curves showing the total response and dotted curves showing the natural response or transient response. Curves are shown for four values of initial condition at $t = 0^-$. They are -0.5 , 0 , 0.5 and 1.5 . All values are normalised ones. A negative initial condition value indicates that the initial current in the inductor at $t = 0^-$ was in a direction opposite to that of forced response. The forced response in all cases is represented by a horizontal line with intercept of unity in the vertical axis.

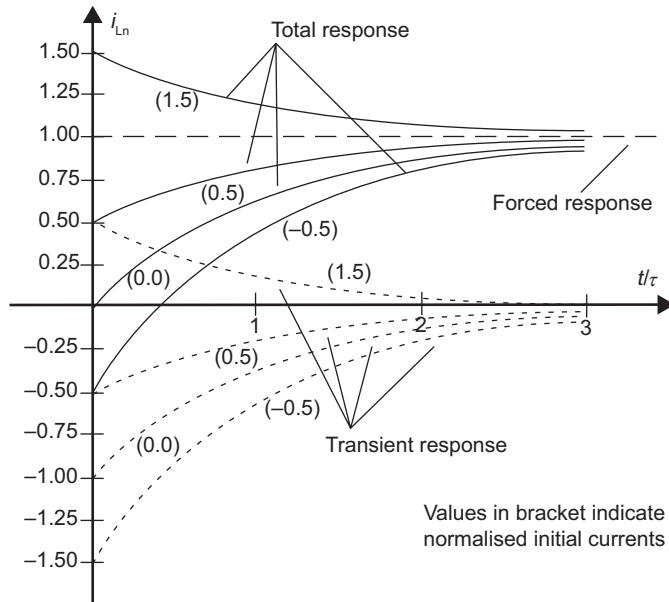


Fig. 10.4-4 Total response and transient response in series RL circuit step response for various initial currents

The waveforms in Fig. 10.4-4 and Eqn. 10.4-6 bring out the following aspects of RL circuit DC switching response with non-zero initial condition.

- The transient response (natural response) of an RL circuit contains two contributions – one from the initial condition specification and another from the value of forced response at $t = 0^+$. The magnitude of the transient term is decided by these two quantities. Thus, the role of transient response is seen clearly to be one of bridging the gap between the initial current in

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the inductor and the final current in the inductor. Transient response thus enforces compliance with the initial condition specification in the circuit.

- (b) The total response is a rising response if the initial current at $t = 0^-$ is less than the final current value. It is a falling response if initial current is more than final current.
- (c) There will be no transient response in the circuit if the initial current specified at $t = 0^-$ is equal to the final current value in magnitude and direction.
- (d) Consider a new current variable defined as $\Delta i_{L_n}(t) = i_{L_n}(t) - I_{0n}$, i.e., the change in inductor current from its initial value. Substituting Eqn. 10.4-6 in this definition, we get

$$\Delta i_{L_n}(t) = (1 - I_{0n})(1 - e^{-t/\tau}) ; \text{ for } t_n \geq 0^+.$$

- (e) Compare this expression for the change in inductor current with the inductor current expression for initially relaxed circuit. We can see that whatever that has been said about time constant becomes applicable in relation to the change in inductor current rather than to the total current when initial current is non zero. The final value of this change is $1 - I_{0n}$ and the change in inductor current rises to 63.2% of its final value in one time constant, 86.5% of its final value in 2τ s etc. Similarly, the change in inductor current covers the 10% to 90% range in 2.2τ s where the percentages are to the base of $1 - I_{0n}$.

10.4.5 Free Response of Series RL Circuit

We consider a special case of an RL circuit with zero forcing function in this sub section. Obviously, the solution for inductor current in this *source-free* RL circuit will contain only complementary solution. The particular integral is zero since forcing function is zero. The complementary solution is of the form $Ae^{-\alpha t}$ where $\alpha = R/L$. Applying initial condition to this solution makes it clear that $A = 0$ unless the initial condition specified at $t = 0^-$ is non-zero. Thus, a source-free RL circuit can have a non-zero solution only if the inductor has some energy trapped in it at $t = 0^-$. This energy storage must have been created by some source prior to $t = 0^-$.

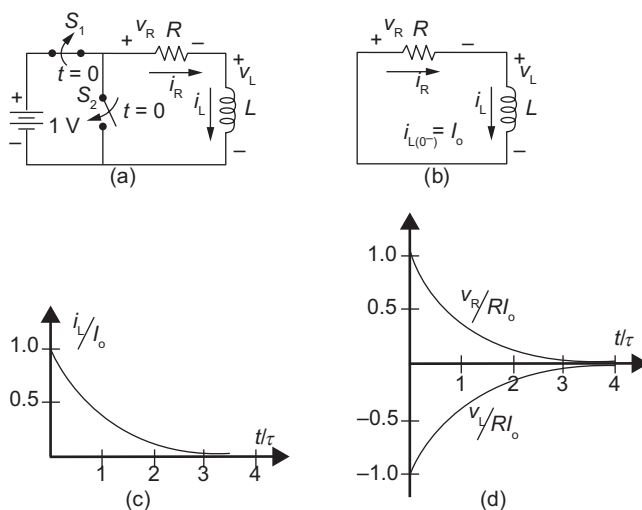


Fig. 10.4-5 Source-free RL circuit and waveforms

Consider the circuit in Fig. 10.4-5 (a). The switch S_1 was closed long back and the circuit has attained the final inductor current value of $1/R$ A by the time $t = 0^-$ is reached. At $t = 0$, the switch S_1 is opened and the switch S_2 is closed simultaneously. Thus, a source-free series RL circuit with an initial current of I_0 (which is equal to $1/R$ A in the circuit in Fig. 10.4-5) is set up at $t = 0$. The circuit in Fig. 10.4-5 (b) is equivalent to the circuit in Fig. 10.4-5 (a) for $t = 0^+$.

The expressions for inductor current and circuit voltages are derived as below follows:

$$i_L(t) = Ae^{-t/\tau} ; \text{ for } t \geq 0^+$$

$$i_L(0^+) = i_L(0^-) \quad (\because \text{infinitely large voltage is neither applied nor supported in the circuit})$$

$$\therefore i_L(0^+) = I_0$$

$$\therefore A = I_0$$

$$\therefore i_L(t) = I_0 e^{-t/\tau} ; \text{ for } t \geq 0^+$$

$$i_R(t) = i_L(t) \quad (\because R \text{ and } L \text{ are in series})$$

$$v_R(t) = RI_0 e^{-t/\tau} ; \text{ for } t \geq 0^+$$

$$v_L(t) = -v_R(t) \quad (\text{By KVL})$$

The current in the circuit decays exponentially from I_0 to zero with a time constant equal to L/R seconds. This is shown in Fig. 10.4-5 (c). The corresponding voltage across inductor is negative valued and decays with the same time constant. The circuit voltages are shown in Fig. 10.4-5 (d).

Example: 10.4-1

Obtain an expression for voltage across the resistor in an initially relaxed series RL circuit for rectangular pulse voltage input defined as $v_s(t) = 1$ V for $0 \leq t \leq T$ and 0 V elsewhere. Plot the response for (i) $T = 0.2\tau$, (ii) $T = \tau$ and (iii) $T = 2\tau$.

Solution

The differential equation for $i_L(t)$ for the interval $[0^+, T^-]$ is

$$\frac{di_L}{dt} + \alpha i_L = \beta \quad \text{where } \alpha = \frac{1}{\tau} = \frac{R}{L} \text{ and } \beta = \frac{1}{L}$$

Particular integral that is valid in the interval $[0^+, T^-]$ is $\beta/\alpha = 1/R$. The complementary function is of the form $Ae^{-t/\tau}$. The circuit is initially relaxed. Applying initial condition to total solution and solving for A , we get the total solution as

$$i_L(t) = \frac{1}{R}(1 - e^{-t/\tau}) \quad \text{for } 0^+ \leq t \leq T^- \quad (10.4-7)$$

The inductor current would have followed this expression till there is a change in input source function or circuit structure. There is a change in the applied voltage at $t = T$ in the present example. The voltage applied for all $t \geq T^+$ is zero. Thus, the circuit is described by the following differential equation for $T^+ \leq t < \infty$.

$$\frac{di_L}{dt} + \alpha i_L = 0 \quad \text{where } \alpha = \frac{1}{\tau} = \frac{R}{L}$$

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The particular integral for this equation is zero. The complementary function is again $A e^{-(t-T)/\tau}$ (but valid only for $t \geq T^+$) with the value of A to be decided. The value of A is found out from the value of current at $t = T^+$. However, since there was no impulse voltage involved in the circuit at $t = T$, the value of current at $t = T^+$ and $t = T^-$ will be same. This value can be obtained by substituting $t = T$ in Eqn. 10.4-7.

$$\begin{aligned} \therefore \text{Initial condition for current at } t = T^+ &= \frac{1}{R}(1 - e^{-T/\tau}) \\ \therefore i_L(t) &= \frac{1}{R}(1 - e^{-T/\tau})e^{-(t-T)/\tau} \text{ A for } T^+ \leq t < \infty \end{aligned} \quad (10.4-8)$$

Therefore, the expression for $v_R(t)$ is

$$\begin{aligned} v_R(t) &= \begin{cases} (1 - e^{-\alpha t}) & \text{for } 0^+ \leq t \leq T^- \\ (1 - e^{-\alpha T}) e^{-\alpha(t-T)} & \text{for } T^+ \leq t < \infty \end{cases} \\ &= \begin{cases} (1 - e^{-t/\tau}) & \text{for } 0^+ \leq t \leq T^- \\ (1 - e^{-T/\tau}) e^{-(t-T)/\tau} & \text{for } T^+ \leq t < \infty \end{cases} \\ &= \begin{cases} (1 - e^{-t_n}) & \text{for } 0^+ \leq t \leq T^- \\ (1 - e^{-T_n}) e^{-(t_n - T_n)} & \text{for } T^+ \leq t < \infty \end{cases} \end{aligned}$$

The subscript ‘ n ’ indicates normalisation with respect to τ . The plots of resistor voltage with normalised time for various T/τ ratios are shown in Fig. 10.4-6.

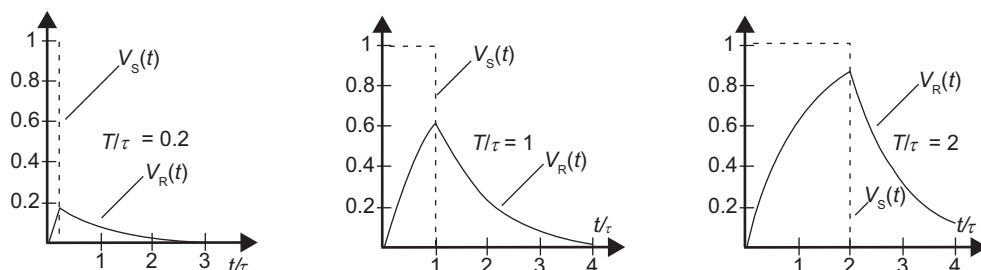


Fig. 10.4-6 Single pulse response of RL circuit in Example: 10.4-1

Example: 10.4-2

Solve for i and v as functions of time in the circuit in Fig. 10.4-7.

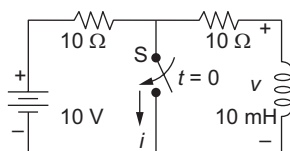


Fig. 10.4-7 Circuit for Example: 10.4-2

Solution

This circuit was already in DC steady-state at $t = 0$. At $t = 0$, the switch closes, thereby forming a source-free RL circuit on the right side and a simple resistive circuit on the left side. These two circuits do not interact after $t = 0$ except that the current

through the switch will be a combination of the currents from these two circuits.

Inductor is a short for DC steady-state. Therefore, the initial current in the inductor at $t = 0^-$ was $10\text{V}/20\Omega = 0.5\text{A}$ from top to bottom. Since the switching at $t = 0$ does not involve impulse voltage, the inductor current remains at 0.5A at $t = 0^+$ too.

Thus, a source-free RL circuit with initial current of 0.5A is set up at $t = 0$. The various current components in the circuit after $t = 0$ are marked in Fig. 10.4-8.

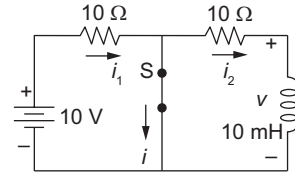


Fig. 10.4-8 Circuit for solving Example: 10.4-2

$$i_1 = \frac{10\text{V}}{10\Omega} = 1\text{A}; i_2 = 0.5e^{-1000t} \text{ A for } t \geq 0^+$$

$$\therefore \text{The current through the switch } i = i_1 - i_2 = (1 - 0.5e^{-1000t}) \text{ A for } t \geq 0^+$$

$$v = 0.01 \frac{di_2}{dt} = -5e^{-1000t} \text{ V for } t \geq 0^+$$

Example: 10.4-3

Solve the circuit in Fig. 10.4-9 for the current through the switch as a function of time.

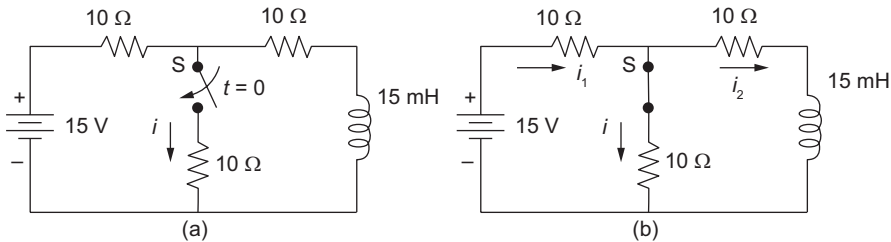


Fig. 10.4-9 Circuits for Example: 10.4-3

Solution

In this example, the two meshes in the circuit interact after $t = 0$. We can solve this circuit in many ways – branch-current method, mesh analysis, Thevenin's equivalent, etc. are some possibilities. First, we solve it by branch-current method.

Various branch currents in the circuit are identified in Fig. 10.4-9 (b). We have to get a differential equation in the current variable i .

Applying KCL at switch node gives us $i_2 = i_1 - i$.

Applying KVL in the first mesh gives us $15 = 10(i_1 + i) \Rightarrow i_1 = 1.5 - i$

$$\therefore i_2 = 1.5 - i - i = 1.5 - 2i$$

Applying KVL in the second mesh gives us $10i = 10i_2 + 0.015 \frac{di_2}{dt} = 15 - 20i - 0.03 \frac{di}{dt}$

Therefore, the differential equation governing i is $\frac{di}{dt} + 1000i = 500$ for $t \geq 0^+$.

Initial condition for i , i.e., its value at $t = 0^+$, is needed. Initial value of i_2 is $15\text{V}/20\Omega = 0.75\text{A}$ since the circuit was in DC steady-state prior to switching. Since $i_2 = 1.5 - 2i$, value of i at $t = 0^+$ will be

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$(1.5 - 0.75)/2 = 0.375$ A. The particular integral of the differential equation for i is $500/1000 = 0.5$ A. Time constant is $1/1000$.

Therefore, $i = (C e^{-1000t} + 0.5)$ A. Evaluating C from initial condition for i at $t = 0^+$, we get $C = 0.375 - 0.5 = -0.125$.

Therefore, the switch current $i = (0.5 - 0.125 e^{-1000t})$ A for $t \geq 0^+$.

Let us solve the same problem by mesh analysis. The relevant circuit with two mesh currents $-I_1$ and I_2 – identified is shown in Fig. 10.4-10 (a).

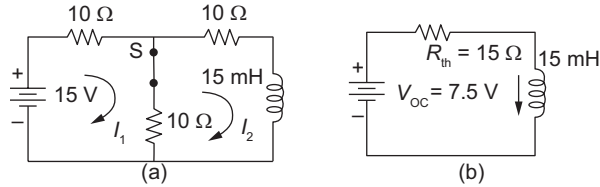


Fig. 10.4-10 Circuits for mesh analysis and Thevenin's equivalent analysis in Example: 10.4-3

The two mesh equations are

$$20I_1 - 10I_2 = 15$$

$$20I_2 - 10I_1 + 0.015 \frac{dI_2}{dt} = 0$$

Eliminating I_1 from second equation using the first equation and simplifying, we get

$$\frac{dI_2}{dt} + 1000I_2 = 500$$

The initial condition for I_2 at $t = 0^+$ is same as the initial condition for inductor current at that instant. This value is 0.75 A. Particular integral is 0.5 A. Time constant is $1/1000$ s.

$$I_2 = (C e^{-1000t} + 0.5) \text{ A for } t \geq 0^+$$

$$C = 0.25 \text{ and } I_2 = (0.5 + 0.25 e^{-1000t}) \text{ A for } t \geq 0^+$$

Using this solution in the first mesh equation we can get

$$I_1 = (1 + 0.125 e^{-1000t}) \text{ A for } t \geq 0^+$$

$$\therefore \text{Current through the switch} = I_1 - I_2 = 0.5 - 0.125 e^{-1000t} \text{ A for } t \geq 0^+$$

This circuit problem can also be solved by using Thevenin's theorem. The circuit portion to the left of inductor may be replaced by its Thevenin's equivalent as shown in Fig. 10.4-10 (b). Inductor current can be obtained from this circuit. Once inductor current is available, we will be able to get back to the switch current using KCL or KVL. This is illustrated now in the following.

The initial current in the inductor is 0.75 A. The circuit in Fig. 10.4-10 (b) is a simple series RL circuit and its particular integral is $7.5/15 = 0.5$ A. Its time constant is $15\text{mH}/15\Omega = 1\text{ms}$. Therefore, its

solution is $= C e^{-1000t} + 0.5$. Evaluating the initial condition constant C and completing the solution, we get, inductor current $= 0.5 + 0.25 e^{-1000t}$ A.

$$i_L = 0.25 e^{-1000t} + 0.5 \text{ A}$$

$$\begin{aligned} \text{Voltage across } 10\Omega \text{ in the switch path} &= 10 \times (0.25 e^{-1000t} + 0.5) + 0.015 \times (-250 e^{-1000t}) \\ &= 5 - 1.25 e^{-1000t} \text{ V} \end{aligned}$$

$$\therefore \text{Current through the switch} = (5 - 1.25 e^{-1000t}) \text{ V} / 10\Omega = 0.5 - 0.125 e^{-1000t} \text{ A for } t \geq 0^+$$

We derived the differential equations governing three variables in the circuit – the branch current in the central limb, second mesh current and the inductor current in the process of solving this circuit. The left-hand side of all the three differential equations had the same coefficients. (Why?)

We also notice that the time constant of the circuit can be easily found as L/R_{th} where R_{th} is the Thevenin's equivalent resistance appearing across the inductor. However, Thevenin's equivalent is found by deactivating all independent sources. Therefore, the time constant of a single-inductor circuit can be found by replacing all independent voltage sources by short-circuits and all independent current sources by open circuits and finding the equivalent resistance connected across the inductor. We illustrate this procedure further in the next example.

Example: 10.4-4

Show that the current in 18mH inductor in the circuit in Fig. 10.4-11 (a) will go to zero as $t \rightarrow \infty$. Also, find the inductor current and currents delivered by the voltage sources as functions of time. Find out how long we have to wait for the inductor current to fall below 100mA.

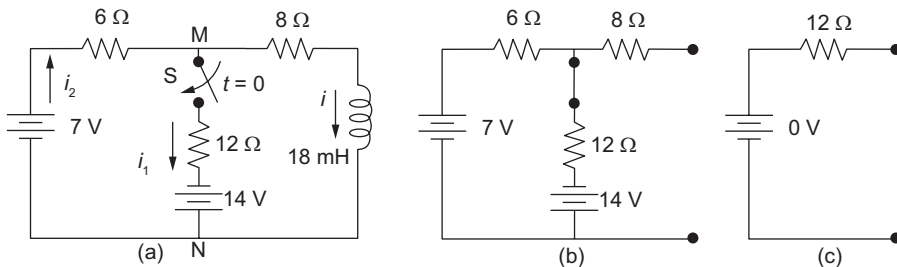


Fig. 10.4-11 Circuits for Example: 10.4-4 (a) Circuit for the problem (b) Circuit for finding Thevenin's equivalent (c) Thevenin's equivalent

Solution

First, we find the time constant effective after $t = 0^+$. The Thevenin's equivalent of the circuit connected across the inductor is evaluated by using the circuit in Fig. 10.4-11 (b) and the resulting equivalent is shown in Fig. 10.4-11 (c). Since the voltage source in Thevenin's equivalent is zero-valued, the inductor current will have a zero steady-state value. The time constant of the circuit is $18\text{mH}/12\Omega = 1.5\text{ms}$.

We find the initial condition for inductor current next. The circuit was in steady-state prior to switching at $t = 0$. Inductor is replaced by a short-circuit DC steady-state. Therefore, the inductor current at $t = 0^-$ must have been $7/14 = 0.5\text{A}$ and it will be 0.5A at $t = 0^+$ since there is no impulse voltage involved in the switching.

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Now, the circuit is a series RL circuit with a known initial condition and DC sources. We know the solution for such a circuit. It is of general format $-A e^{-t/\tau} + C$ – where C is the particular integral (therefore, the DC steady-state value) and A is the arbitrary constant to be found out from initial condition. *This is the general format of solution for any circuit variable in a first-order circuit with DC excitation.*

$$\therefore i(t) = A e^{-t/1.5} + 0; t \text{ in ms}$$

$$i(t) = 0.5 \text{ at } t = 0^+$$

$$\therefore i(t) = 0.5 e^{-t/1.5} \text{ A; } t \text{ in ms and } t \geq 0^+ \quad (10.4-9)$$

$$\therefore \text{Voltage across inductor} = 0.018 \frac{di}{dt} = -6 e^{-t/1.5} \text{ V; } t \text{ in ms and } t \geq 0^+$$

We have two ways to find the currents delivered by the voltage sources $-i_1$ and i_2 . In the first method, we find the voltage across M and N in circuit Fig. (a) as $v_{MN} = 8i + 0.018 \frac{di}{dt}$ and then find i_2 as $(7 - v_{MN})/6$ and i_1 as $(v_{MN} - 14)/12$.

In the second method, we realise that all variables in this circuit will have a $A e^{-t/\tau}$ term and a steady-state term and that we can find the arbitrary constant A if we know the value of the particular variable at 0^+ . Therefore, we set out to find the initial and final (*i.e.*, steady-state) value of the source currents. The inductor current was at 0.5 A at $t = 0^+$. The voltage across inductor at that instant is -6 V from Eqn. 10.4-9. Therefore, $v_{MN} = 8 \times 0.5 - 6 = -2 \text{ V}$ at $t = 0^+$. Therefore, initial value of i_1 at $t = 0^+ = (-2 - (-14))/12 = 1 \text{ A}$. Final current in inductor is zero. Hence, final current in both sources will be same and equal to $(7 + 14) \text{ V} / (6 + 12) \Omega = 7/6 = 1.167 \text{ A}$.

Therefore, our required currents are

$$\therefore i_1(t) = A e^{-t/1.5} + 1.167 \text{ A; } t \text{ in ms with } i_1 = 1 \text{ A at } t = 0^+$$

$$\therefore i_1(t) = 1.167 - 0.167 e^{-t/1.5} \text{ A; } t \text{ in ms and } t \geq 0^+$$

$$i_2(t) = i_1(t) + i(t) = 1.167 - 0.167 e^{-t/1.5} + 0.5 e^{-t/1.5} \text{ A}$$

$$= 1.167 + 0.333 e^{-t/1.5} \text{ A; } t \text{ in ms and } t \geq 0^+$$

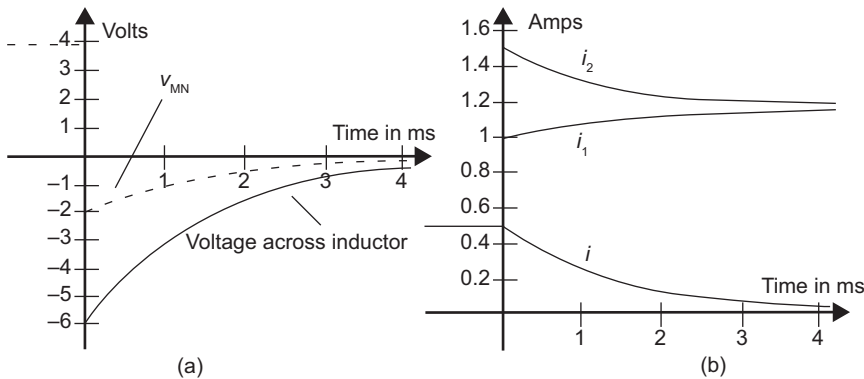


Fig. 10.4-12 Waveforms of (a) voltages and (b) source and inductor currents in Example: 10.4-4

These are plotted in Fig. 10.4-12. The time required for inductor current to go below 100 mA is found as follows:

$$0.1 = 0.5 e^{-t/1.5} \text{ A; } t \text{ in ms, } \therefore -\frac{t}{1.5} = \ln 0.2 = -1.6094 \text{ and } t = 2.414 \text{ ms}$$

10.5 STEADY-STATE RESPONSE AND FORCED RESPONSE

The total response in an RL circuit with any forcing function will consist of two terms – the transient response (or natural response) and the forced response. The transition from initial state of the circuit (which is encoded in a single number in the form of initial inductor current specification at $t = 0^-$) to the final state (in which only forced response will be present) is accomplished with the help of the transient response. Now, we introduce a new term called steady-state response and relate it to the response terms we are already familiar with.

Our study of the solution of differential equation describing the RL circuit has shown us that the total response will always contain two components – the transient response and the forced response. Of course, forced response will be zero if forcing function is zero *i.e.*, in a source-free circuit. Similarly, the transient response term may become zero under certain suitable initial condition values. But these are special situations and, in general, there will be two terms in the total response. This is true not only for an RL circuit but also for any linear circuit described by linear ordinary differential equations with constant coefficients. Such a circuit of higher order will have two groups of terms in its total solution – first group constituting transient response containing one or more terms and the second group constituting forced response containing one or more terms depending on the type of forcing function. *Thus, forced response is a response component which is always present in the total response of a circuit except when the forcing function itself is zero.*

We have seen that the transient response of an RL circuit contains exponential function of the form $e^{-\alpha t}$ where α is a positive number decided by R and L . Such an exponential function with negative real index will taper down towards zero as t approaches ∞ . Hence, we expect the transient response in an RL circuit to vanish with time quite irrespective of the forced response component. Therefore, we expect that there will only be the forced response component active in the circuit in the long run *i.e.*, after sufficient time had been allowed for the transient response to die down. When all the transient response terms in all the circuit variables in the circuit have died down to negligible levels (they never die down to zero) and the only response component in all the circuit variables is the forced response component, we say the circuit has reached the steady-state with respect to the particular forcing function that was applied to the circuit. Notice that under *steady-state* conditions the transient response terms should not be present in any circuit variable at all. Or, in other words, there cannot be a circuit which attains steady-state in some of its variables and does not attain steady-state in yet others.

Therefore, a circuit will reach steady-state if and only if all its transient response terms are of decreasing type. Moreover, the only response that will continue in the circuit after it has reached steady-state is the forced response component. Therefore, *steady-state response* is same as *forced response* with the condition that the steady-state will exist only if all the transient response terms are of damped nature – *i.e.*, decreasing functions of time. Thus, steady-state response is another name for forced response when transient response is assured to die down to negligible levels. Forced response will always be present; but steady-state response need not be.

Consider the circuit in Fig. 10.5-1. The 1V source on the left side has set up an initial current of 1A in the inductor of 1H at $t = 0^-$. The switch S_1 is opened and switch S_2 is closed at $t = 0$ to apply a 1V source right across the inductor. The current in the inductor is shown in Fig. 10.5-1 (b). We note that with a bounded input (1V DC source is a bounded input), we get a current in the inductor which is not bounded. Also, the transient response ($1e^{-0t}$) does not decrease with time. Therefore, there is no steady-state in this circuit though there is a forced response.

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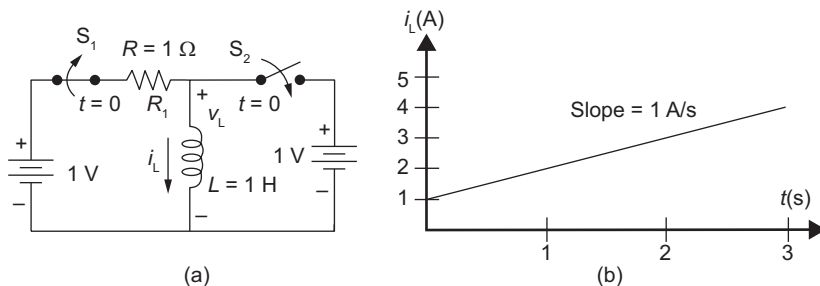


Fig. 10.5-1 A circuit with no steady-state and its step response

10.5.1 The DC Steady-State

It is to be noted is that the steady-state attained by a circuit is intimately connected with the type of forcing function applied to the circuit. We cannot expect the generalisations arrived at based on the steady-state behaviour for a particular forcing function to hold in the case of steady-state behaviour for another forcing function.

For example, consider the steady-state in series RL circuit when input is a unit step voltage. The forced response in this case is a constant current of value $1/R$ A. The voltage across inductor with a constant current through it can only be zero. But zero is indeed a constant. Thus, we see that, under steady-state condition with step input, all the circuit variables become constants. The only constant voltage an inductor can have across it is zero if the current through it is also constrained to remain constant since $v = L di/dt$ for an inductor. Noting the ‘constant’ nature of input voltage and ‘constant’ nature of all circuit variables under steady-state, we name this kind of steady-state as the *DC steady-state*.

Under DC steady-state (provided the circuit can reach such a steady-state – transient response has to die down for that), inductors in a circuit can have any constant valued current; however, their voltages will be constrained to remain at zero. But that is similar to the definition of a short-circuit – except that the voltage across a short-circuit is zero for *any* current. Thus, we can solve for DC steady-state response in RL circuits (containing one or more inductors) by replacing all inductors by short-circuits – provided all independent sources in the circuit are switched DC sources or step functions and a DC steady-state can exist in the circuit. Subject to the above conditions, we can state that ‘*an inductor is a short-circuit under DC steady-state*’.

While we are on the topic of steady-state, we might as well look at the remaining two kinds of steady-state we will need in the analysis of dynamic circuits.

10.5.2 The Sinusoidal Steady-State

Sinusoidal steady-state refers to the steady-state that gets established in a circuit when all the independent sources in the circuit are sinusoids of same angular frequency. Like DC steady-state, this steady-state too can exist only if all the terms in transient response die down to negligible levels with time.

The word ‘*steady*’ has the literal meaning of ‘*unchanging*’. This unfortunately gives an impression that *steady-state* is that state in a circuit in which all circuit variables are *unchanging* in time. This is an error that a beginner in Circuit Analysis has to guard against. *Steady-state* does not necessarily mean that circuit response is *unchanging* in time. It is so only in the case of a DC steady-state.

To understand the meaning of the word '*steady*' in the Circuit Analysis context, we have to look at the input forcing function and find out those features of the forcing function which remain unchanging with time. In the case of DC or step inputs, the input value itself is unchanging in $[0^+, \infty)$. In the case of sinusoidal input forcing function the amplitude of the sinusoid, its angular and cyclic frequencies, its phase and its shape in one period (*i.e.*, sinusoidal shape) remain constant in time. Therefore, if a steady-state exists in a circuit under the action of such a forcing function, we can expect all circuit variables to have sinusoidal shape, fixed amplitudes, fixed frequency *which is same as that of the forcing function* and fixed phase with respect to forcing function. This is what is meant by sinusoidal steady state.

Thus, a circuit excited by one or more sinusoidal forcing functions of *same* frequency is said to have reached *sinusoidal steady-state* if all its transient response components have died down and all its circuit variables have *sinusoidal* waveshape with *same* frequency as that of forcing functions and *fixed* amplitudes and phase angles.

The waveforms in Fig. 10.5-2 show the applied voltage and inductor current in an initially relaxed RL circuit with $R = 0.33 \, \Omega$ and $L = 0.33 \, \text{H}$. A sinusoidal voltage $= 1 \sin(5t) \, \text{V}$ was switched on to the circuit at $t = 0$. The current waveform shows the exponential transient response in the first few seconds clearly. After about 10 s or so, the transient response has decayed to negligible level and the response contains only a sinusoidal waveform that is of same frequency as that of applied voltage. It has fixed amplitude and a fixed phase with respect to the input sine wave. Thus, the circuit has reached sinusoidal steady-state within few time constants ($\tau = 1 \text{ s}$).

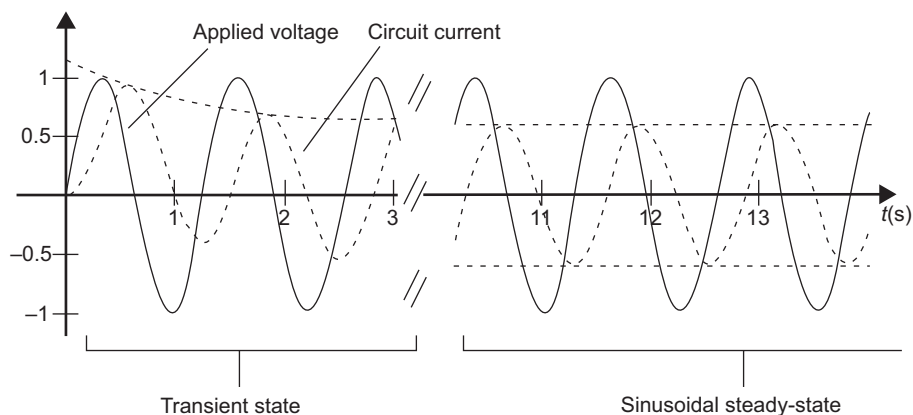


Fig. 10.5-2 Waveforms illustrating sinusoidal steady state

Sinusoidal steady-state is also referred to as *AC steady-state* in Circuit Analysis literature.

10.5.3 The Periodic Steady-State

This is the third kind of steady-state that can come up in linear circuits. Here, the input forcing function is periodic but not sinusoidal. Therefore, a single number like amplitude in the case of a sinusoid is not available for this input. The only aspect that is *steady* about it is its frequency. Thus, we expect the circuit to reach a steady-state (if it can) in which all circuit variables will be *periodic* with same *frequency*

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as that of input. But the waveshape of response variables will not be the same as the waveshape of input forcing function. The sinusoidal steady-state we discussed in the previous sub section is indeed a periodic steady-state; but it is more than that – in sinusoidal steady-state, the response waveform is same as that of forcing function. In a periodic steady-state, such a constraint may not be satisfied.

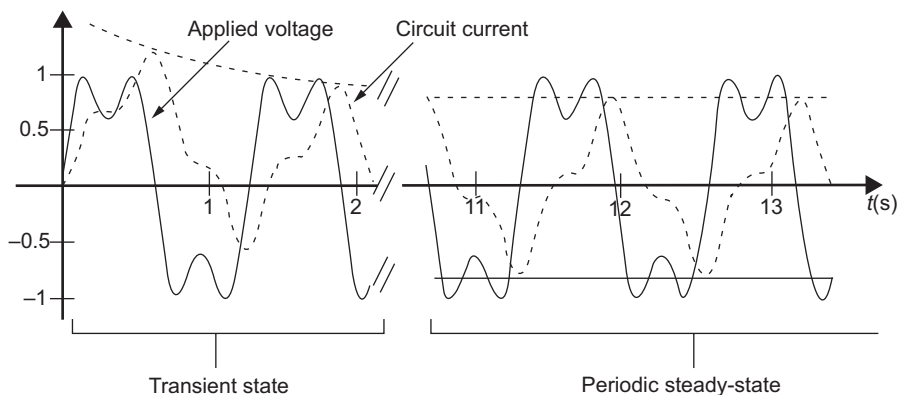


Fig. 10.5-3 Waveforms illustrating periodic steady-state

The waveforms shown in Fig. 10.5-3 illustrate the attainment of periodic steady-state in an RL circuit ($R = 0.33\Omega$, $L = 0.33$ H, $I_0 = 0$, $\tau = 1$ s) driven by a voltage $= (1 \sin 5t + 0.4 \sin 15t)$ V from $t = 0^+$ onwards. Notice that the waveshape of current is not the same as that of applied voltage. However, the current is periodic with same period and its waveshape remains the same in successive periods after it has reached steady-state.

We saw that inductors can be replaced by short-circuits for DC steady-state analysis. However, we notice that the currents in the inductors in circuits are time-varying currents in sinusoidal steady-state and periodic steady-state conditions. *Hence inductors cannot be replaced by short-circuits for all kinds of steady-state analysis – that works only for DC steady-state analysis.*

Example: 10.5-1

Periodic steady-state can come up in RL circuits when one or more sources in the circuit are periodic functions of time. But even when all sources are DC sources there can be a periodic steady-state in the circuit if some parameter in the circuit is varying periodically with time. The circuit for this example appears in Fig. 10.5-4. The switch S in the circuit is operated periodically at a switching frequency of 1 kHz. In every switching cycle, it is kept closed for half the time and then kept open for remaining half of cycle time i.e., 0.5 ms. After a large number of switching cycles a steady repeating pattern of current gets established in the circuit. We describe this steady pattern in this example.

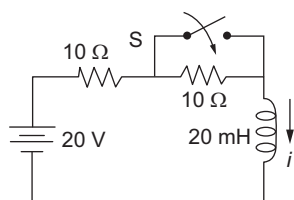


Fig. 10.5-4 Circuit for Example: 10.5-1

Assume that the circuit is in periodic steady-state. Let the current start with a value I_1 when the switch is closed in the beginning of a steady-state cycle. Then, the current increases along an exponential with τ of 2 ms towards 2 A (because 2 is the particular integral under this condition). But the current

is not allowed to reach 2A since the switch goes open after 0.5ms when the current is I_2 . Now, a new transient starts, trying to take the current from this value to 1A (because 1 is the particular integral now) exponentially with a time constant of 1ms. But this exponential is terminated after another 0.5ms and then the cycle starts all over again. *At the end of the second 0.5ms, the current in the circuit will be I_1 if the circuit is in periodic steady-state.* We convert the above reasoning into equations and solve for I_1 and I_2 .

$$I_1 e^{-0.5/2} + 2(1 - e^{-0.5/2}) = I_2 \text{ and } I_2 e^{-0.5/1} + 1(1 - e^{-0.5/1}) = I_1$$

$$\left[I_1 e^{-0.5/2} + 2(1 - e^{-0.5/2}) \right] e^{-0.5/1} + 1(1 - e^{-0.5/1}) = I_1$$

$$\therefore I_1 = 1.254 \text{ A and } I_2 = 1.419 \text{ A}$$

The corresponding values for 500Hz switching also were worked out and they are $I_1 = 1.186 \text{ A}$ and $I_2 = 1.506 \text{ A}$. The plot of inductor current for 500Hz switching and 1kHz switching are shown in Fig. 10.5-5 (a) and (b).

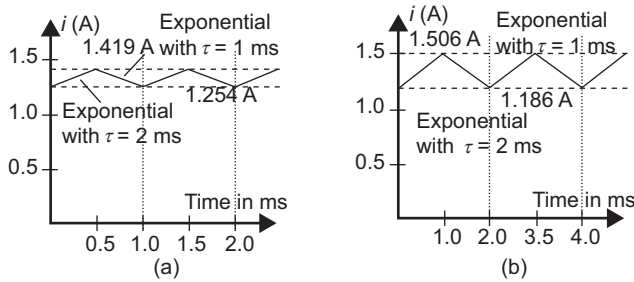


Fig. 10.5-5 Periodic steady-state in circuit in Example: 10.5-1 for (a) 500Hz and (b) 1kHz switching

The total resistance in the circuit was changing abruptly between 10 and 20 Ω . But the current in the circuit does not show any discontinuity. This once again illustrates the fact that inductor smoothes a circuit current. Moreover, we see from Fig. 10.5-5 that, the smoothing effect is more when circuit time constants are larger than the switching cycle period. In fact, the current tends to become almost constant at 1.33A as inductance value in this circuit is increased. (The reader is encouraged to ponder over why it should be 1.33A and why the average value of waveforms in Fig. 10.5-5 also should be 1.33A.)

10.6 LINEARITY AND SUPERPOSITION PRINCIPLE IN DYNAMIC CIRCUITS

We have dealt with DC switching response of an RL circuit with 1V input. How do we get the solution if it is not 1V but V V that is switched onto the circuit? Can we just multiply the unit switching response by V to get the solution for this input?

We know that memory less circuits containing linear passive resistors, linear dependent sources and independent sources will be linear and will obey superposition principle. We examine the issue of linearity of circuits containing one or more energy storage elements along with resistors, linear dependent sources and independent sources in this section. We are already familiar with a particular decomposition of total response in such a circuit in terms of transient response and forced response. We will see in this section that yet another decomposition of total response into the so-called *zero-input response* and *zero-state response* is needed in view of linearity considerations in the circuit.

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An electrical element is called *linear* if its element equation obeys superposition principle. Superposition principle involves two sub-principles – principle of additivity and principle of homogeneity. We have seen earlier that an inductor with an element relation $v = L \, di/dt$ and a capacitor with an element relation $i = C \, dv/dt$ are linear elements. But will an interconnection of such linear elements (and independent sources) into a circuit result in a *linear system*? A linear system is one in which *every* response variable in the system obeys superposition principle.

Intuitively we expect an interconnection of linear elements and independent sources to yield a linear circuit; but mathematically it is not that simple. It requires to be proved. *We accept the result that a circuit formed by interconnecting linear passive elements, linear dependent sources and independent sources will be a linear circuit. Such a linear circuit has to obey superposition principle.*

Therefore, we must be able to get $i_L(t)$ in a series RL circuit with $V \, u(t)$ as its input source function by scaling the unit step response by V . Assuming an initial condition of I_0 at $t = 0^-$, this scaling results in Eqn. 10.3-8 getting multiplied by a dimensionless scalar V to yield

$$i_L(t) = VI_0 e^{-t/\tau} + \frac{V}{R}(1 - e^{-t/\tau}) \quad \text{for } t \geq 0^+$$

as the solution. But this solution is incorrect because the current at $t = 0^+$ is VI_0 according to this equation rather than the correct value of I_0 . It looks as if the principle of homogeneity is not valid here. Let us try to get the solution without resorting to linearity.

The complementary solution is $A e^{-\alpha t}$ with $\alpha = 1/\tau = R/L$ and the particular integral is V/R . Therefore, the total solution is $i_L(t) = A e^{-\alpha t} + V/R$. Substituting the initial condition at $t = 0^+$ and solving for A , we get the final solution as $i_L(t) = (I_0 - V/R)e^{-\alpha t} + V/R = I_0 e^{-\alpha t} + V/R (1 - e^{-\alpha t})$ for $t \geq 0^+$. Thus, the correct solution is,

$$i_L(t) = I_0 e^{-t/\tau} + \frac{V}{R}(1 - e^{-t/\tau}) \quad \text{for } t \geq 0^+ \quad (10.6-1)$$

The first term in Eqn. 10.3-8 does not get multiplied by V when the DC voltage magnitude is scaled by V . The second term in the same equation gets scaled by V . This means that the forced response (the constant term in the solution) obeys the principle of homogeneity.

Now, let us solve the circuit for three situations (a) with V_1 (b) V_2 and (c) $V_1 + V_2$ as the DC voltages with the same initial condition for all the three cases. The expression for $i_L(t)$ in the three cases can be derived as

$$\begin{aligned} \text{Case (a)} \quad i_L(t) &= I_0 e^{-t/\tau} + \frac{V_1}{R}(1 - e^{-t/\tau}) \quad \text{for } t \geq 0^+ \\ \text{Case (b)} \quad i_L(t) &= I_0 e^{-t/\tau} + \frac{V_2}{R}(1 - e^{-t/\tau}) \quad \text{for } t \geq 0^+ \\ \text{Case (c)} \quad i_L(t) &= I_0 e^{-t/\tau} + \frac{V_1 + V_2}{R}(1 - e^{-t/\tau}) \quad \text{for } t \geq 0^+ \end{aligned} \quad (10.6-2)$$

These expressions show that the forced response part obeys the principle of additivity also. In all cases we see that neither the total response nor the transient response (or natural response) obeys superposition principle.

We have arranged the terms in the expression for inductor current in Eqn. 10.6-1 and Eqn. 10.6-2 in a special manner – the dependence on initial condition is contained in the first term and the dependence on step magnitude is contained in the second term. We notice that it is not only the forced

response component which satisfies the superposition principle, but the entire second term which depends on forcing function satisfies the superposition principle. However, both terms contribute to transient response and transient response does not satisfy superposition principle.

We notice further that the first term depends only on initial condition and will be the total response if there is no forcing function, *i.e.*, the first term is the response in a source-free circuit. Similarly, the second term depends on forcing function and does not depend on initial condition. The second term will be the total response if the circuit is initially relaxed. These observations will remain valid for any forcing function. The nature of the second term will change with the nature of forcing function. However, the resolution of total response into two components – one that depends entirely on initial condition and another that depends entirely on forcing function – will be possible for any forcing function in any linear circuit.

Here too we accept a result proved in the general theory of linear systems without worrying about its proof.

The response for $t \geq 0^+$ that results from initial condition alone (that is, with zero input for $t \geq 0^+$) is called '*zero-input response*'. The response for $t \geq 0^+$ that results from application of input for $t \geq 0^+$ with zero initial condition is called '*zero-state response*'.

The total response in a linear time-invariant circuit containing energy storage elements can be found by adding the zero-input response and zero-state response together. Zero-input response will depend only on the initial state of the circuit as encoded in its initial condition specifications. Zero-state response will depend only on forcing function.

Now, we focus on the zero-input response of an *RL* circuit. This is the response in a source-free circuit due to its initial energy alone. It is $I_0 e^{-at}$ A with the usual interpretations for all the symbols. It must be obvious that the zero-input response will scale with I_0 , *i.e.*, when the initial condition value is multiplied by a real constant the zero-input response also gets multiplied by the same constant. Similarly, when two different values of initial condition I_{01} and I_{02} result in two different zero-input responses, the zero-input response with the initial condition value at $I_{01} + I_{02}$ will be the sum of the two zero-input responses observed in the first two cases. Thus, zero-input response of *RL* circuit (and, of all linear time-invariant circuits) obeys superposition principle with respect to initial condition values.

Thus, we see that both zero-state response and zero-input response obey superposition principle individually.

Zero-input response follows superposition principle with respect to initial condition values and zero-state response obeys superposition principle with respect to input source functions.

Therefore, total response will not follow superposition principle with respect to forcing function or initial condition – only its components will obey superposition principle.

Figure 10.6-1 shows the zero-input response and zero-state response components along with the total current response in an *RL* circuit under DC voltage switching condition for various values of normalised initial condition values. Decomposition of total response into transient response and forced response for the same circuit was shown in Fig. 10.4-4. Compare these two decompositions.

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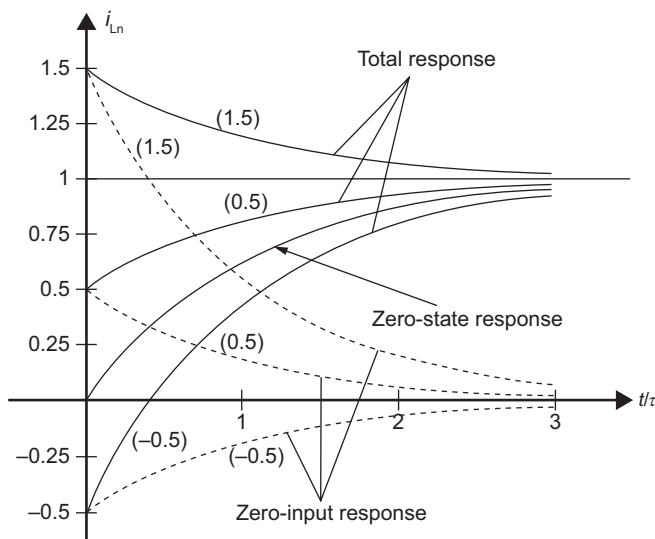


Fig. 10.6-1 Decomposition of total response into zero-input response and zero-state response

Both zero-input response and zero-state response will contain natural response terms. However, the natural response component in zero-state response has amplitude that depends on forcing function value and *does not* depend on initial condition value. Zero-input response and part of zero-state response together will form transient response. The remaining part of zero-state response will be the forced response.

10.7 UNIT IMPULSE RESPONSE OF SERIES RL CIRCUIT

We look at the response of series RL circuit in Fig. 10.1-1 with $v_s(t) = \delta(t)$, the unit impulse voltage. We have discussed the unit impulse function in detail in Chapter 3. We saw that we can view unit impulse function as a limiting case of a rectangular pulse waveform of amplitude $1/\Delta t$ V located between $t = 0$ and $t = \Delta t$ as $\Delta t \rightarrow 0$. The pulse always maintains unit area under it by increasing the amplitude as its duration decreases. We use this interpretation of unit impulse function to analyse impulse response of an RL circuit first. The circuit is assumed to be initially relaxed.

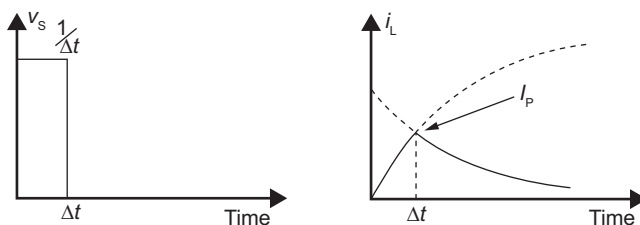


Fig. 10.7-1 Pulse response of series RL circuit

The applied pulse and circuit current responses are shown in Fig. 10.7-1. The response in the interval $[0^+, \Delta t]$ will be the same as the unit step response scaled by $1/\Delta t$. At Δt the applied voltage at input goes to zero, *i.e.*, the input gets shorted and a source-free RL circuit starts executing its zero-input response from that point onwards. The amplitude of this response has to be the initial current value just prior to Δt . This value is marked as I_p in the Fig. 10.7-1. An expression for I_p can be obtained by evaluating the first part of response at Δt as shown in Eqn. 10.7-1.

$$i_L(t) = \begin{cases} \frac{1}{R\Delta t}(1 - e^{-t/\tau}) & \text{for } 0^+ \leq t \leq \Delta t \\ I_p e^{-t/\tau} & \text{for } \Delta t < t < \infty \end{cases} \quad (10.7-1)$$

$$I_p = \frac{1}{R\Delta t}(1 - e^{-\Delta t/\tau})$$

As Δt is decreased, the value of I_p increases and moves towards left in time axis. Eventually, it attains a limit as $\Delta t \rightarrow 0$. This limiting value can be found by using series expansion for exponential function as in Eqn. 10.7-2 the following equation:

$$\begin{aligned} I_p &= \frac{1}{R\Delta t}(1 - e^{-\Delta t/\tau}) \\ &= \frac{1}{R\Delta t} \left[1 - \left\{ 1 - \frac{\Delta t}{\tau} + \frac{1}{2!} \left[\frac{\Delta t}{\tau} \right]^2 - \frac{1}{3!} \left[\frac{\Delta t}{\tau} \right]^3 + \dots \right\} \right] \\ &= \frac{1}{R\tau} \text{ for small } \frac{\Delta t}{\tau} \\ &= \frac{1}{L} \quad (\because \tau = L/R) \end{aligned} \quad (10.7-2)$$

Therefore, as $\Delta t \rightarrow 0$ the first part of response tends to become a jump by $1/L$ A and second part becomes an exponential from $t = 0^+$ onwards with $1/L$ as the starting value. And in the limit, the impulse response becomes

$$i_L(t) = \frac{1}{L} e^{-t/\tau} \text{ A for } t \geq 0^+ \quad (10.7-3)$$

The only *steady* feature the unit impulse function possesses after $t = 0^+$ is *constancy at zero value*. Therefore, we expect the forced response to be zero. The only response component that can be there in the impulse response is the transient response component. And, since the initial condition at $t = 0^-$ is zero (because the voltage applied to the circuit in the interval $(-\infty, 0^-]$ is zero) it follows that zero-input response is zero. Therefore, the impulse response appearing in Eqn. 10.7-3 is the zero-state response for unit impulse application.

Now, consider the series RL circuit with an initial condition of $1/L$ A at $t = 0^-$ and $v_s(t) = 0$ for $t \geq 0^+$. This is a source-free RL circuit with non-zero initial energy. It will have only zero-input response component in its response and its total response will be

$$i_L(t) = \frac{1}{L} e^{-t/\tau} \text{ A for } t \geq 0^+$$

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which is exactly the same as the unit impulse response. Thus, the effect of applying $\delta(t)$ is to change the initial condition of inductor between $t = 0^-$ and $t = 0^+$. Remember, we have always pointed out that initial condition at $t = 0^-$ and $t = 0^+$ will be the same only if no impulse voltage is applied in the circuit or supported in the circuit. Applying $\delta(t)$ amounts to keeping the circuit shorted for $t \leq 0^-$ and $t \geq 0^+$ and applying an undefined voltage at $t = 0$ such that a finite V-s area of 1V-s is dumped into the circuit at $t = 0$. This results in changing the inductor current by 1V-s/L henry = 1/L A between $t = 0^-$ and $t = 0^+$.

The equivalence between non-zero initial condition at $t = 0^-$ and application of suitably sized impulse voltage at $t = 0$ is further clarified by the relations in Eqn. 10.7-4 where v_L and i_L are voltage across an inductor and current through the inductor, respectively.

$$\begin{aligned} i_L(t) &= \frac{1}{L} \int_{-\infty}^t v_L(t) dt \\ &= \underbrace{\frac{1}{L} \int_{-\infty}^{0^-} v_L(t) dt + \frac{1}{L} \int_{0^-}^{0^+} v_L(t) dt}_{=i_{L(0^+)}} + \frac{1}{L} \int_{0^+}^t v_L(t) dt \end{aligned} \quad (10.7-4)$$

Circuit solution for $t \geq 0^+$ requires only the value of initial current at $t = 0^+$. This value is the sum of the first two definite integrals in Eqn. 10.7-4. It does not matter which integral contributes how much as long as the sum of their contributions remain constant as far as $i_L(t)$ after $t = 0^+$ is concerned. First integral gives the initial current in inductor due to all the voltages applied to it in the past. Second integral will be non-zero only if impulse voltage is applied to inductor at $t = 0$. As far as $i_L(t)$ after $t = 0^+$ is concerned these two terms are interchangeable. Therefore, an initial current of I_0 in the inductor at $t = 0^-$ may be replaced by zero initial current at $t = 0^-$ and an impulse voltage $LI_0 \delta(t)$ with correct polarity *in series* with the inductor.

But the voltage variable that appeared in Eqn. 10.7-4 was the voltage across inductor. If we want to replace a non-zero initial current at $t = 0^-$ by an impulse voltage source, we must ensure that impulse source appears fully across inductor and does not lose itself across other elements in the circuit. Therefore, we have to argue that the $\delta(t)$ we applied at the input travels through R and appears fully across L . The definition of unit impulse function avoids defining it at $t = 0$ and makes up for that by providing its area content in an infinitesimally small interval around $t = 0$.

$$\delta(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined} & \text{at } t = 0 \\ 0 & \text{for } 0^+ \leq t < \infty \end{cases} \quad \text{and} \quad \int_{0^-}^{0^+} \delta(t) dt = 1$$

Now, consider applying this to an RL circuit. Refer to Fig. 10.1-1. Applying KVL in the mesh, we get $v_R(t) + v_L(t) = v_S(t)$ for all t . Since this relation is true for all t , we can integrate both sides of the equation between same two limits to get the following relation.

Area under $v_R(t)$ between two instants t_1 and t_2 + Area under $v_L(t)$ between two instants t_1 and t_2 = Area under $v_S(t)$ between two instants t_1 and t_2 . Taking $t_1 = 0^-$ and $t_2 = 0^+$ this relation results in

$$\int_{0^-}^{0^+} v_R(t) dt + \int_{0^-}^{0^+} v_L(t) dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

We need to show that the first integral on the left side is zero. Let us assume that it is zero. Then, second integral is 1. The second integral value divided by L gives the *change* in inductor current over the interval $[0^-, 0^+]$. Therefore, the inductor current change is $1/L$ A and the inductor current is $1/L$ A since circuit was initially relaxed. This is the maximum value that the current can have at $t = 0^+$ since we had assumed that the first integral value is zero. Thus, the current through the circuit during $[0^-, 0^+]$ is confined between 0 and a maximum of $1/L$ A. This implies that though the current during this infinitesimal interval is indeterminate in value (because there is a jump discontinuity in it) it remains upper-bounded and lower-bounded and hence is finite-valued in the interval. *A finite-valued function integrated over infinitesimal interval results in zero integral value.* After all, there is no area under a rectangle if the rectangle is of finite length in one direction and of infinitesimal length in the other direction. Therefore the first integral, *i.e.*, the portion of area content of $\delta(t)$ that gets lost across resistor, is zero and our assumption to that effect is correct. Now, we may similarly assume that this integral is non-zero and prove that the assumption leads to a contradiction. Therefore, all the area content of $\delta(t)$ appears across inductor itself.

Non-zero initial current of I_0 in an inductor L at $t = 0^-$ in a circuit can be replaced by zero initial current at $t = 0^-$ along with an impulse voltage source $LI_0\delta(t)$ in series with the inductor for solving the circuit in the domain $[0^+, \infty)$.

Moreover, we have resolved the small problem we had with linearity and superposition principle in dynamic circuits. In fact, there is no problem. Superposition principle is fully obeyed by linear dynamic circuits. Only that we have to apply it carefully when there are non-zero initial condition values specified. In that case, we have to remember that each initial condition represents a source and that it becomes a multi-source problem. When changes are effected in a source, superposition principle has to be applied to that component of total response contributed by the particular source.

10.7.1 Zero-State Response for Other Inputs from Impulse Response

The *unit impulse response* (hereafter this will be referred to as ‘the impulse response’ – that it is a zero-state response and that unit impulse is applied will be implicit) is the most important response for a linear circuit. We will see in a later chapter that zero-state response for any other well-behaved forcing function can be found from it. In this sub section, we will show that it is an easy matter to get zero-state response for step input, pulse input and ramp input functions from the impulse response by integrating it.

First, we establish the relation between unit impulse function and unit step function. It is a simple one. Running integral of unit impulse function results in unit step function. Unit impulse function is defined as

$$\delta(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined} & \text{at } t = 0 \\ 0 & \text{for } 0^+ \leq t < \infty \end{cases} \quad \text{and} \quad \int_{0^-}^{0^+} \delta(t) dt = 1$$

Consider the following integral.

$$f(t) = \int_{-\infty}^t \delta(t) dt$$

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Obviously, the value of $f(t)$ is zero for $-\infty < t \leq 0^-$ since $\delta(t)$ is zero in that range. But area under impulse function undergoes a rapid change by 1 when t goes from 0^- to 0^+ . Therefore, $f(0^-) = 0$, whereas $f(0^+) = 1$. Therefore, $t = 0$ is a point of discontinuity in $f(t)$ and has to be excluded from its domain. $f(t)$ remains at 1 for $t \geq 0^+$ since $\delta(t)$ is zero for $t \geq 0^+$. Therefore, the description of $f(t)$ coincides with the definition of $u(t)$,

$$u(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0^- \\ \text{undefined} & \text{at } t = 0 \\ 1 & \text{for } 0^+ \leq t < \infty \end{cases}$$

Therefore,

$$u(t) = \int_{-\infty}^t \delta(t) dt.$$

Now, we turn our attention to the differential equation describing series *RL* circuit for zero-state response with some input function $v_s(t)$ switched onto the circuit at $t = 0$. Here, $v_s(t)$ is a function which is defined for all t and may not be zero for $t < 0$. We make the applied voltage equal to zero in the negative time axis and equal to this function in the positive time axis by multiplying it with $u(t)$ to find the zero-state response.

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} v_s(t) u(t) \quad \text{for all } t$$

Let $v_{SU}(t) = v_s(t) u(t)$, then,

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} v_{SU}(t) \quad \text{for all } t$$

Since this equation is true for all t (it is a KVL equation), we can integrate both sides from $-\infty$ to t and write

$$\int_{-\infty}^t \left(\frac{di_L(t)}{dt} \right) dt + \frac{1}{\tau} \int_{-\infty}^t i_L(t) dt = \frac{1}{L} \int_{-\infty}^t v_{SU}(t) dt \quad \text{for all } t$$

Changing the order of integration and differentiation

$$\frac{d}{dt} \int_{-\infty}^t i_L(t) dt + \frac{1}{\tau} \int_{-\infty}^t i_L(t) dt = \frac{1}{L} \int_{-\infty}^t v_{SU}(t) dt \quad \text{for all } t$$

Let $\hat{i}_L(t) = \int_{-\infty}^t i_L(t) dt$ and $\hat{v}_{SU}(t) = \int_{-\infty}^t v_{SU}(t) dt$, then

$$\frac{d\hat{i}_L(t)}{dt} + \frac{\hat{i}_L(t)}{\tau} = \frac{1}{L} \hat{v}_{SU}(t) \quad \text{for all } t$$

The last equation is the differential equation describing the zero-state response for integrated input function.

Therefore, we can state that the zero-state response in a linear circuit for an integrated input function is equal to integral of the zero-state response for the input function.

Strictly speaking, we showed this only for a first-order differential equation; but there is nothing in the proof which limits it to first-order differential equation alone. This result is valid for any linear lumped circuit described by linear differential equation of any order. The integration has to be performed from $-\infty$ in theory. However, since we know that the zero-state response is zero from $-\infty$ to $t = 0^-$, we need to integrate from $t = 0^-$ to t only.

Integral of $\delta(t)$ gives $u(t)$. Therefore, integral of *impulse response* should give *unit step response* (usually referred to as *the step response*). This is verified by carrying out the integration below.

Zero-state unit impulse response of an *RL* circuit $= \frac{1}{L}e^{-t/\tau}$ for $t \geq 0^+$ and 0 for $t \leq 0^-$

$$\begin{aligned} \text{Integrating for a } t > 0^+, \quad & \int_{-\infty}^{0^-} 0 \, dt + \int_{0^-}^{0^+} (a \text{ bounded number}) \, dt + \int_{0^+}^t \frac{1}{L} e^{-t/\tau} \, dt \\ &= \frac{\tau}{L} (1 - e^{-t/\tau}) = \frac{1}{R} (1 - e^{-t/\tau}) \text{ for } t \geq 0^+ \end{aligned}$$

Unit ramp input function is defined as $r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$.

It can be easily verified that unit ramp function is the integral of unit step function. These three basic input functions and their relations are shown in Fig. 10.7-2.

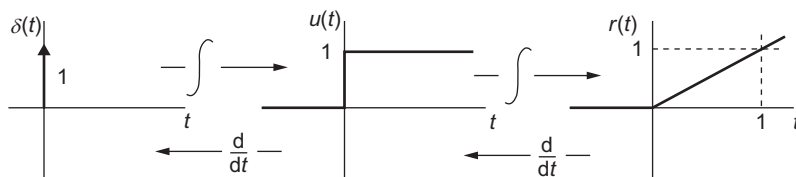


Fig. 10.7-2 Impulse, step and ramp functions and their relations

Therefore, *the ramp response* can be found by integrating the step response as below.

$$\text{Ramp response} = \int_{0^+}^t \frac{1}{R} (1 - e^{-t/\tau}) \, dt = \frac{1}{R} [t - \tau(1 - e^{-t/\tau})] \text{ for } t \geq 0^+$$

Ramp response and its components are shown in Fig. 10.7-3. The voltage across resistor is plotted instead of inductor current.

The unit ramp function has a kink at $t = 0$ and hence it is not differentiable at $t = 0$. However, it is differentiable at all other time instants. Hence, first derivative of $r(t)$ will be a function defined as 0 for $t \leq 0^-$, 1 for $t \geq 0^+$ and *undefined* at $t = 0$. But that is the unit step function. Therefore, unit step function is the first derivative of unit ramp function.

Unit step function is not even continuous at $t = 0$. Obviously, it cannot be differentiated there. However, we raise the question – which function when integrated will yield unit step function? The answer is that it is the impulse function. Therefore, we can consider $\delta(t)$ to be the first derivative of $u(t)$ in the ‘anti-derivative’ sense.

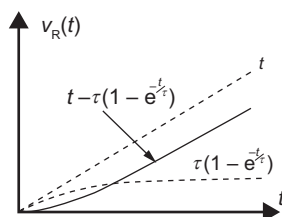


Fig. 10.7-3 Unit ramp response of *RL* circuit

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KVL equations are true for all t . Both sides of an equation which holds for all t can be differentiated with respect to time. From this observation, it is easy to see that the following is true.

The zero-state response in a linear circuit for differentiated input function is equal to the derivative of the zero-state response for the input function.

Therefore, we can get to unit step response and unit impulse response in any linear circuit by successive differentiation of its unit ramp response.

Example: 10.7-1

An inductor, resistor and a unit impulse current source are connected in parallel. Find the zero-state impulse response for resistor current and inductor current in this circuit.

Solution

The impulse current cannot flow through L and hence it flows through R producing an impulse voltage $R\delta(t)$ across the combination. This impulse voltage has an area content of R V-s. R V-s dumped into an inductor of L Henry will produce a change in its current by R/L A. Since initial condition for zero-state response is zero the current of inductor at $t = 0^+$ will be this change amount itself – i.e., R/L A. Thereafter, it is a source-free circuit since impulse current source is an open circuit after $t = 0^+$. Therefore, the current in both R and L will be $(1/\tau)e^{-t/\tau}$ where $\tau = L/R$.

Example: 10.7-2

An inductor, resistor and a 1A DC current source are connected in parallel at $t = 0$. The initial condition for inductor is specified as I_0 . Find the time-domain expressions for current through L and R and the voltage across the combination. Also, find the total energy delivered by the current source, energy consumed by the resistor and the energy stored in the inductor for a special case where the initial condition is zero.

Solution

We use the result from previous example in solving this. The total response for inductor current is the sum of zero-input response and zero-state response. A series RL circuit with some non-zero initial condition gets formed when current source at input is set to zero. Therefore, the zero-input response of this circuit is same as the zero-input response of an RL circuit we have discussed so far. The zero-state response for switched DC current input can be found by integrating the impulse response with current input.

$$\text{Zero-input response } i_{L(zi)} = I_0 e^{-t/\tau} \text{ and Zero-state impulse response } = (1/\tau) e^{-t/\tau}$$

$$\therefore \text{Zero-state step response } i_{L(zs)} = \int_{0^+}^t (1/\tau) e^{-t/\tau} dt = 1 - e^{-t/\tau}$$

$$\therefore \text{Total response } i_L = I_0 e^{-t/\tau} + (1 - e^{-t/\tau}) = 1 - (I_0 - 1) e^{-t/\tau}$$

$$\text{Since } i_L + i_R = 1 \text{ for } t \geq 0^+, i_R = (1 - I_0) e^{-t/\tau} \text{ and } v_R = R(1 - I_0) e^{-t/\tau}$$

The total energy delivered by source is found by integrating the power delivered from 0 to ∞ . The power delivered is given by voltage across the combination $\times$ source current value. Similarly, the energy dissipated in R and stored in L can be found by integrating $v_R i_R$ and $v_L i_L$ from 0 to ∞ . The integral must return $L/2$ in the case of inductor because that is the stored energy in an inductor carrying 1A. Let us call these three energy values E_S , E_R and E_L respectively. Then,

$$\begin{aligned} i_L &= (1 - e^{-t/\tau}) \text{ (with zero initial condition)} \\ i_R &= e^{-t/\tau} \text{ and } v_R = R e^{-t/\tau} \\ \therefore E_S &= \int_0^\infty R e^{-t/\tau} dt = R\tau = \frac{1}{L} \text{ J}; \quad E_R = \int_0^\infty R e^{-2t/\tau} dt = R\tau/2 = \frac{1}{2L} \text{ J} \\ E_L &= \int_0^\infty R e^{-t/\tau} (1 - e^{-t/\tau}) dt = R\tau/2 = \frac{1}{L} - \frac{1}{2L} = \frac{1}{2L} \text{ J} \end{aligned}$$

If the current source had a magnitude of I instead of 1, all the three energy terms are to be multiplied by I^2 . Hence, we see that (i) it takes $LI^2/2$ joules of energy dissipation in the parallel resistance whenever we charge an inductor to IA and this dissipated energy is as much as the energy stored in the inductor and (ii) this energy dissipation is independent of R value. The value of R will decide the time taken to dissipate this amount of energy but not the amount of energy.

Example: 10.7-3

Find the inductor current response for $t \geq 0^+$ with the specified initial condition in the circuit in Fig. 10.7-4 (a). The part of the circuit that is needed to create this initial current in the inductor is not shown in the diagram.

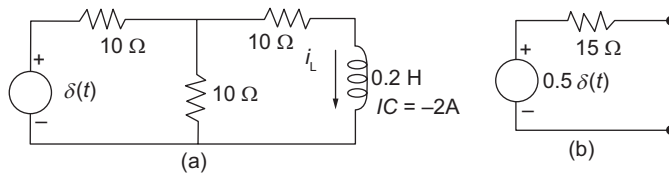


Fig. 10.7-4 Circuit for Example: 10.7-3

Solution

If the unit impulse voltage source is the only source in the circuit, the inductor current at $t = 0^-$ can only be 0. The fact that a non-zero current is specified at $t = 0^-$ indicates that there must have been other sources and elements that were active for $t < 0^-$ in the circuit and that they were switched out before $t = 0^-$, leaving the inductor with a trapped current of 2A.

The time constant of the circuit relevant for $t \geq 0^+$ is found by deactivating sources and finding out the equivalent resistance connected across the inductor. The equivalent resistance is 15Ω and the time constant is 13.33ms. The Thevenin's equivalent of the circuit connected across inductor is shown in Fig. 10.7-4 (b).

The $0.5\delta(t)$ voltage source appears across the inductor forcing its current to change by $0.5/0.2 = 2.5$ A. The inductor current at $t = 0^-$ is specified as 2 A in the opposite direction. Hence, the inductor current at $t = 0^+ = -2 + 2.5 = 0.5$ A. After $t = 0^+$, the circuit is a source-free series RL circuit and the

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inductor current will be $0.5 e^{-75t}$ A for $t \geq 0^+$. The inductor current for $t < 0^-$ cannot be determined since there were unknown elements active (otherwise, the current in the inductor current would not have been non-zero at $t = 0^-$) in the circuit during $t < 0^-$.

Example: 10.7-4

Find the inductor current and voltage across current source as a function of time in the circuit in Fig. 10.7-5 (a). The part of the circuit that is needed to create the initial current in the inductor is not shown in the diagram. Also, find what must be the initial current in the inductor so that the inductor current will be transient-free for $t \geq 0^+$

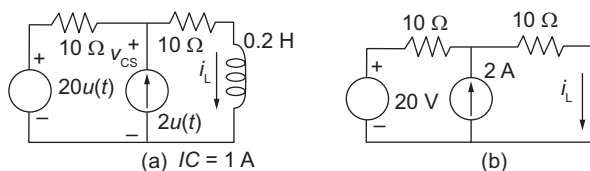


Fig. 10.7-5 Circuits for Example: 10.7-4

Solution

Time constant of the circuit is 0.01s. The zero-input i_L response is $1e^{-100t}$ A. Zero-state response when the voltage source is acting alone is $i_L = (20/20)(1 - e^{-100t})$ A. The zero-state response when the current source is acting alone is $i_L = 1(1 - e^{-100t})$ A.

$\therefore$ Total response for $i_L = 2 - e^{-100t}$ A for $t \geq 0^+$

Zero-input v_{cs} response = voltage across the first 10Ω resistance due to zero-input response in $i_L = -10 e^{-100t}$ A.

Zero-state v_{cs} response when the voltage source is acting alone is $20 - 10 \times$ zero-state i_L for the same condition $= 20 - 10(1 - e^{-100t}) = 10(1 + e^{-100t})$ V

Zero-state v_{cs} response when the current source is acting alone is $10 \times (2 - \text{zero-state } i_L \text{ for the same condition}) = 10 \times (2 - (1 - e^{-100t})) = 10(1 + e^{-100t})$ V

$\therefore$ Total response for $v_{cs} = 20 - 10 e^{-100t}$ V for $t \geq 0^+$

We could have obtained v_{cs} as $(10 \times \text{total response in } i_L + 0.2 \times \text{first derivative of total response in } i_L)$ too.

Inductor current will be transient free if the natural response terms in zero-input response cancel out the natural response terms in *total* zero-state response. The natural response terms in *total* zero-state response in this case is $-2 e^{-100t}$ and hence initial current must be 2A for transient-free inductor current in this circuit.

Another point of view would be that the inductor current will be transient-free if the initial current and final current (*i.e.*, steady-state current for DC steady state) are the same. In general, inductor current will be transient-free if the initial current at $t = 0^+$ and the value of forced response component (this need not be a DC component) at $t = 0^+$ are equal. The DC steady state may be obtained from circuit Fig. 10.7-5 (b). Applying superposition principle, we get the steady-state value of i_L as $(20V/20\Omega) + 2 \text{ A} \times (10\Omega/20\Omega) = 2 \text{ A}$. Thus, the required initial current in the inductor for transient-free response will be 2 A again.

A single-inductor circuit can be described by a first-order differential equation on inductor current. It will be possible to work out the other circuit variables from inductor current alone. Therefore, it follows that if inductor current is transient-free, so will all the other circuit variables be.

10.8 SERIES *RL* CIRCUIT WITH EXPONENTIAL INPUTS

We take up the study of zero-state response of series *RL* circuit for exponential inputs and sinusoidal inputs in this section. We do not worry about the zero-input response anymore since we know that it is $I_0 e^{-t/\tau}$ where I_0 is the initial current specified at $t = 0^-$. The total response is found by adding zero-input response and zero-state response together.

We permit exponential inputs of the form $e^{st} u(t)$ in this section where s can be a complex number $s = -\sigma + j\omega$ where σ and ω are two real numbers. The reason for generalising the exponential input in this manner will be clear soon. We put a -ve sign behind σ since we want the real part of s to be negative when σ is positive. This signal is a complex signal with a real part function and imaginary part function as shown in Eqn. 10.8-1.

$$v_s(t) = e^{st} u(t) = e^{(-\sigma + j\omega)t} u(t) = (e^{-\sigma t} \cos \omega t) u(t) + j(e^{-\sigma t} \sin \omega t) u(t) \quad (10.8-1)$$

We have used Euler's identity in expressing complex exponential function in the rectangular form involving trigonometric functions. No signal like this can be actually generated by a physical system since there can be nothing 'imaginary' about a physical signal. Therefore, this signal does not represent a physical signal and hence it cannot really be applied to any circuit. But it can be the forcing function in a differential equation – the mathematics of differential equation will not complain. Moreover, its real part and imaginary part are real functions only and can be made physically.

We know that zero-state response of a linear circuit obeys superposition principle. Let us apply a signal $x(t)$ to the circuit and let the zero-state response be $i_x(t)$. Similarly let the zero-state response be $i_y(t)$ when a signal $y(t)$ is applied to the same circuit. Now, what is the response when $jy(t)$ is applied where $j = \sqrt{-1}$? By superposition principle it will be $ji_y(t)$. When we apply $x(t) + jy(t)$ as input we must get $i_x(t) + ji_y(t)$ as response by superposition principle again.

The real part of zero-state response for a complex signal input is the zero-state response for the real part of input and the imaginary part of zero-state response for a complex signal input is the zero-state response for the imaginary part of input. This is a direct consequence of linearity of the circuit.

Now we can see why we opted for this complex signal. We can obtain the zero-state response of circuits to sinusoidal inputs by taking the real or imaginary part of zero-state response for complex exponential inputs with $\sigma = 0$. It is easier to deal with complex exponential function than with sinusoids when it comes to solving differential equations. Let us solve for the zero-state response of series *RL* circuit with a complex exponential input.

$$\frac{di_L(t)}{dt} + \frac{i_L(t)}{\tau} = \frac{1}{L} e^{st} u(t) \quad (10.8-2)$$

This equation has to be true for all t and in particular for all $t \geq 0^+$. This is possible only if the shape of functions on both sides of the equation is the same. This will imply that the trial solution for particular integral has to be Ae^{st} . Substituting this trial solution in the differential equation in Eqn. 10.8-2,

$$\begin{aligned} Ae^{st} (s + 1/\tau) &= (1/L) e^{st} \text{ for } t \geq 0^+ \\ \therefore A &= \frac{1}{sL + L/\tau} = \frac{1}{sL + R} \quad (\because \tau = L/R) \\ \therefore i_L(t) &= \frac{1}{sL + R} e^{st} \text{ for } t \geq 0^+ \end{aligned} \quad (10.8-3)$$

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We have to form the total solution by adding this particular integral to complementary solution. Note that finding zero-state response involves finding a particular integral and a suitable complementary function also – because zero-state response involves both forced response and natural response terms.

$$\therefore i_L(t) = B e^{-t/\tau} + \frac{1}{sL+R} e^{st} \text{ for } t \geq 0^+$$

Substituting initial current value at 0^+ ,

$$0 = B + \frac{1}{sL+R} \quad (10.8-4)$$

$$\therefore B = -\frac{1}{sL+R} \text{ and}$$

$$i_L(t) = \frac{1}{sL+R} (e^{st} - e^{-t/\tau}) \text{ for } t \geq 0^+$$

10.8.1 Zero-State Response for Real Exponential Input

We consider a special case of complex exponential with $\omega = 0$. The input source function is then of the form $e^{-\sigma t} u(t)$. With a positive value of σ , this input function has the same format as that of the impulse response of a series RL circuit (impulse response of any first-order circuit for that matter). Hence, the problem we are trying to solve may be thought of as a situation where the impulse response of one circuit is applied to a second circuit as input. Such situations arise when we cascade different circuits. The zero-state response is obtained by putting $s = -\sigma$ in Eqn. 10.8-4 and is expressed as in Eqn. 10.8-5.

$$i_L(t) = \frac{1}{R} \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ for } t \geq 0^+ \text{ where } \alpha = 1/\tau = R/L \quad (10.8-5)$$

We note the $\alpha - \sigma$ in the denominator and raise the question – what is the response when the real exponential input has the same index as that of impulse response of the circuit? It is not *infinite* as Eqn. 10.8-5 would suggest. The correct answer is that this is one situation under which our assumed trial particular integral is incorrect. We have to find the particular integral afresh. There are well-established methods to arrive at particular integral under this kind of situation in mathematics. But we do not take up this case here. We will get at it later chapter and solve it without having to learn a new method for that. Therefore, we qualify the solution in Eqn. 10.8-5 by specifying that solution is valid only if $\alpha \in \sigma$.

$$i_L(t) = \frac{1}{R} \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ for } t \geq 0^+ \text{ where } \alpha = 1/\tau = R/L \text{ and } \alpha \neq \sigma \quad (10.8-6)$$

The input waveform is exponential and we can think of a time constant for this waveform too. Let us denote this time constant as τ_e with the subscript reminding us that this is the time constant of an excitation waveform. Obviously, $\tau_e = 1/\sigma$. Further, we normalise the current in Eqn. 10.8-6 using a normalisation base of $1/R$ A and time by a normalisation base of τ s to put the solution in terms of the normalised variables as in Eqn. 10.8-7 where the subscript n indicates normalised variables.

$$i_{Ln}(t) = \left(\frac{1}{1 - \tau/\tau_e} \right) (e^{-(\tau/\tau_e)t_n} - e^{-t_n}) \text{ for } t \geq 0^+, \tau_e = 1/\sigma, \tau = R/L \text{ and } \tau_e \neq \tau \quad (10.8-7)$$

The input functions and i_L waveforms are shown for two cases in Fig. 10.8-1.

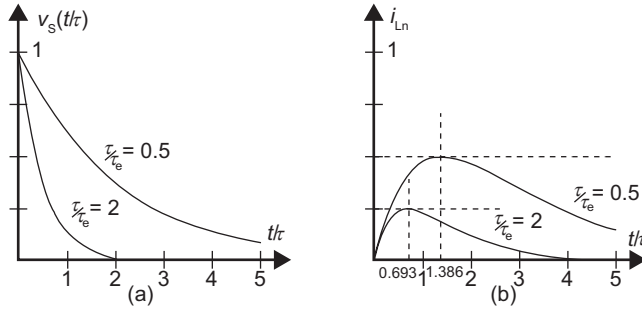


Fig. 10.8-1 Zero state response for exponential input (a) Input wave (b) Normalised inductor current

The time instants at which response reaches maximum and the maximum response are marked. General expressions for these may be derived by differentiating the function in Eqn. 10.8-7 and setting the derivative to zero. Note that sending τ_e to ∞ amounts to applying a unit step input and zero-state response indeed approaches the unit step zero-state response as expected.

10.8.2 Zero-State Response for Sinusoidal Input

The input function in this case is $v_s(t) = \sin \omega t u(t)$ V. We can get sinusoidal zero-state response for sinusoidal input by two methods. In the first method, we use the result in Eqn. 10.8-4 with $s = j\omega$ and take the imaginary part of the result as our desired solution.

$$\begin{aligned}
 \therefore i_L(t) &= \text{Im} \left(\frac{1}{j\omega L + R} (e^{j\omega t} - e^{-t/\tau}) \right) \quad \text{for } t \geq 0^+ \\
 &= \text{Im} \left(\frac{R - j\omega L}{(\omega L)^2 + R^2} \{ (\cos \omega t - e^{-t/\tau}) + j \sin \omega t \} \right) \quad \text{for } t \geq 0^+ \\
 &= \frac{(R \sin \omega t - \omega L \cos \omega t) + \omega L e^{-t/\tau}}{(\omega L)^2 + R^2} \quad \text{for } t \geq 0^+ \quad (10.8-8)
 \end{aligned}$$

$$\text{Let } \phi = \tan^{-1} \frac{\omega L}{R}. \text{ Then } \cos \phi = \frac{R}{\sqrt{R^2 + (\omega L)^2}} \text{ and } \sin \phi = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}.$$

$$\begin{aligned}
 \therefore i_L(t) &= \frac{(\sin \omega t \cos \phi - \cos \omega t \sin \phi) + \sin \phi \cdot e^{-t/\tau}}{\sqrt{R^2 + (\omega L)^2}} \quad \text{for } t \geq 0^+ \\
 &= \frac{1}{\sqrt{R^2 + (\omega L)^2}} (\sin(\omega t - \phi) + \sin \phi \cdot e^{-t/\tau}) \quad \text{for } t \geq 0^+
 \end{aligned}$$

In the second method, we represent the input sine wave by combining two complex exponential functions by using Euler's identity as below.

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

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Then, we get the zero-state response for the two exponential inputs separately and use superposition principle to arrive at the solution for sinusoidal input.

$$\begin{aligned}
 \therefore i_L(t) &= \frac{1}{2j} \left(\frac{1}{R + j\omega L} (e^{j\omega t} - e^{-t/\tau}) - \frac{1}{R - j\omega L} (e^{-j\omega t} - e^{-t/\tau}) \right) \text{ for } t \geq 0^+ \\
 &= \frac{1}{2j} \left\{ \left(\frac{1}{R + j\omega L} + \frac{1}{R - j\omega L} \right) e^{-t/\tau} + \left\{ \frac{e^{j\omega t}}{R + j\omega L} - \frac{e^{-j\omega t}}{R - j\omega L} \right\} \right\} \text{ for } t \geq 0^+ \\
 &= \frac{(R \sin \omega t - \omega L \cos \omega t) + \omega L e^{-t/\tau}}{R^2 + (\omega L)^2} \text{ for } t \geq 0^+ \\
 \therefore i_L(t) &= \frac{(\sin \omega t \cos \phi - \cos \omega t \sin \phi) + \sin \phi \cdot e^{-t/\tau}}{\sqrt{R^2 + (\omega L)^2}} \text{ for } t \geq 0^+ \\
 &= \frac{1}{\sqrt{R^2 + (\omega L)^2}} (\sin(\omega t - \phi) + \sin \phi \cdot e^{-t/\tau}) \text{ for } t \geq 0^+
 \end{aligned} \tag{10.8-9}$$

The angle ϕ in Eqn. 10.8-9 is defined the same way as in Eqn. 10.8-8. Both methods lead to same expression for the final response, as they should. The final expression may be recast in the following form where $k = \omega\tau$ and the current and time are normalised with respect to $1/R$ and τ , respectively.

$$i_{Ln}(t) = \frac{1}{\sqrt{1+k^2}} \left(\sin(k t_n - \tan^{-1} k) + \frac{k}{\sqrt{1+k^2}} e^{-t_n} \right); k = \omega\tau \tag{10.8-10}$$

This waveform for a case with $k = 4$ is shown in Fig. 10.8-2. The number k can be interpreted as a comparison between the characteristic time, *i.e.*, the period of the applied voltage and the characteristic time of the circuit, *i.e.*, its time constant. k can be expressed as $2\pi(\tau/T)$ where T is the period of input. Sinusoids undergo a full cycle of variation in one T and hence the value of T is indicative of the rate of change involved in the waveform, *i.e.*, the *speed* of the waveform. Time constant is a measure of inertia in the system. Therefore, an input sinusoid is too fast for a circuit to follow if its T is smaller than the time constant τ of the circuit. Similarly, if input sinusoid has a T value much larger than time constant of the circuit, the circuit will perceive it as a very slow waveform and will respond almost the same way it does to DC input. These aspects are clearly brought out in Eqn. 10.8-10.

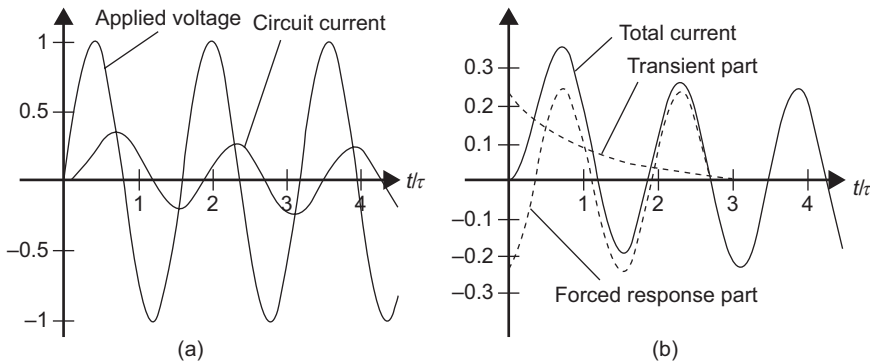


Fig. 10.8-2 Unit sinusoidal response of RL circuit with $k = 4$

The amplitude of forced response component, or equivalently, the amplitude of sinusoidal steady-state response, is a strong function of k . The amplitude decreases with increasing ω or increasing τ . In addition, the steady-state current lags the applied voltage by a phase angle that increases with $\omega\tau$.

Let us imagine that we conduct an experiment. We apply a sinusoidal voltage of 1V amplitude to a series RL circuit and wait for enough time for the transient response to die down. After steady state is satisfactorily established in the circuit, we measure the amplitude of current and its phase with respect to the input sine wave. We repeat this process for various values of frequency of input, keeping its amplitude at 1V always. We ensure that the circuit is in steady-state before we measure the output every time.

The data so obtained can be plotted to show the variation of ratio of output amplitude to input amplitude and phase of steady-state current against $k (= \omega\tau)$. Such a pair of plots will constitute what is called the *AC steady-state frequency response plots* for this RL circuit. The ratio of output amplitude to input amplitude is called *gain* of the circuit. Its dimension will depend on the nature of input and output quantities. If we define a complex function of ω with the magnitude of function equal to the *gain* described here and angle of the function equal to the phase angle by which the steady-state output *leads* the sinusoidal input, the resulting function will be what we termed the *frequency response function* $H(j\omega)$ we described in Chapter 9.

Such an experiment can be performed on any circuit to get its frequency response data. However, if the differential equation of the circuit is known we need not do the experiment. The frequency response plots can be obtained analytically in that case. The frequency response of the series RL circuit is shown in Fig. 10.8-3 as an example.

We make the following observations on the sinusoidal steady-state response of series RL circuit from Eqn. 10.8-10 and Fig. 10.8-3.

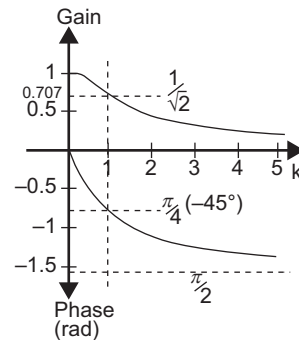


Fig. 10.8-3 Frequency response plots for series RL circuit

- The circuit current under sinusoidal steady-state response is a sinusoid at the same angular frequency ω rad/s as that of input sinusoid.
- The circuit current initially is a mixture of an exponentially decaying unidirectional transient component along with the steady-state sinusoidal component. This unidirectional transient imparts an offset to the circuit current during the initial period.
- The circuit current at its first peak can go close to twice its steady-state amplitude in the case of circuits with $\omega\tau \gg 1$ due to this offset.
- The amplitude of sinusoidal steady-state response is always less than corresponding amplitude when DC input of same amplitude is applied. This is due to the inductive inertia of the circuit. The amplitude depends on the product $\omega\tau$ and decreases monotonically with the $\omega\tau$ product for fixed input amplitude.
- The response sinusoid *lags* behind the input sinusoid under steady-state conditions by a phase angle that increases monotonically with the product $\omega\tau$.
- The frequency at which the circuit gain becomes $1/\sqrt{2}$ times that of DC gain is termed as *cut-off frequency* and since this takes place as we go up in frequency it is called *upper cut-off frequency*.

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Upper cut-off frequency of series RL circuit is seen to be at $\omega = 1/\tau$ rad/s. The phase delay at this frequency will be -45° .

- Circuit current amplitude becomes very small at high frequencies ($\omega\tau \gg 1$) and the current lags the input voltage by $\approx 90^\circ$ at such frequencies.

We assumed that the applied voltage is $v_s(t) = \sin \omega t u(t)$ throughout this analysis. This means that the sinusoidal voltage happened to be crossing the time-axis exactly at the instant at which we closed the switch to apply it to the circuit. Though technically it is possible to do such switching (it is done in some applications that way), that is not the way it takes place in many practical applications. The sinusoid may be at any value between its maximum and minimum when we throw the switch. Therefore, we must analyse the response with $v_s(t) = \sin(\omega t + \theta)u(t)$ for an arbitrary θ .

The function $\sin(\omega t + \theta)$ is the imaginary part of $e^{j(\omega t + \theta)} = \cos(\omega t + \theta) + j \sin(\omega t + \theta)$. The zero-state response when e^{st} is applied to the circuit is given by Eqn. 10.8-4. So, shall we substitute $s = j(\omega + \theta)$ in Eqn. 10.8-4 to get the required output? No, that will be wrong since it is only ω that gets multiplied by t , not θ . We should (i) interpret $e^{j(\omega t + \theta)}$ as $e^{j\theta} e^{j\omega t}$, (ii) solve for the zero-state response for $e^{j\omega t}$, (iii) multiply the response by $e^{j\theta}$ (we apply principle of homogeneity there) and (iv) take the imaginary part. We skip all that basic algebra and give the result below.

$$i_L(t) = \frac{1}{\sqrt{R^2 + (\omega L)^2}} (\sin(\omega t + \theta - \phi) + \sin(\phi - \theta) \cdot e^{-t/\tau}) \quad \text{for } t \geq 0^+ \quad (10.8-11)$$

By substituting $\theta = 90^\circ$, we get the solution for $v_s(t) = \cos \omega t u(t)$ as

$$i_L(t) = \frac{1}{\sqrt{R^2 + (\omega L)^2}} (\cos(\omega t - \phi) - \cos \phi \cdot e^{-t/\tau}) \quad \text{for } t \geq 0^+ \quad (10.8-12)$$

Eqn. 10.8-11 indicates that it is possible to switch on an AC voltage to an initially relaxed series RL circuit in such a way that there is no transient response and circuit immediately goes to steady state – the switching instant must be such that $\theta = \phi$. This principle is used sometimes in switching of heavily inductive power equipment. In such cases, the angle ϕ is close to 90° and transient-free switching is possible if the voltage is switched on to the equipment at positive or negative peak.

Note that the sinusoidal steady-state response part of zero-state response in all the above cases could have been readily obtained by employing phasor method we studied in earlier chapters.

10.9 GENERAL ANALYSIS PROCEDURE FOR SINGLE TIME CONSTANT RL CIRCUITS

We have analysed the simple series RL circuit exhaustively. We would like to arrive at a generalised procedure for analysis of RL circuits that may contain more than one resistor. The circuit may also contain more than one inductor; but in that case we assume that the inductors will enter in series or parallel combinations and can be finally be replaced by a single inductor. If that is not possible, the circuit will have more than one time constant and we cannot handle it by the method we develop here.

A circuit with a single energy storage element can have only first-order dynamics and only one time constant. A circuit with only one time constant can have only one term in its natural response for any circuit variable. That term is of the form $Ae^{-t/\tau}$ where A is to be adjusted for compliance with *initial condition at $t = 0^+$ for that particular circuit variable*.

The single time constant that describes the natural response of the circuit is a property of circuit alone and does not depend on source function values. Therefore, it can be found from the *dead circuit* obtained by *deactivating all independent sources*. We get the *dead circuit* by replacing all independent voltage sources by short-circuits and all independent current sources by open circuits. Now, that circuit will contain only one inductor and possibly many resistors (and dependent sources, if they were present in the original circuit). We can find the equivalent resistance connected across the inductor by series/parallel combinations and by star-delta transformation if necessary. If the circuit contains linear dependent sources we may have employ *unit current injection method* or *unit voltage application method* detailed in Chapter 5 to find the equivalent resistance across inductor. Once we get this resistance, we can find time constant by $\tau = L/R_{eq}$.

Next step is to check whether there are impulse sources in the circuit. If there are, the amount of V-s dumped on the inductor at $t = 0$ has to be evaluated and the *change* in inductor current at $t = 0$ has to be found out. This change in current added to the initial condition specified at $t = 0^-$ will give us the value of inductor current at $t = 0^+$. But we need the initial condition for the particular circuit variable we are solving for. Therefore, we have to carry out a DC circuit analysis at $t = 0^+$ in which the inductor is replaced by a DC current source of value equal to its current at $t = 0^+$ and all the independent sources are replaced by DC sources (if they are not DC sources already) of value equal to their values at $t = 0^+$. Solving the resulting circuit will give us the initial condition for the particular circuit variable of interest.

In the third step, we work out the forced response for all independent sources in the circuit. Since the circuit has a steady-state for DC and AC, we use steady-state analysis for this. In particular, if the sources are DC sources, we replace inductor by short-circuit and solve the resulting resistive circuit for the variable of interest. We may use superposition principle along with mesh or node analysis for this purpose. We employ phasor analysis to solve for steady-state if there are AC sources. The steady-state solution for all the independent sources are added up.

We add the total steady-state solution to the natural response, apply the initial condition for the relevant variable and evaluate the arbitrary constant in the natural response term in the last step. We can also use the *zero-input response plus zero-state response method* instead of *transient response plus forced response method*. These methods are illustrated through a set of examples that follow.

Example: 10.9-1

Find $i_x(t)$ in the circuit in Fig. 10.9-1 (a). The initial condition in the 0.2H inductor is 1A in the direction shown at $t = 0^-$. The circuit elements that create this initial current are not shown in the diagram.

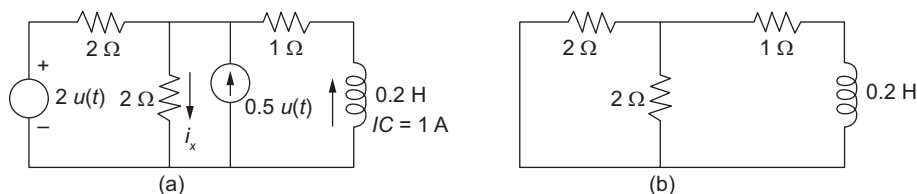


Fig. 10.9-1 Circuit for Example: 10.9-1

Solution

The circuit contains two switched DC sources – a 2V-voltage source and a 0.5A current source. There is only one time constant.

10.50 | First-Order RL Circuits

Step 1 – Find the time constant

The independent voltage source is replaced by a short-circuit and the independent current source is replaced by an open circuit to get the circuit in Fig. 10.9-1 (b). The time constant is found by evaluating the equivalent resistance across the inductor.

$$R_{eq} = 1 + 2 // 2 = 2 \Omega, \therefore \text{Time constant } \tau = \frac{0.2 \text{ H}}{2 \Omega} = 0.1 \text{ s}$$

Step 2 – Find the initial value for i_x at $t = 0^+$

The value of voltage source at $t = 0^+$ is 2V. The value of current source at $t = 0^+$ is 0.5A. The inductor current at $t = 0^+$ is same as at $t = 0^-$ since there is no impulse source in the circuit. Hence, inductor is replaced by a current source of 1A for $t = 0^+$. The resulting circuit appears in Fig. 10.9-2 (a).

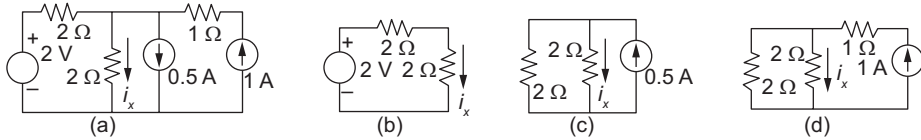


Fig. 10.9-2 Circuits for Step 2 of Example: 10.9-1

This circuit contains three sources and may be solved by using superposition principle. The three circuits in which the sources act one by one are shown in Fig. 10.9-2 (b)-(d). Three contributions to the current i_x may be found from these single-source circuits by simple circuit analysis.

$$i_x \text{ at } t = 0^+ = \frac{2}{2+2} + \frac{0.5 \times 2}{2+2} + \frac{1 \times 2}{2+2} = 1.25 \text{ A}$$

Step 3 – Find the steady-state value of i_x

Both sources in the circuit are DC sources and hence they can be handled together for steady-state calculation. The inductor behaves as a short-circuit for DC steady state. Replacing the inductor by short-circuit and using the known values of source functions we get the circuit in Fig. 10.9-3 (a).

This circuit involves two sources and can be solved by using superposition principle. The sub-circuits needed for that is shown in Fig. 10.9-3 (b) and (c).

$$\text{Steady-state value of } i_x = \frac{2}{2+2//1} \times \frac{1}{1+2} + \frac{0.5 \times 1 // 2 // 2}{2} = 0.25 + 0.125 = 0.375 \text{ A}$$

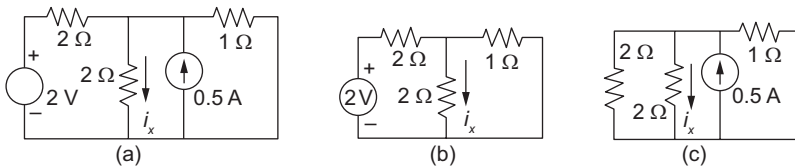


Fig. 10.9-3 Circuits for Step 3 of Example: 10.9-1

Step 3 – Form the total solution and adjust initial condition

The transient response is of the form $A e^{-10t}$ since time constant is 0.1s

$$i_x(t) = A e^{-10t} + 0.375 \text{ for } t \geq 0^+; \text{ initial value of this current at } 0^+ \text{ is } 1.25 \text{ A}$$

$$\therefore A + 0.375 = 1.25 \Rightarrow A = 0.875$$

$$\therefore i_x(t) = 0.375 + 0.875 e^{-10t} \text{ for } t \geq 0^+$$

The current waveform is shown in Fig. 10.9-4.

We have calculated the required current. We obtained the solution in the form of transient response + forced response (or steady-state response). However, this form of solution is not the appropriate form if we are required to answer supplementary questions on the problem. For example, can we work out the current if the initial condition of inductor is changed to 2A? Or if the voltage source value is changed to 5V? We will have to rework the problem almost entirely to get the answer for that. Note that it is not possible to decompose a solution into zero-input response and zero-state response components if the solution is available in the *transient response + forced response* format. However, transient response and forced response can be obtained from the solution in *zero-input response + zero-state response* format. This is where the superiority of *zero-input response + zero-state response* approach becomes evident. We work out the same example to see these two components in the solution in the next example.

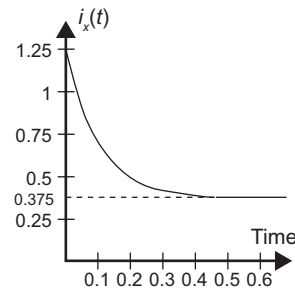


Fig. 10.9-4 The plot of current waveform for Example: 10.9-1

Example: 10.9-2

Obtain the solution in *zero-input response + zero-state response* form for the problem stated in Example 10.9-1 and modify the solution for (i) initial condition changed to 2A, (ii) voltage source magnitude changed to 5V and (iii) voltage source magnitude changed to 4V and current source magnitude changed to 1A along with initial condition changed to 2A.

Solution

When we want the solution in *zero-input response + zero-state response* format, we have to split the given circuit into two sub circuits right at the outset – one containing all independent sources and with zero initial condition for inductor current and the second with all independent sources deactivated and initial condition for inductor current at the specified value.

These two circuits are shown in Fig. 10.9-5 (a) and (b). The solution of Fig. 10.9-5 (b) gives the zero-input response. The solution of Fig. 10.9-5 (a) gives zero-state response. Zero-state response obeys superposition principle. Hence, the zero-state response of circuit in Fig. 10.9-5 (a) with the two sources acting simultaneously can be obtained by summing the zero-state responses when they are acting alone. The circuits required for finding out these individual zero-state responses are given in Fig. 10.9-5 (c) and (d).

Step 1 – Find the time constant

This step is the same as in Example: 10.9-1 and the value of time constant $\tau = 0.1$ s.

Step 2 – Find the zero-input response

The initial value of i_x at $t = 0^+$ is 0.5A from circuit in Fig. 10.9-5 (b). Since there is no source, the particular integral will be zero. Therefore, the zero-input response is $0.5 e^{-10t}$.

Step 3 – Find the zero-state response in circuit in Fig. 10.9-5 (c)

The initial value of i_x at $t = 0^+$ is 0.5A from circuit in Fig. 10.9-5 (c). Final steady-state value is obtained by replacing the inductor by short-circuit and solving the resulting resistive circuit. $2/(2 + 2//1)$ is the current from voltage source. This gets divided between 2Ω and 1Ω . The value is 0.25A. Therefore, the solution is $= 0.25(1 + e^{-10t})$.

10.52 | First-Order RL Circuits

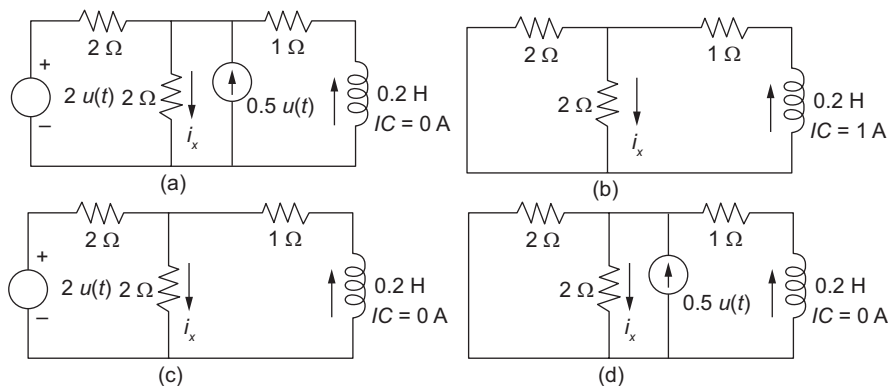


Fig. 10.9-5 Circuits for solving zero-input and zero-state responses in Example: 10.9-2

Step 4 – Find the zero-state response in circuit in Fig. 10.9-5 (d)

The initial value of i_x at $t = 0^+$ is 0.25A from circuit in Fig. 10.9-5 (d). Final steady-state value is obtained by replacing the inductor by short-circuit and solving the resulting resistive circuit. Applying current division principle, we get the value as 0.125A. Therefore, the solution is $= 0.125(1 + e^{-10t})$.

Step 5 – Find the total zero-state response in circuit Fig. 10.9-5 (a)

Total zero-state response is obtained by adding the two zero-state responses obtained in the last two steps. It is $0.25(1 + e^{-10t}) + 0.125(1 + e^{-10t}) = 0.375(1 + e^{-10t})$.

Step 6 – Find the total response in the original circuit

This is obtained by adding the zero-input response obtained in Step-1 and total zero-state response obtained in Step 5. It is $0.5 e^{-10t} + 0.375(1 + e^{-10t}) = 0.375 + 0.875 e^{-10t}$. It is the same expression we obtained in Example 10.9-1.

(i) If initial condition is changed to 2A

This change will affect only zero-input response. Zero-input response obeys superposition principle. Therefore, the zero-input response becomes $(0.5 e^{-10t}) \times 2/1 = e^{-10t}$.

Therefore, $i_x(t) = e^{-10t} + 0.375(1 + e^{-10t}) = 0.375 + 1.375 e^{-10t}$ A for $t \geq 0^+$.

(ii) If voltage source value is changed to 5V

This change will affect the zero-state response contribution from voltage source only. It was $0.25(1 + e^{-10t})$ when voltage source value was 2V. Therefore, applying superposition principle, it will be 2.5 times this function with voltage source value of 5V.

Therefore,

$i_x(t) = 0.5 e^{-10t} + 2.5 \times 0.25(1 + e^{-10t}) + 0.125(1 + e^{-10t}) = 0.75 + 1.25 e^{-10t}$ A for $t \geq 0^+$.

(iii) If voltage source value is changed to 4V, current source value changed to 1A and initial condition changed to 2A

The solution will get affected in all the three components. Zero-input response gets scaled by 2/1, zero-state response from voltage source gets multiplied by 4/2 and zero-state response from current source gets multiplied by 1/0.5.

Therefore,

$i_x(t) = 2 \times 0.5 e^{-10t} + 2 \times 0.25(1 + e^{-10t}) + 2 \times 0.125(1 + e^{-10t}) = 0.75 + 1.75 e^{-10t}$ A for $t \geq 0^+$

Example: 10.9-3

Solve for $v_o(t)$ as a function of time in the circuit in Fig. 10.9-6 (a). Also, identify (i) transient response and steady-state response components, (ii) zero-input response and zero-state response components

in the total response and (iii) contributions to zero-state response from the individual sources. The source functions are

$$v_{s1}(t) = 10 \sin(10t + \pi/4) u(t) \text{ V}$$

$$i_s(t) = 2 \cos(10t - \pi/3) u(t) \text{ A}$$

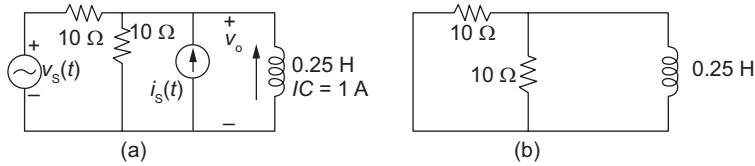


Fig. 10.9-6 Circuit for Example: 10.9-3

Solution

There are two switched sinusoidal sources in the circuit – one independent voltage source and one independent current source. They are of same angular frequency of 10 rad/s. Since we are required to identify the zero-input response and zero-state response components we have to solve two circuits – one with all sources and zero initial condition for inductor and another with all independent sources deactivated and specified initial condition for inductor current.

Step 1 – Find the time constant

The deactivated circuit for finding the time constant is shown in Fig. 10.9-6 (b). The time constant is $0.25/(10/10) = 0.05$ s.

Step 2 – Identify circuits for zero-input response and zero-state response and solve for zero-input response.

These circuits are shown in Fig. 10.9-7. Circuit in Fig. 10.9-7 (a) has to be used for determining zero-state response and circuit in Fig. 10.9-7 (b) has to be used for zero-input response.

The circuit does not apply any impulse to the inductor. Therefore, the inductor current at $t = 0^+$ is same as at $t = 0^-$ and will be 1A in the marked direction. But what we need is the initial condition at $t = 0^+$ for our response variable v_o . The 1A initial current in inductor at $t = 0^+$ gets divided equally between the two equal resistors and develops a potential of 5V across them at that instant. Hence, the required initial condition for $v_o = 5$ V.

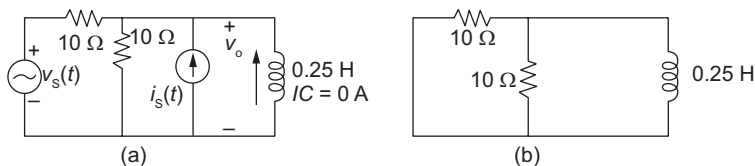


Fig. 10.9-7 Circuits for determining various response components in Example: 10.9-3

This is a single time constant circuit and hence zero-input response in any circuit variable will be of the form $Ae^{-t/\tau}$ where A has to be adjusted for compliance with initial condition on the chosen output variable. Therefore, zero-input response component in v_o is $5e^{-20t}$ V since initial condition is 5V and time constant is 0.05s.

Step 3 – Identify circuits for obtaining zero-state response components and solve them.

Three sub-circuits derived from circuit in Fig. 10.9-7 (a) for evaluating the zero-state response contributions from individual sources are shown in Fig. 10.9-8.

10.54 | First-Order RL Circuits

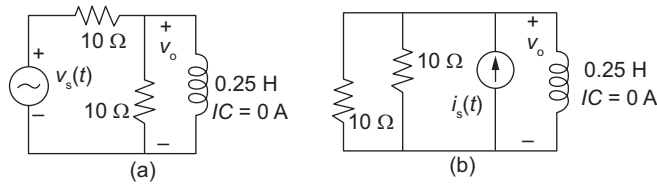


Fig. 10.9-8 Circuits for obtaining zero-state response components in Example: 10.9-3

We employ phasor method to solve for the sinusoidal steady-state response in each circuit, add a transient response term of $Ae^{-t/\tau}$ format and evaluate A by substituting suitable initial condition.

Step 3a – Zero-state response in Fig. 10.9-8 (a)

The circuits for evaluation of initial condition and sinusoidal steady-state response for this circuit is shown in Fig. 10.9-9. Circuit in Fig. 10.9-9 (b) shows the circuit for initial condition evaluation. The voltage source is replaced by a DC source of value same as the value of $v_{s1}(t)$ at $t = 0^+$ i.e., $10 \sin \pi/4 = 7.07$ V. The inductor is replaced by a DC current source of value equal to its initial condition. But in a circuit for evaluating zero-state response initial condition for inductor current will be zero. Therefore, it is a current source of zero value, i.e., it is an open circuit. Therefore, v_o at $t = 0^+$ is $7.07/2 = 3.535$ V.

The angular frequency of $v_{s1}(t)$ is 10 rad/s . Therefore, inductor will have an inductive reactance of $10 \times 0.25 = 2.5 \Omega$. The amplitude of $v_{s1}(t)$ is 10 V. In phasor equivalent circuit, we usually specify rms value of sources. Hence, $10/\sqrt{2}$ at 45° is the voltage source value in the circuit used for sinusoidal steady-state response evaluation. The sinusoidal steady-state response is evaluated as below.

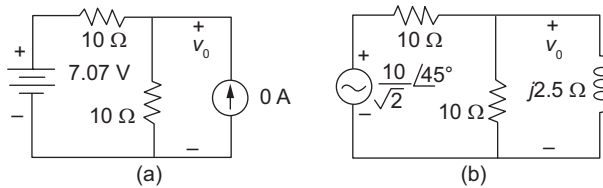


Fig. 10.9-9 Circuits for zero-state response due to voltage source in Example: 10.9-3

$$V_o = \frac{10 / j2.5}{10 + 10 / j2.5} \times \frac{10}{\sqrt{2}} \angle 45^\circ = \frac{2.24}{\sqrt{2}} \angle 108.44^\circ$$

$$v_o(t) = 2.24 \sin(10t + 108.44^\circ) \text{ V } (\because \text{input was sine function})$$

The zero-state response due to voltage source is obtained by adding this solution to transient response term and evaluating the arbitrary constant in the transient response term by using the initial condition for $v_o(t)$ we have already calculated.

$$v_o(t) = Ae^{-20t} + 2.24 \sin(10t + 108.44^\circ)$$

$$v_o(0^+) = 3.535 \text{ V} \Rightarrow A + 2.24 \sin 108.44^\circ = 3.535 \text{ V} \Rightarrow A = 1.41$$

$$\therefore v_o(t) = 1.41e^{-20t} + 2.24 \sin(10t + 108.44^\circ) \text{ V for } t \geq 0^+$$

Step 3b – Zero-state response in Fig. 10.9-8 (b)

The circuits for evaluation of initial condition and sinusoidal steady-state response for this circuit is shown in Fig. 10.9-10. Circuit in Fig. 10.9-10 (a) shows the circuit for initial condition evaluation. The current source is replaced by a DC current source of value same as the value of $i_s(t)$ at $t = 0^+$ i.e.,

$2 \cos(-\pi/3) = 1$ A. The inductor is replaced by a DC current source of value zero. Therefore, v_o at $t = 0^+$ is 5 V.

The angular frequency of $i_s(t)$ is 10-rad/s. Therefore, inductor will have an inductive reactance of $10 \times 0.25 = 2.5 \Omega$. The amplitude of $i_s(t)$ is 2A. Hence, $2/\sqrt{2}$ at -60° is the current source value in the circuit used for sinusoidal steady-state response evaluation. The sinusoidal steady-state response is evaluated as below.

$$V_0 = 10 // 10 // j2.5 \times \frac{2}{\sqrt{2}} \angle -60^\circ = \frac{4.47}{\sqrt{2}} \angle 3.44^\circ$$

$$v_o(t) = 4.47 \cos(10t + 3.44^\circ) \text{ V } (\because \text{input was a cosine function})$$

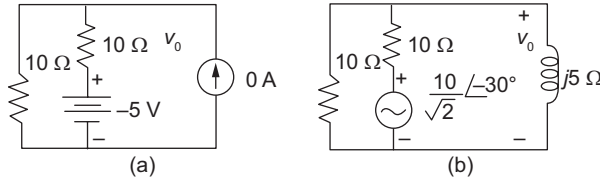


Fig. 10.9-10 Circuits for zero-state response due to current source in Example: 10.9-3

The zero-state response due to voltage source is obtained by adding this solution to transient response term and evaluating the arbitrary constant in the transient response term by using the initial condition for $v_o(t)$ we have already calculated.

$$v_o(t) = Ae^{-20t} + 4.47 \cos(10t + 3.44^\circ)$$

$$v_o(0^+) = 5 \text{ V} \Rightarrow A + 4.47 \cos 3.44^\circ = 5 \Rightarrow A = 0.54$$

$$\therefore v_o(t) = 0.54 e^{-20t} + 4.47 \cos(10t + 3.44^\circ) \text{ A for } t \geq 0^+$$

Step 4 – Get total response and identify various components

The zero-input response = $5 e^{-20t}$ A

The zero-state response due to $v_s(t) = 1.41 e^{-20t} + 2.24 \sin(10t + 108.44^\circ)$ V

The zero-state response due to $i_s(t) = 0.54 e^{-20t} + 4.47 \cos(10t + 3.44^\circ)$ A

$\therefore$ The total response and its components are

$$v_o(t) = 6.95 e^{-20t} + 2.24 \sin(10t + 108.44^\circ) + 4.47 \cos(10t + 3.44^\circ) \text{ V for } t \geq 0^+$$

$$= 6.95 e^{-20t} + 4.45 \cos(10t + 32.52^\circ) \text{ V for } t \geq 0^+$$

Transient component = $3.53 e^{-20t}$ V; steady - state component = $4.45 \cos(10t + 32.52^\circ)$ V

Zero-input response = $5 e^{-20t}$ V; Zero - state response = $1.95 e^{-20t} + 4.45 \cos(10t + 32.52^\circ)$ V

Example: 10.9-4

Find expressions for inductor current and voltage in the circuit in Fig. 10.9-11 (a) for $t \geq 0^+$ and plot them in the range $0^+ \leq t \leq 1$ s.

10.56 | First-Order RL Circuits

Solution

The voltage source in this circuit is specified as $15 u(0.5-t)$. $u(x)$ is 0 for $x \leq 0^-$, 1 for $x \geq 0^+$ and undefined at $x = 0$. Therefore, $u(0.5-t)$ is a time function which is 0 for $t \geq 0.5^+$, 1 for $t \leq 0.5^-$ and is undefined at $t = 0.5$. This waveform is shown in Fig. 10.9-11 (b). Physically, it implies that 15V DC source was connected to the circuit at infinite past and it is removed and instead a short-circuit is put across the circuit at $t = 0.5$ s.

Since the 15V source was connected long back the circuit had enough time to reach DC steady-state by the time t becomes zero. Therefore, the initial inductor current at $t = 0^-$ can be obtained by solving the resistive circuit with inductor replaced by short-circuit. This will be 0.5 A. The relevant circuit appears in Fig. 10.9-12 (a).

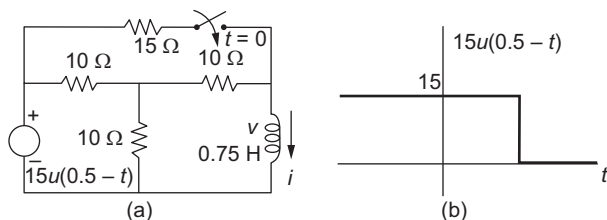


Fig. 10.9-11 Circuit for Example: 10.9-4

The switch closes and puts 15Ω resistance into the circuit at $t = 0$. A new steady-state will be established in the circuit if there is no further change in the circuit. But the circuit cannot anticipate that a structural change or a change in source functions is going to take place in future and modify its response in the present based on such anticipation. Therefore, the evolution of circuit variables will be towards the expected steady-state commensurate with the current nature and values of source functions. In the present example, we know that the circuit will not be allowed to reach steady-state since the input is going to go down to zero at 0.5 s and the circuit will start on a new transient at that instant. But that does not prevent us from asking the question – what would have been the final value had the circuit been allowed to reach there? – Because the circuit is going to move only along that waveform up to 0.5s. This expected final value, had the circuit been allowed to proceed to steady-state, can be worked out by replacing inductor by short-circuit. This circuit along with solution is shown in Fig. 10.9-12 (b).

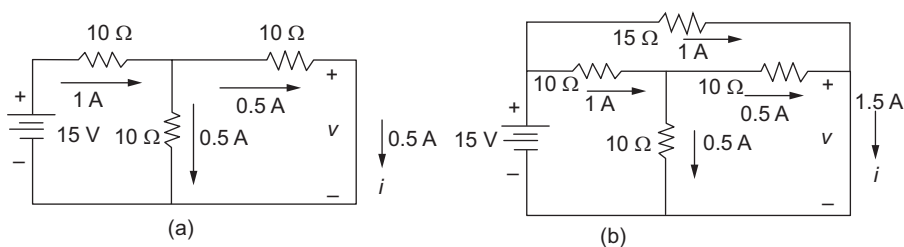


Fig. 10.9-12 Initial and final value evaluation for the circuit in Example: 10.9-4

The time constant relevant for $t \geq 0^+$ can be found by setting the voltage source value to zero with the 15Ω already connected in place in Fig. 10.9-11 (a). The equivalent resistance across inductor will be 7.5Ω and τ will be 0.1s.

Now, the expression for i in the time range $0^+ \leq t \leq 0.5^-$ can be found as below.

$$\begin{aligned}
 i(t) &= Ae^{-10t} + 1.5 \text{ with } i(0^+) = 0.5 \text{ A} \\
 \therefore A + 1.5 &= 0.5 \text{ and } A = -1 \\
 \therefore i(t) &= -e^{-10t} + 1.5 \text{ A for } 0^+ \leq t \leq 0.5^- \\
 \therefore v(t) &= 0.75 \frac{di}{dt} = 7.5e^{-10t} \text{ V for } 0^+ \leq t \leq 0.5^-
 \end{aligned} \tag{10.9-1}$$

At $t = 0.5$ s, the input voltage goes to zero. A new transient in a source-free circuit starts at $t = 0.5^+$. Since there is no impulse voltage involved in the circuit, the inductor current will remain continuous between 0.5^- and 0.5^+ . The value at 0.5^- can be found by evaluating $i(t)$ at that instant using the expression for $i(t)$ in Eqn. 10.9-1. Further, in a source-free circuit there is no forced response term.

$$\begin{aligned}
 i(0.5) &= -e^{-10 \times 0.5} + 1.5 \approx 1.5 \text{ A} \\
 \therefore i(t) &= 1.5e^{-10(t-0.5)} \text{ A for } t \geq 0.5^+ \\
 \therefore v(t) &= 0.75 \frac{di}{dt} = -11.25e^{-10(t-0.5)} \text{ V for } t \geq 0.5^+
 \end{aligned} \tag{10.9-2}$$

The plots of inductor current and voltage across inductor are given in Fig. 10.9-13.

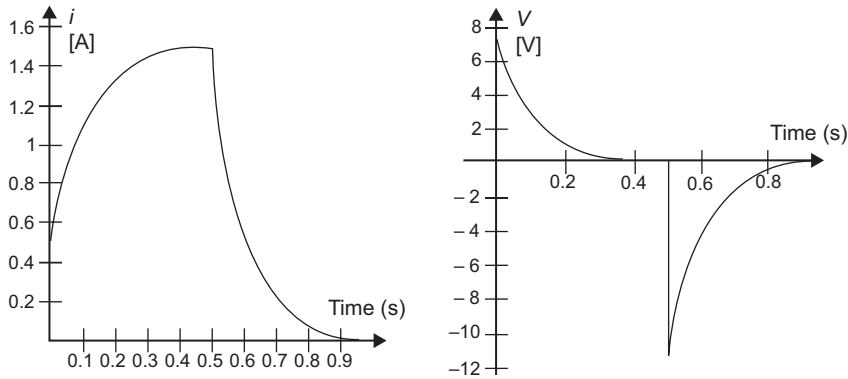


Fig. 10.9-13 Inductor current and voltage in Example: 10.9-4

10.10 SUMMARY

- Circuits containing energy storage elements have memory in time-domain. They are described by linear ordinary differential equations with constant coefficients. We need to know the forcing function from some time instant onwards along with initial conditions in the circuit specified at that instant to solve such differential equations.
- We focussed on series RL circuit in this chapter. RL circuit is described by a first-order linear differential equation. The past history of inductor is contained in a single initial condition specification in an RL circuit.

10.58 | First-Order *RL* Circuits

- The solution of the differential equation describing the inductor current in an *RL* circuit contains two terms – the complementary function and particular integral. Complementary function is the solution of differential equation with zero forcing function. Particular integral is the solution due to the input function and is defined in the domain of input function. The total solution is obtained by adding these two. The complementary function has arbitrary amplitude that should be fixed by ensuring that the *total* solution complies with the specified initial condition.
- The circuit variables in the *RL* circuit will contain two response components – *transient response* (also called *natural response*) and *forced response*. *Natural response* is the way in which the inertia in the circuit reacts to forcing function's command to change. Complementary solution gives the *natural response* and particular integral gives the *forced response* in a circuit.
- The nature of *natural response* of a linear time-invariant circuit is independent of the type or magnitude of forcing function and depends only on circuit parameters and nature of interconnections. *Natural response* in *RL* circuit is exponential of the form $A e^{-t/\tau}$ where $\tau = L/R$ is defined as time constant of the circuit. A is to be fixed for compliance with initial condition.
- The initial current in an *RL* circuit at $t = 0^-$ and $t = 0^+$ are the same if the circuit does not contain impulse sources and it cannot support impulse voltages.
- *Step response* of a circuit is its response when unit step input is applied. In the case of an *RL* circuit, *step response* is a rising exponential, approaching a steady-state value asymptotically as $t \rightarrow \infty$. The *step response* never gets done. However, it may be considered to be over within 5 time constants for practical purposes.
- *Time constant* can be understood as the additional time required from the current instant for the *step response* to reach the final value, assuming that the rate of rise of response is held constant at its current value from that instant onwards.
- *Free response* of an *RL* circuit is its response when input is zero and there is some initial energy trapped in the inductor. It will contain only *natural response* terms. The inductor current in this case falls exponentially towards zero.
- If all the *transient response* terms are of vanishing nature, the only remaining response in the long run will be the *forced response* component. Then, the *forced response* component is termed as the *steady-state response* provided there are constant features describing the forcing function. Three kinds of steady-state are usually studied in circuits – DC steady-state, AC steady-state and periodic steady-state. Inductors can be replaced by short-circuits for DC steady-state analysis. AC steady-state analysis can be carried out using phasor analysis.
- *Impulse response* of a circuit is its response when a unit impulse input is applied to it. *Impulse response* of circuits will contain only *natural response* terms.
- The response due to initial energy and the response due to application of impulse are indistinguishable in an *RL* circuit and hence they can be replaced for each other. An initial current of I_0 in an inductor of value L can be replaced by zero initial condition with a voltage source $LI_0 \delta(t)$ connected in series with the inductor.
- *Step response* and *ramp response* in an *RL* circuit can be obtained by integrating its *impulse response* successively.
- *Zero-input response* of a circuit is its response when there is no input but there is initial energy. It will contain only natural response terms. *Zero-state response* is the response when the circuit is initially at rest (zero initial conditions) and input is applied. It will contain both *natural response* terms and *forced response* terms. The *total response* is given by sum of zero-input response and zero-state response.

- *Forced response* (and hence *steady-state response*) obeys superposition principle with respect to input source functions. But *transient response* and *total response* do not obey superposition principle – neither with respect to initial conditions nor with respect to input source functions. However, *zero-input response* obeys superposition principle with respect to initial conditions and *zero-state response* obeys superposition principle with respect to input source functions.
- *Total response* in single-inductor, multi-resistor circuits can be found with the help of superposition principle and Thevenin's theorem by evaluating *zero-input response* for the entire circuit and *zero-state response* for each source separately.

10.11 PROBLEMS

1. What is the differential equation describing i_L for $t \geq 0^+$ in the circuit in Fig. 10.11-1?

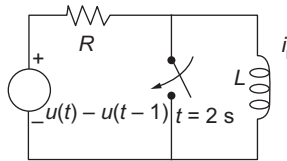


Fig. 10.11-1

2. A series RL circuit with non-zero initial energy is driven by a unit step voltage source. The circuit current is found to reach 75% of its steady-state value in one time constant. Express the initial current in the inductor as a percentage of its steady-state value.
3. A DC voltage source of 1V is switched on to a series RL circuit at $t = 0$. i_L is found to be 50% at $t = \tau$. Was there any initial current in the inductor? If so, what is its magnitude and direction?
4. Find the ratio E/V such that i_L in Fig. 10.11-2 becomes V/R at $t = 0.5 + \tau$ where $\tau = L/R$.

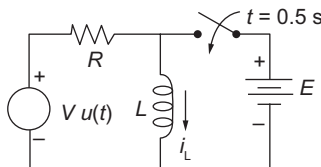


Fig. 10.11-2

5. What is the time required for the energy stored in the inductor in a series RL circuit to reach 90% of its steady-state value in step response?
6. i_L in the circuit in Fig. 10.11-3 is $-2A$ at $t = 0^-$ due to other sources that are not shown, but are switched out at $t = 0$. It is found that $i_L(t) = 0$ for $t \geq 0^+$. What is the value of L ?

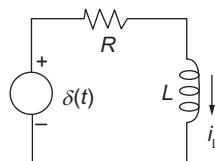


Fig. 10.11-3

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7. i_L in the circuit in Fig. 10.11-4 at $t = 0^-$ is -2 A. It is found that $i_L(t) = 0$ for $t \geq 0^+$. What is the value of time constant of the circuit?

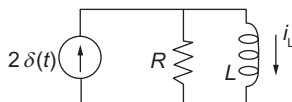


Fig. 10.11-4

8. A series RL circuit with non-zero initial energy is driven by a 1V DC switched source. The inductor current is found to be $(2 - e^{-t})$ A. What are the values of inductance, resistance and initial current?
9. A series RL circuit with non-zero initial energy is driven by a 1V DC switched source. The inductor current is found to be 3A for $t \geq 0^+$. What is the value of initial current and what is its direction relative to the 3A current?
10. An AC voltage source $= V \sin(\omega t)$ is applied to a series RL circuit from $t = 0$. The circuit current is found to be $= 0.7 \sin(\omega t - \pi/3)$ A. Was there any initial current in the inductor? If yes, what is its magnitude and relative direction?
11. A series RL circuit with non-zero initial energy is driven by a 1V DC switched source. The inductor current is found to be 2A. What is the new i_L in the circuit if (i) the initial condition is doubled and (ii) if a 2V DC switched source is applied with no change in initial condition?
12. The inductor current $= 0$ for $t \geq 0^+$ in the circuit in Fig. 10.11-5. Find the values of I_0 and I_S . I_0 is created by other circuit elements that are switched out at $t = 0$. They are not shown in the diagram.

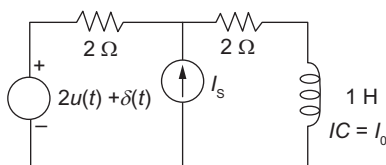


Fig. 10.11-5

13. A series RL circuit is driven by $v_S(t) = \delta(t) - \delta(t - 1)$. Its time constant is 1 s. (i) Starting from impulse response, find the voltage across the resistor in the circuit when driven by the input $v_S(t)$. (ii) Using the result, derive an expression for voltage across the resistor when the circuit is driven by a rectangular pulse of unit amplitude and 1 s duration.
14. Derive expressions for maximum voltage across resistor in series RL circuit when it is driven by $e^{-\alpha t} u(t)$ with $\alpha \in 1/\tau$ where τ is the time constant of the circuit. Also, find an expression for the time instant at which this maximum voltage occurs.
15. An exponential input of $k\delta(t) + 2e^{-2t/\tau} u(t)$ V is applied to a series RL circuit with time constant of τ s. The output for $t \geq 0^+$ is observed to contain only $e^{-2t/\tau}$ waveshape. What is the value of k ?
16. The steady-state voltage across resistor (v_R) in a series RL circuit has an amplitude of 7V when the circuit is driven by a sinusoidal voltage of amplitude 10V and angular frequency ω rad/s. (i) Find the phase angle of v_R with respect to the input sinusoid. (ii) If another sinusoidal voltage of 15V amplitude and 3ω rad/s frequency is applied to the circuit, find the amplitude and phase of v_R under steady-state condition.

17. The desired current in the inductor in a series RL circuit with initial condition as shown in the Fig. 10.11-6 is given by

$$i_L(t) = \begin{cases} 2t & \text{for } t \geq 0^+ \\ 0 & \text{for } t \leq 0 \end{cases}$$

Find the $v_s(t)$ to be applied to the circuit if the initial condition is 1 A as marked in the Fig. 10.11-6. Sketch the required voltage.

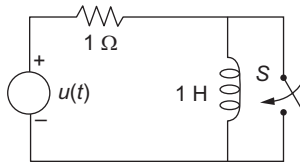


Fig. 10.11-6

18. What is the rise time of inductor current in the step response of circuit in Fig. 10.11-7?

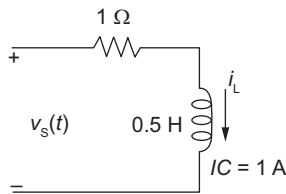


Fig. 10.11-7

19. The switch S_1 in the circuit in Fig. 10.11-8 is closed at $t = 0$ and the switch S_2 is closed at $t = 0.12$ s. Both switches are ideal. (i) Find the current in the inductance as a function of time and plot it. (ii) Find the voltage across inductance and plot it. (iii) Find the voltage across the first resistor and plot it. (iv) What is the time required to attain 90% of final steady-state value of current?

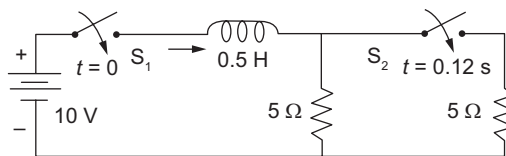


Fig. 10.11-8

20. Initial current at $t = -\infty$ in the inductor in the circuit in Fig. 10.11-9 was zero. Find $i_L(t)$ and $v_L(t)$ for $t \geq 0^+$ and plot them.

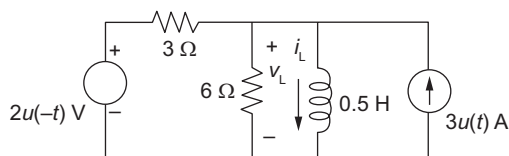


Fig. 10.11-9

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21. The switch S in Fig. 10.11-10 is a two-position switch and starts at position-1 at $t = 0$. It is kept in that position for 10ms and then thrown to position-2. It is kept at 2nd position till the current in the inductor goes to zero. At that instant, it is thrown back to position-1. Then, the whole cycle repeats. (i) Calculate and plot two cycles of $i_L(t)$, $i_{S1}(t)$ and $i_{S2}(t)$. (ii) What is the frequency of switching in the circuit? (iii) What are the average currents in the two sources and inductance? (iv) What are the average power delivered to the 24V source and the average power supplied by 12V source. (v) If the idea was to charge the 24V battery from 12V battery, what is the efficiency of this charger? (vi) Suggest a method to control the average charging current in the 24V battery.

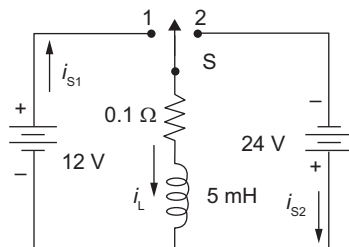


Fig. 10.11-10

22. Repeat the steps in Problem 3 with the circuit in Fig. 10.11-11.

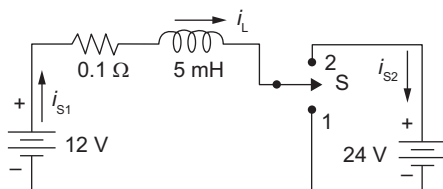


Fig. 10.11-11

23. The switch S in Fig. 10.11-12 starts at position-1 at $t = 0$. It is kept in that position for 10ms and then thrown to position-2. It is kept at 2nd position till the current in the inductor goes to zero. At that instant, it is thrown back to position-1. Then, the whole cycle repeats.

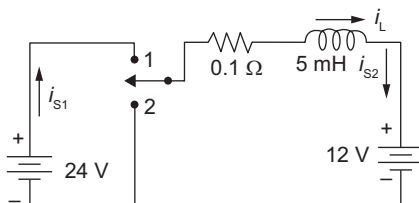
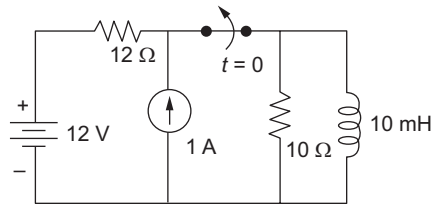
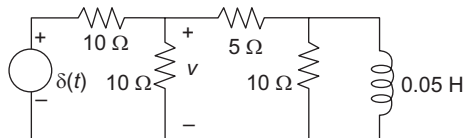


Fig. 10.11-12

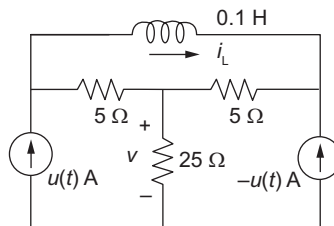
- (i) Calculate and plot two cycles of $i_{S1}(t)$ and $i_{S2}(t)$. (ii) What is the frequency of switching in the circuit? (iii) What are the average currents in the two sources? (iv) What are the average power delivered to the 12V source and the average power supplied by 24V source? (v) If the idea was to charge the 12V battery from 24V battery, what is the efficiency of this charger? (vi) Suggest a method to control the average charging current in the 12V battery.
24. The switch in the circuit in Fig. 10.11-13 was closed for a long time and is opened at $t = 0$. Find and plot the current in inductance and voltage across it as functions of time.

**Fig. 10.11-13**

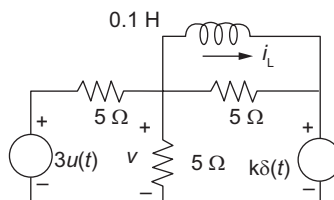
25. Find the impulse response of the voltage variable $v(t)$ in the circuit in Fig. 10.11-14.

**Fig. 10.11-14**

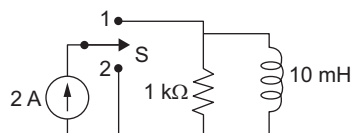
26. (i) Find the zero-input response, zero-state response and total response for $i_L(t)$ and $v(t)$ in the circuit in Fig. 10.11-15. (ii) Obtain $i_L(t)$ and $v(t)$ if the current source on the right side is made $2u(t)$.

**Fig. 10.11-15**

27. What must be the value of k in the circuit in Fig. 10.11-16 if $v(t) = 0$ for $t \geq 0^+$?

**Fig. 10.11-16**

28. The switch S in the circuit in Fig. 10.11-17 operates cyclically at 10kHz spending equal time at both positions in a cycle. Estimate the average power delivered by the current source and the average power dissipated in the resistor. Averages are over switching cycles.

**Fig. 10.11-17**

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29. In Fig. 10.11-18 $V_{ac} = 10\sin(314t)$ V. (i) Find the *steady-state* output voltage waveform for $v_o(t)$ and plot it. (ii) What is the percentage peak-peak ripple with respect to average value in the output voltage? (iii) Repeat (ii) if inductance is changed to 2H. (iv) Justify the following statement – “The steady-state output voltage across resistance in an R - L circuit will be more or less constant at the average value of driving voltage even if the driving voltage has a.c components if the product $\omega L \gg R$ where ω is the angular frequency of AC components.”

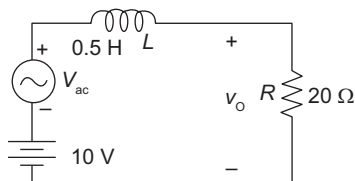


Fig. 10.11-18

30. Let $V_s(t)$ an arbitrary time varying periodic voltage source with a cycle average value of V_{dc} . This means that $V_s(t)$ can be written as $V_{dc} + V_{ac}(t)$ where $V_{ac}(t)$ is a time varying periodic component with equal positive half-cycle and negative half-cycle areas. Let that area be A V-s. This waveform $V_s(t)$ is applied to a series R - L circuit and output voltage is taken across R . Assume that $L/R \gg T$ where T is the period of $V_s(t)$. Show that under periodic steady-state the (i) average value of output voltage is V_{dc} (ii) the peak-to-peak ripple in output voltage $\approx A/\tau$ V where $\tau = L/R$. (iii) calculate the quantities in (i) and (ii) for the three inputs given in Fig. 10.11-19 if τ is 25ms.

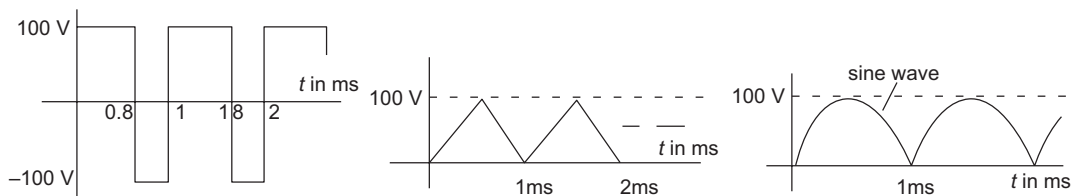


Fig. 10.11-19

31. Two series RL circuits are connected in cascade using a unity gain buffer amplifier as in Fig. 10.11-20. A buffer amplifier is an electronic amplifier that presents infinite resistance at its input and behaves like an ideal voltage at its output. With a unity gain, its output voltage is same as input voltage. Buffer amplifiers are used to interconnect circuits that will interact with each other otherwise. The initial current in the inductor of first stage circuit is 0.5 A and that of second stage is 2 A. Find $v_o(t)$ for $t \geq 0^+$.

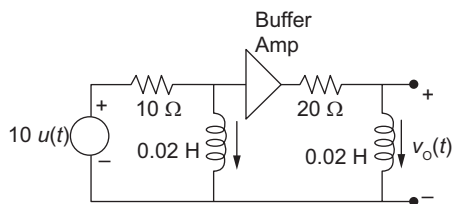


Fig. 10.11-20

Chapter 11

First-Order RC Circuits

CHAPTER OBJECTIVES

- *Impulse, Step and Ramp Response of First-Order RC Circuits*
- *Series RC Circuit with Real Exponential Input*
- *Zero-State Response of Parallel RC Circuit for Sinusoidal Input*
- *The Use of Frequency Response and Linear Distortion*
- *First-Order RC Circuits as Averaging Circuits*
- *Capacitor as a Signal Coupling and Signal Bypassing Element*

INTRODUCTION

A circuit containing a single linear time-invariant capacitor and a linear time-invariant resistor, excited by a voltage source in series or a current source in parallel, will constitute a first-order LTI circuit. We have studied a first-order circuit – namely, the RL Circuit – in great detail and used it to bring out all the important concepts in the time-domain analysis of electrical circuits in the last chapter. Hence, we will be able to move through simple RC circuits rather quickly in this chapter. That does not mean that RC circuits are less important in any sense compared to RL circuits. RL , RC and RLC circuits are basic building blocks of electrical and electronic circuits and are equally important. It is only that we chose to deal with RL circuits first and that in no way makes other dynamic circuits less important.

11.1 RC CIRCUIT EQUATIONS

First-order Series RC Circuit and Parallel RC Circuit are shown in Fig. 11.1-1 with all element variables identified. We choose the voltage across capacitor, $v_C(t)$, as the describing variable in both cases. If $v_C(t)$ is known in the series circuit, the current through capacitor can be obtained by multiplying the first derivative of $v_C(t)$ by C . Then the current through the resistor and voltage across it may be found. Similarly, all other element variables in the Parallel RC Circuit can be found out if $v_C(t)$ in that circuit is known.

11.2 | First-Order RC Circuits

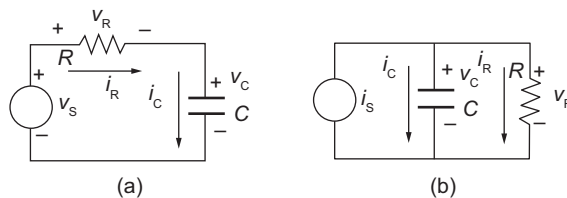


Fig. 11.1-1 1(a) The series RC circuit (b) The parallel RC circuit

The differential equation governing the Series RC Circuit is obtained by applying KCL at the positive terminal of the capacitor along with element equations of resistor and capacitor.

$$\begin{aligned}
 i_R(t) &= i_C(t) \text{ by KCL} \\
 \text{But, } i_R(t) &= \frac{v_s(t) - v_C(t)}{R} \text{ and } i_C(t) = C \frac{dv_C(t)}{dt} \\
 \therefore C \frac{dv_C(t)}{dt} &= \frac{v_s(t) - v_C(t)}{R} \text{ for all } t \\
 \text{i.e., } \frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) &= \frac{1}{RC} v_s(t) \text{ for all } t
 \end{aligned}$$

The differential equation for the Parallel RC Circuit is obtained by applying KCL at the positive terminal of the capacitor.

$$\begin{aligned}
 i_s(t) &= i_R(t) + i_C(t) \text{ by KCL} \\
 \text{But, } i_R(t) &= \frac{v_C(t)}{R} \text{ and } i_C(t) = C \frac{dv_C(t)}{dt} \\
 \therefore C \frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} &= i_s(t) \text{ for all } t \\
 \text{i.e., } \frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) &= \frac{i_s(t)}{C} \text{ for all } t
 \end{aligned}$$

The describing differential equation is a linear first-order equation with constant coefficients in both the cases. We expect the solution to contain a natural response term of $e^{-\alpha t}$ type with $\alpha = 1/RC$. Comparing with the RL circuit equations, we can identify the time constant of RC circuit as RC seconds.

11.2 ZERO-INPUT RESPONSE OF RC CIRCUIT

The differential equations derived above can be used to solve for $v_C(t)$ for all t provided the input source function is known for all t . However, we do not know the input source function for all t . The input source function is usually known only for $t > 0$. It may contain a discontinuity at $t = 0$ too. The input source function is generally unknown for $t < 0$.

Therefore, the effect of all the unknown currents which went through the capacitor from infinite past to $t = 0^-$ is given in a condensed manner in the form of an initial value specification for $v_C(t)$ at $t = 0^-$.

Let us denote this value as V_o . Now we can obtain the total response of the circuit by adding the two response components – zero-input response and zero-state response.

Zero-input response is the response for $t \geq 0^+$ when the input is held at zero from $t = 0$ onwards. *Zero-state response* is the response for $t \geq 0^+$ when the input is held zero from infinite past to $t = 0^-$ (or equivalently, initial condition at $t = 0^-$ is 0) and then a specified input source function is applied from $t = 0$ onwards.

Zero-input response is the same as the so-called *source-free response*. The describing differential equation in this situation is

$$\frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = 0 \text{ for all } t \geq 0^+ \text{ with } v_C(0^-) = V_o \quad (11.2-1)$$

Physically this amounts to connecting a charged capacitor to a resistor to form a closed loop at $t = 0$. The initial voltage across the capacitor appears across the resistor from $t = 0$ onwards. The resistor demands a current of $v_C(t)/R$ and this current flow is of suitable polarity to discharge the capacitor. As the capacitor discharges more and more, its voltage comes down, resulting in the discharge current also going down. Thus the capacitor keeps discharging; but at slower and slower rates as time increases. The discharge current divided by capacitance value gives rate of decrease of voltage across the capacitor. Hence, the initial rate of decrease of voltage is V_o/τ V/s, where τ is the time constant ($=RC$) of the circuit. If the initial rate of decrease were maintained throughout, the voltage across capacitor would have gone to zero in one time constant.

There was no impulse current source in this circuit and hence the voltage across capacitor does not change instantaneously at $t = 0$. Hence $v_C(t)$ at $t = 0^+$ is same as $v_C(t)$ at $t = 0^-$, i.e., V_o . Therefore, the expression for $v_C(t)$ is given by the familiar exponential function.

$$v_C(t) = V_o e^{-t/\tau} \text{ V for } t \geq 0^+ \quad (11.2-2)$$

The graph of this function is only too well known and is not repeated here.

Example: 11.2-1

Practical dielectrics employed in capacitors have non-zero volume conductivity and surface conductivity. This results in a *leakage current* that tends to discharge an initially charged capacitor even when it is left open. This phenomenon is called *self-discharge* of a capacitor. A $1000\mu\text{F}$ Electrolytic capacitor is charged to 400 V and is left open from $t = 0$ onwards. The voltage across capacitor is found to be 40 V at $t = 10$ minutes. What will be the power dissipated in the capacitor if it is connected across a 400 V DC source for a long time?

Solution

The capacitor discharges due to its internal leakage current. This leakage current may be modelled approximately by a resistor in parallel with the capacitor. Hence the charged capacitor undergoes the zero-input response of an RC circuit with a time constant of RC , where R is its shunt resistance equivalent to its self-discharge. The value of R is to be found out. Using Eqn. 11.2-2 with $V_o = 400$ V and $v_C(t) = 40$ V,

11.4 | First-Order RC Circuits

$$\begin{aligned}40 &= 400 e^{-600/\tau} \\ \therefore \tau &= \frac{600}{\ln 10} \text{ s} = 260.6 \text{ s} \\ \therefore R &= \frac{260.6 \text{ s}}{1000 \mu\text{F}} = 0.26 \text{ M}\Omega\end{aligned}$$

Now, if this capacitor is connected across a 400V DC source, this resistor will draw a current of 1.54 mA of current from the source. Hence, the power dissipated by the capacitor when it is connected across a 400V source will be $400 \text{ V} \times 1.54 \text{ mA} = 0.616 \text{ W}$.

Example: 11.2-2

A capacitor C is initially charged to 500 V and is left open for 120 s. The voltage across the capacitor at the end of this time interval is seen to be 400 V. A resistor of 100 k Ω is connected across this capacitor at 120 s. The voltage across the capacitor is found to reach 100 V in 216.4 s after this connection has been made. Find the value of C and its leakage resistance.

Solution

Let R be the leakage resistance of the capacitor. Then, using Eqn. 11.2-2 with suitable values for V_0 and $v_C(t)$,

$$\begin{aligned}400 &= 500 e^{-120/\tau_1} \Rightarrow \tau_1 = \frac{120}{\ln 1.25} = 537.8 \text{ s and} \\ 100 &= 400 e^{-216.4/\tau_2} \Rightarrow \tau_2 = \frac{216.4}{\ln 4} = 156.1 \text{ s} \\ \text{where } \tau_1 &= RC \text{ and } \tau_2 = [R // 100 \text{ k}\Omega]C \\ \therefore \frac{\tau_1}{\tau_2} &= \frac{537.8 \text{ s}}{156.1 \text{ s}} = 3.445 = \frac{R(R+100)}{R \times 100} = \frac{(R+100)}{100} = 0.01R+1; \text{ with } R \text{ in k}\Omega \\ \therefore R &= 244.5 \text{ k}\Omega \text{ and since } RC = 537.8 \text{ s, } C = \frac{537.8 \text{ s}}{244.5 \text{ k}\Omega} = 2.2 \text{ mF} = 2200 \mu\text{F}\end{aligned}$$

11.3 ZERO-STATE RESPONSE OF RC CIRCUITS FOR VARIOUS INPUTS

We consider the response of Series RC Circuit and Parallel RC Circuit for various input source functions in this section. We know that the total response of any circuit to application of input function is obtained by adding the zero-input response and zero-state response together. Hence we consider only the zero-state response part for various input source functions in this section. We begin with impulse response first.

11.3.1 Impulse Response of First-Order RC Circuits

The Series RC Circuit in Fig. 11.3-1 (a) is excited by a unit impulse voltage source. The capacitor cannot absorb the impulse voltage. Hence the resistor absorbs the impulse voltage and as a result an

impulse current containing $1/R$ Coulombs of charge flows through the circuit. This impulse current flow results in sudden dumping of $1/R$ Coulombs of charge on the capacitor plates thereby changing the capacitor voltage from 0 at $t = 0^-$ to $1/RC$ V at $t = 0^+$. The unit impulse voltage source is a short circuit for $t \geq 0^+$. Therefore, the only effect of impulse voltage application is to change the initial condition of the capacitor instantaneously. The circuit effectively becomes a *source-free* circuit with initial energy for $t \geq 0^+$ and executes its zero-input response. The relevant circuit is shown in of Fig. 11.3-1 (b).

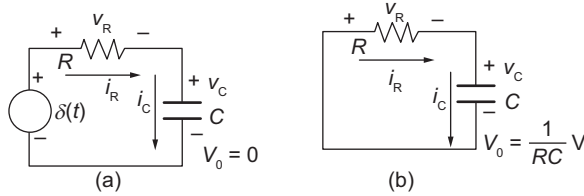


Fig. 11.3-1 Impulse response of series RC circuit

Initial voltage across capacitor is $1/RC$ V and all the voltages and currents in the circuit decay exponentially to zero with a time constant of $\tau = RC$ s.

$$v_C(t) = -v_R(t) = \frac{1}{RC} e^{-t/\tau} \text{ V for } t \geq 0^+$$

$$i_C(t) = i_R(t) = \frac{1}{R^2 C} e^{-t/\tau} \text{ A for } t \geq 0^+$$

We had noticed the equivalence between non-zero initial condition at $t = 0^-$ and the application of impulse at $t = 0$ in our analysis of RL Circuits. We see that it is true in the case of RC circuits too. Specifically, a capacitor with an initial voltage of V_0 V across it at $t = 0^-$ may be replaced by a capacitor with zero initial voltage and a impulse current source of suitable magnitude (CV_0 Coulombs) and polarity connected across it. This equivalence is shown in Fig. 11.3-2.

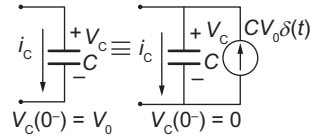


Fig. 11.3-2 Equivalence between non-zero initial voltage and impulse current application in RC circuits

Figure. 11.3-3 shows the application of a unit impulse current to a parallel RC circuit. The resistor cannot support the impulse current. If it were to do so, it would have called for an impulse voltage across it and that will be resisted by the capacitor in parallel. Therefore, all the impulse content goes through the capacitor, changing its voltage by $1/C$ V instantaneously from 0 at $t = 0^-$ to $1/C$ V at $t = 0^+$. The unit impulse current source is effectively an open-circuit after $t = 0^+$. Therefore, the circuit becomes a *source-free* circuit for $t \geq 0^+$ and executes its zero-input response (in Fig. 11.3-3 (b)).

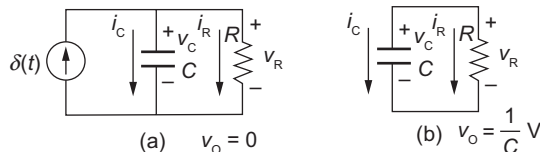


Fig. 11.3-3 Unit impulse response of parallel RC circuit

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Therefore,

$$v_C(t) = v_R(t) = \frac{1}{C} e^{-t/\tau} \text{ V for } t \geq 0^+$$

$$i_R(t) = -i_C(t) = \frac{1}{RC} e^{-t/\tau} \text{ A for } t \geq 0^+$$

Example: 11.3-1

The value of I in the circuit in Fig. 11.3-4 is 3×10^{-6} Coulombs. Find the zero-state response for the current through the 10k resistor in the direction marked in the figure.

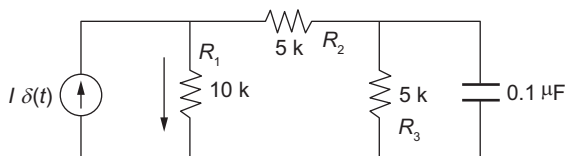


Fig. 11.3-4 Circuit for Example: 11.3-1

Solution

The voltage across the capacitor can, at best, change by a *finite* amount as a result of impulse current flow. This implies that the current through the 5k resistor across the capacitor cannot be an impulse. Therefore, the capacitor effectively shorts the 5k resistor across it as far as the impulse current flow is concerned. Hence the $3 \times 10^{-6} \delta(t)$ gets shared by 10k and the other 5k as per the current division principle in parallel resistors. Thus $2 \times 10^{-6} \delta(t)$ goes through the other 5k resistor and the $0.1 \mu\text{F}$ capacitor. This results in sudden dumping of $2 \mu\text{C}$ of charge across the capacitor, raising its voltage to 20 V at $t = 0^+$. The current source goes open for $t = 0^+$ and the circuit executes its zero-input response.

Time constant of the circuit can be found by obtaining the equivalent resistance connected across the capacitor. The equivalent resistance is $5\text{k}/15\text{k} = 3.75\text{k}$. Therefore, $\tau = 0.375 \text{ ms}$. The discharge current from the capacitor at 0^+ is $20 \text{ V}/3.75 \text{ k}\Omega = 5.333 \text{ mA}$ and $20 \text{ V}/15 \text{ k}\Omega = 1.333 \text{ mA}$ of it goes through the 10k in the direction marked at that instant. Therefore, the zero-state response of this current for $I\delta(t)$ current excitation at input of the circuit = $1.333 e^{-2666.67 t} \text{ mA}$.

11.3.2 Step Response of First-Order RC Circuits

Step Response is understood to be the zero-state response of the circuit when unit step input is applied to it. Hence the initial voltage across the capacitor is zero. It takes an impulse current flow through a capacitor to change its voltage by a non-zero finite amount instantaneously. Since there is no such impulse current flow in the present instance, the voltage across the capacitor remains zero at $t = 0^+$ too.

Consider the Series RC Circuit first. The applied voltage at $t = 0^+$ is 1 V and the voltage across the capacitor is constrained to remain at zero at that instant. And the circuit has to obey Kirchhoff's Voltage Law at that instant. This implies that the voltage across the resistor at that instant has to be 1 V and that a current of $1/R \text{ A}$ has to flow through the circuit at that instant. Since the rate of growth of a capacitor voltage is given by the charging current divided by capacitance value, the rate of change of $v_C(t)$ at $t = 0^+$ will be $1/RC \text{ V/s}$. As the capacitor voltage grows, the voltage available across the resistor decreases;

thereby bringing down the charging current in the circuit. Hence the capacitor keeps charging up with progressively decreasing rate. This is a typical first-order process. The capacitor voltage tends to reach 1 V as $t \rightarrow \infty$ and correspondingly the current through the circuit tends to go to zero.

The detailed solution may be worked out by either '*complementary solution plus particular integral*' format or '*zero-input response plus zero-state response*' format. But we can do better than that. We have already worked out the impulse response of the Series RC Circuit in this section. And we remember that for a *lumped linear time-invariant* circuit, the zero-state response gets integrated when the input source function gets integrated. Unit step function is the integral of Unit impulse function. Therefore, *step response* must be the integral of *impulse response*.

Therefore,

$$\text{Impulse Response, } v_C(t) = \frac{1}{RC} e^{-t/\tau} \text{ V for } t \geq 0^+, \text{ where } \tau = RC$$

$$\therefore \text{ Step Response, } v_C(t) = \int_{0^+}^t \frac{1}{RC} e^{-t'/\tau} dt' = (1 - e^{-t/\tau}) \text{ V for } t \geq 0^+$$

$$\text{and } v_R(t) = 1 - v_C(t) = e^{-t/\tau} \text{ V for } t \geq 0^+$$

$$i_R(t) = i_C(t) = \frac{v_R(t)}{R} = \frac{1}{R} e^{-t/\tau} \text{ A for } t \geq 0^+$$

Now consider the Parallel RC Circuit excited by a unit step current source. The voltage across the capacitor at $t = 0^+$ remains at zero. Therefore all of the source current, *i.e.*, 1 A has to flow through the capacitor. This results in charging up of capacitor with an initial charging rate of $1/C$ V/s. As the capacitor gets charged, the resistor takes its share of current and consequently the rate of rise of voltage comes down. Now we may write down the circuit solution straightaway – $v_C(t)$ must be a rising exponential tending towards R V, $i_C(t)$ must be a decreasing exponential starting at 1 A and $i_R(t)$ must be a rising exponential moving to 1 A. All of them will have the same time constant of $\tau = RC$ s.

$$\therefore \text{ Step Response, } v_C(t) = R(1 - e^{-t/\tau}) \text{ V for } t \geq 0^+$$

$$i_C(t) = e^{-t/\tau} \text{ A and } i_R(t) = (1 - e^{-t/\tau}) \text{ A for } t \geq 0^+$$

These step response waveforms are plotted in Fig. 11.3-5.

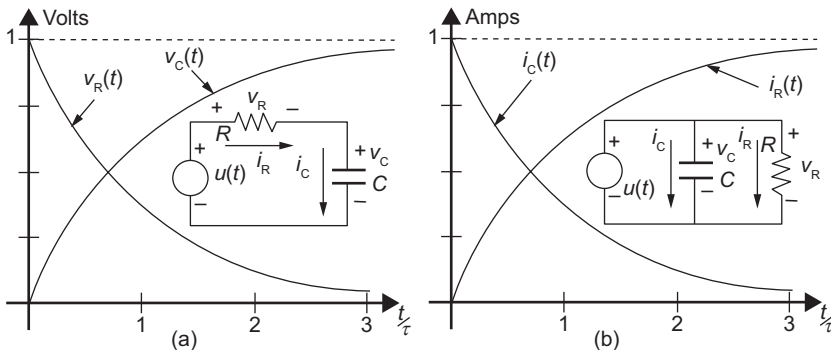


Fig. 11.3-5 Unit step response of RC circuits (a) Series RC circuit (b) parallel RC circuit

11.8 | First-Order RC Circuits

The zero-state responses in both cases contain a transient term (exponential in nature) and a steady-state response term (constant in nature). We had termed the steady-state response term as the *DC steady-state term* earlier. Since the current in a capacitor is proportional to the rate of change of its voltage, the only value of current such that both voltage and current in a capacitor remain constant in time is *zero*. It can have any constant voltage across it; but its current is constrained to be zero under DC steady-state.

A capacitor may be replaced by an open-circuit for the analysis of DC steady-state response.

The DC steady-state current in a Series RC Circuit is zero. This implies that there is energy flow from the DC source only during the charging process. After the capacitor has charged up fully there is no energy drain from the source. Consider the charging of a capacitor to V Volts in a Series RC Circuit using a DC source of V Volts.

$$\text{Total energy delivered by the source} = \int_{0^+}^{\infty} V \times \frac{V}{R} e^{-t/\tau} dt = CV^2 \text{ Joules}$$

$$\text{Total dissipated by the resistor} = \int_{0^+}^{\infty} R [i_R(t)]^2 dt = \int_{0^+}^{\infty} R \left[\frac{V}{R} e^{-t/\tau} \right]^2 dt = \frac{CV^2}{2} \text{ Joules}$$

Thus, the energy spent in charging up a capacitor to V is CV^2 Joules – half of which appears in the capacitor as electrostatic energy storage and the remaining half gets dissipated in the charging resistor. *This conclusion is independent of the value of resistance of the resistor.* However, we should not stretch it to the case where $R = 0$! That is when all those *parasitic elements* that we neglected in modelling a real physical electrical device as a mathematical capacitance will start having their say in the matter.

11.3.3 Ramp Response of Series RC Circuit

Here too we employ the integration method to arrive at the zero-state response to unit ramp input from the zero-state response to unit step input since unit ramp waveform is the integral of unit step waveform.

$$\text{Ramp response, } v_C(t) = \int_{0^+}^t (1 - e^{-t'/\tau}) dt' = t + \tau e^{-t'/\tau} \Big|_{0^+}^t = t - \tau(1 - e^{-t/\tau}) \text{ V for } t \geq 0^+$$

Ramp waveform finds application in many contexts in electronics, instrumentation and signal processing. It is used as the internal time-base waveform in oscilloscopes. It is used in timing and counting applications too. Some dedicated electronic circuitry generates this ramp waveform and the generated waveform is conducted to the application circuit by means of a two-wire connection. This two-wire connection can often be modelled approximately by a Series RC Circuit where the resistor is contributed by the output resistance of the generator circuit (*i.e.*, the resistor that appears in the Thevenin's equivalent of the generator circuit output) and the capacitor is contributed by the input capacitance of the application circuit. The ramp waveform gets modified in this process of transmission from the output of generator circuit to the input of the application circuit. We observe

from the above expression for ramp response of Series *RC* Circuit that the ramp waveform suffers two modifications – in the initial portion, *i.e.*, for $t \ll \tau$, the output fails to follow the linear variation of input and is rather prominently an exponential waveform. For $t \gg \tau$ the output follows the input; but with a constant difference of τ V. Hence, we may conclude that a Series *RC* Circuit will be able to transmit a ramp waveform more or less faithfully if the waveform duration is significantly larger than the time constant of the circuit. The waveforms in Fig. 11.3-6 show this clearly.

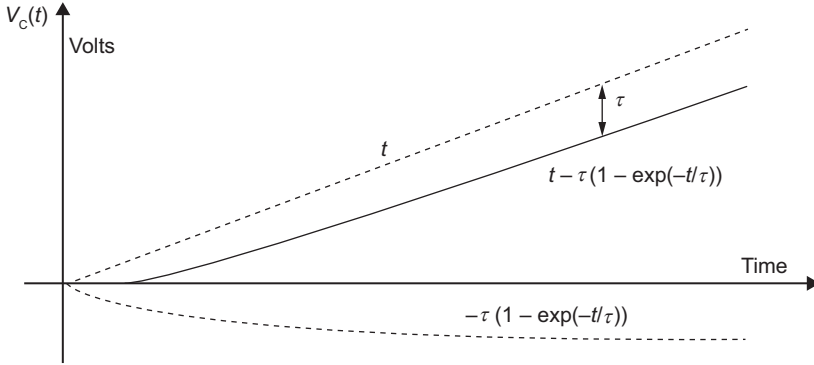


Fig. 11.3-6 Unit ramp response of series *RC* circuit

11.3.4 Series *RC* Circuit with Real Exponential Input

Consider a Series *RC* Circuit excited at the input by a real exponential voltage source of the form $v_s(t) = e^{-\sigma t} u(t)$ V. The zero-state response under real exponential excitation in the case of a Series *RL* circuit was described in Section 10.8 in Chapter 10. The reader is referred to Eqn. 10.8-5 in Chapter 10. The zero-state response of capacitor voltage in the present case is written down by analogy from that equation as

$$v_C(t) = \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ V for } t \geq 0^+ \text{ where } \alpha = 1/RC \text{ and } \alpha \neq \sigma. \quad (11.3-1)$$

This may also be written as

$$v_C(t) = \left(\frac{1}{1 - \tau/\tau_e} \right) (e^{-(\tau/\tau_e)t_n} - e^{-t_n}) \text{ V for } t \geq 0^+; \tau = RC, \tau_e = 1/\sigma, t_n = t/\tau \text{ and } \tau \neq \tau_e$$

in terms of circuit time constant and excitation time constant.

The input source functions and corresponding capacitor voltage waveforms are shown in Fig. 11.3-7. Time is normalised to the base of circuit time constant in this figure.

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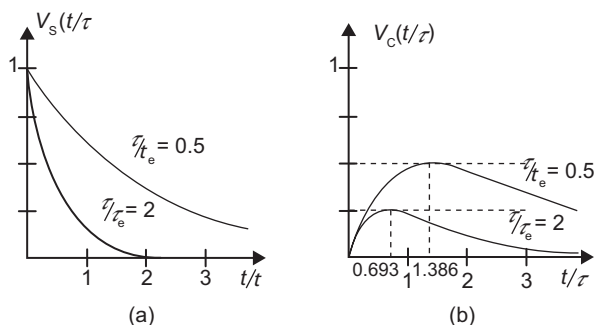


Fig. 11.3-7 Zero-state response of RC circuit for exponential input
(a) Input wave (b) Capacitor voltage

The case with $\alpha = \sigma$ was avoided in Chapter 10 with a promise that it will be taken up later in a different context. We do not avoid it any longer. The case with $\alpha = \sigma$ is taken up now as a limiting case of Eqn. 11.3-1 as $\alpha \rightarrow \sigma$.

$$\begin{aligned}
 v_C(t) &= \left(\frac{\alpha}{\alpha - \sigma} \right) (e^{-\sigma t} - e^{-\alpha t}) \text{ for } t \geq 0^+ \\
 &= \left(\frac{\alpha}{\alpha - \sigma} \right) \left[\left(1 - \sigma t + \frac{(\sigma t)^2}{2!} - \frac{(\sigma t)^3}{3!} + \frac{(\sigma t)^4}{4!} - \dots \right) - \left(1 - \alpha t + \frac{(\alpha t)^2}{2!} - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!} - \dots \right) \right] \\
 &= \left(\frac{\alpha}{\alpha - \sigma} \right) \left[(\alpha - \sigma)t - \frac{(\alpha^2 - \sigma^2)}{2!} t^2 + \frac{(\alpha^3 - \sigma^3)}{3!} t^3 - \frac{(\alpha^4 - \sigma^4)}{4!} t^4 + \dots \right] \\
 &= (\alpha t) \left[1 + \sum_{i=2}^{\infty} (-1)^{i-1} \frac{(\alpha^i - \sigma^i)}{(\alpha - \frac{\alpha}{\alpha - \sigma}) i!} t^{i-1} \right]
 \end{aligned}$$

It may be verified by long division that,

$$(\alpha^i - \sigma^i) = (\alpha - \sigma) \underbrace{(\alpha^{i-1} + \alpha^{i-2}\sigma + \alpha^{i-3}\sigma^2 + \dots + \alpha\sigma^{i-2} + \sigma^{i-1})}_{i \text{ terms}}$$

$$\therefore \text{As } \alpha \rightarrow \sigma, \frac{(\alpha^i - \sigma^i)}{(\alpha - \sigma)} \rightarrow i \times \alpha^{i-1} \text{ and } v_C(t) \rightarrow (\alpha t) \left[1 + \sum_{i=2}^{\infty} (-1)^{i-1} \frac{i \times \alpha^{i-1}}{i!} t^{i-1} \right]$$

$$\text{i.e., } v_C(t) \rightarrow (\alpha t) \left[1 - \alpha t + \frac{(\alpha t)^2}{2!} - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!} - \dots \right]$$

$$\therefore v_C(t) = \alpha t e^{-\alpha t} = \left(\frac{t}{\tau} \right) e^{-\left(\frac{t}{\tau} \right)} \text{ V for } t \geq 0^+$$

The waveshape of output voltage will be the same as in Fig. 11.3-7. The output in this case will reach a maximum of $1/e = 0.3678$ V at $t = \tau$ s.

Example: 11.3-2

Two first-order Series RC Circuits are cascaded using a unity gain buffer amplifier as shown in Fig. 11.3-8. Find the output $v_o(t)$ as a function of time.

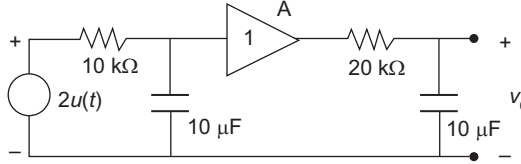


Fig. 11.3-8 Circuit for Example: 11.3-2

Solution

The unity gain buffer amplifier in between prevents any interaction between the two RC circuits. This implies that the response of the first RC stage is independent of the presence of the second stage. The first stage produces a voltage across its capacitor that is accepted by second stage as its input source function as if it is coming from an ideal independent voltage source. Let $v_1(t)$ be the response voltage at the terminals of the first capacitor. Then $v_1(t)$ is twice the zero-state response to unit step input (*i.e.*, step response). The time constant of first stage is $10\text{k}\Omega \times 10\mu\text{F} = 100\text{ms} = 0.1\text{ s}$.

$$\therefore v_1(t) = 2(1 - e^{-10t}) \text{ V for } t \geq 0^+$$

This voltage is the input to the second stage since the gain of the buffer amplifier is unity. This input may be treated as the sum of two inputs $-2u(t)$ and $-2e^{-10t}u(t)$. Zero-state response of a lumped linear time-invariant circuit obeys superposition principle and hence the responses to these inputs may be found out individually and be superposed to get the desired response.

The time constant of the second stage is $20\text{k}\Omega \times 10\mu\text{F} = 200\text{ms} = 0.2\text{ s}$. The component contributed to $v_o(t)$ by $2u(t)$ is $2(1 - e^{-5t})\text{ V}$. The contribution to $v_o(t)$ by $-2e^{-10t}u(t)$ is obtained by using Eqn. 11.3-1 with $\alpha = 5$ and $\sigma = 10$. This contribution is $-2 \times -(e^{-10t} - e^{-5t})\text{ V for } t \geq 0^+$.

Therefore, $v_o(t) = 2(1 - e^{-5t}) + 2(e^{-10t} - e^{-5t})\text{ V for } t \geq 0^+$

$$= 2 - 4e^{-5t} + 2e^{-10t} \text{ V for } t \geq 0^+$$

The first stage and second stage output voltage waveforms are shown in Fig. 11.3-9.

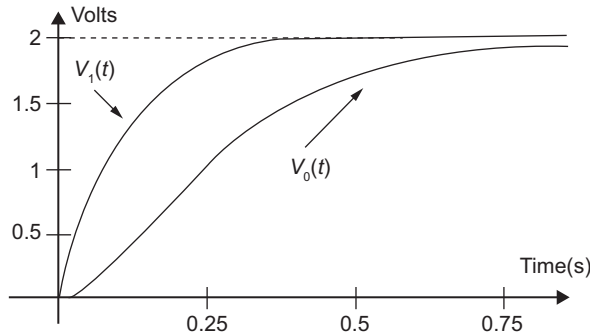


Fig. 11.3-9 Output waveforms across RC stage capacitors in Example: 11.3-2

Example: 11.3-3

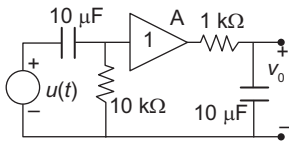


Fig. 11.3-10 Circuit for Example: 11.3-3

Two first-order Series RC Circuits are cascaded non-interactively by employing a unity gain buffer amplifier as shown in Fig. 11.3-10. The voltage across the resistor of the first RC stage is the input to the second stage and the voltage across the capacitor of the second stage is the desired output. Find the step response of the system.

Solution

Let $v_1(t)$ be the voltage across the $10\text{k}\Omega$ resistor in the first stage. We know that the zero-state step response of capacitor voltage in a Series RC Circuit is $(1 - e^{-t/\tau})$, where τ is the time constant of the circuit given by RC product. The voltage across the capacitor and the voltage across resistor will have to add up to 1 V for all $t \leq 0^+$ in step response of a Series RC Circuit. Therefore, the step response of voltage across the $10\text{k}\Omega$ resistor is $[1 - (1 - e^{-t/\tau})] = e^{-t/\tau}$ V. The time constant of the first stage is 0.1 s.

$$\therefore v_1(t) = e^{-10t} \text{ V for } t \geq 0^+$$

This voltage is the input to the second stage. Its time constant is 0.01 s. We use Eqn. 11.3-1 to obtain $v_o(t)$ with $\alpha = 100$ and $\sigma = 10$.

$$\therefore v_o(t) = \left(\frac{100}{100 - 10} \right) (e^{-10t} - e^{-100t}) = 1.111 (e^{-10t} - e^{-100t}) \text{ V for } t \geq 0^+$$

Note that the steady-state value of step response is zero. This is so because the first capacitor effectively opens the circuit for DC under steady-state. All the DC content of the source voltage will be found across the first capacitor in the steady-state.

Example: 11.3-4

Show that the two circuits shown of Fig. 11.3-11 (a) and (b) have the same step response except for a sign change. The operational amplifier may be treated as an ideal one. Compare the currents drawn from the voltage source by the circuits.

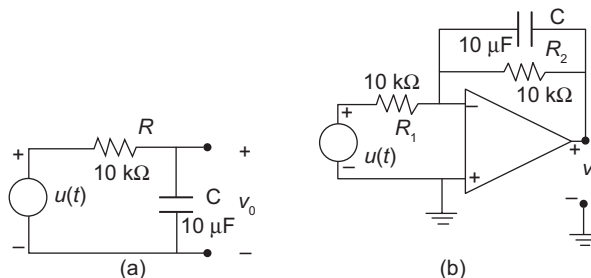


Fig. 11.3-11 Circuits for Example: 11.3-4

Solution

The time constant of the circuit in Fig. 11.3-11 (a) is $100 \text{ ms} = 0.1 \text{ s}$ and its step response is $(1 - e^{-10t}) \text{ V}$ for $t \geq 0^+$.

Consider the circuit in (b). The Opamp is connected with negative feedback and we further assume that the input voltage applied is of such magnitude that the Opamp does not enter voltage saturation at its output. Moreover, we assume that the Opamp has sufficiently large slew rate capability such that it never enters rate-limited operation. With these assumptions, we can analyse the Opamp using its ideal model.

The non-inverting terminal of Opamp is grounded and by *virtual short* principle the inverting input terminal is also *virtually grounded*. Therefore, the current that flows through R_1 is $u(t)/R_1$. Since the current into the input terminals of an ideal Opamp is zero, this current flows into the $R_2//C$ combination connected in the feedback path of the Opamp. The voltage developed across this parallel combination is nothing but a scaled version of the step response of a Parallel RC Circuit with step current excitation. The scaling factor is $1/R_1$. This step response is $R_2(1 - e^{-10t})$ since the time constant involved is 0.1 s . Therefore, the voltage developed across the parallel combination in the feedback path is $(R_2/R_1)(1 - e^{-10t}) \text{ V}$ with its positive polarity at the inverting input of operational amplifier. Since the inverting input is at *virtual ground*, the voltage of output terminal with respect to ground (reference point) is the negative of this voltage.

$$\therefore v_o(t) = -\frac{R_2}{R_1}(1 - e^{-10t}) = -(1 - e^{-10t}) \text{ V for } t \geq 0^+ \text{ since } R_1 = R_2 = 10k\Omega$$

Therefore, the two circuits in Fig. 11.3-11 have the same step response (and hence same dynamic behaviour) except for a change in sign.

The voltage across R in the circuit of Fig. 11.3-11 (a) is $[1 - (1 - e^{-10t})] = e^{-10t} \text{ V}$ and therefore the current drawn from the unit step voltage source by this circuit is $0.1 e^{-10t} \text{ mA}$ for $t \geq 0^+$. But the current drawn by the second circuit is $u(t)/R_1 = 0.1 \text{ mA}$ for $t \geq 0^+$. Thus, the second circuit presents a constant input resistance level to the applied voltage source whereas the first circuit presents a time-varying input resistance level to the source. If the voltage source is not an ideal one, *i.e.*, if it has a non-zero internal resistance, the time constant of circuit in Fig. 11.3-11 (a) will change and hence the shape of its step response will change. However, the shape of step response will not change in the case of circuit Fig. 11.3-11 (b); but the final magnitude will change due to change in the ratio (R_2/R_1) .

11.3.5 Zero-State Response of Parallel RC Circuit for Sinusoidal Input

The zero-state response for sinusoidal input for any linear time-invariant circuit can be obtained in three ways.

1. Let the input be $\sin \omega t u(t)$. Obtain the zero-state response of the circuit for a complex exponential input function e^{st} , where s is a complex number. Substitute $s = j\omega$ in the solution and accept the *imaginary part of the solution* as the zero-state response for $\sin \omega t u(t)$.
2. Express $\sin(\omega t)$ as $[(e^{j\omega t} - e^{-j\omega t})/2j]$ by using Euler's formula, get the zero-state responses for the two exponential functions separately and use superposition principle.
3. Use phasor method to obtain the sinusoidal steady-state response and add a transient response term such that the total response satisfies initial conditions. Initial conditions will be zero valued since we are dealing with zero-state response.

The first two methods were already illustrated in the context of sinusoidal response of RL circuits in Chapter 10. We use the third method here to obtain the zero-state response for the voltage

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appearing across a Parallel RC Circuit excited by a sinusoidal current source with source function of $\sin \omega t$ A.

The circuit in time-domain and phasor-domain are shown in of Fig. 11.3-12 (a) and (b), respectively.

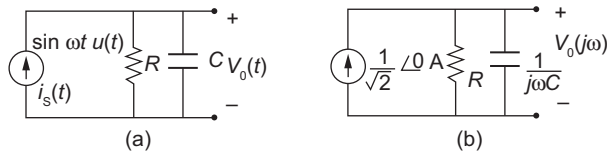


Fig. 11.3-12 Parallel RC circuit with sinusoidal excitation and its phasor equivalent

$$\begin{aligned}
 V_o(j\omega) &= R // \left(\frac{1}{j\omega C} \right) \times \frac{1}{\sqrt{2}} \angle 0 = \frac{R}{1 + j\omega RC} \times \frac{1}{\sqrt{2}} \angle 0 = \frac{R}{1 + jk} \times \frac{1}{\sqrt{2}} \angle 0; k = \omega RC = \omega\tau \\
 &= \frac{R}{\sqrt{1 + k^2}} \angle -\phi \times \frac{1}{\sqrt{2}} \angle 0; \phi = \tan^{-1} k
 \end{aligned}$$

Going back to time-domain, we get the steady-state component of voltage across the Parallel RC Circuit as

$$\frac{R}{\sqrt{1 + k^2}} \sin(\omega t - \tan^{-1} k) \text{ with } k = \omega\tau.$$

Now we add a transient component of known form $Ae^{-t/\tau}$ and evaluate A such that the total solution is zero at $t = 0^+$. We get,

$$A = \frac{R}{\sqrt{1 + k^2}} \sin(\tan^{-1} k)$$

Since $\tan^{-1} k$ lies in first quadrant, $\sin(\tan^{-1} k) = \frac{k}{\sqrt{1 + k^2}}$

$$\therefore A = \frac{R}{\sqrt{1 + k^2}} \frac{k}{\sqrt{1 + k^2}} \text{ and}$$

$$v_o(t) = \frac{R}{\sqrt{1 + k^2}} \left(\sin(\omega t - \tan^{-1} k) + \frac{k}{\sqrt{1 + k^2}} e^{-t/\tau} \right) \text{ V for } t \geq 0^+$$

We normalise the time variable using the circuit time constant as the base and the output voltage by using the value of R as the base value and obtain the following expression for normalised voltage $v_{on}(t)$ as a function of normalised time t_n .

$$v_{on}(t) = \frac{1}{\sqrt{1 + k^2}} \left(\sin(kt_n - \tan^{-1} k) + \frac{k}{\sqrt{1 + k^2}} e^{-t_n} \right) \text{ V for } t_n \geq 0^+ \quad (11.3-2)$$

$$\text{where } v_{on}(t) = \frac{v_o(t)}{R} \text{ and } t_n = \frac{t}{\tau}$$

This waveform for a case with $k = 4$ is shown in Fig. 11.3-13. We had noted under a similar context (Section 10.8 in Chapter 10 on RL Circuits) that the number k can be interpreted as a comparison between the characteristic time, *i.e.*, the period of the applied input and the characteristic time of the circuit, *i.e.*, its time constant. k can be expressed as $2\pi(\tau/T)$, where T is the period of input. The value of T is indicative of the rate of change involved in the waveform, *i.e.*, the *speed* of the waveform and time-constant is a measure of inertia in the system. Therefore, an input sinusoid is too fast for a circuit to follow, if its T is smaller than the time constant τ of the circuit. Similarly, if input sinusoid has a T value much larger than time constant of the circuit, the circuit will perceive it as a very slow waveform and will respond almost the same way it does to DC input. These aspects are clearly brought out in the expression in Eqn. 11.3-2.

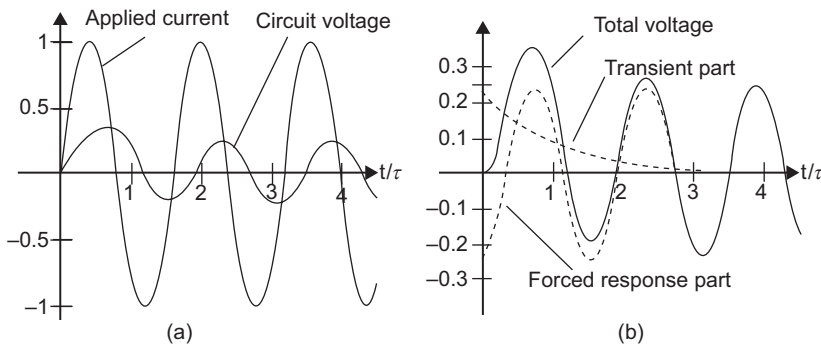


Fig. 11.3-13 Unit sinusoidal response (normalised) of a parallel RC circuit with $k = 4$

We make the following observations on the sinusoidal steady-state response of Parallel RC Circuit with *current source excitation* from Eqn. 11.3-2:

- The circuit voltage under sinusoidal steady-state response is a sinusoid at the same angular frequency ω rad/s as that of input current sinusoid.
- The circuit voltage initially is a mixture of an exponentially decaying unidirectional transient component along with the steady-state sinusoidal component. This unidirectional transient imparts an offset to the circuit voltage during the initial period.
- The circuit voltage at its first peak can go close to twice its steady-state amplitude in the case of circuits with $\omega\tau \gg 1$ due to this offset.
- The amplitude of sinusoidal steady-state response is always less than the corresponding amplitude when DC input is applied. This is due to the capacitive inertia of the circuit. When input is a current, capacitor in a circuit behaves as electrical inertia, and, when input is a voltage, inductance in a circuit behaves as electrical inertia. The amplitude depends on the product $\omega\tau$ and decreases monotonically with the $\omega\tau$ product for fixed input amplitude.
- The response sinusoid (circuit voltage) *lags* behind the input sinusoid (applied current) under steady-state conditions by a phase angle that increases monotonically with the product $\omega\tau$.
- The frequency at which the circuit gain becomes $1/\sqrt{2}$ times that of DC gain is termed as *cut-off frequency* and since this takes place as we go up in frequency it is called *upper cut-off frequency*. Upper cut-off frequency of Parallel RC Circuit is at $\omega = 1/\tau$ rad/s. The phase at this frequency will be -45° .
- Circuit voltage amplitude becomes very small at high frequencies ($\omega\tau \gg 1$) and the voltage lags the input current by $\approx 90^\circ$ at such frequencies.

11.4 PERIODIC STEADY-STATE IN A SERIES RC CIRCUIT

We address the issue of finding the zero-state response to *repetitive input* in RC circuits in this section and consider a specific example for doing so. Refer to Fig. 11.4-1.

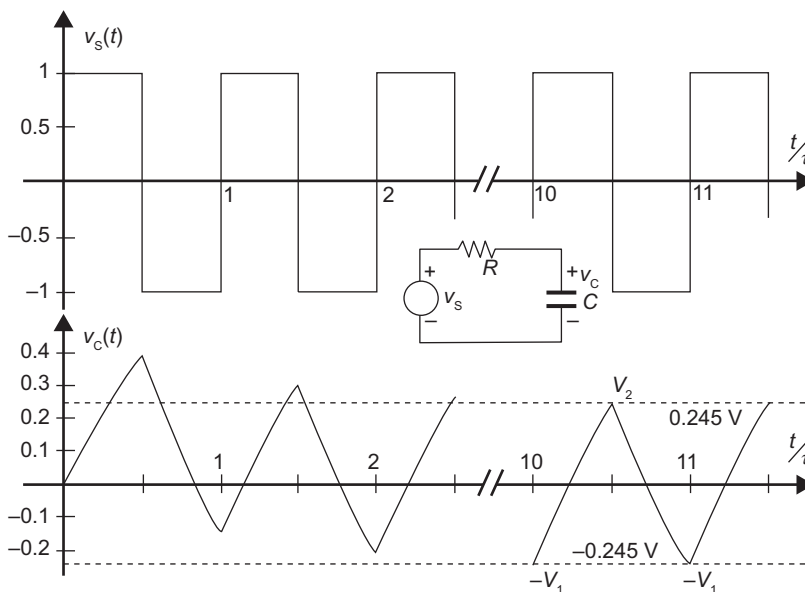


Fig. 11.4-1 Series RC circuit with repetitive square wave input

A symmetric square wave voltage with 1 V amplitude is applied from $t = 0$ to an initially relaxed Series RC Circuit. The period of the square wave is assumed to be equal to the time constant and the time scale is marked in terms of t/τ .

A step response starts at $t = 0$ and takes the output to $(1 - e^{-0.5}) = 0.3935$ V at $t/\tau = 0.5$. Another zero-state step response starts in the negative direction at that point along with a zero-input response corresponding to an initial voltage of 0.3935 V on the capacitor. Thus the output waveform can be expressed as $= 0.3935 e^{-(t-0.5\tau)/\tau} - (1 - e^{-(t-0.5\tau)/\tau})$ for the time range $0.5 \leq t/\tau \leq 1$. This expression may be evaluated at $t/\tau = 1$ to get the initial condition at that point and a new expression valid for $1 \leq t/\tau \leq 1.5$ may be obtained as a superposition of zero-input response and a new zero-state step response. This way the solution may be taken forward.

It may be observed that the values of capacitor voltage at the beginning and at the end of a cycle are not the same in the first few cycles of operation. However, after a few cycles, the circuit settles down to a mode of operation in which the output starts out with a particular value at the beginning of the cycle and returns to the same value at the end of that cycle only to repeat that process again in the next cycle. This means that the output too reaches a *repetitive* pattern after a few initial cycles. When the circuit reaches this kind of *repetitive* operation under the influence of any *repetitive* input, it is said to have reached a *periodic steady-state* with respect to that input. Figure. 11.4-1 shows that the capacitor voltage oscillates between 0.245 V and -0.245 V under periodic steady-state in the present instance, where T – the period of input – has been taken to be equal to τ , the time constant of the circuit.

It is not necessary to start at the beginning and march forward till the circuit reaches the periodic steady-state in order to find out the amplitude under steady-state condition. We can proceed in the following manner to develop an expression for this amplitude.

Let $-V_1$ and $+V_2$ be the negative and positive amplitudes in a cycle as shown in Fig. 11.4-1. Then the circuit would have reached steady-state when the output value at the end of the cycle turns out to be exactly $-V_1$.

$$v_C(t) = -V_1 e^{-t/\tau} + \left(1 - e^{-t/\tau}\right) V \text{ for } 0 \leq t \leq T/2 \text{ with } t \text{ measured from the beginning of cycle}$$

after the circuit has reached periodic steady-state

$$\therefore V_2 = -V_1 e^{-0.5T/\tau} + \left(1 - e^{-0.5T/\tau}\right) V$$

$$v_C(t) = V_2 e^{-(t-0.5T)/\tau} - \left(1 - e^{-(t-0.5T)/\tau}\right) V \text{ for } T/2 \leq t \leq T$$

This expression evaluated with $t=T$ should be equal to $-V_1$ under periodic steady-state.

$$\therefore -V_1 = V_2 e^{-0.5T/\tau} - \left(1 - e^{-0.5T/\tau}\right) V$$

Substituting for V_2 in terms of V_1 and solving for V_1 , we get ,

$$V_1 = \frac{1 - e^{-T/2\tau}}{1 + e^{-T/2\tau}} V \text{ and } V_1 = V_2$$

This expression evaluated with $T/\tau = 1$ gives $V_1 = V_2 = 0.245 V$ for a 1 V amplitude square wave input.

The key to the above derivation was our knowledge of step response of Series RC Circuit. The input could be thought of as a sequence of unit steps and hence the output could be strung together employing step response and zero-input response segments. Square waves and more generalised versions of it (the so-called *rectangular pulse waveforms* that are ‘squarish’ waveforms with unequal half-cycle duration and unequal positive and negative amplitudes) appear very frequently in Pulse Electronics Applications and Digital Electronics. And, Series RC Circuits are routinely used to model the transmission channel that takes such signals from one location to another location in the electronic system. Hence periodic steady-state of Series RC Circuit under rectangular pulse waveforms is of crucial significance in Analog and Digital Electronics. This is the motivation behind this section on periodic steady-state.

The method described above for finding out steady-state amplitudes under repetitive excitation will work only if we can identify the repetitive waveform as a sequence of some well-known shape like a step or ramp or sinusoid. However, in practice, we will be called upon to solve for periodic steady-state even when the period of input is of a complex shape. How do we proceed? The answer lies in *frequency response function* $H(j\omega)$ of the circuit.

11.5 FREQUENCY RESPONSE OF FIRST ORDER RC CIRCUITS

The concept of *sinusoidal steady-state frequency response* was already introduced in Chapters 9 and 10. Essentially, we apply a sinusoidal input of suitable amplitude to a circuit and wait for enough time

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for the transient response to die down. After steady-state is satisfactorily established in the circuit, we measure the amplitude of output and its phase with respect to the input sine wave. We repeat this process for various values of frequency of input. We ensure that the circuit is in steady-state before we measure the output every time. The data so obtained are plotted to show the variation of ratio of output amplitude to input amplitude and phase of steady-state voltage against $k (= \omega\tau)$. Such a pair of plots will constitute what is called the *AC steady-state frequency response plots* for this circuit. The ratio of output amplitude to input amplitude is called the *gain* of the circuit. Its dimension will depend on the nature of input and the output quantities.

The same data can be obtained from the analytical model of the circuit if such a model exists. Consider the Series RC Circuit and its phasor model shown in Fig. 11.5-1.

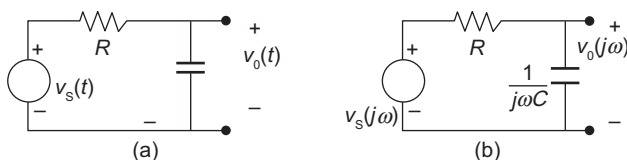


Fig. 11.5-1 Series RC circuit and its phasor model

Frequency response function,

$$H(j\omega) = \frac{V_o(j\omega)}{V_s(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega\tau} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \angle -\tan^{-1}(\omega\tau) \quad (11.5-1)$$

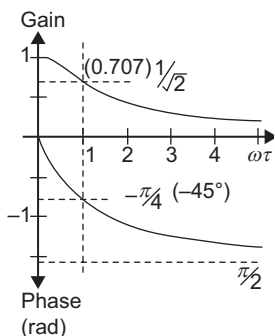


Fig. 11.5-2 Frequency response plots for series RC circuit

The Gain and Phase plots for the circuit are shown in Fig. 11.5-2. The gain goes to 70.7% level at $\omega = 1/\tau$ rad/s and phase delay at that frequency is 45° .

11.5.1 The Use of Frequency Response

Frequency response information helps us to find the *steady-state output* when the input is a *mixture of sinusoids of different frequencies*. A sinusoid with a particular angular frequency is a periodic waveform. But a sum of many sinusoids with arbitrary frequencies need not be periodic. A special case is where the sinusoids in the additive mixture

are of frequencies which are related *harmonically* -i.e., when all the frequencies are integer multiples of some basic frequency value. This is an important practical case.

We have seen in Chapter 9 that a periodic non-sinusoidal waveform can be expanded as a sum of a DC component (which may be zero as a special case) and infinitely many sinusoids (may be finite as a special case) with harmonically related frequencies. Therefore, the periodic steady-state solution in a circuit excited by a non-sinusoidal periodic input can be obtained by using the Fourier Series of the waveform along with the frequency response data for the circuit. That makes frequency response an extremely important mathematical description of a linear circuit.

Before we proceed to employ frequency response to solve circuits excited by sum of sinusoids, let us settle an issue regarding superposition principle. We know that zero-state responses due to multiple sources acting simultaneously can be obtained by superposition. But does it work for steady-state response component too?

Zero-state response contains two components – the transient response part and the steady-state response part. The transient response components, whether from zero-state response or zero-input response, will vanish with time if the circuit is passive and stable. Therefore, superposition principle can be applied on steady-state response components.

Now, if the input contains many sinusoids with different frequencies, the steady-state response component due to each sinusoid may be obtained from frequency response plots and these components may be added up to obtain the complete steady-state response. This procedure is illustrated in the case of a Series RC Circuit with a time constant of 1 s and with an input of $v_s(t) = (\sin t + 0.33 \sin 3t + 0.2 \sin 5t) u(t)$ V. The output is taken across the capacitor.

Consider the first sinusoid that has an angular frequency of 1 rad/s. The gain of the circuit at this frequency is 0.707 and the phase delay is 45° (0.79 rad) (either from Eqn. 11.5-1 or from Fig. 11.5-2 with $\tau = 1$ s). Therefore, the steady-state component due to this sinusoid is $0.707 \sin(t - 0.79)$ V. The second sinusoid of $0.33 \sin 3t$ with an angular frequency of 3 rad/s meets with a gain of 0.3162 and phase delay of 71.57° (1.25 rad). Therefore the steady-state component due to this sinusoid is $0.104 \sin(3t - 1.25)$ V. Similarly, the third sinusoid of $0.2 \sin 5t$ with an angular frequency of 5 rad/s meets with a gain of 0.1961 and phase delay of 78.7° (1.37 rad). Therefore, the steady-state component due to this sinusoid is $0.04 \sin(5t - 1.37)$ V. Therefore, $v_o(t) = 0.707 \sin(t - 0.79) + 0.104 \sin(3t - 1.25) + 0.04 \sin(5t - 1.37)$ V. The input and output waveforms are shown in Fig. 11.5-3.

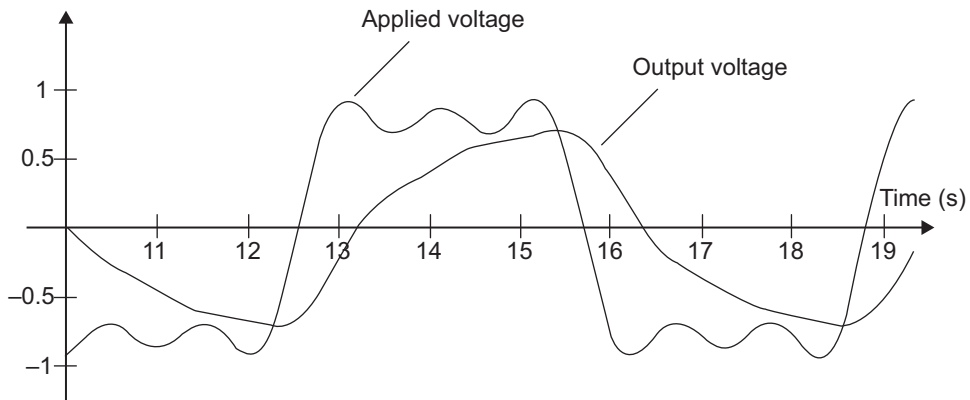


Fig. 11.5-3 Steady-state response of series RC circuit for mixed sinusoidal input

11.5.2 Frequency Response and Linear Distortion

We observe that our Series RC Circuit has meted out differential treatment to various sinusoids in the input mixture. It has shown a clear preference to the sinusoid with lowest frequency (with 1 rad/s) and passed it on with only about 30% loss of amplitude whereas the remaining two sinusoids with 3 rad/s and 5 rad/s frequencies suffered 68.5% and 80% loss of amplitude, respectively. Circuits that preferentially pass low frequency sinusoids to the output and curtail high frequency sinusoids are called *low-pass filters*. Low-pass filters will have a frequency response with a gain function that tapers

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down to zero as frequency goes up. Thus a Series RC Circuit with the output taken across the capacitor is a first-order low-pass filter. It shows a tendency to remove high frequency components in the input.

Further, we observe from Fig. 11.5-3 that the waveshape of output voltage is considerably different from that of input. This is inevitable in a filtering context. After all, some frequency components get removed or attenuated considerably in a filtering process and therefore the output cannot but look different compared to input! When the waveshape of output under steady-state in a circuit is different from the waveshape of input, the circuit is said to have *distorted* the signal. Thus, *distortion* invariably follows *filtering*. When the change in waveshape is the desired outcome, we call it *filtering*; when the change in waveshape is the undesired outcome we call it *distortion*.

This distortion of waveshape arises out of two reasons. Sinusoids at different frequencies meet with different gains in the circuit and therefore the mix of amplitudes, *i.e.*, the relative ratio of amplitudes of various sinusoids, will be different at output and input. In the example we considered, the ratio was 1:0.33:0.2 at input and 1:0.147:0.057 at the output. Waveshape changes due to this change in amplitude mix. Distortion arising out of this mechanism is called *amplitude distortion* and it is due to the gain response part of frequency response.

The second cause of distortion comes from phase response. Each sinusoid suffers a time delay when it goes through the circuit – the time delay is measured between zero crossing of that sinusoid in the input and in the output. Phase delay is equal to time delay multiplied by angular frequency. Thus the 1 rad/s component in the previous example underwent a delay of 0.79 s, the 3 rad/s component suffered a delay of $1.25 \text{ rad/s} / 3 \text{ rad/s} = 0.42 \text{ s}$ and the 5 rad/s component was subjected to a delay of $1.79 \text{ rad/s} / 5 \text{ rad/s} = 0.36 \text{ s}$.

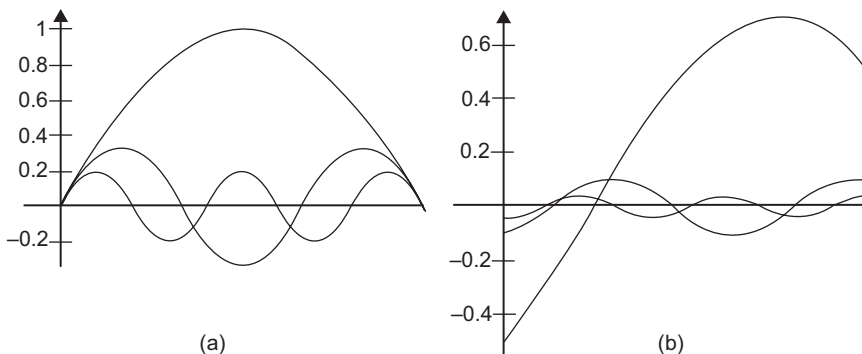


Fig. 11.5-4 Illustrating phase distortion due to dispersion

All the three cross the time-axis simultaneously at the input. But at the output they do not cross the time-axis simultaneously – the 5 rad/s crosses first followed by the 3 rad/s component and the 1 rad/s component is the last one to cross the time-axis. Thus, they get *dispersed*. This *dispersion* results in change in waveshape. Refer to Fig. 11.5-4. The three components in the input are shown in Fig. 11.5-4 (a) and the corresponding components in the output are shown in Fig. 11.5-4 (b). The dispersion in zero-crossing instants is clearly brought out in Fig. 11.5-4 (b). The distortion resulting from dispersion of components brought about by *unequal time delays* suffered in going through the circuit is termed as *phase distortion*. Of course, in any distortion context these two – amplitude distortion and phase distortion – are mixed up and cannot be separated out.

Phase distortion arises essentially due to phase response part of frequency response. If all the sinusoids are delayed by *same time delay* there will be no change in waveshape (assuming there is

no amplitude distortion). The entire input waveshape will get bodily shifted in time-axis by a definite delay and will appear as output in that case. Therefore, either zero time delay for all frequencies or constant time delay for all frequencies will prevent phase distortion. A constant time delay implies that the phase delay must be a *linear function of ω* .

The conditions to be satisfied by a circuit such that there is no waveshape distortion when a signal passes through it must be evident now – its frequency response must have a gain that is flat with ω and a phase which is either zero or linear on ω , *i.e.*, of the form $\phi = -k\omega$ where k is a real number.

Obviously only a memoryless circuit can satisfy this. Hence a circuit which contains at least one inductor or capacitor will cause waveshape distortion in general. Similarly, we conclude that a memoryless circuit cannot function as a *filter*; we will need inductors and capacitors for that.

We observe that, in the example we analysed in this section, the input contained three sinusoids of 1 rad/s, 3 rad/s and 5 rad/s and the output contained exactly three sinusoidal components with the same frequencies as in the input. In short, the circuit did not change the frequency of sinusoids. Neither did it generate a sinusoid with a frequency that was not there in the input. This, in fact, is a property of any lumped linear time-invariant (LTI) system. They can only scale, differentiate or integrate signals. And these three mathematical operations cannot produce a sinusoid with a frequency that is different from that of input.

Therefore, a single frequency sinusoid cannot suffer waveshape distortion in passing through a linear time-invariant circuit.

However, a non-linear circuit can change the waveshape of a single frequency sinusoid. Apply about 10mV of 1kHz sinusoidal voltage to a 741 Operational Amplifier non-inverting pin after grounding the inverting pin. The Operational Amplifier is in the open loop and its large gain results in output getting saturated. We will observe a waveform that is almost a square wave at the output. That is *non-linear distortion*. The waveshape distortion we observed in the example in this section was not due to non-linearity. It occurred due to differential treatment experienced by various sinusoidal components in a mixture of sinusoids when they went through the circuit. The distortion which occurs due to frequency response of a linear circuit is termed as *linear distortion* in order to distinguish this kind of distortion from distortion due to non-linearity. *Amplitude Distortion* and *Phase Distortion* are the two inseparable components of *Linear Distortion*.

One should not be under the impression that the Series *RC* Circuit can function only as a low-pass filter. In fact, the kind of filter realised by a given circuit will strongly depend on where exactly is the input applied and where exactly is the output taken. A Series *RC* Circuit excited by a voltage source at the input with output taken across the capacitor is a low-pass filter. The same circuit with the same excitation but with the output taken across the resistor is a high-pass filter that passes the high frequency sinusoids to the output and curtails the low frequency components including DC.

11.5.3 First-Order *RC* Circuits as Averaging Circuits

A Series *RC* Circuit with voltage excitation and output taken across the capacitor is a low-pass filter. Similarly, a Parallel *RC* Circuit with current excitation and output taken across the parallel combination is also a low-pass filter.

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Averaging is a signal processing application that appears often in analog and digital signal processing. The waveform to be averaged is typically a rectangular pulse waveform with a slowly varying DC content. An *averaging circuit* produces an output that is the DC content of input signal.

Consider the rectangular pulse waveform shown in Fig. 11.5-5. It has an amplitude of 1 V, a frequency of 1 kHz and a duty ratio of 0.2. This waveform can be thought of as the sum of a DC voltage of value 0.2 V and a pure alternating waveform with zero full-cycle area (*i.e.*, zero DC content). These two components are also shown in Fig. 11.5-5.

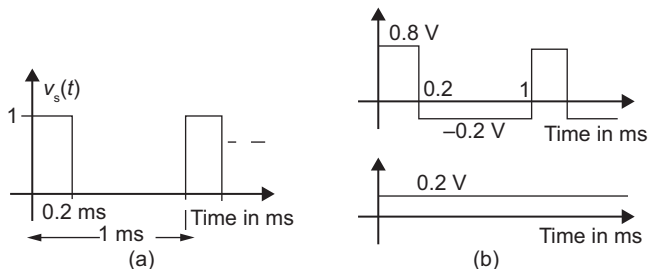


Fig. 11.5-5 A rectangular pulse waveform and its components

Assume that this waveform is applied to a Series RC Circuit with $R = 10 \text{ k}\Omega$ and $C = 1 \mu\text{F}$ from $t = 0$. The time constant is 10 ms. Output is taken across the capacitor. The total zero-state response can be obtained by using superposition principle. Therefore, we expect the standard step response scaled by 0.2 to be present along with other components in the output. This step response component will reach steady-state in about 5τ , *i.e.*, in 50ms and contribute 0.2 V steady component to the output after that.

The pure alternating component of applied voltage also will reach a periodic steady-state at the output after about 50ms. We remember that this alternating component can be thought of as the sum of infinitely many sinusoids of frequencies that are integer multiples of 1 kHz. The lowest frequency component will be 1 kHz. The phasor impedance of capacitor and the resistor share the sinusoidal voltage under steady-state and we are interested in the voltage absorbed by the capacitor. Let us calculate the phasor impedance of capacitor at 1 kHz. It is $-j159.2 \Omega$. We see that this is only about 1.6% of the resistor value (10 k Ω) and hence we expect the capacitor to absorb only a very small percentage of sinusoidal voltage at 1 kHz – most of it will appear across the resistor. This will be more so for other sinusoidal components with frequencies higher than 1 kHz since phasor impedance of capacitor goes down with frequency.

As a first approximation, we assume that the alternating component that appears across capacitor is negligible when we calculate the alternating component of current in the circuit. Thus, the alternating component of applied voltage is assumed to appear across 10k Ω almost entirely, thereby resulting in a current whose waveshape will be the same as that of alternating component of voltage. This current will vary between 0.08 mA and -0.02 mA . Now we work out the small voltage that appears across capacitor due to this alternating current flow.

The half-cycle area of this current is $0.08 \text{ mA} \times 0.2 \text{ ms}$ (or $0.02 \text{ mA} \times 0.8 \text{ ms}$) = $0.016 \mu\text{C}$. Hence, the peak-to-peak voltage across the capacitor due to the alternating current flow will be $0.016 \mu\text{C} / 1 \mu\text{F} = 0.016 \text{ V}$. Hence the total capacitor voltage will vary in the range $0.2 \pm 0.008 \text{ V}$. The variation is $\pm 4\%$ of the desired average value of 0.2 V.

This approximate solution is confirmed by the accurate solution worked out using the method to solve for periodic steady-state explained earlier in this section. This is shown in Fig. 11.5-6. The output

waveform segments are actually exponential; but they appear nearly straight-line segments confirming the validity of assumption employed in our approximate reasoning.

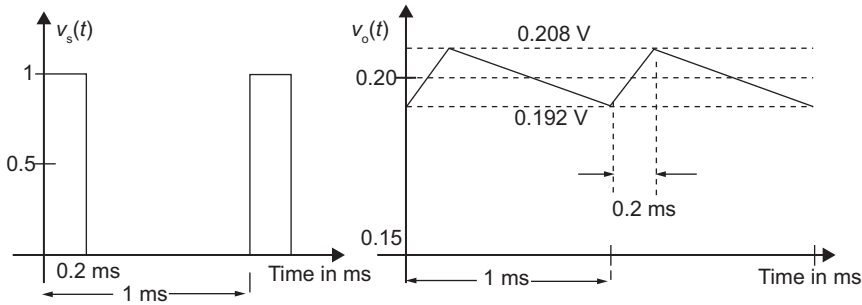


Fig. 11.5-6 Input and output waveforms of series RC averaging circuit

We should not ever forget that a circuit reaches steady-state only after covering the transient period. This reasonably clean average value appears only after 50 ms of applying the input. We can increase the time constant of the circuit to higher levels in order to make the output appear cleaner; but there is a price to pay. *The cleaner output will take longer to establish.* The average value of input is not likely to remain constant forever in a practical application of averaging circuit. In fact, this value may be used to code some information and will consequently change slowly. Typically, the duty ratio of the input wave changes slowly while its frequency is kept constant. And the averaging circuit is expected to track the change in average value faithfully. Obviously there is a conflict between the requirement of a clean output and the requirement of a fast response to changing DC content at input. The time constant of the circuit must be selected in such a way that it has enough speed to catch up with the average value variation. And, if the ripple in the output is excessive with that value of time constant, we better look for some other better technique to do averaging!

Parallel RC Circuit can be employed for averaging current signals subject to similar constraints. Essentially, *averaging* is only a special case of low-pass filtering. Good averaging performance requires that $\tau \gg T$, where T is the period of the input signal or its characteristic time of variation if a regular period cannot be identified.

11.5.4 Capacitor as a Signal Coupling Element

Another application context involving the Series RC Circuit is the ‘*signal coupling problem*’ in electronic amplifiers. We abstract the problem as follows.

At a certain input point in the electronic amplifier a certain value of DC voltage has to be established using a DC source and resistors. This DC voltage is needed to fix the operating point of transistors in the amplifier at suitable levels. Yet, we want to connect an AC signal source to that input point without upsetting the DC potential there. Interposing a suitably sized capacitor between the signal source and input point achieves this objective. The capacitor used for this purpose is called *coupling capacitor* and amplifiers employing this form of signal coupling are called *RC-Coupled Amplifiers* in the study of Electronic Circuits. Notice that the RC circuit is used as a *high-pass filter* (or *average absorber*) in this application.

This application is explained further in the following example.

Example: 11.5-1

Consider the signal-coupling problem presented in Fig. 11.5-7. The signal $v_s(t)$ is a mixture of sinusoids and may contain sinusoids of frequency from 20 Hz to 20 kHz (the standard audio range). It is desired that 95% of signal amplitude appear at the point A for the entire frequency range without affecting the DC content of voltage at that point. (i) Calculate the value of C needed for this purpose. (ii) Assume $v_s(t)$ is a 500 mV amplitude 20 Hz sine wave and plot the potential at A and the voltage across the coupling capacitor with the value of C calculated.

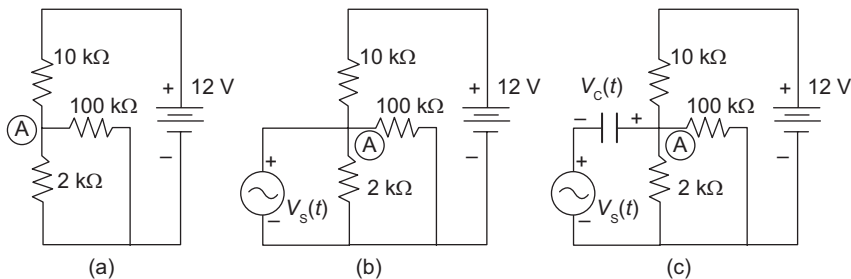


Fig. 11.5-7 Circuits for RC signal-coupling example

Solution

Consider the circuit in Fig. 11.5-7 (a). The $100\text{k}\Omega$ resistor produces only negligible loading at the point A and hence the current drawn by it may be ignored. Then the DC voltage at that point is $12\text{ V} \times (2/10+2) = 2\text{ V}$ with respect to the negative of DC source.

Now if the AC signal source is connected straight at point A as in Fig. 11.5-7 (b) the DC voltage at that point gets affected. In fact it goes to zero if $v_s(t)$ has no DC content. This is a two-source problem and can be solved by applying superposition principle. When the DC source alone is considered, the AC source has to be shorted (assuming it is an ideal independent voltage source). This short will reduce the DC potential at point A to zero once the signal source is connected. Hence direct coupling will not work.

The circuit Fig. 11.5-7 (c) is also excited by two sources. We know that the steady-state responses due to many sources obey superposition principle in a linear time-invariant circuit. The two equivalent circuits needed for calculating the steady-state response are shown in Fig. 11.5-8.

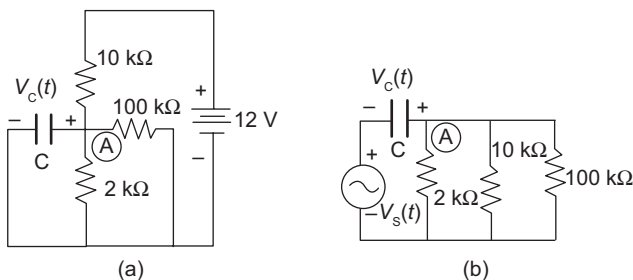


Fig. 11.5-8 Circuits for applying superposition in Example: 11.5-1

The DC component of a steady-state response at point A is 2 V (neglecting the effect of $100\text{k}\Omega$ resistor) since a capacitor behaves as an open-circuit under DC steady-state, and the DC content in $v_c(t)$ is also 2 V.

As far as steady-state component of AC signal is concerned, we want 95% of input amplitude to appear at point A even at 20 Hz. The parallel combination of the three resistors is 1.64 k Ω . The signal voltage gets divided between the phasor impedance of capacitor, $1/(j2\pi \times 20C)$, and 1.64 k Ω .

$$\text{Percentage of signal amplitude reaching point A at 20Hz} = \left| \frac{1640}{1640 + \frac{1}{j40\pi C}} \right| \times 100\%$$

We desire this to be $\geq 95\%$.

$$\therefore 0.95 = \frac{1640 \times 40\pi C}{\sqrt{(1640 \times 40\pi C)^2 + 1}} \Rightarrow C = 14.8 \mu F$$

Therefore the capacitance value required is 14.8 μF .

The ratio between voltage at A to voltage at input at 20 Hz has a gain value of 0.95.

$$\text{Its phase will be angle of } \frac{1640}{1640 + \frac{1}{j40\pi C}}.$$

Substituting $C = 14.8 \mu F$ we get the phase angle as $18.15^\circ (0.317 \text{ rad})$

Therefore with 500 mV amplitude sine wave at input there will be a steady-state component of $(500 \times 0.95 \text{ mV}) \sin(40\pi t + 0.317)$ at point A.

Therefore total steady-state voltage at A = $2 + 0.475 \sin(40\pi t + 0.317)$ V.

$$\frac{V_C(j\omega)}{V_S(j\omega)} = -\frac{\frac{1}{j40\pi C}}{1640 + \frac{1}{j40\pi C}} = -\frac{-j537.7}{1640 - j537.7} = -0.312 \angle -1.254 \text{ rad}$$

$$\begin{aligned} \therefore \text{The ac component of steady-state voltage across capacitor} &= -0.312 \times 0.5 \sin(40\pi t - 1.254) \\ &= -0.156 \sin(40\pi t - 1.254) \end{aligned}$$

Therefore total steady-state voltage across capacitor, $v_C(t) = 2 - 0.156 \sin(40\pi t - 1.254)$ V.

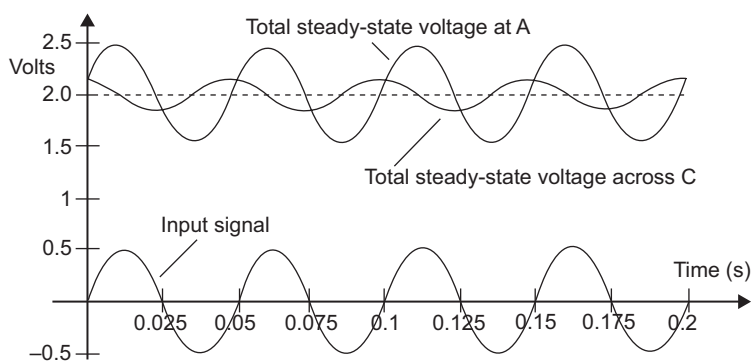


Fig. 11.5-9 Waveform plots for Example: 11.5-1

11.5.5 Parallel RC Circuit for Signal Bypassing

The need for making a resistor offer its full resistance to DC current flow and near-zero resistance to signal (*i.e.*, alternating component) current flow arises often in Electronic Circuit applications. Constructing a Parallel RC Circuit by adding a capacitor in parallel to the concerned resistor is the standard solution to this ‘*signal bypassing*’ problem. The capacitor is sized suitably such that it offers a phasor impedance with very small magnitude compared to the resistance value of the resistor to be bypassed even for the lowest frequency sinusoidal component that is present in the signal current.

Parallel RC Circuit is used as a low-pass filter with its bandwidth much lower than the lowest frequency present in the signal in *signal bypassing* application.

The DC component of total current flowing into the parallel combination goes through the resistor under steady-state. But all the AC components flow almost entirely through the capacitor since it offers much lower impedance than the parallel resistor at the frequencies concerned. The voltage across the combination remains nearly DC. The next example illustrates this application of a Parallel RC Circuit.

Example: 11.5-2

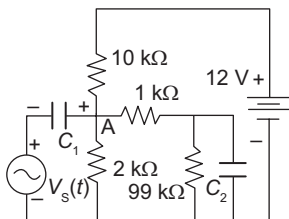


Fig. 11.5-10 Circuit for signal bypassing problem in Example: 11.5-2

Consider the ‘*signal bypassing*’ context in Fig. 11.5-10. The signal $v_s(t)$ is a mixture of sinusoids and may contain sinusoids of frequency from 20 Hz to 20 kHz (the standard audio range). The coupling capacitor C_1 may be assumed to be so large that it is effectively a short-circuit at all frequencies except at DC. It is desired that more than 95% of applied signal voltage appear across the $1\text{ k}\Omega$ resistor for the entire frequency range without affecting the DC content of current through that resistor. (i) Calculate the value of C needed for this purpose. (ii) Assume $v_s(t)$ is a 10 mV amplitude 20 Hz

sine wave and plot the waveforms of total current through $1\text{ k}\Omega$, $99\text{ k}\Omega$ and the capacitor with the value of C calculated above.

Solution

The equivalent circuits needed for calculating the steady-state response contributions from the DC source and signal source are shown in of Fig. 11.5-11 (a) and (b), respectively.

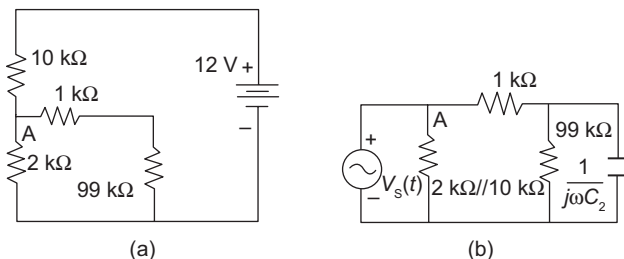


Fig. 11.5-11 Steady-state equivalent circuits in Example: 11.5-2

Solving the circuit in Fig. 11.5-11 (a), we get the DC potential at the point A as 1.97 V, the DC current through 1 k Ω and 99 k Ω resistors as 0.0197 mA and the DC potential across 99 k Ω resistor (and hence across the capacitor C_2) as 1.95 V.

Consider the circuit in of Fig. 11.5-11 (b). We desire the amplitude of signal voltage across 1 k Ω to be $\geq 95\%$ of amplitude of applied signal voltage even when the signal is a 20 Hz sinusoid.

$$\text{Ratio of phasor voltage across } 1\text{k}\Omega \text{ to applied phasor voltage} = \frac{1000}{1000 + 99000 // \frac{1}{j40\pi C_2}}$$

We want the magnitude of this ratio to be 0.95. Solving for C_2 we get $C_2 = 24.5 \mu\text{F}$.

Substituting this value of C_2 in the phasor ratio above, we get the ratio as $0.95 \angle 0.32 \text{ rad}$.

$$\text{Applied signal voltage} = 0.01 \sin(40\pi t) \text{ V}$$

$$\therefore \text{Signal voltage across } 1\text{k}\Omega = 0.0095 \sin(40\pi t + 0.32) \text{ V}$$

$$\therefore \text{Signal current through } 1\text{k}\Omega = 0.0095 \sin(40\pi t + 0.32) \text{ mA}$$

$$\therefore \text{Total current through } 1\text{k}\Omega = 0.0197 + 0.0095 \sin(40\pi t + 0.32) \text{ mA}$$

$$\text{Phasor impedance of } 24.5 \mu\text{F capacitor at } 20 \text{ Hz} = -j325 \Omega$$

$$\text{Parallel Combination of } 24.5 \mu\text{F capacitor at } 20 \text{ Hz and } 99\text{k}\Omega = 325 \angle -1.567 \text{ rad } \Omega$$

$$\begin{aligned} \therefore \text{Signal voltage across parallel combination} &= 0.0095 \times 0.325 \sin(40\pi t + 0.32 - 1.567) \text{ V} \\ &= 0.00309 \sin(40\pi t - 1.247) \text{ V} \end{aligned}$$

$$\therefore \text{Signal current through } 99\text{k}\Omega = 0.00003 \sin(40\pi t - 1.247) \text{ mA}$$

$$\therefore \text{Total current through } 99\text{k}\Omega = 0.0197 + 0.00003 \sin(40\pi t - 1.247) \text{ mA}$$

$$\begin{aligned} \text{Signal current through } 24.5 \mu\text{F capacitor} &= \frac{0.00309 \text{ V}}{0.325 \text{ k}\Omega} \sin(40\pi t - 1.247 + \pi/2) \\ &= 0.0095 \sin(40\pi t + 0.323) \text{ mA} \end{aligned}$$

$$\text{Total voltage across parallel combination} = 1.95 + 0.00309 \sin(40\pi t - 1.247) \text{ V}$$

Almost the entire signal current flowing through 1 k Ω resistor goes into the capacitor. Thus the 99 k Ω resistor has been bypassed very effectively by a 24.5 μF even at 20 Hz. The relevant waveforms are shown in Fig. 11.5-12. Voltages shown are with respect to the negative terminal of the DC source.

11.28 | First-Order RC Circuits

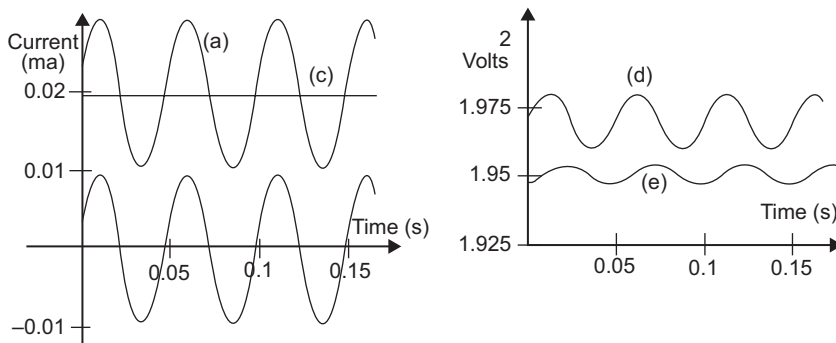


Fig. 11.5-12 Waveforms for Example: 11.5-2 (a) Total current in $1\text{k}\Omega$ resistor (b) Total current in the capacitor (c) Total current in $99\text{k}\Omega$ resistor (d) Total voltage at A (e) Total voltage across capacitor

11.6 SUMMARY

- Series and parallel RC circuits are described by a first-order linear differential equation. The past history of the circuit is contained in a single initial condition specification for capacitor voltage in RC circuits.
- The solution of the differential equation describing the capacitor voltage in an RC circuit contains two terms – the complementary function and particular integral. Complementary function is the solution of differential equation with zero-forcing function. Particular integral is the solution of the differential equation *due to input function*. The total solution is obtained by adding these two. The complementary function has arbitrary amplitude that should be fixed by ensuring that the *total* solution complies with the specified initial condition.
- The circuit variables in the RC circuit will contain two response components – *transient response* (also called *natural response*) and *forced response*. *Natural response* is the way in which the inertia in the circuit reacts to forcing function's command to change. Complementary solution gives the *natural response* and particular integral gives the *forced response* in a circuit.
- The *natural response* of a circuit is independent of the type or magnitude of forcing function and depends only on circuit parameters and the nature of interconnections. *Natural response* in RC circuit is exponential of the form $A e^{-t/\tau}$, where $\tau = RC$ is defined as time constant of the circuit. A is to be fixed by complying with initial condition.
- The initial capacitor voltage in an RC circuit at $t = 0^-$ and $t = 0^+$ are the same if the circuit does not contain impulse sources.
- In the case of RC circuit, *step response* is a rising exponential, approaching a steady-state value asymptotically as $t \rightarrow \infty$. The *step response* never gets done. But for practical purposes it may be considered to be over within 5 time constants.
- Free-response* of an RC circuit is its response when input is zero and there is some initial energy trapped in the capacitor. It will contain only *natural response* terms. The capacitor voltage in this case falls exponentially towards zero.
- The response due to initial energy and application of impulse are indistinguishable in an RC circuit and hence they can be replaced for each other. An initial voltage of V_0 in a capacitor of value C can be replaced by zero initial condition with a current source $CV_0 \delta(t)$ connected in *parallel* with the capacitor.

- *Step response* and *ramp response* in a RC circuit can be obtained by integrating its *impulse response* successively.
- Waveshape distortion occurs in linear circuits due to differential treatment experienced by various sinusoidal components in a mixture of sinusoids when they go through the circuit. The conditions to be satisfied by a circuit such that there is no waveshape distortion when a signal passes through it, is that its frequency response must have a gain that is flat with ω and a phase which is either zero or linear on ω , i.e., of the form $\phi = -k\omega$, where k is a real number.
- A Series RC Circuit excited by a voltage source at the input of the series combination with output taken across the capacitor is a low-pass filter with a cut-off frequency of $1/\tau$ rad/s and a monotonically decreasing gain. The same circuit with same excitation but with output taken across the resistor is a high-pass filter with a cut-off frequency of $1/\tau$ rad/s and a monotonically increasing gain.
- Series RC circuit with output taken across capacitor can be used as averaging filter for voltage signals and parallel RC circuit with output taken across the combination can be used as averaging filter for current signals. Good averaging performance requires that $\tau \gg T$, where T is the period of the input signal or its characteristic time of variation if a regular period cannot be identified.

11.7 PROBLEMS

[Circuits are initially relaxed unless specified otherwise]

1. What is the differential equation describing $v_C(t)$ for $t \geq 0^+$ in the circuit in Fig. 11.7-1?

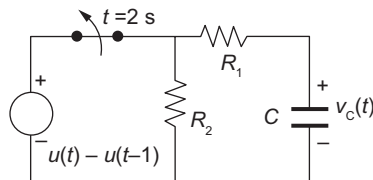


Fig. 11.7-1

2. A parallel RC circuit with non-zero initial energy is driven by a 1A DC current source from $t = 0$. $v_C(t)$ is found to reach 75% of its steady-state value in one time constant. Express the initial voltage across capacitor as a percentage of its steady-state value.
3. A parallel RC circuit is driven by a 1A DC current source from $t = 0$. $v_C(t)$ is found to be 50% at $t = \tau$. Was there any initial voltage across C ? If so, what is its magnitude and polarity?
4. Find and plot the $v_C(t)$ and voltage across the current source as functions of time in Fig. 11.7-2. Is there any exponential function in the expression for $v_C(t)$? If not why?

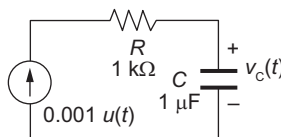


Fig. 11.7-2

11.30 | First-Order RC Circuits

- Derive an expression for v_C in an initially relaxed series RC circuit when it is driven by $v_s(t) = t$ for $0^+ \leq t \leq \tau$ and 0 for all other t , where $\tau = RCt$.
- What is the time required for the energy stored in C in a series RC circuit with voltage source input to reach 99% of its steady-state value in step response?
- v_C in the circuit in Fig. 11.7-3 is -10 V at $t = 0^-$. If $v_C(t) = 0$ for $t \geq 0^+$, find C .

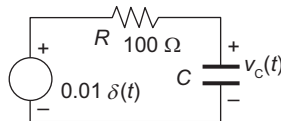


Fig. 11.7-3

- A parallel RC circuit is driven by a 1A DC current source from $t = 0$. The resistor voltage is found to be $(10 - 5e^{-t})$ V for $t \leq 0^+$. What are the values of C , R and initial voltage?
- v_C in the circuit in Fig. 11.7-4 at $t = 0^-$ is -12 V. It is found that $v_C(t) = 0$ for $t \leq 0^+$. What is the rise time of step response of the circuit?

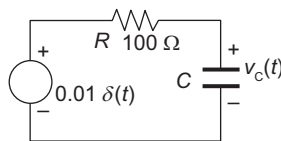


Fig. 11.7-4

- A parallel RC circuit with non-zero initial energy is driven by a 1A DC current source from $t = 0$. The capacitor voltage is found to be 15 V for $t \leq 0^+$. What is the value of initial voltage and what is its polarity relative to the observed voltage? What is the value of R in the circuit?
- A parallel RC circuit with non-zero initial energy is driven by a 1A DC current source from $t = 0$. The resistor current is found to be 12 mA under steady-state. What is the new steady-state value of this current if (i) the initial condition is doubled (ii) if $2u(t)$ is applied with no change in initial condition?
- The capacitor voltage $v_C(t) = 0$ for $t \leq 0^+$ in the circuit in Fig. 11.7-5. Find the values of V_0 and I_s . The circuit elements that produced initial energy storage in the capacitor are not shown.

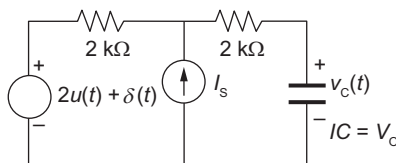


Fig. 11.7-5

- A series RC circuit with zero initial current is driven by $v_s(t) = \delta(t) - \delta(t-1)$. Its time constant is 1 s. (i) Starting from impulse response, find the voltage across the resistor in the circuit when driven by the input $v_s(t)$. (ii) Using the result derive an expression for voltage across the resistor when the circuit is driven by a rectangular pulse of unit amplitude and 1 s duration.
- Derive expressions for maximum voltage across resistor in an initially relaxed series RC circuit when it is driven by $e^{\alpha t} u(t)$ A with $\alpha \neq -1/\tau$, where τ is the time constant of the circuit. Also find an expression for the time instant at which this maximum voltage occurs.

15. An input of $k\delta(t) + 2e^{-2t/\tau}\text{V}$ is applied to an initially relaxed series RC circuit with time constant of τ s. The output across capacitor for $t \leq 0^+$ is observed to contain only $e^{-2t/\tau}$ waveshape. What is the value of k ?
16. The desired voltage across a parallel RC with initial condition as shown in the Fig. 11.7-6 is given by $v(t) = 2t$ for $t \leq 0$ and 0 for $t < 0$. Find the $i_s(t)$ to be applied to the circuit if the initial condition is 5 V . Sketch the required $i_s(t)$.

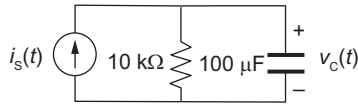


Fig. 11.7-6

17. The switches in Fig. 11.7-7 are ideal. (i) Find the voltage across C and plot it. (ii) Find the current through C and plot it. (iii) Find the voltage across the first resistor and plot it.

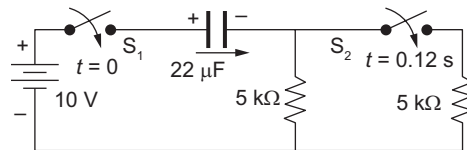


Fig. 11.7-7

18. Initial voltage at $t = -\infty$ in the capacitor in the circuit in Fig. 11.7-8 was zero. Find the voltage across capacitor and current through it for $t \leq 0^+$ and plot them.

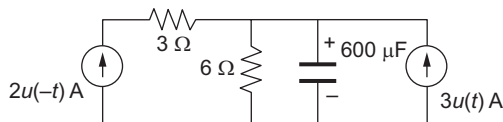


Fig. 11.7-8

19. The switch in the circuit in Fig. 11.7-9 was closed for a long time and is opened at $t = 0$. Find and plot the current in C and voltage across C as functions of time.

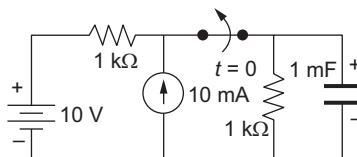


Fig. 11.7-9

20. Find the impulse response of the voltage variable $v(t)$ in the circuit in Fig. 11.7-10.

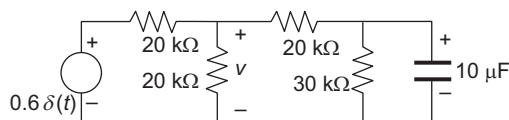


Fig. 11.7-10

11.32 | First-Order RC Circuits

21. $v_C(t)$ at 0^- in the circuit in Fig. 11.7-11 is $-V$. The switch S is closed at $t = 0$. Show that $v_C(t)$ will cross zero at $t = 0.69RC$ s.

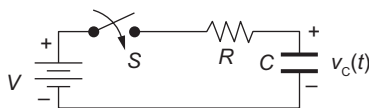


Fig. 11.7-11

22. A pulse of height V V and width T s is applied to a series RC circuit with zero initial condition as in Fig. 11.7-12. (i) Plot the output for (a) $RC = T$ (b) $RC = 10T$ and (c) $RC = 0.1T$ (ii) Find the relation between RC and T if the area of output pulse after T s is to be less than 10% of area of output pulse from 0 to T s.

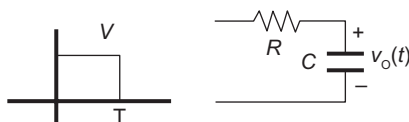


Fig. 11.7-12

23. What must be the value of k in the circuit in Fig. 11.7-13, if $v(t) = 0$ for $t \leq 0^+$?

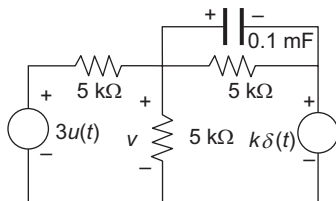


Fig. 11.7-13

24. A single pulse defined as $v_s(t) = 10t$ for $0 \leq t \leq 1$ and 0 otherwise is applied to a series RC circuit with a time constant of 0.3 s. Find and plot the capacitor voltage.
25. Find the time constants for each circuit in Fig. 11.7-14 for voltage excitation and current excitation. All resistors are $1\text{ k}\Omega$ and all capacitors are $1\mu\text{F}$.

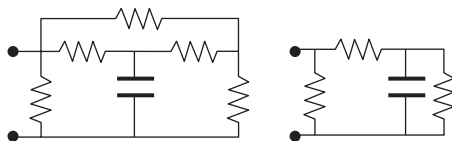


Fig. 11.7-14

26. The Inverter in the circuit in Fig. 11.7-15 is a digital electronic gate circuit and its input and output behaviour is as shown in the waveforms. Each inverter gate has 15 pF input capacitance. The gate circuit will draw only zero current from $+V$ supply if its input is held at V V or 0 V steadily.

The output of one such gate is connected as input to 4 such gates. Calculate (i) the power dissipation in the driving gate and (ii) the average power supply current drawn by the driving gate when the input to driving gate is a square wave varying between $+V$ and 0 with a frequency f for (a) $V = 5V$, $f = 100\text{kHz}$ (b) $V = 5V$, $f = 10\text{MHz}$ (c) $V = 15V$, $f = 100\text{kHz}$ (d) $V = 15V$, $f = 10\text{MHz}$.

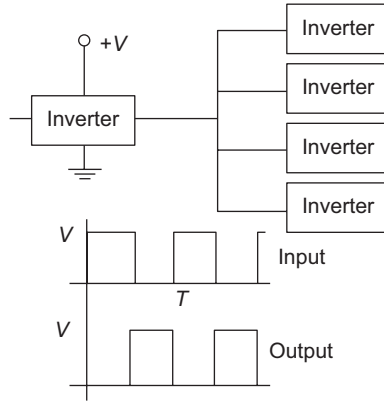


Fig. 11.7-15

27. A symmetric square wave of $\pm V$ amplitude and period of T s is applied to a high pass RC circuit as shown in Fig. 11.7-16. After a few cycles of initial transient the output waveform settles to a periodic steady-state as shown where V'_1 and V'_2 will be equal to V_1 and V_2 , respectively, under periodic steady-state. The slanting portion of output wave will be exponential with time constant $= RC$. However, if $RC \gg T$ (equivalently, if the cut-off frequency of the circuit is much less than the square wave frequency) we can approximate the exponential by straight-line segments. Use this approximation and find expressions for V_1 and V_2 . Also find an expression for the so-called “percentage tilt” defined as $100 \times (V_1 - V_2)/V$.

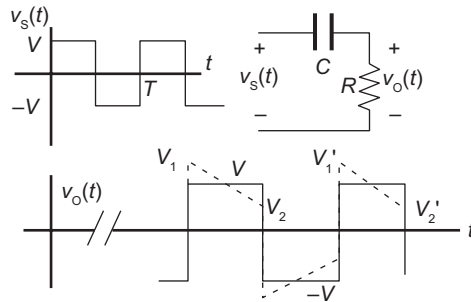


Fig. 11.7-16

28. Let $V_s(t)$ be an arbitrary time varying periodic voltage source with a cycle average value of V_{dc} . This means that $V_s(t)$ can be written as $V_{dc} + V_{ac}(t)$, where $V_{ac}(t)$ is a time varying periodic component with equal positive half cycle and negative half cycle areas. Let that area be A V-s.

11.34 | First-Order RC Circuits

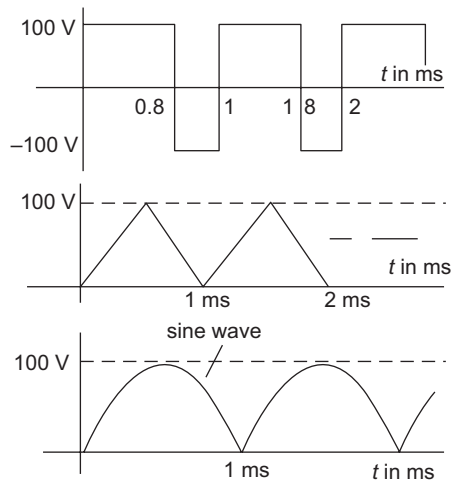


Fig. 11.7-17

This waveform $V_s(t)$ is applied to a series RC circuit and output voltage is taken across C . Assume that $RC \gg T$ where T is the period of $V_s(t)$. Show that under periodic steady-state the (i) average value of output voltage is V_{dc} (ii) the peak-to-peak ripple in output voltage $\approx A/\tau V$, where $\tau = RC$. (iii) Calculate the quantities in (i) and (ii) for the three inputs given in Fig. 11.7-17, if τ is 20ms. (The Series RC Circuit can be used to extract average value of the input. The basic issue involved in this application is the tradeoff between ripple in the average value V_s response time)

29. The switch S_1 in Fig. 11.7-18 is closed at $t = 0$ with zero initial condition in the capacitor. The switch S_2 is kept open for T_1 s and closed for $(T - T_1)$ s periodically. Obtain the voltage across capacitor and plot it assuming the following data. $V = 12V$, $R = 12 \text{ k}\Omega$, $C = 1\mu F$, $T_1 = 10\text{ms}$, $T = 11\text{ms}$ and Resistance of S_2 when it is ON = 100Ω .

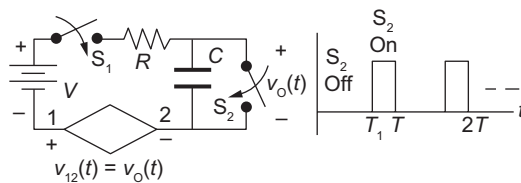


Fig. 11.7-18

30. The source-free response in an RC circuit takes the capacitor voltage from 10 V to 3V in 20ms. What is the bandwidth of a low-pass RC filter made using the same components?
31. The steady-state voltage across resistor (v_R) in a series RC circuit has an amplitude of 7.07V when the circuit is driven by an AC voltage of amplitude 10V and angular frequency ω rad/s. (i) Find the phase angle of v_R with respect to the input sinusoid. (ii) If another sinusoidal voltage of 15V amplitude and 3ω rad/s frequency is applied to the circuit, find the amplitude and phase of v_R under steady-state condition.

32. The bandwidth of a low-pass RC filter is 2kHz. If a 10V DC source is applied to this series RC circuit (capacitor is uncharged before applying this source) at $t = 107\text{ms}$, find t at which the voltage across resistor will be 3V.
33. A Series RC circuit is excited by a sinusoidal voltage source at 3kHz. The voltage across the resistor is seen to lead the input voltage by 30° . If this RC circuit undergoes source-free response, how much time will it take to dissipate 50% of its initial stored energy in the resistance?
34. A Series RC circuit is excited by a sinusoidal voltage source at 1kHz. The voltage across the capacitor is seen to lag the input voltage by 30° . What is the cyclic frequency of input sinusoid such that the voltage across the resistor will lead the input by 30° ?
35. A Series RC circuit is excited by a sinusoidal voltage source at 10kHz. The voltage across the capacitor is seen to be 50% of the input voltage in amplitude. What is the cyclic frequency of input sinusoid such that the voltage across the resistor will be 50% of the input voltage in amplitude?
36. A Series RC circuit is excited by a sinusoidal voltage source at 100kHz. The voltage across the capacitor is seen to be 30% of the input voltage in amplitude. What is the cyclic frequency of input sinusoid such that the voltage across the resistor will lead the input voltage by 60° ?
37. An AC voltage source $= V \sin \omega t$ is applied to a series RC circuit from $t = 0$. The circuit current is found to be $= 0.7 \sin (\omega t - \pi/3)$ A for $t \leq 0^+$. Was there any initial voltage across the capacitor? If so, what is its magnitude and relative polarity?
38. The fall time of a series RC circuit is 6.6 ms. If the same components are used to make a parallel RC circuit excited by a sinusoidal current source $i_s(t) = 0.5 \sin 300\pi t$ A, find the steady-state current in the resistor in the circuit.
39. A parallel RC circuit has $C = 10\mu\text{F}$. It is excited by an AC current source of 100mA amplitude and 5kHz frequency. The sinusoidal voltage across the combination is seen to lag the applied input by 60° . Find the value of R and the amplitude of voltage across the combination.
40. A current signal $i_s(t) = (2 \sin \omega t - 1.2 \sin 3\omega t + 0.8 \sin 5\omega t)u(t)$ mA is applied to a parallel RC circuit of bandwidth 0.8ω . Find and plot the steady-state waveform of voltage across the circuit.

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Chapter 12

Series and Parallel *RLC* Circuits

CHAPTER OBJECTIVES

- *Source-free Response of Series RLC Circuit*
- *The Series LC Circuit – A Special Case*
- *The Series LC Circuit with Small Damping – Another Special Case*
- *Standard Formats for Second-Order Circuit Zero-Input Response*
- *Sinusoidal Forced-Response of RLC Circuits from Differential Equation*
- *Frequency Response of RLC Circuits from Phasor Equivalent Circuit*
- *Qualitative Discussion on Frequency Response of Series RLC Circuit*
- *A More Detailed Look at the Band-pass Output of Series RLC Circuit – Series Resonance*
- *Quality Factor of Practical Inductors and Capacitors*
- *Zero-Input Response and Zero-State Response of Parallel RLC Circuit*
- *Sinusoidal Steady-State Frequency Response of Parallel RLC Circuit – Parallel Resonance*

INTRODUCTION

Circuits containing one energy storage element – either electrostatic energy storage element or magnetic energy storage element – will show first-order dynamics. The natural response of such circuits will be monotonically decreasing real exponential signals. In fact, the natural response of a circuit containing *only one kind of energy storage elements – either electrostatic or magnetic* – will contain only monotonically decreasing real exponential signals. That is, circuits with only one kind of energy storage elements are incapable of generating *oscillatory natural response*. We need competing energy storage elements with different kinds of energy storage – e.g. a capacitor and an inductor – in a circuit for that circuit to feature oscillatory natural response. Such a response involves exchange of stored energy from one form to another form. This does not imply that all circuits containing at least one inductor and one capacitor will necessarily produce oscillatory transients. But, *they can*.

12.2 | Series and Parallel *RLC* Circuits

Whereas circuits that contain only inductors or circuits that contain only capacitors as energy storage elements *can not*. We deal with circuits that can show oscillatory transients with minimum number of energy storage elements – one inductor and one capacitor; the series *RLC* and parallel *RLC* circuits – in this chapter. These circuits are applied extensively in communication systems, power systems, electronic filtering, etc. We meet with linear second-order differential equations in our attempt to describe these circuits and we go into responses of such a second-order systems to various standard input signals in detail subsequently. We also look at *resonance* phenomenon in series and parallel *RLC* circuits in this chapter.

12.1 THE SERIES *RLC* CIRCUIT – ZERO-INPUT RESPONSE

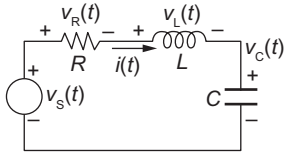


Fig. 12.1-1 Series *RLC* circuit with voltage source excitation

We begin our study of *RLC* circuits with series *RLC* circuit in this section. Figure. 12.1-1 shows a series *RLC* circuit with voltage excitation with all instantaneous variables identified.

We choose to develop the differential equation governing the circuit in terms of the voltage across the capacitor $v_C(t)$, *i.e.*, we select $v_C(t)$ as the describing variable for the circuit. We obtain all other variables in terms of this variable once the solution is obtained.

$$i(t) = C \frac{dv_C(t)}{dt} \quad (\text{By element equation of a capacitor})$$

$$-v_s(t) + Ri(t) + L \frac{di(t)}{dt} + v_C(t) = 0 \quad (\text{By KVL})$$

Substituting for $i(t)$ in terms of $v_C(t)$ and rearranging terms,

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{1}{LC} v_s(t)$$

All voltages and currents in a dynamic circuit are, in general, functions of time. Even if the variables do not include (t) explicitly, we understand them to be functions of time. Hence we write the describing differential equation of series *RLC* circuit in the following manner.

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{LC} v_s \quad (12.1-1)$$

This equation is true for all t . However, we know the input source function $v_s(t)$ only from $t = 0^+$ onward. There may also be a discontinuity in $v_s(t)$ at $t = 0$. Therefore, we have to have additional information summarising all that happened to the inductor and capacitor from $t = -\infty$ to $t = 0^-$ in the form of initial condition specification for inductor current and capacitor voltage in order to solve this differential equation for $t \geq 0^+$. The two initial conditions needed to solve a second-order differential equation are the values of its zeroth and first derivatives at $t = 0^-$. But, if we know the initial condition for inductor current to be I_o at $t = 0^-$, we can obtain the initial value of first derivative of $v_C(t)$ as I_o/C since $i(t) = C dv_C/dt$. Thus, the differential equation that we attempt to solve is restated as

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{LC} v_s \text{ for } t \geq 0^+ \quad (12.1-2)$$

with $v_C(0^-) = V_o$ V and $\left. \frac{dv_C}{dt} \right|_{(0^-)} = \frac{I_o}{C}$ V/s

12.1.1 Source-Free Response of Series *RLC* Circuit

We focus on the zero-input response (*i.e.*, source-free response) of *RLC* circuit first. $v_s(t) = 0$ for $t \geq 0^+$ for this analysis. Initial energy storage in inductor (evidenced by a non-zero I_o) and/or initial energy storage in capacitor (evidenced by a non-zero V_o) are responsible for non-zero response under this condition. The differential equation describing zero-input response is

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = 0 \text{ for } t \geq 0^+ \quad (12.1-3)$$

with $v_C(0^-) = V_o$ V and $\left. \frac{dv_C}{dt} \right|_{(0^-)} = \frac{I_o}{C}$ V/s

The complementary function of homogeneous differential equations with constant coefficients is $Ae^{\gamma t}$. Substituting this trial solution in the differential equation in Eqn. 12.1-3,

$$\left(\gamma^2 + \frac{R}{L} \gamma + \frac{1}{LC} \right) A e^{\gamma t} = 0 \Rightarrow \gamma^2 + \frac{R}{L} \gamma + \frac{1}{LC} = 0$$

Let the two solutions of this algebraic equation be called α_1 and α_2 . Then,

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \text{ and } \alpha_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\therefore v_C(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

where A_1 and A_2 are two arbitrary constants to be fixed by initial conditions.

The initial conditions are $v_C(0^-) = V_o$ and $\left. \frac{dv_C}{dt} \right|_{(0^-)} = \frac{I_o}{C}$.

But we need the initial conditions at $t = 0^+$ since the differential equation being solved is valid only for $t \geq 0^+$. There is no impulse current flowing into the capacitor in this circuit. And there is no impulse voltage appearing across inductor in this circuit. Hence the initial conditions at $t = 0^+$ are the same as at $t = 0^-$.

$$\therefore v_C(0^+) = V_o \text{ and } \left. \frac{dv_C}{dt} \right|_{(0^+)} = \frac{I_o}{C}.$$

Applying these two initial conditions on the solution,

$$A_1 + A_2 = V_o$$

$$\alpha_1 A_1 + \alpha_2 A_2 = \frac{I_o}{C}$$

Solving for A_1 and A_2 ,

$$A_1 = \frac{\alpha_2 V_o - \frac{I_o}{C}}{\alpha_2 - \alpha_1} \text{ and } A_2 = \frac{\frac{I_o}{C} - \alpha_1 V_o}{\alpha_2 - \alpha_1}$$

12.4 | Series and Parallel *RLC* Circuits

Substituting for A_1 and A_2 in the equation for $v_C(t)$ and collecting terms we get,

$$v_C(t) = V_o \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_o}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \text{ V for } t \geq 0^+ \quad (12.1-4)$$

$$\text{where } \alpha_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

We observe that there are two natural response terms with exponential format in the solution. We observe further that there are two contributions in the zero-input response for $v_C(t)$ – one from initial energy storage in the capacitor and the second from initial energy storage in the inductor.

When we take a square root, we need to be concerned about the sign of quantity under the radical! Therefore, we identify three situations based on the sign of quantity under the radical in the expressions for α_1 and α_2 .

Case-1 α_1 and α_2 real, negative and distinct

This case is a straightforward one and occurs when $R > 2\sqrt{L/C}$. The two distinct roots of characteristic equation of the homogeneous differential equation are real and negative and are arranged on either side of $R/2L$ magnitude-wise. The equation for $v_C(t)$ is as given in Eqn. 12.1-4. The capacitor voltage in this case is an additive mixture of two decaying exponential functions – one with a time constant that is less than $L/2R$ s and another with a time constant that is more than $L/2R$ s. Not only the capacitor voltage but also all the circuit variables will have these two exponential functions in them. The following example illustrates this case. A 0.5 F capacitor 1 H inductor are assumed in the example. 1 H inductance is a practical value. However, 0.5 F capacitor is hard to come by in practice. Capacitors used in practical engineering usually range from pF to mF. A 0.5 F capacitor is perfectly legitimate in a numerical example aimed at illustrating theoretical concepts – at least it keeps the numbers simple!

Example: 12.1-1

A series *RLC* circuit has $R = 3 \Omega$, $L = 1$ H and $C = 0.5$ F. The capacitor is initially charged to 2 V and the initial current in the inductor is 1 A at $t = 0^-$. Find the zero-input response of capacitor voltage and circuit current.

Solution

The differential equation governing the capacitor voltage $v_C(t)$ is

$$\frac{d^2 v_C}{dt^2} + 3 \frac{dv_C}{dt} + 2v_C = 0 \text{ for } t \geq 0^+$$

The characteristic equation is $\gamma^2 + 3\gamma + 2 = 0$ and its roots $\alpha_{1,2} = -1 \text{ s}^{-1}$ and -2 s^{-1} .

Therefore, the general solution for $v_C(t) = A_1 e^{-t} + A_2 e^{-2t}$ V. Applying initial conditions at $t = 0^+$,

$$A_1 + A_2 = 2 \text{ V and } -1 \times A_1 + -2 \times A_2 = 1 \text{ A}/0.5 \text{ F} = 2 \text{ V/s}$$

$$\therefore A_1 = 6 \text{ and } A_2 = -4$$

$$\therefore v_C(t) = 6 e^{-t} - 4 e^{-2t} \text{ V for } t \geq 0^+.$$

And $i(t) = C dv_C/dt = -3e^{-t} + 4e^{-2t}$ A for $t \geq 0^+$.

The capacitor voltage and circuit current contain two decaying exponential transients with time constants of 1 s and 0.5 s. Both exponential terms decay down to zero thereby taking the circuit to zero-energy condition in about 5 to 6 s.

The circuit contained a total initial energy storage of 1.5 J (1 J in the capacitor and 0.5 Joule in the inductor). Both capacitor voltage and current inductor current approach zero as $t \rightarrow \infty$. Therefore total energy storage in the circuit goes to zero with time. Then the total energy dissipated in the resistor from $t = t = 0^+$ to ∞ must be 1.5 J. This is verified as follows.

$$\begin{aligned} \text{Total energy dissipation in } 3\Omega \text{ resistor} &= \int_0^{\infty} 3 \times (-3e^{-t} + 4e^{-2t})^2 dt \\ &= \int_0^{\infty} 3 \times (9e^{-2t} + 16e^{-4t} - 24e^{-3t}) dt \\ &= -\frac{27}{2}e^{-2t} \Big|_0^{\infty} - 12e^{-4t} \Big|_0^{\infty} + 24e^{-3t} \Big|_0^{\infty} = 1.5 \text{ J} \end{aligned}$$

The time-variation of all the circuit variables under zero-input response conditions is shown in Fig. 12.1-2.

Both the natural response terms are decaying exponential functions. But that, by no means, implies that all the circuit variables will decay monotonically from $t = 0^+$ onwards. Notice that the voltage across capacitor increases from its initial value of 2 V to 2.25 V first before it starts decaying. This is so because the initial current in inductor charges up the capacitor further.

The difference between two decaying exponential functions can exhibit a maximum or minimum. In the present example, $v_C(t)$ reaches a maximum and $i(t)$ reaches a minimum before they settle down to zero as $t \rightarrow \infty$.

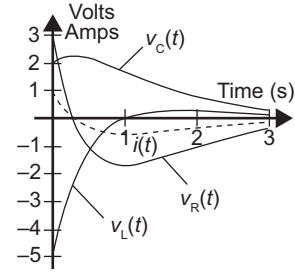


Fig. 12.1-2 Zero-input response of series *RLC* circuit in Example: 12.1-1

Case-2 α_1 and α_2 real, negative and equal

This case occurs when $R = 2\sqrt{L/C}$ and then $\alpha_1 = \alpha_2 = -R/2L$. The limiting form of expression for $v_C(t)$ given in Eqn. 12.1-4 has to be found in this case. We had found out such a limiting form in Section 11.3 in Chapter 11 in the context of zero-state response of a *RC* circuit excited by an exponential signal with a time constant equal to the time constant of the circuit itself. We had employed the series expansion of exponential function for that purpose. Here we try a different approach. We denote α_2 as $\alpha_1 + \Delta\alpha$ and determine the limit of $v_C(t)$ as $\Delta\alpha \rightarrow 0$.

The expression for $v_C(t)$ in a general case is repeated in Eqn. 12.1-5.

$$v_C(t) = V_o \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_o}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \text{ V for } t \geq 0^+ \quad (12.1-5)$$

This equation is first recast in the following equivalent form.

$$\begin{aligned}
 v_C(t) &= V_o \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t} + \alpha_1 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_o}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \text{ V for } t \geq 0^+ \\
 &= V_o \left[e^{\alpha_1 t} + \alpha_1 \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \right] - \frac{I_o}{C} \left[\frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \right] \text{ V for } t \geq 0^+ \\
 &= V_o \left[e^{\alpha_1 t} - \alpha_1 \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_1 - \alpha_2} \right] + \frac{I_o}{C} \left[\frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_1 - \alpha_2} \right] \text{ V for } t \geq 0^+
 \end{aligned}$$

Now, let $\alpha_1 = \alpha$ and $\alpha_2 = \alpha + \Delta\alpha$. Then,

$$\begin{aligned}
 v_C(t) &= V_o \left[e^{\alpha t} - \alpha_1 \frac{e^{\alpha t} - e^{(\alpha+\Delta\alpha)t}}{-\Delta\alpha} \right] + \frac{I_o}{C} \left[\frac{e^{\alpha t} - e^{(\alpha+\Delta\alpha)t}}{-\Delta\alpha} \right] \text{ V for } t \geq 0^+ \\
 e^{\alpha t} - e^{(\alpha+\Delta\alpha)t} &= e^{\alpha t} [1 - e^{\Delta\alpha t}] \approx -e^{\alpha t} \times \Delta\alpha t \text{ for small } \Delta\alpha \\
 \lim_{\Delta\alpha \rightarrow 0} v_C(t) &= V_o e^{\alpha t} + \left(\frac{I_o}{C} - \alpha V_o \right) t e^{\alpha t} \text{ V for } t \geq 0^+ \text{ where } \alpha = -\frac{R}{2L} \text{ sec}^{-1}
 \end{aligned}$$

Therefore, the circuit solution in this case is

$v_C(t) = V_o e^{-(R/2L)t} + \left(\frac{I_o}{C} + \frac{R}{2L} V_o \right) t e^{-(R/2L)t}$ V for $t \geq 0^+$. The other circuit variables may be readily obtained now. The next example illustrates this case.

Example: 12.1-2

A series *RLC* circuit has $R = 2 \Omega$, $L = 1$ H and $C = 1$ F. The capacitor is initially charged to 2 V and the initial current in the inductor is 2 A at $t = 0^-$. Find the zero-input response of capacitor voltage and circuit current.

Solution

The differential equation governing the capacitor voltage $v_C(t)$ is

$$\frac{d^2 v_C}{dt^2} + 2 \frac{dv_C}{dt} + v_C = 0 \text{ for } t \geq 0^+.$$

The characteristic equation is $\gamma^2 + 2\gamma + 1 = 0$ and its roots $\alpha_{1,2} = -1 \text{ s}^{-1}$ and -1 s^{-1} . The trial solution to be attempted is of the form $v_C(t) = A_1 e^{-t} + A_2 t e^{-t}$ V. Applying initial conditions at $t = 0^+$, we get two equations in A_1 and A_2 .

$$v_C(0^+) = A_1 = 2 \text{ V}; \quad \left. \frac{dv_C}{dt} \right|_{(0^+)} = \frac{I_o}{C} = 2 \text{ V/s} = \left[-A_1 e^{-t} - A_2 t e^{-t} + A_2 e^{-t} \right] \Big|_{(0^+)} = -A_1 + A_2$$

$$\therefore A_1 = 2 \text{ V and } A_2 = 4 \text{ V s}^{-1}$$

$$\therefore v_C(t) = 2e^{-t} + 4te^{-t} \text{ V for } t \geq 0^+ \text{ and } i(t) = C \frac{dv_C}{dt} = -2e^{-t} - 4te^{-t} + 4e^{-t} = 2e^{-t} - 4te^{-t} \text{ A for } t \geq 0$$

$$v_L(t) = L \frac{di}{dt} = 1 \frac{di}{dt} = -2e^{-t} + 4te^{-t} - 4e^{-t} = -6e^{-t} + 4te^{-t} \text{ V for } t \geq 0^+$$

$$v_R(t) = Ri = 2i = 4e^{-t} - 8te^{-t} \text{ V for } t \geq 0^+$$

These waveforms are shown in Fig. 12.1-3.

Case-3 α_1 and α_2 complex and conjugates with negative real parts

Now we come up against the most interesting case of all. This occurs when $R < 2\sqrt{L/C}$. The quantity $\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)$ is negative under this condition and the roots of characteristic equation become

$$\alpha_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

We define three new symbols – mainly for convenience at present. But they will turn out to be important parameters in circuit studies soon. They are defined as in Eqn. 12.1-6.

$$\text{Let } \xi = \frac{R}{2\sqrt{L/C}}, \omega_n = \frac{1}{\sqrt{LC}} \text{ and } \omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{1 - \xi^2} \omega_n \quad (12.1-6)$$

The roots of characteristic equation can be expressed in terms of these new symbols as $\alpha_1 = -\xi\omega_n + j\sqrt{1 - \xi^2}\omega_n$ and $\alpha_2 = -\xi\omega_n - j\sqrt{1 - \xi^2}\omega_n$.

$$v_C(t) = V_o \frac{\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}}{\alpha_2 - \alpha_1} - \frac{I_o}{C} \frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} \text{ V for } t \geq 0^+$$

$$\alpha_{1,2} = -\xi\omega_n \pm j\sqrt{1 - \xi^2}\omega_n = -\xi\omega_n \pm j\omega_d \Rightarrow \alpha_2 - \alpha_1 = -2j\omega_d$$

$$\begin{aligned} & \text{Simplifying } \frac{\alpha_2 e^{\alpha_1 t}}{\alpha_2 - \alpha_1} \\ &= \frac{-1}{2j\sqrt{1 - \xi^2}\omega_n} \left[(-\xi\omega_n - j\omega_d) e^{(-\xi\omega_n + j\omega_d)t} - (-\xi\omega_n + j\omega_d) e^{(-\xi\omega_n - j\omega_d)t} \right] \end{aligned}$$

Now we use Eqn. 12.1-4 for the zero-input response for $v_C(t)$ and substitute these expressions for α_1 and α_2 and employ Euler's formula for algebraic simplification of the resulting expression.

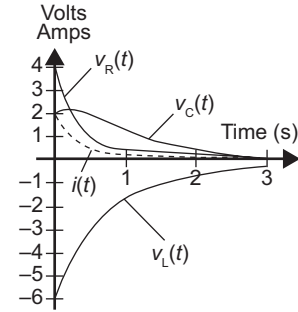


Fig. 12.1-3 Zero-input response waveforms of a series *RLC* circuit in Example: 12.1-2

$$\begin{aligned}
&= \frac{-e^{-\xi\omega_n t}}{2j\sqrt{1-\xi^2}\omega_n} \left[(-\xi\omega_n - j\omega_d)e^{(j\omega_d)t} - (-\xi\omega_n + j\omega_d)e^{(-j\omega_d)t} \right] \\
&= \frac{-e^{-\xi\omega_n t}}{2j\sqrt{1-\xi^2}\omega_n} \left[-\xi\omega_n \left(e^{(j\omega_d)t} - e^{(-j\omega_d)t} \right) - j\omega_d \left(e^{(j\omega_d)t} + e^{(-j\omega_d)t} \right) \right] \\
&= \frac{-e^{-\xi\omega_n t}}{2j\sqrt{1-\xi^2}\omega_n} \left[-\xi\omega_n (2j \sin \omega_d t) - j\omega_d (2 \cos \omega_d t) \right] \text{ (by Euler's formula)} \\
&= e^{-\xi\omega_n t} \left[\frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t + \cos \omega_d t \right] \left(\because \omega_d = \sqrt{1-\xi^2}\omega_n \right)
\end{aligned}$$

Similarly simplifying $\frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1}$,

$$\begin{aligned}
\frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_2 - \alpha_1} &= -\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}\omega_n} \sin \omega_d t \\
\therefore v_C(t) &= V_o e^{-\xi\omega_n t} \left[\frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t + \cos \omega_d t \right] + \frac{I_o}{C} \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}\omega_n} \sin \omega_d t
\end{aligned}$$

Substituting $\omega_n = \frac{1}{\sqrt{LC}}$ in the second term ,

$$v_C(t) = V_o e^{-\xi\omega_n t} \left[\frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t + \cos \omega_d t \right] + I_o \sqrt{\frac{L}{C}} e^{-\xi\omega_n t} \frac{1}{\sqrt{1-\xi^2}} \sin \omega_d t \quad \text{V}$$

Therefore, the zero-input response for capacitor voltage in this case with $0 \leq \xi < 1$ is given by the following expression in terms of ξ and ω_n .

$$v_C(t) = e^{-\xi\omega_n t} \left[\left(\frac{V_o \xi + I_o \sqrt{\frac{L}{C}}}{\sqrt{1-\xi^2}} \right) \sin \sqrt{1-\xi^2}\omega_n t + V_o \cos \sqrt{1-\xi^2}\omega_n t \right] \quad \text{V for } t \geq 0^+ \quad (12.1-7)$$

$$\text{where } \omega_n = \frac{1}{\sqrt{LC}}, \xi = \frac{R}{2\sqrt{L/C}}$$

That, indeed, looks slightly complicated. We will take it up through special cases soon. But before that, here is how it looks for a series *RLC* circuit with $L = 1$ H, $C = 1$ F, $R = 0.5 \Omega$, $V_o = 2$ V and $I_o = 2$ A. The value of ξ is 0.25 and ω_n is 1.

The zero-input response has an *exponentially damped sinusoidal shape*. All the circuit variables oscillate with the amplitude of oscillation decreasing with time. An exponential function governs the decrease in amplitude with time. The numbers ξ and ω_n decide the decay rate of this exponential function. They also decide the time-interval between successive zero-crossings of circuit variables.

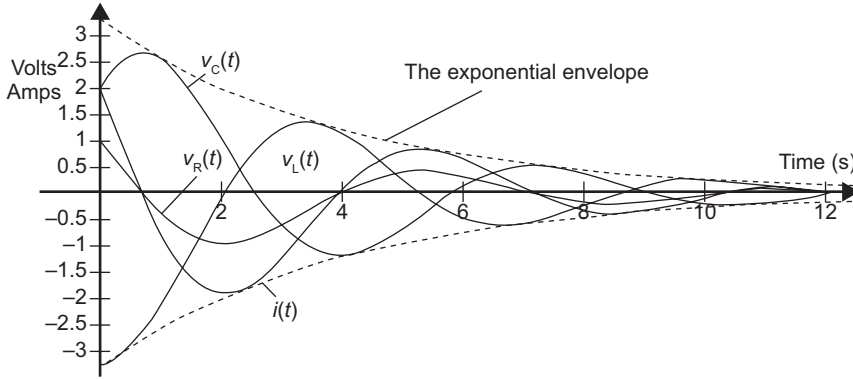


Fig. 12.1-4 ZIR Underdamped zero-input response of a series RLC circuit with $R = 0.5\Omega$, $L = 1\text{H}$, $C = 1\text{F}$, $V_o = 2\text{V}$ and $I_o = 2\text{A}$

12.2 THE SERIES LC CIRCUIT – A SPECIAL CASE

We consider a special case of series RLC circuit – the one with resistance equal to zero. Hence $\xi = 0$, $\omega_n = 1/\sqrt{LC}$ and $\omega_d = \omega_n$. Hence Eqn. 12.1-7 reduces to

$$v_C(t) = V_o \cos \omega_n t + I_o \sqrt{\frac{L}{C}} \sin \omega_n t \quad \text{V for } t \geq 0^+ \quad (12.2-1)$$

Further,

$$\begin{aligned} i(t) &= C \frac{dv_C(t)}{dt} = I_o \cos \omega_n t - V_o \sqrt{\frac{C}{L}} \sin \omega_n t \quad \text{A for } t \geq 0^+ \\ v_L(t) &= -v_C(t) = -V_o \cos \omega_n t - I_o \sqrt{\frac{L}{C}} \sin \omega_n t \quad \text{V for } t \geq 0^+ \end{aligned} \quad (12.2-2)$$

Eqns 12.2-1 and 12.2-2 show that the voltage variables and current variable in a pure LC circuit are sinusoidal in nature. The amplitudes of sinusoids are decided by the initial voltage across the capacitor and initial current through the inductor and the circuit parameters. But the initial voltage across capacitor and initial current through the inductor in turn decide the total *energy storage* in the LC circuit at $t = 0^-$. Since there are no impulse voltages and currents in the LC circuit under source-free condition, the initial condition values at $t = 0^-$ and $t = 0^+$ are the same. Therefore, the energy storage in the elements also will be the same at $t = 0^-$ and $t = 0^+$.

Thus, the source-free response (equivalently the zero-input response) of a pure LC circuit will contain undying sinusoids with steady amplitudes. The amplitude of sinusoidal waveforms is decided by the total initial energy storage in the circuit and the circuit parameters. Circuit parameters, *i.e.*, L and C decide the angular frequency of oscillations too – it is $\sqrt{1/LC}$ rad/s.

We may recast the expressions that involve sum of two sinusoidal functions in Eqns 12.2-1 and 12.2-2 as single sinusoidal functions by employing trigonometric identities in the following manner.

12.10 | Series and Parallel *RLC* Circuits

$$\begin{aligned}
 v_C(t) &= \sqrt{V_o^2 + \frac{LI_o^2}{C}} \cos(\omega_n t - \phi) \text{ V for } t \geq 0^+ \\
 i(t) &= -\sqrt{I_o^2 + \frac{CV_o^2}{L}} \sin(\omega_n t - \phi) \text{ A for } t \geq 0^+ \\
 \text{where } \phi &= \tan^{-1} \left(\frac{I_o \sqrt{L/C}}{V_o} \right)
 \end{aligned} \tag{12.2-3}$$

These waveforms are shown in Fig. 12.2-1 for $L = 1 \text{ H}$, $C = 1 \text{ F}$, $V_o = 2 \text{ V}$ and $I_o = 1 \text{ A}$.

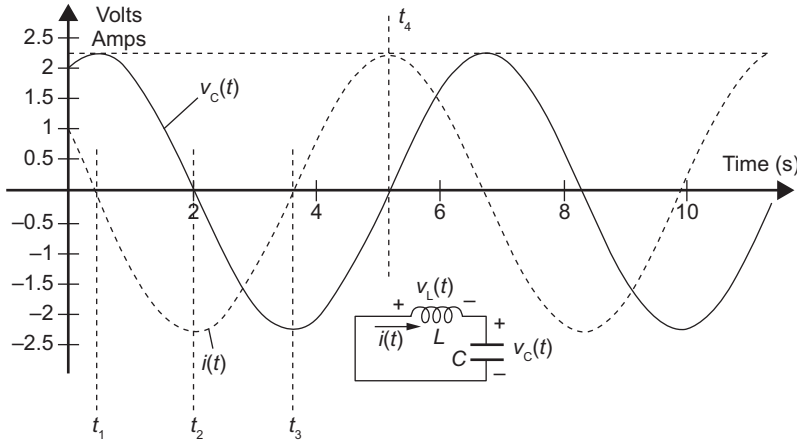


Fig. 12.2-1 Zero-input response of a LC circuit ($L = 1 \text{ H}$, $C = 1 \text{ F}$, $V_o = 2 \text{ V}$ and $I_o = 1 \text{ A}$)

The initial voltage of 2 V across the capacitor appears across the inductor at $t = 0^+$ with a polarity such that the inductor current starts decreasing at the rate of $2\text{V}/1\text{H} = 2 \text{ A/s}$ from its initial value of 1 A. However, the circuit current which flows from left to right charges the capacitor. The capacitor voltage increases while the inductor current decreases. Under the action of increasing reverse voltage, the inductor current decreases more rapidly to reach zero at the instant t_1 . At that instant, the current and hence the energy storage in inductor are zero. The inductor had an initial energy of 0.5 J and the capacitor had an initial energy of 2 J. There was no dissipation in the circuit. Therefore, when the circuit current reaches zero, the capacitor must hold the total initial energy of 2.5 J in it. It will require $\sqrt{5} \text{ V}$ across it (since $C = 1 \text{ F}$ and energy = $0.5 CV^2$). Equation 12.2-3 predicts exactly this value as the amplitude of $v_C(t)$. When circuit current goes through zero, capacitor voltage must go through a positive or negative peak due to two reasons – firstly, the current through a capacitor is proportional to rate of change of voltage across it and secondly that is the instant at which it will contain the maximum possible energy equal to the total initial energy. Therefore $v_C(t)$ reaches a positive peak at t_1 .

With such a large reverse voltage across it, the inductor has to continue its current build up in the negative direction. But, with the current changing its direction, the capacitor enters discharge mode and its voltage decreases. Hence, during (t_1, t_2) the circuit current builds up in the negative direction with progressively decreasing rate and voltage across capacitor decreases to zero at t_2 . When capacitor voltage reaches zero, its energy storage also reaches zero. Therefore, the inductor must be containing all the

circuit energy in it at that instant. Hence the circuit current has to be $-\sqrt{5}$ A at that instant (since $L = 1$ H and energy = $0.5LI^2$). Equation 12.2-3 predicts exactly this value as the amplitude of $i(t)$. When voltage across capacitor (and hence voltage across inductor) goes through zero, circuit current must go through a positive or negative peak due to two reasons – firstly, the voltage across an inductor is proportional to rate of change of current through it and secondly that is the instant at which it will contain the maximum possible energy equal to the total initial energy. Therefore $i(t)$ reaches a negative peak at t_2 .

Now the large discharge current flowing out of the capacitor will result in build-up of capacitor voltage in the negative direction during the interval (t_2, t_3) . The consequent change in polarity of inductor voltage will result in the circuit current turning back from its negative peak. The circuit current decreases in magnitude and capacitor voltage increases in magnitude. At t_3 the circuit current reaches zero and capacitor voltage reaches negative peak of $-\sqrt{5}$ V.

The total initial stored energy gets shunted out from inductor to capacitor and back in this manner periodically. This initial energy can not be dissipated since there is no resistance in the circuit. The circuit tries to get rid of its initial energy but fails to do so. This results in voltage and current oscillations.

In this case, the sum of energy storage in capacitor and inductor must be equal to the total initial energy storage in the circuit. Calculating the sum as $\frac{Li(t)^2}{2} + \frac{Cv_C(t)^2}{2}$ by substituting the expressions for $v_C(t)$ and $i(t)$ from Eqn. 12.2-3 will indeed show that the sum is equal to $\frac{LI_o^2}{2} + \frac{CV_o^2}{2}$, which is the total initial energy.

The frequency of voltage and current oscillations in a pure LC circuit is $1/\sqrt{LC}$ rad/s or $1/2\pi\sqrt{LC}$ Hz and period of oscillations is $2\pi\sqrt{LC}$ sec. The reader may convince himself of the dimensional consistency of these expressions. We had defined ω_n to be equal to $1/\sqrt{LC}$ earlier. Thus, ω_n of a RLC circuit is the angular frequency of pure sinusoidal oscillations that will take place in source-free conditions if the dissipation in the circuit is reduced to zero – in the case of a series RLC circuit this amount to reducing the resistance to zero. Energy loss mechanisms cause *damping* of oscillations in systems. Thus, resistance is a damping element in an electrical circuit. ω_n is the angular frequency of oscillations in an oscillating electrical circuit when *damping* is zero in the circuit – hence it is called the *undamped natural frequency of oscillations*.

Since the energy storage in capacitor is proportional to the square of its voltage, the frequency of oscillation of stored energy will be double that of voltage waveform. Moreover, the stored energy function will always be positive valued and hence will have a positive average value. It will oscillate about a positive average value with a frequency of $1/\pi\sqrt{LC}$ Hz. These aspects are shown clearly in Fig. 12.2-2 that shows the time-variation of stored energy in the capacitor and inductor along with the time-variation of total stored energy in the circuit for the circuit referred in Fig. 12.2-1.

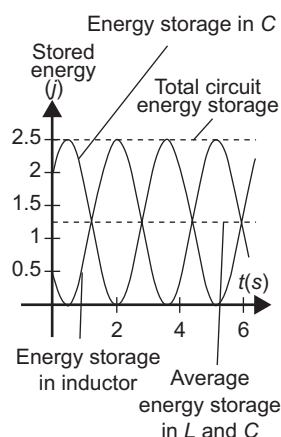


Fig. 12.2-2 Time-variation of stored energy under zero-input response condition in an LC circuit

12.12 | Series and Parallel *RLC* Circuits

Energy storage in both elements varies between 0 to 2.5 J sinusoidally. The average energy storage in both elements is 1.25 J. The total stored energy in the circuit is a constant as expected.

Example: 12.2-1

A pure LC series circuit has $L = 0.2$ H, $C = 10,000$ μ F, $V_o = 100$ V and $I_o = 10$ A. (i) Find the natural frequency of oscillation and its period. (ii) Predict the amplitude of voltage and current oscillations in the source-free response of the circuit employing energy-based arguments. (iii) Obtain $v_C(t)$ and $i(t)$ in the circuit ?

Solution

$$\text{Natural frequency of oscillation} = \sqrt{\frac{1}{0.2 \times 10,000 \times 10^{-6}}} = 22.36 \text{ rad/s} = 3.56 \text{ Hz}$$

$$\text{Period of oscillations} = \frac{1}{3.56} = 0.281 \text{ sec}$$

$$\text{Total initial stored energy} = \frac{0.2 \times 10 \times 10}{2} + \frac{10^{-2} \times 100 \times 100}{2} = 60 \text{ J}$$

The capacitor must hold this much energy when its voltage reaches positive or negative peak.

$\therefore$ Amplitude of $v_C(t) = \sqrt{\frac{60 \times 2}{10^{-2}}} = 109.54$ V. Similarly, the inductor must hold 60 J of energy when its current reaches positive or negative peak. $\therefore$ Amplitude of $i(t) = \sqrt{\frac{2 \times 60}{0.2}} = 24.5$ A.

$\therefore v_C(t) = 109.54 \cos(22.36t + \phi)$ V where ϕ is to be found out using initial conditions. Substituting initial condition for $v_C(t)$, $100 = 109.54 \cos(\phi) \Rightarrow \phi = \pm 0.42$ rad. We use the initial condition for inductor to fix the sign of this phase angle.

$$i(t) = C \frac{dv_C(t)}{dt} = -10^{-2} \times 109.54 \times 22.36 \times \sin(22.36t + \phi) = -24.5 \sin(22.36t + \phi) \text{ A}$$

Applying initial condition $\Rightarrow -24.5 \sin \phi = 10 \Rightarrow \phi = -0.42$ rad. Therefore,

$$v_C(t) = 109.54 \cos(22.36t - 0.42) \text{ V for } t \geq 0^+$$

$$i(t) = -24.5 \sin(22.36t - 0.42) \text{ A for } t \geq 0^+$$

Of course, we could have used Eqn. 12.2-3 directly to obtain the solution. But then, that will require committing that equation to memory. Moreover, committing too many equations to memory does not take an engineer very far in his professional career.

Example: 12.2-2

Show that the ratio of amplitude of voltage oscillation to the amplitude of current oscillation in a LC circuit zero-input response will be $\sqrt{L/C}$.

Solution

Let V_o and I_o be the initial values for capacitor voltage and inductor current in the LC circuit. And let V_m and I_m be the amplitudes of voltage oscillation and current oscillation respectively.

$$\text{Total initial stored energy} = \frac{LI_o^2}{2} + \frac{CV_o^2}{2}$$

$$\text{Maximum stored energy in the Capacitor} = \frac{CV_m^2}{2}$$

These two have to be equal.

$$\therefore V_m = \sqrt{V_o^2 + \frac{L}{C} I_o^2}$$

$$\text{Similarly the maximum stored energy in the inductor} = \frac{LI_m^2}{2}$$

This has to be equal to the total initial energy storage.

$$\therefore I_m = \sqrt{I_o^2 + \frac{C}{L} V_o^2}$$

There is yet another way to arrive at I_m .

$$i(t) = C \frac{dv_C(t)}{dt}$$

$$\therefore \text{Amplitude of current oscillation} = \omega_n C \times \text{amplitude of voltage oscillation}$$

$$\therefore I_m = C \times \sqrt{\frac{1}{LC}} \times \sqrt{V_o^2 + \frac{L}{C} I_o^2} \times \sqrt{I_o^2 + \frac{C}{L} V_o^2}$$

Both approaches lead to the final conclusion that $\frac{V_m}{I_m} = \sqrt{\frac{L}{C}}$. The reader may verify that this ratio has dimension of resistance. This resistance is at times referred to as the *characteristic resistance* of LC circuit.

We observe that the amplitude of oscillation in a LC circuit depends only on the *magnitude* of initial conditions and not on their *polarity*. This is indeed expected since stored energy is a square function of initial conditions. The *polarity* of initial conditions decides the phase angle of oscillations.

Absence of any dissipation mechanism dooms a pure LC circuit with non-zero energy storage to oscillate forever. But suppose there is a little resistance in series – so little that there is no marked change in the shape of $v_C(t)$ and $i(t)$ in the first cycle. However, in that case, the current which flows through the resistance must dissipate some energy cycle after cycle (however minute that may be) leading to reduction of total available stored energy in the circuit. Moreover, that must surely lead to gradual loss of amplitude of voltage and current oscillation. Sometime or other circuit must settle down to a zero-energy state.

Therefore, we throw in a little damping in a LC circuit in the next section.

12.3 THE SERIES LC CIRCUIT WITH SMALL DAMPING – ANOTHER SPECIAL CASE

Now we need the full expression for $v_C(t)$ given in Eqn. 12.1-7 to proceed with our analysis.

$$v_C(t) = e^{-\xi \omega_n t} \left[\left(\frac{V_o \xi + I_o \sqrt{\frac{L}{C}}}{\sqrt{1 - \xi^2}} \right) \sin \sqrt{1 - \xi^2} \omega_n t + V_o \cos \sqrt{1 - \xi^2} \omega_n t \right] \text{ for } t \geq 0^+ \quad (12.3-1)$$

$$\text{where } \omega_n = \frac{1}{\sqrt{LC}}, \xi = \frac{R}{2\sqrt{L/C}}$$

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The resistance is small since damping is taken to be small. We note from the equation above that, for given L and C values the number ξ decides the rate of loss of amplitude of oscillations. Therefore, ξ has to be small in a case in which damping is small. Therefore, the assumption of small damping is equivalent to the assumption that $R \ll 2\sqrt{L/C}$, i.e., small resistance compared to $2\sqrt{L/C}$. This resistance, $2\sqrt{L/C}$, which is a characteristic value for a given series *RLC* circuit is termed as its *critical resistance*. Since amplitude loss is basically due to *damping* and since the factor ξ governs the rate of loss of amplitude, this factor ξ is aptly termed as *damping factor* in circuit studies. Thus, we are dealing with series *RLC* circuit with $\xi \ll 1$ in this section. We make use of this fact to simplify the expression in Eqn. 12.3-1 considerably.

We employ the following approximations in this context.

$$\frac{V_o \xi + I_o \sqrt{\frac{L}{C}}}{\sqrt{1 - \xi^2}} \approx I_o \sqrt{\frac{L}{C}} \text{ for } \xi \ll 1$$

$$\omega_d = \sqrt{1 - \xi^2} \omega_n \approx \omega_n \text{ for } \xi \ll 1$$

$$\therefore v_C(t) \approx e^{-\xi \omega_n t} \left[I_o \sqrt{\frac{L}{C}} \sin \omega_n t + V_o \cos \omega_n t \right]$$

$$\text{and } i(t) = C \frac{dv_C(t)}{dt} = e^{-\xi \omega_n t} \left[I_o \cos \omega_n t - V_o \sqrt{\frac{C}{L}} \sin \omega_n t \right] \quad (12.3-2)$$

The algebra involved in the second step in Eqn. 12.3-2 has been skipped. Reader is urged to verify the derivation of $i(t)$.

Expressing the expressions for $v_C(t)$ and $i(t)$ in Eqn. 12.3-2 as single sinusoids, we have,

$$v_C(t) \approx \sqrt{V_o^2 + \frac{L I_o^2}{C}} e^{-\xi \omega_n t} \cos(\omega_n t - \phi) \text{ V for } t \geq 0^+$$

$$i(t) \approx -\sqrt{I_o^2 + \frac{C V_o^2}{L}} e^{-\xi \omega_n t} \sin(\omega_n t - \phi) \text{ A for } t \geq 0^+ \quad (12.3-3)$$

$$\text{where } \phi = \tan^{-1} \left(\frac{I_o \sqrt{L/C}}{V_o} \right)$$

We see that the expressions for $v_C(t)$ and $i(t)$ in the case of series *RLC* circuit with small damping is different from corresponding functions in the case of an undamped LC circuit only by the factor $e^{-\xi \omega_n t}$ appearing as a multiplier. Compare Eqns 12.2-3 and 12.3-3.

Let us examine how the total stored energy in the circuit evolves in time in this case.

$$\begin{aligned}
 \text{Total stored energy} &= 0.5Li(t)^2 + 0.5Cv_C(t)^2 \\
 &= e^{-2\xi\omega_n t} \left[0.5L \left(I_o^2 + \frac{CV_o^2}{L} \right) \cos^2(\omega_n t - \phi) + 0.5C \left(V_o^2 + \frac{LI_o^2}{C} \right) \sin^2(\omega_n t - \phi) \right] \\
 &= (0.5LI_o^2 + 0.5CV_o^2) e^{-2\xi\omega_n t}
 \end{aligned}$$

It is not a constant anymore. Starting at the initial value (equal to the total initial energy storage in L and C), it decreases exponentially with time and goes to zero as $t \rightarrow \infty$. Once again, the damping factor ξ governs the rate at which the total stored energy decreases with time.

It is instructive to look at the amount of energy lost in one oscillation period. *However, is there a period of oscillation in this case?*

The product of exponential function and sinusoidal function is not a sinusoid. It will not be a periodic waveform at all. However, the time interval between the successive zero-crossings of the waveform will remain constant since this is decided by the sinusoid in the product. Therefore, we can not talk about a *period* in the case of damped oscillations; but we can talk about the time interval between two successive zero-crossings in the same direction. Sometimes this interval is called ‘period’ of damped oscillation in loose manner.

Let us find out the energy loss incurred over one such interval and express it as a fraction of the total energy storage at the beginning of that interval.

Let the variable $\omega_n t$ be $k\pi$ at the beginning of one such interval (i.e., at a zero-crossing) where k is an integer. The value of total energy storage at this point is $(0.5LI_o^2 + 0.5CV_o^2) e^{-2k\pi\xi}$. The next similar zero crossing will take place at $\omega_n t + 2\pi$ rad. The value of total energy storage at that time point will be $(0.5LI_o^2 + 0.5CV_o^2) e^{-2(k+2)\pi\xi}$ J.

$$\begin{aligned}
 \therefore \text{Fractional loss of stored energy over one oscillation} &= \frac{e^{-2k\pi\xi} - e^{-2(k+2)\pi\xi}}{e^{-2k\pi\xi}} \\
 &= 1 - e^{-4\pi\xi} = \left(1 - 1 - 4\pi\xi + \frac{(4\pi\xi)^2}{2!} + \dots \right) \\
 &= 4\pi\xi \text{ for } \xi \ll 1
 \end{aligned}$$

By a similar reasoning one can see that the fractional loss of amplitude of voltage and current oscillations in the circuit over one oscillation period $\approx 2\pi\xi$, i.e., the amplitude of oscillation in the $(n+1)^{\text{th}}$ cycle of oscillation will be $\approx (1 - 2\pi\xi)$ times the amplitude of n^{th} oscillation.

All the voltage variables and the circuit current variable during the zero-input response of a weakly damped series RLC circuit with $L = 1$ H, $C = 1$ F, $R = 0.08 \Omega$, $V_o = 2$ V and $I_o = 1$ A are plotted in Fig. 12.3-1. The damping factor ξ in this case is 0.04 and the undamped natural frequency ω_n is 1 rad/s. The successive positive peaks of $v_C(t)$ are 2.22 V and 1.71 V and negative peaks are 1.95 V and 1.51 V. The fractional loss of amplitude over first cycle of oscillation calculated from these numbers are 0.23 and 0.226 and are almost equal to $2\pi\xi = 0.25$. The exponential envelope function has a time constant of $1/0.04 = 25$ s. One full oscillation takes about 6.3 s and hence there will be roughly about 9 oscillations before the amplitude of oscillation get damped down to less than 10% of their starting values.

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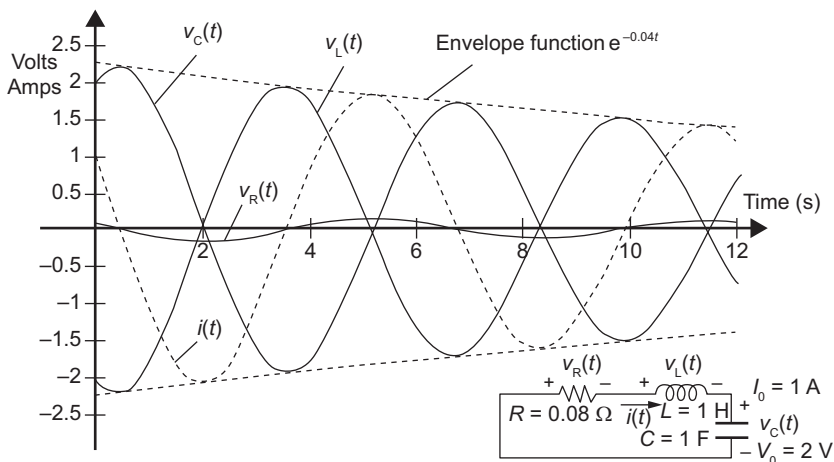


Fig. 12.3-1 Zero-input response of a weakly damped *RLC* circuit ($L = 1 \text{ H}$, $C = 1 \text{ F}$, $R = 0.08 \Omega$, $V_0 = 2 \text{ V}$ and $I_0 = 1 \text{ A}$)

The time-variation of various energy storage functions for zero-input response of this circuit is plotted in Fig. 12.3-2.

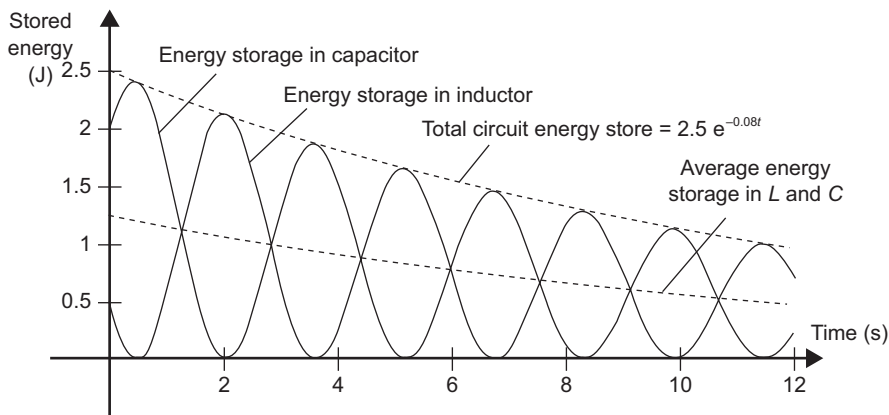


Fig. 12.3-2 Energy storage functions of a weakly damped *RLC* circuit ($L = 1 \text{ H}$, $C = 1 \text{ F}$, $R = 0.08$, $V_0 = 2 \text{ V}$ and $I_0 = 1 \text{ A}$)

This *exponential decay* of total energy storage in a series *RLC* circuit is applicable only when $\xi \ll 1$. Total energy storage will come down with time for other values of ξ too; but not as per a clean exponential law. This is illustrated in Fig. 12.3-3 for a circuit with $\xi = 0.25$. $v_C(t)$ and $i(t)$ for the circuit referred in this figure was already plotted in Fig. 12.1-4.

The nature of zero-input response when ξ is not small is qualitatively similar to that of the weakly damped case. However, the time-interval in the zero crossings (*period* of oscillation in a loose sense) is not $2\pi/\omega_n$ – it is rather $2\pi/\omega_d$ where $\omega_d = \sqrt{1 - \xi^2} \omega_n$. This number is termed as ‘*damped natural frequency*’ in circuit studies.

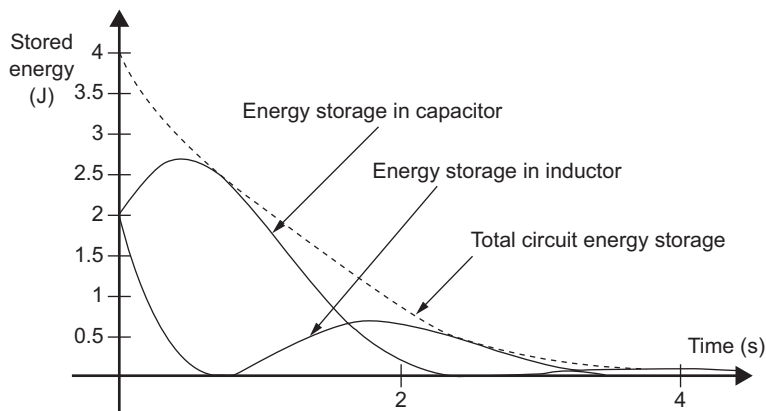


Fig. 12.3-3 Energy storage functions of a moderately damped ($\xi = 0.25$) RLC circuit ($L = 1$ H, $C = 1$ F, $R = 0.5\Omega$, $V_o = 2$ V and $I_o = 2$ A)

12.4 STANDARD FORMATS FOR SECOND-ORDER CIRCUIT ZERO-INPUT RESPONSE

Second-order circuits embody all the three kinds of responses possible in any linear system. It is a widely studied and applied response. Hence certain standard formats have emerged in dealing with second-order system response in the areas of circuit theory, linear control systems, linear electronic circuits, linear system analysis, etc. We have already introduced many terms associated with this standardised format and we complete the rest in this section.

A second-order circuit will have a homogeneous differential equation of the following type where $x(t)$ is some circuit variable chosen as the describing variable for the circuit.

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_2x = 0 \text{ for } t \geq 0^+ \text{ with } x(0^-) = x_o \text{ and } \left. \frac{dx}{dt} \right|_{(0^-)} = x'_o \quad (12.4-1)$$

The coefficients of this differential equation, a_1 and a_2 , will be decided by circuit parameters like R , L and C . The same differential equation will be obtained quite independent of which particular circuit variable happens to be chosen as the describing variable $x(t)$.

The first standard format for this equation is

$$\frac{d^2x}{dt^2} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2x = 0 \text{ for } t \geq 0^+ \text{ with } x(0^-) = x_o \text{ and } \left. \frac{dx}{dt} \right|_{(0^-)} = x'_o \quad (12.4-2)$$

$$\text{where } \omega_n = \sqrt{a_2} \text{ and } \xi = \frac{a_1}{2\sqrt{a_2}}$$

The second standard format is

$$\frac{d^2x}{dt^2} + \frac{\omega_n}{Q} \frac{dx}{dt} + \omega_n^2x = 0 \text{ for } t \geq 0^+ \text{ with } x(0^-) = x_o \text{ and } \left. \frac{dx}{dt} \right|_{(0^-)} = x'_o \quad (12.4-3)$$

$$\text{where } \omega_n = \sqrt{a_2} \text{ and } Q = \frac{\sqrt{a_2}}{a_1} = \frac{1}{2\xi}$$

12.18 | Series and Parallel *RLC* Circuits

The ξ factor was seen to be responsible for damping of oscillations in series *RLC* circuit and was called *damping factor*. ω_n was seen to be the angular frequency of voltage and current sinusoidal oscillations in an *undamped LC* circuit and was called *undamped natural frequency*. These interpretations for ξ and ω_n are applicable to any second-order circuit or linear system.

The new factor that has appeared in Eqn. 12.4-3 – the *Q* factor – is called the *quality factor* of the second-order circuit or system and is inversely proportional to its *damping factor*. Damping factor and Quality factor are used interchangeably in linear system related studies with electrical filter designers preferring quality factor to damping factor.

Reduced damping leads higher values for *Q* factor. An undamped system has a quality factor that tends to infinity.

Four cases are identified depending on the value of ξ .

1. Case (i) – $\xi > 1$. This is called the *over-damped circuit*. The roots of characteristic equation for such a circuit will be two distinct negative real numbers. Its zero-input response will consist of two decaying real exponential functions of time.
2. Case (ii) – $\xi = 1$. This is called the *critically damped circuit*. The roots of characteristic equation for such a circuit will be a negative real number with multiplicity of two. In general, its zero-input response will consist of one decaying real exponential function and another function that is the product of time and this real exponential function.
3. Case (iii) – $0 < \xi < 1$. This is called the *under-damped circuit*. The roots of characteristic equation of such a circuit are complex conjugate numbers with negative real part. The zero-input response of such a circuit will contain an *exponentially damped sinusoidal oscillation*. The oscillations will not be periodic in the strict sense of periodicity. However, the zero crossings of oscillations will take place at regular intervals and periodicity is understood in this sense. Then the angular

frequency will be given by $\omega_d = \sqrt{1 - \xi^2} \omega_n = \sqrt{1 - \left(\frac{1}{2Q}\right)^2} \omega_n$ rad/s. ω_d is called the *damped natural frequency* in circuit studies.

4. Case (iv) – $\xi = 0$. This is called the *undamped circuit*. The roots of characteristic equation are pure imaginary and conjugates. The zero-input response consists of steady sinusoidal oscillation at angular frequency of ω_n .

There is an unambiguous periodicity in the response in the case of an undamped circuit. The response in the case of an under-damped circuit is also periodic in the sense that there are oscillations and the number of oscillation cycles in a given interval will be correctly given by $\omega_d/2\pi$ Hz. However, the time-functions will not be periodic in the mathematical sense.

The response in the over-damped and critically damped cases are not oscillations and the concept of frequency (if it is understood as the number of times something repeats in one second) is not relevant in those cases. However, linear system theorists decided to call the index of the exponential functions describing the zero-input response of a linear system as its *natural frequencies* long back.

Thus, the roots of characteristic equation of a circuit are called its *natural frequencies*. We should not get confused here trying to count the number of oscillations in a decaying real exponential function. The word ‘frequency’ in the term *natural frequency* has absolutely nothing to do with the idea of *repetitiveness*. Those numbers give us the index of exponential functions that describe the *natural* source-free response of our circuit – nothing more or nothing less. The emphasis is on *natural* and not on *frequency*!

Since roots of characteristic equation of a circuit can be complex, natural frequencies also can be complex. Let $s = \sigma + j\omega$ denote a natural frequency of a linear circuit. Then,

The natural frequency $s = \sigma + j\omega$ stands for a complex exponential signal e^{st} in time-domain. This signal will represent one of the many components present in the zero-input response of the circuit that has s as one of its natural frequencies. Thus, *natural frequency* is a stand-in for e^{st} .

However, $e^{st} = e^{\sigma t} (\cos \omega t + j \sin \omega t)$ is a complex function of time. A physical electrical circuit can not have a non-physical complex time function in its zero-input response. That is why, if $\sigma + j\omega$ is one of its natural frequencies, then, $\sigma - j\omega$ will also be one of its natural frequencies. Natural frequencies of a linear circuit, if complex, appear in conjugates. The complex exponential response components due to conjugate pairs will always add up to yield a real physical waveform.

We complete this section by listing the solution of Eqn. 12.4-1 for the under-damped case in both formats. They are

$$x(t) = e^{-\xi \omega_n t} \left[\left(\frac{x_o \xi + \frac{x'_o}{\omega_n}}{\sqrt{1 - \xi^2}} \right) \sin \omega_d t + x_o \cos \omega_d t \right] \text{ for } t \geq 0^+$$

$$x(t) = e^{-0.5 \frac{\omega_n t}{Q}} \left[\left(\frac{x_o + \frac{2Qx'_o}{\omega_n}}{\sqrt{4Q^2 - 1}} \right) \sin \omega_d t + x_o \cos \omega_d t \right] \text{ for } t \geq 0^+$$

where $\omega_d = \sqrt{1 - \xi^2} \omega_n = \frac{\sqrt{4Q^2 - 1}}{2Q} \omega_n$

12.5 IMPULSE RESPONSE OF SERIES *RLC* CIRCUIT

Impulse response of a circuit is the zero-state response with unit impulse input. The relevant circuit is shown in Fig. 12.5-1 (a).

The capacitor can not absorb the impulse voltage. It can not absorb even a finite voltage change without infinite current flow, let alone impulse voltage. Resistor can not absorb impulse voltage, for in that case the circuit current will become infinite and the inductor won't permit that. Hence, the entire impulse voltage appears across the inductor. This results in 1 Wb-T of flux linkage getting dumped into the inductor instantaneously, resulting in a sudden change in its current from zero to $1/L$ A. Therefore $i(t)$ at $t = 0^+$ becomes $1/L$ A and $v_C(t)$ at $t = 0^+$ remains at 0. And the input source becomes short-circuit for all $t \geq t = 0^+$. Thus, the circuit effectively becomes a source-free circuit shown in (b) Fig. 12.5-1 (b), with initial energy of $1/2L$ J in inductor and 0 J in the capacitor. Thereafter the response is similar to zero-input response.

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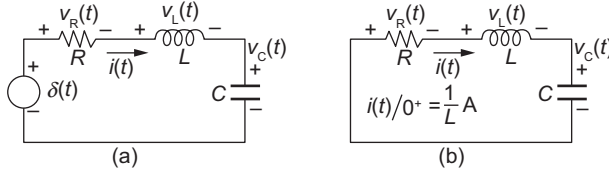


Fig. 12.5-1 Series *RLC* circuit with unit impulse voltage input

$$\begin{aligned}
 v_C(t) &= \left(\frac{\omega_n}{\sqrt{1-\xi^2}} \right) e^{-\xi\omega_n t} \sin \sqrt{1-\xi^2} \omega_n t \text{ V for } t \geq 0^+ \\
 i(t) &= C \frac{dv_C(t)}{dt} = \frac{1}{L} \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos \left(\sqrt{1-\xi^2} \omega_n t + \tan^{-1} \frac{\xi}{\sqrt{1-\xi^2}} \right) \text{ amps for } t \geq 0^+
 \end{aligned} \tag{12.5-1}$$

12.6 STEP RESPONSE OF SERIES *RLC* CIRCUIT

Step response of any linear electrical circuit is the zero-state response with unit step input. This can be found out by integrating the impulse response of the same circuit. Only the under-damped case is considered here.

Impulse Response of a under-damped Series *RLC* Circuit is

$$\begin{aligned}
 v_C(t) &= \left(\frac{\omega_n}{\sqrt{1-\xi^2}} \right) e^{-\xi\omega_n t} \sin \sqrt{1-\xi^2} \omega_n t \text{ V for } t \geq 0^+ \\
 &= \left(\frac{\omega_n}{\sqrt{1-\xi^2}} \right) \frac{e^{(-\xi\omega_n + \sqrt{1-\xi^2})t} - e^{(-\xi\omega_n - \sqrt{1-\xi^2})t}}{2j} \text{ V for } t \geq 0^+
 \end{aligned}$$

$\therefore$ Step Response of the same circuit is

$$v_C(t) = \left(\frac{\omega_n}{2j\sqrt{1-\xi^2}} \right) \int_0^t \left(e^{(-\xi\omega_n + \sqrt{1-\xi^2})t} - e^{(-\xi\omega_n - \sqrt{1-\xi^2})t} \right) dt$$

Carrying out the required integration and simplifying the result, we get,

$$v_C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos \left(\sqrt{1-\xi^2} \omega_n t - \tan^{-1} \frac{\xi}{\sqrt{1-\xi^2}} \right) \text{ V for } t \geq 0^+ \tag{12.6-1}$$

The step response waveforms of a series *RLC* circuit with $L = 1$ H and $C = 1$ F are shown in Fig. 12.6-1 for various values of resistance resulting in damping factors of 0.5, 0.2, 0.05 and 0. Solid curve shows $v_C(t)$ and dotted curve shows $i(t)$ in all cases.

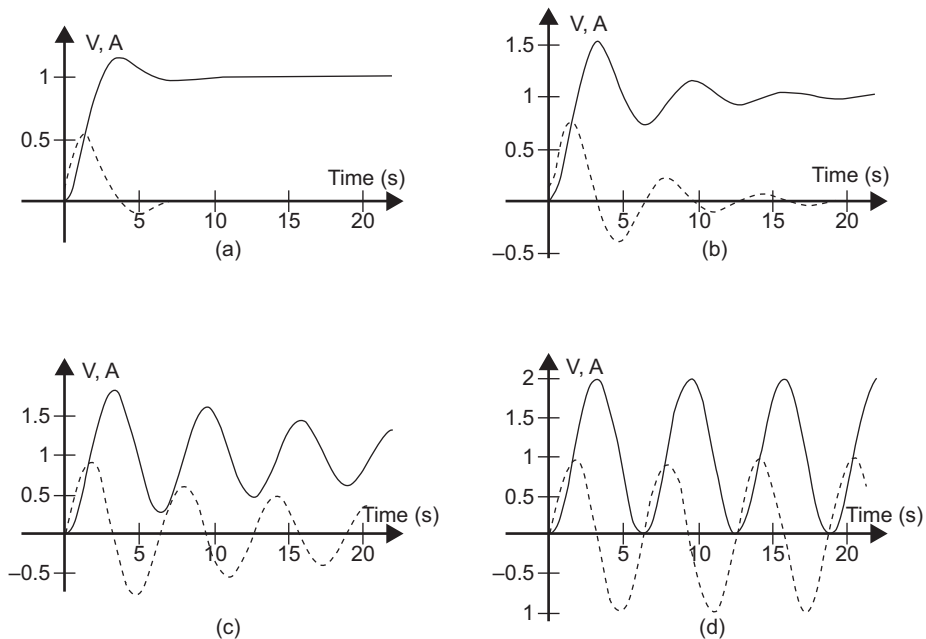


Fig. 12.6-1 Step response of series RLC circuit ($L=1H$, $C=1F$): (a) $\xi = 0.5$, (b) $\xi = 0.2$ (c) $\xi = 0.05$ (d) $\xi = 0$

12.7 STANDARD TIME-DOMAIN SPECIFICATIONS FOR SECOND-ORDER CIRCUITS

Certain measures of speed and nature of response of a second-order system, defined with respect to step response, are in wide use in various areas of electrical and electronic engineering. These measures are as follows:

1. Delay time, t_d – is the time required by step response to cross 50% of final value for the first time.
2. Rise time, t_r – is the time required for step response to reach 100% of final value for the first time starting from 0% in the case of under-damped response. In the case of over-damped response and critically damped response, it is the time required for step response to reach 90% of final value from 10% of final value.
3. Peak time, t_p – is the time required for step response to reach its first peak.
4. Settling time, t_s – is the time required for the step response to reach a $\pm 5\%$ band around the final value and remain within that band thereafter.
5. Maximum (percentage) overshoot, M_p – is the difference between the maximum peak value attained by the step response and the final value of step response expressed as a fraction or percentage of the final value of step response.

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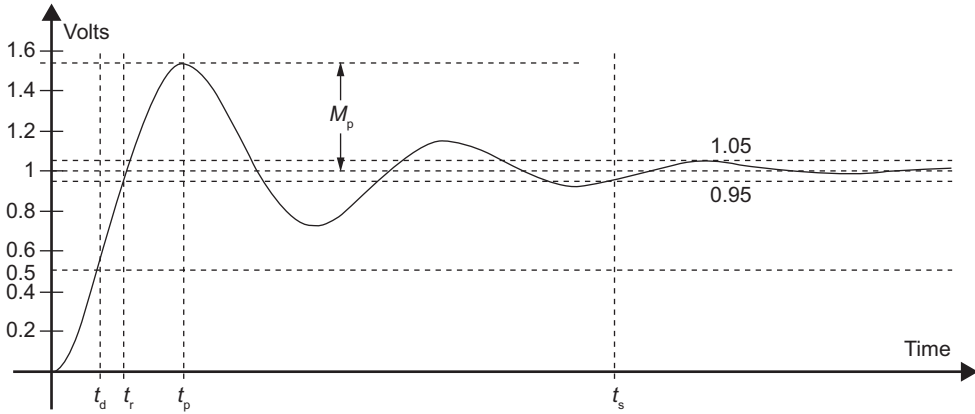


Fig. 12.7-1 Time-domain specifications for a second order circuit

These measures are illustrated on a typical step response plot of a series *RLC* circuit (capacitor voltage is plotted) in the Fig. 12.7-1.

It is possible to derive expressions for these measures starting from Eqn. 12.6-1. The algebra involved is skipped here and the final expressions are given in Eqn. 12.7-1.

$$t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}}{\sqrt{1-\xi^2} \omega_n}, \quad t_p = \frac{\pi}{\sqrt{1-\xi^2} \omega_n}, \quad t_s = \frac{3}{\xi \omega_n}, \quad M_p = e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}} \quad (12.7-1)$$

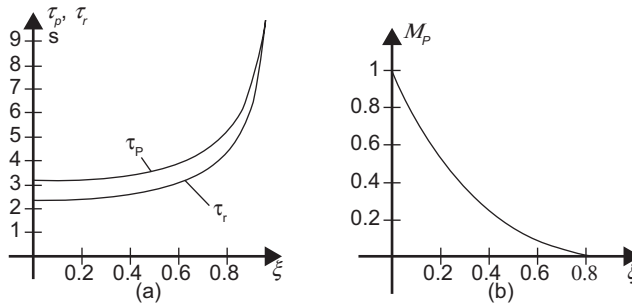


Fig. 12.7-2 Rise time, peak time and maximum overshoot in a Second-order circuit vs. damping factor

The relationship between various time-domain specifications and the damping factor is highly non-linear. Refer Fig. 12.7-2 that shows the variation of rise time (the lower curve), peak time (the upper curve) and maximum overshoot in step response of capacitor voltage in a series *RLC* circuit with an undamped natural frequency of 1 rad/s. $\xi \approx 0.7$ is a critical value. When ξ increases above 0.7, the response becomes *sluggish*. When ξ decreases below 0.7, the maximum *overshoot* increases rapidly without much change in the rise time. Thus, $\xi = 0.7$ is a value of damping factor that results in a step response which has optimal speed without incurring too heavy a penalty in the form of excessive overshoot.

12.8 EXAMPLES ON IMPULSE AND STEP RESPONSE OF SERIES *RLC* CIRCUITS

The circuit variables and polarity conventions for specifying initial conditions in the following examples are the same as in the series *RLC* circuit in Fig. 12.1-1.

Example: 12.8-1

A series *RLC* circuit has $L = 10$ mH, $C = 1000$ μ F and $R = 10$ Ω . The capacitor is initially charged with 10 V. Find an expression for the voltage across the capacitor when the circuit is excited by $10 u(t)$.

Solution

This is a problem that involves both zero-input response and zero-state response. The zero-input response component arises due to the initial voltage across the capacitor. The zero-state response component is nothing but step response scaled by 10.

Zero-Input Response Component

Characteristic equation is $\gamma^2 + 10^3 \gamma + 10^5 = 0$

Natural frequencies of the circuit = -112.7 and -887.3 . This is an over-damped circuit.

$$\therefore v_C(t) = A_1 e^{-112.7 t} + A_2 e^{-887.3 t}$$

$$\text{and } i(t) = 10^{-3} \times (-112.7 A_1 e^{-112.7 t} - 887.3 A_2 e^{-887.3 t})$$

Applying initial conditions of 10 V and 0 A at 0^+ ,

$$A_1 + A_2 = 10 \text{ and } -0.1127 A_1 - 0.8873 A_2 = 0 \Rightarrow A_1 = 11.455 \text{ and } A_2 = -1.455$$

$\therefore v_C(t) = 11.455 e^{-112.7 t} - 1.455 e^{-887.3 t}$ V for $t \geq 0^+$ is the zero-input response.

Zero-State Response Component

The initial conditions at $t = 0^-$ are set to zero for determining zero-state response. The final value of step response in a series *RLC* circuit is 1 V across the capacitor. Hence the final value of response here is 10 V across the capacitor in this example. The format of transient response is known from the values of natural frequencies. Therefore, $v_C(t) = 10 + A_1 e^{-112.7 t} + A_2 e^{-887.3 t}$ is the format of zero-state response for $10 u(t)$ input.

The applied voltage suddenly becomes 10 V at $t = 0^+$. Since the inductor current can not change instantaneously unless impulse voltage comes across it and capacitor voltage can not change instantaneously unless impulse current flows through it, the capacitor voltage and inductor current remain at zero values at $t = 0^+$. The 10 V step appears across the inductor first, thereby changing the initial value of derivative of current. But, we need initial values only for capacitor voltage and inductor current to solve for A_1 and A_2 in the zero-state response above. Applying these initial values on

$$v_C(t) = 10 + A_1 e^{-112.7 t} + A_2 e^{-887.3 t} \text{ and } i(t) = -0.1127 A_1 e^{-112.7 t} - 0.8873 A_2 e^{-887.3 t},$$

we get $A_1 + A_2 = -10$ and $-0.1127 A_1 - 0.8873 A_2 = 0 \Rightarrow A_1 = -11.455$ and $A_2 = 1.455$

$\therefore v_C(t) = 10 - 11.455 e^{-112.7 t} + 1.455 e^{-887.3 t}$ V for $t \geq 0^+$ is the zero-state response.

$\therefore$ Total response of $v_C(t)$ = zero-input response + zero-state response

$$= 11.455 e^{-112.7 t} - 1.455 e^{-887.3 t} + 10 - 11.455 e^{-112.7 t} + 1.455 e^{-887.3 t}$$

$$= 10 \text{ V for } t \geq 0^+$$

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We didn't have to go through all this to see that $v_C(t) = 10$! If the expected final state of a circuit is same as the given initial state, then, there is no transient response in the circuit.

Note the method employed in solving this problem. We could have obtained the zero-input response by direct substitution in Eqn. 12.1-5. But such direct substitutions in equations should be avoided as far as possible and problems should be worked out employing basic principles governing the circuit.

Example: 12.8-2

A series *RLC* circuit has $L = 10$ mH, $C = 100$ μ F and $R = 10$ Ω . The capacitor is initially charged with -10 V and the inductor has 1 A flowing in it at $t = 0^-$. Find an expression for the voltage across the capacitor when the circuit is excited by $10 u(t)$.

Solution

This is a problem that involves both zero-input response and zero-state response. The zero-input response component arises due to the initial voltage across the capacitor and initial current through inductor. The zero-state response component is nothing but step response scaled by 10.

Zero-Input Response Component

Characteristic equation is $\gamma^2 + 10^3 \gamma + 10^6 = 0$

Natural frequencies of the circuit = -500 and -500 . This is an critically damped circuit.

$$\therefore v_C(t) = A_1 e^{-500t} + A_2 t e^{-500t}$$

$$\text{and } i(t) = 10^{-4} \times (-500 A_1 e^{-500t} - 500 A_2 t e^{-500t} + A_2 e^{-500t})$$

Applying initial conditions of -10 V and 1 A at 0^+ ,

$$A_1 = -10 \text{ and } -0.05 A_1 + 0.0001 A_2 = 1 \Rightarrow A_1 = -10 \text{ and } A_2 = 5000$$

$$\therefore v_C(t) = -10 e^{-500t} + 5000 t e^{-500t} \text{ V for } t \geq 0^+ \text{ is the zero-input response.}$$

Zero-State Response Component

The initial conditions at $t = 0^-$ are set to zero values for determining zero-state response. The final value of step response in a series *RLC* circuit is 1 V across the capacitor. Hence the final value of response here is 10 V across the capacitor in this example. The format of transient response is known from the values of natural frequencies. $\therefore v_C(t) = 10 + A_1 e^{-500t} + A_2 t e^{-500t}$ is the format of zero-state response for $10 u(t)$ input.

The initial values of $v_C(t)$ and $i(t)$ remain at zero at $t = 0^+$ as explained in the previous example. Applying these initial values on

$$v_C(t) = 10 + A_1 e^{-500t} + A_2 t e^{-500t} \text{ and } i(t) = -0.05 A_1 e^{-500t} - 0.05 A_2 t e^{-500t} + 0.0001 A_2 e^{-500t},$$

$$\text{we get } A_1 = -10 \text{ and } -0.05 A_1 + 0.0001 A_2 = 0 \Rightarrow A_1 = -10 \text{ and } A_2 = -5000$$

$$\therefore v_C(t) = (10 - 10 e^{-500t} - 5000 t e^{-500t}) \text{ V for } t \geq 0^+ \text{ is the zero-state response.}$$

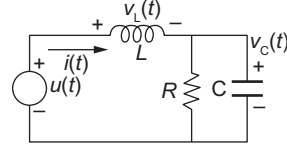
$\therefore$ Total response of $v_C(t)$ = zero-input response + zero-state response

$$= -10 e^{-500t} + 5000 t e^{-500t} + 10 - 10 e^{-500t} - 5000 t e^{-500t}$$

$$= (10 - 20 e^{-500t}) \text{ V for } t \geq 0^+$$

Example: 12.8-3

Develop the differential equation governing $v_C(t)$ in the circuit in Fig. 12.8-1 and find its step response for $L = 16 \text{ mH}$, $C = 10 \text{ } \mu\text{F}$ and $R = 100 \text{ } \Omega$. Obtain the rise time, peak time, settling time and maximum percentage overshoot for this response.

**Fig. 12.8-1** Circuit for Example: 12.8-3**Solution**

Applying KCL at the junction between R , L and C

along with the element equation of a capacitor yields $i(t) = C \frac{dv_C(t)}{dt} + \frac{v_C(t)}{R}$.

Applying KVL in the first mesh yields $L \frac{di(t)}{dt} + v_C(t) = u(t)$.

Substituting the first equation in the second one results in

$$LC \frac{d^2 v_C(t)}{dt^2} + \frac{L}{R} \frac{dv_C(t)}{dt} + v_C(t) = u(t)$$

$$\text{i.e., } \frac{d^2 v_C(t)}{dt^2} + \frac{1}{RC} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{1}{LC} u(t)$$

The critical resistance of this circuit is $\frac{1}{2} \sqrt{\frac{L}{C}}$.

Substituting the parameter values,

$$\frac{d^2 v_C(t)}{dt^2} + 10^3 \frac{dv_C(t)}{dt} + 2500^2 v_C(t) = 2500^2 \text{ for } t \geq 0^+$$

$$\omega_n = 2500 \text{ rad/s, } 2\xi\omega_n = 10^3 \Rightarrow \xi = 0.2$$

$$\therefore \omega_d = \sqrt{1 - \xi^2} \omega_n = 2449.5 \text{ rad/s}$$

The final value of step response is obtained by replacing the capacitor by open-circuit and the inductor by short-circuit. This shows that the final value is 1 V.

The initial conditions required for solving the differential equation are the values of capacitor voltage at $t = 0^+$ and of its first derivative at $t = 0^+$.

$$v_C(0^+) = v_C(0^-) = 0$$

$$\frac{dv_C(t)}{dt} = \frac{i_C(t)}{C} = \frac{i(t) - \frac{v_C(t)}{R}}{C}$$

$$i(0^+) = i(0^-) = 0$$

$$\therefore \left. \frac{dv_C(t)}{dt} \right|_{(0^+)} = 0$$

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The step response for $v_C(t)$ will have the following format with A_1 and A_2 decided by initial conditions.

$$v_C(t) = 1 + e^{-500t} (A_1 \cos 2449.5t + A_2 \sin 2449.5t). \text{ Then,}$$

$$v_C'(t) = -500 e^{-500t} (A_1 \cos 2449.5t + A_2 \sin 2449.5t) + e^{-500t} (-2449.5 A_1 \sin 2449.5t + 2449.5 A_2 \cos 2449.5t)$$

Substituting initial conditions,

$$A_1 = -1 \text{ and } -500 A_1 + 2449.5 A_2 = 0 \Rightarrow A_1 = -1 \text{ and } A_2 = -0.2041$$

$$\therefore \text{Step response } v_C(t) = 1 - e^{-500t} (\cos 2449.5t + 0.2041 \sin 2449.5t)$$

$$= [1 - 1.0206 e^{-500t} \cos (2449.5t - 0.2013 \text{ rad})] \text{ V for } t \geq 0^+$$

$$\text{Rise time, } t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}}{\sqrt{1-\xi^2} \omega_n} = \frac{\pi - 1.37 \text{ rad}}{0.978 \times 2500} = 0.725 \text{ ms}$$

$$\text{Peak time, } t_p = \frac{\pi}{\sqrt{1-\xi^2} \omega_n} = \frac{\pi}{0.978 \times 2500} = 1.29 \text{ ms}$$

$$\text{Settling time, } t_s = \frac{3}{\xi \omega_n} = \frac{3}{0.2 \times 2500} = 6 \text{ ms}$$

$$\text{Percentage overshoot, } M_p = 100 \times e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}} = 100 \times e^{-0.6425} = 52.6\%$$

12.9 FREQUENCY RESPONSE OF SERIES *RLC* CIRCUIT

Consider a series *RLC* circuit excited by a voltage source $v_s(t) = 1 \sin \omega t$ V for $t \geq 0^+$ with a set of specified initial conditions. We continue to use $v_C(t)$, the voltage across capacitor, as the describing variable for the circuit.

The total response will contain natural response terms of exponentially damped sinusoidal nature and forced response term. We expect the natural response terms to vanish when time increases without limit. Only the forced response term acts in the long run, and of course, it is termed steady-state response. The sinusoidal steady-state response of series *RLC* circuit in this section.

12.9.1 Sinusoidal Forced-Response from Differential Equation

The differential equation governing $v_C(t)$, the capacitor voltage, in a series *RLC* circuit excited by a voltage source $v_s(t)$ was derived earlier in this chapter.

$$\frac{d^2 v_C(t)}{dt^2} + 2\xi \omega_n \frac{dv_C(t)}{dt} + \omega_n^2 v_C(t) = \omega_n^2 v_s(t) \quad (12.9-1)$$

where $\omega_n = \sqrt{1/LC}$, $\xi = \frac{R}{2\sqrt{L/C}}$

The steady-state component of response when $v_s(t) = 1 \sin \omega t$ $u(t)$ is obtained by ‘method of undetermined coefficients’. We assume a trial solution of the form $v_C(t) = A \sin \omega t + B \cos \omega t$ and determine the values of A and B by substituting the assumed solution in the differential equation and equating coefficients of $\sin \omega t$ and $\cos \omega t$ on both sides of the resulting equation.

Trial Solution, $v_C(t) = A \sin \omega t + B \cos \omega t$

Substituting in the differential equation and collecting terms,

$$[(\omega_n^2 - \omega^2)A - 2\xi\omega_n B] \sin \omega t + [(\omega_n^2 - \omega^2)B + 2\xi\omega_n A] \cos \omega t = \omega_n^2 \sin \omega t$$

The only way this equation can remain true independent of t is by the coefficients of $\sin \omega t$ (and $\cos \omega t$) becoming equal on both sides of the equation. Therefore,

$$\begin{aligned} (\omega_n^2 - \omega^2)A - 2\xi\omega_n B &= \omega_n^2 \\ (\omega_n^2 - \omega^2)B + 2\xi\omega_n A &= 0 \end{aligned}$$

Solving these two equations simultaneously we get,

$$A = \frac{\omega_n^2(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}, \quad B = \frac{-2\xi\omega_n^2(\omega_n\omega)}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}$$

Substituting these in the assumed solution and simplifying the solution to a ‘single sinusoid with phase’ form we get,

$$v_C(t) = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \sin(\omega t + \phi_C) \quad (12.9-2)$$

where $\phi_C = -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2}$

Therefore, the magnitude part of the frequency-response function for $v_C(t)$ is

$$\frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \text{ and the phase part of frequency-response function is } -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2}.$$

12.9.2 Frequency Response from Phasor Equivalent Circuit

We remember at this point that phasor analysis is another way to arrive at the sinusoidal steady-state response of linear circuits. We verify the result in Eqn. 12.9-2 by employing phasor analysis. The series *RLC* circuit with unit amplitude sinusoidal excitation and its phasor equivalent circuit are shown in Fig. 12.9-1.

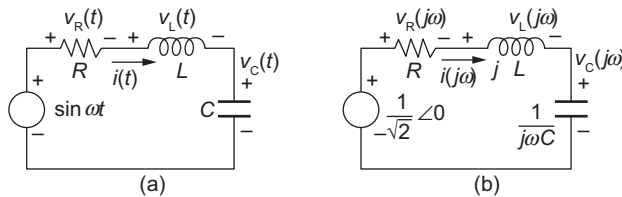


Fig. 12.9-1 Series *RLC* circuit and phasor equivalent

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The frequency-response function for any circuit variable is the ratio of output phasor to the input phasor. This ratio will be a complex ratio and its magnitude part will give the amplitude ratio and its angle part will give the phase angle by which the output sine wave *leads* the input sine wave under steady-state condition. We obtain three phasor ratios for the voltage variables in this circuit by employing voltage division principle in a series circuit.

$$\frac{V_C(j\omega)}{V_S(j\omega)} = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{1}{1 - \omega^2 LC + j\omega RC} = \frac{1/LC}{1/LC - \omega^2 + j\omega R/L} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega}.$$

This ratio can be written in polar form as

$$\frac{V_C(j\omega)}{V_S(j\omega)} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_C \text{ where } \phi_C = -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad} \quad (12.9-3)$$

We see that the frequency-response obtained by solving the differential equation is the same as the one obtained by employing phasor equivalent circuit as expected. Similar evaluation of phasor ratios leads to the frequency-response functions for the remaining two voltage variables in the circuit. They are,

$$\begin{aligned} \frac{V_R(j\omega)}{V_S(j\omega)} &= \frac{j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega} \\ &= \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_R \text{ where } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad} \end{aligned} \quad (12.9-4)$$

$$\begin{aligned} \frac{V_L(j\omega)}{V_S(j\omega)} &= \frac{(j\omega)^2}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega} \\ &= \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_L \text{ where } \phi_L = \pi - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad} \end{aligned} \quad (12.9-5)$$

The remaining variable, $i(t)$, is directly related to $v_R(t)$ and hence its frequency-response need not be obtained separately.

12.10 RESONANCE IN SERIES *RLC* CIRCUIT

The ratio of voltage appearing across an element in a series circuit to the source voltage is equal to the ratio between the phasor impedance of that element to sum of all the impedances in series. The impedance of an inductor increases linearly with angular frequency and impedance of a capacitor decreases in inverse proportion to angular frequency. We use these basic principles to discuss the shape of frequency-response plots for the three possible outputs in the series *RLC* circuit.

12.10.1 The Voltage Across Resistor – The Band-pass Output

At zero frequency (*i.e.*, for DC steady state) the inductor appears as a short and capacitor appears as open. Therefore, the magnitude part of frequency-response is 0 for $v_R(t)$, and $v_L(t)$ and 1 for $v_C(t)$ at this frequency.

The inductor appears as impedance of infinite magnitude and capacitor appears as impedance of zero magnitude as ω increases without limit. Therefore, all the high frequency voltage will appear across the inductor. Thus, as $\omega \rightarrow \infty$, the magnitude part of frequency-response is 0 for $v_R(t)$, $i(t)$, and $v_C(t)$ and 1 for $v_L(t)$.

The sign of impedance of inductor is positive and the sign of impedance of capacitor is negative for any ω . Thus, they tend to cancel each other partially in the sum at all frequencies. The cancellation is 100% at one particular frequency. The value of frequency at which this happens is when $\omega L = 1/\omega C \Rightarrow \omega = 1/\sqrt{LC}$. But this frequency was named as *undamped natural frequency* earlier. Thus, we conclude that, the reactance part of the total series impedance of a series *RLC* circuit goes to zero at ω_n and the circuit appears purely resistive under steady-state conditions at that frequency. Therefore, the current in the circuit at that frequency will be $v_s(t)/R$ and will be in phase with the input voltage. The power factor of the circuit will be unity at that frequency.

The cancellation between the inductive reactance and capacitive reactance is only partial at all other frequencies. Hence, the magnitude of total series impedance of the circuit at any frequency other than ω_n will be more than R and amplitude of current will be less than $1/R$ A (assuming unit amplitude excitation) at all other frequencies. *Thus, in a series RLC circuit, the impedance is a minimum and amplitude of current (and hence amplitude of voltage across the resistor) is a maximum at ω_n .* Moreover, the current in the circuit will be at unity power factor at that frequency. This condition in the series *RLC* circuit is called the *resonance* condition and the frequency at which this happens is called the *resonant frequency*. Obviously, in a series *RLC* circuit, the *resonant frequency* and the *undamped natural frequency* are the same.

- In general, in a circuit excited by a single sinusoidal voltage source (current source) across a pair of terminals, *resonance* is the condition under which the current drawn at the terminals (voltage appearing across the terminals) is in phase with the source voltage (current). Equivalently, *resonance* is the condition under which the input impedance (admittance) offered to the sinusoidal source is resistive.
- Resonance frequency is the frequency of sinusoidal excitation for which the circuit presents pure resistive impedance. It is equal to the undamped natural frequency of the circuit. It is $\frac{1}{2\pi\sqrt{LC}}$ Hz for series *RLC* circuit.
- In a series *RLC* circuit under resonance condition,
 - (a) The input impedance is a minimum and is purely resistive at R Ohms.
 - (b) The input current is in phase with applied voltage. The power factor is unity.
 - (c) For fixed amplitude of excitation, the circuit draws maximum amplitude current when the excitation frequency is resonance frequency. The power dissipated in the resistor will be a maximum under this condition.
 - (d) The voltage across resistor will be equal to the applied voltage in amplitude and phase.
- The input impedance of a series *RLC* circuit is capacitive for $\omega < \omega_n$ and inductive for $\omega > \omega_n$. The circuit has *lead* power factor for $\omega < \omega_n$ and *lag* power factor for $\omega > \omega_n$.

The impedance of series *RLC* circuit can be expressed in terms of the critical resistance $R_{cr} = 2\sqrt{\frac{L}{C}}$, normalised frequency $x = \frac{\omega}{\omega_n}$ and damping factor $\xi = \frac{R}{R_{cr}}$ as follows.

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$$\begin{aligned}
 \mathbf{Z} &= R + j\omega L + \frac{1}{j\omega C} \\
 &= R + j \frac{(\omega^2 LC - 1)}{\omega C} \\
 &= R - j\omega_n L \frac{1-x^2}{x} \text{ where } x = \frac{\omega}{\omega_n} \\
 &= \xi R_{cr} - j \frac{R_{cr}}{2} \frac{1-x^2}{x} \left(\because \omega_n L = \sqrt{\frac{L}{C}}, R_{cr} = 2\sqrt{\frac{L}{C}} \text{ and } \xi = \frac{R}{R_{cr}} \right) \\
 \therefore \frac{|\mathbf{Z}|}{R_{cr}} &= \sqrt{\xi^2 + \frac{(1-x^2)^2}{4x^2}}
 \end{aligned}$$

The magnitude of series *RLC* circuit impedance normalised to the base of critical resistance of the circuit is shown in Fig. 12.10-1. The impedance reaches a minimum value at resonance frequency for all damping factors.

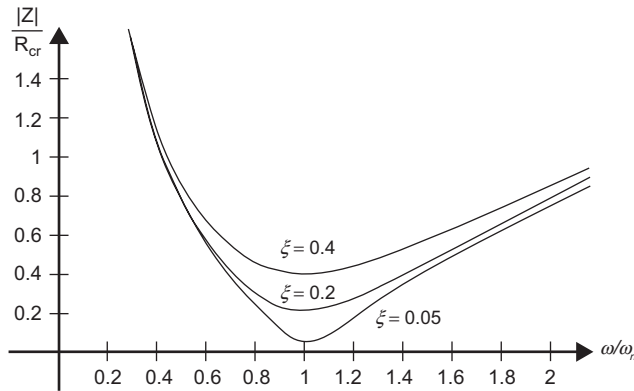


Fig. 12.10-1 Normalised impedance of a series *RLC* circuit for various damping factors

The amplitude of voltage appearing across the resistor in a series *RLC* circuit under resonance condition is same as the amplitude of input. Therefore, the magnitude of frequency-response for $v_R(t)$ begins with zero at zero frequency, goes to unity at ω_n and tapers down to zero as $\omega \rightarrow \infty$. The total reactance in a series *RLC* circuit is capacitive for $\omega < \omega_n$ and it is inductive for $\omega > \omega_n$. Therefore, the voltage across resistor leads the input voltage for frequencies lower than resonant frequency and lags the input voltage for frequencies higher than resonant frequency. Thus, the phase of frequency-response of $v_R(t)$ starts at 90° at $\omega = 0$, becomes zero at $\omega = \omega_n$ and decreases to -90° as $\omega \rightarrow \infty$.

Equation 12.9-4 confirms all these conclusions. The shape of magnitude response and phase response for the voltage across resistor is plotted against ω/ω_n ratio for various damping factors in Fig. 12.10-2.

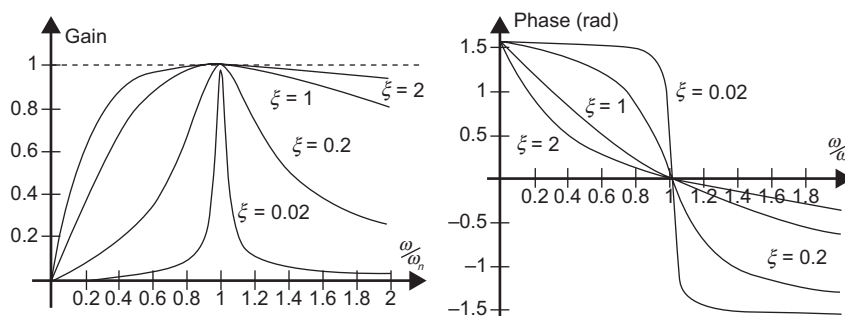


Fig. 12.10-2 Frequency response of resistor voltage in a series *RLC* circuit

The magnitude curve is found to become narrower as the damping ratio is reduced. The resistor voltage in a series *RLC* circuit exhibits the so-called *band-pass* characteristic. A frequency-response is said to be of *band-pass* nature when it attenuates low-frequency sinusoids and high-frequency sinusoids considerably and passes on mid-frequency sinusoids preferentially. Resistor voltage in a series *RLC* circuit is a band-pass output for all values of damping factor – *i.e.*, even an over-damped series *RLC* circuit behaves as a band-pass filter if the output is taken across the resistor. However, the *band-pass* characteristic becomes sharper and sharper when the damping in the circuit is reduced. That is, the circuit becomes highly *frequency-selective* as ξ approaches zero. In conclusion, series *RLC* circuit with output taken across *R* is a *narrow band-pass filter* for low values of ξ and it is a *wide band-pass filter* for high value of ξ .

Another point of great significance is that the output in this band-pass filter is *in-phase* with the input at a frequency that is at the centre of the band – *i.e.*, at ω_n . Output signal undergoes a phase change by about 180° when its frequency varies in a small band around ω_n if ξ is very small. See the phase curve for $\xi = 0.02$ in Fig. 12.10-2. This kind of rapid variation of phase of output over a small frequency range has considerable negative implications in designing control systems for systems that involve *RLC* circuits.

12.10.2 The Voltage Across Capacitor – The Low-pass Output

The magnitude response for this output starts at unity at zero frequency and goes to zero as $\omega \rightarrow \infty$. In between it may be a monotonically decreasing function or it may attain a maximum depending on the damping present in the circuit. The frequency-response function shown in Eqn. 12.9-3 is plotted in Fig. 12.10-3 for the various values of ξ .

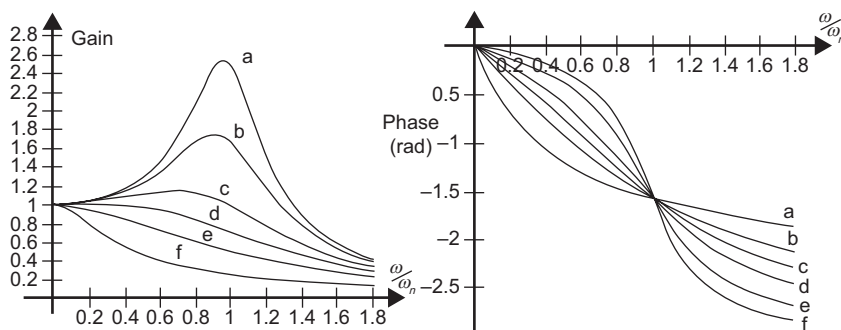


Fig. 12.10-3 Frequency response of capacitor voltage in a series *RLC* circuit: (a) $\xi = 0.2$, (b) $\xi = 0.3$, (c) $\xi = 0.5$, (d) $\xi = 0.7$, (e) $\xi = 1$ and (f) $\xi = 2$

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This output is essentially a *low-pass* output. However, it is a bad low-pass filter if it is too under-damped or too over-damped. This is so because we want the magnitude response of a low-pass filter to be reasonably flat till a particular value of ω and then fall to zero more or less rapidly. Only curves (c) and (d) in Fig. 12.10-3 look good from this point of view. In fact, the value of ξ used in filter design is 0.7 typically and this corresponds to curve (d).

Observe that for $\xi < 0.7$ the magnitude response exhibits a peak. The ratio of this peak gain value to DC gain value is called *resonant peak factor*. The frequency at which the resonant peak in gain occurs is *not* the frequency at which resonance occurs in the circuit. Resonance occurs in the circuit at ω_n and the voltage across the resistor goes to a maximum value at that frequency. But the frequency at which the capacitor voltage goes to a maximum for fixed input amplitude is different from ω_n and it also depends on ξ too. This frequency is less than ω_n and shifts more to the left with increase in ξ . The expression for magnitude response (Eqn. 12.9-3) may be differentiated with respect to ω and set to zero to find the frequency at which maximum takes place (if at all there is a maximum) and the value of the maximum. The results will be

$$\omega_{cp} = \omega_n \sqrt{1 - 2\xi^2} \quad \text{and} \quad R_{cp} = \frac{1}{2\xi\sqrt{1 - \xi^2}} = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

where ω_{cp} is the frequency at which gain maximum takes place and R_{cp} is the resonant peak factor. The expressions reveal that there is a resonant peak only for $\xi < 1/\sqrt{2} \approx 0.7$.

The gain for capacitor voltage at ω_n is $1/2\xi$ as shown by Eqn. 12.9-3 with $\omega = \omega_n$. This predicts large amplitude voltage across the capacitor when the damping factor is small. For example, the amplitude of voltage across the capacitor is 50 V when a 1 V sinusoid is applied to the circuit at resonant frequency if the ξ factor is 0.01 (equivalently, Q factor is 50). How does this voltage amplification take place?

We have seen that the series circuit impedance is resistive and a minimum at ω_n . The reactance of L and C cancel each other at that frequency. Hence R decides the amplitude of current and the reactance of C multiplies this current amplitude to convert it into voltage amplitude. Therefore, the ratio of amplitude of capacitor voltage to amplitude of input voltage at ω_n must be $1/\omega_n RC$. Similarly, the ratio of amplitude of inductor voltage to amplitude of input voltage at ω_n must be $\omega_n L/R$. But, both these ratios are equal to the Q factor of the circuit as seen in the following equations.

$$\begin{aligned} \frac{1}{\omega_n RC} &= \frac{\sqrt{LC}}{RC} = \frac{\sqrt{L/C}}{R} = \frac{1}{2\xi} = Q \\ \frac{\omega_n L}{R} &= \frac{L}{\sqrt{LC}R} = \frac{\sqrt{L/C}}{R} = \frac{1}{2\xi} = Q \end{aligned}$$

The voltage amplification factor for capacitor voltage and inductor voltage in a series *RLC* circuit at resonant frequency is its Q factor.

Resonant amplification of voltage in a series *RLC* circuit is one method used in high voltage engineering to generate high AC voltages from low voltage sources. Such amplification becomes possible because at resonant frequency the voltage across L will be of same amplitude and opposite phase as that of the voltage across C and therefore they cancel each other without absorbing any portion of input voltage to sustain them.

12.10.3 The Voltage Across Inductor – The High-Pass Output

The magnitude response for this output starts at zero at zero frequency and goes to unity as $\omega \rightarrow \infty$. In between it may be a monotonically increasing function or it may attain a maximum depending on the damping present in the circuit. The frequency-response function shown in Eqn. 12.9-5 is plotted in Fig. 12.10-4 for various values of ξ .

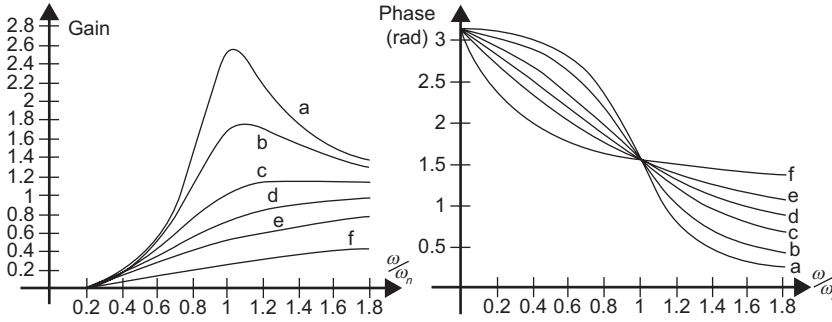


Fig. 12.10-4 Magnitude and phase plots for inductor voltage frequency-response in a series *RLC* circuit: (a) $\xi = 0.2$, (b) $\xi = 0.3$, (c) $\xi = 0.5$, (d) $\xi = 0.7$, (e) $\xi = 1$, (f) $\xi = 2$

The plots as well as our qualitative discussion reveal that the inductor voltage in a series *RLC* circuit has *high-pass nature* under sinusoidal steady state. But it is a bad high-pass filter if the series *RLC* circuit is only lightly damped. A good high-pass filter is expected to have near-zero gain at low frequencies till a particular frequency and a gain that rapidly rises to unity and remains there after that frequency. The prominent resonant peaks in the gain evident in magnitude response plot in Fig. 12.10-4 for low ξ values are not acceptable in a good high-pass filter. Such resonant peaks will lead to considerable distortion even for signals that do not have any low frequency sinusoids in them.

The voltage across inductor in series *RLC* circuit *leads* the input voltage by a phase angle that varies from 180° to 0° with ω . The phase angle is 90° at ω_n .

The angular frequency at which the voltage across the inductor peaks for given input voltage amplitude is *not* ω_n . It is always greater than ω_n and moves further to the right of ω_n as ξ increases. Exact expression for the angular frequency at which the magnitude response peaks and the value of the peak (if such resonant peaking occurs at all) can be obtained by differentiating the magnitude expression in Eqn. 12.9-5 with respect to ω and setting the derivative to zero. The result will be

$$\omega_{lp} = \frac{\omega_n}{\sqrt{1-2\xi^2}} \text{ and } R_{lp} = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{Q}{\sqrt{1-1/4Q^2}}$$

where ω_{lp} is the frequency at which gain maximum takes place and R_{lp} is the resonant peak factor. The expressions reveal that there is a resonant peak only for $\xi < 1/\sqrt{2} \approx 0.7$.

We observe that $\omega_{cp}\omega_{lp} = \omega_n^2$ and $R_{cp} = R_{lp}$. Thus, the geometric mean of resonant peak frequencies at high-pass and low-pass outputs is equal to the resonant peak frequency at the band-pass output.

The magnitude response for all the three outputs are shown in Fig. 12.10-5 for $\xi = 0.3$. The points brought out in the discussion on resonant peaks and the frequencies at which resonant peaks occur are illustrated in this plot.

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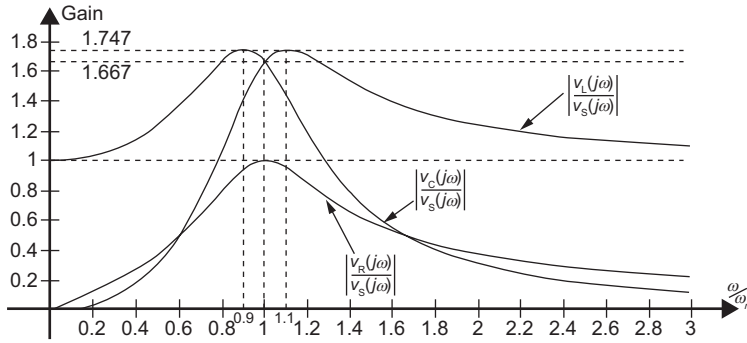


Fig. 12.10-5 Magnitude response for low-pass, band-pass and high-pass outputs in a series *RLC* circuit with damping factor = 0.3

The capacitor voltage attains a peak amplitude of 1.747 V at $\sim 0.9 \omega_n$ and the inductor voltage attains a peak amplitude of 1.747 V at $\sim 1.1 \omega_n$ for an input amplitude of 1 V. Both attain amplitude of 1.667 V at resonant frequency ω_n . The Q factor of the circuit is 1.67.

12.10.4 Bandwidth Versus Quality Factor of Series *RLC* Circuit

The frequency-response at the band-pass output of series *RLC* circuit (*i.e.*, the voltage across the resistor) was obtained as

$$\begin{aligned} \frac{V_R(j\omega)}{V_S(j\omega)} &= \frac{j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \\ &= \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_R \text{ where } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad} \end{aligned} \quad (12.10-1)$$

It was shown that this output has a maximum gain of unity and that occurs at $\omega = \omega_n$. We wish to develop a measure for the *frequency selectivity* exhibited by this output in this subsection.

The resistor gets all of the input voltage if input frequency is resonant frequency. Therefore, the resistor receives maximum power for fixed amplitude input if the input frequency is resonant frequency. The power dissipated at all other frequencies will be less than this value. There will be one or more frequencies at which the power dissipated in the resistor is exactly 50% of the power that will be dissipated if input is at resonant frequency (assuming the input amplitude is kept fixed). These frequencies have been used traditionally to define measures of frequency selectivity in band-pass circuits. They are called *half-power frequencies* for obvious reason.

Power dissipated in a resistor becomes 50% when voltage developed across it becomes $1/\sqrt{2} \approx 0.707$. Therefore, *half-power frequencies* are the angular frequencies at which the magnitude response of the circuit output become 70.7% of some reference gain value. The reference gain value in the case of low-pass circuit is the DC gain, in the case of band-pass circuit it is the maximum gain and in the case of high-pass circuit it is the gain as $\omega \rightarrow \infty$.

The frequency at which the gain of a band-pass circuit reaches maximum is termed *centre frequency* in filter studies. A typical band-pass response is shown in Fig. 12.10-6. The half power frequencies ω_1 and ω_2 are marked in the magnitude plot.

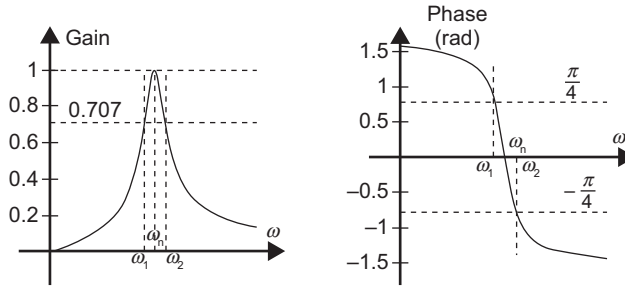


Fig. 12.10-6 A typical band-pass circuit frequency-response

The difference between the two half-power frequencies is called the *bandwidth* of the band-pass circuit. We develop an expression for bandwidth of a *narrow band-pass* circuit in the following and develop interesting insight into the relation between the bandwidth and quality factor of the circuit.

$$\frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{(2\xi\omega\omega_n)^2}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2} = \frac{1}{2}$$

$$\text{Let } x = \frac{\omega}{\omega_n}. \text{ Then, } \frac{4\xi^2x^2}{(1-x^2)^2 + 4\xi^2x^2} = \frac{1}{2}$$

$$\therefore (1-x^2)^2 - 4\xi^2x^2 = 0 \Rightarrow (1-x^2) = \pm 2\xi x$$

$$\therefore x^2 \pm 2\xi x - 1 = 0 \Rightarrow x = \mp \xi \pm \sqrt{1+\xi^2}$$

$$\text{Taking only positive values for } \omega, \omega_{2,1} = (\sqrt{1+\xi^2} \pm \xi)\omega_n \Rightarrow bw = 2\xi\omega_n$$

$$\therefore \frac{\text{Centre frequency}}{\text{Bandwidth}} = \frac{1}{2\xi} = Q \quad (12.10-2)$$

This Q factor (equivalently, damping factor) has indeed turned out to be an important parameter for series *RLC* circuit. Eqn. 12.10-2 is the third interpretation for Q . We had seen earlier that, in a weakly damped series *RLC* circuit, the fractional loss of total stored energy in the circuit over one cycle of oscillation is given by $4\pi\xi$. Since $Q = 1/2\xi$,

$$Q = 2\pi \frac{\text{Total stored energy in the source-free circuit}}{\text{Energy lost in one cycle of free response}}.$$

Second interpretation is based on the same energy ratio under sinusoidal steady-state conditions at resonant frequency. Let the circuit be at resonance with 1 V amplitude input. Then,

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$$v_S(t) = 1 \sin \omega_n t, \therefore i(t) = \frac{1}{R} \sin \omega_n t \quad \text{and} \quad v_C(t) = \frac{-1}{\omega_n RC} \cos \omega_n t$$

$$\begin{aligned} \text{Total stored energy} &= \frac{Li(t)^2}{2} + \frac{Cv_C(t)^2}{2} = \frac{L}{2R^2} \sin^2 \omega_n t + \frac{C}{2\omega_n^2 R^2 C^2} \cos^2 \omega_n t \\ &= \frac{L}{2R^2} (\because \omega_n^2 = \frac{1}{LC}) \end{aligned}$$

$$\text{Energy dissipated in one cycle} = \frac{1}{\omega_n} \int_0^{2\pi} R \times \left(\frac{1}{R} \sin \omega_n t \right)^2 d(\omega_n t) = \frac{\pi}{\omega_n R}$$

$$\therefore \frac{\text{Total stored energy}}{\text{Energy dissipated in one cycle}} = \frac{1}{2\pi} \frac{\omega_n L}{R} = \frac{1}{2\pi} \frac{\sqrt{L/C}}{R} = \frac{1}{2\pi} \frac{1}{2\xi} = \frac{Q}{2\pi}$$

$$\therefore Q = 2\pi \frac{\text{Total stored energy under resonance condition}}{\text{Energy dissipated in one cycle under resonance condition}}$$

12.10.5 Quality Factor of Inductor and Capacitor

We are already familiar with the concept of Q factor for a series RLC circuit. We discuss a similar factor for the elements themselves in this section.

We had assumed till now that the inductor and capacitor used in the series RLC circuit are ideal elements and that they have no parasitic elements associated with them. This is not true in practice. The non-zero parasitic elements associated with inductor and capacitor will affect the performance of RLC circuits considerably in narrow band-pass circuit applications.

An inductor has a non-zero wire resistance that goes along with its inductance in series. Further, if the inductor uses iron core, there will be hysteresis and eddy current losses in the iron core due to time-varying magnetic fields in the core. These losses are strongly dependent on frequency of operation and flux level in the core. Core loss is usually modelled by a resistance in parallel to the inductance. However, due to its complex dependence on frequency of operation, it can not be satisfactorily modelled by a single value of resistance at all frequencies.

In addition, a physical inductor will have distributed capacitance of winding shunting its inductance value. Thus, a practical inductor is more like the equivalent circuit shown in Fig. 12.10-7 (a).

However, in frequency-response studies, the distributed capacitance C_p is usually ignored since its value is generally too small to affect the circuit performance. Circuit equivalent in (b) is usually employed in studying the effect of losses in the inductor on the Q factor of a circuit employing this inductor. R_p can, at the best, represent the core losses in the inductor only for a small band of frequencies around a frequency value at which it was measured.

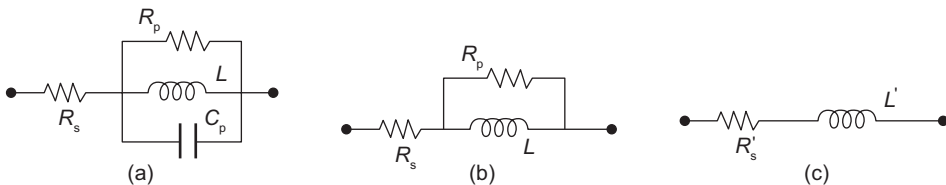


Fig. 12.10-7 Equivalent circuits for a practical inductor

The circuit in (b) can not be reduced to the circuit in (c) such that (c) remains equivalent to (b) at all frequencies. However, (b) can be reduced to (c) – i.e., a value of L' and R'_s can be found such that (c) will have same phasor impedance as that of (b) – at some particular frequency. Usually, the value of L' will be close to L and it is approximated that way in practice. R'_s will include the effects of R_p and R_s together. Since, in any case, a specific value of R_p is valid only over a small band around a specific frequency, the circuit in (b) can be equivalenced to circuit in (c) subject to the condition that it can be expected to give reasonably accurate results only over a small band of frequencies around the specific frequency at which R_p and R'_s are measured or calculated. This is satisfactory in the case of resonance studies in under-damped circuits since the frequency range of interest is a small band of frequencies around ω_n . The value of R'_s is usually indicated in an indirect manner by specifying it through a ratio. That ratio is the *Q factor of Inductor* and it is defined as the ratio of reactance of the inductor at ω to the resistance value R'_s relevant to that frequency. Q factor of an inductor will change with frequency. Therefore,

$$\text{Q factor of an inductor at } \omega = \frac{\omega L}{R'_s \text{ measured at } \omega}$$

A commercial Q-Meter that is available in any well-equipped laboratory will have features that permit measurement of *Q* of inductors at various frequencies.

A practical capacitor also has three parasitic elements associated with it. The foil resistance and lead inductance come in series with the capacitance. The leakage current that flows through the imperfect dielectric employed in the capacitor is modelled by a resistance in parallel to the capacitor. The loss mechanisms in the capacitor are also frequency dependent and hence an equivalent circuit for a practical capacitor will be valid only for a small band of frequencies around a specific frequency at which the measurement is carried out.

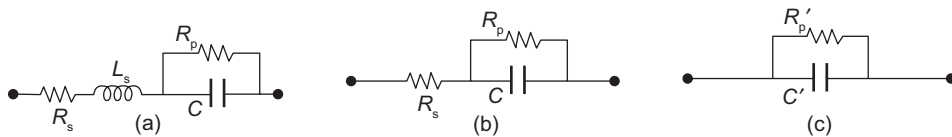


Fig. 12.10-8 Equivalent circuits for a practical capacitor

The series inductance L_s in the detailed equivalent circuit in Fig. 12.10-8 (a) is usually ignored (or absorbed along with the inductor in series) in studies on under-damped resonant circuits. And, the equivalent circuit in (b) is approximated by the circuit in (c) with $C' \approx C$ and R'_p measured by a Q-meter. The equivalent circuit in (c) is understood to be valid only for a small band ($\approx \pm 20\%$ max) around the frequency at which R'_p was measured. The Q-meter measures it indirectly and displays the *Quality factor of capacitor*. *Q factor* for a capacitor is defined as the ratio of resistance value R'_p at ω to the reactance of the capacitor at ω . Q factor of a capacitor will change with frequency. Therefore, *Q* – factor of an capacitor at $\omega = \omega C \times R'_p$ measured at ω .

Example: 12.10-1

Series *RLC* circuit is widely employed as *averaging filters* in switched mode power supplies, DC–DC Converters, DC–AC Inverters, etc. The load appears across the capacitor of the averaging filter.

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There will be no external resistor in series with the inductor. However, the winding resistance and core losses of inductor along with other series dissipating mechanisms present in the power electronic circuitry behind the filter stage will provide damping for the otherwise undamped LC filter circuit. We look at an example of use of LC circuit as an averaging filter in a DC–DC converter stage.

The power electronics in the converter converts a fixed DC voltage of 24 V into a switched wave as shown in Fig. 12.10-9 (b).

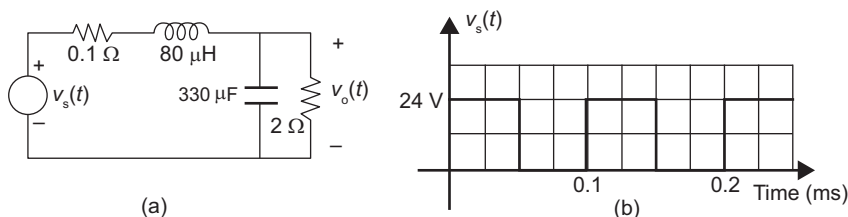


Fig. 12.10-9 Circuit and input waveform for Example: 12.10-1

The average value of the input is 12 V and it has a frequency of 10 kHz. The circuit used to extract the average value from this input and deliver it to a load resistance of 2 Ω is shown in Fig. 12.10-9 (a). Calculate and plot the output voltage under steady state.

[Hint - A symmetric square wave of ± 1 V amplitude and f Hz frequency can be approximated by the truncated series $\frac{4}{\pi} \left(\sin 2\pi ft + \frac{1}{3} \sin 6\pi ft + \frac{1}{5} \sin 10\pi ft + \frac{1}{7} \sin 14\pi ft \right)$.]

Solution

When the rectangular waveform is switched on to the filter circuit the system undergoes the usual transient response and settles down to a steady state in the long run. We look at only the steady-state output in this example.

The waveform shown in Fig (b). 12.10-9 can be resolved into two components – the first component is a DC of 12 V (*i.e.*, the cyclic average value of $v_s(t)$) and the second component is a symmetric square wave of ± 12 V amplitude and 10 kHz frequency.

Steady-state responses due to simultaneous application of two sources obey superposition principle. Therefore, $v_o(t)$ can be found as superposition of DC steady-state response and steady-state response to square wave input. The DC steady-state response is $12 \times 2/2.1 = 11.43$ V since inductor is short and capacitor is open under DC steady state.

The ± 12 V square wave can be approximated by sum of four sinusoids as given in the hint. Therefore, we have frequency-response problem at hand. We use phasor equivalent circuit for this purpose.

The lowest frequency we are interested in is 10 kHz. The other frequencies of interest are 30 kHz, 50 kHz and 70 kHz. The resonant frequency, ω_n , of the circuit is $\frac{1}{2\pi} \frac{1}{\sqrt{80 \times 10^{-6} \times 330 \times 10^{-6}}} \text{ Hz} = 979.5 \text{ Hz}$.

The phasor impedances at 10 kHz are $j5.026 \Omega$ for 80 μH inductor and $-j0.0482 \Omega$ for 330 μF capacitor.

$$\therefore \text{Gain at 10 kHz} = \frac{2 // -j0.0482}{0.1 + j5.026 + 2 // -j0.0482} = 0.00968 \angle -177.5^\circ$$

Amplitude of input at this frequency = $12 \times 4/\pi = 15.28$ V

$\therefore$ 10 kHz component in steady-state output = $15.28 \times 0.00968 \sin(2\pi \times 10^4 t - 177.5^\circ)$ V.

The next input component to be considered is $5.09 \sin(6\pi \times 10^4 t)$ V. The gain at 30 kHz is obtained by noting the inductor impedance as $j5.026 \times 3 = j15.08 \Omega$ and the capacitor impedance as $-j0.0482/3 = -j0.016 \Omega$. Compared to the 2Ω in parallel, the impedance of capacitor is so small in magnitude that the parallel combination is practically $-j0.016 \Omega$ itself. Therefore, the required gain is

$$\frac{-j0.016}{0.1 + j15.08 - j0.016} \approx \frac{-j0.016}{j15.064} \approx -0.00106.$$

$\therefore$ 30 kHz component in steady-state output = $5.09 \times 0.00106 \sin(2\pi \times 10^4 t - 177.5^\circ)$ V.

Similar calculations show that the gain at 50 kHz is -0.0004 and the gain at 70 kHz is -0.0002 . Thus, the total steady-state output due the square wave input component is constructed as follows.

$$\begin{aligned} &= 15.28 \times 0.00968 \sin(2\pi \times 10^4 t - 177.5^\circ) - 5.09 \times 0.00106 \sin 6\pi \times 10^4 t \\ &\quad - 3.06 \times 0.0004 \sin 10\pi \times 10^4 t - 2.2 \times 0.0002 \sin 14\pi \times 10^4 t \\ &= 0.15 \sin(2\pi \times 10^4 t - 177.5^\circ) - 0.005 \sin 6\pi \times 10^4 t - 0.001 \sin 10\pi \times 10^4 t - 0.0004 \sin 14\pi \times 10^4 t \end{aligned}$$

All the components, except the first component, are really negligible and we need not even have calculated them. We will accept the first two components and neglect the last two. Thus, the total steady-state voltage at the output = $11.43 + 0.15 \sin(2\pi \times 10^4 t - 177.5^\circ) + 0.005 \sin 6\pi \times 10^4 t$ V. The AC component of output is plotted in Fig. 12.10-10 and it is clear that it is heavily dominated by the 10 kHz component.

The peak-to-peak ripple in DC output is 0.3 V and represents 2.6% of the DC value of 11.43 V.

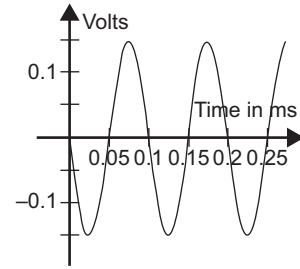


Fig. 12.10-10 The ripple component of output voltage in Example: 12.10-1

12.10.6 LC Circuit as an Averaging Filter

Let us see how the low-pass filter in Example 12.10-1 has achieved this extent of attenuation of AC component. We noted that the resonant frequency of the circuit is about 0.98 kHz. Thus, the ratio between the lowest frequency present in the input, *i.e.*, 10 kHz, and the resonant frequency of the circuit is about 10. We now show that the gain of a LC low-pass filter for a sinusoidal input at f Hz is $\propto (f_n/f)^2$, where f_n is the resonant frequency in Hz unit and $f \gg f_n$. We show it in two ways.

Refer to Eqn. 12.9-3 that gives the frequency-response function for capacitor voltage in a standard series *RLC* circuit. It is reproduced in the following.

$$\begin{aligned} \frac{V_C(j\omega)}{V_S(j\omega)} &= \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2 \omega_n^2 \omega^2}} \angle \phi_c \quad \text{where } \phi_c = -\tan^{-1} \frac{2\xi \omega_n \omega}{\omega_n^2 - \omega^2} \text{ rad} \\ \left| \frac{V_C(j\omega)}{V_S(j\omega)} \right| &= \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2 \omega_n^2 \omega^2}} \\ &= \frac{1}{\sqrt{(1 - x^2)^2 + 4\xi^2 x^2}} \quad \text{where } x = \frac{\omega}{\omega_n} = \frac{f}{f_n} \\ &\approx \frac{1}{\sqrt{(-x^2)^2 + 4\xi^2 x^2}} \approx \frac{1}{\sqrt{(-x^2)^2}} = \frac{1}{x^2} = \left(\frac{\omega_n}{\omega} \right)^2 = \left(\frac{f_n}{f} \right)^2 \quad \text{for } x \gg 1 \end{aligned}$$

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The second way to appreciate the second-order gain characteristic of LC low-pass filter is by considering the following qualitative argument based on phasor impedances. The purpose of the averaging filter is to eliminate AC components in the output. Some intervening element has to absorb the AC component in the input if it is not to appear at output. An element which can take large AC voltage across it while keeping current low is inductance (since its impedance increases with frequency) and a capacitor across the output will shunt out whatever AC current that tries to get into the load resistance (because capacitor has low impedance at high frequency). Thus, the inductor *chokes* the high frequency current while absorbing almost all the input AC voltage content and the capacitor located across the output *absorbs* whatever AC current that appears even after the inductor *chokes* it. This is how a LC filter does averaging. There is a two-fold action – a series element that makes it more and more difficult for AC current to flow as frequency increases and a shunt element which makes it more and more difficult for AC voltage to develop across it as frequency increases. This explains the inverse square dependence of gain on frequency at high frequency values.

At a sufficiently high frequency, any resistance or inductor that is connected in parallel with a capacitor may be ignored for approximate calculation. Similarly, at a sufficiently high frequency any resistance or current connected in series with an inductor may be ignored for approximate calculations. For example, see the phasor equivalent of circuit in this example at 10kHz in Fig. 12.10-11 (a).

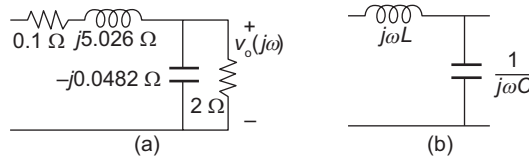


Fig. 12.10-11 Phasor equivalent circuits for Example: 12.10-1

Obviously, that 2Ω and 0.1Ω can be safely ignored. Therefore, the circuit can be approximated by the circuit in (b) for high frequencies. Then the gain function is

$$= \left| \frac{1/j\omega C}{j\omega L + 1/j\omega C} \right| = \left| \frac{1}{1 - \omega^2 LC} \right| = \frac{1}{|1 - \omega^2 / \omega_n^2|} \approx \left(\frac{\omega_n}{\omega} \right)^2 = \left(\frac{f_n}{f} \right)^2 \text{ for } f \gg f_n. \text{ Current in the circuit}$$

(b) lags voltage by 90° at high frequency because the net reactance is inductive at $\omega > \omega_n$. And the capacitance voltage lags behind circuit current by 90° . Therefore, the output voltage will be 180° out of phase with respect to input at high frequencies.

We could have used this approximate gain expression to solve this example. The gain at 10 kHz will be $= (0.98/10)^2 = 0.0096$, at 30 kHz $= (0.98/30)^2 = 0.00106$ etc. Phase of output will be 180° at all these frequencies.

Note that a LC filter will offer superior performance in averaging applications compared to *RC* circuit. This is due to the fact that the gain of an *RC* averaging circuit falls off in inverse proportion to ω for large values of ω (large compared to $1/\tau$), whereas the gain of an LC filter falls in inverse proportion to ω^2 for large values of ω (large compared to ω_n). Moreover, averaging by an LC filter is more efficient since an inductor does not dissipate power. Hence averaging at high power levels (few 10's of watts and up) is usually done by LC filters. *RC* averaging is commonly used in low-power signal processing applications.

Example: 12.10-2

Standard test signals such as square wave, triangular wave, sinusoidal wave, etc. are routinely used in electronics, communication and allied areas for a variety of purposes. It is rather easy to generate high quality square waves in electronic circuits. It is not so easy to make pure sinusoids. One of the commonly used methods to generate low power sinusoidal signals of high waveform quality is to generate a good square wave and pass it through a narrow band-pass filter with a large Q factor. The centre frequency of filter is adjusted to be equal to the frequency of square wave.

This example deals with applying a series *RLC* circuit for this purpose. Consider the circuit in Fig. 12.10-12.

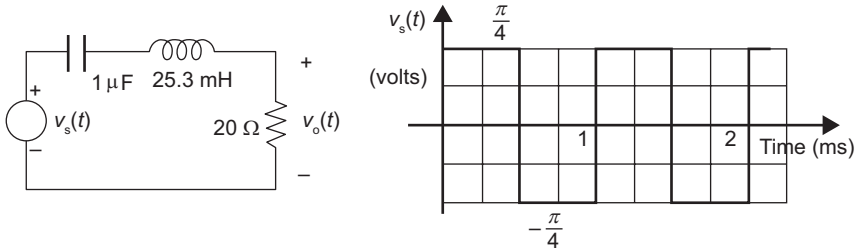


Fig. 12.10-12 Circuit and input waveform for Example: 12.10-2

This circuit is used to filter the square wave shown and deliver a sine wave to the 20Ω load resistance. We will accept the fact that the square wave shown can be expressed in the form of a series as follows.

$$v_s(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin 2\pi \times 10^3 t$$

- Find and plot the steady-state output waveform expected from the circuit.
- If the quality factor of the inductor used was measured to be 50 at 1 kHz and the test square wave was obtained from a function generator which has an output resistance of 50Ω , find and plot the steady-state output voltage waveform that will be observed.

Solution (a)

The circuit is the familiar series *RLC* circuit and the frequency-response function for resistor voltage in such a circuit has been shown to be

$$\begin{aligned} \frac{V_R(j\omega)}{V_S(j\omega)} &= \frac{j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \\ &= \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_R \text{ where } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad} \end{aligned}$$

We define $x = \omega/\omega_n$ and express the above function as

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$$\frac{V_R(j\omega)}{V_S(j\omega)} = \frac{j2\xi x}{(1-x^2) + j2\xi x}.$$

The input signal is a sum of sinusoids with odd multiples of 1 kHz as their frequencies. Thus, the separation between frequencies is large and it is quite possible that x may turn out to be quite large compared to 1 for many of the sinusoidal components. We calculate ω_n to verify this.

$$f_n = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi \times \sqrt{25.3 \times 10^{-3} \times 1 \times 10^{-6}}} = 1 \text{ kHz}$$

$$\text{Critical resistance} = 2\sqrt{\frac{L}{C}} = 318.2 \Omega$$

$$\therefore \text{Damping factor } \xi = \frac{20}{318.2} = 0.063 \text{ and Quality factor } Q = \frac{1}{2\xi} = 7.95$$

Therefore, the values of x of interest to us are the odd integers. For $x \geq 3$, we can approximate the gain and phase expressions as shown in the following.

$$\begin{aligned} \frac{V_R(j\omega)}{V_S(j\omega)} &= \frac{j2\xi x}{(1-x^2) + j2\xi x} \\ \left| \frac{V_R(j\omega)}{V_S(j\omega)} \right| &= \frac{2\xi x}{\sqrt{(1-x^2)^2 + 4\xi^2 x^2}} \approx \frac{2\xi}{x} \text{ for } x \gg 1 \\ \text{Phase} &= \frac{\pi}{2} - \left(\pi - \tan^{-1} \frac{2\xi x}{x^2 - 1} \right) \text{ for } x > 1 \\ &\approx -\frac{\pi}{2} + \tan^{-1} \frac{2\xi}{x} = -\frac{\pi}{2} + \frac{2\xi}{x} \text{ for } x \gg 1 \end{aligned}$$

The following table shows gain and phase of output for x values from 1 to 11.

x	1	3	5	7	9	11
Gain	1	0.042	0.025	0.018	0.014	0.012
Phase(rad)	0	-1.53	-1.545	-1.553	-1.557	-1.56

Now the first six terms of output may be constructed using the amplitude information.

$$\begin{aligned} v_o(t) &= 1 \sin 2\pi \times 10^3 t + 0.014 \sin(6\pi \times 10^3 t - 1.53) + 0.005 \sin(10\pi \times 10^3 t - 1.545) \\ &\quad + 0.0026 \sin(14\pi \times 10^3 t - 1.553) + 0.0013 \sin(18\pi \times 10^3 t - 1.557) \\ &\quad + 0.0011 \sin(22\pi \times 10^3 t - 1.56) \end{aligned}$$

Obviously, the first three terms are the ones which are significant. All the others can easily be ignored (including the higher frequency components that we did not evaluate).

$$\therefore v_o(t) \approx 1 \sin 2\pi \times 10^3 t + 0.014 \sin(6\pi \times 10^3 t - 1.53) + 0.005 \sin(10\pi \times 10^3 t - 1.545) \text{ V}$$

This waveform has only about 1.5% of other frequency content. It is almost pure sinusoid at 1 kHz.

Solution (b)

The 25.3 mH inductor has a reactance of $2\pi \times 10^3 \times 25.3 \times 10^{-3} = 159 \Omega$ at 1 kHz and since its Q at that frequency is given as 50, it has a series resistance of $159/50 = 3.18 \Omega$. The function generator that provides the square wave contributes 50Ω . Thus, the total series resistance in the series RLC circuit is 73.18Ω . The value of f_n of the circuit remains unchanged at 1 kHz. But its ξ factor becomes 0.23 now. Also, the total voltage that develops across 73.18Ω has to be multiplied by $20/73.18 = 0.273$ to calculate the voltage across the load resistance. Thus, the frequency-response expression now is

$$\frac{V_R(j\omega)}{V_S(j\omega)} = \frac{j0.547\xi x}{(1-x^2) + j2\xi x}$$

We continue to use the same approximations for $x \gg 1$.

The following table shows gain and phase of output for x values from 1 to 11.

x	1	3	5	7	9	11
Gain	0.273	0.042	0.025	0.018	0.014	0.011
Phase(rad)	0	-1.418	-1.479	-1.505	-1.52	-1.53

Now the first six terms of output may be constructed using the amplitude information.

$$\begin{aligned} v_o(t) = & 0.273 \sin 2\pi \times 10^3 t + 0.014 \sin(6\pi \times 10^3 t - 1.418) + 0.005 \sin(10\pi \times 10^3 t - 1.479) \\ & + 0.0026 \sin(14\pi \times 10^3 t - 1.505) + 0.0015 \sin(18\pi \times 10^3 t - 1.52) \\ & + 0.001 \sin(22\pi \times 10^3 t - 1.53) \end{aligned}$$

Higher frequency terms are neglected. The output waveforms for the two cases are plotted in Fig. 12.10-13.

The output in case (b) shows distortion clearly. This example illustrates the importance of high quality factor in filtering a square wave to a high Q sine wave. It also tells us that an otherwise high Q factor circuit may appear to be a low Q circuit if we forget about the output resistance of the signal generator that we use to test the circuit. The test signal should be passed on to the test circuit through a unity gain buffer amplifier with negligible output resistance.

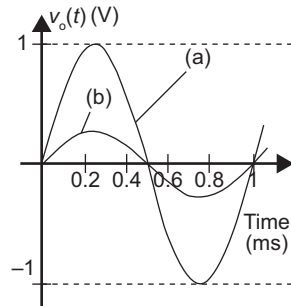


Fig. 12.10-13 Output waveforms in Example: 12.10-2

12.11 THE PARALLEL RLC CIRCUIT

We have dealt with the series RLC circuit in great detail and have developed considerable insight into the time-domain behaviour of second-order circuits in general. Moreover, we have also studied the frequency-response of second-order circuits using series RLC circuit as an example. This will help us to draw parallels between the behaviour of series RLC circuit with another equally important, if not more important, circuit – the parallel RLC circuit.

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Parallel *RLC* circuit finds application in almost all communications equipment (starting from radio receiver), sinusoidal oscillators, low-power and high-power filters and electrical power systems. In fact, one can even state that analog communications will be impossible without using parallel *RLC* circuit

12.11.1 Zero-Input Response and Zero-State Response of Parallel *RLC* Circuit

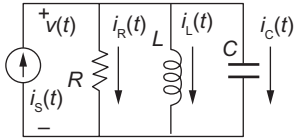


Fig. 12.11-1 The parallel *RLC* circuit

Figure 12.11-1 shows a parallel *RLC* circuit excited by a current source $i_s(t)$. There are four other circuit variables apart from $i_s(t)$ – they are the three current variables and one common voltage variable.

We choose $i_L(t)$ as the variable for deriving the differential equation. However, the variable that is commonly used as output variable in practice is $v(t)$.

All the three possible output voltage variables are used in practical applications in the case of series *RLC* circuit. But in the case of parallel *RLC* circuit, it is $v(t)$ that is almost used invariably. However, $v(t)$ can be obtained easily once we solve for $i_L(t)$.

$$v(t) = L \frac{di_L(t)}{dt} \text{ (by element equation of inductance)}$$

$$\therefore i_R(t) = \frac{L}{R} \frac{di_L(t)}{dt} \text{ and } i_C(t) = LC \frac{d^2 i_L(t)}{dt^2} \text{ (By element equations of R and C)}$$

Now we apply KCL at the positive node of current source and make use of expressions for $i_C(t)$ and $i_R(t)$ in terms of $i_L(t)$.

$$i_C(t) + i_R(t) + i_L(t) = i_s(t) \text{ for } t \geq 0^+$$

$$\therefore LC \frac{d^2 i_L(t)}{dt^2} + \frac{L}{R} \frac{di_L(t)}{dt} + i_L(t) = i_s(t) \text{ for } t \geq 0^+$$

$$\therefore \frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC} \frac{di_L(t)}{dt} + \frac{1}{LC} i_L(t) = \frac{1}{LC} i_s(t) \text{ for } t \geq 0^+$$

$$\xi \text{ and } \omega_n \text{ are identified as } \omega_n = \frac{1}{\sqrt{LC}} \text{ and } 2\xi\omega_n = \frac{1}{RC} \Rightarrow \xi = \frac{\frac{1}{2}\sqrt{\frac{L}{C}}}{R} = \frac{R_{cr}}{R}$$

$$\text{Critical Resistance } R_{cr} = \frac{1}{2} \sqrt{\frac{L}{C}} \Omega$$

We see that if we write the differential equations in the standard format using ξ and ω_n , the differential equations for $v_C(t)$ in series *RLC* circuit and $i_L(t)$ in parallel *RLC* Circuit are identical except that $v_C(t)$ gets replaced by $i_L(t)$ and $v_S(t)$ gets replaced by $i_S(t)$. The only point we have to

remember is that the damping factor is $R/2\sqrt{L/C}$ in the series *RLC* circuit, whereas it is $\frac{1}{2}\sqrt{L/C}/R$

for parallel *RLC* Circuit. Thus, increasing resistance will increase damping in series *RLC* circuit, whereas it will decrease damping in parallel *RLC* Circuit. The initial conditions needed for solving the differential equation remain the same – inductor current I_o at $t = 0^-$ and capacitor voltage V_o at $t = 0^-$.

The parallel *RLC* circuit is over-damped for $\xi > 1$, critically damped for $\xi = 1$ and under-damped for $\xi < 1$.

Zero-input response and zero-state response of parallel RLC circuit is illustrated by a series of examples that follow. The polarity conventions for specifying initial conditions is as per Fig. 12.11-1.

Example: 12.11-1

A parallel RLC circuit with $L = 10$ mH, $C = 100$ μ F, $R = 2.5$ Ω , $V_o = 10$ V and $I_o = -1$ A is allowed to execute its free-response from $t = 0$. Obtain and plot all the circuit variables as functions of time.

Solution

Undamped natural frequency, $\omega_n = \frac{1}{\sqrt{LC}} = 1000$ rad/sec

Critical resistance $= \frac{1}{2} \sqrt{\frac{L}{C}} = 5$ Ω

$\therefore$ Damping factor $\xi = 5 / 2.5 = 2$

$\therefore$ Natural frequencies $= (-\xi \pm \sqrt{\xi^2 - 1})\omega_n = -0.268 \times 10^3$ and -3.732×10^3

Hence, this circuit is an over-damped one with two real negative natural frequencies corresponding to two time constants of 3.732 ms and 0.268 ms. There is no forced response component. Therefore total response for all variables will contain sum of two decaying exponential functions with time constants calculated above.

$$\therefore i_L(t) = A_1 e^{-0.268 \times 10^3 t} + A_2 e^{-3.732 \times 10^3 t} \quad (12.11-1)$$

A_1 and A_2 have to be evaluated from initial conditions. One of the initial conditions – inductor current – can be directly applied on this equation. The initial current given is specified at $t = 0^-$. But since there is no impulse voltage in this circuit, the inductor current can not change instantaneously. Therefore, the value of inductor current at $t = 0^+$ and at $t = 0^-$ will be the same. Applying this initial condition at $t = 0^+$ on the assumed solution gives one equation on A_1 and A_2 .

$$A_1 + A_2 = I_o = -1$$

The second initial condition, the value of capacitor voltage at $t = 0^-$, can be employed to obtain the second equation needed to solve for A_1 and A_2 . We recognise that (i) in a parallel RLC circuit, voltage is the common variable and hence the initial value of voltage across inductor is same as V_o and (ii) the voltage across an inductor is proportional to first derivative of current.

$$\therefore \left. \frac{di_L(t)}{dt} \right|_{0^+} = \left. \frac{v_L(t)}{L} \right|_{(0^+)} = \left. \frac{v_C(t)}{L} \right|_{(0^+)}$$

$$\text{But } \left. \frac{v_C(t)}{L} \right|_{(0^+)} = \left. \frac{v_C(t)}{L} \right|_{(0^-)} \quad \text{if there is no impulse current flow in the capacitor at } t=0$$

$$\therefore \left. \frac{di_L(t)}{dt} \right|_{0^+} = \frac{V_o}{L} \text{ A/sec}$$

Differentiate the assumed response in Eqn. 12.11-1 and apply this initial condition to get the second equation on A_1 and A_2 .

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$$-268 A_1 - 3732 A_2 = 10 / 0.01 = 1000 \text{ A/sec}$$

Solving for A_1 and A_2 ,

$$A_1 = -0.789 \text{ and } A_2 = -0.211$$

$$\therefore i_L(t) = -0.789 e^{-268t} - 0.211 e^{-3732t} \text{ A for } t \geq 0^+$$

$$v(t) = L \frac{di_L(t)}{dt} = 2.11 e^{-268t} + 7.89 e^{-3732t} \text{ V for } t \geq 0^+$$

$$i_C(t) = C \frac{dv(t)}{dt} = -0.0565 e^{-268t} - 2.945 e^{-3732t} \text{ A for } t \geq 0$$

$$i_R(t) = \frac{v(t)}{R} = 0.846 e^{-268t} + 3.156 e^{-3732t} \text{ A for } t \geq 0^+$$

The sum of the three currents should be zero and it is verified.

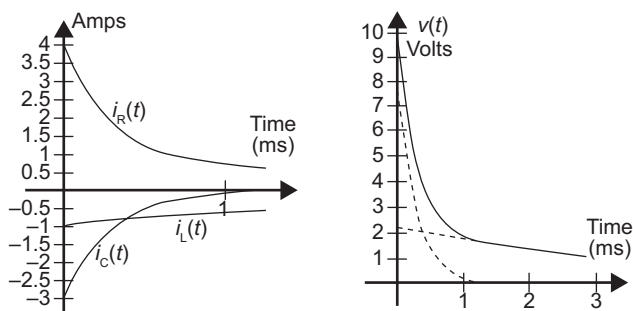


Fig. 12.11-2 Current and voltage waveforms for Example: 12.11-1

Figure 12.11-2 shows the three currents and the common voltage waveforms. The two exponential functions that make the voltage waveform are also shown. In all cases, the exponential with lower time constant (0.268 ms) dominates the initial behaviour, whereas the exponential with the higher time constant (3.732 ms) affects the behaviour after 1 ms.

Example: 12.11-2

A parallel *RLC* circuit with $L = 1 \text{ mH}$, $C = 1000 \text{ }\mu\text{F}$, $R = 2.5 \text{ }\Omega$, $V_o = 0 \text{ V}$ and $I_o = 0 \text{ A}$ is driven by a single pulse of current of amplitude 100 A lasting for $10 \text{ }\mu\text{s}$. Obtain and plot all the circuit variables as functions of time.

Solution

A single rectangular pulse can be expressed as sum of two step functions. Let the pulse height be I , pulse duration be t_o and let it start at $t = 0$. Then this pulse can be expressed as $I[u(t) - u(t - t_o)]$ where $u(t - t_o)$ is a unit step which is delayed by t_o s in the time-axis. Thus, what we have at hand looks like a step-response problem.

Undamped natural frequency $\omega_n = \frac{1}{\sqrt{LC}} = 1000$ rad/sec

Critical resistance $= \frac{1}{2}\sqrt{\frac{L}{C}} = 0.5 \Omega \quad \therefore$ Damping factor $\xi = 0.5 / 2.5 = 0.2$

$\therefore$ Natural frequencies $= (-\xi \pm j\sqrt{1-\xi^2})\omega_n = -200 + j979.8$ and $-200 - j979.8$

Hence the circuit is an under-damped one and its natural response terms will be exponentially damped sinusoidal functions of time.

We note that the time constant of the exponential which damps the sinusoid is $1/200 = 5$ ms. The period of the sinusoid term will be $2\pi/979.8 = 6.413$ ms. Observe that the duration of current pulse applied to the circuit is only 0.01 ms. It is very small compared to 5 ms and 6.413 ms. An approximate solution to the circuit problem can usually be obtained under such situations by approximating the driving pulse as an impulse with an area content equal to the area content of the pulse being approximated. The pulse can be arbitrary in shape – only that the pulse duration has to be much less than the characteristic times involved in the circuit natural response. Hence this problem can be solved by obtaining the impulse response and scaling it by the magnitude to the impulse used to approximate the rectangular pulse, *i.e.*, $100 \text{ A} \times 10 \mu\text{s} = 1000 \mu\text{C}$.

No portion of an impulse current can flow through the resistor in a parallel RLC circuit since that will result in impulse voltage across the combination. Capacitor will not let that happen. No portion of impulse current can flow through the inductor since inductor does not even permit a finite change in current over infinitesimal time duration. Hence, the entire input impulse current has to flow through the capacitor at $t = 0$. Therefore, the capacitor voltage changes instantaneously to $1000 \mu\text{C}/1000 \mu\text{F} = 1 \text{ V}$ from 0. The impulse current source is open (because a current source which is zero-valued is an open-circuit) for all $t \geq 0^+$. The only effect of impulse excitation is that it changes the initial condition abruptly. The subsequent response is the free-response of the circuit.

Now the problem has got reduced to finding the zero-input response (*i.e.*, free-response) of the circuit with $V_o = 1 \text{ V}$ and $I_o = 0$.

$$i_L(t) = e^{-200t} (A_1 \sin 979.8t + A_2 \cos 979.8t) \text{ A}$$

$$i_L(0^+) = 0 \text{ and } \left. \frac{di_L(t)}{dt} \right|_{(0^+)} = \frac{V_o}{L} = 1000 \text{ A/s}$$

Differentiating $i_L(t)$ and applying initial conditions, $A_2 = 0$ and $979.8A_1 - 200A_2 = 1000$. Solving for A_1 and A_2 , $A_1 = 1.0206$ and $A_2 = 0$

$$\therefore i_L(t) = 1.0206 e^{-200t} \sin 979.8t \text{ A for } t \geq 0^+$$

$$\begin{aligned} v(t) &= L(i_L(t))' = e^{-200t} (-0.204 \sin 979.8t + \cos 979.8t) \text{ V} \\ &= 1.0206 e^{-200t} \cos(979.8t + 0.2 \text{ rad}) \text{ V for } t \geq 0^+ \end{aligned}$$

$$\begin{aligned} i_C(t) &= C(v(t))' = e^{-200t} [-0.204 \cos(979.8t + 0.2 \text{ rad}) - \sin(979.8t + 0.2 \text{ rad})] \\ &= -1.0206 e^{-200t} \sin(979.8t + 0.4 \text{ rad}) \text{ A for } t \geq 0^+ \end{aligned}$$

$$i_R(t) = \frac{v(t)}{R} = 0.408 e^{-200t} \cos(979.8t + 0.2 \text{ rad}) \text{ A for } t \geq 0^+$$

Figure 12.11-3 shows the current and voltage waveforms.

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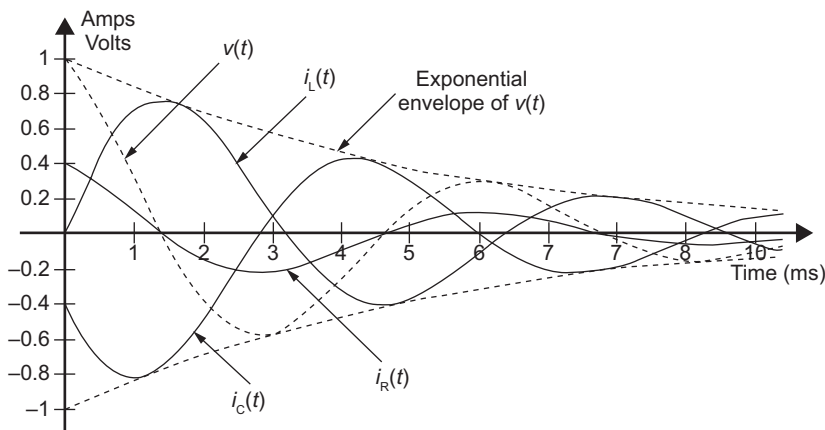


Fig. 12.11-3 Current and voltage waveforms for Example: 12.11-2

Example: 12.11-3

A parallel *RLC* circuit with $R \rightarrow \infty$ has an initial voltage of V_o V across the capacitor and I_o A in the inductor at $t = 0^-$. Find expressions for all variables under free-response conditions.

Solution

A series *RLC* circuit with $R = 0$ and a parallel *RLC* circuit with $R \rightarrow \infty$ will be the same circuit and hence the free-response in this case is same as the free-response in the case of undamped series *RLC* circuit. It is given in Eqn. 12.2-3 and is reproduced in the following.

$$v(t) = \sqrt{V_o^2 + \frac{LI_o^2}{C}} \cos(\omega_n t - \phi) \quad \text{A for } t \geq 0^+$$

$$i_C(t) = -i_L(t) = -\sqrt{I_o^2 + \frac{CV_o^2}{L}} \sin(\omega_n t - \phi) \quad \text{A for } t \geq 0^+ \text{ where } \phi = \tan^{-1} \left(\frac{I_o \sqrt{L/C}}{V_o} \right)$$

The ratio of amplitude of voltage to amplitude of current is $\sqrt{\frac{L}{C}}$. Total energy storage in the circuit will be $\frac{LI_o^2}{2} + \frac{CV_o^2}{2}$ J and it will remain constant at that value.

Example: 12.11-4

Find the unit step response of a current excited parallel *RLC* circuit with $L = 1$ mH, $C = 1000$ μ F and $R = 2.5$ Ω .

Solution

$$\text{Undamped natural frequency } \omega_n = \frac{1}{\sqrt{LC}} = 1000 \text{ rad/s, critical resistance} = \frac{1}{2} \sqrt{\frac{L}{C}} = 0.5 \Omega$$

$$\therefore \text{Damping factor, } \xi = 0.5 / 2.5 = 0.2$$

$$\therefore \text{Natural frequencies} = (-\xi \pm j\sqrt{1-\xi^2})\omega_n = -200 + j979.8 \text{ and } -200 - j979.8$$

Capacitor behaves as a short-circuit and inductor behaves as an open-circuit under DC steady-state condition. Therefore, the 1 A current in the unit step will go through the inductor under steady state. Hence, the solution for $i_L(t)$ can be assumed as

$$i_L(t) = 1 + e^{-200t} (A_1 \sin 979.8t + A_2 \cos 979.8t) \text{ with } i_L(0^+) = 0 \text{ and } i_L'(0^+) = V_o/L = 0$$

The initial conditions are zero valued since it is a step-response problem. We get the two equations needed to solve for A_1 and A_2 by applying initial conditions. They are $A_2 = -1$ and $979.8A_1 - 200A_2 = 0$. Solving for A_1 and A_2 , $A_1 = -0.204$ and $A_2 = -1$

$$\therefore i_L(t) = 1 - e^{-200t} (0.204 \sin 979.8t + \cos 979.8t) \text{ A for } t \geq 0^+$$

$$= 1 - 1.0206 e^{-200t} \cos(979.8t - 0.2\text{rad}) \text{ A for } t \geq 0^+$$

$$v(t) = L(i_L'(t))' = e^{-200t} [0.204 \cos(979.8t - 0.2\text{rad}) + \sin(979.8t - 0.2\text{rad})]$$

$$= 1.0206 e^{-200t} \sin 979.8t \text{ V for } t \geq 0^+$$

$$i_R(t) = v(t) / 2.5 = 0.255 e^{-200t} \sin 979.8t \text{ A for } t \geq 0^+$$

$$i_C(t) = 1 - i_R(t) - i_L(t) = 1.0206 e^{-200t} \cos(979.8t - 0.2\text{rad}) - 0.255 e^{-200t} \sin 979.8t$$

$$= e^{-200t} [-0.051 \sin 979.8t + \cos 979.8t] = e^{-200t} \cos(979.8t + 0.05 \text{ rad}) \text{ A for } t \geq 0^+$$

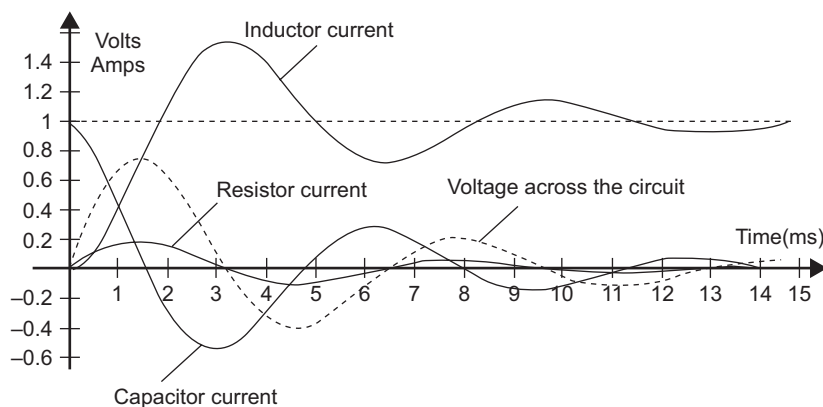


Fig. 12.11-4 Voltage and current waveforms in Example: 12.11-4

All the input current goes through the capacitor at $t = 0^+$ since neither the voltage across the circuit nor the current through inductor can become non-zero at that instant. Inductor current shows 52.7% overshoot which is the value predicted by Eqn. 12.7-1 for $\xi = 0.2$.

12.11.2 Frequency Response of Parallel *RLC* Circuit

The parallel *RLC* circuit and its phasor equivalent circuit are shown in Fig. 12.11-5.

12.50 | Series and Parallel *RLC* Circuits

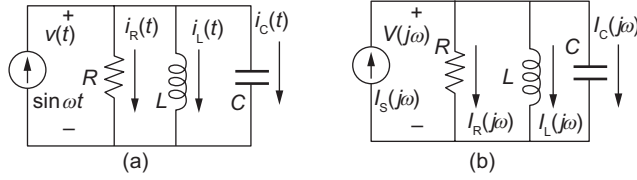


Fig. 12.11-5 Parallel *RLC* circuit and its phasor equivalent circuit

Almost the entire source current flows through the inductor at low frequency since inductor is a short-circuit at DC and low impedance for low frequency AC. Similarly, almost the entire source current passes through the capacitor at high frequencies since capacitor impedance approaches zero as frequency increases without limit. Thus, the magnitude response (*i.e.*, gain) of inductor current must be a low-pass function. Magnitude response of capacitor current must be a high-pass function and that of resistor current (and hence that of circuit voltage $v(t)$) must be a band-pass function. These frequency-response functions are obtained by applying current division principle to the parallel *RLC* circuit phasor equivalent circuit.

$$\begin{aligned}
 \frac{I_L(j\omega)}{I_S(j\omega)} &= \frac{1/j\omega L}{\frac{1}{R} + j\omega C + 1/j\omega L} \\
 &= \frac{1}{1 - \omega^2 LC + j\omega L/R} \\
 &= \frac{1/LC}{1/LC - \omega^2 + j\omega 1/RC} \\
 &= \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega}
 \end{aligned}$$

This ratio can be written in polar form as

$$\frac{I_L(j\omega)}{I_S(j\omega)} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_L \text{ where } \phi_L = -\tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad}$$

Thus, the frequency-response function for $i_L(t)$ in a parallel *RLC* circuit is found to be the same as the frequency-response function for $v_C(t)$ in series *RLC* circuit. It is a low-pass output.

The frequency-response of $i_C(t)$ and $i_R(t)$ are also obtained similarly.

$$\begin{aligned}
 \frac{I_R(j\omega)}{I_S(j\omega)} &= \frac{j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega} \\
 &= \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_R \text{ where } \phi_R = \frac{\pi}{2} - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad}
 \end{aligned}$$

$$\begin{aligned}\frac{I_C(j\omega)}{I_S(j\omega)} &= \frac{(j\omega)^2}{(\omega_n^2 - \omega^2) + j2\xi\omega_n\omega} \\ &= \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \angle \phi_L \text{ where } \phi_L = \pi - \tan^{-1} \frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \text{ rad}\end{aligned}$$

The frequency-response of voltage developed across the circuit is the frequency-response function for $i_R(t)$ multiplied by R . It will be a band-pass function. Hence, a parallel *RLC* circuit with high Q factor (*i.e.*, low ξ factor) will work as a narrow band-pass filter if it is excited by a current signal and the voltage across the circuit is accepted as the output. And, that is the most frequently used application of a parallel *RLC* circuit.

These frequency-response functions have been dealt with in detail in the context of series *RLC* circuit and nothing further need to be added. Whatever that has been stated with respect to capacitor voltage in series circuit can be applied directly to inductor current in the parallel circuit and so on.

Resonance in Parallel *RLC* Circuit takes place when input frequency is ω_n . Under resonant condition the input admittance (and impedance) of Parallel *RLC* Circuit becomes purely resistive and equal to $1/R$ Siemens. This is so since that frequency susceptance of inductor and capacitor are exactly equal in magnitude and opposite in sign and they cancel each other when added. They do not cancel completely at any other frequency and hence the admittance of a parallel *RLC* circuit is a *minimum* of $1/R$ at resonant frequency.

All the current from the source flows through R under resonance conditions. Thus, amplitude of voltage across the parallel combination is a maximum of R V (assuming unit amplitude for source current) at ω_n . The amplitude of current through capacitor at that frequency will then be $\omega_n RC$ V. The amplitude of current through inductor at resonant frequency will then be $R/\omega_n L$ V. Thus the current amplification factor at resonance in a parallel *RLC* circuit, defined as the ratio of amplitude of current in capacitor or inductor to the amplitude of source current, is

$$= \omega_n RC = \frac{RC}{\sqrt{LC}} = \frac{1}{\sqrt{L/C}/R} = \frac{1}{2\xi} = Q.$$

Thus, a high Q circuit will carry very high amplitude currents in L and C even when the source current amplitude is small if the source frequency is equal to or near about the circuit resonant frequency. These currents cancel themselves due to their phase opposition and they do not consume any portion of the source current. Entire source current flows through the resistance under resonant condition.

Example: 12.11-5

A parallel LC circuit used in a tuned amplifier circuit has $L = 25.3 \mu\text{H}$ and $C = 1 \text{ nF}$. There is no resistance load across it. A frequency-response test on the circuit revealed that the band-pass output (*i.e.*, the voltage across the circuit when the excitation is a sinusoidal current source) has a centre frequency that is approximately the expected value. But its bandwidth was found to be 6 kHz. (i) Find the Q factor of the inductor at the centre frequency. Losses in the capacitor may be ignored. (ii) What is the resistance that has to be connected across the LC combination such that the band-pass filter has a bandwidth of 10 kHz?

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Solution

A pure parallel LC circuit excited by a current source should have a bandwidth that goes to zero. This is due to the fact that for a narrow band-pass filter, the bandwidth and centre frequency are related by $bw = \omega_n / Q$. The Q of a pure LC parallel circuit is ∞ since its damping factor is zero.

The fact that the experiment conducted revealed a bandwidth of 6 kHz implies that there is damping in the circuit. Winding and core losses in the inductor produce damping in the circuit. So does the dielectric losses in the capacitor; but the effect of capacitor losses is usually small compared to the effect of inductor losses. Hence, we assume that, the limiting of bandwidth observed is essentially due to losses in the inductor. We can obtain the effective resistance that has come across the inductor at and around resonant frequency using the data provided.

$$\text{Undamped natural frequency of the circuit} = \frac{1}{2\pi\sqrt{LC}} = 1 \text{ MHz}$$

$\therefore$ Center frequency observed = 1 MHz (because that is the expected value and the experiment confirmed it)

Observed bandwidth = 6 kHz

$\therefore$ The Q factor that is effective in the circuit = $1000/6 = 166.7$

$\therefore$ The ξ factor that is effective in the circuit = $1/2Q = 0.003$

$$\therefore \text{The parallel resistance that is effectively there in the circuit} = \frac{1}{2\xi} \sqrt{\frac{L}{C}} = 26.5 \text{ k}\Omega$$

This resistance represents the losses at 1 MHz in the inductor in the form of a resistance in parallel with inductance.

The Q factor of a reactive element is defined as the ratio between maximum energy storage in the reactance to the energy lost in one cycle under steady-state operation at the frequency at which Q is calculated.

From this definition it follows that Q of an inductor = $\frac{\omega L}{R_s} = \frac{R_p}{\omega L}$ where R_s is the resistance that comes in series to represent the losses in the inductor and R_p is the resistance in parallel to the inductor if the losses are to be represented by such a parallel resistance.

The reactance of 25.3 μH inductor at 1 MHz is 159 Ω .

Therefore Q of the inductor = $26500/159 = 166.7$.

If the bandwidth is to be raised to 10 kHz, the Q of the circuit has to be lowered to 100. Then the damping factor has to be 0.005 and the effective parallel resistance has to be = $\frac{1}{2\xi} \sqrt{\frac{L}{C}} = 15.9 \text{ k}\Omega$. There is already 26.5 $\text{k}\Omega$ effective parallel resistance from losses in the inductor. Let R_1 be the extra resistance that has to be connected in parallel. Then $R_1 // 26.5 \text{ k}\Omega$ has to be 15.9 $\text{k}\Omega$. Therefore R_1 is 39.75 $\text{k}\Omega$.

12.12 SUMMARY

- Series and Parallel *RLC* circuits are described by second-order linear differential equation with constant coefficients. The natural response of such circuits contains two exponential functions. Three different types of natural response terms are possible in such circuits depending on the size of elements.

- A pure LC circuit with initial energy oscillates sinusoidally forever with a frequency of $\omega_n = 1/LC$ rad/s. $v_C(t)$ will have an amplitude of $\sqrt{2E_o/C}$, where E_o is the total initial energy storage in the circuit, i.e., $E_o = (LI_o^2/2) + (CV_o^2/2)$, where V_o and I_o are the initial capacitor voltage and inductor current respectively. $i(t)$ will have an amplitude of $\sqrt{2E_o/L}$ and the ratio between voltage amplitude and current amplitude is $\sqrt{L/C}$.
- The parameter called damping factor, ξ , decides the nature of natural response in RLC circuits. $\xi = \frac{R}{2\sqrt{L/C}}$ for series RLC circuit and $\xi = \frac{\frac{1}{2}\sqrt{L/C}}{R}$ for parallel RLC circuit. If $\xi > 1$, the circuit is an over-damped one and its natural response will contain two decaying real exponential functions. If $\xi = 1$, the circuit is a critically damped one and its natural response will contain an exponential function and a product of time with same exponential function. If $\xi < 1$, the circuit will be an under-damped one and its natural response will contain exponentially damped sine function and cosine function.
- The natural frequency $s = \sigma + j\omega$ stands for a complex exponential signal e^{st} in time-domain. This signal will represent one of the many components present in the zero-input response of the circuit that has s as one of its natural frequencies. Thus, *natural frequency* is a stand-in for e^{st} .
- Natural frequencies for an RLC circuit are : $(-\xi \pm \sqrt{\xi^2 - 1})\omega_n$ if the circuit is over-damped, $-\xi\omega_n$ with multiplicity of 2 if the circuit is critically damped and $(-\xi \pm j\sqrt{1 - \xi^2})\omega_n$ if the circuit is under-damped.
- Quality factor Q of a RLC circuit is another parameter that quantifies damping in the circuit. It is related to ξ through the relationship $Q = 1/2\xi$. In lightly damped RLC circuits, the fractional loss of energy in one oscillation in zero-input response is given by $2\pi/Q$ or $4\pi\xi$.
- The capacitor voltage in a series RLC circuit is a low-pass output. This leads to application of series RLC circuit as a good averaging filter.
- The resistor voltage in a series RLC circuit is a band-pass output with centre frequency at ω_n and a bandwidth of ω_n/Q or $2\xi\omega_n$. The two half-power frequencies are asymmetrically located around centre frequency in general. However, in a narrow band-pass case (i.e., $Q > 5$ or $\xi < 0.1$), they are more or less symmetric about ω_n .
- In a circuit excited by a single sinusoidal voltage source (current source) across a pair of terminals, *resonance* is the *sinusoidal steady-state condition* under which the current drawn at the terminals (voltage appearing across the terminals) is in phase with the source voltage (current). Equivalently, *resonance* is the condition under which the input impedance (admittance) offered to the sinusoidal source is resistive.
- Series RLC circuit is resonant at ω_n . At that frequency, the impedance of the circuit is *minimum* at R and is resistive. Circuit draws current at unity power factor. Voltage across capacitor and inductor will be of equal amplitude, but opposite in phase. The voltage amplification factor at resonance will be Q or $1/2\xi$.
- Parallel RLC circuit is resonant at ω_n . At that frequency, the impedance of the circuit is a *maximum* at R and is resistive. Circuit develops voltage at unity power factor. Current through capacitor and inductor will be of equal amplitude, but opposite in phase. The current amplification factor at resonance will be Q or $1/2\xi$.
- The voltage across a parallel RLC circuit is a band-pass output with centre frequency at ω_n and a bandwidth of ω_n/Q or $2\xi\omega_n$. The two half power frequencies are asymmetrically located around

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the centre frequency in general. However, in a narrow band-pass case (*i.e.*, $Q > 5$ or $\xi < 0.1$), they are more or less symmetric about ω_n .

- Quality factor Q of a *RLC* circuit is 2π times the ratio between total energy storage in the beginning of an oscillation cycle to the energy lost during that cycle under source-free condition. Equivalently, it is 2π times the ratio of total energy stored in the circuit to the energy dissipated in one cycle when the circuit is in sinusoidal steady state at resonant frequency.
- However, quality factor of an element like L or C is 2π times the ratio between maximum energy stored in reactive part of that element to the energy lost in one cycle in that element under steady-state operation at a particular ω . It will be a frequency-dependent number.

12.13 PROBLEMS

1. A pure LC circuit with $L = 100$ mH and $C = 10$ μ F has 0.05 Joules in the capacitor and 0.2 Joules in the inductor at $t = 0^-$. The circuit is allowed to carry out its free-response from $t = 0$. (i) Find the frequency of oscillations and amplitude of voltage and current oscillations. (ii) If the voltage across the capacitor is found to be 150 V at a particular instant, what is the current in the circuit at that instant?
2. A pure LC circuit in free-response is found to have 15 V amplitude voltage and 1.5 A current sinusoidal oscillations with a frequency of $2\pi \times 10^3$ rad/s. (i) Find L and C . (ii) Find the total initial energy storage. (iii) Can the initial voltage across capacitor and initial current through inductor be found out from this data? Explain.
3. A lightly damped series *RLC* circuit uses 100 mH inductor and 1 μ F capacitor and starts its free-response at $t = 0$ with an initial energy of 0.1 J. The voltage across capacitor is 200 V and current through the circuit is 0.4 A after 4 ms. Find ξ , Q and ω_d .
4. The Q factor of a 1 mH inductor at 10 kHz is measured to be 25. (i) What is the value of capacitor needed to make a series *RLC* circuit resonant at 10 kHz using this inductor? (ii) What will be the maximum percentage overshoot in the step response of the series circuit made with this inductor and calculate the value of capacitance?
5. Show that the amplitude of steady-state voltage across capacitor in a series *RLC* circuit is $\propto 1/\omega^2$ for $\omega \gg \omega_n$ where ω is the angular frequency of sinusoidal input voltage and ω_n is the undamped natural frequency of the circuit.
6. Show that the amplitude of steady-state voltage across inductor in a series *RLC* circuit is $\propto \omega^2$ for $\omega \ll \omega_n$ where ω is the angular frequency of sinusoidal input voltage and ω_n is the undamped natural frequency of the circuit.
7. Show that the amplitude of steady-state voltage across resistor in a series *RLC* circuit is $\propto 1/\omega$ for $\omega \gg \omega_n$ where ω is the angular frequency of sinusoidal input voltage and ω_n is the undamped natural frequency of the circuit.
8. The total energy storage in a parallel *RLC* circuit in free-response mode is found to be 70% of total initial energy storage after three full oscillations. The input admittance of the circuit is found to be resistive with a value of 0.01 siemens at 100 kHz. If this circuit is used as a band-pass filter, find the centre frequency, half-power frequencies and bandwidth of the filter.
9. A series *RLC* circuit with zero damping (*i.e.*, with $R = 0$) is driven by a sinusoidal voltage source of unit amplitude. The angular frequency of the source coincides with the value of ω_n of the circuit. Show that the zero-state response for capacitor voltage in this case is $= 0.5 \sin \omega_n t - 0.5 (\omega_n t) \cos \omega_n t$ if input is unit amplitude sine function. Does the circuit reach a steady state?

[Hint – Solve for a general value of ω and then use the following limit.

$$\lim_{x \rightarrow y} \frac{\sin x - \sin y}{x - y} = \cos x.]$$

10. A series RLC circuit with $L = 16$ mH, $C = 16$ μ F and $R = 10$ Ω has initial energy storage of 0.0016 J in the inductor and 0.0032 J in the capacitor. It is driven by a 10 V DC source switched on at $t = 0$. Find the total response for the resistor voltage in the circuit.
11. A series LC circuit uses $L = 4$ mH and $C = 64$ μ F. The Q factor of inductor is measured to be 80 at 1975 rad/s and the Q factor of capacitor at that frequency is found to be 300. If the circuit starts its free-response with total initial energy storage at 0.3 J what is the time at which the total stored energy in the circuit is 0.1 J?
12. (i) Derive the differential equation governing the output voltage $v_o(t)$ in the circuit in Fig. 12.13-1. (ii) Obtain expressions for ω_n and ξ of the circuit and calculate them for $L = 1$ mH, $C = 100$ μ F, $R_1 = 0.5$ Ω and $R_2 = 10$ Ω . (iii) Find the zero-input response of output voltage if the initial value of capacitor voltage is 10 V.

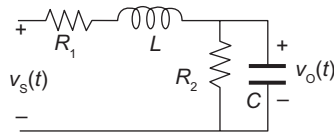


Fig. 12.13-1

13. A DC power supply of 12 V is connected to an electronic circuit by means of a two-wire connection that has an inductance of 10 μ H and resistance of 0.5 Ω . The circuit 0.2 A from the power supply may be modelled as a resistor. A 0.22 μ F ceramic capacitor is connected directly across the electronic circuit to hold its power supply constant. The circuit can withstand only ± 2 V variation on supply and will get damaged if the power supply exceeds 15 V. Examine whether the circuit will get damaged when the 12 V supply is switched on. If so, what solution will you suggest for the problem? [Hint – Consider the maximum percentage overshoot in the step response of RLC circuit model.]
14. Series LC filters are frequently employed at the output stage of DC power supplies to reduce the ripple content in the DC output. One such filter in a 12 V DC supply (see Fig. 12.13-2) uses $L = 4$ mH and $C = 220$ μ F. The inductor has a series resistance of 1 Ω . Two loads (electronic circuits) which can be modelled as 5 Ω and 20 Ω resistors are connected across the capacitor of the filter. The system was working under steady state for long time. Suddenly, the load resistance of 5 Ω is switched off. Find the power supply output voltage as a function of time. The maximum voltage that the electronic circuit represented by 20 Ω can withstand is 15 V. Will it get damaged when the other load is thrown open?

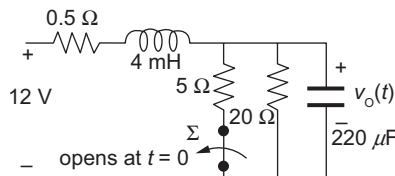


Fig. 12.13-2

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15. A series *RLC* circuit is resonant at 1 kHz and is found to have critically damped step response if a $10\ \Omega$ resistance is used in the circuit. (i) Find L and C . (ii) If $R = 1\ \Omega$ is actually used, find rise time, peak time, settling time and maximum percentage overshoot in step response of capacitor voltage.
16. A series *RLC* circuit with $R = 5\ \Omega$ was found to draw 5 W power when driven by a 10 V, 2 kHz source. It has 60% maximum percentage overshoot in its step response. (i) Find L and C . (ii) Find the additional resistance that has to be included in the circuit to make it critically damped. (iii) Find the voltage across the resistor under critical damping condition if the circuit with zero initial energy storage is excited by a 10 V DC source from $t = 0$.
17. A series *RLC* circuit was driven by a 10 V amplitude sinusoidal wave with variable frequency. The voltage across R was found to be of 10 V amplitude at 5 kHz and of 7.07 V amplitude at 4.9 kHz and 5.1 kHz. (i) Find L and C , (ii) Find ω_n , ω_d , ξ and Q of the circuit. (iii) Find the amplitude of voltage across C and voltage across L at 5 kHz if input amplitude is 10 V. (iv) Find the maximum amplitude of voltage that appears across L and C and the corresponding frequencies if the input amplitude is 10 V.
18. A series *RLC* circuit is resonant at 1 kHz and has 50% maximum overshoot in step response. It is driven by $v_s(t) = 1 \sin \omega t$ V. Find the steady-state voltage across L and C as functions of time approximately for (a) $\omega = 10$ kHz and (b) $\omega = 100$ Hz.
19. The switch in Fig. 12.13-3 was open for a long time and closes at $t = 0$. Find $v_C(t)$ and $i_L(t)$ for $t \geq 0^+$.

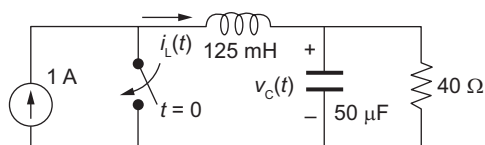


Fig. 12.13-3

20. The circuit in Fig. 12.13-4 is initially relaxed. Find $v_C(t)$ and $i_L(t)$ for $t \geq 0^+$. [Hint – Natural frequencies of a circuit will not depend on the excitation source value.]

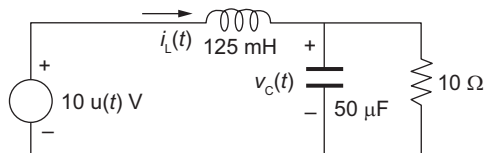


Fig. 12.13-4

21. The circuit in Fig. 12.13-5 is initially relaxed. Find $v_C(t)$ for $t \geq 0^+$. [Hint – Refer to Problem 20]

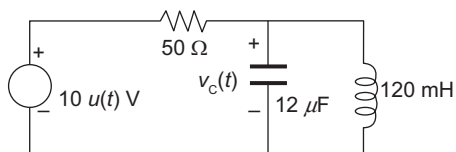


Fig. 12.13-5

22. A parallel LC circuit was driven by a sinusoidal current source of 10 mA amplitude. The frequency of the source was variable. Maximum voltage amplitude observed across the LC combination was 10 V and this happened at 20 kHz. Moreover, the voltage amplitude was seen to be 7.07 V at 20.1 kHz as well as at 19.9 kHz. (i) Find the values of L , C and Q factor of inductor at 20 kHz. The capacitor may be assumed as loss-less. (ii) If a bandwidth of 2 kHz is desired, calculate the extra resistance that has to be put in parallel with the LC parallel combination.
23. A parallel RLC Circuit is resonant at 0.8 kHz and has 60% maximum overshoot in step response of inductor current. It is driven by $i_s(t) = 2 \cos \omega t$ A. Find the steady-state current through L and C as functions of time approximately for (a) $f = 8$ kHz and (b) $f = 80$ Hz.

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Chapter 13

Analysis of Dynamic Circuits by Laplace Transforms

CHAPTER OBJECTIVES

- Expansion of right-sided time-domain waveforms in terms of $e^{\sigma t} \cos(\omega t + \phi)$.
- Definition of Laplace transform and inverse integral.
- Various properties of Laplace transform of a real function of time.
- Convergence of inverse integral and region of convergence in s -plane.
- Basic Laplace transform pairs.
- Solution of differential equations using Laplace transforms.
- Method of partial fractions for inversion of Laplace transforms.
- S -domain equivalent circuit and applying them for circuit analysis.
- System function $H(s)$ and Laplace transform of impulse response.
- Network functions and pole-zero plots.
- Graphical interpretation of frequency response function.

INTRODUCTION

We solved first-order and second-order circuits using the differential equation model in earlier chapters. Differential equation model can be used to solve multi-mesh, multi-node circuits containing R , L , C , M , linear dependent sources and independent sources too.

A linear time-invariant circuit containing R , L , C , M , linear dependent sources and a single input source is described by a linear ordinary differential equation with constant coefficients. The differential equation for such a circuit can be expressed in a standard format as below.

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x$$

13.2 | Analysis of Dynamic Circuits by Laplace Transforms

The variable y is any circuit voltage or current variable chosen as the *describing variable* for the circuit and x is the input source function. Standard mesh analysis or nodal analysis technique along with variable elimination will help us to arrive at this equation. However, the variable elimination involved can be considerably tricky in the case of large circuits containing many energy storage elements. This is a serious shortcoming of time-domain analysis by differential equation model.

The *order* of a circuit is equal to the order of the differential equation that describes it. Order of the circuit will be equal to the total number of independent inductors and capacitors – (number of all-inductor nodes + number of all-capacitor loops in the circuit).

The order of a circuit depends also on the kind and location of input. The same circuit will have different order if voltage source input is replaced by current source input.

The coefficients $a_{n-1}\dots a_0$ and $b_m\dots b_0$ are decided by the circuit parameters. They are real-valued. $a_{n-1}\dots a_0$ are positive real numbers if the circuit is passive *i.e.*, if it contains only R , L , C and M . They can be zero or negative real numbers if the circuit contains dependent sources. $b_m\dots b_0$ can be positive or negative or zero in all circuits.

The format of left-side of the differential equation that describes a circuit is independent of the particular circuit variable chosen as the describing variable in general. That is, $a_{n-1}\dots a_0$ will remain the same even if some other circuit variable is used as the variable y . However, $b_m\dots b_0$ will depend on the variable chosen.

This n^{th} -order differential equation requires n initial values for solving it if the input function is known only for a range of values of t rather than over entire t -axis. The required initial values are

$$y(0^+), \left. \frac{dy}{dt} \right|_{(0^+)}, \left. \frac{d^2y}{dt^2} \right|_{(0^+)}, \dots, \left. \frac{d^{n-1}y}{dt^{n-1}} \right|_{(0^+)}$$

The differential equation can be solved using $x(t)$ for $t \geq 0^+$ and these initial values.

The initial values available in a circuit are the initial current values for all inductors and initial voltage values for all capacitors. It requires considerable effort to translate these values into the required initial values in the case of a circuit containing many energy storage elements. This is another serious shortcoming of the differential equation approach.

Laplace transformation technique converts a linear differential equation with constant coefficients into an algebraic equation. Thus, the task of solving a collection of simultaneous differential equations will be reduced to a much simpler task of solving a set of simultaneous algebraic equations involving Laplace transforms of input signals and Laplace transform of desired output signal. This set of equations may be solved for the Laplace transform of output and the time-function may be obtained by inverting the transformation. Moreover, we will see later that the initial conditions specified for inductor currents and capacitor voltages can be used directly in Laplace transform method of solving a circuit. This makes it sound as if the technique of Laplace transforms is just a mathematical artifice for solving linear differential equations. Just as Logarithm is not a mathematical artifice for converting multiplication into addition, Laplace transform is not just a mathematical artifice to make solution of differential equations easier.

There is a very compelling reason why we take Laplace transform of a function. The reason is that (i) complex exponential signals are eigen functions of linear time-invariant circuits, (ii) linear time-invariant circuits obey superposition principle and (iii) Laplace transform expresses a given arbitrary input function as a sum of complex exponential signals.

Therefore, we commence our study of Laplace transform method of solving a circuit by examining the circuit response to a complex exponential input.

13.1 CIRCUIT RESPONSE TO COMPLEX EXPONENTIAL INPUT

Let the n^{th} -order differential equation describing an n^{th} -order linear time-invariant circuit be

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \quad (13.1-1)$$

where y is some circuit variable identified as the output variable and x is some independent voltage/current source function. Let $x(t) = 1 e^{st}$ be a complex exponential function of unit amplitude and complex frequency $s = \sigma + j\omega$. Let $y = A e^{st}$ be the trial solution, where A is a complex number to be determined. Substituting the trial solution in Eqn. 13.1-1, we get

$$\begin{aligned} [s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0] A e^{st} &= [b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0] e^{st} \\ \therefore A &= \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} \end{aligned}$$

Thus, when the input to a linear time-invariant circuit is a complex exponential function e^{st} , the output is the same complex exponential function multiplied by a complex number. Therefore, the complex frequency of output *remains same as that of the input*. The output will have a different phase compared to that of input since A is a complex number in general and has an angle. The value of this complex scaling factor depends on the coefficients of circuit differential equation (*i.e.*, on the circuit parameters) and the complex frequency s of the input. In consonance with the symbol $H(j\omega)$ used for a similar complex number that relates the output to an input of $e^{j\omega t}$, we use the symbol $H(s)$ to represent this number A from this point onwards. Therefore,

When $x(t) = e^{st}$ in a linear time-invariant circuit, $y(t) = H(s) e^{st}$, where

$$H(s) = \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}$$

However, which component of response is this? Since $x(t) = 1 e^{st}$, the complex exponential function was taken to be applied to the circuit from $t = -\infty$ onwards. Therefore, there is only one component in response and that is the forced response. Therefore, the response given above is the forced response as well as the total response. But if $x(t) = 1 e^{st} u(t)$, then, the above expression yields the forced response component only. The natural response terms in zero-state response and the natural response terms in zero-input response have to be found out from initial conditions. However, those terms too will be complex exponential functions since natural response terms of a linear time-invariant circuit are complex exponential functions.

The complex function $H(s)$ of a complex variable s can also be written in polar form as $|H(s)| \angle \theta$ and in exponential form as $|H(s)| e^{j\theta}$, where θ is its angle. Therefore, the output $y(t)$ can be expressed as $y(t) = |H(s)| e^{st + j\theta} = |H(s)| e^{\sigma t} e^{j(\omega t + \theta)}$. $H(s)$ may be viewed as a generalised frequency response function. Its magnitude gives the ratio between the amplitudes of output complex exponential function and input complex exponential function. Its angle gives the phase angle by which the output complex exponential function leads the input complex exponential function.

The signal e^{st} with a complex value for s goes through a linear time-invariant circuit and comes out as a scaled replica of itself. The scaling factor is $H(s)$. An input function that goes through a system

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and comes out as a scaled copy of itself is called an *eigen function* of the system. The scaling factor is called an *eigen value* of the system.

Complex exponential signals of e^{st} format with a complex-valued s are *eigen functions* of linear time-invariant circuits.

13.2 EXPANSION OF A SIGNAL IN TERMS OF COMPLEX EXPONENTIAL FUNCTIONS

The set of e^{st} format signals for all possible values of s and can be represented as a collection of points in a two-dimensional space. The horizontal axis of this plane represents the real part of s and the vertical axis represents the imaginary part of s . Then a point $s = \sigma + j\omega$ in this plane will stand for a signal $e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$. Such a point which acts as a stand-in for a complex exponential signal is called a *signal point*. The complex number representing that point *i.e.*, s , is called the *complex frequency* of the signal e^{st} . The real part of s has nepers/s as its unit and the imaginary part has radians/s as its unit. Collection of all such signal points – *i.e.*, the plane – is called the *signal plane* or *signal space*. The same space is also called the *s-domain* in circuit studies. The shape of signal for various signal point locations in *s-plane* is shown in Fig. 13.2-1.

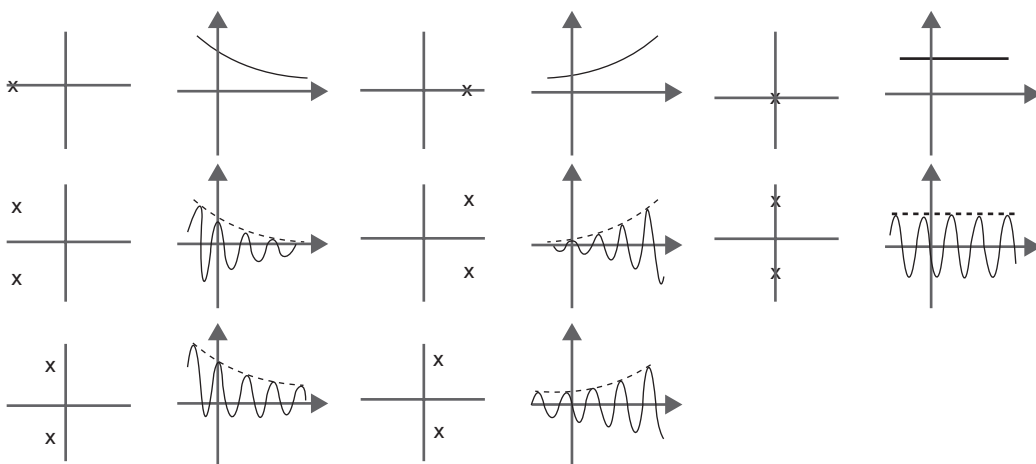


Fig. 13.2-1 Signal point in *s*-domain versus signal shape

Let $v(t) = f(t) u(t)$ (*i.e.*, right side of some function that is possibly two-sided) and $|v(t)| < Me^{\alpha t}$ for some M and α , then $V(s) = \int_{0^-}^{\infty} v(t) e^{-st} dt$ is its Laplace transform, where $s = \sigma + j\omega$ is the general complex frequency with $\sigma \geq \alpha$. The Laplace transform exists and the inverse integral converges to $v(t)$ only for those values of s that have $\text{Re}(s) > \alpha$. The region formed by all those values of s in the *s*-plane for which the Laplace transform of a time-function is defined and is convergent is called the region of convergence (ROC) of the Laplace transform. Obviously the ROC of Laplace transform of a right-sided function is the region to the right of $\text{Re}(s) = \alpha$ line. This is a vertical straight line parallel to $j\omega$ axis and crossing σ -axis at α .

$$v(t) = \frac{1}{j2\pi} \int_{\text{on } \text{Re}(s)=\sigma \text{ line}} V(s) e^{st} ds$$

The time-function can be obtained from its Laplace transform by carrying out the inversion integral given below.

The Laplace transform defined this way returns the right-side of the underlying function $f(t)$ on inversion. The left-side returned will be zero. In this sense, this Laplace transform may be termed as a *unilateral Laplace transform*. We deal with only unilateral Laplace transform in this chapter.

Note that the evaluation of inversion integral has to be performed on a line parallel to $j\omega$ -axis in s -plane with the line crossing the σ -axis within the ROC of the Laplace transform.

Let $v(t)$ be a right-sided function that is bounded by $Me^{\alpha t}$ with some finite value of M and α . Then the Laplace transform pair is defined as

$$V(s) = \int_0^{\infty} v(t)e^{-st} dt \quad \text{-- The Analysis Equation} \quad (13.2-1)$$

$$v(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} V(s)e^{st} ds \quad \text{-- The Synthesis Equation} \quad (13.2-2)$$

where $s = \sigma + j\omega$ is the complex frequency variable standing for the complex exponential function e^{st} with $\sigma \geq \alpha$. The ROC of $V(s)$ is the entire plane to the right of $\text{Re}(s) = \alpha$ line.

13.2.1 Interpretation of Laplace Transform

The Laplace transform $V(s)$ is a ‘complex amplitude density function’. Equation 13.2-2 makes it clear that Laplace transform expresses the given time-function as a sum of infinitely many complex exponential functions of infinitesimal complex amplitudes. Thus, Laplace transform is an expansion of $v(t)$ in terms of complex exponential functions. The entire ROC is available for evaluating Laplace transform. We will consider an example to clarify this matter.

Example: 13.2-1

Find the Laplace transform of $v(t) = u(t)$.

Solution

$$V(s) = \int_0^{\infty} e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_0^{\infty} = \frac{1}{s} \text{ for } \text{Re}(s) > 0$$

Therefore, $V(s) = 1/s$ with ROC of $\text{Re}(s) > 0$.

Thus, the inversion integral can be evaluated on any vertical straight-line on the right-half in s -plane. But does not that mean that a steady function like $u(t)$ is being synthesised from oscillations that grow with time? It precisely means that. The synthesis Eqn. 13.2-2 reveals that infinite growing complex exponential functions of infinitesimal amplitudes, which start at $-\infty$ and go up to $+\infty$ in time, participate in making the transient time-function $u(t)$. The contribution from a band of complex frequencies around a complex frequency value s is approximately $V(s) \times \Delta s \times e^{st}$, where Δs is the width of complex frequency band. A similar contribution comes from the band located around s^* . These two contributions together will form a growing sinusoidal function as shown below.

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$$\begin{aligned}
 &= \frac{1}{\sigma + j\omega} \times \Delta\omega \times e^{(\sigma + j\omega)t} + \frac{1}{\sigma - j\omega} \times \Delta\omega \times e^{(\sigma - j\omega)t} \\
 &= \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} \Delta\omega
 \end{aligned}$$

Thus, similarly located bands in the two half-sections of the vertical line on which the inversion integral is being evaluated result in a real valued contribution as shown above. Now the inversion integral for $1/s$ can be written as

$$\begin{aligned}
 v(t) &= \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} \frac{1}{s} e^{st} ds \\
 &= \frac{1}{j2\pi} \int_0^\infty \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} (jd\omega) \\
 &= \frac{1}{2\pi} \int_0^\infty \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} d\omega
 \end{aligned} \tag{13.2-3}$$

Thus, infinitely many exponentially growing sinusoids of frequencies ranging from zero to infinity, each with infinitesimal amplitude, interfere with each other constructively and destructively from $t = -\infty$ to $t = +\infty$ to synthesise the unit step waveform. Moreover, the exponentially growing sinusoids that participate in this waveform construction process are not unique. The value of σ can be any number > 0 . Therefore, each vertical line located in the right-half of s -plane yields a distinct set of infinitely many exponentially growing sinusoids which can construct the unit step waveform.

That infinitely many exponentially growing sinusoids interfere with each other to produce a clean zero for all $t < 0$ and a clean 1 for all $t > 0$ is indeed counter-intuitive and quite surprising when heard first. The inversion integral in Eqn. 13.2-3 was evaluated using a short computer program for various values of σ and over finite length sections on the vertical line. In effect, the program calculated the partial integral of the form

$v(t) \approx \frac{1}{2\pi} \int_0^{\omega_0} \frac{e^{\sigma t} [2\sigma \cos \omega t + 2\omega \sin \omega t]}{\sigma^2 + \omega^2} d\omega$ for various values of σ and ω_0 . Figure 13.2-2 shows the resulting waveforms for $\sigma = 0.1$ and $\omega_0 = 10, 20$ and 50 .

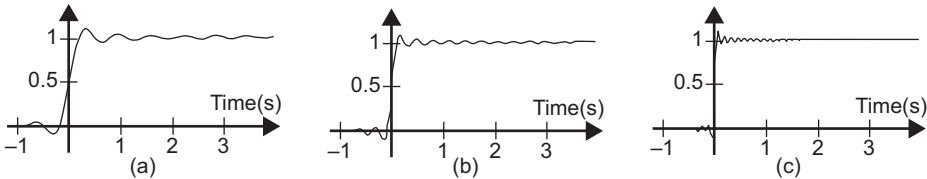


Fig. 13.2-2 Partial inversion integral for unit step function for $\sigma = 0.1$ and (a) $\omega_0 = 10$ (b) $\omega_0 = 20$ (c) $\omega_0 = 50$

Even a short range of 10 rad/s shows the tendency of the integral to approach step waveform. With $\omega_0 = 50$ rad/s the integral has more or less yielded step waveform – at least in the range -1 s to 4 s. We also observe the familiar Gibb's oscillations at discontinuities. Figure 13.2-3 shows the results of partial evaluation of inversion integral for $\sigma = 1$ and $\omega_0 = 10, 20$ and 50 .

This set of simulation result shows that we have to include more and more components in the partial integral to converge to unit step waveshape in a given time-interval as we let the components

grow at a faster rate, *i.e.*, for higher values of σ . And, keeping σ at a fixed value, we would need to include more and more frequency components when we increase the time-range over which we want convergence. However, we have infinite components at our disposal and it will be possible to include enough of them to recover the $u(t)$ shape up to any finite t however large it may be.

Therefore, Laplace transform expands a transient right-sided time-function in terms of infinitely many complex exponential functions of infinitesimal amplitudes. The ROC of such a Laplace transform will include portions of right-half of s -plane and hence the time-domain waveform gets constructed by growing complex exponential functions though that appears counter-intuitive.

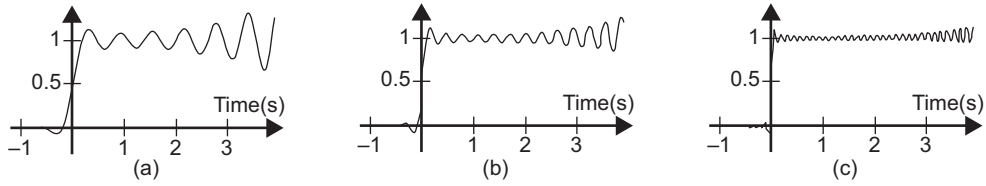


Fig. 13.2-3 Partial inversion integral for step function for $\sigma = 1$ and (a) $\omega_0 = 10$ (b) $\omega_0 = 20$ (c) $\omega_0 = 50$

13.3 LAPLACE TRANSFORMS OF SOME COMMON RIGHT-SIDED FUNCTIONS

Integral of sum of two functions is sum of integrals of each function. Thus, Laplace transformation is a linear operation. If $v_1(t)$ and $v_2(t)$ are two right-sided functions and a_1 and a_2 are two real numbers, then, $a_1 v_1(t) + a_2 v_2(t) \Leftrightarrow a_1 V_1(s) + a_2 V_2(s)$ is a Laplace transform pair. This is called **Property of Linearity of Laplace transforms**. Now we work out the Laplace transforms for many commonly used right-sided functions using the defining integral and property of linearity.

Let $v(t) = e^{s_o t} u(t)$ be a right-sided complex exponential function with a complex frequency of s_o . Then,

$$V(s) = \int_0^{\infty} e^{s_o t} e^{-st} dt = \int_0^{\infty} e^{-(s-s_o)t} dt = \frac{e^{-(s-s_o)t} \Big|_0^{\infty}}{(s-s_o)} = \frac{1}{(s-s_o)} \text{ with ROC } \text{Re}(s) > \sigma_o$$

Therefore, $e^{s_o t} u(t) \Leftrightarrow 1/(s-s_o)$ is a Laplace transform pair with ROC $\text{Re}(s) > \text{Re}(s_o)$.

The special case of $v(t) = u(t)$ is covered by this transform pair with $s_o = 0$.

Therefore, $u(t) \Leftrightarrow 1/s$ is a Laplace transform pair with ROC $\text{Re}(s) > 0$.

The special case of $v(t) = \cos \omega_o t u(t)$ is covered by expressing $v(t)$ as $(e^{j\omega_o t} + e^{-j\omega_o t})/2$ by employing Euler's formula and then applying property of linearity of Laplace transforms.

$$\therefore V(s) = \frac{0.5}{s - j\omega_o} + \frac{0.5}{s + j\omega_o} = \frac{s}{s^2 + \omega_o^2}$$

Therefore, $\cos \omega_o t u(t) \Leftrightarrow \frac{s}{s^2 + \omega_o^2}$ is a Laplace transform pair with ROC $\text{Re}(s) > 0$.

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Similarly, $\sin \omega_o t u(t) \Leftrightarrow \frac{\omega_o}{s^2 + \omega_o^2}$ is a Laplace transform pair with ROC $\text{Re}(s) > 0$.

Consider $v(t) = e^{-\alpha t} \cos \beta t u(t)$. This can be expressed as $= [e^{(-\alpha + j\beta)t} + e^{(-\alpha - j\beta)t}] / 2$ by Euler's formula. Then,

$$\therefore V(s) = \frac{0.5}{s + \alpha - j\beta} + \frac{0.5}{s + \alpha + j\beta} = \frac{(s + \alpha)}{(s + \alpha)^2 + \beta^2}$$

Therefore, $e^{-\alpha t} \cos \beta t u(t) \Leftrightarrow \frac{(s + \alpha)}{(s + \alpha)^2 + \beta^2}$ is a Laplace transform pair with ROC of $\text{Re}(s) > -\alpha$.

Similarly, $e^{-\alpha t} \sin \beta t u(t) \Leftrightarrow \frac{\beta}{(s + \alpha)^2 + \beta^2}$ is a Laplace transform pair with ROC of $\text{Re}(s) > -\alpha$.

Now consider $v(t) = \frac{e^{(s_o + \Delta s)t} - e^{s_o t}}{\Delta s} u(t)$

The Laplace transform of this function can be found from the defining integral as

$$V(s) = \frac{1}{\Delta s} \left[\frac{1}{s - s_o - \Delta s} + \frac{1}{s - s_o} \right] = \frac{1}{(s - s_o - \Delta s)(s - s_o)}$$

Now we send $v(t)$ to a limit as $\Delta s \rightarrow 0$.

$$\lim_{\Delta s \rightarrow 0} v(t) = \lim_{\Delta s \rightarrow 0} \frac{e^{(s_o + \Delta s)t} - e^{s_o t}}{\Delta s} u(t) = \left(\lim_{\Delta s \rightarrow 0} \frac{e^{(s_o + \Delta s)t} - e^{s_o t}}{\Delta s} \right) u(t) = t e^{s_o t} u(t)$$

Therefore, Laplace transform of $t e^{s_o t} u(t) = \lim_{\Delta s \rightarrow 0} \frac{1}{(s - s_o - \Delta s)(s - s_o)} = \frac{1}{(s - s_o)^2}$

Therefore, $t e^{s_o t} u(t) \Leftrightarrow 1 / (s - s_o)^2$ is a Laplace transform pair with ROC $\text{Re}(s) > \text{Re}(s_o)$.

The special case of $v(t) = t u(t)$ is covered by this transform pair with $s_o = 0$.

Therefore, $t u(t) \Leftrightarrow 1 / s^2$ is a Laplace transform pair with ROC $\text{Re}(s) > 0$.

And finally, we consider $v(t) = \delta(t)$. $V(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) e^0 dt = 1$. Thus, $\delta(t) \Leftrightarrow 1$ is a Laplace transform pair with ROC of entire s -plane. It requires all complex exponential functions with equal intensity to synthesise an impulse function in time-domain.

These commonly used Laplace transform pairs are listed in Table 13.3-1. Some of them have been derived in this section. Others will be taken up later.

Table 13.3-1 Basic Laplace Transform Pairs

Time-Function	Laplace Transform	Region of Convergence
$\delta(t)$	1	Entire s -plane
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
$e^{s_o t} u(t)$	$\frac{1}{s - s_o}$	$\text{Re}(s) > \text{Re}(s_o)$
$e^{j\omega_o t} u(t)$	$\frac{1}{s - j\omega_o}$	$\text{Re}(s) > 0$
$e^{-\alpha} u(t)$	$\frac{1}{s + \alpha}$	$\text{Re}(s) > -\alpha$
$\cos \omega_o t u(t)$	$\frac{s}{s^2 + \omega_o^2}$	$\text{Re}(s) > 0$
$\sin \omega_o t u(t)$	$\frac{\omega_o}{s^2 + \omega_o^2}$	$\text{Re}(s) > 0$
$e^{-\alpha} \cos \beta t u(t)$	$\frac{(s + \alpha)}{(s + \alpha)^2 + \beta^2}$	$\text{Re}(s) > -\alpha$
$e^{-\alpha} \sin \beta t u(t)$	$\frac{\beta}{(s + \alpha)^2 + \beta^2}$	$\text{Re}(s) > -\alpha$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
$t^n u(t), n=1, 2, \dots$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
$te^{s_o t} u(t)$	$\frac{1}{(s - s_o)^2}$	$\text{Re}(s) > \text{Re}(s_o)$
$t^n e^{s_o t} u(t), n=1, 2, \dots$	$\frac{n!}{(s - s_o)^{n+1}}$	$\text{Re}(s) > \text{Re}(s_o)$

13.4 THE s -DOMAIN SYSTEM FUNCTION $H(s)$

We saw in Section 13.1 that when an input e^{st} is applied to a linear time-invariant circuit described by an n^{th} -order differential equation

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x$$

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the response is given by $H(s)e^{st}$ where

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$$

$H(s)$ in this context is the ratio of complex amplitude of forced response component in output to the complex amplitude of input complex exponential function with a complex frequency of s . There is only forced response in this context and forced response itself is the total response.

In Section 13.2, we observed that a right-sided function $x(t)$ can be expressed as a sum of infinitely many complex exponential functions of frequency between $\sigma - j\infty$ to $\sigma + j\infty$ with the line $\text{Re}(s) = \sigma$ falling within the ROC of Laplace transform of $x(t)$. We combine these two facts along with superposition principle to arrive at the zero-state response of a linear time-invariant circuit to a right-sided input function.

Consider a particular value of complex frequency s and a small band of complex frequency Δs centered on it. This band contributes complex exponential functions of frequencies between $(s - 0.5\Delta s)$ and $(s + 0.5\Delta s)$. For sufficiently small Δs , we may take all these complex exponential functions to be evolving approximately at the centre frequency of the band, i.e., at s itself. In that case, all the infinitesimal contributions coming from this band may be consolidated into a signal $\approx X(s) \Delta s e^{st}$.

This single complex frequency component with complex amplitude of $X(s) \Delta s$ will produce a total response component of $H(s) X(s) \Delta s e^{st}$ in the output. We get the zero-state response of the circuit by adding all such contributions over the line $\text{Re}(s) = \sigma$ falling within the ROC of $X(s)$ and sending the sum to a limit by making $\Delta s \rightarrow 0$. The result will be the following integral.

$$\therefore y(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} H(s)X(s)e^{st} ds \quad (13.4-1)$$

Compare Eqn. 13.4-1 with the synthesis equation of Laplace transform given by Eqn. 13.2-2. It is evident that Eqn. 13.4-1 is the synthesis equation of the Laplace transform $H(s)X(s)$. But then, a synthesis equation which returns $y(t)$ must be synthesising it from the Laplace transform $Y(s)$ of the time-function $y(t)$. Therefore, $Y(s) = H(s)X(s)$. This important result requires restatement.

The Laplace transform of zero-state response = Laplace transform of input source function $\times$ Ratio of complex amplitude of forced response to the complex amplitude of input complex exponential function at a complex frequency of s .

Now comes a definition. *The ratio of Laplace transform of zero-state response to Laplace transform of input source function is defined as the s -domain System Function.* And these two are seen to be the same.

$\therefore$ The s -domain System Function,

$$H(s) = \frac{Y_{zsr}(s)}{X(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (13.4-2)$$

Note carefully that System Function is independent of initial conditions in the circuit since it is the zero-state response to a right-sided input that is employed in its definition. This function is also called a *Transfer Function* when both x and y are similar quantities, i.e., when x and y are voltages or x and y are currents and is denoted by $T(s)$. It is called an *Input Impedance Function* and is denoted by $Z_i(s)$

if y is the voltage across a terminal pair and x is the current entering the positive terminal. It is called an *Input Admittance Function* and is denoted by $Y_i(s)$ if y is the current into a terminal pair and x is the voltage across the terminal pair. These two, i.e., $Z_i(s)$ and $Y_i(s)$, together are at times referred to as *immittance functions*.

If the quantities x and y are voltage–current or current–voltage pair and they refer to different terminal pairs in the circuit, we call the s -domain System Function a *Transfer Impedance Function* or *Transfer Admittance Function*, as the case may be. They are represented by $Z_m(s)$ and $Y_m(s)$, respectively.

We have an expression for $H(s)$ as a ratio of rational polynomials in s in Eqn. 13.4-2. Rational polynomials are polynomials containing only integer powers of the independent variable. However, there is another more interesting interpretation possible for $H(s)$.

Let us try to find the impulse response of the circuit by this transform technique. We remember that ‘impulse response’ means ‘zero-state response to unit impulse input’ by definition. Hence, we can use the System Function to arrive at the Laplace transform of impulse response as $H(s)X(s)$. But $x(t) = \delta(t)$ and therefore $X(s) = 1$. Hence, for a linear time-invariant circuit, the following statement holds.

Laplace transform of impulse response = s -domain System Function, and, Impulse response = inverse Laplace transform of s -domain system function. This result was anticipated in naming the System Function as $H(s)$.

Once the System Function and Laplace transform of input source function are known, one can obtain the Laplace transform of zero-state response by inverting the product of input transform and System Function. We will take up the task of inverting Laplace transforms in later sections.

13.5 POLES AND ZEROS OF SYSTEM FUNCTION AND EXCITATION FUNCTION

$H(s)$ is the System Function, $X(s)$ is the excitation function and $Y(s)$ is the output function referred to in this section.

We observed that $H(s) = \frac{Y_{zsr}(s)}{X(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$ is a ratio of rational polynomials

in complex frequency variable s . Further, we observe from Table 13.3-1 that the excitation functions corresponding to commonly employed input source functions are also in the form of ratio of rational polynomials in s . Thus the output function also turns out to be a ratio of rational polynomials in s . Therefore, we can write

$$H(s) = \frac{Q(s)}{P(s)}, X(s) = \frac{Q_e(s)}{P_e(s)} \text{ and } Y(s) = \frac{Q(s)}{P(s)} \times \frac{Q_e(s)}{P_e(s)} \text{ where } Q(s) \text{ is an } m^{\text{th}}\text{-order polynomial on } s \text{ and}$$

$P(s)$ is an n^{th} -order polynomial on s . They are the numerator polynomial and denominator polynomial of System Function, respectively. Similarly, $Q_e(s)$ and $P_e(s)$ are the numerator and denominator polynomials on s for the excitation function.

Let the n roots of $P(s)$ be represented as $p_1, p_2, \dots, p_n$ and the m roots of $Q(s)$ be represented as $z_1, z_2, \dots, z_m$. These roots can be complex in general. $p_1, p_2, \dots, p_n$ are the n values of complex frequency s at which the System Function goes to infinity. They are defined as *poles* of System Function. $z_1, z_2, \dots, z_m$ are the m values of complex frequency s at which the System Function goes to zero value. They are defined as the *zeros* of System Function.

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Similarly the values of s at which $X(s)$ goes to infinity are called the *excitation poles* and the values of s at which $X(s)$ goes to zero are called the *excitation zeros*. They are the same as roots of $P_e(s)$ and $Q_e(s)$, respectively.

Obviously, the System Function poles and excitation function poles together will form the poles of output function. Similarly, the System Function zeros and excitation function zeros together will form the output function zeros. These statements assume that no pole-zero cancellation takes place.

A diagram that shows the complex signal plane, *i.e.*, the s -plane, with all poles of a Laplace transform marked by 'x' symbol and all zeros marked by 'o' symbol is called the *pole-zero plot* of that Laplace transform. Some poles and zeros may have multiplicity greater than 1. In that case, the multiplicity is marked near the corresponding pole or zero in the format ' $r = k$ ' where r indicates the multiplicity and k is the actual value of multiplicity. The default value of $r = 1$ is not marked.

Example: 13.5-1

Obtain the pole-zero plot of the transfer function $V_o(s)/V_s(s)$, excitation function and the output function in the circuit shown in Fig. 13.5-1 with $v_s(t) = 10 e^{-1.5t} \cos 2t u(t)$ V.

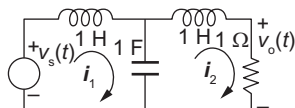


Fig. 13.5-1 Circuit for Example: 13.5-1

Solution

The differential equation describing the second mesh current in this circuit is derived below. The two mesh equations are

$$\begin{aligned} \frac{di_1}{dt} + \int_{-\infty}^t (i_1 - i_2) dt &= v_s(t) \\ \int_{-\infty}^t (i_2 - i_1) dt + \frac{di_2}{dt} + i_2 &= 0 \end{aligned}$$

Adding the two mesh equations results in

$$\frac{di_1}{dt} + \frac{di_2}{dt} + i_2 = v_s(t) \Rightarrow \frac{di_1}{dt} = v_s(t) - \frac{di_2}{dt} - i_2$$

Differentiating the second mesh equation twice with respect to time and using the above equation in the result gives us

$$\frac{d^3 i_2}{dt^3} + \frac{d^2 i_2}{dt^2} + 2 \frac{di_2}{dt} + i_2 = v_s(t)$$

$v_o(t)$ is numerically equal to $i_2(t)$ and hence the differential equation governing $v_o(t)$ is

$$\frac{d^3 v_o}{dt^3} + \frac{d^2 v_o}{dt^2} + 2 \frac{dv_o}{dt} + v_o = v_s(t)$$

The characteristic equation is $s^3 + s^2 + 2s + 1 = 0$ and its roots are $s_1 = -0.2151 + j1.307$, $s_2 = -0.2151 - j1.307$ and $s_3 = -0.5698$. The roots of a polynomial of degree higher than 2 will normally require the help of root-finding software or numerical methods. There is a pair of *complex conjugate* roots.

The System Function $H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{s^3 + s^2 + 2s + 1}$

Laplace transform of $e^{-1.5t} \cos 2t u(t)$ is $\frac{s+1.5}{(s+1.5)^2 + 4}$

Therefore, $V_s(s) = \frac{s+1.5}{(s+1.5)^2 + 4}$ and $V_o(s) = \frac{1}{s^3 + s^2 + 2s + 1} \times \frac{10(s+1.5)}{(s+1.5)^2 + 4}$

We observe that the denominator polynomial is the same as the left side of the characteristic equation of the governing differential equation. This will always be so. *Hence poles of System Function (they are also called ‘system poles’) will be the same as the natural frequencies of the circuit for any linear time-invariant circuit.*

Therefore the system poles are $p_1 = -0.2151 + j1.307$, $p_2 = -0.2151 - j1.307$ and $p_3 = -0.5698$.

The numerator polynomial of System Function in this case is trivial and there are no ‘system zeros’.

The excitation poles are at $p_{e1} = -1.5 + j2$ and $p_{e1} = -1.5 - j2$ and excitation zero is at $z_{e1} = -1.5$.

The pole-zero plots are shown in Fig. 13.5-2.

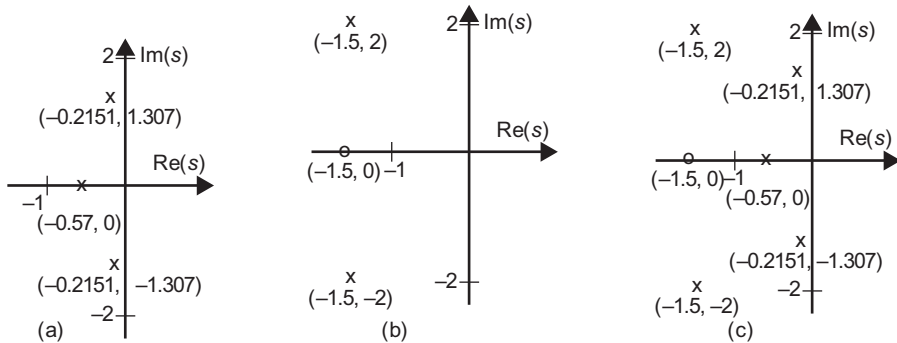


Fig. 13.5-2 Pole-zero plots in Example: 13.5-1 (a) for System Function (b) for excitation function (c) for output function

13.6 METHOD OF PARTIAL FRACTIONS FOR INVERTING LAPLACE TRANSFORMS

Any Laplace transform can be inverted by evaluating the synthesis integral in Eqn. 13.2-2 on a suitably selected vertical line extending from $-\infty$ to ∞ in the s -plane within the ROC of the transform being inverted. However, simpler methods based on Residue Theorem in Complex Analysis exist for special Laplace transforms. We do not take up the detailed analysis based on Residue Theorem here. However, the reader has to bear in mind the fact that the ‘method of partial fractions’ for inverting certain special types of Laplace transforms is based on Residue Theorem in Complex Analysis.

Linear time-invariant circuits are described by linear constant-coefficient ordinary differential equations. All the coefficients are real. Such a circuit will have only real-valued natural frequencies or complex-conjugate natural frequencies. Thus, the impulse response of such a circuit will contain only complex exponential functions. Each complex exponential function will have a Laplace transform of the form $\frac{k}{s - s_o}$ where s_o is the complex frequency of that particular term. Laplace transformation is a

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linear operation. Hence, Laplace transform of the sum of impulse response terms will be the sum of Laplace transform of impulse response terms. Therefore, Laplace transform of impulse response of a linear time-invariant circuit will be the sum of finite number of terms of the $\frac{k}{s - s_o}$ type. Such a sum will finally become a ratio of rational polynomials in s . The order of denominator polynomial will be the same as the number of first-order terms of $\frac{k}{s - s_o}$ type that entered the sum.

Many of the normally employed excitation functions in linear time-invariant circuits are also of complex exponential nature. Input functions that can be expressed as linear combinations of complex exponential functions will have Laplace transforms that are ratios of rational polynomials in s as explained in the last paragraph.

Product of Laplace transforms that are ratios of rational polynomials in s will result in a new Laplace transform which will also be a ratio of rational polynomials in s . Hence, the Laplace transform of output of a linear time-invariant circuit excited by an input source function, that can be expressed as a linear combination of complex exponential functions, will be a ratio of rational polynomials in s .

A Laplace transform that is in the form of a ratio of rational polynomials in s can be inverted by the method of partial fractions.

Let $Y(s) = Q(s)/P(s)$ be such a Laplace transform. Let the degree of denominator polynomial be n and that of numerator be m . The degree of numerator polynomial will usually be less than n . If the Laplace transform of output of a linear time-invariant circuit shows $n \leq m$, it usually implies that the circuit model employed to model physical processes has been idealised too much. We assume that $m < n$ in this section. If m is equal to n or more than n , then $Y(s)$ can be written as

$Y(s) = k_1 s^{m-n} + \dots + k_{m-n} + \frac{Q'(s)}{P(s)}$, and we employ method of partial fractions on $\frac{Q'(s)}{P(s)}$ only.

Let $p_1, p_2, \dots, p_n$ be the n roots of denominator polynomial. They may be real or complex. If there is a complex root, the conjugate of that root will also be a root of the polynomial. We identify two cases. In the first case all the n roots (*i.e.*, poles of $Y(s)$) are distinct.

Case-1 All the n roots of $P(s)$ are distinct

Then we can express $Y(s)$ as a sum of first-order factors as below.

$$Y(s) = \frac{A_1}{(s - p_1)} + \frac{A_2}{(s - p_2)} + \dots + \frac{A_n}{(s - p_n)} \quad (13.6-1)$$

Each term in this expansion is a partial fraction. The value of A_i appearing in the numerator of i^{th} partial fraction is the 'residue at the pole p_i '. The problem of partial fractions involves the determination of these residues. Multiply both sides of Eqn. 13.6-1 by $(s - p_i)$, where p_i is the pole at which the residue A_i is to be evaluated. Remember that $Y(s)$ will contain $(s - p_i)$ as a factor in the denominator. Hence, the multiplication by $(s - p_i)$ results in cancellation of this factor in $Y(s)$.

$$(s - p_i)Y(s) = \frac{A_1(s - p_i)}{(s - p_1)} + \frac{A_2(s - p_i)}{(s - p_2)} + \dots + \frac{A_2(s - p_i)}{(s - p_i)} + \dots + \frac{A_n(s - p_i)}{(s - p_n)} \quad (13.6-2)$$

Now we evaluate both sides of Eqn. 13.6-2 at $s = p_i$ to get $A_i = (s - p_i)Y(s)|_{s=p_i}$. This calculation is repeated for $i = 1$ to n to complete all the partial fractions.

Each partial fraction of the type $\frac{A_i}{s - p_i}$ can be recognised as the Laplace transform of $A_i e^{p_i t} u(t)$ by consulting relevant entry in Table 13.3-1. But, though we know that $e^{p_i t} u(t)$ has a Laplace transform

of $\frac{1}{s-p_i}$, how do we know that is the only time-function that will have $\frac{1}{s-p_i}$ as its Laplace transform?

It is vital to be sure about that if we want to assert that the time-function is $e^{p_i t} u(t)$ whenever we see a Laplace transform $\frac{1}{s-p_i}$. The ‘Theorem of Uniqueness of Laplace transforms’ states that a Laplace transform pair is unique. That is, if we have, by some method or other, found out that $F(s)$ is the Laplace transform of $f(t)$, then this theorem assures us that only $f(t)$ will have this $F(s)$ as its Laplace transform and no other function will have $F(s)$ as its Laplace transform. Therefore, whenever we see a $\frac{1}{s-p_i}$, we can write $e^{p_i t} u(t)$ as its inverse.

Therefore, $y(t) = (A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_n e^{p_n t}) u(t)$ where $A_i = (s-p_i)Y(s)\big|_{s=p_i}$.

Case-2 One root of multiplicity r and $n-r$ distinct roots for $P(s)$

In this case the partial fraction expansion is as shown below.

$$Y(s) = \frac{A_1}{(s-p)^r} + \frac{A_2}{(s-p)^{r-1}} + \dots + \frac{A_r}{(s-p)} + \frac{A_{r+1}}{(s-p_{r+1})} + \dots + \frac{A_n}{(s-p_n)} \quad (13.6-3)$$

The first root p is assumed to repeat r times. It may be real or complex. The remaining $(n-r)$ roots are designated as $p_{r+1}, p_{r+2}, \dots, p_n$. That Eqn. 13.6-3 is the partial fraction expansion in this case can be shown by an application of Residue Theorem. We take this as a fact and proceed.

The procedure for evaluating the $(n-r)$ residues at the $(n-r)$ non-repeating poles of $Y(s)$ is the same as in Case-1. Therefore,

$$A_i = (s-p_i)Y(s)\big|_{s=p_i} \quad \text{for } i = r+1 \text{ to } n$$

We multiply both sides of Eqn. 13.6-3 by $(s-p)^r$ for evaluating the r residues at the repeating pole. The result is

$$(s-p)^r Y(s) = A_1 + A_2(s-p) + \dots + A_r(s-p)^{r-1} + \frac{A_{r+1}(s-p)^r}{(s-p_{r+1})} + \dots + \frac{A_n(s-p)^r}{(s-p_n)} \quad (13.6-4)$$

$(s-p)^r$ is a factor of denominator of $Y(s)$. Therefore, multiplication of $Y(s)$ by $(s-p)^r$ will cancel this factor in the denominator. Now, evaluating both sides with $s=p$, we get

$$A_1 = (s-p)^r Y(s)\big|_{s=p}$$

Now, we differentiate Eqn. 13.6-4 on both sides with respect to s and substitute $s=p$ to get

$$A_2 = \frac{d[(s-p)^r Y(s)]}{ds}\bigg|_{s=p}$$

Similarly, successive differentiation with respect to s and substitution of $s=p$ leads to

$$A_3 = \frac{d^2[(s-p)^r Y(s)]}{ds^2}\bigg|_{s=p}, A_4 = \frac{1}{2!} \frac{d^3[(s-p)^r Y(s)]}{ds^3}\bigg|_{s=p}, \dots, A_r = \frac{1}{r!} \frac{d^{r-1}[(s-p)^r Y(s)]}{ds^{r-1}}\bigg|_{s=p}$$

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The reader may verify that the partial fraction terms corresponding to the non-repeating roots will contribute only zero values in all stages of this successive differentiation.

Once all residues are calculated, Eqn. 13.6-4 is inverted to get the following time-function:

$$y(t) = \left(A_1 \frac{t^{r-1}}{(r-1)!} e^{pt} + A_2 \frac{t^{r-2}}{(r-2)!} e^{pt} + \cdots + A_r e^{pt} + A_{r+1} e^{p_{r+1}t} \cdots + A_n e^{p_n t} \right) u(t)$$

We have used the Laplace transform pair $t^k e^{pt} u(t) \Leftrightarrow \frac{k!}{(s-p)^{k+1}}$ in arriving at this result. This Laplace transform pair will be proved in a later section.

Example: 13.6-1

Determine (i) the impulse response (ii) the step response and (iii) the zero-state response when $v_s(t) = 2e^{-2t} u(t)$ for $v_o(t)$ in the circuit in Fig. 13.6-1.

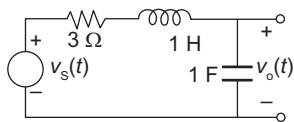


Fig. 13.6-1 Circuit for Example: 13.6-1

Solution

The mesh equation of the circuit is $3i + \frac{di}{dt} = v_s(t) - v_o(t)$, where i is the current flowing in the mesh. But $i = 1 \times \frac{dv_o(t)}{dt}$ since i flows in the capacitor and $v_o(t)$ is the voltage across capacitor.

Therefore, $3 \frac{dv_o(t)}{dt} + \frac{d^2 v_o(t)}{dt^2} = v_s(t) - v_o(t)$. Therefore, the differential equation governing the output voltage is $\frac{d^2 v_o(t)}{dt^2} + 3 \frac{dv_o(t)}{dt} + v_o(t) = v_s(t)$.

$$\text{The System Function } H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{s^2 + 3s + 1}.$$

The roots of denominator polynomial are -2.618 and -0.382 . The factors of the denominator polynomial are $(s + 2.618)$ and $(s + 0.382)$.

(i) The impulse response of a linear time-invariant circuit is the same as the inverse transform of its System Function.

$$\therefore h(t) = \text{Inverse of } H(s)$$

$$= \text{Inverse of } \frac{1}{(s + 2.618)(s + 0.382)}$$

$$\frac{1}{(s + 2.618)(s + 0.382)} = \frac{A_1}{(s + 2.618)} + \frac{A_2}{(s + 0.382)}$$

$$A_1 = (s + 2.618) \times \frac{1}{(s + 2.618)(s + 0.382)} \Big|_{s=-2.618} = \frac{1}{(s + 0.382)} \Big|_{s=-2.618} = -0.4472$$

$$A_2 = (s + 0.382) \times \frac{1}{(s + 2.618)(s + 0.382)} \Big|_{s=-0.382} = \frac{1}{(s + 2.618)} \Big|_{s=-0.382} = 0.4472$$

$$\therefore \frac{1}{(s+2.618)(s+0.382)} = \frac{-0.4472}{(s+2.618)} + \frac{0.4472}{(s+0.382)}$$

$$\therefore h(t) = 0.4472(e^{-0.382t} - e^{-2.618t})u(t) \text{ V}$$

- (ii) $v_s(t) = u(t) \Rightarrow V_s(s) = 1/s$. Therefore the Laplace transform of step response = $\frac{1}{s(s+2.618)(s+0.382)}$.
Expressing this in partial fractions,

$$\frac{1}{s(s+2.618)(s+0.382)} = \frac{A_1}{s} + \frac{A_2}{(s+2.618)} + \frac{A_3}{(s+0.382)}$$

$$A_1 = s \times \frac{1}{s(s+2.618)(s+0.382)} \Big|_{s=0} = \frac{1}{2.618 \times 0.382} = 1$$

$$A_2 = (s+2.618) \times \frac{1}{s(s+2.618)(s+0.382)} \Big|_{s=-2.618} = \frac{1}{-2.618 \times -2.236} = 0.1708$$

$$A_3 = (s+0.382) \times \frac{1}{s(s+2.618)(s+0.382)} \Big|_{s=-0.382} = \frac{1}{-0.382 \times 2.236} = -1.1708$$

$$\therefore \frac{1}{s(s+2.618)(s+0.382)} = \frac{1}{s} + \frac{0.1708}{(s+2.618)} + \frac{-1.1708}{(s+0.382)}$$

$$\therefore \text{Step response of } v_o(t) = (1 + 0.1708e^{-2.618t} - 1.1708e^{-0.382t})u(t) \text{ V}$$

- (iii) $v_s(t) = 2e^{-2t}u(t) \Rightarrow V_s(s) = 2/(s+2)$. The Laplace transform of zero-state response is given by product of System Function and Laplace transform of input function.

$$\therefore V_o(s) = \frac{2}{(s+2)(s+2.618)(s+0.382)}$$

Expressing this in partial fractions,

$$V_o(s) = \frac{2}{(s+2)(s+2.618)(s+0.382)} = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2.618)} + \frac{A_3}{(s+0.382)}$$

$$A_1 = (s+2) \times \frac{2}{(s+2)(s+2.618)(s+0.382)} \Big|_{s=-2} = \frac{2}{0.618 \times -1.618} = -2$$

$$A_2 = (s+2.618) \times \frac{2}{(s+2)(s+2.618)(s+0.382)} \Big|_{s=-2.618} = \frac{2}{-0.618 \times -2.236} = 1.4474$$

$$A_3 = (s+0.382) \times \frac{2}{(s+2)(s+2.618)(s+0.382)} \Big|_{s=-0.382} = \frac{2}{1.618 \times 2.236} = 0.5528$$

$$\therefore \frac{1}{(s+2)(s+2.618)(s+0.382)} = \frac{-2}{(s+2)} + \frac{1.4474}{(s+2.618)} + \frac{0.5528}{(s+0.382)}$$

$$\therefore v_o(t) = (-2e^{-2t} + 1.4474e^{-2.618t} + 0.5528e^{-0.382t})u(t) \text{ V}$$

Example: 13.6-2

The resistor value in Fig. 13.6-1 under Example 13.6-1 is changed to 2Ω . (i) Find the step response of $v_o(t)$ (ii) Determine the zero-state response of $v_o(t)$ if $v_s(t) = e^{-t} u(t)$ V.

Solution

The differential equation governing the output voltage with 2Ω resistor instead of 3Ω is $\frac{d^2 v_o(t)}{dt^2} + 2 \frac{dv_o(t)}{dt} + v_o(t) = v_s(t)$.

$$\text{The System Function } H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}.$$

The roots of denominator polynomial are -1 and -1 . The factors of the denominator polynomial are $(s+1)$ and $(s+1)$. Therefore, the root at -1 has a multiplicity of 2.

(i) With $v_s(t) = u(t)$ the Laplace transform of output is $V_o(s) = \frac{1}{s(s+1)^2}$. Expressing this in partial fractions, $V_o(s) = \frac{1}{s(s+1)^2} = \frac{A_1}{s} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)}$. We can find out the residues by applying the expressions developed earlier in this section. Or we may proceed as below.

$$\frac{1}{s(s+1)^2} = \frac{A_1}{s} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)} = \frac{A_1(s+1)^2 + A_2s + A_3s(s+1)}{s(s+1)^2} = \frac{s^2(A_1 + A_3) + s(2A_1 + A_2 + A_3) + A_1}{s(s+1)^2}$$

Now comparing the coefficients of various powers of s in the numerator, we get

$$A_1 = 1; 2A_1 + A_2 + A_3 = 0 \text{ and } A_1 + A_3 = 0$$

Solving these equations, we get, $A_1 = 1$, $A_2 = -1$ and $A_3 = -1$.

$$\therefore \frac{1}{s(s+1)^2} = \frac{1}{s} + \frac{-1}{(s+1)^2} + \frac{-1}{(s+1)}$$

$\therefore$ The step response $v_o(t) = (1 - te^{-t} - e^{-t}) u(t)$ V

(ii) With $v_s(t) = e^{-t} u(t)$ the Laplace transform of input is $\frac{1}{(s+1)}$ the Laplace transform of output is $V_o(s) = \frac{1}{(s+1)^3}$. There is no need for partial fractions in this case.

$$v_o(t) = \frac{t^2}{2} e^{-t} u(t) \text{ V.}$$

Example: 13.6-3

Find the impulse response of the circuit in Fig. 13.5-1 under Example 13.5-1 in this chapter.

Solution

The System Function was shown to be $\frac{V_o(s)}{V_s(s)} = \frac{1}{s^3 + s^2 + 2s + 1}$ in Example 13.5-1.

The characteristic equation is $s^3 + s^2 + 2s + 1 = 0$ and its roots are $s_1 = -0.215 + j1.307$, $s_2 = -0.215 - j1.307$ and $s_3 = -0.57$. There is a pair of *complex conjugate* roots.

Impulse response is obtained by inverting the System Function.

$$H(s) = \frac{1}{s^3 + s^2 + 2s + 1} = \frac{1}{(s + 0.215 - j1.307)(s + 0.215 + j1.307)(s + 0.57)}$$

$$\therefore H(s) = \frac{A_1}{(s + 0.215 - j1.307)} + \frac{A_2}{(s + 0.215 + j1.307)} + \frac{A_3}{(s + 0.57)}$$

$$A_1 = \left. \frac{1}{(s + 0.215 + j1.307)(s + 0.57)} \right|_{s=-0.215+j1.307} = \frac{1}{j2.614 \times (0.355 + j1.307)} = 0.283 \angle -164.8^\circ$$

A_2 will be conjugate of A_1 and therefore $A_2 = 0.283 \angle 164.8^\circ$

$$A_3 = \left. \frac{1}{(s + 0.215 - j1.307)(s + 0.215 + j1.307)} \right|_{s=-0.57} = \frac{1}{(0.355 - j1.307)(0.355 + j1.307)} = 0.545$$

$$\begin{aligned} \therefore h(t) &= 0.283e^{-j164.8^\circ} e^{(-0.215+j1.307)t} + 0.283e^{j164.8^\circ} e^{(-0.215-j1.307)t} + 0.545e^{-0.57t} \\ &= 0.283e^{-0.215t} [e^{(j1.307t-164.8^\circ)} + e^{-(j1.307t-164.8^\circ)}] + 0.545e^{-0.57t} \\ &= 0.566e^{-0.215t} \cos(1.307t - 164.8^\circ) + 0.545e^{-0.57t} \end{aligned}$$

13.7 SOME THEOREMS ON LAPLACE TRANSFORMS

The property of linearity of Laplace transforms was already noted and made use of in earlier sections. We look at other interesting properties of Laplace transform in this section.

13.7.1 Time-Shifting Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$ then $v_d(t) = v(t - t_d) = f(t - t_d) u(t - t_d)$ has a Laplace transform $V_d(s) = V(s)e^{-st_d}$.

The shifting operation implied in this theorem is illustrated in Fig. 13.7-1. Note that there is a difference between $f(t - t_d) u(t)$ and $f(t - t_d) u(t - t_d)$.

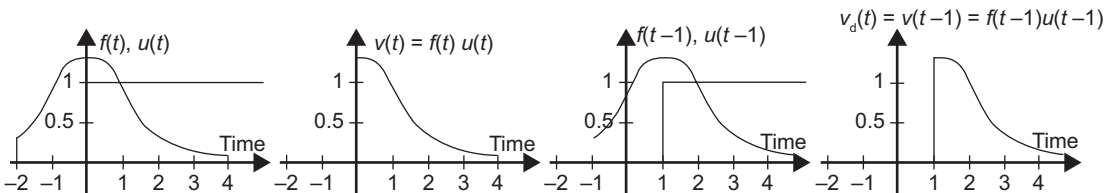


Fig. 13.7-1 Illustrating the time-shift operation envisaged in shifting theorem on laplace transforms

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This theorem follows from the defining equation for Laplace transforms.

$$V_d(s) = \int_{0^-}^{\infty} v_d(t) e^{-st} dt = \int_{0^-}^{\infty} v(t - t_d) e^{-st} dt$$

Use variable substitution $\tau = t - t_d$

$$V_d(s) = \int_{-t_d}^{\infty} v(\tau) e^{-s(\tau+t_d)} d\tau = e^{-st_d} \int_{-t_d}^{\infty} v(\tau) e^{-s\tau} d\tau$$

But $v(\tau)$ is zero for all $\tau \leq 0^-$. $\therefore V_d(s) = e^{-st_d} \int_{0^-}^{\infty} v(\tau) e^{-s\tau} d\tau = e^{-st_d} V(s)$

Example: 13.7-1

Find the zero-state response of a series RC circuit with a time constant of 2 s excited by a rectangular pulse voltage of 10 V height and 2 s duration starting from $t = 0$. The voltage across the capacitor is the output variable.

Solution

The differential equation governing the voltage across capacitor in a series RC circuit excited by a voltage source is $\frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} v_s$, where v is the voltage across the capacitor and v_s is the source voltage.

In this case, the equation is $\frac{dv}{dt} + 0.5v = 0.5v_s$. Therefore, the System Function is $H(s) = 0.5 / (s + 0.5)$.

The rectangular pulse voltage can be expressed as the sum of $10u(t)$ and $-10u(t-2)$. i.e., $v_s = 10[u(t) - u(t-2)]$. Therefore, its Laplace transform is $\frac{10}{s} - \frac{10e^{-2s}}{s} = \frac{10[1 - e^{-2s}]}{s}$.

$$\text{Therefore, } V(s) = \frac{0.5}{s + 0.5} \times \frac{10[1 - e^{-2s}]}{s} = \frac{5}{s(s + 0.5)} [1 - e^{-2s}]$$

We express $\frac{5}{s(s + 0.5)}$ in partial fractions as $\frac{A_1}{(s + 0.5)} + \frac{A_2}{s}$ and determine A_1 and A_2 as

$$A_1 = (s + 0.5) \times \frac{5}{s(s + 0.5)} \Big|_{s=-0.5} = -10 \text{ and } A_2 = (s) \times \frac{5}{s(s + 0.5)} \Big|_{s=0} = 10.$$

$$\therefore V(s) = \frac{5}{s(s + 0.5)} [1 - e^{-2s}] = \frac{10[1 - e^{-2s}]}{s} - \frac{10[1 - e^{-2s}]}{(s + 0.5)}.$$

Inverse transform of $-10e^{-2s} / s$ is $-10u(t-2)$ by Time-shifting Theorem. Similarly, inverse transform of $-10e^{-0.5t} / (s + 0.5)$ is $-10 e^{-0.5(t-2)}$. Therefore, the output voltage is given by

$$\begin{aligned} v(t) &= 10[u(t) - u(t-2)] - 10[e^{-0.5t}u(t) - e^{-0.5(t-2)}u(t-2)] \\ &= 10(1 - e^{-0.5t})u(t) - 10(1 - e^{-0.5(t-2)})u(t-2) \end{aligned}$$

Figure 13.7-2 shows the two components of response in dotted curves.

13.7.2 Frequency-Shifting Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$, then, $v_d(t) = v(t)e^{s_0 t}$ has a Laplace transform $V_d(s) = V(s - s_0)$.

This theorem follows from the defining equation for Laplace transforms.

$$V_d(s) = \int_{0^-}^{\infty} v_d(t) e^{-st} dt = \int_{0^-}^{\infty} v(t) e^{s_0 t} e^{-st} dt = \int_{0^-}^{\infty} v(t) e^{-(s-s_0)t} dt = V(s-s_0)$$

13.7.3 Time-Differentiation Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$, then, $v_d(t) = \frac{dv(t)}{dt}$ has a Laplace transform $V_d(s) = sV(s) - v(0^-)$. Note that $\frac{dv(t)}{dt} \neq \frac{df(t)}{dt} \times u(t)$.

$$\begin{aligned} V_d(s) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\ \frac{d[v(t)e^{-st}]}{dt} &= -sv(t)e^{-st} + \frac{dv(t)}{dt} e^{-st} \\ \therefore \frac{dv(t)}{dt} e^{-st} &= \frac{d[v(t)e^{-st}]}{dt} + sv(t)e^{-st} \\ \therefore V_d(s) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt = s \int_{0^-}^{\infty} v(t) e^{-st} dt + \int_{0^-}^{\infty} \frac{d[v(t)e^{-st}]}{dt} dt \\ &= sV(s) + v(t)e^{-st} \Big|_{0^-}^{\infty} \end{aligned}$$

The function $v(t)e^{-st}$ will be a decaying function for any value of s in the ROC of $V(s)$. Otherwise, the Laplace transform will not converge for that value of s . Therefore, it will go to zero as $t \rightarrow \infty$.

$$\therefore V_d(s) = sV(s) - v(0^-)$$

Now, by using mathematical induction, we may show that,

$$\text{Laplace transform of } \frac{d^2 v(t)}{dt^2} = s^2 V(s) - sv(0^-) - \frac{dv(t)}{dt} \Big|_{(0^-)}$$

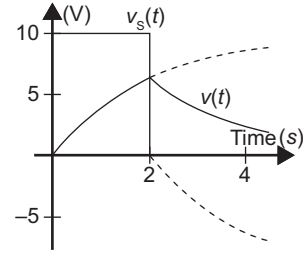


Fig. 13.7-2 Output response and its components in the circuit in Example: 13.7-1

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and that, in general, Laplace transform of

$$\frac{d^n v(t)}{dt^n} = s^n V(s) - s^{n-1} v(0^-) - s^{n-2} \left. \frac{dv(t)}{dt} \right|_{(0^-)} - \dots - \left. \frac{d^{n-1} v(t)}{dt^{n-1}} \right|_{(0^-)}$$

13.7.4 Time-Integration Theorem

If $v(t) = f(t)$ $u(t)$ has a Laplace transform $V(s)$, then, $v_i(t) = \int_{0^-}^t v(t) dt$ has a Laplace transform

$$V_i(s) = \frac{V(s)}{s}.$$

$$\begin{aligned} V_i(s) &= \int_{0^-}^{\infty} \left(\int_{0^-}^t v(t) dt \right) e^{-st} dt \\ \frac{d \left(\int_{0^-}^t v(t) dt \right) e^{-st}}{dt} &= v(t) e^{-st} - s \left(\int_{0^-}^t v(t) dt \right) e^{-st} \\ \therefore V_i(s) &= \int_{0^-}^{\infty} \left(\int_{0^-}^t v(t) dt \right) e^{-st} dt = \frac{1}{s} \int_{0^-}^{\infty} v(t) e^{-st} dt + \frac{1}{s} \int_{0^-}^{\infty} \frac{d \left(\int_{0^-}^t v(t) dt \right) e^{-st}}{dt} dt \\ &= \frac{1}{s} \int_{0^-}^{\infty} v(t) e^{-st} dt + \frac{1}{s} \left(\int_{0^-}^t v(t) dt \right) e^{-st} \Big|_{0^-}^{\infty} \end{aligned}$$

The function $v(t)$ is stated to possess a Laplace transform. This implies that there is an exponential function $Me^{\alpha t}$ with some positive value of M and some real value for α such that $|v(t)| < Me^{\alpha t}$. Otherwise, $v(t)$ would not have a Laplace transform. Therefore, the function $\int_{0^-}^t v(t) dt$ will satisfy the inequality $|\int_{0^-}^t v(t) dt| < |M/\alpha(e^{\alpha t} - 1)|$ and therefore is bounded. Therefore, the Laplace transform of $\int_{0^-}^t v(t) dt$ will exist. That is, it is possible to select a value for s such that the function $\int_{0^-}^t v(t) dt \times e^{-st}$ is a decaying function. For such an s , i.e., for a value of s in the ROC of Laplace transform of $\int_{0^-}^t v(t) dt$, the value of $\left(\int_{0^-}^t v(t) dt \right) e^{-st}$ will go to zero as $t \rightarrow \infty$. And, the value of $\left(\int_{0^-}^t v(t) dt \right) e^{-st}$ at $t = 0^-$ is zero in any case.

$$\therefore V_i(s) = \frac{1}{s} \int_{0^-}^{\infty} v(t) e^{-st} dt = \frac{V(s)}{s}$$

Example: 13.7-2

Find the Laplace transform of $t^n u(t)$.

Solution

The function $tu(t)$ is the integral of $u(t)$. Therefore, $tu(t) \Leftrightarrow 1/s^2$. Now the function $t^2 u(t)$ is 2 times the integral of $tu(t)$. Therefore, $t^2 u(t) \Leftrightarrow 2/s^3$. Proceeding similarly to the power n , we get, $t^n u(t) \Leftrightarrow n!/s^{n+1}$.

13.7.5 s-Domain-Differentiation Theorem

If $v(t) = f(t)$ has a Laplace transform $V(s)$, then, $-tv(t)$ has a Laplace transform $\frac{dV(s)}{ds}$.

We show this by determining the Laplace transform of $-tv(t)$ from the defining integral.

$$\int_{0^-}^{\infty} -tv(t)e^{-st} dt = \int_{0^-}^{\infty} -v(t)[te^{-st}] dt$$

We have repeatedly used the limit $\lim_{\Delta\alpha \rightarrow 0} \frac{e^{(\alpha+\Delta\alpha)t} - e^{\alpha t}}{\Delta\alpha} = te^{\alpha t}$ many times before. We use this limit again with $\alpha = -s$ within the integral.

$$\therefore \int_{0^-}^{\infty} -tv(t)e^{-st} dt = \int_{0^-}^{\infty} -v(t) \left[\lim_{\Delta s \rightarrow 0} \frac{e^{-(s+\Delta s)t} - e^{-st}}{-\Delta s} \right] dt$$

Since the limiting operation is on s and integration is on t we may interchange the order of these two operations.

$$\begin{aligned} \therefore \int_{0^-}^{\infty} -tv(t)e^{-st} dt &= \lim_{\Delta s \rightarrow 0} \int_{0^-}^{\infty} -v(t) \left[\frac{e^{-(s+\Delta s)t} - e^{-st}}{-\Delta s} \right] dt \\ &= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \int_{0^-}^{\infty} v(t) [e^{-(s+\Delta s)t} - e^{-st}] dt \\ &= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \left(\int_{0^-}^{\infty} v(t)e^{-(s+\Delta s)t} dt - \int_{0^-}^{\infty} v(t)e^{-st} dt \right) \\ &= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} (V(s+\Delta s) - V(s)) \\ &= \lim_{\Delta s \rightarrow 0} \frac{V(s+\Delta s) - V(s)}{\Delta s} \\ &= \frac{dV(s)}{ds} \end{aligned}$$

13.7.6 s-Domain-Integration Theorem

If $v(t) = f(t)$ has a Laplace transform $V(s)$ and $\lim_{t \rightarrow 0} \frac{v(t)}{t}$ is finite, then, $\frac{v(t)}{t}$ has a Laplace transform $\int_s^{\infty} V(s)ds$.

$$\int_s^{\infty} V(s)ds = \int_s^{\infty} \left[\int_{0^-}^{\infty} v(t)e^{-st} dt \right] ds = \int_{0^-}^{\infty} v(t) \left[\int_s^{\infty} e^{-st} ds \right] dt = \int_{0^-}^{\infty} v(t) \left[\frac{e^{-st} - e^{-\infty t}}{t} \right] dt$$

If the integration $\int_s^{\infty} V(s)ds$ is carried out in the right-half of s -plane (ROC of right-sided functions will have at least a part of right-half s -plane in it), then $e^{-\infty t}$ in the last step in the equation above will vanish. Then, $\int_s^{\infty} V(s)ds = \int_{0^-}^{\infty} \frac{v(t)}{t} e^{-st} dt = \text{Laplace Transform of } \frac{v(t)}{t}$.

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13.7.7 Convolution Theorem

If $x(t)$ and $y(t)$ are two right-sided time-functions with Laplace transforms $X(s)$ and $Y(s)$ respectively and $Z(s) = X(s)Y(s)$, then, $z(t) = x(t) * y(t) = \int_0^t x(\tau)y(t - \tau)d\tau$.

We use the inverse integral to show this.

$$\begin{aligned}
 z(t) &= \frac{1}{j2\pi} \int_{-\infty}^{\infty} X(s)Y(s)e^{st} ds \\
 &= \frac{1}{j2\pi} \int_{-\infty}^{\infty} \left[\int_0^{\infty} x(\tau)e^{-s\tau} d\tau \right] Y(s)e^{st} ds \quad [\text{Substituting for } X(s)] \\
 &= \frac{1}{j2\pi} \int_0^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} Y(s)e^{st}e^{-s\tau} ds \right] d\tau \quad [\text{Reversing the order of integration}] \\
 &= \int_0^{\infty} x(\tau) \left[\frac{1}{j2\pi} \int_{-\infty}^{\infty} Y(s)e^{s(t-\tau)} ds \right] d\tau \\
 &= \int_0^{\infty} x(\tau)y(t - \tau) d\tau
 \end{aligned}$$

The integral $\int_0^{\infty} x(\tau)y(t - \tau)d\tau$ is called the *Convolution Integral* between $x(t)$ and $y(t)$ and is denoted by $x(t)*y(t)$. Linear System Theory predicts that convolving the impulse response of a linear time-invariant circuit with its input source function gives the zero-state response. Hence, the convolution theorem stated here corroborates the fact that Laplace transform of zero-state response is given by the product of Laplace transform of input source function and Laplace transform of impulse response. We had termed the Laplace transform of impulse response as the s -domain System Function.

13.7.8 Initial Value Theorem

If $v(t) = f(t) u(t)$ has a Laplace transform $V(s)$ and $\lim_{s \rightarrow \infty} sV(s)$ exists, then, $\lim_{s \rightarrow \infty} sV(s) = v(0^+)$.

We know that the Laplace transform of $\frac{dv(t)}{dt}$ is $sV(s) - v(0^-)$. Therefore,

$$sV(s) - v(0^-) = \int_0^{\infty} \frac{dv(t)}{dt} e^{-st} dt$$

We assume that $s \rightarrow \infty$ with its real part is always positive. This is consistent with the fact that ROC of Laplace transform of right-sided functions will contain the right-half of s -plane or at least portions of right-half s -plane. Evaluation of the term e^{-st} with $t \rightarrow 0$ and $s \rightarrow \infty$ simultaneously is to be avoided. Hence, we write the integral as below.

$$\begin{aligned}
 sV(s) - v(0^-) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\
 &= \int_{0^-}^{0^+} \frac{dv(t)}{dt} e^{-st} dt + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\
 &= \int_{0^-}^{0^+} \frac{dv(t)}{dt} e^0 dt + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\
 &= v(0^+) - v(0^-) + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt
 \end{aligned}$$

Therefore, $sV(s) = v(0^+) + \int_{0^+}^{\infty} \frac{dv(t)}{dt} e^{-st} dt$. Now we apply the limit $s \rightarrow \infty$ with its real part always positive. Then the integral vanishes. Therefore, $\lim_{s \rightarrow \infty} sV(s) = v(0^+)$.

13.7.9 Final Value Theorem

If $v(t) = f(t)$ has a Laplace transform $V(s)$ and $\lim_{s \rightarrow 0} sV(s)$ exists and all the poles of $sV(s)$ have negative real part, then, $\lim_{s \rightarrow 0} sV(s) = v(\infty)$.

We know that the Laplace transform of $\frac{dv(t)}{dt}$ is $sV(s) - v(0^-)$. Therefore,

$$\begin{aligned}
 sV(s) - v(0^-) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^{-st} dt \\
 \lim_{s \rightarrow 0} sV(s) - v(0^-) &= \int_{0^-}^{\infty} \frac{dv(t)}{dt} e^0 dt = v(t) \Big|_{0^-}^{\infty} = v(\infty) - v(0^-) \\
 \therefore \lim_{s \rightarrow 0} sV(s) &= v(\infty)
 \end{aligned}$$

One has to be very careful in applying this theorem. This theorem works only if all the poles of $sV(s)$ are in the open left-half plane in s -domain. That is, all the poles of $sV(s)$ must have negative real part. Only then will the function $v(t)$ reach a unique and steady final value with time. Otherwise, the value returned by the application of this theorem will not be the final value of $v(t)$. For that matter $v(t)$ may not have a final value at all. Let $v(t)$ be $\sin \omega t$. Then $V(s) = \omega/(s^2 + \omega^2)$ and $sV(s) = s\omega/(s^2 + \omega^2)$. Application of final value theorem says that $v(\infty) = 0$. But there is no unique final value for a sinusoidal waveform. This conflict comes about because of wrong application of the theorem. The $sV(s)$ function in this case has poles on $j\omega$ -axis and hence the final value theorem is not applicable.

13.8 SOLUTION OF DIFFERENTIAL EQUATIONS BY USING LAPLACE TRANSFORMS

One of the important applications of Laplace transform is in solving linear constant-coefficient ordinary differential equations with initial conditions. The procedure is illustrated below through an example.

Example: 13.8-1

Find $y(t)$ for $t > 0^+$ for $x(t) = \delta(t)$ in the given differential equation with $y(0^-) = 1$, $y'(0^-) = -1$ and $y''(0^-) = 0$.

$$\frac{d^3 y}{dt^3} + 2.5 \frac{d^2 y}{dt^2} + 2.5 \frac{dy}{dt} + 1.5 y = x$$

Solution

Since the differential equation is an equation, both sides of it can be multiplied by e^{-st} . Since the differential equation is satisfied at all instants of time in a given interval, both sides of it can be integrated with respect to time from 0^- to ∞ . In short, the Laplace transform operation can be carried out on both sides. Laplace transformation is a linear operation and hence Laplace transform of a sum of terms is equal to sum of Laplace transforms of individual terms. Therefore,

$$\text{LT of } \frac{d^3 y}{dt^3} + 2.5 \times \text{LT of } \frac{d^2 y}{dt^2} + 2.5 \times \text{LT of } \frac{dy}{dt} + 1.5 \times \text{LT of } y = \text{LT of } x$$

Now we apply the ‘Differentiation in Time Theorem on Laplace transforms’ to get

$$[s^3 + 2.5s^2 + 2.5s + 1.5]Y(s) = X(s) + y''(0^-) + y'(0^-)[s + 2.5] + y(0^-)[s^2 + 2.5s + 2.5]$$

$$\begin{aligned} \therefore Y(s) = & \underbrace{\left\{ \frac{1}{(s^3 + 2.5s^2 + 2.5s + 1.5)} X(s) \right\}}_{\text{Zero-State Response Terms}} \\ & + \underbrace{\left\{ \frac{y''(0^-) + (s + 2.5)y'(0^-) + (s^2 + 2.5s + 2.5)y(0^-)}{(s^3 + 2.5s^2 + 2.5s + 1.5)} \right\}}_{\text{Zero-Input Response Terms}} \end{aligned}$$

The transform terms that depend only on the input function result in zero-state response. The transform terms that depend only on the initial conditions on output and its derivatives result in zero-input response.

Since $x(t) = \delta(t)$, $X(s) = 1$ in this example. Substituting the values for initial conditions and Laplace transform of $x(t)$, we get

$$Y(s) = \underbrace{\left\{ \frac{1}{(s^3 + 2.5s^2 + 2.5s + 1.5)} \right\}}_{\text{Zero-State Response Terms}} + \underbrace{\left\{ \frac{(s^2 + 1.5s)}{(s^3 + 2.5s^2 + 2.5s + 1.5)} \right\}}_{\text{Zero-Input Response Terms}}$$

We need to factorise the denominator polynomial in order to arrive at the partial fraction expansion. It is a third-order polynomial with real coefficients. Complex roots, if any, will have to occur in conjugate pairs for such a polynomial. Therefore, a polynomial of odd degree with real coefficients will necessarily possess a real-valued root. We try to locate that real root by the method of bisection.

Try two different values of s such that the polynomial evaluates to a positive and a negative number.

$s^3 + 2.5s^2 + 2.5s + 1.5$ evaluates to 1.5 for $s = 0$ and -1.5 for $s = -2$. Therefore, there must be root between 0 and -2 . We try the mid-value of -1 and see that the polynomial evaluates to 0.5. Therefore, the root must be between -1 and -2 . Hence we try the mid-value -1.5 . The polynomial evaluates to 0. Hence the root is $s = -1.5$. In practice, many iterations may be needed to arrive at a root by this technique.

Now we know that $(s + 1.5)$ is a factor of $s^3 + 2.5s^2 + 2.5s + 1.5$. Therefore, we get the remaining second-order factor by long division as $s^2 + s + 1$. The roots of this factor are $-0.5 \pm j0.866$.

Now we expand each response term in partial fractions. Normally we expand a transform in terms of first-order partial fractions. However, a second-order factor with complex conjugate roots may be expanded more conveniently in a form illustrated below.

$$\begin{aligned}\frac{1}{(s^3 + 2.5s^2 + 2.5s + 1.5)} &= \frac{A}{(s + 1.5)} + \frac{Bs + C}{(s^2 + s + 1)} = \frac{A(s^2 + s + 1) + (Bs + C)(s + 1.5)}{s^2(s + 1.5)(s^2 + s + 1)} \\ &= \frac{(A + B)s^2 + (A + 1.5B + C)s + (A + 1.5C)}{s^2(s + 1.5)(s^2 + s + 1)}\end{aligned}$$

Comparing the coefficients of various powers of s in the numerator, we get $A + B = 0$, $A + 1.5B + C = 0$ and $A + 1.5C = 1$. Solving for the unknowns, we get $A = 0.5715$, $B = -0.5715$ and $C = 0.2857$.

Therefore, the zero-state response = Inverse of $\frac{0.5715}{s + 1.5} + \frac{-0.5715s + 0.2857}{s^2 + s + 1}$

Inverse of $\frac{0.5715}{s + 1.5}$ is $0.5715e^{-1.5t}u(t)$

$$\begin{aligned}&\text{Inverse of } \frac{-0.5715s + 0.2857}{s^2 + s + 1} \\ &= \text{Inverse of } \frac{-0.5715s + 0.2857}{(s + 0.5)^2 + (\sqrt{0.75})^2} \\ &= \text{Inverse of } -0.5715 \frac{s + 0.5 - 0.5}{(s + 0.5)^2 + (0.866)^2} + \frac{0.2857}{(s + 0.5)^2 + (0.866)^2} \\ &= \text{Inverse of } -0.5715 \frac{s + 0.5}{(s + 0.5)^2 + (0.866)^2} + \frac{0.5715}{(s + 0.5)^2 + (0.866)^2} \\ &= -0.5715e^{-0.5t} \cos 0.866t u(t) + \frac{0.5715}{0.866} e^{-0.5t} \sin 0.866t u(t) \\ &= -0.5715e^{-0.5t} \cos 0.866t u(t) + 0.66e^{-0.5t} \sin 0.866t u(t)\end{aligned}$$

$\therefore$ Zero - state response = $[0.5715e^{-1.5t} - 0.5715e^{-0.5t} \cos 0.866t + 0.66e^{-0.5t} \sin 0.866t]; t \geq 0^+$

The zero-input response is given by the inverse of $\frac{(s^2 + 1.5s)}{(s^3 + 2.5s^2 + 2.5s + 1.5)}$.

$$\begin{aligned}&\frac{(s^2 + 1.5s)}{(s^3 + 2.5s^2 + 2.5s + 1.5)} \\ &= \frac{s(s + 1.5)}{(s + 1.5)(s^2 + s + 1)} \\ &= \frac{s}{(s^2 + s + 1)} \\ &= \frac{s + 0.5}{(s + 0.5)^2 + 0.866^2} - \frac{0.5}{0.866} \frac{0.866}{(s + 0.5)^2 + 0.866^2}\end{aligned}$$

$\therefore$ Zero-input response = $[e^{-0.5t} \cos 0.866t - 0.5774e^{-0.5t} \cos 0.866t]; t \geq 0^+$

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Total response $y(t)$ is the sum of zero-input response and zero-state response. Therefore,

$$\begin{aligned}\text{Total Response } y(t) &= [0.5715e^{-1.5t} + 0.4285e^{-0.5t} \cos 0.866t + 0.0856e^{-0.5t} \sin 0.866t]; t \geq 0^+ \\ &= [0.5715e^{-1.5t} + 0.437e^{-0.5t} \cos(0.866t - 11.3^\circ)]; t \geq 0^+\end{aligned}$$

If we can solve a differential equation with non-zero initial conditions completely in one stroke using Laplace transforms, then, we can indeed solve linear time-invariant circuits with non-zero initial conditions for their total response using Laplace transforms. We have to derive the n^{th} order linear constant-coefficient differential equation describing the circuit in terms of a single chosen variable first. In the second step, we have to determine the initial values for that chosen circuit variable and its $(n-1)$ derivatives from the known initial values of inductor currents and capacitor voltages in the circuit. Then we are ready to employ Laplace transform technique to solve for zero-input response and zero-state response in one step as illustrated in this example.

However, the derivation of differential equation and determination of initial values of chosen variable and its derivatives are the toughest tasks in a circuit analysis problem. Can't Laplace transform technique help us to simplify these two stages of circuit analysis?

13.9 THE s -DOMAIN EQUIVALENT CIRCUIT

Yes, it can. Laplace transform technique tells us that we do not even have to derive the circuit differential equation and initial values required to solve it. Let us see how.

The circuit equations arising out of applying KVL in loops and KCL at nodes are equations that remain true at all values of t . Therefore, such equations can be differentiated and integrated with respect to time without changing their truth content. Moreover, of course, being equations they can be multiplied by the same constant or function on both sides.

We choose to multiply all KCL and KVL equations in a linear time-invariant circuit by a function e^{-st} , where s is a complex frequency value drawn from s -plane, with a real part of suitable value such that each term in the equation is converted into an absolutely integrable function of time (so that Laplace transform for that term will converge). Then we choose to integrate the equations from 0^- to ∞ in time-domain. We apply the principle that integral of sum of terms is sum of integrals of individual terms.

The result will be a conclusion – (i) *the algebraic sum of Laplace transforms of element voltages in any loop in a circuit is zero* (ii) *the algebraic sum of Laplace transforms of element currents at any node in a circuit is zero*.

The Laplace transforms of voltage variables and current variables in a linear time-invariant circuit obey KVL and KCL, respectively.

Now, suppose we know the relation between the Laplace transform of element voltage and Laplace transform of element current for all circuit elements. Then, we can write the node equations and mesh equations in terms of Laplace transforms of variables straightaway – *i.e.*, we can write the circuit equations in s -domain straightaway instead of writing them in time-domain and transforming them into s -domain at the end of solution process. Hence, we derive the element relationships in s -domain first.

13.9.1 s -Domain Equivalents of Circuit Elements

A resistor R is described by the time-domain element equation $v_R(t) = R i_R(t)$. Multiplying both sides by e^{-st} and integrating from 0^- to ∞ , we get $V_R(s) = R I_R(s)$. Note that we use upper case letters for

Laplace transforms. This relation makes it clear that a resistor is represented by a multiplying factor of R that connects the Laplace transform of current through it to the Laplace transform of voltage across it. The ratio of Laplace transform of voltage across an element to the Laplace transform of current through it is defined as its ‘ s -domain impedance’. Thus, s -domain impedance of a resistor is R itself.

An inductor L is described by the time-domain element equation $v_L(t) = L \frac{di_L(t)}{dt}$, where $v_L(t)$ and $i_L(t)$ are its voltage and current variables as per passive sign convention. Applying Laplace transformation on both sides of this element equation, we get $V_L(s) = sLI_L(s) - Li_{L(0^-)}$ where $i_{L(0^-)}$ is the initial current in the inductor at $t = 0^-$. The Laplace transform $Li_{L(0^-)}$ represents an impulse voltage $Li_{L(0^-)} \delta(t)$ in time-domain and hence it is consistent with the fact that a non-zero initial condition in an inductor can be replaced with an impulse voltage source in series with the inductor. This s -domain equation for an inductor suggests the following s -domain equivalent circuit shown in Fig. 13.9-1 (a) for an inductor with sL as its s -domain impedance function. The second equivalent circuit shown in Fig. 13.9-1 (b) follows from $I_L(s) = \frac{V_L(s)}{sL} + \frac{i_{L(0^-)}}{s}$ and indicates the fact that the s -domain admittance of an inductor is $1/sL$ and that non-zero initial condition in an inductor can be replaced with a step current source in parallel with it.

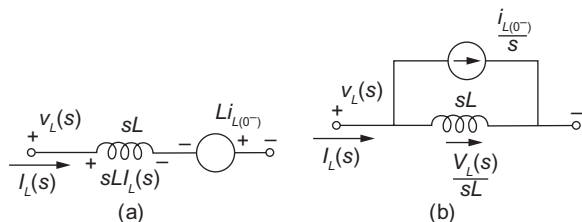


Fig. 13.9-1 s -Domain equivalent circuits for an inductor

Note that we use the same graphic symbol for inductor in s -domain as the one we used in time-domain. This will be the case with all other circuit elements too.

A capacitor C is described by the time-domain element equation $i_C(t) = C \frac{dv_C(t)}{dt}$, where $v_C(t)$ and $i_C(t)$ are its voltage and current variables as per passive sign convention. Applying Laplace transformation on both sides of this element equation, we get $I_C(s) = sCV_C(s) - Cv_{C(0^-)}$ where $v_{C(0^-)}$ is the initial voltage across the capacitor at $t = 0^-$. The Laplace transform $Cv_{C(0^-)}$ represents an impulse current $Cv_{C(0^-)} \delta(t)$ in time-domain and hence it is consistent with the fact that a non-zero initial condition in a capacitor can be replaced with an impulse current source in parallel with the capacitor. This s -domain equation for a capacitor suggests the following s -domain equivalent circuit shown in Fig. 13.9-2 (a) for a capacitor with sC as its s -domain admittance function. The second equivalent circuit shown in Fig. 13.9-2 (b) follows from $V_C(s) = \frac{I_C(s)}{sC} + \frac{v_{C(0^-)}}{s}$ and indicates the fact that the s -domain impedance of a capacitor is $1/sC$ and that non-zero initial condition in a capacitor can be replaced with a step voltage source in series with it.

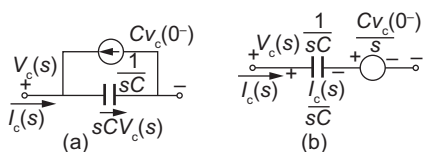


Fig. 13.9-2 s -Domain equivalent circuits for a capacitor

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s -domain impedance is assigned the unit of Ω (ohms) and s -domain admittance is assigned the unit of S (Siemens).

Now we can construct the *s-domain equivalent circuit* for a circuit in time-domain by replacing all sources by their Laplace transforms and replacing all other circuit elements by their s -domain equivalents. The resulting equivalent circuit will have Laplace transforms of voltages and Laplace transforms of currents as the circuit variables instead of the time-domain variables. Each energy storage element will result in an extra independent source representing its initial condition in s -domain equivalent circuit.

Applying KVL and KCL in this circuit will result in *algebraic equations involving Laplace transforms of voltages and currents, respectively*. Thus the problem of solving a coupled set of integro-differential equations involving functions of time in the time-domain circuit is translated to solving a coupled set of algebraic equations involving Laplace transforms of variables in the s -domain equivalent circuit. The time-functions may be determined by inverting the Laplace transforms after they are obtained.

Transforming a time-domain circuit into an s -domain circuit makes it similar to a memoryless circuit with DC excitation since both are described by algebraic equations. Thus, all concepts and techniques developed in the context of analysis of memoryless circuits (and used later in the analysis of phasor equivalent circuits under sinusoidal steady-state) will be directly applicable in the analysis of s -domain equivalent circuits too.

In particular, (i) the concepts of series and parallel equivalent impedances apply without modification (ii) the concepts of input resistance (*i.e.*, driving-point resistance) and input conductance (*i.e.*, driving-point conductance) apply without modification except that it is '*input impedance function $Z_i(s)$* ' and '*input admittance function $Y_i(s)$* ' in the case of s -domain circuits.

Moreover, the techniques of nodal analysis and mesh analysis can be applied in s -domain circuits. All the circuit theorems, except maximum power transfer theorem, can be applied in the context of s -domain equivalent circuits.

However, Laplace transform of instantaneous power is not equal to product of Laplace transforms of voltage and current. In fact, the s -domain convolution of $V(s)$ and $I(s)$ gives the Laplace transform of $p(t) = v(t) i(t)$. Therefore, dealing with power and energy variables in the s -domain is better avoided. They are better dealt with in time-domain itself.

We observe that the s -domain equivalent circuit makes use of the stated initial values of inductor currents and capacitor voltages right at the start. The s -domain equivalent circuit takes care of these initial values in the form of additional source transforms. Therefore, the circuit solution arrived at by the analysis of s -domain equivalent circuit will contain both zero-input response components and zero-state response components in one step. Thus, Laplace transform technique yields the total response in a single-step solution process.

13.9.2 Is s -domain Equivalent Circuit Completely Equivalent to Original Circuit?

No, it is not. It is equivalent to the original circuit only for $t \geq 0^+$.

Given a non-zero initial value for current at $t = 0^-$ in an inductor we have two ways of taking that into account when we try to solve the circuit for $t \geq 0^+$. We may adopt the view that unknown voltage was applied across the inductor in the past which resulted in this initial current. We accept the possibility of other circuit elements that were switched out at $t = 0$. We can solve for the current for $t \geq 0^+$ using this initial condition and the known input function. However, we will specify the range of applicability of the circuit solution as $t \geq 0^+$ specifically.

The second view that we may adopt is that the inductor current was zero at $t = 0^-$; but an impulse voltage of LI_0 V-s was applied in series with the inductor with suitable polarity. This view will explain the initial current in the inductor becoming I_0 though it was taken to be zero at $t = 0^-$. Hence, the circuit solution will be the same as the one we obtained by adopting the first point of view.

However, in the second point of view, we are fixing the voltage applied to the inductor in the past at zero value (since the impulse voltage is zero valued for $t < 0^-$). Thus, we are assuming that the circuit was the same from $t = -\infty$ onwards, no sources were active in it till $t = 0^-$ and some impulse sources acted at $t = 0$ to change the initial energy storage in some elements abruptly. Thus, the circuit and the sources active in the circuit are completely known from $t = -\infty$ onwards in this point of view. Therefore, the circuit solution is valid from $t = -\infty$. In fact, the solution will be zero for $(-\infty, 0^-]$. This is usually denoted by multiplying all time-functions that appear in the circuit solution by $u(t)$. The solution will be correct for all t as far as the fictitious circuit that we assumed in this point of view is concerned. However, that is not the actual circuit that we wanted to solve. The actual circuit that we wanted to solve is the one we described under first point of view. Therefore, time-functions multiplied by $u(t)$ cannot be the solution in the actual circuit. We cannot solve the actual circuit for $t < 0^-$ since the input is really unknown in this time-range. Therefore, only the right side of the solution arrived at from the fictitious circuit, that takes into account initial conditions by bringing in impulse sources, should be accepted as the solution for the actual circuit. Hence, the circuit solution should be specified as time-functions with the range of applicability specified as $t \geq 0^+$. The solution for $t < 0^-$ is left unspecified. It is understood that the circuit cannot be solved for $t < 0^-$ with the given data.

s -Domain equivalent of a circuit uses the second point of view described above. Hence, it is equivalent to the circuit only for $t \geq 0^+$. Circuit solution from s -domain equivalent circuit is obtained by inverting Laplace Transforms. That will yield time-functions multiplied by $u(t)$. We have to replace $u(t)$ by ‘for $t \geq 0^+$ ’ before we accept the solution from s -domain equivalent circuit as the solution for actual time-domain circuit.

However, in practice, this step is skipped often and the solution is left in the ‘time-function $\times u(t)$ ’ format itself. This does not lead to errors in practice since we are usually interested in circuit variables for $t \geq 0^+$ only. However, in the strict sense, it is a bad practice.

13.10 TOTAL RESPONSE OF CIRCUITS USING s -DOMAIN EQUIVALENT CIRCUIT

The application of s -domain equivalent circuit in obtaining the total response of a circuit is illustrated through a set of examples in this section.

Example: 13.10-1

The inductor L_1 has an initial current of 1 A and the inductor L_2 has an initial current of 1 A in the directions marked in the circuit in Fig. 13.10-1 (i) Find the voltage transfer function $V_o(s)/V_s(s)$ and the input impedance function $V_s(s)/I(s)$ (ii) Determine the total response of $v_o(t)$ if $v_s(t) = 2u(t)$ V.

Solution

(i) Transfer functions and immittance functions are defined in the s -domain equivalent circuit. They are defined under

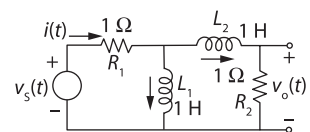


Fig. 13.10-1 Circuit for Example: 13.10-1

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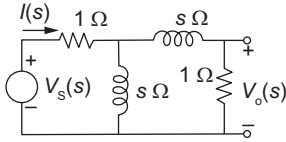


Fig. 13.10-2 The transformed equivalent circuit for circuit with zero initial conditions

zero-state response conditions. They are defined as ratios of Laplace transforms of relevant quantities under zero-state response conditions. Therefore, they are defined under zero initial conditions. The s -domain equivalent circuit with zero initial conditions is shown in Fig. 13.10-2.

Series-parallel equivalents and voltage-current division principle may be employed to arrive at the required ratios. Input impedance function $Z_i(s) = V_s(s) / I(s)$.

$$Z_i(s) = (s+1) // s + 1 = \frac{(s+1)s}{2s+1} + 1 = \frac{s^2 + 3s + 1}{2s+1} \Omega$$

The transformed current in the second 1Ω resistor may be found out in terms of $I(s)$. Then $V_o(s)$ may be obtained from that current by multiplying by 1Ω . Let this current transform be called $I_o(s)$. Then,

$$I_o(s) = I(s) \frac{s}{s+s+1} = I(s) \frac{s}{2s+1}$$

$$\text{But } I(s) = \frac{V_s(s)}{Z_i(s)} = V_s(s) \frac{2s+1}{s^2 + 3s + 1}$$

$$\text{Therefore, } \frac{V_o(s)}{V_s(s)} = \frac{I_o(s) \times 1\Omega}{V_s(s)} = \frac{I(s)}{V_s(s)} \times \frac{s}{2s+1} = \frac{V_s(s)}{V_s(s)} \frac{2s+1}{s^2 + 3s + 1} \times \frac{s}{2s+1} = \frac{s}{s^2 + 3s + 1}$$

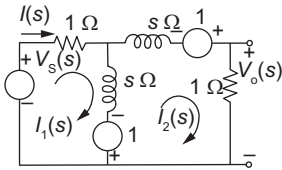


Fig. 13.10-3 Transformed equivalent circuit of circuit in Fig. 13.10-1 with initial condition sources included

The poles of transfer function are at $s = -2.618$ and $s = -0.382$. The zero is at $s = 0$.

(ii) The total response of $v_o(t)$ with $v_s(t) = 2u(t)$ V may be solved by mesh analysis or by applying superposition principle. Both methods are illustrated below. The s -domain equivalent circuit with initial conditions accounted and mesh current transforms identified is shown in Fig. 13.10-3.

The mesh equations are

$$-V_s(s) + I_1(s)[s+1] + I_2(s)[-s] - 1 = 0$$

$$1 + I_1(s)[-s] + I_2(s)[2s+1] - 1 = 0$$

These are expressed in matrix form as below.

$$\begin{bmatrix} s+1 & -s \\ -s & 2s+1 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s) + 1 \\ 0 \end{bmatrix}$$

Solving for $I_2(s)$, we get

$$I_2(s) = \frac{s(V_s(s) + 1)}{(s+1)(2s+1) - s^2} = \underbrace{\frac{sV_s(s)}{s^2 + 3s + 1}}_{\text{Zero-state Response}} + \underbrace{\frac{s}{s^2 + 3s + 1}}_{\text{Zero-input Response}}$$

$$\text{Since } V_o(s) = 1 \times I_2(s), V_o(s) = \underbrace{\frac{sV_s(s)}{s^2 + 3s + 1}}_{\text{Zero-state Response}} + \underbrace{\frac{s}{s^2 + 3s + 1}}_{\text{Zero-input Response}}$$

The roots of denominator polynomial (*i.e.*, poles of transfer function) are at $s = -2.618$ and $s = -0.382$. The input transform $V_s(s) = 2/s$.

$$\text{Therefore, } \frac{sV_s(s)}{s^2 + 3s + 1} = \frac{2}{s^2 + 3s + 1} = \frac{A}{s + 2.618} + \frac{B}{s + 0.382}$$

$$A = (s + 2.618) \times \frac{2}{s^2 + 3s + 1} \Big|_{s=-2.618} = -0.8945$$

$$B = (s + 0.382) \times \frac{2}{s^2 + 3s + 1} \Big|_{s=-0.382} = 0.8945$$

$$\begin{aligned} \text{Therefore, zero-state response} &= \text{Inverse of } \frac{-0.8945}{s + 2.618} + \frac{0.8945}{s + 0.382} \\ &= 0.8945(e^{-0.382t} - e^{-2.618t})u(t) \text{ V} \end{aligned}$$

The second component of output is expanded in partial fractions as below.

$$\frac{s}{s^2 + 3s + 1} = \frac{A}{s + 2.618} + \frac{B}{s + 0.382}$$

$$A = (s + 2.618) \times \frac{s}{s^2 + 3s + 1} \Big|_{s=-2.618} = 1.1708$$

$$B = (s + 0.382) \times \frac{s}{s^2 + 3s + 1} \Big|_{s=-0.382} = -0.1708$$

$$\begin{aligned} \text{Therefore, zero-input response} &= \text{Inverse of } \frac{1.1708}{s + 2.618} - \frac{0.1708}{s + 0.382} \\ &= (1.1708e^{-2.618t} - 0.1708e^{-0.382t})u(t) \text{ V} \end{aligned}$$

Total response in the actual time-domain circuit is the sum of zero-state response and zero-input response accepted for only $t \geq 0^+$ and is given by

$$v_o(t) = (0.2763e^{-0.382t} + 0.7237e^{-2.618t}) \text{ V for } t \geq 0^+$$

It is not necessary to split the two components of the response this way always. It was done here only to demonstrate how the Laplace transform technique brings out both together in one step. Inverting the total response transform $\frac{2+s}{s^2+3s+1}$ would have resulted in total response straightaway. Indeed

$$\frac{2+s}{s^2+3s+1} = \frac{0.2763}{s+2.618} + \frac{0.7237}{s+0.382}.$$

The same solution can be arrived at by using Superposition Theorem. This theorem can be applied only for the zero-state response components due to various inputs. But then, all the initial conditions get translated into sources in the transformed equivalent circuit and hence the circuit analysis problem in the s -domain is always a zero-state response problem. Therefore, superposition principle can be freely applied in transformed equivalent circuits. The solution term due to initial condition sources will be understood as the zero-input response once we get back to time-domain.

There are three sources in this transformed equivalent circuit. The component circuits required to find out the individual response components are shown in Fig. 13.10-4.

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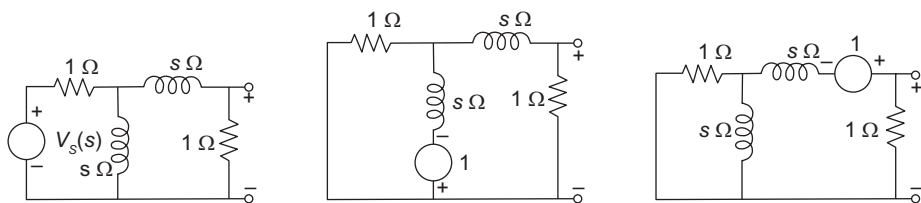


Fig. 13.10-4 Component circuits for applying superposition theorem in Example: 13.10-1

The transfer function of the first circuit is already known as $\frac{s}{s^2 + 3s + 1}$ and $V_s(s) = \frac{2}{s}$. Therefore, the output transform in the first circuit is $\frac{2}{s^2 + 3s + 1}$.

$(s + 1)/1 \Omega$ shares -1 with $s \Omega$ in series in the second circuit. Therefore, the voltage transform across $(s + 1) \Omega$ is $\frac{(s + 1)/1}{s + (s + 1)/1} \times -1 = \frac{-(s + 1)}{s^2 + 3s + 1}$. This voltage transform is divided between $s \Omega$ and 1Ω to yield $\frac{-(s + 1)}{s^2 + 3s + 1} \times \frac{1}{s + 1} = \frac{-1}{s^2 + 3s + 1}$ at the output.

$s + 1/s \Omega$ shares 1 in series with 1Ω to produce $\frac{1}{1 + s + 1/s} = \frac{s + 1}{s^2 + 3s + 1}$ across the output.

The total output voltage transform is given by the sum of three output voltage transforms. Therefore, $V_o(s) = \frac{2}{s^2 + 3s + 1} + \frac{-1}{s^2 + 3s + 1} + \frac{s + 1}{s^2 + 3s + 1} = \frac{2 + s}{s^2 + 3s + 1}$. This is the same output transform we obtained in the mesh analysis too.

The time-domain function will be $v_o(t) = (0.2763e^{-0.382t} + 0.7237e^{-2.618t})u(t)$ V.

Example: 13.10-2

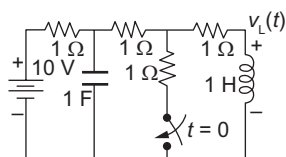


Fig. 13.10-5 Circuit for Example: 13.10-2

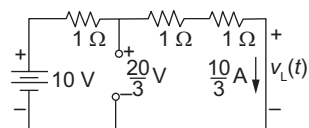


Fig. 13.10-6 Circuit under DC steady-state for $t < 0$

The circuit in Fig. 13.10-5 was in DC steady-state at $t = 0$. The switch in the circuit closes at $t = 0$ introducing a new 1Ω into the circuit. Determine the voltage across the inductor as a function of time.

Solution

The circuit was in DC steady-state prior to switching at $t = 0$. A capacitor can be modelled by an open-circuit and an inductor by a short-circuit for DC steady-state analysis. Therefore, the circuit for DC steady-state prior to $t = 0$ is as shown in Fig. 13.10-6.

Therefore, the voltage across the capacitor at $t = 0^-$ was $20/3$ V and current through the inductor at $t = 0^-$ was $10/3$ A. The circuit solution after $t = 0$ can be obtained by solving a new circuit with $10u(t)$ as input, $v_C(0^-) = 20/3$ V and $i_L(0^-) = 10/3$ A.

The new circuit can be analysed by mesh analysis or nodal analysis techniques. We opt for nodal analysis since the desired output is a node voltage variable straightaway. It will be convenient to use a current source in parallel with capacitor and a current source in parallel with inductor to account for initial conditions since we have opted for nodal analysis. The transformed equivalent circuit required is shown in Fig. 13.10-7.

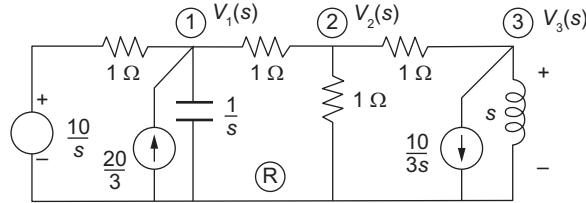


Fig. 13.10-7 Transformed equivalent circuit in Example: 13.10-2

The first source in series with $1\ \Omega$ may be replaced by a current source in parallel with $1\ \Omega$. The node equations in matrix form will be

$$\begin{bmatrix} 2+s & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1+\frac{1}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} \frac{10}{s} + \frac{20}{3} \\ 0 \\ -\frac{10}{3s} \end{bmatrix}$$

The determinant of Nodal Admittance Matrix is

$$\Delta(s) = (2+s) \left[3 \left(1 + \frac{1}{s} \right) - 1 \right] - (-1) \left[-1 \left(1 + \frac{1}{s} \right) \right] = (2+s) \left(\frac{2s+3}{s} \right) - \left(\frac{s+1}{s} \right) = \frac{2s^2 + 6s + 5}{s}$$

Solving for $V_3(s)$

$$\begin{aligned} V_3(s) &= \frac{-10(s+2)}{3(2s^2+6s+5)} = \frac{-1.667(s+2)}{s^2+3s+2.5} \\ &= \frac{-1.667(s+1.5+0.5)}{(s+1.5)^2+(0.5)^2} = \frac{-1.667(s+1.5)}{(s+1.5)^2+(0.5)^2} + \frac{-1.667(0.5)}{(s+1.5)^2+(0.5)^2} \end{aligned}$$

$$\begin{aligned} \text{Therefore, } v_3(t) &= -1.667e^{-1.5t}(\cos 0.5t + \sin 0.5t) \text{ V for } t \geq 0^+ \\ &= -2.36e^{-1.5t}(\cos 0.5t - 45^\circ) \text{ V for } t \geq 0^+ \end{aligned}$$

The voltage across the inductor is the same as $v_3(t)$. Note that the final value of inductor voltage is zero. This is expected since under DC steady-state condition the inductor behaves like a short-circuit.

Example: 13.10-3

Verify the initial value theorem and final value theorem on Laplace transforms for $i(t)$ and $v_o(t)$ in the initially relaxed circuit shown in Fig. 13.10-8 when driven by $v_s(t) = u(t)$.

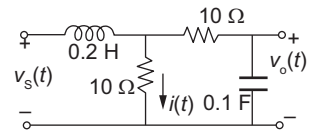


Fig. 13.10-8 Circuit for Example: 13.10-3

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Solution

The s -domain equivalent circuit required for analysis is shown in Fig. 13.10-9.

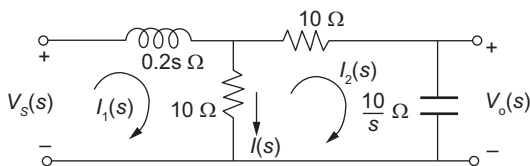


Fig. 13.10-9 The s -domain equivalent circuit of the circuit in Fig. 13.10-8

Two mesh current transforms are identified in the s -domain equivalent circuit. The mesh equations in matrix form is written by inspection by using the rule that diagonal entry is the sum of all impedances in the corresponding mesh and off-diagonal entries are negative of the sum of impedances shared by the two meshes in question.

$$\begin{bmatrix} 0.2s + 10 & -10 \\ -10 & 20 + \frac{10}{s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s) \\ 0 \end{bmatrix}$$

Solving for $I_1(s)$ and $I_2(s)$ by Kramer's rule, we get

$$I_1(s) = \frac{\left(20 + \frac{10}{s}\right)V_s(s)}{(0.2s + 10)\left(20 + \frac{10}{s}\right) - 100} = \frac{(5s + 2.5)V_s(s)}{s^2 + 25.5s + 25}$$

$$I_2(s) = \frac{10V_s(s)}{(0.2s + 10)\left(20 + \frac{10}{s}\right) - 100} = \frac{2.5sV_s(s)}{s^2 + 25.5s + 25}$$

Now,

$$I(s) = I_1(s) - I_2(s) = \frac{(5s + 2.5)V_s(s)}{s^2 + 25.5s + 25} - \frac{2.5sV_s(s)}{s^2 + 25.5s + 25} = \frac{2.5(s + 1)V_s(s)}{s^2 + 25.5s + 25}$$

$$V_o(s) = \frac{10}{s} \times I_2(s) = \frac{25V_s(s)}{s^2 + 25.5s + 25}$$

The input is an $u(t)$ function and its transform is $1/s$. Therefore,

$$I(s) = \frac{2.5(s + 1)}{s(s^2 + 25.5s + 25)}$$

$$V_o(s) = \frac{25}{s(s^2 + 25.5s + 25)}$$

$$\text{Initial value of } i(t) \text{ at } t = 0^+ = \lim_{s \rightarrow \infty} sI(s) = s \times \frac{2.5(s + 1)}{s(s^2 + 25.5s + 25)} = 0$$

The poles of $sI(s)$ are in the left-half of s -plane and hence the final value theorem on Laplace transforms is applicable to $I(s)$.

$$\therefore \text{Final value of } i(t) = \lim_{s \rightarrow 0} sI(s) = s \times \frac{2.5(s+1)}{s(s^2 + 25.5s + 25)} = \frac{2.5}{25} = 0.1 \text{ A}$$

$$\text{Initial value of } v_o(t) \text{ at } t = 0^+ = \lim_{s \rightarrow \infty} sV_o(s) = s \times \frac{25}{s(s^2 + 25.5s + 25)} = 0$$

The poles of $sV_o(s)$ are in the left-half of s -plane and hence the final value theorem on Laplace transforms is applicable to $V_o(s)$.

$$\therefore \text{Final value of } v_o(t) = \lim_{s \rightarrow 0} sV_o(s) = s \times \frac{25}{s(s^2 + 25.5s + 25)} = \frac{25}{25} = 1 \text{ V}$$

The initial current at $t = 0^-$ through the inductor was zero and initial voltage across the capacitor at that instant was zero. There was no impulse content in voltage at input. Therefore, the inductor current at $t = 0^+$ remains at zero. There was no impulse current in the circuit. Therefore, the voltage across the capacitor remains at zero at $t = 0^+$. The time-domain equivalent circuit at $t = 0^+$ is shown in Fig. 13.10-10 (a). The initial value of current $i(t)$ at 0^+ is clearly zero and the initial value of $v_o(t)$ is also zero.

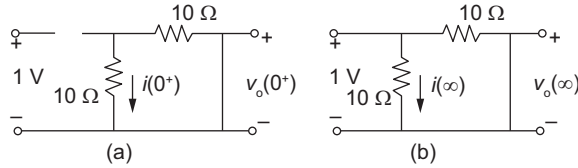


Fig. 13.10-10 Equivalent circuits at (a) $t = 0^+$ and (b) $t \rightarrow \infty$

The inductor is replaced by a short-circuit and the capacitor by an open-circuit under DC steady-state conditions. The resulting circuit is shown in Fig. 13.10-10 (b). Hence, the final value of $i(t)$ $1/10 = 0.1$ A and the final value of $v_o(t)$ is equal to input voltage *i.e.*, 1 V.

Hence the initial value theorem and the final value theorem on Laplace transforms are verified for $v_o(t)$ and $i(t)$.

Example: 13.10-4

(a) Find the transfer function $\frac{V_o(s)}{V_s(s)}$ in the Opamp circuit shown in

Fig. 13.10-11 assuming ideal Opamp. (b) Show its pole-zero plot for $k = 2.9$ and $k = 3.1$ and find its zero-state response to $v_s(t) = 0.01\delta(t)$ in both cases with $R = 10\text{k}\Omega$ and $c = 1\mu\text{F}$. (c) What is the maximum value of k that can be used in the circuit without making it an unstable one?

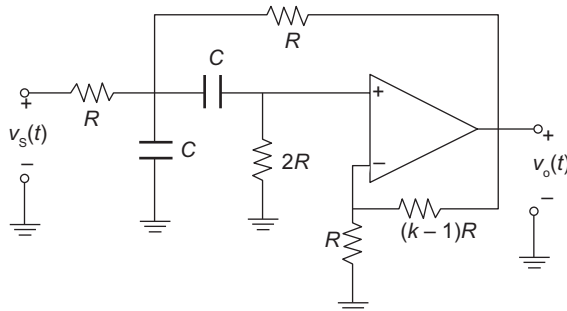


Fig. 13.10-11 The Opamp circuit in Example: 13.10-4

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Solution

(a) The Opamp circuit from its non-inverting input to its output is a simple non-inverting amplifier of gain k and can be replaced with a dependent source as shown in the s -domain equivalent circuit in Fig. 13.10-12.

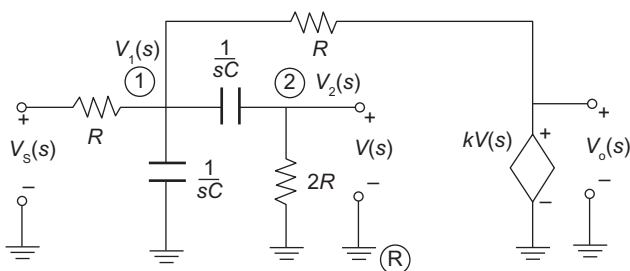


Fig. 13.10-12 Transformed equivalent circuit of the circuit in Fig. 13.10-11

The node voltage transforms $V_1(s)$ and $V_2(s)$ are identified in the transformed equivalent circuit. Writing the node equations at these two nodes, we get

$$\text{Second node: } sC[V_2(s) - V_1(s)] + \frac{V_2(s)}{2R} = 0$$

$$\text{Therefore, } V_1(s) = V_2(s) + \frac{V_2(s)}{2sCR} = V_2(s) \left[1 + \frac{1}{2sCR} \right]$$

$$\text{First node: } \frac{V_1(s) - V_s(s)}{R} + sCV_1(s) + sC[V_1(s) - V_2(s)] + \frac{V_1(s) - kV_2(s)}{R} = 0$$

Substituting for $V_1(s)$ in terms of $V_2(s)$ and simplifying, we get

$$\frac{V_2(s)}{V_s(s)} = \frac{s^{1/RC}}{s^2 + s^{(3-k)/RC} + (1/RC)^2} \quad \text{and} \quad \frac{V_o(s)}{V_s(s)} = \frac{kV_2(s)}{V_s(s)} = \frac{s^{k/RC}}{s^2 + s^{(3-k)/RC} + (1/RC)^2}$$

This is the transfer function of the circuit.

(b) The denominator polynomial with $R = 10 \text{ k}\Omega$, $C = 1 \text{ }\mu\text{F}$ and $k = 2.9$ is $s^2 + 10s + 10000$. Therefore, the poles are at $s = -5 \pm j 99.88$. The zero of the transfer function is at $s = 0$, i.e., at the origin in the s -plane.

With $k = 3.1$, the denominator polynomial is $s^2 - 10s + 10000$ and the poles are at $s = 5 \pm j 99.88$. The zero of the transfer function is again at $s = 0$. The pole-zero plots are shown in Fig. 13.10-13.

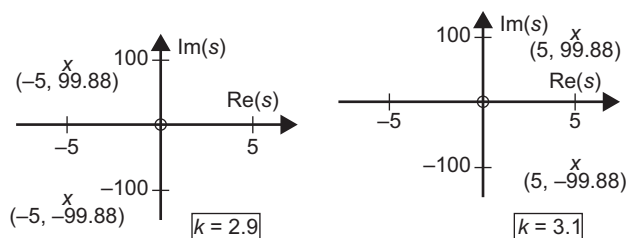


Fig. 13.10-13 Pole-zero plots for the Opamp circuit in Fig. 13.10-11 with $k = 2.9$ and $k = 3.1$

The zero-state response to $v_s(t) = 0.01\delta(t)$ is nothing but the impulse response of the circuit scaled by 0.01.

With $k = 2.9$

The transfer function $H(s) = \frac{290s}{s^2 + 10s + 10000}$. Inverse transform of $H(s)$ gives the impulse response of the circuit. We complete the squares in the denominator and identify the inverse transforms as shown below.

$$\begin{aligned}\frac{290s}{s^2 + 10s + 10000} &= 290 \frac{(s+5) - 5}{(s+5)^2 + 99.88^2} \\ &= 290 \left[\frac{(s+5)}{(s+5)^2 + 99.88^2} - \frac{5}{99.88} \frac{99.88}{(s+5)^2 + 99.88^2} \right] \\ \therefore h(t) &= 290 e^{-5t} (\cos 99.88t - 0.05 \sin 99.88t) u(t) \\ &= 290.36 e^{-5t} \cos(99.88t + 2.86^\circ) u(t)\end{aligned}$$

This is a stable impulse response since $\lim_{t \rightarrow \infty} h(t) = 0$

The zero-state response to $0.01\delta(t) = 2.9 e^{-5t} \cos(99.88t + 2.86^\circ) u(t)$.

With $k = 3.1$

The transfer function $H(s) = \frac{310s}{s^2 - 10s + 10000}$. Inverse transform of $H(s)$ gives the impulse response of the circuit. We complete the squares in the denominator and identify the inverse transforms as shown below.

$$\begin{aligned}\frac{310s}{s^2 - 10s + 10000} &= 310 \frac{(s-5) + 5}{(s-5)^2 + 99.88^2} \\ &= 310 \left[\frac{(s-5)}{(s-5)^2 + 99.88^2} + \frac{5}{99.88} \frac{99.88}{(s-5)^2 + 99.88^2} \right] \\ \therefore h(t) &= 310 e^{5t} (\cos 99.88t + 0.05 \sin 99.88t) u(t) \\ &= 310.39 e^{5t} \cos(99.88t - 2.86^\circ) u(t)\end{aligned}$$

This is an unstable impulse response since it is unbounded. The circuit is an unstable one as evidenced by its poles located in the right-half of s -plane.

The zero-state response to $0.01\delta(t) = 3.1 e^{5t} \cos(99.88t - 2.86^\circ) u(t)$.

(c) The transfer function $\frac{V_o(s)}{V_s(s)} = \frac{s^{k/RC}}{s^2 + s^{(3-k)/RC} + (1/RC)^2}$ has poles on $j\omega$ -axis when $k = 3$. The poles

will lie on the right-half of s -plane for all values of $k > 3$. Therefore, $k < 3$ is the constraint on k value for stability in the circuit.

The circuit is marginally stable at $k = 3$. It can function as a sinusoidal oscillator with $k = 3$. But additional circuitry will be needed to stabilize its amplitude of oscillation.

Note that the circuit is a pure RC circuit with one dependent source in it. A passive RC circuit, *i.e.*, a circuit containing only resistors and capacitors and no dependent sources, will have all its poles in the negative real axis. The dependent source is responsible for making the poles complex conjugate

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in such a circuit. Complex conjugate poles are often necessary in filter circuits to tailor the filter frequency response function suitably to meet filtering specifications.

This circuit is used as a band-pass filter in practice. The value of k will be decided by the bandwidth required in the band-pass filter and will be < 3 at any rate.

Example: 13.10-5

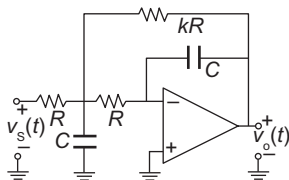


Fig. 13.10-14 The Opamp-rc filter circuit in Example: 13.10-5

(a) Obtain the transfer function of the filter circuit shown in Fig. 13.10-14 and identify the type of filter. (b) Determine the zero-state response for $v_s(t) = 0.1\delta(t)$. $R = 10\text{k}\Omega$, $C = 1\mu\text{F}$ and $k = 1$

Solution

The transformed equivalent circuit is shown in Fig. 13.10-15. Initial condition sources are not required since the circuit is needed for determining transfer function and for evaluating zero-state response. The Opamp is assumed to be ideal. There is only one node that has a free node voltage variable. This node and the assigned node voltage transform are also indicated in the equivalent circuit.

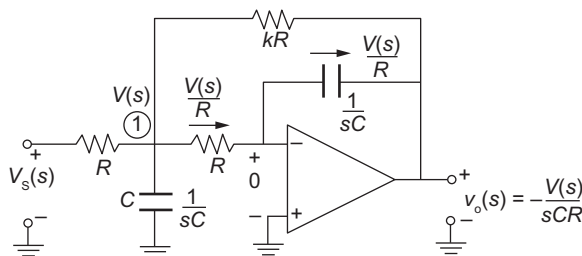


Fig. 13.10-15 The transformed equivalent circuit of circuit in Fig. 13.10-14

Virtual short across the Opamp input terminals and zero input current drawn by the Opamp inverting pin makes the current in the feedback capacitor equal to $V(s)/R$. This results in the output voltage transform becoming equal to $-V(s)/sCR$.

Now, we write the node equation at the node-1 marked in Fig. 13.10-15.

$$\frac{V(s)}{R} + sCV(s) + \frac{V(s)}{R} + \frac{V(s) - (-\frac{V(s)}{sRC})}{kR} = \frac{V_s(s)}{R}$$

Algebraic simplification leads to
$$\frac{V(s)}{V_s(s)} = \frac{skRC}{k(sRC)^2 + (2k+1)(sRC) + 1}$$

The output transform $V_o(s) = -V(s)/sCR$. Therefore,

$$\frac{V_o(s)}{V_s(s)} = \frac{-k}{k(sRC)^2 + (2k+1)(sRC) + 1} = \frac{-k \times \frac{1}{k}(\frac{1}{RC})^2}{s^2 + s(2 + \frac{1}{k})(\frac{1}{RC}) + \frac{1}{k}(\frac{1}{RC})^2}$$

The denominator has its roots in the left-half of s -plane for all positive values of k since a second-order polynomial with positive coefficients will have both roots in left-half of s -plane. Therefore, the impulse response is stable and will be absolutely integrable. Therefore, the impulse response will have a Fourier transform. If Fourier transform of a time-function exists, then, the Laplace transform of the same function evaluated on $j\omega$ -axis is its Fourier transform. Fourier transform of impulse response is the frequency response function. *Therefore, the sinusoidal steady-state frequency response function of a stable circuit is given by its Laplace transform evaluated with $s = j\omega$.*

Therefore, $\frac{V_o(j\omega)}{V_s(j\omega)} = (-k) \frac{\frac{1}{k}(\frac{1}{V_{RC}})^2}{-\omega^2 + (2 + \frac{1}{k})(\frac{1}{V_{RC}})j\omega + \frac{1}{k}(\frac{1}{V_{RC}})^2}$. This frequency response function has

a magnitude of unity at $\omega = 0$ and 0 at $\omega = \infty$. Therefore, it is a low-pass filter.

Substituting the numerical values, we get

$$\frac{V_o(s)}{V_s(s)} = \frac{-10^4}{s^2 + 300s + 10^4}$$

The impulse response of the circuit is obtained by inverting the transfer function. The roots of the denominator are at $s = -261.8$ and $s = -38.2$.

$$\frac{V_o(s)}{V_s(s)} = \frac{-10^4}{s^2 + 300s + 10^4} = \frac{A}{s + 261.8} + \frac{B}{s + 38.2}$$

A and B can be evaluated as -44.72 and 44.72 , respectively.

Therefore, $h(t) = 44.72(e^{-38.2t} - e^{-261.8t})u(t)$

Therefore, zero-state response for $v_s(t) = 0.1\delta(t)$ is $4.47(e^{-38.2t} - e^{-261.8t})u(t)$ V

Example: 13.10-6

A single Opamp, a resistor and a capacitor can form a good differentiator circuit. This circuit is shown in Fig. 13.10-16. (i) Derive the transfer function of the circuit and show that it is a differentiator.

(ii) An ideal Opamp is too good to be true. A practical Opamp suffers from many non-idealities. The particular non-ideality that compromises the circuit performance will vary depending on circuit function. For example, we found that integrator circuit is severely compromised by offsets in a practical Opamp. We will see in this example that it is the limited gain and bandwidth of Opamp that compromises the performance of differentiator circuit.

An amplifier contains many capacitors – intentional as well as parasitic – in it and hence represents a high-order dynamic circuit. However, some Opamps like IC 741 can be modelled approximately as a single-time constant

amplifier. That is, its gain function is of the form $\frac{A}{1 + s\tau}$ with $A \approx 250,000$ and $\tau \approx 4$ ms. Obtain the transfer function of differentiator circuit with $R = 10\text{k}\Omega$ and $C = 1\mu\text{F}$ using IC 741 and find its impulse response.

(iii) Suggest a method to modify the oscillatory impulse response to critically damped impulse response.

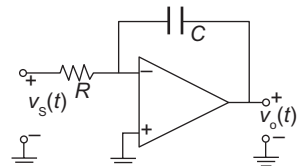


Fig. 13.10-16 An Opamp differentiator circuit

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Solution

(i) This is essentially an inverting amplifier structure. The transfer function = $-\frac{\text{Impedance in feedback path}}{\text{Impedance in input line}} = -sRC$. It is a differentiator since multiplication by s in s -domain is equivalent to differentiation in time-domain according to time-differentiation theorem on Laplace transforms. It is an inverting differentiator.

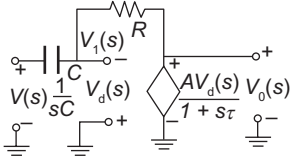


Fig. 13.10-17 Transformed equivalent circuit for differentiator circuit in Fig. 13.10-16

(ii) The Opamp is to be modelled as a dependent source that senses the voltage transform between non-inverting pin and inverting pin and produces a voltage transform $\frac{AV_d(s)}{1+s\tau}$ at its output with respect to ground, where $V_d(s)$ is the transform of voltage of non-inverting pin with respect to inverting pin. The s -domain equivalent circuit incorporating this model for Opamp is shown in Fig. 13.10-17.

Let the node voltage transform at the inverting pin be $V_1(s)$. Writing the node equation at inverting pin, we get

$$sC[V_1(s) - V(s)] + \frac{1}{R} \left[V_1(s) - \left(-\frac{AV_1(s)}{1+s\tau} \right) \right] = 0$$

Simplifying this equation results in $\frac{V_1(s)}{V(s)} = \frac{sC}{sC + \frac{1}{R} \left(1 + \frac{A}{1+s\tau} \right)}$

$$V_o(s) \text{ is } -\frac{A}{1+s\tau} V_1(s).$$

Therefore, $\frac{V_o(s)}{V(s)} = \frac{\frac{-AsC}{1+s\tau}}{sC + \frac{1}{R} \left(1 + \frac{A}{1+s\tau} \right)} = (-sRC) \frac{\frac{A}{RC\tau}}{s^2 + s \frac{1}{\tau} + \frac{A+1}{RC\tau}}$

The DC gain A of any practical Opamp is in thousands and hence $A+1 \approx A$. Therefore,

$$\frac{V_o(s)}{V(s)} = (-sRC) \frac{\frac{A}{RC\tau}}{s^2 + s \frac{1}{\tau} + \frac{A}{RC\tau}}$$

Note that the order of the circuit is two. The Opamp contributes one extra order to the circuit. Substituting $R = 10\text{k}\Omega$, $C = 1\mu\text{F}$, $A = 250000$ and $\tau = 4\text{ms}$, we get

$$\begin{aligned} \frac{V_o(s)}{V(s)} &= (-0.01s) \frac{79060^2}{s^2 + 250s + 79060^2} \\ &= (-0.01s) \frac{79060^2}{(s+125)^2 + 79057^2} \\ &= (-62.5 \times 10^6) \frac{s}{(s+125)^2 + 79057^2} \end{aligned}$$

Step response is the inverse of transfer function multiplied by $1/s$.

$$(-62.5 \times 10^6) \frac{1}{(s+125)^2 + 79057^2} = \frac{(-62.5 \times 10^6)}{79057} \frac{79057}{(s+125)^2 + 79057^2}$$

$$\therefore v_o(t) \approx (-62.5 \times 10^6) e^{-125t} \cos 79057t u(t) \text{ V}$$

Of course, if a unit step is really applied to this circuit, the output of Opamp will saturate. But what is to be noted is that the response is *highly under-damped*. The oscillation is at 12.6 kHz and oscillation period is about 0.08 ms. But the time constant of damping exponential is 40 ms. It takes about 5 time constants for an exponential transient to settle down. That implies that the 12.6 kHz transient oscillations will last for about 200 ms before they die down. That is a bad transient performance.

(iii) The solution is to add a little damping by means of a resistor in series with the input capacitor. This will make the differentiator imperfect however.

13.11 NETWORK FUNCTIONS AND POLE-ZERO PLOTS

Network function is a ratio of Laplace transforms. It is the ratio of Laplace transform of zero-state response to the Laplace transform of the right-sided input function causing this response. We called it s -domain System Function till now. We use the name '*Network Function*' synonymously from this section onwards.

But ratio of Laplace transform of zero-state response to Laplace transform of *which* input function? There should only be one! That is, a network function can be defined and evaluated only in an *s-domain equivalent circuit containing one input source transform*. Thus, the circuit should have only one independent source active when a network function is evaluated. Hence, there cannot be initial condition sources too. That is why the definition specifies that it is the ratio of Laplace transform of *zero-state response* to Laplace transform of input function.

The response may be measured across or through any circuit element or combinations of such elements in general. Hence, a variety of network functions are defined in a circuit. In particular, when the response variable and excitation variable pertain to same terminal pair, the network function can only be of driving-point impedance or driving-point admittance type. The two together are referred as immittance functions.

13.11.1 Driving-Point Functions and Transfer Functions

Input impedance function or driving-point impedance function is $Z_i(s) = \frac{V(s)}{I(s)}$ where $V(s)$ and $I(s)$ are transforms of voltage and current in a terminal pair as per passive sign convention. Input admittance function or driving-point admittance function is $Y_i(s) = \frac{I(s)}{V(s)}$ where $V(s)$ and $I(s)$ are transforms of voltage and current in a terminal pair as per passive sign convention. They are reciprocals of each other at the same terminal pair. These functions are a special class of network functions.

Transfer impedance function is $Z_m(s) = \frac{V_{ij}(s)}{I_{pq}(s)}$ where $V_{ij}(s)$ is the transform of voltage developed at i^{th} terminal with respect to j^{th} terminal due to a current source $I_{pq}(s)$ delivering current to p^{th} terminal from q^{th} terminal. The terminal pairs $p-q$ and $i-j$ are not the same. Transfer admittance function is $Y_m(s) = \frac{I_{pq}(s)}{V_{ij}(s)}$

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where $I_{pq}(s)$ is the transform of current developed from q^{th} terminal to p^{th} terminal due to a voltage source $V_{ij}(s)$ between i^{th} terminal and j^{th} terminal. The terminal pairs $p-q$ and $i-j$ are not the same though they may share a common terminal. These functions are a special class of network functions.

Voltage transfer function is $A_v(s) = \frac{V_{pq}(s)}{V_{ij}(s)}$ where $V_{pq}(s)$ is the Laplace transform of zero-state voltage response developed across terminal pair $p-q$ due to a voltage source transform $V_{ij}(s)$ applied across terminal pair $i-j$. The terminal pairs $p-q$ and $i-j$ are not the same though they may share a common terminal.

Current transfer function is $A_i(s) = \frac{I_{pq}(s)}{I_{ij}(s)}$ where $I_{pq}(s)$ is the Laplace transform of zero-state current response developed through terminal pair $p-q$ due to a current source transform $I_{ij}(s)$ applied through terminal pair $i-j$. The terminal pairs $p-q$ and $i-j$ are not the same though they may share a common terminal.

Thus there are two types of driving-point network functions and four types of transfer network functions.

We will use the symbol $H(s)$ when we refer to network functions in general and use $Z_i(s)$, $Y_i(s)$, $Z_m(s)$, $Y_m(s)$, $A_v(s)$ and $A_i(s)$ when we refer to specific network functions.

13.11.2 The Three Interpretations for a Network Function $H(s)$

The first interpretation is the definition of a network function itself. That is, a network function is the ratio of Laplace transform of *zero-state response* to the Laplace transform of input source function causing the response.

Two circuit variables are clearly identified in the definition of network function – they are the variable used to measure the zero-state response and the variable that was decided by the input source function. These two variables can be identified in the s -domain equivalent circuit, and circuit analysis in s -domain using nodal analysis or mesh analysis can be performed to arrive at the desired network function. The result will be $H(s)$ in the form of a ratio of rational polynomials in s . Thus, from this point of view, we expect to get an $H(s)$ in the following format. We have chosen to make the coefficient of highest power in s in the denominator 1.

$$H(s) = \frac{b'_{m'} s^{m'} + b'_{m'-1} s^{m'-1} + \dots + b'_1 s + b'_0}{s^{n'} + a'_{n'-1} s^{n'-1} + \dots + a'_1 s + a'_0} \quad (13.11-1)$$

The second interpretation for $H(s)$ comes from the meaning of Laplace transform. Laplace transform of a right-sided function is an expansion of that function in terms of functions of e^{st} type with value of s ranging from $\sigma - j\infty$ to $\sigma + j\infty$. The value of σ is such that the expansion converges to the time-function at all t . The components in expansion, i.e., signals of type e^{st} are from $-\infty$ to ∞ in time-domain. Thus, Laplace transform converts a right-sided input into the sum of *everlasting* complex exponential inputs. Therefore, the problem of zero-state response with a right-sided input is translated into that of forced response with everlasting complex exponential inputs. And, the ratio of Laplace transform of zero-state response to Laplace transform of input source function must then be the same as the ratio between forced response to input when input is e^{st} (not $e^{st} \times u(t)$).

Forced response to an everlasting complex exponential input of $1e^{st}$ was seen to be a scaled version e^{st} itself; the scaling factor being a complex number that depends on the complex frequency value s . (Refer Section 13.1)

The time-domain circuit can be analysed using nodal analysis or mesh analysis to arrive at the n^{th} -order differential equation relating the response variable (y) to excitation variable (x). The result will be

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x$$

Then the scaling factor connecting an input of $1e^{st}$ to the output is $\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$.

Therefore,

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \quad (13.11-2)$$

In this expression for $H(s)$, the coefficients come from the coefficients of differential equation governing the circuit. In the expression for $H(s)$ in Eqn. 13.11-1, the coefficients were the result of circuit analysis in s -domain. We conclude that $n' = n$, $m' = m$, all a' values are equal to corresponding a values and all b' values are equal to corresponding b values.

The third interpretation comes from the definition of network function itself. If the input source function is assumed to be $\delta(t)$, then $H(s)$ becomes a Laplace transform – it becomes the Laplace transform of impulse response. Thus, $H(s)$ is a *ratio of Laplace transforms* and a *Laplace transform* at the same time. It is a Laplace transform when we invert it in order to find the impulse response.

The three faces of $H(s)$

$H(s)$, the network function, is a *Laplace transform* if we *invert* it to find the impulse response. $H(s)$, the network function, is a *complex gain* if we *evaluate* it at a particular value of s . In that case, it gives the complex amplitude of the forced response with an input of e^{st} with the value of s same as the value at which $H(s)$ was evaluated. $H(s)$, the network function, functions as a *ratio of Laplace transforms* when we multiply it by the Laplace transform of input source function and invert the product to determine the zero-state response in time-domain.

13.11.3 Poles and Zeros of $H(s)$ and Natural Frequencies of the Circuit

A network function goes to infinite magnitude at certain values of s . These values are obviously the values of s at which the denominator polynomial evaluates to zero, *i.e.*, at the roots of denominator polynomial. These values of s are called *poles* of the network function. Thus, *poles* are roots of denominator polynomial of a network function. Similarly, a network function attains zero magnitude at certain values of s . They are roots of numerator polynomial. They are called *zeros* of the network function.

A diagram showing the *pole* points by '×' marking and *zero* points by 'o' marking in complex signal plane (*i.e.*, s -plane) is called the *pole-zero plot* of the network function.

We note from the discussion in the previous subsection that the denominator polynomial of a network function apparently has the same order and same coefficients as that of the characteristic polynomial of differential equation describing the linear time-invariant circuit. The roots of the characteristic polynomial have been defined as the natural frequencies of the circuit. Does this mean that (i) the degree of denominator polynomial in a network function is the same as the degree of characteristic polynomial (ii) the poles and natural frequencies are the same?

The order of a differential equation is the order of highest derivative of dependent variable. The order of a circuit and order of the describing differential equation are the same. It will also be equal

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to the total number of independent inductors and capacitors – (number of all-capacitor-voltage source loops + number of all-inductor-current source nodes).

The order of a network function is the degree of denominator polynomial, *i.e.*, the highest power of s appearing in the denominator polynomial.

Thus we are raising the question – is the order of a network function in a linear time-invariant circuit same as the order of the circuit?

The characteristic polynomial of a differential equation is quite independent of right-hand side of differential equation. But, a network function is very much dependent on the right-hand side of the differential equation. Therefore, there exists a possibility of cancellation of some of the denominator factors by numerator factors in the case of a network function. Therefore, the order of a network function can be lower than the order of the circuit. It cannot, however, be higher. This will also imply that the order of two network functions defined within the same network need not be the same.

For instance, let the differential equation describing a linear time-invariant circuit be

$$\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = \frac{dx}{dt} + x$$

The characteristic equation is $s^2 + 3s + 2 = 0$ and the order of circuit is 2. The natural frequencies are $s = -1$ and $s = -2$. The zero-input response *can* contain e^{-t} and e^{-2t} terms. But it may contain only one of them for certain combination of initial conditions. Consider $y(0) = 1$ and $y'(0) = -1$. Then $y(t) = e^{-t}$ and it will not contain e^{-2t} . *Therefore, not all natural response terms need be present in all circuit variables under all initial conditions.*

Now consider the network function. It is $\frac{Y(s)}{X(s)} = \frac{(s+1)}{(s^2+3s+2)} = \frac{(s+1)}{(s+2)(s+1)} = \frac{1}{(s+2)}$

The order of network function is 1. It has one pole at $s = -2$. Therefore, zero-state response to *any* input will *not* contain e^{-t} term. This is the effect that a pole-zero cancellation in a network function has on circuit response. But, note that the same circuit may have other network functions that may not involve such pole-zero cancellation. It is only this particular circuit variable denoted by y that refuses to have anything to do with the natural response term e^{-t} .

Therefore, we conclude the following:

- The order of a network function and the order of the circuit can be different due to possible pole-zero cancellations in a particular network function.
- Poles of any network function defined in a linear time-invariant circuit will be natural frequencies of the circuit.
- However, all natural frequencies need not be present as poles in all network functions defined in that circuit.
- However, all natural frequencies will appear as poles in some network function or other.
- Thus, poles of a network function is a sub-set of natural frequencies of the circuit and natural frequencies will be a union-set of poles of all possible network functions in the circuit.
- A complex frequency that is not a natural frequency of the circuit cannot appear as a pole in any network function in that circuit.
- Both the denominator polynomial and the numerator polynomial of a network function in a linear time-invariant circuit have real coefficients. Therefore, poles and zeros of a network function either will be real-valued or will occur in complex conjugate pairs.

13.11.4 Specifying a Network Function

A network function $H(s)$ is specified in three ways. In the first method, it is specified as a ratio of rational polynomials in s .

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (13.11-3)$$

In the second method, it is specified as the ratio of product of first-order factors in numerator and denominator with a gain factor multiplying the entire ratio.

$$H(s) = K \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} \quad (13.11-4)$$

There are m factors in the numerator and n factors in the denominator. $z_1, z_2, \dots, z_m$ are the *zeros* of the network function and $p_1, p_2, \dots, p_n$ are the *poles* of the network function. Note that though the degree of denominator polynomial is shown as n , which is the order of the circuit, pole-zero cancellation may take place leaving the denominator polynomial of network function with a degree less than n .

In the third method of specifying a network function, the *pole-zero plot* along with the *gain factor* K is given. The gain factor K may be directly given or indirectly in the form of value of $H(s)$ evaluated at a particular value of s .

Example: 13.11-1

The circuit shown in Fig. 13.11-1 is the small signal equivalent circuit of a transistor amplifier for analysis of its behaviour for sinusoidal input at high frequency. Obtain the transfer function between the output voltage and input source voltage.

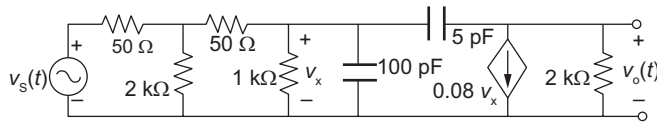


Fig. 13.11-1 Small-signal equivalent circuit of a transistor amplifier in Example: 13.11-1

Solution

We find the Norton's equivalent of the circuit to the left of 100pF capacitor first.

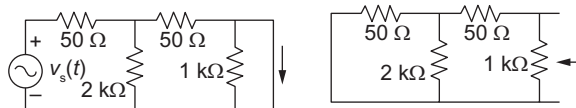


Fig. 13.11-2 Sub-circuits for determining Norton's equivalent

The sub-circuits needed for determining this equivalent are shown in Fig. 13.11-2. The short-circuit current in the first circuit is $\frac{v_s(t)}{50 + 2000 // 50} \times \frac{2000}{2050} = 9.876 \times 10^{-3} v_s(t)$. The Norton's equivalent resistance is $[(50 \Omega // 2 \text{ k} \Omega) + 50 \Omega] // 1 \text{ k} \Omega = 89.9 \Omega$. Thus the required Norton's equivalent is $9.876 \times 10^{-3} v_s(t)$ A in parallel with 89.9Ω . The original circuit with this Norton's equivalent in place is shown in

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circuit of Fig. 13.11-3 (a) with $R_1 = 89.9 \Omega$, $R_2 = 2 \text{ k}\Omega$, $C_1 = 100 \text{ pF}$, $C_2 = 5 \text{ pF}$ and $g_m = 0.08$. The corresponding s -domain equivalent circuit is shown in circuit (b) of Fig. 13.11-3.

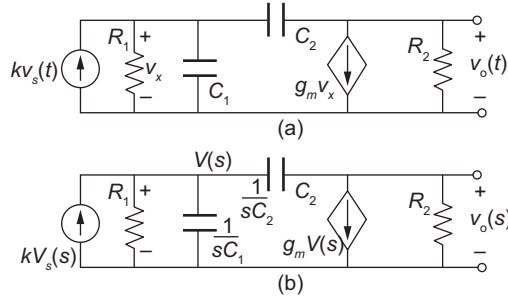


Fig. 13.11-3 (a) Reduced version of circuit in Fig. 13.11-1 and (b) Its s -domain equivalent
The node equations written for the two node voltage transforms $V(s)$ and $V_o(s)$ are as follows:

$$\begin{aligned}
 V(s) \left[s(C_1 + C_2) + \frac{1}{R_1} \right] + V_o(s) [-sC_2] &= kV_s(s) \\
 V(s) [-sC_2 + g_m] + V_o(s) \left[sC_2 + \frac{1}{R_2} \right] &= 0 \\
 \begin{bmatrix} s(C_1 + C_2) + \frac{1}{R_1} & -sC_2 \\ -sC_2 + g_m & sC_2 + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V(s) \\ V_o(s) \end{bmatrix} &= \begin{bmatrix} kV_s(s) \\ 0 \end{bmatrix}
 \end{aligned}$$

Solving for $V_o(s)$ and simplifying the expression, we get

$$H(s) = \frac{V_o(s)}{V_s(s)} = \frac{k}{C_1} \frac{(s - g_m/C_2)}{s^2 + s \left(\frac{1+g_m R_1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} \right) + \left(\frac{1}{R_1 R_2 C_1 C_2} \right)}$$

Substituting the numerical values for various parameters, we get

$$H(s) = \frac{98.76 \times 10^6 (s - 1.6 \times 10^{10})}{s^2 + 1.016 \times 10^9 s + (105.47 \times 10^6)^2}$$

The poles are at $s = -10^9$ nepers/s and $s = -11.07 \times 10^6$ nepers/s. The zero is at $s = 1.6 \times 10^{10}$ nepers/s.

Note that compared to the pole at -11.07×10^6 the other pole and the zero are located two orders away from it. The natural response term contributed by the pole at $s = -11.07 \times 10^6$ nepers/s will have a time constant of 90.3 ns whereas the natural response term contributed by the pole at $s = -10^9$ nepers/s will have a time constant of 1 ns. Thus the natural response term contributed by the pole at $s = -10^9$ nepers/s will disappear in about 5% of the time constant of the other term. Therefore, the time constant of 90.3 ns is the *dominant time constant* in this amplifier and the corresponding pole at -11.07×10^6 is the *dominant pole*. The amplifier transfer function can be approximated by neglecting the zero and the non-dominant pole to

$$H(s) = \frac{98.76 \times 10^6 (s - 1.6 \times 10^{10})}{(s + 10^9)(s + 11.07 \times 10^6)} \approx \frac{-1.58 \times 10^9}{(s + 11.07 \times 10^6)} = -142.7 \left(\frac{11.07 \times 10^6}{s + 11.07 \times 10^6} \right)$$

Example: 13.11-2

Find (i) Input impedance function and (ii) $V_o(s)/V_s(s)$ in the circuit shown in Fig. 13.11-4.

Solution

$$Z_i(s) = (1+s) // \left(1 + \frac{1}{s} \right) = \frac{(1+s) \left(1 + \frac{1}{s} \right)}{2 + s + \frac{1}{s}} = \frac{(s+1)(s+1)}{s^2 + 2s + 1} = 1\Omega$$

This is a case of cancellation of all poles by zeros leaving a real value for a network function. The input impedance of the circuit is purely resistive at all values of s . This implies that the current drawn by the circuit behaves as in a memoryless circuit. But, this does not mean that the order of the circuit is zero.

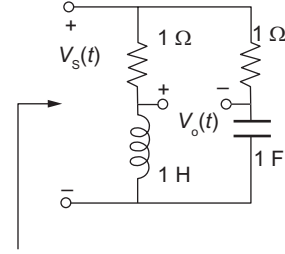


Fig. 13.11-4 Circuit for Example: 13.11-2

$$\frac{V_o(s)}{V(s)} = \frac{s}{s+1} - \frac{\frac{1}{s}}{1 + \frac{1}{s}} = \frac{s-1}{s+1}$$

The voltage transfer function has a pole at $s = -1$ and zero at $s = 1$.

This innocuous circuit challenges our notions on the order of a circuit. It contains two energy storage elements. Hence it must be a second-order circuit. But no network function defined in this circuit will be second-order function if the excitation is a voltage source. Even the zero-input response obtained by shorting the voltage source with initial conditions on inductor and capacitor will contain only e^{-t} . The reader is encouraged to analyse the general situation that develops when many sub-circuits with same set of poles in their input admittance functions are connected in parallel and driven by a common voltage source. Similarly, he is encouraged to ponder over the order of a circuit resulting from series connection of many sub-circuits with the same set of poles in their input impedance functions driven by a common current source. The reader may also note that the current in the circuit in Fig. 13.11-4 will have e^{-t} and $t e^{-t}$ terms in zero-input response due to initial energy storage in inductor and capacitor with input open-circuited (*i.e.*, zero-input response for current source excitation), thereby confirming it is a second-order circuit.

13.12 IMPULSE RESPONSE OF NETWORK FUNCTIONS FROM POLE-ZERO PLOTS

Let $H(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$ be a network function defined in a linear time-invariant circuit. Then the impulse response of this network function is given by its inverse transform. The transform $H(s) \times 1$ (1 is the Laplace transform of $\delta(t)$) can be expressed in partial fractions as below.

$$H(s) = K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} = \frac{A_1}{(s - p_1)} + \frac{A_2}{(s - p_2)} + \cdots + \frac{A_n}{(s - p_n)} \quad (13.12-1)$$

We have assumed that all poles are non-repeating ones. If there are repeating poles we may assume that the poles are slightly apart by Δp and evaluate the limit of $h(t)$ as $\Delta p \rightarrow 0$ after we complete the inversion. We will need a familiar limit $\lim_{\Delta p \rightarrow 0} \frac{e^{(p+\Delta p)t} - e^{pt}}{\Delta p} = te^{pt}$ for this. This strategy will help us to view all poles as non-repeating ones at the partial fractions stage.

$$\therefore H(s) = \sum_{i=1}^n \frac{A_i}{(s - p_i)}; \quad A_i = K \left. \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_{i-1})(s - p_{i+1}) \cdots (s - p_n)} \right|_{s=p_i} \quad (13.12-2)$$

Thus each pole contributes a complex exponential function to impulse response. The complex frequency of the complex exponential function contributed by a pole to impulse response is the same as the value of the pole frequency itself.

A point s in the complex signal space (*i.e.*, the s -plane) stands for the complex exponential signal e^{st} for all t . But, when a point s is marked out as a pole of a network function by a '×' mark, that signal point contributes $e^{st} u(t)$ to the impulse response and not e^{st} . Thus, a point in signal space stands for a two-sided complex exponential signal in general and stands for a right-sided complex exponential signal when that point is specified as a pole of a network function.

The evaluation of residue A_i at the pole p_i involves the evaluation of product of terms like $(p_i - z_1) \cdots (p_i - z_m)$ and $(p_i - p_1) \cdots (p_i - p_{i-1})(p_i - p_{i+1}) \cdots (p_i - p_n)$. Each of these factors will be a complex number. For instance, consider $(p_i - z_1)$. This is a complex number that can be represented by a *directed line* drawn from the point $s = z_1$ in s -plane to the point $s = p_i$ in the s -plane with the arrow of the line at $s = p_i$. The length of this line gives the magnitude of the complex number $(p_i - z_1)$ and the angle of the complex number $(p_i - z_1)$ is given by the angle the line makes with the positive real axis in the counter-clockwise direction. The magnitude of product of complex numbers is product of magnitudes of individual numbers. The angle of product of complex numbers is the sum of angles of individual complex numbers. Therefore, evaluation of residue A_i at the pole p_i reduces to determining certain lengths and angles in the pole-zero plot of the network function.

The reasoning employed in the paragraph above also reveals the roles of poles and zeros of a network function in deciding the impulse response terms. *The poles decide the number of terms in impulse response and their complex frequencies. The zeros along with the poles and gain factor K decide the amplitude of each impulse response term.*

A network function is a stable one if its impulse response decays to zero with time. This is equivalent to stating that its impulse response must be absolutely integrable, *i.e.*, $\int_0^\infty |h(t)| dt$ must be finite. Therefore, a network function is stable if all the impulse response terms are damped ones. That is, all the poles must have negative real values or complex values with negative real parts. *Therefore, a network function is stable if and only if all its poles are in the left-half of s -plane excluding the $j\omega$ -axis.*

Note that a stable network function in a linear time-invariant circuit does not necessarily imply that the circuit itself is stable. A linear time-invariant circuit is stable only if *all* the network functions that can be defined in it are stable ones. That is, a stable circuit will have only stable network functions in it. But, an unstable circuit can have both stable and unstable network functions in it.

The graphical interpretation adduced to impulse response coefficients in this section is illustrated in the examples that follow.

Example: 13.12-1

Obtain the pole-zero plots for (i) $H(s) = \frac{\alpha}{s + \alpha}$ (ii) $H(s) = \frac{s}{s + \alpha}$ (iii) $H(s) = \frac{s - \alpha}{s + \alpha}$ for positive and negative values of α and sketch the impulse response for $\alpha = \pm 1$.

Solution

These are standard first-order network functions. They are stable ones for positive values of α and unstable ones for negative values of α . They are important, yet simple, functions.

- (i) $H(s) = \frac{\alpha}{s + \alpha}$. Therefore, $h(t) = \alpha e^{-\alpha t} u(t)$. The pole-zero plot and impulse response are shown in Fig. 13.12-1 for $\alpha = \pm 1$.

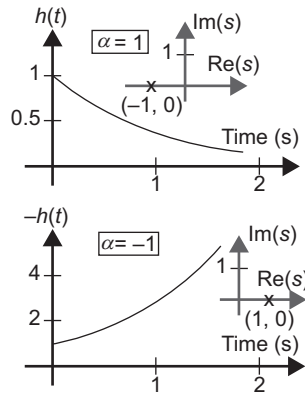


Fig. 13.12-1 Pole-zero plot and impulse response for $\alpha = \pm 1$

$$(ii) \quad H(s) = \frac{s}{s + \alpha} = 1 - \frac{\alpha}{s + \alpha}$$

$$\therefore h(t) = \delta(t) - \alpha e^{-\alpha t} u(t)$$

The pole-zero plots and impulse responses for $\alpha = \pm 1$ are shown in Fig. 13.12-2.

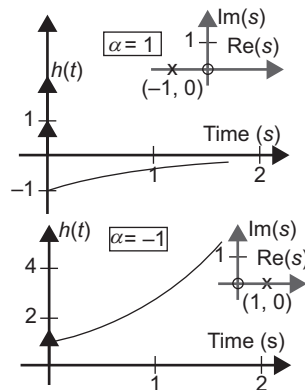


Fig. 13.12-2 Pole-zero plot and impulse response for $\alpha = \pm 1$

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- (iii) $H(s) = \frac{s - \alpha}{s + \alpha} = 1 - \frac{2\alpha}{s + \alpha}$. The pole-zero plots and impulse responses are shown in Fig. 13.12-3.
 $\therefore h(t) = \delta(t) - 2\alpha e^{-\alpha t} u(t)$

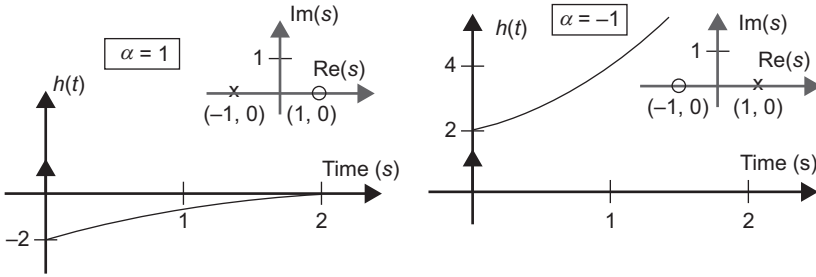


Fig. 13.12-3 Pole-zero plot and impulse response for $\alpha = \pm 1$

Example: 13.12-2

A second-order low-pass network function in standard form is given as $H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$ where ξ is the damping factor and ω_n is the undamped natural frequency as defined in Section 11.6 in Chapter 11. Obtain expressions for impulse response of the network function for positive and negative values of ξ in the range $-1 < \xi < 1$.

Solution

The poles are at $s = -\xi\omega_n \pm j\sqrt{1-\xi^2}\omega_n$. They are complex conjugate poles for $-1 < \xi < 1$. They are located in the right-half s -plane for $-1 < \xi < 0$ and in the left-half s -plane for $0 < \xi < 1$. They are located on $j\omega$ -axis at $\pm j\omega_n$ when $\xi = 0$. The poles have a magnitude of ω_n for all values of ξ in the range $(-1, 1)$. The pole line of the pole $s = -\xi\omega_n + j\sqrt{1-\xi^2}\omega_n$ makes an angle of $\cos^{-1}(\xi)$ with negative real axis in the case of positive ξ and with positive real axis in the case of negative ξ . Thus the damping factor magnitude is given by cosine of pole angle. See Fig. 13.12-4.

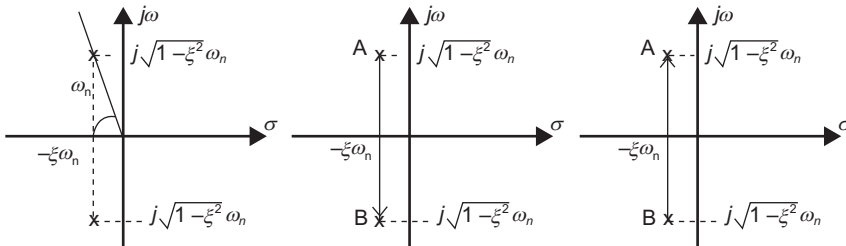


Fig. 13.12-4 Pole-zero plots for a standard second-order low-pass network function with positive damping

The residue at the pole marked as B is given by ω_n^2 divided by the complex number represented by the line connecting A and B in Fig. 13.12-4 with arrow towards B . This line is seen to be $2(\sqrt{1-\xi^2})\omega_n$

in length and it makes -90° with positive real axis. Similarly, the residue at the pole marked as B is given by ω_n^2 divided by the complex number represented by the line connecting A and B in Fig. 13.12-4 with arrow towards B . This line is seen to be $2(\sqrt{1-\xi^2})\omega_n$ in length and it makes 90° with positive real axis.

Therefore,

$$\begin{aligned} h(t) &= \frac{\omega_n^2}{2(\sqrt{1-\xi^2})\omega_n} \left[e^{j90^\circ} e^{(-\xi+j\sqrt{1-\xi^2})\omega_n t} + e^{-j90^\circ} e^{(-\xi-j\sqrt{1-\xi^2})\omega_n t} \right] u(t) \\ &= \frac{\omega_n e^{-\xi t}}{2(\sqrt{1-\xi^2})} \left[e^{j(\sqrt{1-\xi^2}\omega_n t + 90^\circ)} + e^{-j(\sqrt{1-\xi^2}\omega_n t + 90^\circ)} \right] u(t) \\ &= \frac{\omega_n e^{-\xi t}}{\sqrt{1-\xi^2}} \cos(\sqrt{1-\xi^2}\omega_n t + 90^\circ) u(t) = \frac{-\omega_n e^{-\xi t}}{\sqrt{1-\xi^2}} \sin \sqrt{1-\xi^2}\omega_n t u(t) \end{aligned}$$

This response is shown in Fig. 13.12-5 for $\omega_n = 1$ and $\xi = 0.7, 0.3, 0.1$ and 0.05 .

We observe that as the poles get closer and closer to $j\omega$ -axis, the impulse response oscillations become more and more under-damped and last for many cycles.

The impulse response for $\xi = -0.05$ in Fig. 13.12-6 shows the unbounded nature of impulse response of a network function with poles on right-half s -plane.

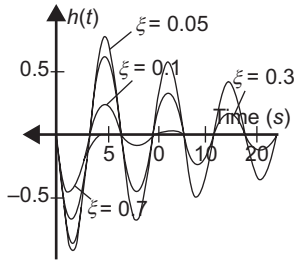


Fig. 13.12-5 Impulse response of standard second-order network function for various damping factors

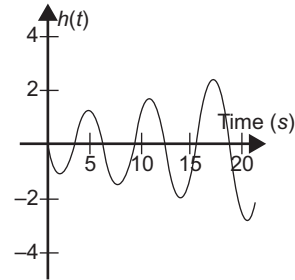


Fig. 13.12-6 Impulse response of standard second-order network function for $\xi = -0.05$

13.13 SINUSOIDAL STEADY-STATE FREQUENCY RESPONSE FROM POLE-ZERO PLOTS

Let $H(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$ be a network function defined in a linear time-invariant

circuit and let all the poles of this network function be in the left-half of s -plane excluding $j\omega$ -axis. The zeros can be located anywhere in s -plane. We know that $H(s)$, the network function, is a *complex gain* if we *evaluate* it at a particular value of s . In that case, it gives the complex amplitude of the forced response with an input of e^{st} with the value of s same as the value at which $H(s)$ was evaluated. Sinusoidal steady-state frequency response function of a stable circuit is given by the complex gain

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offered by the circuit to $e^{j\omega t}$ signal. Therefore, $H(s)$ evaluated with $s = j\omega$ gives the frequency response function provided the network function is stable. Thus,

$$H(j\omega) = K \frac{(j\omega - z_1)(j\omega - z_2) \cdots (j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2) \cdots (j\omega - p_m)} \quad (13.13-1)$$

$H(j\omega)$ is a complex function of a real variable ω . The plots of $|H(j\omega)|$ versus ω and $\angle H(j\omega)$ versus ω yield the *frequency response plots* of the network function. The first is called the *magnitude plot* and the second is called the *phase plot*.

13.13.1 Three Interpretations for $H(j\omega)$

We saw that we can interpret the network function $H(s)$ in three ways in Section 13.11. Three interpretations of $H(j\omega)$ follow from this.

- (i) $H(j\omega)$ is the ratio of complex amplitudes of output complex exponential and input complex exponential when input complex exponential is of the form $Ae^{j\omega t}$. Equivalently, $H(j\omega)$ is the complex amplitude of output when input is $e^{j\omega t}$ (not $e^{j\omega t}u(t)$). The signal $e^{-j\omega t}$ is a signal that is different from $e^{j\omega t}$. Hence, $H(j\omega)$ is a *two-sided* function from this point of view.

If input is $e^{j\omega t}u(t)$, then, $H(j\omega) e^{j\omega t}$ gives the forced response component (same as the steady-state response component).

- (ii) $H(j\omega)$ is the ratio of Laplace transform of zero-state response to Laplace transform of input when input is of the form $Ae^{j\omega t}u(t)$. The signal $Ae^{-j\omega t}u(t)$ is not the same as $Ae^{j\omega t}u(t)$. Hence, $H(j\omega)$ is a *two-sided* function from this point of view too.
- (iii) $H(j\omega)$ is the expansion of the impulse response $h(t)$ of the circuit in terms of complex exponential signals drawn from $j\omega$ axis in s -plane. It expands the time-domain signal into the sum of complex exponential functions of $e^{j\omega t}$ type (i.e., essentially in terms of sinusoids) with ω value ranging from $-\infty$ to ∞ . Hence, $H(j\omega)$ is a *two-sided* function from this point of view too.

There are two ways to solve the problem of finding the *steady-state output* when input variable is $\cos \omega_o t u(t)$. The first method is to express $\cos \omega_o t u(t)$ as $\text{Re}(e^{j\omega_o t})u(t)$ and express the output as $\text{Re}[H(j\omega_o)e^{j\omega_o t}]u(t)$. This method was called *Phasor Method* in Chapter 8. This method results in the steady-state response component and is based on the first interpretation of $H(j\omega)$.

$\text{Re}[H(j\omega_o)e^{j\omega_o t}] = \text{Re}[|H(j\omega_o)| e^{\angle H(j\omega_o)} e^{j\omega_o t}] = |H(j\omega_o)| \cos[\omega_o t + \angle H(j\omega_o)]$. Now, there is no harm if $H(j\omega)$ is thought of as a *single-sided function of ω* provided we interpret the magnitude of $H(j\omega)$ as the amplitude of output sinusoidal waveform with input amplitude of 1 and the phase of $H(j\omega)$ as the phase angle by which the output sinusoidal waveform leads the input sinusoidal waveform.

The second way is to express $\cos \omega_o t u(t)$ as $0.5 e^{j\omega_o t}u(t) + 0.5 e^{-j\omega_o t}u(t)$ by applying Euler's formula and express the *steady-state output* as $0.5[H(j\omega_o)e^{j\omega_o t} + H(-j\omega_o)e^{-j\omega_o t}]$. We have seen in Chapter 7 that $H(-j\omega) = [H(j\omega)]^*$. Therefore the steady-state output will be $\text{Re}[H(j\omega_o)]\cos \omega_o t - \text{Im}[H(j\omega_o)]\sin \omega_o t = |H(j\omega_o)| \cos[\omega_o t + \angle H(j\omega_o)]$; same as in the first method. This method also is based on the first interpretation of $H(j\omega)$ but uses a two-sided version of $H(j\omega)$.

Frequency response function is not new to us. We had dealt with the frequency response of first-order circuits and second order circuits in detail earlier in the book. But the observation that $H(j\omega)$ can be evaluated by evaluating $H(s)$ on $j\omega$ -axis leads to a graphical interpretation for sinusoidal steady-state frequency response function based on the pole-zero plot of $H(s)$. This interpretation affords an insight into the variation of magnitude and phase of $H(j\omega)$ without evaluating it at all values of ω . It helps us to visualise the salient features of frequency response function without extensive calculations.

13.13.2 Frequency Response from Pole-Zero Plot

Each factor of the form $(j\omega - z_i)$ in the numerator of Eqn. 13.13-1 is a complex number that can be thought of as a line directed from $s = z_i$ to $s = j\omega$ in s -plane. The magnitude of $(j\omega - z_i)$ is equal to the length of the line and the angle of $(j\omega - z_i)$ is the angle that the line makes with the positive real axis in counter-clockwise direction. Similar interpretation is valid for factors of the type $(j\omega - p_i)$ in the denominator of Eqn. 13.13-1 too. Let d_{z_i} be the length of the line joining the zero at $s = z_i$ to the excitation signal point $s = j\omega$ on the imaginary axis in s -plane. Let the angle that the line makes with the positive real axis in counter-clockwise be θ_{z_i} . Similar quantities for a pole at $s = p_i$ are d_{p_i} and θ_{p_i} . Then, the frequency response function $H(j\omega)$ can be expressed in terms of these lengths and angles as

$$H(j\omega) = K \frac{d_{z_1} d_{z_2} \cdots d_{z_m}}{d_{p_1} d_{p_2} \cdots d_{p_n}} \angle (\theta_{z_1} + \theta_{z_2} + \cdots + \theta_{z_m}) - (\theta_{p_1} + \theta_{p_2} + \cdots + \theta_{p_n})$$

$$= K |H(j\omega)| \angle \theta(j\omega)$$

$$\text{where } |H(j\omega)| = K \frac{d_{z_1} d_{z_2} \cdots d_{z_m}}{d_{p_1} d_{p_2} \cdots d_{p_n}} \text{ and } \theta(j\omega) = \sum_{i=1}^m \theta_{z_i} - \sum_{i=1}^n \theta_{p_i}$$

$$H(j\omega) = \text{Gain Factor} \times \frac{\text{Product of zero-distances}}{\text{Product of pole-distances}} \angle (\text{Sum of zero angles}) - (\text{Sum of pole angles})$$

Hence, we can make a rough sketch of the frequency response function by visualising how the various zero-distances and pole-distances vary when the ω is taken from $-\infty$ to $+\infty$ in s -plane.

Example: 13.13-1

Obtain the frequency response plots for (i) $H(s) = \alpha / (s + \alpha)$ (ii) $H(s) = s / (s + \alpha)$ using geometrical interpretation of frequency response and obtain expressions for bandwidth in both cases.

Solution (i)

Figure 13.13-1 shows the pole-zero plot and frequency response function. The pole distance and the pole angle are marked in the Fig. 13.13-1. Obviously, the gain magnitude goes to $1/\sqrt{2}$ times initial gain when $\omega = \alpha$ and the phase at that point is -45° . Therefore bandwidth is α rad/s and the function is a low-pass function.

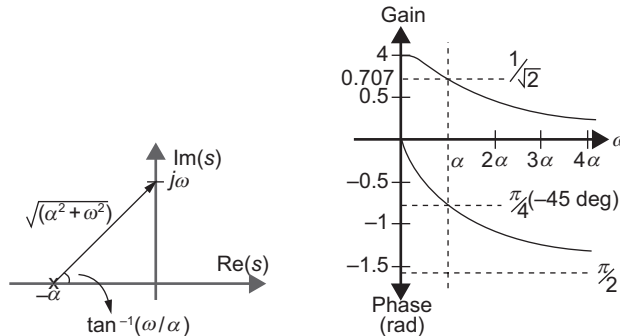


Fig. 13.13-1 Pole-zero plot and frequency response of $H(s) = \alpha/(s + \alpha)$

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Solution (ii)

Figure 13.13-2 shows the pole-zero plot and frequency response function. The pole distance and the pole angle are marked in Fig. 13.13-2. The zero distance is same as the excitation frequency value ω and the zero angle is 90° . Obviously, the gain magnitude goes to $1/\sqrt{2}$ times the final gain when $\omega = \alpha$ and the phase at that point is 45° . Therefore bandwidth is α rad/s and the function is a high-pass function.

$$|H(j\omega)| = \frac{\omega}{\sqrt{\alpha^2 + \omega^2}} \text{ and } \theta(j\omega) = \left(\frac{\pi}{2} - \tan^{-1} \frac{\omega}{\alpha} \right) \text{ rad}$$

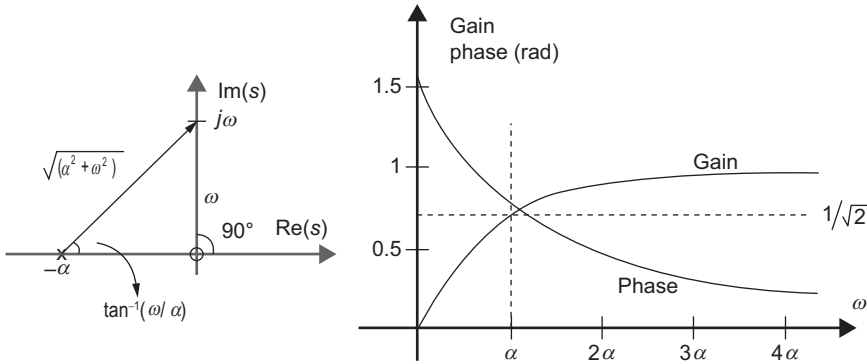


Fig. 13.13-2 Gain and phase plots of $H(s) = s/(s + \alpha)$

Example: 13.13-2

The biquadratic network function $H(s) = \frac{as^2 + bs + c}{s^2 + 2\xi\omega_n s + \omega_n^2}$ is a second-order low-pass function if $a = 0$, $b = 0$ and $c = \omega_n^2$. It is a second-order high-pass function if $a = 1$ and $b = c = 0$. It is a band-pass function if $a = c = 0$ and $b = 2\xi\omega_n$ and it is a band-reject function if $a = 1$, $b = 0$ and $c = \omega_n^2$. The frequency response function for low-pass, high-pass and band-pass second-order functions were studied in detail in Section 11.11 in Chapter 11 in the context of frequency response of Series RLC circuit.

Consider the band-pass function and band-reject function and sketch their frequency response plots for $\xi < 1$.

Solution

The poles of the function are at $s = -\xi\omega_n \pm j\sqrt{1 - \xi^2}\omega_n$. Let us consider the band-pass function first.

$$H(s) = \frac{2\xi\omega_n s}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The zero of this function is at $s = 0$.

The distance of the pole at $s = -\xi\omega_n + j\sqrt{1 - \xi^2}\omega_n$ to the excitation frequency point $j\omega$ is denoted by d_1 and the distance of the pole at $s = -\xi\omega_n - j\sqrt{1 - \xi^2}\omega_n$ to $j\omega$ is d_2 . The distance of zero at $s = 0$ to $j\omega$ is ω itself. The pole angles θ_1 and θ_2 are also shown in the pole-zero diagram in Fig. 13.13-3.

The magnitude function then is $2\xi\omega_n\omega/(d_1d_2)$ and the phase function is $(\pi/2) - (\theta_1 + \theta_2)$. The distances d_1 and d_2 are equal to ω_n at $\omega = 0$ and the sum of the angles θ_1 and θ_2 at that frequency is 360° . Therefore, the gain at zero frequency is 0 (due to the zero-distance of zero) and angle is 90° . As $\omega \rightarrow \infty$, all the three distances go to ∞ and hence magnitude goes to zero. θ_1 and θ_2 go to 90° as $\omega \rightarrow \infty$ and hence the phase angle of frequency response function goes to -90° as $\omega \rightarrow \infty$.

As ω increases from 0, the distance d_1 decreases and the distances d_2 and d_3 increase. Consider a pair of ω values equal to $[\sqrt{(1-\xi^2)} - \xi]\omega_n$ and $[\sqrt{(1-\xi^2)} + \xi]\omega_n$ i.e., two ω values separated by $\pm$ real part of the pole from the imaginary part of the pole. The distance d_1 undergoes a variation from $\sqrt{2}\xi\omega_n$ to $\xi\omega_n$ and again to $\sqrt{2}\xi\omega_n$ as ω varies from $[\sqrt{(1-\xi^2)} - \xi]\omega_n$ to $[\sqrt{(1-\xi^2)} + \xi]\omega_n$ passing through the point $[\sqrt{(1-\xi^2)}]\omega_n$. The distances d_2 and d_3 also vary. However, if $\xi \ll 1$, the variation in these two quantities will be negligible over this frequency range and approximation $d_2 \approx 2\sqrt{(1-\xi^2)}\omega_n$ and $d_3 \approx \omega_n$ will be satisfactory.

Therefore, the magnitude of frequency response function will vary from $2\xi\omega_n\omega_n \div 2\sqrt{2}\xi\omega_n\sqrt{(1-\xi^2)}\omega_n \approx 1/\sqrt{2}$ to $2\xi\omega_n\omega_n \div \xi\omega_n \times 2\sqrt{(1-\xi^2)}\omega_n \approx 1$ and again to $1/\sqrt{2}$ as ω varies from $[\sqrt{(1-\xi^2)} - \xi]\omega_n$ to $[\sqrt{(1-\xi^2)} + \xi]\omega_n$ passing through the point $[\sqrt{(1-\xi^2)}]\omega_n$. The imaginary part of poles can be taken as approximately ω_n itself for $\xi \ll 1$. Therefore, the maximum gain is 1 at $\omega = \omega_n$ and gain goes through $1/\sqrt{2}$ at $\omega_{1,2} = (1 \mp \xi)\omega_n$. The phase angle at $\omega = \omega_n$ is zero. See Fig. 13.13-3.

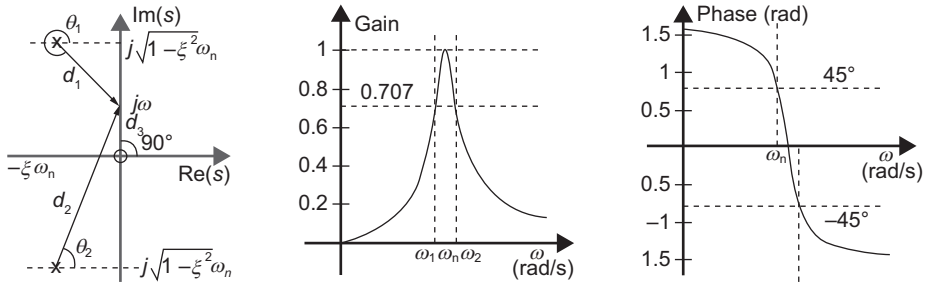


Fig. 13.13-3 Pole-zero plot and frequency response plot for a second order band-pass function

The center frequency of the narrow band-pass function is seen to be $\approx \omega_n$ and the bandwidth is $\approx 2\xi\omega_n$. Thus the ratio of center frequency to bandwidth of a narrow band-pass second-order network function is $1/2\xi$ or Q of the denominator polynomial.

Let us consider the band-reject function now.

$$H(s) = \frac{s^2 + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The poles are at $s = -\xi\omega_n \pm j\sqrt{1-\xi^2}\omega_n$ and zeros are at $s = \pm j\omega_n$. The distance of the pole at $s = -\xi\omega_n + j\sqrt{1-\xi^2}\omega_n$ to the excitation frequency point $j\omega$ is denoted by d_1 and the distance of the pole at $s = -\xi\omega_n - j\sqrt{1-\xi^2}\omega_n$ to $j\omega$ is d_2 . The distance of zero at $s = j\omega_n$ to $j\omega$ is d_3 and the distance of zero at $s = -j\omega_n$ to $j\omega$ is d_4 . The pole angles θ_1 and θ_2 are also shown in the pole-zero diagram in

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Fig. 13.13-4. The zero-angles are -90° and 90° for all $\omega < \omega_n$ and 90° and 90° for all $\omega > \omega_n$. The gain is given by $d_3 d_4 / d_1 d_2$ and starts at 1 at $\omega = 0$ since $d_1 = d_2$ and $d_3 = d_4$. The gain goes to 1 as $\omega \rightarrow \infty$ since $d_1 \approx d_2$ and $d_3 \approx d_4$ under that condition. The gain is zero at $\omega = \omega_n$ since d_3 is zero under that condition. Therefore, it is a band-reject function.

The pole-zero plots and frequency response plots are shown in Fig. 13.13-4. For $\xi \ll 1$, it may be shown that the gain crosses $1/\sqrt{2}$ level at $\omega_{1,2} = (1 \mp \xi)\omega_n$ and that the phase angles at those frequencies are -45° and 45° . The centre frequency of the narrow band-reject function is seen to be $\approx \omega_n$ and the bandwidth is $\approx 2\xi\omega_n$. Thus the ratio of centre frequency to bandwidth of a narrow band-reject second order network function is $1/2\xi$ or Q of the denominator polynomial.

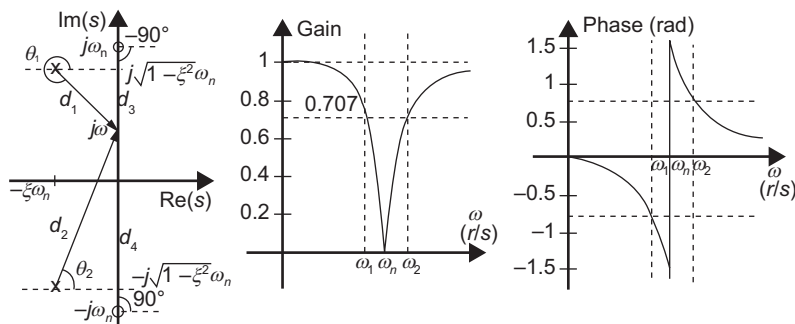


Fig. 13.13-4 Pole-zero plot and frequency response plot for a second-order band-pass function

The frequency response of higher order network functions can similarly be sketched. A higher order $H(j\omega)$ can be expressed as the product of first-order factors and biquadratic factors. The magnitude response for first-order factors and biquadratic factors may be sketched separately first and then multiplied together to get the magnitude response curve of $H(j\omega)$. Phase curves will have to be added.

Poles on negative real axis contribute a monotonically decreasing magnitude response. Poles close to $j\omega$ -axis render resonant peaks in magnitude response and zeros on $j\omega$ -axis produce zero gain response at the excitation frequencies equal to the zero locations. Thus graphical interpretation of frequency response function is a valuable aid to a circuit designer who wants to locate poles and zeros of a network function to tailor the frequency response function to meet design specifications.

13.14 SUMMARY

- Let $v(t)$ be a right-sided function that is bounded by $Me^{\alpha t}$ with some finite value of M and α . Then the Laplace transform pair is defined as

$$V(s) = \int_{0^-}^{\infty} v(t)e^{-st} dt \text{ — the analysis equation}$$

$$v(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} V(s)e^{st} ds \text{ — the synthesis equation}$$

where $s = \sigma + j\omega$ is the complex frequency variable standing for the complex exponential function e^{st} with σ value $\geq \alpha$. The ROC of $V(s)$ is the entire plane to the right of $\text{Re}(s) = \alpha$ line.

Table 13.14-1 Some Important Properties of Laplace Transforms

Signal / Property	Laplace Transform
$v(t) = f(t) u(t)$	$V(s)$ – Definition
$av_1(t) + bv_2(t)$	$aV_1(s) + bV_2(s)$ – Linearity
$v(t-t_d)$	$e^{-js t_d} V(s)$ – Time-shifting
$e^{s_o t} v(t)$, s_o is a complex number	$V(s - s_o)$ – Multiplication by a complex exponential In time-domain – Frequency Shifting
$\frac{dv(t)}{dt}$	$sV(s) - v(0^-)$ Time-domain differentiation
$\int_0^t v(t) dt$	$\frac{1}{s} V(s)$ – Time-domain integration
$-t v(t)$	$\frac{dV(s)}{ds}$ – Frequency-domain differentiation
$\frac{v(t)}{t}$	$\int_s^\infty V(s) ds$
$v_1(t) * v_2(t)$	$V_1(s) V_2(s)$ – Time-domain convolution
$v(0^+)$	$\lim_{s \rightarrow \infty} sV(s) = v(0^+)$, if the limit exists.
$v(\infty)$	$\lim_{s \rightarrow 0} sV(s) = v(\infty)$, if the limit exists and all poles of $sV(s)$ are in the left-half of s -plane.

- Laplace transform expands a transient right-sided time-function in terms of infinitely many complex exponential functions of infinitesimal amplitudes. The ROC of such a Laplace transform will include right-half of s -plane and hence the time-domain waveform gets constructed by growing complex exponential functions.

- For a linear time-invariant circuit described by $\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x$, the ratio of rational polynomials in s defined as

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \text{ has three interpretations.}$$

- It may be viewed as a generalised frequency response function. Its magnitude gives the ratio between the amplitude of output complex exponential function and input complex exponential function when input is of the form Ae^{st} . Its angle gives the phase angle by which the output complex exponential function leads the input complex exponential function.
- It is also the ratio of Laplace transform of zero-state response to Laplace transform of input source function called ‘the s -domain System Function’.
- Further, it is the Laplace transform of Impulse Response

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- Laplace transforms can be inverted by the method of partial fractions.
- Laplace transformation of both sides of a linear constant-coefficient ordinary differential equation converts it into an algebraic equation on transforms. Thus Laplace transform affords a convenient way to solve such differential equations with initial conditions.
- All circuit elements have s -domain equivalents. The s -domain equivalent of the complete circuit can be constructed by replacing each element with its s -domain equivalent. KVL and KCL are directly applicable to the transformed quantities.
- s -domain equivalent circuits may be analysed by nodal analysis or mesh analysis procedures. All circuit theorems developed in the context of memoryless circuits are applicable to s -domain equivalent circuits.
- The s -domain System Function is also called the network function. Immittance functions, transfer functions and transfer immittance functions are three classes of network functions usually employed in circuit analysis. Those complex frequency values at which a network function goes to ∞ are called its *poles* and those complex frequency values at which the network function goes to zero are called its *zeros*.
- Pole-zero plot along with a gain factor K will specify a network function uniquely. The impulse response of the network function may be obtained from its pole-zero plot. The poles decide the number of terms in impulse response and their complex frequencies. The zeros along with the poles and gain factor K decide the amplitude of each impulse response term.
- A network function is a stable one if its impulse response decays to zero with time. This is equivalent to stating that its impulse response must be absolutely integrable, i.e., $\int_0^\infty |h(t)| dt$ must be finite. Therefore, a network function is stable if all the impulse response terms are damped ones. That is, all the poles must have negative real values or complex values with negative real parts. Therefore, a network function is stable if and only if all its poles are in the left-half of s -plane excluding the $j\omega$ -axis.
- Sinusoidal steady-state frequency response function $H(j\omega)$ can be obtained by evaluating $H(s)$ on $j\omega$ -axis. This evaluation can be carried out in a graphical manner too.
- The DC steady-state gain of a network function is $H(0)$.

13.15 PROBLEMS

1. Evaluate the voltage across the capacitor in a series RC circuit with $R = 1 \text{ k}\Omega$ and $C = 1000 \text{ }\mu\text{F}$ at $t = 0^+$ and $t = 20 \text{ sec}$ if the applied voltage to the circuit is (a) $2 e^{0.01t} \cos t \text{ V}$ (b) $2 e^{0.01t} \cos u(t) \text{ V}$. Use differential equation approach.
2. A current source with $i_s(t) = f(t) u(t) \text{ A}$ is applied to a parallel RL circuit with $R = 1 \text{ }\Omega$ and $L = 1 \text{ H}$. The voltage across the combination is found to be $= 2e^{-t} \sin t u(t) \text{ V}$. Find $f(t)$ and the initial current in the inductor. Do not use Laplace transform technique.
3. The output variable y in a linear time-invariant circuit is related to the input variable x by the following differential equation $\frac{d^3 y}{dt^3} + 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = \frac{dx}{dt} + x$. Its zero-input response is found to contain $e^{-0.5t}$ term among other terms. If $x(t) = 3e^{0.01t} \sin t$, find the instantaneous value of y at $t = 10 \text{ s}$. Do not use Laplace transform technique.
4. A voltage source of $v_s(t) = 2e^{-0.2t} \cos(t - 45^\circ) \text{ V}$ is applied to a series RLC circuit with $R = 1 \text{ }\Omega$, $L = 1 \text{ H}$ and $C = 1 \text{ F}$ from $t = 0^+$. The circuit is initially relaxed. Determine the total response of current in the circuit by solving the circuit differential equation without using Laplace transforms.

5. A bounding exponential $Me^{\alpha t}$ is to be determined for each of the functions listed below. Find the minimum value of α and the corresponding value of M for each. (i) $tu(t)$ (ii) $u(t)-u(t-2)$ (iii) $e^{3t}u(t)$ (iv) $e^{-3t}u(t)$ (v) $e^{3t}u(t-3)$ (vi) $e^{-3t}u(t-2)$ (vii) $e^{-2t}\cos 2t u(t)$ (viii) $e^{2t}\cos 2t u(t)$.
6. The signals listed in Problem 5 are applied to a parallel RL circuit with $R = 2 \Omega$ and $L = 1$ H as current sources. Find the instantaneous voltage across the combination at $t = 2$ sec in each case by using Laplace transforms.
7. The Laplace transform of impulse response of a linear time-invariant circuit is given by $H(s) = \frac{2}{s^2 + 3s + 2}$. (i) Find the differential equation describing the circuit assuming that the output variable is y and the input variable is x . (ii) Find the total response of the circuit if $x(t) = 3$ for $t \geq 0^+$ with $y(0^-) = 1$ unit and $y'(0^-) = 1$ unit/s by Laplace transform technique.
8. The Laplace transform of current drawn by an initially relaxed dynamic circuit from an unit impulse voltage source is $I(s) = \frac{2s}{2s^2 + 2s + 2}$. (i) Find the differential equation relating the current drawn by the circuit to voltage applied. (ii) Find the total response of current if the circuit is initially relaxed and $v_s(t) = 2e^{-0.5t}\cos 2t$ V for $t \geq 0^+$ is applied to it by using Laplace transforms.
9. Find the total response of current in the circuit in Problem 8 if the circuit is initially relaxed at $t = 0$ and $v_s(t) = 2e^{-0.5t}\cos 2t u(t-0.5)$ V is applied to it.
10. Let $f(t)$ be a periodic waveform with a period of T s. Let $v(t) = f(t)u(t)$ and $v_p(t) = f(t)[u(t)-u(t-T)]$. That is, $v(t)$ is the right-side of a periodic waveform and $v_p(t)$ is one period of $f(t)$. Develop an expression for Laplace transform of $v(t)$ in terms of Laplace transform of $v_p(t)$ using time-shifting theorem. What is the ROC of Laplace transform of $v(t)$?
11. Find the Laplace transforms of (i) a symmetric square wave of unit amplitude and unit period (ii) a rectangular pulse waveform of unit amplitude with first pulse located between 0 sec and 0.5 sec and pulses repeating every 2 sec using the result derived in Problem 10.
12. Solve the system of differential equation $\frac{di_1}{dt} + 2i_1 - i_2 = v_s$ and $\frac{di_2}{dt} + 3i_2 - i_1 = 0$ for $v_s(t) = 2\sin 3t$ V for $t \geq 0^+$ with $i_1(0^-) = 1$ and $i_2(0^-) = 1$.
13. If $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} = 0$ and $y(0^-) = 0$, $y'(0^-) = 1$ and $y''(0^-) = -1$, find $y(t)$ by using Laplace transform technique.
14. Let $x(t) = 3tu(t)$ and $y(t) = 2u(t)$. Find $x(t)*y(t)$ by inverting $X(s)Y(s)$ and verify by time-domain convolution.
15. The impulse response of a linear time-invariant circuit is $2e^{-0.05t}u(t)$. Find the zero-state response when input is $3e^{-0.1t}$ by convolution theorem on Laplace transforms.
16. Let $x(t) = t[u(t)-u(t-2)]$ and $y(t) = [u(t)-u(t-2)]$. Find $x(t)*y(t)$ by inverting $X(s)Y(s)$ and verify by time-domain convolution.
17. Find the Laplace transform of (i) $\cos \pi [u(t)-u(t-2)]$ (ii) $2\sinh 0.2t [u(t)-u(t-1)]$ by using shifting theorem.
18. Find the Laplace transform of (i) $2t^3e^{-t}u(t)$ (ii) $\frac{(e^{-3t}-e^{-t})}{t}u(t)$.
19. Let $V(s) = \frac{4}{s^2 + 4}$. (i) Find $\int_s^\infty V(s)ds$ and its inverse transform. (ii) Find $\frac{dV(s)}{ds}$ and its inverse transform.
20. Using Laplace transforms find the value of R in the circuit in Fig. 13.15-1 such that the damping factor of the circuit for voltage input is 0.2. Find the step response for $v_o(t)$ with this R and verify the initial value and final value theorems on step response.

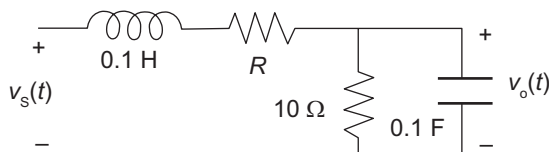


Fig. 13.15-1

21. (i) Obtain the input impedance function and input admittance function for the circuits shown in Fig. 13.15-2 and prepare pole-zero plots for these immittance functions. (ii) Determine the zero-state input current as a function of time when input is $u(t)$ V and verify initial value and final value theorems in each case. (iii) Determine the zero-state input voltage as a function of time when input is $u(t)$ A. All circuit elements have unit values.

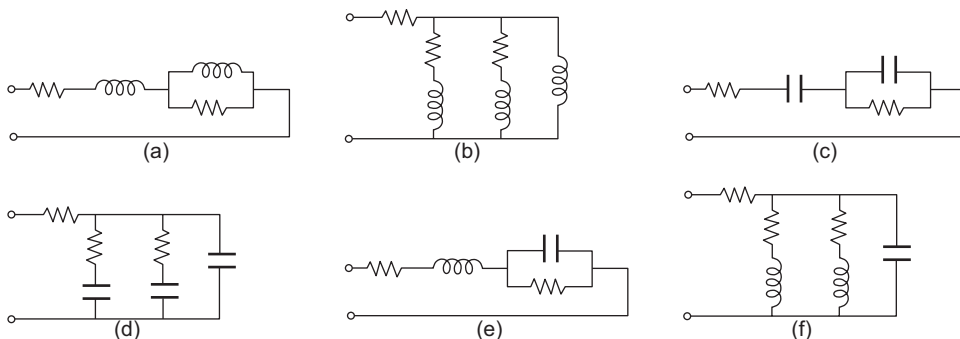


Fig. 13.15-2

22. (a) Find the voltage transfer function $V_o(s)/V_s(s)$ and driving-point impedance function in the circuit in Fig. 13.15-3. (b) Prepare the pole-zero plot for both network functions. (c) Determine the step response for $v_o(t)$ and $i_s(t)$ (d) Verify initial value theorem and final value theorem on Laplace transforms in the case of $v_o(t)$ and $i_s(t)$.

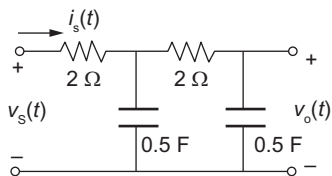


Fig. 13.15-3

23. The initial current in the inductor is 0.5 A and the initial voltage across the capacitor is 1 V in the circuit in Fig. 13.15-4. A single rectangular pulse of current is applied to the circuit as shown in the figure. Solve for $v_o(t)$ by s -domain equivalent circuit method.

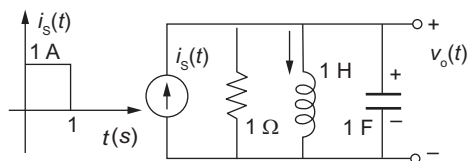


Fig. 13.15-4

24. (i) Show that the voltage transfer function in the circuit in Fig. 13.15-5 is a real number if $R_1 C_1 = R_2 C_2$. (ii) Obtain the input impedance function with $R_1 C_1 = R_2 C_2$.

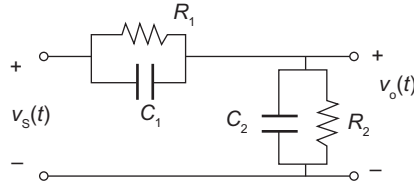


Fig. 13.15-5

25. The impulse response of i_x in the circuit in Fig. 13.15-6 contains a real exponential term that has a time constant of 1.755 s. (i) Show the pole-zero plots for $I_x(s)$ and $V_o(s)$ when $v_s(t) = u(t)$ V and the circuit is initially relaxed. (ii) Find $i_x(t)$ and $v_o(t)$ for $t \geq 0^+$ if $v_s(t) = u(t)$ and both capacitors have 1 V across them with the bottom plate positive at $t = 0^-$ and inductor has zero current at $t = 0^-$.

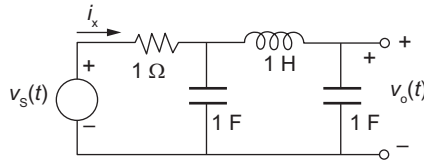


Fig. 13.15-6

26. The impulse response of input current in the circuit in Fig. 13.15-7 contains a $(1/6)\delta(t)$ component. (i) Find the value of R . (ii) Find the driving-point impedance function and show its pole-zero plot. (iii) Find the time-function describing the current delivered by source for $t \geq 0^+$ if $v_s(t) = 2\cos(2t + 30^\circ)u(t)$, $i_1(0^-) = 1$ A and $i_2(0^-) = -1$ A.

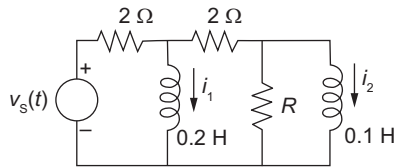


Fig. 13.15-7

27. Find the zero-state response for $v_x(t)$ by nodal analysis in s -domain in the circuit in Fig. 13.15-8 if $v_{s1}(t) = 2u(t)$ V and $v_{s2}(t) = 2e^{-t}u(t)$ V.

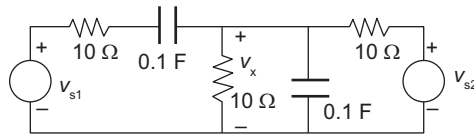


Fig. 13.15-8

28. Find the zero-input response for $v_x(t)$ by mesh analysis in s -domain in the circuit in Fig. 13.15-8 if first capacitor has 1 V across it at $t = 0^-$ with the left plate positive and the second capacitor has 1 V across it with bottom plate positive at $t = 0^-$.

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29. Pole-zero plots of some transfer functions are shown in Fig. 13.15-9. Find the transfer functions and their impulse responses. The DC gain for all the transfer functions is unity.

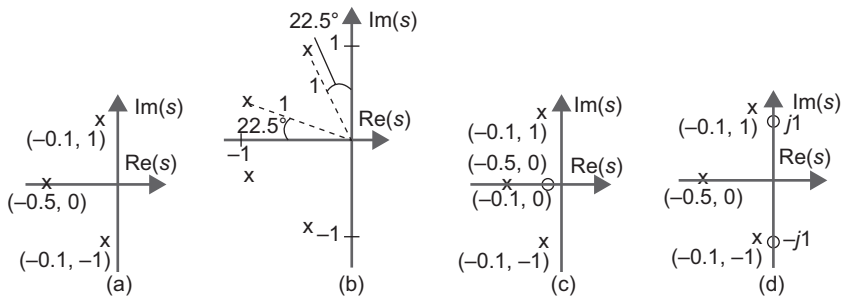


Fig. 13.15-9

30. Obtain the voltage transfer function in the circuit in Fig. 13.15-10 in terms of k and determine the range of values for k such that the transfer function is stable. Use mesh analysis in s -domain.

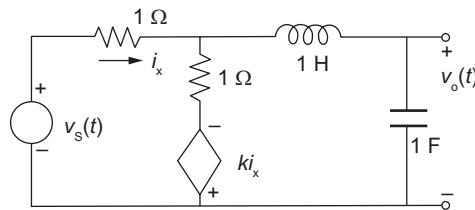


Fig. 13.15-10

31. The value of RC product in the circuit in Fig. 13.15-11 is $1\mu\text{s}$. (i) Derive the voltage transfer function for the circuit and determine the maximum value of A for which the circuit will be stable. (ii) If the value of actually used is $1/10^{\text{th}}$ of this value, calculate the poles and zeros of the voltage transfer function and show the pole-zero plot. (iii) Sketch the frequency response plots for the above condition by geometrical interpretation of frequency response.

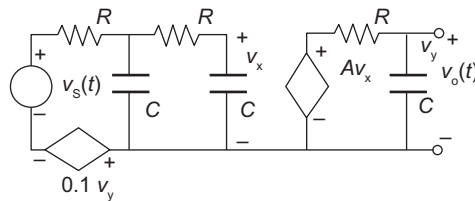


Fig. 13.15-11

32. The impulse response of a voltage transfer function in a linear time-invariant circuit is found to contain two waveshapes $e^{-0.5t}$ and $e^{-t} \sin 2t$. The steady-state step response of the same circuit is 0.7 V. Find the voltage that must be applied to the circuit if the desired steady-state output is $10 \sin (4t + 45^\circ)$ V by geometrical calculations in s -plane. Assume that the transfer function has no zeros.
33. Mark the pole-zero plot of the transfer function $H(s) = \frac{s^2 - \sqrt{2}s + 1}{s^2 + \sqrt{2}s + 1}$ and obtain its frequency response plot by geometrical calculations in pole-zero plot.

34. Obtain the voltage transfer function in the Opamp circuit in Fig. 13.15-12 and show that the circuit can work as a band-pass filter. Select the values for R_1C_1 and R_2C_2 such that the filter has a centre frequency of 1000 rad/s and bandwidth of 100 rad/s.

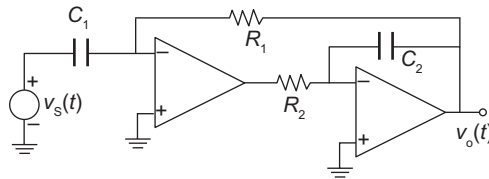


Fig. 13.15-12

35. Sketch the frequency response plots for the transfer functions with pole-zero plots as in Fig. 13.15-13 (a) through (g) approximately by using geometrical interpretation in s -plane. 'r' indicates the multiplicity number. The maximum gain is unity in all cases.

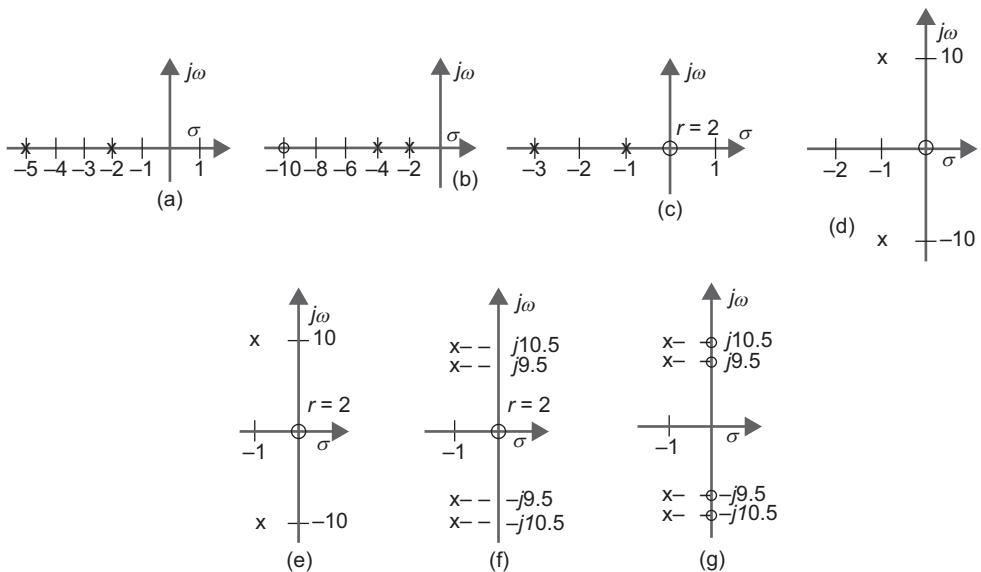


Fig. 13.15-13

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Chapter 14

Magnetically Coupled Circuits

CHAPTER OBJECTIVES

- *To introduce mutual inductance element*
- *To explain dot polarity convention*
- *To explain equivalent circuits for mutually coupled coil systems*
- *To define a perfectly coupled linear transformer*
- *To define an ideal transformer*
- *To define and explain steady-state analysis of coupled coil systems*
- *To explain transformers for impedance matching*
- *To explain what is meant by single-tuned and Double-tuned Band-pass Filters*
- *To explain transient response of coupled coil systems*
- *To explain constant flux linkage theorem*

INTRODUCTION

There are many different ways in which two or more circuits can get coupled without any conductive connections among them. Two separate circuits interacting with each other by means of optical signals is an example.

Another form of coupling is through thermal effect. The heat produced by dissipation in one circuit may affect the temperature of components in a nearby circuit and change their parameters. Change in parameter values affects the circuit operation in the nearby circuit.

Circuits can be designed to cause mechanical motion of some part of a component when input is applied to the circuit. This mechanical motion may result in variation of some circuit parameter in a nearby circuit, thereby affecting its operation.

There is no conductive connection (*i.e.*, a wire connection) between the interacting circuits. They are *galvanically isolated* – yet they interact.

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A practically important interaction between two or more galvanically isolated circuits takes place through the medium of *shared flux linkage* in circuits. Current in a coil in one circuit produces magnetic flux at locations where coils belonging to other circuits are placed. This results in interaction between various circuits through magnetic fields and gives rise to mutual flux linkages and mutually induced electromotive forces (emfs). We take up transient and steady-state analysis of such circuits in this chapter.

14.1 THE MUTUAL INDUCTANCE ELEMENT

Flux linkage in a coil can be produced in two ways. The first method is to send a current through the coil. The flux linkage produced in the coil will be proportional to current through it and the proportionality constant is its self-inductance, L . We now call this inductance as self-inductance because we need to distinguish it from mutual inductance. However, the qualifier ‘self’ is often dropped in practice and the word ‘inductance’ alone is taken to mean ‘self-inductance’ by default.

The flux linkage will be time-varying when the current in the coil is a time varying one. Time-varying flux linkage produces induced emf in the coil. This induced emf, if viewed as a voltage drop across element, is given by $v = L di/dt$ where L is the self-inductance of the coil.

The second way flux linkage in a coil can be produced is to place it near some other coil that is carrying a current. The flux produced by the current in the second coil links the first coil too due to the coils sharing same magnetic path. Now, two possibilities arise. The flux in the first coil due to the second coil may be crossing the area of cross section of the coil in the same direction as the flux produced in the coil by a current flowing in it did. Or, it could be in the opposite direction. That depends on how the coils are wound. The magnitude of flux linkage produced in the first coil by the current in the second coil is proportional to that current and the proportionality constant is defined as mutual inductance M_{12} .

Both self-induced as well as mutually induced emfs are neglected in modeling a physical electrical system by two-terminal elements except in the case of those devices that have been specifically designed to strengthen induced emf components. The self-induced emf in such a device was modeled by a two-terminal inductance element in Chapter 1.

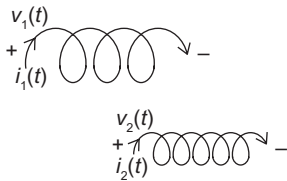


Fig. 14.1-1 A two-coil coupled system

Consider a two-coil system shown in Fig. 14.1-1. Assume for a moment that $i_2(t)$ is zero. Then, the induced emf in the first coil is entirely due to induced electric field created by its own current. Or, equivalently, the flux linkage in the first coil is entirely due to its own current. The flux linkage per ampere in the first coil per unit current in it will give its self-inductance. Let it be L_1 . Then, the voltage that appears across first coil

with the second coil kept open is $v_1(t) = L_1 \frac{di_1(t)}{dt}$ V and the flux linkage of the first coil with the second coil kept open is $\psi_1(t) = L_1 i_1(t)$ Wb-T. Similarly, the voltage that appears across

the second coil with the first coil kept open is $v_2(t) = L_2 \frac{di_2(t)}{dt}$ V and the flux linkage of the first coil

with the second coil kept open is $\psi_2(t) = L_2 i_2(t)$ Wb-T, where L_2 is the self-inductance of the second coil. *Self-inductance of a coil in a multi-coil system is measured by measuring the inductance across the coil with all other coils kept open.*

Time-varying current flow in the second coil will produce induced electric field at all points inside the first coil. Therefore, there will be an induced emf in the first coil due to $i_2(t)$ in the second

coil even when the first coil is kept open. This emf will appear as a potential difference across its terminals. This emf is called the mutually induced emf in coil-1 due to coil-2. It will be proportional to $\frac{di_2(t)}{dt}$ and the proportionality constant is defined as M_{12} , the *mutual inductance between coil-1 and coil-2*. The induced electric field created at a point in coil-1 by an increasing current in the second coil may add to the induced electric field created at the same point coil-1 by an increasing current in coil-1 itself or may subtract from it. That depends on relative winding directions in the two coils.

The process of mutual induction may also be understood from the point of view of flux linkage. There is a flux linkage in the first coil due to the current in the second coil even when the first coil is kept open. This flux linkage is called the *mutual flux linkage in coil-1 due to coil-2*. The value of this mutual flux linkage will be proportional to $i_2(t)$ and the proportionality constant is the mutual inductance M_{12} . Thus, M_{12} can be understood as the mutual flux linkage in coil-1 per unit current in coil-2. Rate of change of the mutual flux linkage gives the emf induced in coil-1 by current in coil-2. The flux linkage created in coil-1 by coil-2 may add to or subtract from the flux linkage created in coil-1 by its own current.

Hence, the voltage that appears across the coil-1 when both coils are carrying current will be $v_1(t) = L_1 \frac{di_1(t)}{dt} \pm M_{12} \frac{di_2(t)}{dt}$ V and total flux linkage in coil-1 is $\psi_1(t) = L_1 i_1(t) \pm M_{12} i_2(t)$ Wb-T. The sign connecting the two terms, + or –, will be decided by whether the flux produced in the first coil by the second coil current is aiding the flux produced by its own current or opposing it.

Similarly, the voltage that appears across coil-2 is $v_2(t) = L_2 \frac{di_2(t)}{dt} \pm M_{21} \frac{di_1(t)}{dt}$ V and total flux linkage in coil-2 is $\psi_2(t) = L_2 i_2(t) \pm M_{21} i_1(t)$ Wb-T where M_{21} is the *mutual inductance between coil-2 and coil-1*.

The value of mutual inductance M_{12} can be measured by measuring the voltage that appears across the open-circuited coil-1 with a known time-varying voltage applied to coil-2. Similarly, the value of mutual inductance M_{21} can be measured by measuring the voltage that appears across the open-circuited coil-2 with a known time-varying voltage applied to coil-1.

Mutual induction arises out of magnetic coupling between two coils. Now, we make an important assumption. We assume that the two-coil system is constructed in such a way that there is magnetic coupling only between them and there is no magnetic coupling between these coils and any other element in the circuit or with the circuit loop itself. This is possible only if the two-coil system is designed to confine the magnetic field almost entirely within itself. A closed core structure employing a magnetic material with high magnetic permeability will be needed in practice to achieve this. Both the coils will be wound around the same core.

With this assumption of *confinement of magnetic flux linkage entirely within the device itself*, we can model a two-coil system by an *ideal four-terminal element model*. The symbol of the model is shown in Fig. 14.1-2. L_1 and L_2 are the self-inductance of the coils. M is the mutual inductance between them. The two parallel lines between the coils indicate that they share a common core. The two ‘dot’ points marked by the side of coils help to decide the polarity of mutual emf in relation to the self-induced emf. *Increasing current*

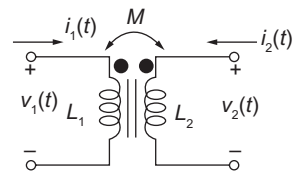


Fig. 14.1-2 A four-terminal element model for a two-coil system

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entering the dot point in one coil generates a mutual emf in other coil with positive polarity at its dot point.

14.1.1 Why Should M_{12} Be Equal to M_{21} ?

Consider an experiment with the two-coil system shown in Fig. 14.1-2. We want to set up a DC current of I_1 in coil-1 and a DC current of I_2 in coil-2 starting from zero-current condition in both coils. We do this by applying an independent current source i_{s1} to coil-1 and the independent current source i_{s2} to coil-2. Refer to Fig. 14.1-3.

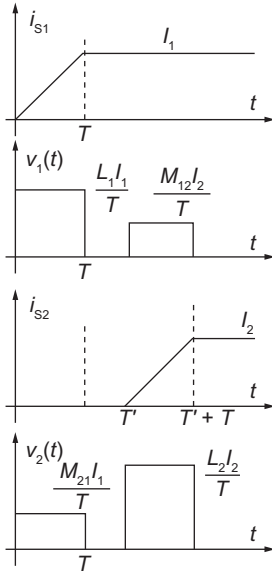


Fig. 14.1-3 Current source waveforms and voltage appearing across them during the current-build up in a two-coil system.

The current in the first coil ramps up linearly with a slope of I_1/T A/s. This results in $v_1(t) = L_1 I_1 / T$ V and $v_2(t) = M_{21} I_1 / T$ V during $[0, T]$ interval. But the current in the second coil during that interval is zero and hence there is no power delivered to the coil-2 during this interval. There is power delivered to coil-1 during this interval. The total energy delivered to coil-1 during $[0, T]$ is $L_1 I_1^2 / 2$ J.

The current in coil-2 ramps up linearly during the interval $[T', T' + T]$ with a slope of I_2/T A/s. This results in $v_1(t) = M_{12} I_2 / T$ V and $v_2(t) = L_2 I_2 / T$ V during $[T', T' + T]$ interval. The total energy delivered to coil-2 is $L_2 I_2^2 / 2$ J. But, now the first coil current is at a constant level of I_1 and hence the mutually induced voltage across the first coil results in additional energy input into the first coil from the current source i_{s1} . This additional energy

input is $\frac{M_{12} I_2}{T} \times I_1 \times T = M_{12} I_1 I_2$ J.

Therefore, the total energy delivered to the two-coil system in building up the currents is =

$$\frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{12} I_1 I_2 \text{ J.}$$

Now, assume that we built up the currents in the coils to the same levels using a current source i_{s1} which had a waveform that is delayed by T' second for coil-1 and a current source i_{s2} which had a waveform that started at $t = 0$ for coil-2. Then, the total energy required to build up the currents will

$$\text{be} = \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{21} I_1 I_2 \text{ J.}$$

The total energy delivered to the system must be the same in these two situations. Otherwise, we can build up the currents using the procedure resulting in lower energy input and subsequently reduce the currents to zero using the other process (with suitably constructed current sources) to receive some net energy out of the system. That is an attempt to violate conservation law for energy. Therefore,

$$\frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{12} I_1 I_2 = \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + M_{21} I_1 I_2$$

$$\therefore M_{12} = M_{21}$$

Now, we know that the total energy storage in a two-coil system shown in Fig. 1.8-2 is given by $E(t) = \frac{L_1[i_1(t)]^2}{2} + \frac{L_2[i_2(t)]^2}{2} + M[i_1(t)][i_2(t)]$ J. Two-coil system is a *passive* system. Therefore, the total energy storage in it cannot be non-negative for any permissible voltage–current condition. Therefore,

$$\begin{aligned} \frac{L_1[i_1(t)]^2}{2} + \frac{L_2[i_2(t)]^2}{2} + M i_1(t) i_2(t) &\geq 0 \\ \therefore L_1[i_1(t)]^2 + L_2[i_2(t)]^2 + 2M i_1(t) i_2(t) &\geq 0 \\ \text{i.e., } \left[\sqrt{L_1} i_1(t) + \sqrt{L_2} i_2(t) \right]^2 + 2(M - \sqrt{L_1 L_2}) i_1(t) i_2(t) &\geq 0 \end{aligned}$$

This has to be true for any $(i_1(t), i_2(t))$. Let us choose $\sqrt{L_1} i_1(t) + \sqrt{L_2} i_2(t) = 0$. This implies that $i_1(t)$ and $i_2(t)$ have opposite polarities for this choice. Therefore, $i_1(t) i_2(t)$ will be negative. Therefore, $\sqrt{L_1 L_2}$ has to be $\geq M$ for the last inequality to be true. Therefore, $M \leq \sqrt{L_1 L_2}$

The equality sign applies only if the magnetic coupling is so tight that there is no leakage of magnetic flux from the common core and windings. That is, the fluxes linking the two coils are equal. The coupling coefficient goes to unity under this situation. No physical coil system can have k equal to 1. However, coil systems in power transformers approach this value very closely.

14.1.2 Dot Polarity Convention

One does not want to physically examine every coupled coil-set to find out the relative winding directions in order to decide the sign connecting the self-induced voltage drop and mutually induced voltage drop in each coil. Therefore, the manufacturer marks this information in the form of two *dots* – one each on each coil in a coupled set of two coils. He/She has a choice of two terminals in the first coil to put a *dot* on. He/She chooses that arbitrarily. That choice will fix the terminal of the second coil at which the second coil *dot* has to be marked. The following rule helps us to write the voltage equation in a set of coupled coils.

Increasing current entering a *dot* point in one coil will produce a mutual emf with its positive polarity at the *dot* point in the other coil.

Increasing current leaving a *dot* point in one coil will produce a mutual emf with its negative polarity at the *dot* point in the other coil.

With this interpretation for dot polarity assignment, the mutual inductance between a pair of coils is always a positive quantity.

Self-induced voltage is always positive at the current entry point for both coils.

Various possibilities for mutual emf polarity in a two-coil coupled system are shown in Fig. 14.1-4. A double-headed arrow indicates magnetic coupling with M value marked by the side.

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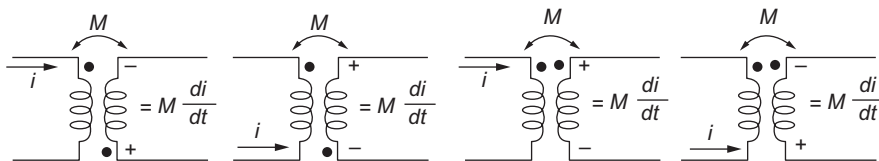


Fig. 14.1-4 Dot polarity convention and sign of mutually induced voltage in a two-coil system

But markings can fade with time. And people can forget to do things they are supposed to do. Therefore, the following experimental method can help us to put the *dots* if they are not known already.

Let A and B be the two terminals of the first coil and C and D be the two terminals of the second coil. Join B and D. Apply a sinusoidal source of suitable amplitude and frequency between A and B as shown in Fig. 14.1-5. Use three rms reading voltmeters to record the readings V_1 , V_2 and V_3 . Then, V_2 will be close to either $V_1 + V_3$ or $|V_1 - V_3|$. The *dot point* assignment for these two cases are shown in Fig. 14.1-5.

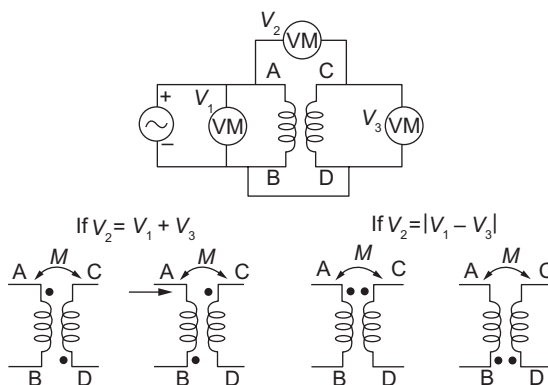


Fig. 14.1-5 Experimental determination of dot point assignment in a two-coil system

Dot polarity convention can be applied in the case of more than two coupled coils too. Consider three mutually coupled coils with self-inductance L_1 , L_2 and L_3 . Let M_1 be the mutual inductance between coil-1 and coil-2, M_2 be the mutual inductance between coil-2 and coil-3 and M_3 be the mutual inductance between coil-1 and coil-3. We need three pairs of dots to indicate the relative polarity of mutually induced emf in the three coils. Each coil has a self-induced emf and two mutual emf from the remaining two coils. A possible dot polarity assignment is shown in Fig. 14.1-6.

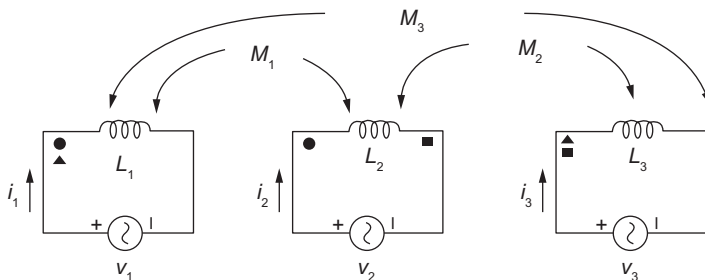


Fig. 14.1-6 Dot polarity assignment in a three coil system

The KVL equations (assuming that there is no resistance in the coils) can be obtained as follows:

$$\begin{aligned} v_1(t) &= L_1 \frac{di_1(t)}{dt} + M_1 \frac{di_2(t)}{dt} + M_3 \frac{di_3(t)}{dt} \\ v_2(t) &= M_1 \frac{di_1(t)}{dt} + L_2 \frac{di_2(t)}{dt} - M_2 \frac{di_3(t)}{dt} \\ v_3(t) &= M_3 \frac{di_1(t)}{dt} - M_2 \frac{di_2(t)}{dt} + L_3 \frac{di_3(t)}{dt} \end{aligned}$$

14.1.3 Maximum Value of Mutual Inductance and Coupling Coefficient

The maximum value of mutual inductance that a two-coil coupled system can have is the geometric mean of self-inductances of the two coils – i.e., $\sqrt{L_1 L_2}$. This maximum value is realised when the flux produced by a current in one coil links totally with the other coil. This is impossible to achieve physically and can only be approached in practice. Winding both the coils over a common core made of some high permeability material like laminated iron makes M very close to (but less than) $\sqrt{L_1 L_2}$ in practice.

The ratio between the actual mutual inductance realised to the theoretical maximum value is defined as *coupling coefficient* and is symbolised by k . Coupling coefficient is a positive number between 0 and 1. It is difficult to achieve anything more than few tenths for k -value in air-cored coils; however, coils wound on common core made of iron can have k -value very close to unity.

$$M = k \sqrt{L_1 L_2} \text{ H; } 0 \leq k \leq 1$$

14.2 THE TWO-WINDING TRANSFORMER

A system of two coils with constant values of L_1 , L_2 and M with two pairs of terminals identified for application of excitation and/or measurement of response is called a *two-winding linear transformer*. Such a two-winding transformer will be referred to simply as a *transformer* in this section.

Fig. 14.2-1 shows a two-coil system with self-inductance values of L_1 , L_2 and mutual inductance value of M . Also shown are two independent voltage sources driving the circuit at two pairs of terminals.

The mesh equations of the circuit are written assuming that the coils have zero winding resistance values.

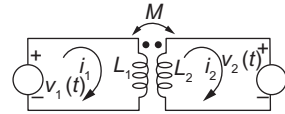


Fig. 14.2-1 A transformer with voltage source drive at both ports

$$\begin{aligned} L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} &= v_1(t) \\ -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} &= v_2(t) \end{aligned}$$

The current i_2 leaves the *dot* in the second coil and therefore the mutually induced voltage in the first coil appears with negative polarity at the *dot*. Hence, the negative sign for mutually induced voltage in the first mesh equation. The current i_1 enters the *dot* of the first coil and therefore the mutually induced voltage in the second coil appears with positive polarity at the *dot*. When mesh

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equation is written by traversing the loop in clockwise direction the self-induced voltage and mutually induced voltages will enter with opposite signs. Hence, the negative sign for mutually induced voltage in the second mesh equation.

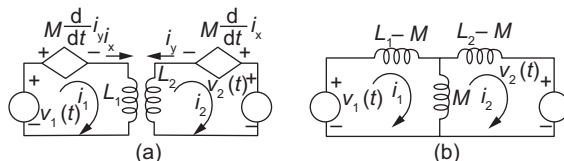


Fig. 14.2-2 Two circuit models for a transformer (a) model using linear dependent sources (b) Conductive equivalent model

Refer to Fig. 14.2-2. The reader may easily verify that the circuit in Fig. 14.2-2 (a) and the circuit (b) have the same mesh equations as those of the circuit in Fig. 14.2-1. Hence, the coupled set of coils may be replaced by two decoupled coils and two linear dependent sources as in circuit model Fig. 14.2-2 (a) *as far as the v - i behaviour at the terminals is concerned*. This circuit model preserves the conductive decoupling – i.e., the galvanic isolation – that exists between the two sides of the circuit.

A coupled set of two coils can also be replaced by a T-shaped equivalent circuit comprising three pure decoupled inductors as in circuit model in Fig. 14.2-2 (a) *as far as the v - i behaviour at the terminals is concerned*. This circuit model hides the galvanic isolation that is present in the transformer. Therefore, it is called the *conductive equivalent circuit* of coupled coils. Note that one inductance (either $L_1 - M$ or $L_2 - M$) can be negative-valued for sufficiently large value of coupling coefficient. There is no negative inductance in the physical world. But then, the inductors that appear in the conductive equivalent circuit of coupled coils are *not* physical inductors – they are *mathematical inductors* that are arranged to result in same set of circuit equations as those of the coupled coils. Therefore, they can assume negative values – we do not have to construct them!

Note that a certain *dot* polarity was assumed for the equivalent models established above. The equivalent models for the second relative *dot* polarity are shown in Fig. 14.2-3.

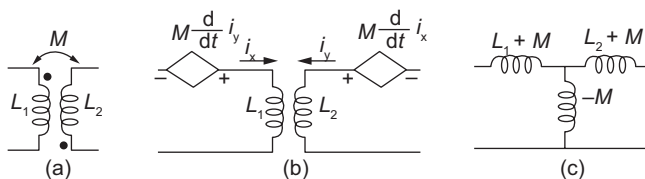


Fig. 14.2-3 Circuit models for coupled coils

The circuit in Fig. 14.2-3 (b) shows the dependent source based circuit model for the two-coil system in (a) and the circuit in Fig. 14.2-3 (c) shows the conductive equivalent circuit of the two-coil system in Fig. 14.2-3 (a). Note the reversal of polarity in the case of dependent sources and change in inductance values in the case conductive equivalent circuit.

Differentiation in time is replaced by multiplication by $j\omega$ in the phasor equivalent circuit for sinusoidal steady-state analysis in the equivalent circuit employing dependent sources. Winding resistance can be included as series resistors on both sides in both equivalent circuits.

Example: 14.2-1

The applied voltage is $v_s(t) = 100\sqrt{2} \cos 10^3 t$ V in the circuit shown in Fig. 14.2-4. (a) Find the primary (the side with source connection) and secondary (the side with no source connection) currents and the average power delivered to the resistors for (i) $k = 0$, (ii) $k = 0.1$, (iii) $k = 0.5$ and (iv) $k = 1$. (b) Repeat calculations with $k = 1$ for $L_1 = 25$ H and $L_2 = 100$ H.

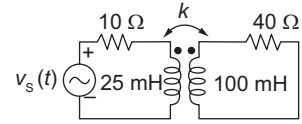


Fig. 14.2-4 Circuit for Example: 14.2-1

Solution

(a) The conductive equivalent circuit for coupled coils is used to obtain the phasor equivalent circuits for the four values of coupling coefficients. The phasor equivalent circuits are shown in Fig. 14.2-5. The solution process is illustrated for $k = 0.1$.

$V_s = 100\angle 0^\circ$ V rms. The two mesh equations are expressed in matrix form as

$$\begin{bmatrix} 10 + j25 & -j5 \\ -j5 & 40 + j100 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 100\angle 0^\circ \\ 0 \end{bmatrix}$$

Solving these equations, we get $I_1 = 3.737\angle -67.86^\circ$ A rms and $I_2 = 0.174\angle -46^\circ$ A rms.

Average power dissipated in $10\ \Omega$ resistor is $3.737^2 \times 10 = 139.65$ W

Average power dissipated in $40\ \Omega$ resistor is $0.174^2 \times 40 = 1.21$ W

Average power delivered by source = $100 \times 3.737 \times \cos 67.9^\circ = 140.83$ W.

Impedance seen by source = $Z_{in} = 100\angle 0^\circ \div 3.737\angle 67.86^\circ = 10.09 + j24.78\ \Omega$

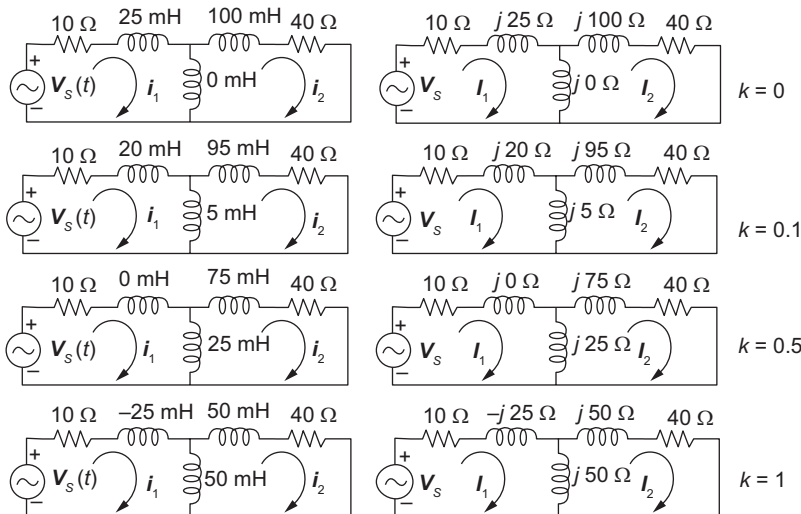


Fig. 14.2-5 Phasor equivalent circuits of the circuit in for different values of coupling coefficient

The table that follows summarises the results of calculations for all four values of k .

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K	I_1 (A rms)	I_2 (A rms)	$P_{10\Omega}$ (W)	$P_{40\Omega}$ (W)	PF	Z_{in} (Ω)
0	$3.714\angle-68.2^\circ$	0	137.94	0	0.371	$10 + j25$
0.1	$3.74\angle-67.86^\circ$	$0.174\angle-46^\circ$	139.65	1.21	0.376	$10.09 + j24.78$
0.5	$4.334\angle-58.2^\circ$	$1.006\angle-36.4^\circ$	187.83	40.48	0.527	$12.16 + j19.61$
1	$5.281\angle-10.49^\circ$	$2.451\angle11.31^\circ$	278.89	240.29	0.983	$18.62 + j3.45i$

(b) We observe from the table of results that as the coupling coefficient increases (i) the primary and secondary currents increase in magnitude and become more active in content than reactive, (ii) power delivered to the resistor in the secondary side increases, (iii) the power factor of circuit increases and (iv) the source experiences a decreasing input impedance that becomes more and more resistive.

Now, we raise the self-inductance level of the coils by 1000 fold. But we keep the ratio between them at 4 as before. The mesh equations for $k = 1$ are

$$\begin{bmatrix} 10 + j25 \times 10^3 & -j50 \times 10^3 \\ -j50 \times 10^3 & 40 + j10^5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 100\angle 0^\circ \\ 0 \end{bmatrix}$$

We observe that the reactive terms in the matrix entries overpower the resistive terms. The solution is obtained as $I_1 = 5\angle-0.01^\circ$ A rms and $I_2 = 2.5\angle0.01^\circ$ A rms.

Average power dissipated in $10\ \Omega$ resistor is $5^2 \times 10 = 250$ W

Average power dissipated in $40\ \Omega$ resistor is $2.5^2 \times 40 = 250$ W

Average power delivered by source = $100 \times 5 \times \cos 0.01^\circ = 500$ W.

Impedance seen by source = $Z_{in} = 100\angle 0^\circ \div 5\angle-0.01^\circ = 20 + j0.004\ \Omega$

The input impedance as seen by the source is almost purely resistive at $20\ \Omega$. The resistor that is in series with the source in the primary side must be contributing $10\ \Omega$ to the input impedance. Therefore, the transformer and its secondary load of $40\ \Omega$ resistor contributes $10\ \Omega$ to input impedance. Thus, the equivalent impedance of the transformer plus $40\ \Omega$ load seems to be an almost pure resistance of $10\ \Omega$. The ratio between $10\ \Omega$ and $40\ \Omega$ happens to coincide with the ratio $L_1 : L_2$ – that is a coincidence that calls for further analysis.

Let us calculate the voltage appearing across primary and secondary windings of the transformer.

Voltage across primary winding = $100\angle 0^\circ - 10 \times 5\angle-0.01^\circ \approx 50\angle 0^\circ$ V rms

Voltage across primary winding = $40 \times 2.5\angle 0.01^\circ \approx 100\angle 0^\circ$ V rms

Thus, primary voltage: secondary voltage $\approx 1:2 = \sqrt{L_1} : \sqrt{L_2}$ and using the solution for currents, primary current: secondary current $\approx 2:1 = \sqrt{L_2} : \sqrt{L_1}$. That makes it too many coincidences to pass over lightly. Therefore, we analyse the transformer with $k = 1$ in greater detail now.

14.3 THE PERFECTLY COUPLED TRANSFORMER AND THE IDEAL TRANSFORMER

Figure 14.3-1 shows a two-winding transformer with primary self-inductance of L_1 and secondary self-inductance of L_2 with coupling coefficient of unity. We neglect the winding resistance for the time being. The two circuits shown in this figure differ only in the *dot* polarity in the transformer.

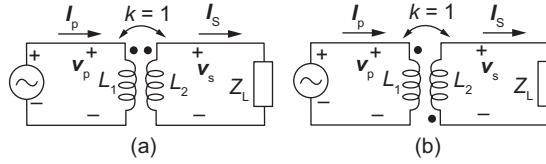


Fig. 14.3-1 A passively terminated two-winding transformer with $k = 1$

The primary is driven by a source $= \sqrt{2}V \cos \omega t$ V and secondary is *passively terminated in an impedance* $Z_L = R_L + jX_L \Omega$. Passive termination implies that there are no sources of any kind in the load circuit.

The mesh currents are I_p and I_s themselves and the mesh equations for circuit in Fig. 14.3-1 (a) in matrix form are

$$\begin{bmatrix} j\omega L_1 & -j\omega M \\ -j\omega M & Z_L + j\omega L_2 \end{bmatrix} \begin{bmatrix} I_p \\ I_s \end{bmatrix} = \begin{bmatrix} V \angle 0^\circ \\ 0 \end{bmatrix}$$

Since $k = 1$, the value of $M = \sqrt{L_1 L_2}$. Solving for primary and secondary current phasors using this value for M , we get

$$\begin{aligned} I_p &= \frac{V \angle 0}{j\omega L_1} + \left(\frac{L_2}{L_1} \right) \frac{V \angle 0}{Z_L} = V_p \left[\frac{1}{j\omega L_1} + \frac{L_2}{L_1} \frac{1}{Z_L} \right] \\ I_s &= V \angle 0 \left(\sqrt{\frac{L_2}{L_1}} \right) \frac{1}{Z_L} = V_p \left(\sqrt{\frac{L_2}{L_1}} \right) \frac{1}{Z_L} \\ V_s &= I_s Z_L = V_p \left(\sqrt{\frac{L_2}{L_1}} \right) \\ \therefore \frac{V_s}{V_p} &= \sqrt{\frac{L_2}{L_1}} \end{aligned}$$

We have used the fact that the source voltage phasor $V \angle 0^\circ$ itself is the primary voltage phasor V_p to make substitutions in the equation above.

We note that the secondary voltage and primary voltage are in phase and related by a real scaling factor of $\sqrt{\frac{L_2}{L_1}}$. However, this number is nothing but $\frac{n_2}{n_1}$, where n_1 and n_2 are the number of turns employed in primary and secondary windings. This is so since self-inductance of a coil is proportional to the square of the number of turns in the coil. We represent this ratio as *turns ratio* ' n '.

Therefore, secondary voltage magnitude in a transformer with unity coupling coefficient is turns ratio times the primary voltage magnitude.

The equation for I_p is the sum of two terms. Each term involves a division of primary voltage phasor by impedance. Therefore, the equation for I_p suggests that the input impedance of the circuit is a parallel combination of two impedances – an inductor of L_1 and another impedance of value $\frac{L_1}{L_2} Z_L = \frac{Z_L}{n^2}$.

Therefore, a transformer with unity coupling coefficient *reflects* the secondary load impedance Z_L onto the primary side as a *turns-ratio transformed impedance* of value Z_L/n^2 where n = secondary turns/primary turns.

Thus, the entire transformer and load circuit can be replaced by the equivalent circuit shown in Fig. 14.3-2 as far as the input side v - i behaviour under sinusoidal steady-state condition is concerned. We will see later that this is true not only for sinusoidal steady-state analysis but also for time-domain analysis too.

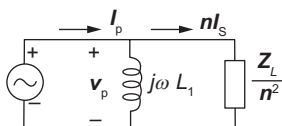


Fig. 14.3-2 Input equivalent circuit of transformer with $k = 1$

The primary winding resistance is included as a series resistance in the primary side, whereas the secondary winding resistance is included as a part of R_L in the load for analysis purposes.

Now, we consider the circuit in Fig. 14.3-1 (b). Reader may easily verify that the mesh equations for this circuit will be,

$$\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & Z_L + j\omega L_2 \end{bmatrix} \begin{bmatrix} I_p \\ I_s \end{bmatrix} = \begin{bmatrix} V \angle 0^\circ \\ 0 \end{bmatrix}$$

Both the off-diagonal terms in the mesh impedance matrix have changed sign. The solution for primary and secondary current phasors with $k = 1$ will be,

$$\begin{aligned} I_p &= \frac{V \angle 0}{j\omega L_1} + \left(\frac{L_2}{L_1} \right) \frac{V \angle 0}{Z_L} = V_p \left[\frac{1}{j\omega L_1} + \frac{L_2}{L_1} \frac{1}{Z_L} \right] = V_p \left[\frac{1}{j\omega L_1} + \frac{n^2}{Z_L} \right] \\ I_s &= -V \angle 0 \left(\sqrt{\frac{L_2}{L_1}} \right) \frac{1}{Z_L} = -n V_p \frac{1}{Z_L} \\ V_s &= I_s Z_L = -V_p \left(\sqrt{\frac{L_2}{L_1}} \right) = -n V_p \end{aligned}$$

There is no difference in solution *except* for 180° phase shift in secondary voltage. That is anyway expected due to the change in relative polarity of windings. The voltage magnitude ratio remains at turns ratio n .

But the most important aspect to be noted is that the same equivalent circuit shown in Fig. 14.3-2 will describe the transformer in this case too. See the equation for primary current phasor. In fact, this is true even if $k \neq 1$.

The input impedance of a passively terminated transformer is independent of its *dot* point assignment.

There are many important applications of transformers with $k \approx 1$ in which we make use of the inductor L_1 that appears in the input impedance to advantage. A tuned amplifier widely employed in communication circuits is such an application example.

There are many applications of transformers with $k \approx 1$ in electrical power engineering and electronics engineering where that inductance in the input impedance of transformer is not desirable.

Therefore, we minimise the effect of that inductance by making L_1 (and consequently L_2 and M too) very large compared to the maximum value of Z_L that we may use. This is done by using a large cross-section iron core and a large number of turns in both windings keeping the turns ratio at the desired value. An *ideal transformer* is an idealised model for such a transformer.

An ideal transformer is a two-winding transformer with perfect coupling ($k = 1$) and infinite self and mutual inductance values ($L_1 \rightarrow \infty$, $L_2 \rightarrow \infty$, $M \rightarrow \infty$). Complex power is conserved in an ideal transformer.

Thus, the equations for an ideal transformer are:

$$\begin{aligned}\frac{V_s}{V_p} &= n; \frac{I_s}{I_p} = \frac{1}{n} \\ \therefore V_p I_p^* &= V_s I_s^* \\ Z_{in} &= \frac{Z_L}{n^2}\end{aligned}$$

Complex power is conserved in an ideal transformer. That is, an ideal transformer absorbs zero active power, zero reactive power and zero complex power. Whatever active power and reactive power go into the primary winding go out of secondary winding into the load. An ideal transformer simply changes the voltage and current levels at which complex power transfer takes place. Moreover, an ideal transformer is an *impedance transformer*. It scales the secondary load impedance by a scaling factor of $\frac{1}{n^2}$ and presents it at the primary input. In fact, those two are the major applications of transformers – efficient voltage/current level translation in power engineering and impedance level translation (called *impedance matching*) in electronics engineering.

14.4 IDEAL TRANSFORMER AND IMPEDANCE MATCHING

Consider a power (or energy) delivery application in which a power delivery circuit delivers power to a load circuit. A typical case in point is an audio power amplifier. The power delivery block here is a cascade of low-power and high-power transistor amplifiers that amplify the audio input. The power delivery circuit maybe equivalenced to its Thevenin's equivalent in the form of a signal source with a series impedance. This impedance is called the *output impedance* of the power delivery circuit. As per Maximum Power Theorem, the load impedance has to be the conjugate of output impedance of power delivery circuit for maximum power transfer into the load. But we may not be able to make the load impedance satisfy this requirement since the load is designed with other factors in mind. For instance, loud speakers that are designed for good acoustic performance generally have low impedance and amplifiers that are designed for efficient and distortion-free amplification of audio signals have high output impedance. Thus, we have to interpose a circuit element that will translate the load impedance to the value demanded by maximum power transfer condition. A tightly coupled two-winding transformer with small winding resistance can do this. Transformers used for this purpose are called *Impedance Matching Transformers*.

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Example: 14.4-1

The sinusoidal voltage source that is driving a load of impedance $25 + j10 \Omega$ at 1 MHz has a source impedance of $1 - j1 \Omega$ at 1 MHz. An ideal transformer is introduced between the source and load such that the power transferred to load is a maximum. Find the turns ratio of the transformer and value and nature of additional components needed, if any.

Solution

The circuit diagram detailing the application context is shown in Fig. 14.4-1.

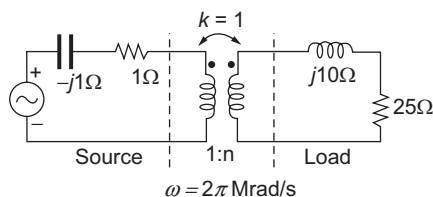


Fig. 14.4-1 Circuit for impedance matching example in Example: 14.4-1

The transformer should reflect the load impedance in such a way that the *conjugate impedance* criterion for maximum power transfer is met with. Therefore, the 25Ω resistance in the load should appear as 1Ω resistance when reflected to primary. This fixes the turns ratio n as $\sqrt{25} = 5$.

But with $n = 5$, the $j10 \Omega$ inductive reactance in the load will appear as $j0.4 \Omega$ to the primary side. As per *conjugate impedance matching*, we want it to appear as $-(-j1) = j1 \Omega$ to the primary side. Turns ratio cannot be adjusted any more. Therefore, we need to connect an extra reactance in series with the secondary winding such that the reflected total reactance becomes $j1 \Omega$. Therefore, we need an extra inductive reactance of $(j1 - j0.4) \times 5^2 = j15 \Omega$ in series with the secondary winding. The final matched network is shown in Fig. 14.4-2.

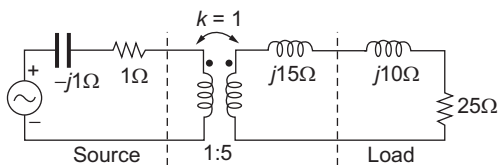


Fig. 14.4-2 Solution for impedance matching problem in Example: 14.4-1

14.5 TRANSFORMERS IN SINGLE-TUNED AND DOUBLE-TUNED FILTERS

Tuned amplifiers are transistor amplifiers that combine amplification with band-pass filtering. They are used extensively in electronic communication circuits.

In *carrier modulated* communication systems, some feature (amplitude, frequency or phase) of a high-frequency sinusoidal carrier waveform is made to vary according to the information-bearing signal (for example, an audio signal). The resulting signal will usually have its frequency content concentrated in a small band around the carrier frequency. For instance, a medium wave radio station transmitting at 1 MHz will transmit a signal that contains frequency components in the frequency band

between 995 kHz and 1005 kHz. The band-pass filters in the RF stage in the radio receiver have to be tuned to 1 MHz centre frequency with a bandwidth of 10 kHz to receive this station. Tuned amplifiers make use of inductances in the signal coupling transformer and variable capacitors to realise the required band-pass filter.

14.5.1 Single-Tuned Amplifier

Single-tuned amplifier uses a tightly coupled transformer and a capacitor. The capacitor is connected in parallel to the primary of the transformer. Next stage amplifier is connected to the secondary and it may be modeled by the input resistance of the next stage. The transformer primary in parallel with the capacitor forms a parallel resonant circuit and the transistor drives the signal *current* into this parallel combination.

Example: 14.5-1

An RF transformer with $k \approx 1$ and $n = 0.5$ is passively terminated by a resistor of 800Ω . It is driven by a current source $i_s(t) = I_m \cos \omega t$ A in parallel with a capacitor of 220 pF at the primary side. The primary self-inductance of the transformer is $115 \mu\text{H}$. Derive an expression for the ratio of output voltage phasor to input current phasor.

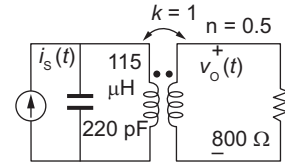


Fig. 14.5-1 Circuit for Example: 14.5-1

Solution

The phasor equivalent circuit of the circuit is shown in Fig. 14.5-2. The transformer is replaced by its equivalent circuit in this diagram by reflecting the 800Ω in the secondary as $800/(0.5)^2 = 3.2 \text{ k}\Omega$ in the primary.

The ratio of output voltage phasor to input current phasor is just the impedance of parallel combination of inductor, capacitor and resistor. Using the parameter symbols R , L and C ,

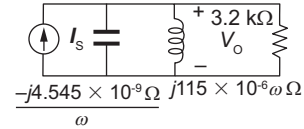


Fig. 14.5-2 Phasor equivalent circuit for the circuit in Fig. 14.5-1

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{j\omega L + R - \omega^2 LCR}{j\omega RL} = \frac{(1 - \omega^2 LC) + j\omega L/R}{j\omega L}$$

$$\therefore \frac{V_o}{I_s} = Z = \frac{j\omega L}{(1 - \omega^2 LC) + j\omega L/R}$$

Substituting the numerical values for circuit parameters,

$\frac{V_o}{I_s} = \frac{j722.6f}{(1 - f^2) + j0.226f}$, where f is the cyclic frequency of input in MHz. The magnitude and phase of this ratio function is given by

$$\text{Magnitude of } Z = \frac{722.6f}{\sqrt{(1 - f^2)^2 + (0.226f)^2}}; f \text{ in MHz}$$

$$\text{Phase of } Z = \left(\frac{\pi}{2} - \tan^{-1} \frac{0.226f}{1 - f^2} \right); f \text{ in MHz}$$

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$\omega = \frac{1}{\sqrt{LC}}$ is the angular frequency at which the susceptance of inductor and capacitor will be equal in magnitude but opposite in sign. Therefore, these two susceptance values will cancel each other at that frequency and leave R as the impedance value. At all other frequencies, some susceptance – inductive or capacitive – will shunt R thereby bringing down the magnitude of impedance below R . Thus, impedance is a maximum of R at $\omega = \frac{1}{\sqrt{LC}}$. Normalising the impedance function by dividing it by value of R ,

$$\left| \frac{Z}{R} \right| = \frac{722.6f}{\sqrt{(1-f^2)^2 + (0.226f)^2}} \text{ and } \angle \text{of } \frac{Z}{R} = \theta = \left(\frac{\pi}{2} - \tan^{-1} \frac{0.226f}{1-f^2} \right); f \text{ in MHz}$$

Fig. 14.5-3 shows the plots for these functions.

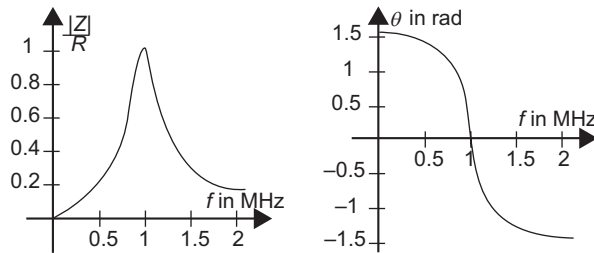


Fig. 14.5-3 Magnitude and phase plots of normalised impedance function in Example: 14.5-1

The plots reveal that it is a highly *frequency-selective* circuit. The circuit produces large magnitude output if the frequency of sinusoidal current source is at 1 MHz or nearby. The response is small if the frequency is far away from 1 MHz on either side. Such a response characteristic is called a *band-pass characteristic* or *tuned characteristic*. A circuit with band-pass nature can extract some frequency components selectively from a mixture of sinusoidal waveforms at different frequencies presented to it at the input. Such circuits find wide application in communication and signal processing circuits. The circuit we discussed in this example illustrates the principle of *tuned amplifiers* that are used in all sorts of electronic communication circuits starting from the common transistor radio receiver. The current source is realised by a transistor circuit in a tuned amplifier.

Single-tuned amplifier is an example where the inductance of a transformer is intentionally designed to resonate with a chosen capacitor at a desired frequency.

The cut-off frequencies of a band-pass filter are the frequencies at which the magnitude of transfer gain of the filter is $\frac{1}{\sqrt{2}} = 0.707$ times the value at centre frequency. They are 0.9 MHz and 1.1 MHz in this example. The difference between the two cut-off frequencies is defined as the bandwidth of the band-pass filter. Hence, the band-pass filter in this example has centre frequency of 1MHz and bandwidth of 0.2 MHz.

14.5.2 Double-Tuned Amplifier

Consider a modulated waveform with a carrier frequency of 1MHz with signal frequency components in the range between 0.9MHz and 1.1MHz. The single-tuned amplifier in the previous example can amplify this modulated signal since the centre frequency and bandwidth of that amplifier are 1MHz

and 0.2 MHz, respectively. However, the signal frequency components in 0.9MHz to 1.1MHz range get a variable gain – varying from 100% to 70.7%. This will lead to waveform distortion in the demodulated waveform. The signal after demodulation will be a distorted version of signal that was transmitted. The gain offered to components in the frequency range 0.9MHz to 1.1MHz must be a constant if this kind of signal distortion is to be avoided. We can achieve this by placing two resonant humps at two frequencies that are nearby in frequency response of the tuned amplifier. This is achieved in a double-tuned amplifier.

A double-tuned amplifier is a voltage-driven amplifier and uses a *weakly coupled transformer* and two capacitors – one in series with the primary and one in series with the secondary and load. We consider a 1:1 transformer with equal capacitors in the primary and secondary for illustration. The secondary is loaded resistively with a resistor equal in value to the source resistance in the primary side. We ignore the winding resistances for simplifying the analysis. The double-tuned stage is shown in Fig. 14.5-4. Transistor circuitry provides the input voltage signal to this filter.

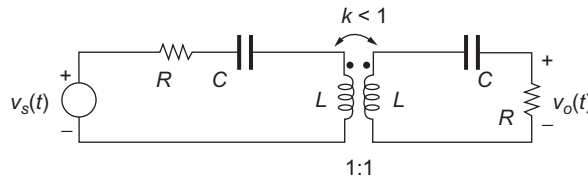


Fig. 14.5-4 Doubled-tuned band-pass filter

We first study the AC steady-state input impedance of the transformer to understand how two tuning frequencies appear in the circuit. We assume that the resistor R is small compared to $\sqrt{\frac{L}{C}}$ and ignore the resistor in this study. See Fig. 14.5-5.

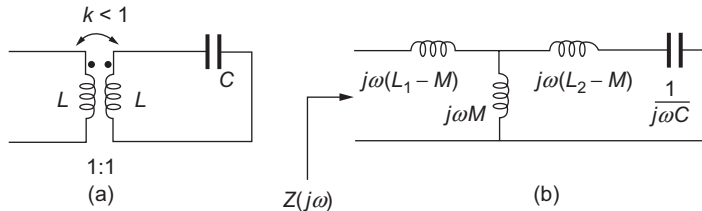


Fig. 14.5-5 Simplified circuit for study of input impedance

The circuit in Fig. 14.5-5 (a) is translated to the phasor equivalent circuit in Fig. 14.5-5 (b) using conductive equivalent circuit for the transformer. We require the input impedance of the circuit in Fig. 14.5-5 (b) for sinusoidal steady-state. This may be obtained by applying impedance series parallel combination rules. The result is given in the following:

$$Z(j\omega) = \frac{j\omega L[1 - \omega^2 LC(1 - k^2)]}{[1 - \omega^2 LC]}$$

We have used the relation $M = k\sqrt{L_1 L_2} = kL$ in arriving at this impedance function. The impedance is always reactive. It is an inductive reactance for $0 \leq \omega < \sqrt{\frac{1}{LC}}$. It is a capacitive reactance for

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$\sqrt{\frac{1}{LC}} < \omega < \sqrt{\frac{1}{LC(1-k^2)}}$. It is an inductive reactance again for $\omega < \sqrt{\frac{1}{LC(1-k^2)}}$. It becomes zero at $\omega = \sqrt{\frac{1}{LC(1-k^2)}}$. It changes from an infinite valued inductive reactance to infinite valued capacitive reactance as ω crosses $\sqrt{\frac{1}{LC}}$ from left to right.

The plot of this reactance function is shown in Fig. 14.5-6. Also shown in the same figure in dotted curve is the magnitude of capacitive reactance of the capacitor connected in series with the primary. $L = 115\mu\text{H}$, $C = 200\text{pF}$ and $k = 0.5$ for the circuit for which these reactance curves were prepared. The resonant frequency of L and C is 1 MHz.

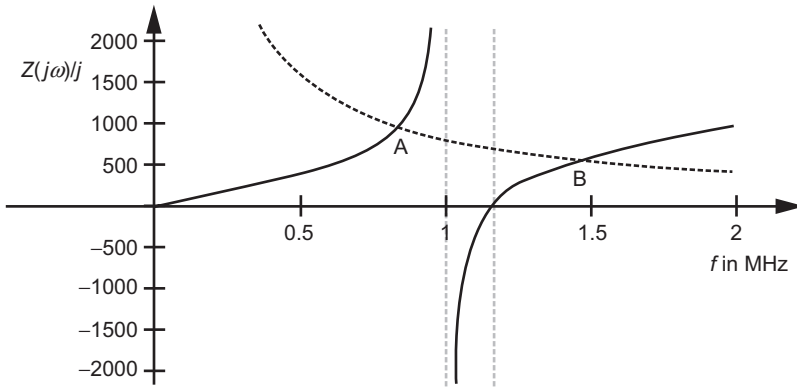


Fig. 14.5-6 Input reactance in a double-tuned circuit with no load resistance

The input impedance of the circuit in Fig. 14.5-5 (a) is the sum of solid curve and negative of the dotted curve in Fig. 14.5-6. This sum will go zero at frequencies corresponding to A and B in Fig. 14.5-6. The corresponding frequencies are 0.835 MHz and 1.45 MHz. The circuit is resonant at these two frequencies.

When the load resistance R is connected in the secondary, the resonant frequencies will shift. However, the shift will be small if $R \ll \sqrt{\frac{L}{C}}$.

The circuit in Fig. 14.5-4 can be analysed to obtain the steady-state frequency response function $H(j\omega)$ relating the output voltage to input voltage. It is possible to show that the frequency response function between load voltage and source voltage is

$$H(j\omega) = k \frac{-j\omega^3(LC)(RC)}{[1 - 2\omega^2LC + R^2C^2] + \omega^4(LC)^2(1 - k^2) - 2j\omega(RC)(1 - \omega^2LC)}$$

The term within the square brackets in the denominator goes to zero at $\omega_1 \approx \sqrt{\frac{1-k}{LC}}$ and $\omega_2 \approx \sqrt{\frac{1+k}{LC}}$ if $R \ll \sqrt{\frac{L}{C}}$. The gain magnitude of $H(j\omega)$ will exhibit a peak value of 0.5 at these two frequencies.

The plot of magnitude of frequency response function between source voltage and output voltage for single-tuned filter and double-tuned filter is shown in Fig. 14.5-7. The source resistance was 700Ω and load resistance was 700Ω in both cases. Desired centre frequency is 1.05 MHz and desired bandwidth

is 165 kHz. The single-tuned design makes use of a 1:1 transformer with $k \approx 1$, primary and secondary inductance of $8 \mu\text{H}$ each and a 2.8nF tuning capacitor connected across primary. Secondary is loaded with a 700Ω resistor. Gain magnitude for this filter is shown in dotted curve in Fig. 14.5-7. The double-tuned filter uses a 1:1 transformer with $k = 0.11$, primary and secondary inductance of 1.045mH and two capacitors of value 22pF each in series with the windings. A 700Ω in series with the capacitor and secondary winding load the filter. Gain magnitude for this filter is shown in solid curve in Fig. 14.5-7.

Two resonant peaks due to double tuning can be seen clearly in the frequency response plot. Note that while keeping the centre frequency and bandwidth the same as in single-tuned filter, pass-band gain variation in double-tuned filter is lesser than in the case of single-tuned filter. The skirts on either side of pass-band are steeper. The stop-band performance of double-tuned filter too is superior to that of single-tuned filter.

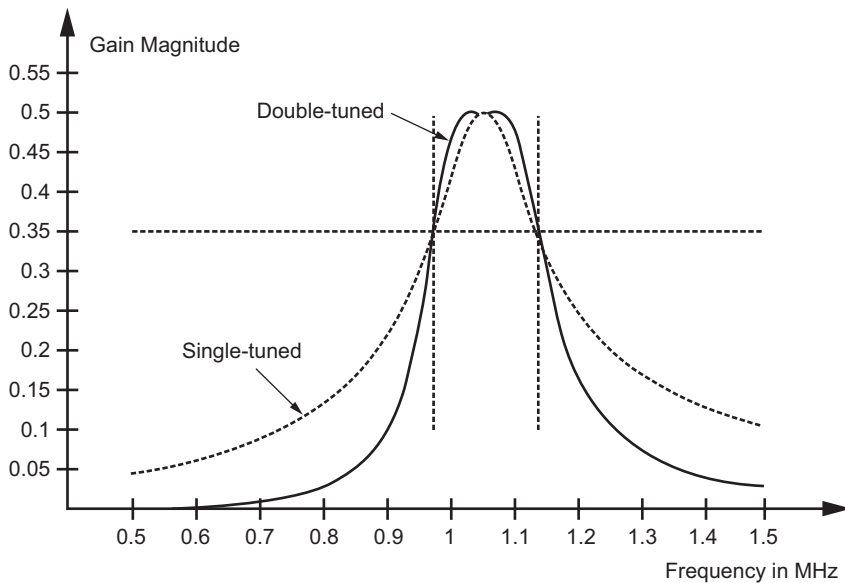


Fig. 14.5-7 Magnitude of frequency response function for single-tuned and double-tuned band-pass filters

14.6 ANALYSIS OF COUPLED COILS USING LAPLACE TRANSFORMS

In the first part of this section, we show that (i) the input impedance function of a *passively terminated* two-winding transformer is independent of relative polarity of windings, (ii) the secondary voltage of a transformer with unity coupling coefficient ($k = 1$) and zero winding resistance is turns ratio times primary voltage, quite independent of the waveshape of input, (iii) a transformer with $k = 1$ reflects an impedance $Z(s)$ connected in the secondary side to the primary side as $Z(s)/n^2$ where n is the ratio of secondary turns to primary turns and (iv) a practical two-winding transformer with finite inductance values, imperfect coupling ($k < 1$) and non-zero winding resistance will be a band-pass system with a lower cut-off frequency decided by winding resistance and an upper cut-off frequency decided by coupling coefficient.

In the second part of this section, we take up the issue of instantaneous changes in currents in a coupled-coil system and show that such changes are possible in inductors involved in a coupled-coil

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system under certain specific circuit conditions. We show that a coil with zero winding resistance kept shorted will *expel magnetic flux* in the common magnetic path. This is the principle that works behind *electromagnetic shielding* of electrical and electronic equipment.

In the third part of this section, we look briefly at a practical problem involving coupled-coils – that of breaking the primary side circuit while it is carrying current. This turns out to be a stressful operation for the load connected at the secondary side and specific design measures are incorporated in the load circuit to protect it against failure under this condition.

14.6.1 Input Impedance Function of a Two-Winding Transformer

Consider a two-winding transformer with *zero winding resistance*. Let the self-inductance of primary and secondary windings be L_p and L_s . Let n be the ratio of secondary turns to primary turns. Let the secondary be terminated in a load circuit that has an input impedance of $Z(s)$. See Fig 14.6-1.

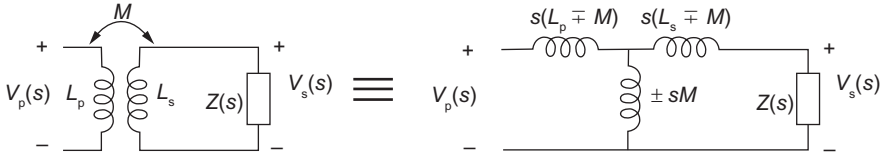


Fig. 14.6-1 A passively terminated two-winding transformer (winding resistance negligible)

We do not specify the relative polarity. Instead, we cover both options by using proper signs for the inductance values in the equivalent circuit shown.

Now, the primary-side input impedance function $Z_p(s)$ can be derived as shown below.

$$\begin{aligned}
 Z_p(s) &= s(L_p \mp M) + \frac{[s(L_s \mp M) + Z(s)] \times [\pm sM]}{s(L_s \mp M) + Z(s) \pm sM} \\
 &= s(L_p \mp M) + \frac{[s(L_s \mp M) + Z(s)] \times [\pm sM]}{sL_s + Z(s)} \\
 &= \frac{s(L_p \mp M) \times sL_s + s(L_p \mp M) \times Z(s) \pm s(L_s \mp M) \times sM \pm sMZ(s)}{sL_s + Z(s)} \\
 &= \frac{s^2(L_p L_s - M^2) + sL_p Z(s)}{sL_s + Z(s)}
 \end{aligned}$$

We observe that this impedance function is independent of relative polarity of windings.

Now, assume that the coupling coefficient is unity. Thus, for a perfectly coupled two-winding transformer with zero winding resistance, the input impedance function is given by,

$$\begin{aligned}
 Z_p(s) &= \frac{sL_p Z(s)}{sL_s + Z(s)} = \frac{sL_p Z(s)}{s(n^2 L_p) + Z(s)} \quad (\because \text{self-inductance} \propto \text{square of number of turns}) \\
 &= \frac{sL_p [Z(s) / n^2]}{sL_p + [Z(s) / n^2]} = \text{Parallel combination of } L_1 \text{ and } Z(s)
 \end{aligned}$$

This impedance is shown in Fig. 14.6-2.

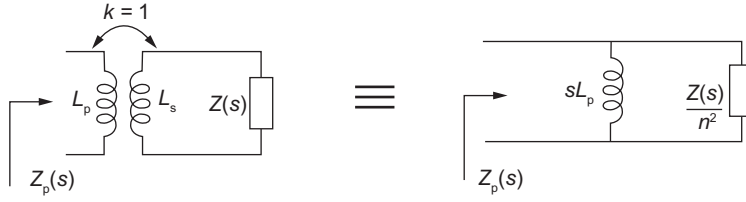


Fig. 14.6-2 Reflection of secondary impedance function to primary with $k = 1$

Now, we can incorporate the primary winding resistance r_p as extra impedance in series at the input and the secondary winding resistance r_s as a part of the load.

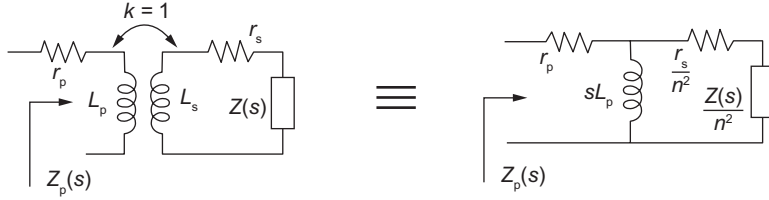


Fig. 14.6-3 Primary-side input impedance in a perfectly coupled transformer

A perfectly coupled transformer is called an ideal transformer if L_p , L_s and M have infinitely high values. Practical iron-cored transformers can be modeled approximately by an ideal transformer. Such an ideal transformer will have input impedance that is a turns-ratio ratio transformed version of the secondary load impedance. We have discussed the application of such transformers in impedance matching applications earlier in this chapter.

14.6.2 Transfer Function of a Two-Winding Transformer

Now, we look at the voltage transfer function of a two-winding transformer. Let $I_p(s)$ and $I_s(s)$ be the Laplace transforms of the primary and secondary mesh currents in the s -domain equivalent circuit shown in Fig. 14.6-1. Then, the mesh equations in matrix form is

$$\begin{bmatrix} sL_p & \mp sM \\ \mp sM & sL_s + Z(s) \end{bmatrix} \begin{bmatrix} I_p(s) \\ I_s(s) \end{bmatrix} = \begin{bmatrix} V_p(s) \\ 0 \end{bmatrix}$$

$$\text{Solving for } I_s(s), \quad I_s(s) = \frac{\pm sMV_p(s)}{s^2(L_pL_s - M^2) + sL_pZ(s)}.$$

$$\text{Since } V_s(s) = Z(s) \times I_s(s),$$

$$\frac{V_s(s)}{V_p(s)} = \frac{\pm sMZ(s)}{s^2(L_pL_s - M^2) + sL_pZ(s)} = \frac{\pm sMZ(s)}{s(L_pL_s - M^2) + L_pZ(s)} \text{ for } r_p = r_s = 0 \quad (14.6-1)$$

$L_pL_s = M^2$ for a perfectly coupled transformer (i.e., $k = 1$). Therefore, for a perfectly coupled transformer with $r_p = r_s = 0$, the voltage transfer function is given by

$$\frac{V_s(s)}{V_p(s)} = \frac{\pm sMZ(s)}{sL_pZ(s)} = \pm \frac{M}{L_p} = \pm \frac{\sqrt{L_pL_s}}{L_p} = \pm \sqrt{\frac{L_s}{L_p}} \pm n \text{ for } k = 1, r_p = r_s = 0 \quad (14.6-2)$$

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Eqn. 14.6-2 indicates that there is a polarity inversion in secondary voltage with respect to primary voltage for one relative polarity in windings.

Various factors prevent a practical transformer from meeting this ideal. Winding resistance and imperfect coupling are two such factors. If the mesh equations are written again with r_p and r_s included, the transfer function can be shown as

$$H(s) = \frac{V_s(s)}{V_p(s)} = \frac{\pm sMZ(s)}{s^2(L_pL_s - M^2) + s(L_pZ(s) + r_sL_p + r_pL_s) + r_p(r_s + Z(s))} \quad (14.6-3)$$

for a special load – resistive load. Let $Z(s) = R$. Then,

$$H(s) = \frac{V_s(s)}{V_p(s)} = \frac{\pm sMR}{s^2(L_pL_s - M^2) + s(L_pR + r_sL_p + r_pL_s) + r_p(r_s + R)} \quad (14.6-4)$$

This is a second order band-pass function (note the s term in the numerator polynomial).

We express this transfer function as the product of the ideally expected value ($= \pm n$) and a factor involving s as follows.

$$H(s) = \frac{V_s(s)}{V_p(s)} = \frac{\pm M}{L_p} \times \frac{s}{s^2 \left(\frac{(1-k^2)L_s}{R} \right) + s \left(1 + \frac{r_s}{R} + \frac{r_p}{R} \frac{L_s}{L_p} \right) + \frac{r_p}{L_p} \left(\frac{r_s}{R} + 1 \right)} \quad (14.6-5)$$

$$M = k\sqrt{L_pL_s}. \text{ Therefore, } \frac{M}{L_p} = k\sqrt{\frac{L_s}{L_p}} = nk. \text{ Therefore,}$$

$$H(s) = \pm n \times k \times \frac{s}{s^2 \left(\frac{(1-k^2)L_s}{R} \right) + s \left(1 + \frac{r_s}{R} + \frac{r_p}{R} \frac{L_s}{L_p} \right) + \frac{r_p}{L_p} \left(\frac{r_s}{R} + 1 \right)} \quad (14.6-6)$$

We want to determine the low-frequency cut-off and high-frequency cut-off for the sinusoidal steady-state frequency response function of this transfer function. In general, these two frequencies will be some complex functions of k , n , r_s , r_p , R and L_p . However, we can use some approximations in the case of a practically important iron-cored transformer.

In an iron-cored transformer, the coupling coefficient is nearly unity. The load resistance is usually much larger than the transformer's winding resistance. Hence, the effect of leakage flux and the effect of winding resistance are only second-order effects. The aggregate result of two second-order effects can be taken as the product of effects when each is acting alone. Therefore, when we consider the effect of winding resistance we assume that $k = 1$ and when we consider the effect of imperfect coupling we assume that $r_p = r_s = 0$.

$$H(s) (\text{with } k = 1) = \pm n \frac{s}{s \left(1 + \frac{r_s}{R} + \frac{r_p}{R} \frac{L_s}{L_p} \right) + \frac{r_p}{L_p} \left(\frac{r_s}{R} + 1 \right)}$$

Further algebraic manipulation of the above expression leads to

$$H(s) = n \times \frac{1}{1 + \frac{r_s}{R} + n^2 \frac{r_p}{R}} \times \frac{s}{r_p \left(1 + \frac{r_s}{R}\right) s + \frac{L_p \left(1 + \frac{r_s}{R} + n^2 \frac{r_p}{R}\right)}{s}} = n \times \beta \times \frac{s}{s + \alpha} \quad (14.6-7)$$

β is a voltage reduction factor due to part of the applied voltage getting dropped across the winding resistance. $-\alpha$ is the pole of the first order transfer function. The transfer function is seen to be a first order high-pass function. The DC gain (evaluated by substituting $s = j0$) – i.e., the ratio between the steady-state response under DC conditions to the DC input value – is zero. The upper cut-off frequency is α rad/s where

$$\alpha = \frac{r_p \left(1 + \frac{r_s}{R}\right)}{L_p \left(1 + \frac{r_s}{R} + n^2 \frac{r_p}{R}\right)} \approx \frac{r_p}{L_p} \text{ for practical values of } R$$

We observe that both the voltage reduction factor β and the cut-off frequency α can be obtained from the equivalent circuit shown in Fig. 14.6-3 with $Z(s)$ replaced with R .

The winding resistances prevent a two-winding transformer from passing DC voltage and low frequency AC voltage from primary to secondary. Hence, a signal containing low-frequency content will suffer waveform distortion when it goes through a transformer.

The high-frequency behaviour of a tightly coupled transformer is studied after neglecting the winding resistances. Then, Eqn. 14.6-6 simplifies to

$$H(s) = \pm n \times k \times \frac{1}{s \left(\frac{(1-k^2)L_s}{R} \right) + 1} = \pm n \times k \times \frac{\frac{R}{(1-k^2)L_s}}{s + \frac{R}{(1-k^2)L_s}} = \pm n \times k \times \frac{\lambda}{s + \lambda} \quad (14.6-8)$$

This is a low-pass transfer function with a cut-off frequency of $\lambda = \frac{R}{(1-k^2)L_s} = \frac{R/n^2}{(1-k^2)L_p}$ rad/s.

Imperfect magnetic coupling prevents a two-winding transformer from passing high frequency AC voltage from primary to secondary. The two-winding transformer exhibits low-pass behaviour in the high-frequency end due to imperfect magnetic coupling. Therefore, a signal containing high-frequency content will suffer waveform distortion when it goes through a transformer.

The product of right sides of Eqn. 14.6-7 and Eqn. 14.6-8 provides an approximation for Eqn. 14.6-6 in the case of a tightly coupled two-winding transformer with winding resistances very small compared to load resistance. Thus,

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$$H(s) = \text{ideal gain expected} \times \begin{matrix} \pm n \\ \text{voltage-drop factor} \\ \text{due to imperfect} \\ \text{magnetic coupling} \end{matrix} \times \begin{matrix} k \\ \text{voltage-drop factor} \\ \text{due to winding} \\ \text{resistances} \end{matrix} \times \begin{matrix} \beta \\ \text{high-pass} \\ \text{factor} \end{matrix} \times \begin{matrix} \frac{s}{s+\alpha} \\ \text{low-pass} \\ \text{factor} \end{matrix} \times \begin{matrix} \frac{\lambda}{s+\lambda} \end{matrix} \quad (14.6-9)$$

This transfer function has a band-pass frequency response function with lower cut-off frequency of α and upper cut-off frequency of λ . Primary winding resistance and primary winding inductance decide the lower cut-off frequency. The upper cut-off frequency is decided by coupling coefficient, primary winding inductance and load resistance.

The bandwidth of a transformer can be increased by reducing the winding resistances and increasing the coupling coefficient. Bandwidth of a transformer is load dependent.

14.7 FLUX EXPULSION BY A SHORTED COIL

One does not make a good transformer and keep its secondary shorted! However, there are situations and applications in which an electrical system can be modeled as a transformer with a low resistance secondary winding kept shorted. Therefore, we look at the input impedance of a transformer with the secondary winding shorted as in Fig.14.7-1.

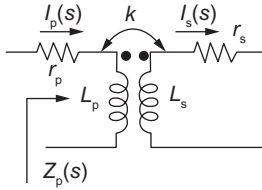


Fig. 14.7-1 A transformer with secondary shorted

$Z_p(s)$ may be determined by mesh analysis. The result will be,

$$Z_p(s) = \frac{s^2 L_p L_s (1 - k^2) + s(L_p r_s + L_s r_p) + r_p r_s}{r_s + s L_s} \quad (14.6-10)$$

The impedance at DC frequency is $r_p \Omega$.

The impedance at high frequency is $\approx s(1-k^2)L_p$, indicating that the transformer behaves as an inductance of $(1-k^2)L_p$ H at high frequencies when its secondary is kept shorted.

Assume that the shorted coil has negligibly small resistance. Taking $r_s = 0$, we get,

$$Z_p(s) = r_p + s(1 - k^2)L_p.$$

Now, if the transformer is perfectly coupled (*i.e.*, $k = 1$), then the input impedance becomes equal to primary winding resistance at all frequencies. Hence, the short-circuit impedance of a good transformer with $k = 1$, $r_p = r_s = 0$ is zero at all frequencies.

The expressions for $I_p(s)$ and $I_s(s)$ with $r_2 = 0$ may be derived as

$$I_p(s) = \frac{V_p(s)}{s(1 - k^2)L_p + r_p} \text{ and } I_s(s) = \frac{\frac{M}{L_s} V_p(s)}{s(1 - k^2)L_p + r_p} = \frac{k}{n} I_p(s)$$

Currents entering the dots generate positive flux linkages in coils. Let ψ_p be the flux linkage in primary coil and ψ_s be the flux linkage in the secondary coil. Then,

$$\begin{aligned}
\psi_p(s) &= L_p I_p(s) - M I_s(s) \\
&= L_p I_p(s) - \frac{k}{n} M I_p(s) \\
&= L_p I_p(s) - \frac{k^2}{n} \sqrt{L_p L_s} I_p(s) \\
&= L_p I_p(s) [1 - k^2] \\
\psi_2(s) &= -L_s I_s(s) + M I_p(s) = \frac{-M V_p(s)}{s(1 - k^2) L_p + r_p} + \frac{M V_p(s)}{s(1 - k^2) L_p + r_p} = 0
\end{aligned}$$

Therefore, the secondary coil that started with zero flux linkage continues to hold zero flux linkage at all time after that. Thus,

A shorted coil with zero resistance expels flux completely at all frequencies in the magnetic path where it is located.

It does this by allowing suitable current to flow in it in order to cancel the mutual flux generated by current in the driven coil. Both primary and secondary currents rise exponentially with a time constant of $\frac{(1 - k^2) L_p}{r_p}$ s in step response.

But what if the windings are perfectly coupled in addition? Then,

$$I_p(s) = \frac{V_p(s)}{r_p} \text{ and } I_s(s) = \frac{k}{n} \frac{V_p(s)}{r_p}$$

If $v_p(t)$ is a unit step function, the current *rises instantaneously from 0 to $1/r_p$ in the primary winding and from 0 to k/nr_p in the secondary winding*. Further, the flux linkages in both windings continue to be at zero.

There was no impulse voltage in the circuit. How was it possible for the inductor currents to change instantaneously without impulse voltages in the circuit?

Currents in individual coils can change instantaneously in a coupled-coil system. Faraday's law indicates that the induced emf across a coil is proportional to the rate of change of flux linkage in it. Therefore, flux linkage of a coil cannot change instantaneously unless impulse voltage is applied to the coil. This reduces to the statement that current in an inductor cannot change instantaneously unless impulse voltage is applied in the case of inductors that are not magnetically coupled. However, there is a distinction between instantaneous change of flux linkage and current if the flux linkage in a coil can be produced by more than one current as in the case of coupled inductors. Therefore, the more general principle of 'no instantaneous change in flux linkage in a coil unless impulse voltage is applied' should be applied in the case of coupled-coils.

The flux linkage in a shorted coil with zero resistance remains constant in time. This statement is sometimes referred to as 'Constant Flux Linkage Theorem'.

Note that in the step response of a perfectly coupled transformer with shorted secondary ($r_s = 0$), the flux linkage of secondary winding remained at zero though the current increased instantaneously.

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The magnetic coupling was perfect, and the primary cannot have flux linkage in it in that case unless secondary has it. Therefore, the primary winding flux linkage also remained at zero despite step change in primary current.

We may derive a relationship between the instantaneous changes in primary and secondary windings as follows:

$$\psi_p(t) = L_p i_p(t) - M i_s(t)$$

$$\psi_s(t) = -L_s i_s(t) + M i_p(t)$$

Therefore

$$\Delta \psi_p(t) = L_p \Delta i_p(t) - M \Delta i_s(t)$$

$$\Delta \psi_s(t) = -L_s \Delta i_s(t) - M \Delta i_p(t)$$

But, there cannot be instantaneous changes in flux linkages in the absence of impulse voltages. Hence, $\Delta \psi_p(t)$ and $\Delta \psi_s(t)$ are zero. Therefore,

$$\begin{bmatrix} L_p & -M \\ M & -L_s \end{bmatrix} \begin{bmatrix} \Delta i_p \\ \Delta i_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant of the matrix on the left side is nonzero for all $k < 1$. Therefore, Δi_p and Δi_s can only be zero with imperfect coupling. That is, there can be no instantaneous change in coil currents in a two-coil system if the coils are imperfectly coupled and there is no impulse voltage applied or supported somewhere in the system.

However, if $k = 1$, the determinant of the square matrix in the left side of the equation is zero and there can be a non-zero solution for Δi_p and Δi_s . In fact, any pair of values that satisfy the constraint that $\Delta i_s = \Delta i_p/n$ will be permitted. The exact value by which the primary current jumps will be decided by the jump in primary voltage and the primary resistance. Since the flux linkage in primary winding does not change, all the primary voltage will have to be absorbed by the primary resistance at all t .

Its resistance compromises the effectiveness of flux expulsion from shorted coil. The expression for flux linkage in shorted coil when $k \neq 1$ and $r_s \neq 0$ is

$$\frac{\psi_s(s)}{V_p(s)} = \frac{Mr_s}{s^2(1-k^2)L_pL_s + s(L_pr_s + L_sr_p) + r_pr_s}$$

DC value of this ratio is M/r_p . Therefore, DC flux will not be expelled under steady state. Similarly, low frequency AC flux will also manage to get into the shorted coil under steady state conditions. However, the ratio goes to low values at high frequency. Therefore non-zero resistance in shorted coil results in DC and low-frequency fluxes penetrating into the coil. High-frequency flux is expelled more or less effectively.

The principle of flux expulsion detailed in this section is employed in *shielding* sensitive electronic equipment from electromagnetic interference.

14.8 BREAKING THE PRIMARY CURRENT IN A TRANSFORMER

Let I_p and I_s be the currents in primary winding and secondary winding of the transformer at the instant at which the switch S is opened in the circuit shown in Fig. 14.8-1.

Let us denote the instant of switching as $t = 0$. Then, the flux linkages in the windings just before switching are

$$\psi_p(0^-) = L_p I_p - M I_s$$

$$\psi_s(0^-) = -L_s I_s + M I_p$$

and the flux linkages at $t = 0^+$ are

$$\psi_p(0^+) = L_p 0 - M i_s(0^+) = -M i_s(0^+)$$

$$\psi_s(0^+) = -L_s i_s(0^+) + M 0 = -L_s i_s(0^+)$$

Since there is no impulse in the closed secondary loop, the flux linkage in the secondary winding will have to be continuous. Therefore,

$$\psi_s(0^+) = \psi_s(0^-) \Rightarrow -L_s I_s + M I_p = -L_s i_s(0^+)$$

$$\text{Therefore, } i_s(0^+) = I_s - \frac{M}{L_s} I_p = I_s - \frac{I_p}{n}$$

This current is usually in a direction opposite to that of $i_s(0^-)$.

The corresponding flux linkage in primary winding at $t = 0^+$ is $\psi_p(0^+) = -M i_s(0^+) = -M I_s + \frac{M^2}{L_s} I_p$.

The initial flux linkage in this coil was $\psi_p(0^-) = L_p I_p - M I_s$. Therefore, the instantaneous decrease in flux linkage that took place in this coil is $(L_p I_p - M I_s) - \left(-M I_s + \frac{M^2}{L_s} I_p \right) = I_p \left(L_p - \frac{M^2}{L_s} \right) = (1 - k^2) L_p I_p$.

Hence, an impulse voltage of area content equal to $(1 - k^2) L_p I_p$ V-s will appear across the primary winding and will have to be supported by the switch. In fact, there will be arcing across the switch due to this very large voltage trying to establish across it.

However, arcing involves energy. Where does the energy come from? It is possible to show that the energy that gets dissipated across the switch is $0.5(1 - k^2) L_p I_p^2$ J. This comes from the magnetic energy storage in the coupled-coil system.

The sudden change of secondary current from I_s to $I_s - \frac{I_p}{n}$ results in a sudden change across the load voltage from $R I_s$ to $R I_s - \frac{R I_p}{n}$. There are practical applications in which the second term dominates and makes the load voltage a negative voltage with enough magnitude to destroy the load if the load happens to be a sensitive electronic equipment.

This large negative voltage decays exponentially with a time constant of L_s/R s. By the time the transient is over, the remaining initial magnetic energy stored in the coil system would have got dissipated in R .

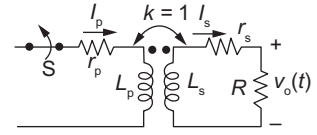


Fig. 14.8-1 Breaking the primary current in a transformer

14.9 SUMMARY

- Self-inductance of coil is the flux linkage in it when 1A flows in it with all other coils in the vicinity kept open. Mutual inductance between two coils is the flux linkage in one coil when it is kept open with 1A flowing in the other coil.

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- For two coupled coils with self-inductance L_1 and L_2 , the mutual inductances M_{12} and M_{21} are equal. The maximum value mutual inductance can have is $\sqrt{L_1 L_2}$ H. The ratio between actual value of mutual inductance and its maximum possible value is defined as the *magnetic coupling coefficient*.
- A system of two coils with constant values of L_1 , L_2 and M , with two pairs of terminals identified for application of excitation and/or measurement of response, is called a *two-winding linear transformer*.
- Secondary voltage magnitude in a transformer with unity coupling coefficient is turns ratio times the primary voltage magnitude. A transformer with unity coupling coefficient *reflects* the secondary load impedance Z_L onto the primary side as a *turns-ratio transformed impedance* of value Z_L/n^2 where n = secondary turns/primary turns.
- An *ideal transformer* is a two-winding transformer with perfect coupling ($k = 1$) and infinite self- and mutual inductance values ($L_1 \rightarrow \infty$, $L_2 \rightarrow \infty$, $M \rightarrow \infty$). The input impedance of such a transformer is Z_L/n^2 . Complex power is conserved in an ideal transformer. Practical transformers that approach ideal transformer are employed in voltage/current level conversion and impedance matching applications.
- The input impedance function of a *passively terminated* two-winding transformer is independent of relative polarity of windings.
- The secondary voltage of a transformer with unity coupling coefficient ($k = 1$) and zero winding resistance is turns ratio times primary voltage, quite independent of the waveshape of input.
- A transformer with $k = 1$ reflects an impedance $Z(s)$ connected in the secondary side to the primary side as $Z(s)/n^2$ where n is the ratio of secondary turns to primary turns, quite independent of the waveshape of input.
- A practical two-winding transformer with finite inductance values, imperfect coupling ($k < 1$) and non-zero winding resistance will be a band-pass system with a lower cut-off frequency decided by winding resistance and upper cut-off frequency decided by coupling coefficient.
- The flux linkage in a shorted coil with zero resistance remains constant in time. If the shorted coil has non-zero resistance, DC and low-frequency flux will penetrate into the coil. However, high-frequency flux will be expelled from the coil.
- There can be instantaneous change in coil currents in a coupled-coil system even in a circuit that does not apply or support impulse voltage if the coils are perfectly coupled. However, there cannot be an instantaneous change in flux linkage of any coil, whether coupled or uncoupled, unless the circuit applies or supports impulse voltage.

14.10 PROBLEMS

1. A current source with $i_s(t) = 10t$ A for $0 \leq t \leq 1$ s and zero for all other t is connected to one of the coils of a two-coil system with the second coil kept open. The voltage across current source is seen to be a rectangular pulse of amplitude 10 V and duration 1 s. The voltage across the other coil is seen to be a rectangular pulse of amplitude 7 V and duration 1 s. Find the self-inductance of the first coil and mutual inductance between the coils.
2. A two-coil system with self-inductance values of 1H and 4H carries 2A in 1H coil and 1A in 4H coil. The total stored energy in the system is seen to be zero under this condition. Find the mutual inductance between the coils and coupling coefficient between them.
3. A two-coil system with $L_1 = 100$ mH, $L_2 = 300$ mH and $k = 0.8$ has $i_1(t) = 1.2 \sin(100\pi t)$ A and $i_2(t) = 0.5 \cos(100\pi t)$ A. Assuming the self-flux linkage and mutual flux linkage aid each other, (a) find time-domain expressions for flux linkage in coil-1 and coil-2 and (b) find time-domain expressions for voltage across both coils. (c) Repeat (a) and (b) assuming that the self-induced emf and mutually induced emf oppose each other in the coils.

4. A sinusoidal source of $200\sqrt{2} \sin 100t$ V is applied at terminals A and B in the circuit in Fig. 14-10.1 with terminals C and D kept open. The coils are lossless. The rms reading meters read $A_1 = 1$ A, $V_1 = 200$ V, $V_2 = 100$ V, and $V_3 = 100$ V. When the same source is connected across C and D with terminals A and B kept open, the ammeter A_2 is seen to read 0.5 A rms. Find the position of the missing dot, L_1 , L_2 , M and k .

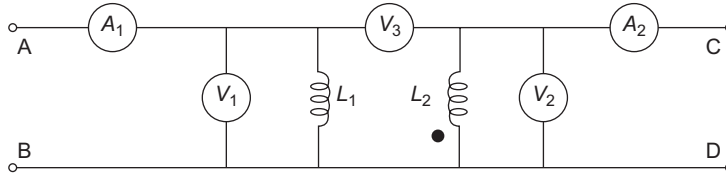


Fig. 14-10.1

5. A sinusoidal source of $200\sqrt{2} \sin 1000t$ V is applied at terminals A and B in the circuit in Fig. 14-10.2 with terminals C and D kept open. The coils are lossless. The rms reading meters read $V_1 = 200$ V, $V_2 = 100$ V, $V_3 = 100$ V, $V_4 = 200$ V and $V_5 = 100$ V. When the same source is connected across C and D with terminals A and B kept open, the voltmeters V_2 and V_4 are seen to read 100 V each. Obtain a dot polarity assignment for this coil system using three distinct symbols for coil 1-coil 2, coil 2-coil 3 and coil 1-coil 3 mutual coupling.

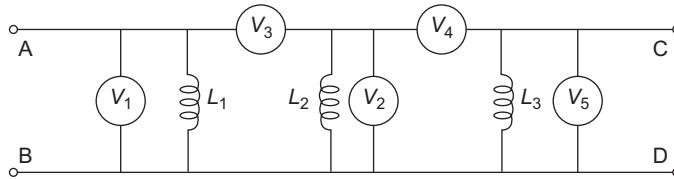


Fig. 14-10.2

6. Two coupled coils can be connected in series in two ways. The connection (a) in Fig. 14-10.3 is *differential series* connection and connection (b) is *cumulative series* connection. Find the equivalent inductance between A and B in both cases.

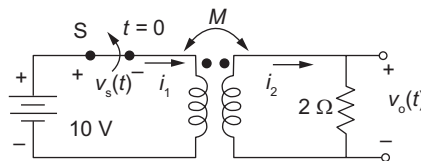


Fig. 14-10.3

7. Find the equivalent inductance between A and B in the two possible parallel connections of two coupled inductors in Fig. 14-10.4.

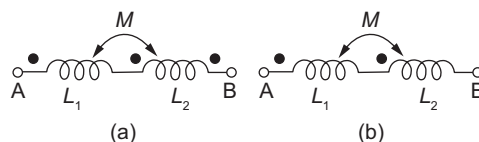


Fig. 14-10.4

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8. The complex power input into the primary side of a lossless transformer with perfect coupling is $100 + j100$ kVA at 230V rms and 50Hz. The load in the secondary side is seen to consume 90kVA of reactive power along with active power. (i) Find the complex power consumed by the load and its power factor. (ii) Find the primary winding self-inductance.
9. (i) What is the turns ratio and coupling coefficient of the transformer in the circuit in Fig. 14-10.5? (ii) Find the primary and secondary currents and voltage across $10\ \Omega$ resistor as functions of time if the source has 110 V rms value and 60 Hz frequency.

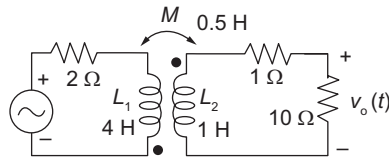


Fig. 14-10.5

10. Find all angular frequency values for which the input impedance of the circuit shown in Fig. 14-10.6 is purely resistive.

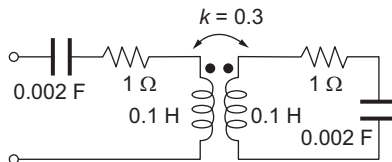


Fig. 14-10.6

11. Find the power dissipated in the resistors and complex power delivered by the source in the circuit in Fig. 14-10.7 if the voltage source has 110 V rms value at 60 Hz,

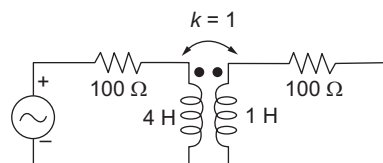


Fig. 14-10.7

12. What is the turns ratio n of the ideal transformer in the circuit in Fig. 14-10.8 if the reactive power delivered by the source (230 V rms at 50 Hz) is zero?

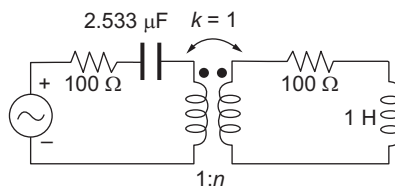


Fig. 14-10.8

13. What should the value of turns ratio n if maximum power is to be transferred to the $8\ \Omega$ load in the circuit in Fig. 14-10.9? If the source has 110 V rms value, what is the amount of power transferred to $8\ \Omega$ with this turns ratio?

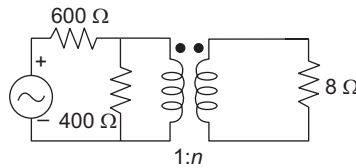


Fig. 14-10.9

14. The sinusoidal voltage source that is driving a load of impedance $16 - j10\ \Omega$ at 1 MHz has source impedance of $1 + j1\ \Omega$ at 1 MHz. An ideal transformer is introduced between the source and load such that the power transferred to load is a maximum. Find the turns ratio of the transformer and value and nature of additional components needed, if any.
15. A transformer used in a DC power supply has 100 turns in the primary and 25 turns in the secondary. The primary winding resistance is $0.1\ \Omega$ and secondary winding resistance is $0.15\ \Omega$. The step response of input current with secondary open is found to have a rise time of 0.22 s. The coupling coefficient is 0.98. (a) Find the self-inductance of windings and the mutual inductance between them. (b) Find the input admittance function and voltage transfer function when a resistive load of $2\ \Omega$ is connected across the secondary. (c) Prepare the pole-zero plot for the transfer function. (d) Sketch the frequency response plots for output voltage across the load resistance and estimate the cut-off frequencies and bandwidth.
16. Obtain the step response of output voltage for the transformer in the Problem 15 with a $2\ \Omega$ load in the secondary. What are the flux linkages in coils and total energy stored in the transformer under step response steady-state condition?
17. The transformer used in the circuit in Fig. 14-10.10 is the same as the one in Problem 15. Switch was in closed position for a long time prior to $t = 0$ and it is opened at $t = 0$. (i) Find and plot $i_2(t)$, $v_o(t)$ and $v_s(t)$ for $t \geq 0^+$ and (ii) Calculate the energy dissipated in the switch and the $2\ \Omega$ resistance after at $t = 0$.

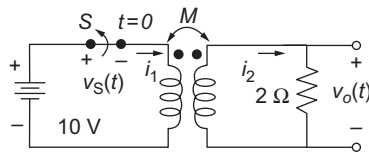


Fig. 14-10.10

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