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Tadeusz Kaczorek
Krzysztof Rogowski

Fractional Linear Systems and Electrical Circuits

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Tadeusz Kaczorek · Krzysztof Rogowski

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Tadeusz Kaczorek
Faculty of Electrical Engineering
Białystok University of Technology
Białystok
Poland

Krzysztof Rogowski
Faculty of Electrical Engineering
Białystok University of Technology
Białystok
Poland

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Preface

This monograph covers some selected problems of positive and fractional electrical circuits composed of resistors, coils, capacitors and voltage (current) sources.

The monograph consists of 8 chapters, 4 appendices and a list of references.

Chapter 1 is devoted to fractional standard and positive continuous-time and discrete-time linear systems without and with delays. The state equations and their solutions of linear continuous-time and discrete-time linear systems are presented. Necessary and sufficient conditions for the internal and external positivity of the linear systems are given. Solutions of the descriptor standard and fractional linear systems using the Weierstrass–Kronecker decomposition and Drazin inverse matrix method are also presented. In chapter 2 the standard and positive fractional electrical circuits are considered. The fractional electrical circuits in transient states are analyzed. The reciprocity theorem, equivalent voltage source theorem and equivalent current source theorem are presented. Descriptor linear electrical circuits and their properties are investigated in chapter 3. The Weierstrass–Kronecker decomposition method and the shuffle algorithm method are discussed. The regularity, pointwise completeness and pointwise degeneracy of descriptor electrical circuits is analyzed. The descriptor fractional standard and positive electrical circuits are also investigated. Chapter 4 is devoted to the stability of fractional standard and positive linear electrical circuits. It is shown that the electrical circuits with resistances can be also unstable. The reachability, observability and recontsructability of fractional positive electrical circuits and their decoupling zeros are analyzed in chapter 5. Necessary and sufficient conditions for the reachability, observability and reconstructability are established and illustrated by examples of electrical circuits. The decompositions of the pairs (A, B) and (A, C) of the electrical circuits are given and the decoupling zeros of the positive electrical circuits are proposed.

The fractional linear electrical circuits with feedbacks are considered in chapter 6. The zeroing of the state vector of electrical circuits by state and output-feedbacks is discussed. In chapter 7 the problem of minimum energy control for standard and fractional systems with and without bounded inputs has been solved. In chapter 8 the fractional continuous-time 2D linear systems described by the Roesser type models are investigated. The fractional derivatives and integrals of 2D functions are introduced. The descriptor fractional 2D model is proposed and its solution is derived. The standard fractional 2D Roesser type models are also investigated.

In Appendix A some basic definitions and theorems on Laplace transforms and $\mathcal{Z}$-transforms are given. The elementary row and column operations on matrices are recalled in Appendix B. In Appendix C some properties of the nilpotent matrices are given. Definition of Drazin inverse of matrices and its properties are presented in Appendix D.

The monograph contains some original results of the authors, most of which have already been published. It is dedicated to scientists and Ph.D. students from the field of electrical circuits theory and control systems theory.

We would like to express our gratitude to Professors Mikołaj Busłowicz, Krzysztof Latawiec and Wojciech Mitkowski for their invaluable remarks, comments and suggestions which helped to improve the monograph.

Białystok, June 2014

Tadeusz Kaczorek
Krzysztof Rogowski

Contents

List of Symbols

$\Gamma(x)$	the Euler gamma function
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of real nonnegative numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{R}^{n\times m}$	the set of $n \times m$ real matrices
$\mathbb{R}^n = \mathbb{R}^{n\times 1}$	the set of n-components real vectors
$\mathbb{R}_+^{n\times m}$	the set of $n \times m$ real matrices with nonnegative entries
$\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$	the set of n-rows real vectors with nonnegative components
$\mathbb{I}_n$	the $n \times n$ identity matrix
M_n	the set of $n \times n$ Metzler matrices
$a \in A$	a is an element of the set A
$\det A$	determinant of the matrix A
$\mathrm{rank} A$	rank of the matrix A
$\mathrm{Adj} A$	adjoint matrix
$\deg P(s)$	degree of polynomial (polynomial matrix) $P(s)$

A^{-1}	inverse matrix
A^D	Drazin inverse matrix
A^T	transpose matrix
$\mathrm{Im}A$	image of the matrix A
$\ker A$	kernel of the matrix A
$n!$	the factorial of natural number n
$\mathrm{diag}[a_1, \ldots, a_n]$	diagonal matrix with $a_1, \ldots, a_n$ on diagonal
$\Re(x)$	real part of complex number x
$\Im(x)$	imaginary part of complex number x
$E_\alpha(z)$	one parameter Mittag-Leffler function of z
$E_{\alpha,\beta}(z)$	two parameters Mittag-Leffler function of z
${}_0I_t^n f(t)$	n-multiple integral of the function $f(t)$ on the interval $(0, t)$
${}_0I_t^\alpha f(t)$	Riemann-Liouville fractional (α-order) integral
${}_0^{RL}D_t^\alpha f(t)$	Riemmann-Liouville fractional (α-order) derivative-integral
${}_0^C D_t^\alpha f(t)$	Caputo fractional (α-order) derivative-integral
${}_0\Delta_k^n x_i$	n-order backward difference of x_i on the interval $[0, k]$
$\mathcal{L}\left[f(t)\right] = F(s)$	Laplace transform of the function $f(t)$
$\mathcal{Z}\left[f_k\right] = F(z)$	$\mathcal{Z}$-transform of the function f_k
$\dot{f}(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t}$	first order derivative of continuous-time function $f(t)$
$f^{(n)}(t) = \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$	n-order derivative of continuous-time function $f(t)$
$f(t) * g(t)$	the convolution of functions $f(t)$ and $g(t)$
$\delta(t)$	Dirac impulse function
$\mathbb{1}(t)$	unit step function

Chapter 1
Fractional Differential Equations

1.1 Definition of Euler Gamma Function and Its Properties

There exist the following two definitions of the Euler gamma function.

Definition 1.1. A function given by the integral [51, 154, 163]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \quad \Re(x) > 0 \tag{1.1}$$

is called the Euler gamma function.

The Euler gamma function can be also defined by

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}, \quad x \in \mathbb{C}\backslash\{0, -1, -2, \ldots\},$$

where $\mathbb{C}$ is the field of complex numbers.

We shall show that $\Gamma(x)$ satisfies the equality

$$\Gamma(x+1) = x\Gamma(x). \tag{1.2}$$

Proof. Using (1.1), we obtain

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} \mathrm{d}t = -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^{x-1} e^{-t} \mathrm{d}t = x\Gamma(x).$$

□

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Example 1.1. From (1.2) we have for:

$$x = 1: \qquad \Gamma(2) = 1 \cdot \Gamma(1) = 1, \quad \text{since} \quad \Gamma(1) = \int_0^\infty e^{-t} \mathrm{d}t = 1,$$
$$x = 2: \qquad \Gamma(3) = 2 \cdot \Gamma(2) = 1 \cdot 2 = 2!,$$
$$x = 3: \qquad \Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot \Gamma(2) = 3!.$$

In general case, for $x \in \mathbb{N}$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots(1) = n!.$$

The gamma function is also well-defined for x being any real (complex) numbers. For example we have for:

$$x = 1.5: \qquad \Gamma(2.5) = 1.5 \cdot \Gamma(1.5) = 1.5 \cdot 0.5\Gamma(0.5),$$
$$x = -0.5: \qquad \Gamma(0.5) = -0.5 \cdot \Gamma(-0.5) = -0.5 \cdot (-1.5)\Gamma(-1.5).$$

1.2 Mittag-Leffler Function

The Mittag-Leffler function is a generalization of the exponential function $e^{s_i t}$ and it plays important role in solution of the fractional differential equations.

Definition 1.2. A function of the complex variable z defined by [51, 154, 163]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \tag{1.3}$$

is called the one parameter Mittag-Leffler function.

Example 1.2. For $\alpha = 1$ we obtain

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

i.e. the classical exponential function.

An extension of the one parameter Mittag-Leffler function is the following two parameters function.

Definition 1.3. A function of the complex variable z defined by [51, 154, 163]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \tag{1.4}$$

is called the two parameters Mittag-Leffler function.

For $\beta = 1$ from (1.4) we obtain (1.3).

1.3 Definitions of Fractional Derivative-Integral

1.3.1 Riemann-Liouville Definition

It is well known that to reduce N-multiple integral to 1-tiple integral the following formula

$$
\begin{aligned}
{}_0I_x^N f(u) &= \int_0^x \int_0^{u_1} \cdots \int_0^{u_{N-1}} f(u_N)\mathrm{d}u_N \mathrm{d}u_{N-1} \cdots \mathrm{d}u_1 \\
&= \frac{1}{(N-1)!} \int_0^x (x-u)^{N-1} f(u)\mathrm{d}u
\end{aligned} \tag{1.5}
$$

can be used, where $f(u)$ is a given function.

Using the equality $(N-1)! = \Gamma(N)$, the formula (1.5) can be extended for any $N \in \mathbb{R}$ and we obtain Riemann-Liouville fractional integral

$$
{}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \tag{1.6}
$$

where $\alpha \in \mathbb{R}_+ \backslash \{0\}$ is the order of integral.

Definition 1.4. The function defined by [51, 154, 163]

$$
\begin{aligned}
{}^{RL}_{0}D_t^\alpha f(t) &= \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) = \frac{\mathrm{d}^N}{\mathrm{d}t^N}\left[{}_0I_t^{(N-\alpha)} f(t)\right] \\
&= \frac{1}{\Gamma(N-\alpha)} \frac{\mathrm{d}^N}{\mathrm{d}t^N} \int_0^t (t-\tau)^{N-\alpha-1} f(\tau)\mathrm{d}\tau,
\end{aligned} \tag{1.7}
$$

where $N-1 \le \alpha < N$, $N \in \mathbb{N}$ is called Riemann-Liouville fractional derivative-integral.

Note, that from (1.7), for $\alpha = 0$ we obtain

$$
{}^{RL}_{0}D_t^0 f(t) = \frac{1}{\Gamma(1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(\tau)\mathrm{d}\tau = \frac{\mathrm{d}^0}{\mathrm{d}t^0} f(t) = f(t)
$$

and for $\alpha = 1$ we have

$$
{}^{RL}_{0}D_t^1 f(t) = \frac{1}{\Gamma(1)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t f(\tau)\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t} f(t).
$$

Therefore, by induction, Definition 1.4 is true for $\alpha \in \mathbb{N}$.

Example 1.3. Consider the unit-step function

$$f(t) = \mathbb{1}(t) = \begin{cases} 1 & \text{for} \quad t \geq 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

Using (1.7), we obtain

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha}\mathbb{1}(t) =& \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_0^t (t-\tau)^{N-\alpha-1} d\tau \\ =& \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \left[\frac{-1}{N-\alpha} (t-\tau)^{N-\alpha} \right]_0^t = \frac{1}{\Gamma(N-\alpha)} \frac{1}{N-\alpha} \frac{d^N}{dt^N} t^{N-\alpha} \\ =& \frac{1}{\Gamma(N-\alpha)} \frac{1}{N-\alpha} (N-\alpha)(N-\alpha-1)\cdots(1-\alpha) t^{-\alpha} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned}$$

Therefore, the α order Riemann-Liouville derivative of unit-step function is a decreasing in time function.

Theorem 1.1. *The Riemann-Liouville derivative-integral operator is a linear operator satisfying the relation*

$${}^{RL}_{0}D^{\alpha}_{t}[\lambda f(t) + \mu g(t)] = \lambda\, {}^{RL}_{0}D^{\alpha}_{t} f(t) + \mu\, {}^{RL}_{0}D^{\alpha}_{t} g(t), \quad \lambda, \mu \in \mathbb{R}.$$

Proof.

$$\begin{aligned} {}^{RL}_{0}D^{\alpha}_{t}(\lambda f(t) + \mu g(t)) =& \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_0^t (t-\tau)^{N-\alpha-1} [\lambda f(\tau) + \mu g(\tau)] d\tau \\ =& \frac{\lambda}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_0^t (t-\tau)^{N-\alpha-1} f(\tau) d\tau \\ &+ \frac{\mu}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_0^t (t-\tau)^{N-\alpha-1} g(\tau) d\tau \\ =& \lambda\, {}^{RL}_{0}D^{\alpha}_{t} f(t) + \mu\, {}^{RL}_{0}D^{\alpha}_{t} g(t). \end{aligned}$$

□

Theorem 1.2. *The Laplace transform of the derivative-integral* (1.7) *for* $N-1 < \alpha < N$ *has the form*

$$\mathcal{L}\left[{}^{RL}_{0}D^{\alpha}_{t} f(t)\right] = s^\alpha F(s) - \sum_{k=1}^{N} s^{k-1} f^{(\alpha-k)}(0^+), \tag{1.8}$$

where $f^{(\alpha-k)}(0^+) = {}^{RL}_{0}D^{\alpha-k}_{t} f(t)\big|_{t=0}$

Proof. Using (1.6), (1.7) and (A.5), (A.6) (see Appendix A.1) for $N-1<\alpha<N$ we obtain

$$\begin{aligned}
\mathcal{L}\left[{}^{RL}_{0}D^{\alpha}_{t}f(t)\right] =&\mathcal{L}\left\{\frac{\mathrm{d}^N}{\mathrm{d}t^N}\left[\frac{1}{\Gamma(N-\alpha)}\int\limits_0^t(t-\tau)^{N-\alpha-1}f(\tau)\mathrm{d}\tau\right]\right\}\\
=&\mathcal{L}\left\{\frac{\mathrm{d}^N}{\mathrm{d}t^N}\left[{}_0I_t^{N-\alpha}f(t)\right]\right\}\\
=&\frac{s^N F(s)}{s^{N-\alpha}}-\sum_{k=1}^{N}s^{N-k}\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\left[{}_0I_t^{N-\alpha}f(t)\right]_{t=0}\\
=&s^{\alpha}F(s)-\sum_{k=1}^{N}s^{N-k}\ {}^{RL}_{0}D_t^{k-N-1+\alpha}f(t)\big|_{t=0}\\
=&s^{\alpha}F(s)-\sum_{k=1}^{N}s^{k-1}\ {}^{RL}_{0}D_t^{\alpha-k}f(t)\big|_{t=0}\,.
\end{aligned}$$

□

1.3.2 Caputo Definition

Definition 1.5. The function defined by [51, 154, 163]

$${}^{C}_{0}D^{\alpha}_{t}f(t)=\frac{1}{\Gamma(N-\alpha)}\int\limits_0^t\frac{f^{(N)}(\tau)}{(t-\tau)^{\alpha+1-N}}\mathrm{d}\tau,\quad f^{(N)}(\tau)=\frac{\mathrm{d}^N f(\tau)}{\mathrm{d}\tau^N}\tag{1.9}$$

is called the Caputo fractional derivative-integral, where $N-1\le\alpha<N$, $N\in\mathbb{N}$.

Remark 1.1. From Definition 1.5 it follows that the Caputo derivative of constant is equal to zero.

Theorem 1.3. *The Caputo derivative-integral operator is linear satisfying the relation*

$${}^{C}_{0}D^{\alpha}_{t}[\lambda f(t)+\mu g(t)]=\lambda{}^{C}_{0}D^{\alpha}_{t}f(t)+\mu{}^{C}_{0}D^{\alpha}_{t}g(t).$$

Proof. The proof is similar to the proof of Theorem 1.1. □

Theorem 1.4. *The Laplace transform of the derivative-integral* (1.9) *for* $N-1<\alpha<N$ *has the form*

$$\mathcal{L}\left[{}^{C}_{0}D^{\alpha}_{t}f(t)\right]=s^{\alpha}F(s)-\sum_{k=1}^{N}s^{\alpha-k}f^{(k-1)}(0^+).\tag{1.10}$$

Proof. Using Definitions 1.5 and A.2, equations (A.3), (A.5) for $N-1<\alpha<N$ we obtain

$$\begin{aligned}
\mathcal{L}\left[{}_0^C D_t^\alpha f(t)\right] &= \mathcal{L}\left[\frac{1}{\Gamma(N-\alpha)}\int_0^t (t-\tau)^{N-\alpha-1} f^{(N)}(\tau)\mathrm{d}\tau\right] \\
&= \frac{1}{\Gamma(N-\alpha)}\mathcal{L}\left[t^{N-\alpha-1}\right]\mathcal{L}\left[f^{(N)}(t)\right] \\
&= \frac{1}{\Gamma(N-\alpha)}\frac{\Gamma(N-\alpha)}{s^{N-\alpha}}\left[s^N F(s) - \sum_{k=1}^{N} s^{N-k} f^{(k-1)}(0^+)\right] \\
&= s^\alpha F(s) - \sum_{k=1}^{N} s^{\alpha-k} f^{(k-1)}(0^+).
\end{aligned}$$

□

1.4 Solutions of the Fractional State Equation of Continuous-Time Linear System

Consider the continuous-time linear system described by the equations [52]

$$ {}_0^C D_t^\alpha x(t) = \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{1.11a}$$

$$ y(t) = Cx(t) + Du(t), \tag{1.11b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Theorem 1.5. *The solution of the equation* (1.11a) *has the form*

$$ x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\mathrm{d}\tau, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{1.12}$$

where

$$ \Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{1.13}$$

$$ \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{1.14}$$

and $E_\alpha(At^\alpha)$ is the Mittag-Leffler function and $\Gamma(x)$ is the Euler gamma function.

Proof. Applying the Laplace transform to (1.11a) and taking into account that

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}\mathrm{d}t, \quad U(s) = \mathcal{L}[u(t)],$$
$$\mathcal{L}\left[{}_0^C D^\alpha x(t)\right] = s^\alpha X(s) - s^{\alpha-1}x_0 \quad \text{for} \quad 0 < \alpha < 1$$

we obtain

$$X(s) = [\mathbb{I}_n s^\alpha - A]^{-1}\left[s^{\alpha-1}x_0 + BU(s)\right]. \tag{1.15}$$

It is easy to show that

$$[\mathbb{I}_n s^\alpha - A]^{-1} = \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}, \tag{1.16}$$

since

$$[\mathbb{I}_n s^\alpha - A]\left(\sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}\right) = \mathbb{I}_n.$$

Substituting of (1.16) to (1.15), we obtain

$$X(s) = \sum_{k=0}^{\infty} A^k s^{-(k\alpha+1)}x_0 + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}BU(s). \tag{1.17}$$

Using the inverse Laplace transform and the convolution theorem (Appendix A.1) to (1.17) we obtain

$$\begin{aligned} x(t) =& \mathcal{L}^{-1}[X(s)] = \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}\left[s^{-(k\alpha+1)}\right]x_0 + \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}\left[s^{-(k+1)\alpha}BU(s)\right] \\ =& \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\mathrm{d}\tau, \end{aligned}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}\left[s^{-(k\alpha+1)}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)},$$
$$\Phi(t) = \mathcal{L}^{-1}\left\{[\mathbb{I}_n s^\alpha - A]^{-1}\right\} = \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}\left[s^{-(k+1)\alpha}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.$$

□

Theorem 1.6. *The matrix* $\Phi_0(t)$ *defined by* (1.13) *satisfies the equality*

$$\frac{\mathrm{d}^\alpha \Phi_0(t)}{\mathrm{d}t^\alpha} = A\Phi_0(t).$$

Proof. By Theorem 1.5 for $u(t) = 0$, $t \geq 0$, the matrix $\Phi_0(t)$ satisfies the equation (1.11a). □

Remark 1.2. From (1.13) and (1.14) for $\alpha = 1$ we have

$$\Phi_0(t) = \Phi(t) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(k+1)} = e^{At}.$$

Remark 1.3. From classical Cayley-Hamilton theorem [39, 44, 115] it follows that if

$$\det\left[\mathbb{I}_n s^\alpha - A\right] = (s^\alpha)^n + a_{n-1}(s^\alpha)^{n-1} + \cdots + a_1 s^\alpha + a_0,$$

then

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 \mathbb{I}_n = 0.$$

Example 1.4. Find the solution of the equation (1.11a) for $0 < \alpha < 1$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u(t) = \mathbb{1}(t).$$

Using (1.13) and (1.14), we obtain

$$\begin{aligned} \Phi_0(t) &= \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} = \mathbb{I}_2 + \frac{At^\alpha}{\Gamma(\alpha+1)}, \\ \Phi(t) &= \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \mathbb{I}_2 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}. \end{aligned} \tag{1.18}$$

Substituting (1.18) and $u(t) = \mathbb{1}(t)$ into (1.12), we obtain

$$\begin{aligned} x(t) =& x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \int_0^t \left[\frac{B}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{AB}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1}\right] \mathrm{d}\tau \\ =& x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \frac{Bt^\alpha}{\Gamma(\alpha+1)} + \frac{ABt^{2\alpha}}{\Gamma(2\alpha+1)} \\ =& \begin{bmatrix} 1 + \dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \dfrac{t^\alpha}{\Gamma(\alpha+1)} \end{bmatrix}, \end{aligned}$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Theorem 1.7. *If the Caputo Definition 1.5 is used, then the solution of the equation* (1.11a) *for $N-1 < \alpha < N$ has the form*

$$x(t) = \sum_{l=1}^{N} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau) B u(\tau) \mathrm{d}\tau, \tag{1.19}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha+l-1}}{\Gamma(k\alpha+l)}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.$$

Proof. Taking into account (A.1), (1.10) we obtain the Laplace transform of (1.11a)

$$X(s) = [\mathbb{I}_n s^\alpha - A]^{-1} \left[\sum_{l=1}^{N} s^{\alpha-l} x^{(l-1)}(0^+) + BU(s) \right], \quad U(s) = \mathcal{L}[u(t)]. \tag{1.20}$$

Substitution of (1.16) into (1.20) yields

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} \left[\sum_{l=1}^{N} s^{\alpha-l} x^{(l-1)}(0^+) + BU(s) \right] \\ &= \sum_{k=0}^{\infty} \sum_{l=1}^{N} A^k s^{-(k\alpha+l)} x^{(l-1)}(0^+) + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} BU(s). \end{aligned} \tag{1.21}$$

Applying the inverse Laplace transform and the convolution theorem (Appendix A.1) to (1.21), we obtain

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \sum_{l=1}^{N} A^k \mathcal{L}^{-1} \left[s^{-(k\alpha+l)} \right] x^{(l-1)}(0^+) + \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1} \left[s^{-(k+1)\alpha} BU(s) \right] \\ &= \sum_{l=1}^{n} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) \mathrm{d}\tau, \end{aligned}$$

where

$$\begin{aligned} \Phi_l(t) &= \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1} \left[s^{-(k\alpha+l)} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha+l-1}}{\Gamma(k\alpha+l)}, \\ \Phi(t) &= \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1} \left[s^{-(k+1)\alpha} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \end{aligned}$$

□

Theorem 1.8. *If the Riemann-Liouville Definition 1.4 is used, then the solution of the equation* (1.11a) *for* $N-1 \le \alpha \le N$ *has the form*

$$x(t) = \sum_{l=1}^{N} \Phi_l(t) x^{(\alpha-l)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) \mathrm{d}\tau, \tag{1.22}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - l}}{\Gamma[(k+l)\alpha - l + 1]}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]}.$$

Proof. Taking into account (A.1) and (1.8), from (1.11a) we obtain

$$X(s) = [\mathbb{I}_n s^\alpha - A]^{-1} \left[\sum_{l=1}^{N} s^{l-1} x^{(\alpha - l)}(0^+) + BU(s) \right], \quad U(s) = \mathcal{L}[u(t)]. \tag{1.23}$$

Substitution of (1.16) to (1.23) yields

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} \left[\sum_{l=1}^{N} s^{l-1} x^{(\alpha - l)}(0^+) + BU(s) \right] \\ &= \sum_{k=0}^{\infty} \sum_{l=1}^{N} A^k s^{-(k+1)\alpha + l - 1} x^{(\alpha - l)}(0^+) + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} BU(s). \end{aligned} \tag{1.24}$$

Applying the inverse Laplace transform and the convolution theorem (Appendix A.1) to (1.24), we obtain

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \sum_{l=1}^{N} A^k \mathcal{L}^{-1} \left[s^{-(k+1)\alpha + l - 1} \right] x^{(\alpha - l)}(0^+) \\ &\quad + \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1} \left[s^{-(k+1)\alpha} BU(s) \right] \\ &= \sum_{l=1}^{N} \Phi_l(t) x^{(\alpha - l)}(0^+) + \int_0^t \Phi(t - \tau) B u(\tau) \mathrm{d}\tau, \end{aligned}$$

where

$$\begin{aligned} \Phi_l(t) &= \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1} \left[s^{-(k+1)\alpha + l - 1} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - l}}{\Gamma[(k+l)\alpha - l + 1]}, \\ \Phi(t) &= \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1} \left[s^{-(k+1)\alpha} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]}. \end{aligned}$$

□

Remark 1.4. From comparison of (1.19) and (1.22) it follows that the component of the solution corresponding to $u(t)$ is the same.

1.5 Positivity of the Fractional Systems

Definition 1.6. [47, 71] The fractional system (1.11) is called (internally) positive if the state vector $x(t) \in \mathbb{R}_+^n$ and the output vector $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$ for all initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Definition 1.7. [47, 71] A real square matrix $A = [a_{ij}]$ is called Metzler matrix if its off diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$.

Lemma 1.1. Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha \leq 1$. Then

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0, \tag{1.25}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0 \tag{1.26}$$

if and only if A is a Metzler matrix.

Proof. Necessity. From

$$\begin{aligned} \Phi_0(t) =& \mathbb{I}_n + \frac{At^\alpha}{\Gamma(\alpha+1)} + \cdots, \\ \Phi(t) =& \mathbb{I}_n \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots \end{aligned}$$

it follows that $\Phi_0(t) \in \mathbb{R}_+^{n\times n}$ and $\Phi(t) \in \mathbb{R}_+^{n\times n}$ for small value $t > 0$ only if A is a Metzler matrix.
Sufficiency. It is well-known [47] that

$$e^{At} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0 \tag{1.27}$$

if and only if A is a Metzler matrix.

Using (1.25), we can write

$$\Phi_0(t) - e^{At^\alpha} = \sum_{k=0}^{\infty} \left(\frac{(At^\alpha)^k}{\Gamma(k\alpha+1)} - \frac{(At^\alpha)^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{k! - \Gamma(k\alpha+1)}{\Gamma(k\alpha+1)} \cdot \frac{(At^\alpha)^k}{k!} \tag{1.28}$$

for $t \geq 0$. From (1.27) and (1.28) we have $\Phi_0(t) \geq e^{At^\alpha} \geq 0$ for $t \geq 0$, since $k! \geq \Gamma(k\alpha+1)$ for $0 < \alpha \leq 1$.

The proof for (1.26) is similar. □

Theorem 1.9. *The fractional continuous-time system* (1.11) *is (internally) positive if and only if*

$$A \in M_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{1.29}$$

Proof. Sufficiency. By Theorem 1.5 the solution (1.11a) has the form (1.12) and $x(t) \in \mathbb{R}_+^n$, $t \geq 0$, if the condition (1.29) is satisfied, since $\Phi_0 \in \mathbb{R}_+^{n\times n}$, $x_0 \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m$ for $t \geq 0$.

Necessity. Let $u(t) = 0$, $t \geq 0$ and $x_0 = e_i$ (i-th column of the identity matrix $\mathbb{I}_n$). The trajectory does not leave the orthant $\mathbb{R}_+^n$ only if the derivative of order α, $x^\alpha(0) = Ae_i \geq 0$, what implies $a_{ij} \geq 0$ for $i \neq j$. The matrix A is a Metzler matrix. From the same reason for $x_0 = 0$ we have $x^\alpha(0) = Bu(0) \geq 0$, what implies $B \in \mathbb{R}_+^{n\times m}$, since $u(0) \in \mathbb{R}_+^m$ can be arbitrary. From (1.11b) for $u(t) = 0$, $t \geq 0$ we have $y(0) = Cx_0 \geq 0$ and $C \in \mathbb{R}_+^{p\times n}$, since $x_0 \in \mathbb{R}_+^n$ can be arbitrary. In a similar way assuming $x_0 = 0$, we obtain $y(0) = Du(0) \geq 0$ and $D \in \mathbb{R}_+^{p\times m}$, since $u(0) \in \mathbb{R}_+^m$ is arbitrary. □

1.6 External Positivity of the Fractional Systems

Definition 1.8. [71] The fractional system (1.11) is called externally positive if for all $u(t) \in \mathbb{R}_+^m$, $t \geq 0$ and zero initial conditions $x_0 = 0$ the output vector $y(t) \in \mathbb{R}_+^p$, $t \geq 0$.

Definition 1.9. [71] Output of the fractional SISO (single-input single-output) system with zero initial conditions for Dirac impulse $u(t) = \delta(t)$ is called the impulse response of the system. In a similar way we define the matrix of impulse response of the MIMO (multiple-input multiple-output) fractional system (1.11).

Lemma 1.2. Matrix of the impulse responses $g(t)$ of the fractional system (1.11) is given by

$$g(t) = C\Phi(t)B + D\delta(t), \quad t \geq 0, \tag{1.30}$$

where $\delta(t)$ is the Dirac delta defined by

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0, \\ 0 & \text{for } t \neq 0, \end{cases}$$

which satisfies the condition

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1.$$

Proof. Substituting (1.12) into (1.11b) and taking into account $x_0 = 0$, $u(t) = \delta(t)$, $y(t) = g(t)$ we obtain

$$g(t) = \int_0^t C\Phi(t-\tau)B\delta(\tau)\mathrm{d}\tau + D\delta(t) = C\Phi(t)B + D\delta(t).$$

□

Theorem 1.10. *The fractional system* (1.11) *is externally positive if and only if*

$$g(t) \in \mathbb{R}_+^{p\times m} \quad \text{for} \quad t \geq 0. \tag{1.31}$$

Proof. Sufficiency. The output $y(t)$ of the system (1.11) with zero initial conditions for the input $u(t)$ is given by

$$y(t) = \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau. \tag{1.32}$$

If the condition (1.31) is satisfied then from (1.32) we have $y(t) \in \mathbb{R}_+^p$, $t \geq 0$.
Necessity. The necessity follows immediately from the fact that the matrix of impulse responses in a particular case of the output of the system for $u(t) = \delta(t)$ and $\delta(t)$ is nonnegative for $t \geq 0$. □

Corollary 1.1. The matrix of impulse responses (1.30) of internally positive system (1.11) is nonnegative for $t \geq 0$.

Between the internal and external positivity we have the following relationship.

Corollary 1.2. Every fractional continuous-time (internally) positive system (1.11) is always externally positive.

1.7 Positive Continuous-Time Linear Systems with Delays

Consider the continuous-time linear system with h delays described by the state equations [58]

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0x(t) + \sum_{k=1}^{h} A_kx(t-d_k) + Bu(t), \tag{1.33a}$$

$$y(t) = Cx(t) + Du(t), \tag{1.33b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A_k \in \mathbb{R}^{n\times n}$, $k = 0, 1, \ldots, h$; $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and d_k ($d_k \geq 0$), $k = 1, 2, \ldots, h$ are delays.
Initial conditions for (1.33) have the form

$$x(t) = x_0(t) \quad \text{for} \quad t \in [-d, 0], \quad d = \max(d_k),$$

where $\max(d_k)$ denotes the maximal element of the set $\{d_1, d_2, \ldots, d_h\}$ and $x_0(t) \in \mathbb{R}^n$ is given.

Definition 1.10. The system (1.33) is called (internally) positive if $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for any $x_0(t) \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 1.11. *The system* (1.33) *is (internally) positive if and only if*

$$\begin{aligned} A_0 \in M_n, \quad A_k \in \mathbb{R}_+^{n\times n}, \quad k = 1,\ldots,h; \\ B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \end{aligned}$$

Proof. Necessity. The equation (1.33a) for $x(t-d_k) = 0$, $t \in [d,0]$, $k = 1,\ldots,h$ and $u(t) = 0$, $t \geq 0$ takes the form

$$\dot{x}(t) = A_0 x(t), \quad t \in [0,d]. \tag{1.34}$$

It is well-known [30, 47], that $x(t) \in \mathbb{R}_+^n$ of (1.34) if and only if $A_0 \in M_n$. Assuming in (1.33a) $u(t) = 0$, $t \geq 0$, $x_0(-d_k) = e_i$, $i = 1,\ldots,n$ (i-th column of the identity matrix $\mathbb{I}_n$), $x(-d_j) = 0$, $j = 0,1,\ldots,k-1,k+1,\ldots,h$ for $t = 0$ we obtain $\dot{x}(0) = A_k e_i = A_{ki} \in \mathbb{R}_+^n$, where A_{ki} is i-th column of $A_k \in \mathbb{R}_+^{n\times n}$, $k = 1,\ldots,h$.

From (1.33a) for $t = 0$ and $x_0(t) = 0$, $t \in [-d,0]$ we have $\dot{x}(0) = Bu(0)$ and $B \in \mathbb{R}_+^{n\times m}$, since by definition $u(0) \in \mathbb{R}_+^m$ is arbitrary. The necessity of $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ can be shown in a similar way as for positive systems without delays [30, 47].

Sufficiency. The solution of the equation (1.33a) for $t \in [0,d]$ has the form

$$x(t) = e^{A_0 t}x(0) + \int_0^t e^{A_0(t-\tau)} \left(\sum_{k=1}^{h} A_k x_0(\tau - d_k) + Bu(\tau) \right) \mathrm{d}\tau. \tag{1.35}$$

Taking into account that $e^{A_0 t} \in \mathbb{R}_+^{n\times n}$, $t \geq 0$, for $A_0 \in M_n$, and the condition (1.34), from (1.35) we obtain $x(t) \in \mathbb{R}_+^n$, $t \in [0,d]$, since $x_0(t) \in \mathbb{R}_+^n$, $t \in [-d,0]$ and $u(t) \in \mathbb{R}_+^m$, $t \geq 0$. From (1.33b) we have $y(t) \in \mathbb{R}_+^p$, $t \in [0,d]$, since $x(t) \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m$.

Using the step method we can extend the considerations for the intervals $[d,2d]$, $[2d,3d]$, ... □

Definition 1.11. Let to the asymptotically stable positive system (1.33) a constant input $u(t) = u \in \mathbb{R}_+^m$ be applied. The vector x_e satisfying the equation

$$0 = \sum_{k=0}^{h} A_k x_e + Bu \tag{1.36}$$

is called the equilibrium point (state) of the system (1.33) for constant input u.

If the positive system (1.33) is asymptotically stable, then the matrix

$$A = \sum_{k=0}^{h} A_k \in M_n$$

is nonsingular and from (1.36) we obtain

$$x_e = -A^{-1}Bu. \tag{1.37}$$

Remark 1.5. For positive asymptotically stable system (1.33) we have

$$-A^{-1} \in \mathbb{R}_+^{n\times n}. \tag{1.38}$$

This remark follows immediately from (1.37), since $x_0 \in \mathbb{R}_+^n$ and $Bu \in \mathbb{R}_+^m$ is arbitrary [30, 47].

Theorem 1.12. *The equilibrium point x_e for positive asymptotically stable system* (1.33) *is strictly positive, i.e. $x_e > 0$ if $Bu > 0$.*

Proof. If $A \in M_n$ and $Bu > 0$ then from (1.36) we have $x_e \in \mathbb{R}_+^n$. The hypothesis will be proved by contradiction. Assume that $x_e = 0$ then from (1.36) we have $Bu = 0$. This contradicts that $Bu > 0$. This completes the proof. □

These considerations can be extended to linear systems with delays in control and positive fractional continuous-time linear systems with delays in states and controls.

1.8 Positive Linear Systems Consisting of *n* Subsystems with Different Fractional Orders

1.8.1 Linear Differential Equations with Different Fractional Orders

Following [69] let us consider a fractional linear system described by the equation

$$\begin{bmatrix} \frac{\mathrm{d}^{\alpha_1} x_1}{\mathrm{d}t^{\alpha_1}} \\ \vdots \\ \frac{\mathrm{d}^{\alpha_n} x_n}{\mathrm{d}t^{\alpha_n}} \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u, \tag{1.39}$$
$$p_k - 1 < \alpha_k < p_k, \quad p_k \in \mathbb{N} = \{1, 2, \dots\}, \quad k = 1, \dots, n;$$

where $x_k \in \mathbb{R}^{\bar{n}_k}$, $k = 1, \dots, n$ are the state vectors, $A_{kj} \in \mathbb{R}^{\bar{n}_k \times \bar{n}_j}$, $B_k \in \mathbb{R}^{\bar{n}_k \times m}$; $k, j = 1, \dots, n$ and $u \in \mathbb{R}^m$ is the input vector.

Initial conditions for (1.39) have the form

$$x_k^{(j)}(0) = x_{k0}^{(j)} \in \mathbb{R}^{\bar{n}_k}, \quad k = 1, \dots, n; \quad j = 0, 1, \dots, p_k - 1. \tag{1.40}$$

Theorem 1.13. *The solution of the equation* (1.39) *for* $p_k - 1 < \alpha_k < p_k$, $k = 1, \dots, n$ *with initial conditions* (1.40) *has the form*

$$x(t) = \int_0^t \left[\Phi_1(t-\tau)B_{10} + \cdots + \Phi_n(t-\tau)B_{n0}\right] u(\tau)\mathrm{d}\tau$$

$$+ \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix}, \tag{1.41}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^N, \quad N = \bar{n}_1 + \cdots + \bar{n}_n, \quad x_0 = \begin{bmatrix} x_{10} \\ \vdots \\ x_{n0} \end{bmatrix}, \tag{1.42a}$$

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, B_{n0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_n \end{bmatrix}, \tag{1.42b}$$

$$\Phi_1(t) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \frac{t^{(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n-1}}{\Gamma[(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n]},$$

$$\vdots$$

$$\Phi_n(t) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \frac{t^{k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n-1}}{\Gamma[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]}, \tag{1.42c}$$

and

$$T_{k_1\ldots k_n} = \begin{cases} \mathbb{I}_N & \textit{for} \quad k_1 = \cdots = k_n = 0, \\ \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \textit{for} \quad \begin{matrix} k_1 = 1, \\ k_2 = \cdots = k_n = 0, \end{matrix} \\ \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ A_{i1} & \cdots & A_{in} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \textit{for} \quad \begin{matrix} k_1 = \cdots = k_{i-1} = 0, \\ k_i = 1, \\ k_{i+1} = \cdots = k_n = 0, \end{matrix} \\ \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} & \textit{for} \quad \begin{matrix} k_1 = \cdots = k_{n-1} = 0, \\ k_n = 1, \end{matrix} \\ T_{10\ldots0}T_{01\ldots1} + \cdots + T_{0\ldots01}T_{1\ldots10} & \textit{for} \quad k_1 = \cdots = k_n = 1, \\ \vdots & \\ \begin{matrix} T_{10\ldots0}T_{k_1-1,k_2,\ldots,k_n} \\ + \cdots + T_{0\ldots01}T_{k_1,k_{n-1},k_n-1} \end{matrix} & \textit{for} \quad k_1 + \cdots + k_n > 0, \\ 0 \qquad \textit{for at last one} & k_i < 0, i = 1,\ldots,n. \end{cases} \tag{1.42d}$$

Proof. Using the Laplace transforms

$$X_k(s) = \mathcal{L}[x_k(t)], \quad k = 1,\ldots,n; \quad U(s) = \mathcal{L}[u(t)],$$

and (1.10) we may write the equations (1.39) for $p_k - 1 < \alpha < p_k$; $p_k \in \mathbb{N}$, $k = 1\ldots,n$ in the form

$$\begin{bmatrix} \mathbb{I}_{\bar{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & \mathbb{I}_{\bar{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$
$$= \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1-j_1}x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n-j_n}x_{n0}^{(j_n-1)} \end{bmatrix}. \tag{1.43}$$

From (1.43) we have

$$\begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} \mathbb{I}_{\bar{n}_1} s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & \mathbb{I}_{\bar{n}_n} s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1}$$
$$\times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}.$$

Comparing the coefficients at the same powers of $s^{-\alpha_k}$ it is easy to verify that

$$\begin{bmatrix} \mathbb{I}_{\bar{n}_1} - A_{11} s^{-\alpha_1} & \cdots & -A_{1n} s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1} s^{-\alpha_n} & \cdots & \mathbb{I}_{\bar{n}_n} - A_{nn} s^{-\alpha_n} \end{bmatrix}$$
$$\times \left[\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \ldots k_n} s^{-(k_1\alpha_1 + \cdots + k_n\alpha_n)} \right] = \mathbb{I}_N, \tag{1.44}$$

where matrices $T_{k_1 \ldots k_n}$ are defined by (1.42d).

Using (1.44) we obtain

$$\begin{bmatrix} \mathbb{I}_{\bar{n}_1} s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & \mathbb{I}_{\bar{n}_n} s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1} =$$
$$= \left\{ \begin{bmatrix} \mathbb{I}_{\bar{n}_1} s^{\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{I}_{\bar{n}_n} s^{\alpha_n} \end{bmatrix} \begin{bmatrix} \mathbb{I}_{\bar{n}_1} - A_{11} s^{-\alpha_1} & \cdots & -A_{1n} s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1} s^{-\alpha_n} & \cdots & \mathbb{I}_{\bar{n}_n} - A_{nn} s^{-\alpha_n} \end{bmatrix} \right\}^{-1}$$
$$= \begin{bmatrix} \mathbb{I}_{\bar{n}_1} - A_{11} s^{-\alpha_1} & \cdots & -A_{1n} s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1} s^{-\alpha_n} & \cdots & \mathbb{I}_{\bar{n}_n} - A_{nn} s^{-\alpha_n} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{I}_{\bar{n}_1} s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{I}_{\bar{n}_n} s^{-\alpha_n} \end{bmatrix} \tag{1.45}$$
$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \ldots k_n} s^{-(k_1\alpha_1 + \cdots + k_n\alpha_n)} \begin{bmatrix} \mathbb{I}_{\bar{n}_1} s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{I}_{\bar{n}_n} s^{-\alpha_n} \end{bmatrix}.$$

Substitution of (1.45) into (1.44) yields

$$\begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} s^{-(k_1\alpha_1+\dots+k_n\alpha_n)} \begin{bmatrix} \mathbb{I}_{\bar{n}_1} s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{I}_{\bar{n}_n} s^{-\alpha_n} \end{bmatrix}$$
$$\times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}$$
$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \left\{ \left[B_{10} s^{-[(k_1+1)\alpha_1 + k_2\alpha_2 + \dots + k_n\alpha_n]} \right.\right.$$
$$\left. + \dots + B_{n0} s^{-[k_1\alpha_1+\dots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]} \right] U(s)$$
$$\left. + \, s^{-(k_1\alpha_1+\dots+k_n\alpha_n)} \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{-j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}. \tag{1.46}$$

Applying the inverse Laplace transform and the convolution theorem to (1.46) we obtain

$$\mathcal{L}^{-1} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \mathcal{L}^{-1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \left\{ \left[B_{10} s^{-[(k_1+1)\alpha_1 + k_2\alpha_2 + \dots + k_n\alpha_n]} \right.\right.$$
$$\left. + \dots + B_{n0} s^{-[k_1\alpha_1+\dots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]} \right] U(s)$$
$$\left. + \, s^{-(k_1\alpha_1+\dots+k_n\alpha_n)} \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{-j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\},$$

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \int_0^t \left[\Phi_1(t-\tau)B_{10} + \cdots + \Phi_n(t-\tau)B_{n0}\right] u(\tau)\mathrm{d}\tau$$

$$+ \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} \begin{bmatrix} \displaystyle\sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \displaystyle\sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix},$$

since $\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^\alpha}{\Gamma(\alpha+1)}$. □

In a particular case if $0 < \alpha_k < 1$, $k = 1, \ldots, n$; $(p_1 = \cdots = p_n = 1)$, then

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} \begin{bmatrix} \displaystyle\sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \displaystyle\sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix} = \Phi_0(t)x_0,$$

where

$$\Phi_0(t) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+1)}.$$

1.8.2 Positive Fractional Systems with Different Fractional Orders

Definition 1.12. [71] The fractional system (1.39) is called positive if $x_k(t) \in \mathbb{R}_+^{\bar{n}_k}$, $k = 1, \ldots, n$, $t \geq 0$ for any initial conditions $x_{k0} \in \mathbb{R}_+^{\bar{n}_k}$, $k = 1, \ldots, n$ and all input vectors $u \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 1.14. *The fractional system* (1.39) *for* $p_k - 1 < \alpha < p_k$, $p_k \in \mathbb{N}$, $k = 1, \ldots, n$ *is positive if and only if*

$$A = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix} \in M_N, \tag{1.47a}$$

$$\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \in \mathbb{R}_+^{N \times m}. \tag{1.47b}$$

Proof. To simplify the notation the proof will be given for $n = 2$. First we shall show that

$$\Phi_k(t) \in \mathbb{R}_+^{\bar{n} \times \bar{n}}, \quad (\bar{n} = \bar{n}_1 + \bar{n}_2) \quad \text{for} \quad k = 0, 1, 2 \quad \text{and} \quad t \geq 0,$$

only if (1.47a) holds.

From the expansion (1.42c) we have

$$\begin{aligned} \Phi_0(t) &= \begin{bmatrix} \mathbb{I}_{\bar{n}_1} & 0 \\ 0 & \mathbb{I}_{\bar{n}_2} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \\ &\quad + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \cdots \ , \qquad (1.48a) \\ \Phi_1(t) &= \begin{bmatrix} \mathbb{I}_{\bar{n}_1} & 0 \\ 0 & \mathbb{I}_{\bar{n}_2} \end{bmatrix} \frac{t^{\alpha_1 - 1}}{\Gamma(\alpha_1)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{2\alpha_1 - 1}}{\Gamma(2\alpha_1)} \\ &\quad + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1 + \alpha_2)} + \cdots \ , \qquad (1.48b) \\ \Phi_2(t) &= \begin{bmatrix} \mathbb{I}_{\bar{n}_1} & 0 \\ 0 & \mathbb{I}_{\bar{n}_2} \end{bmatrix} \frac{t^{\alpha_2 - 1}}{\Gamma(\alpha_2)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1 + \alpha_2)} \\ &\quad + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{2\alpha_2 - 1}}{\Gamma(2\alpha_2)} + \cdots \ . \qquad (1.48c) \end{aligned}$$

From (1.48) it follows that $\Phi_k(t) \in \mathbb{R}_+^{\bar{n} \times \bar{n}}$, $k = 0, 1, 2$ for small value of $t > 0$ only if the condition (1.47a) is satisfied.

In a similar way as in [52, 57], it can be shown that if (1.47) holds then

$$\Phi_0(t) \in \mathbb{R}_+^{\bar{n} \times \bar{n}}, \quad t \geq 0,$$

and

$$\Phi_1(t)B_{10} + \Phi_2(t)B_{01} \in \mathbb{R}_+^{\bar{n} \times \bar{n}}, \quad t \geq 0.$$

In this case from (1.41) we have $x(t) \in \mathbb{R}_+^{\bar{n}}$, $t \geq 0$, since by definition $x_0 \in \mathbb{R}_+^{\bar{n}}$ and $u(t) \in \mathbb{R}_+^m$, $t \geq 0$. The remaining part of the proof is similar as in [52, 57]. □

1.9 Descriptor Fractional Continuous-Time Linear Systems

Consider a descriptor (singular) fractional linear system described by the state equations [71, 72]

$$E\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) = Ax(t) + Bu(t), \tag{1.49a}$$

$$y(t) = Cx(t) + Du(t), \tag{1.49b}$$

where $\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha}$ is the α-order $(0 < \alpha < 1)$ fractional derivative described by (1.9), $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

The initial condition for (1.49a) is given by

$$x(0) = x_0. \tag{1.50}$$

It is assumed that the pencil of the pair $(E,\ A)$ is regular, i.e.

$$\det[Es^\alpha - A] \neq 0, \tag{1.51}$$

for some $s \in \mathbb{C}$.

1.9.1 Solution of the Descriptor Fractional Systems

It is well-known [39, 50], that if the pencil is regular, then there exists a pair of nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$P[Es - A]Q = \begin{bmatrix} \mathbb{I}_{n_1} & 0 \\ 0 & N \end{bmatrix} s - \begin{bmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n_2} \end{bmatrix}, \tag{1.52}$$

where n_1 is equal to the degree of the polynomial $\det[Es - A]$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with the index μ (i.e. $N^\mu = 0$ and $N^{\mu-1} \neq 0$) and $n_1 + n_2 = n$.

Applying the Laplace transform ($\mathcal{L}$) to the state equation (1.49a) with zero initial conditions $x_0 = 0$, we obtain

$$[Es^\alpha - A]\, X(s) = BU(s),$$

where $X(s) = \mathcal{L}[x(t)]$, $U(s) = \mathcal{L}[u(t)]$.

By the assumption (1.51), the pencil $[Es^\alpha - A]$ is regular and we may apply the decomposition (1.52) to equation (1.49a).

Premultiplying (1.49a) by the matrix $P \in \mathbb{R}^{n\times n}$ and introducing the new state vector

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Q^{-1}x(t),$$

where $Q \in \mathbb{R}^{n\times n}$, $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, we obtain

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x_1(t) = A_1x_1(t) + B_1u(t), \tag{1.53a}$$

$$N\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x_2(t) = \quad x_2(t) + B_2u(t), \tag{1.53b}$$

where

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{n_1 \times m}, \quad B_2 \in \mathbb{R}^{n_2 \times m}. \tag{1.53c}$$

Using (1.12) we obtain the solution to the equation (1.53a) in the form

$$x_1(t) = \Phi_{10}(t)x_{10} + \int_0^t \Phi_{11}(t-\tau)B_1 u(\tau)\mathrm{d}\tau, \tag{1.54a}$$

where

$$\Phi_{10}(t) = \sum_{k=1}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{1.54b}$$

$$\Phi_{11}(t) = \sum_{k=1}^{\infty} \frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{1.54c}$$

and $x_{10} \in \mathbb{R}^{n_1}$ is the initial condition for (1.53a) defined by

$$\begin{bmatrix} x_{10} \\ x_{11} \end{bmatrix} = Q^{-1}x_0, \quad x_0 = x(0). \tag{1.54d}$$

To find the solution of equation (1.53b), we apply the Laplace transform and obtain

$$Ns^{\alpha}X_2(s) - Ns^{\alpha-1}x_{20} = X_2(s) + B_2U(s), \tag{1.55a}$$

since by (1.10), for $0 > \alpha > 1$,

$$\mathcal{L}\left[\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}x_2(t)\right] = s^{\alpha}X_2(s) - s^{\alpha-1}x_{20}, \tag{1.55b}$$

where $X_2(s) = \mathcal{L}[x_2(t)]$.

From (1.55a) we have

$$X_2(s) = [Ns^{\alpha} - \mathbb{I}_{n_2}]^{-1}\left[B_2U(s) + Ns^{\alpha-1}x_{20}\right]. \tag{1.56}$$

It is easy to check that

$$[Ns^{\alpha} - \mathbb{I}_{n_2}]^{-1} = -\sum_{i=0}^{\mu-1} N^i s^{i\alpha}, \tag{1.57}$$

since

$$[Ns^{\alpha} - \mathbb{I}_{n_2}]\left(-\sum_{i=0}^{\mu-1} N^i s^{i\alpha}\right) = \mathbb{I}_{n_2}$$

and $N^i = 0$ for $i = \mu, \mu+1, \ldots$.

Substitution of (1.57) into (1.56) yields

$$X_2(s) = -B_2U(s) - \frac{Nx_{20}}{s^{1-\alpha}} - \sum_{i=0}^{\mu-1}\left[N^iB_2s^{i\alpha}U(s) + N^{i+1}s^{(i+1)\alpha-1}x_{20}\right]. \tag{1.58}$$

Applying the inverse Laplace transform ($\mathcal{L}^{-1}$) to (1.58) and then the convolution theorem we obtain, for $1-\alpha > 0$,

$$\begin{aligned} x_2(t) = \mathcal{L}^{-1}[X_2(s)] = &- B_2u(t) - Nx_{20}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \\ &- \sum_{i=0}^{\mu-1}\left[N^iB_2\frac{\mathrm{d}^{i\alpha}}{\mathrm{d}t^{i\alpha}}u(t) + N^{i+1}\frac{\mathrm{d}^{(i+1)\alpha-1}}{\mathrm{d}t^{(i+1)\alpha-1}}x_{20}\right], \end{aligned} \tag{1.59}$$

since

$$\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^\alpha}{\Gamma(1+\alpha)}$$

for $\alpha + 1 > 0$.

Therefore, the following theorem has been proved.

Theorem 1.15. *The solution to the state equation* (1.49a) *with the initial conditions* (1.50) *has the form*

$$x(t) = Q\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{1.60}$$

where $x_1(t)$ *and* $x_2(t)$ *are given by* (1.54) *and* (1.59), *respectively.*

Knowing the solution (1.60), we can find the output $y(t)$ of the system using the formula

$$y(t) = CQ\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t). \tag{1.61}$$

1.9.2 Drazin Inverse Method for the Solution of Fractional Descriptor Continuous-Time Linear Systems

Following [93] consider the fractional descriptor continuous-time linear system described by the equation (1.49a). It is assumed that $\det E = 0$ and the pencil of the matrices (E, A) is regular, i.e. the condition (1.51) is met.

Assuming that for some chosen $c \in \mathbb{R}$, $\det[Ec - A] \neq 0$ and premultiplying (1.49a) by $[Ec - A]^{-1}$ we obtain

$$\bar{E}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}x(t) = \bar{A}x(t) + \bar{B}u(t), \tag{1.62a}$$

where

$$\bar{E} = [Ec - A]^{-1} E, \quad \bar{A} = [Ec - A]^{-1} A, \quad \bar{B} = [Ec - A]^{-1} B. \tag{1.62b}$$

Note that the equations (1.49a) and (1.62a) have the same solution $x(t)$.

Theorem 1.16. *The solution of the equation* (1.62a) *(and* (1.49a)*) is given by*

$$\begin{aligned} x(t) =& \Phi_0(t)v + \bar{E}^D \int_0^t \Phi(t-\tau)\bar{B}u(\tau)\mathrm{d}\tau \\ &+ \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{A}^D\right)^k \bar{A}^D\bar{B}u^{(k\alpha)}(t), \end{aligned} \tag{1.63a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \Phi(t) = \sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{1.63b}$$

$$u^{(k\alpha)} = \frac{\mathrm{d}^{k\alpha}u(t)}{\mathrm{d}t^{k\alpha}} \tag{1.63c}$$

and the vector $v \in \mathbb{R}^n$ is arbitrary.

Proof. The proof will be accomplished by showing that the solution (1.63) satisfies the equation (1.62a).

Substituting (1.63a) in the left side of the equation (1.62a), using (1.63b), Definition D.2 and Lemma D.1 we obtain

$$
\begin{aligned}
\bar{E}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) =& \bar{E}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left[\Phi_0(t)v + \bar{E}^D\int_0^t \Phi(t-\tau)\bar{B}u(\tau)\mathrm{d}\tau\right.\\
&\left. + \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{A}^D\right)^k \bar{A}^D\bar{B}u^{(k\alpha)}(t)\right]\\
=& \bar{E}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left[v + \sum_{k=1}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^k t^{k\alpha}}{\Gamma\left(k\alpha+1\right)}v + \bar{E}^D\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\bar{B}u(\tau)\mathrm{d}\tau\right.\\
&+ \bar{E}^D\int_0^t\sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^{k+1}(t-\tau)^{(k+2)\alpha}}{\Gamma\left[(k+2)\alpha\right]}\bar{B}u(\tau)\mathrm{d}\tau\\
&\left. + \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{A}^D\right)^k \bar{A}^D\bar{B}u^{(k\alpha)}(t)\right]\\
=& \sum_{k=0}^{\infty}\frac{\bar{E}\left(\bar{E}^D\bar{A}\right)^{k+1}t^{k\alpha}}{\Gamma\left(k\alpha+1\right)}v + \bar{E}^D\bar{B}u(t) + \left(\bar{E}^D\right)^2\bar{A}\int_0^t \Phi(t-\tau)\bar{B}u(\tau)\mathrm{d}\tau\\
&+ \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{A}^D\right)^k \bar{A}^D\bar{B}u^{(k\alpha)}(t)\\
=& \bar{A}x(t) + \bar{B}u(t),
\end{aligned}
$$

since

$$
\begin{aligned}
&\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}v = 0, \quad \bar{E}\left(\bar{E}^D\bar{A}\right)^{k+1} = \bar{A}^{k+1}\left(\bar{E}^D\right)^k,\\
&\Phi(t) = \sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^k t^{(k+1)\alpha-1}}{\Gamma\left[(k+1)\alpha\right]} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^{k+1}t^{(k+2)\alpha-1}}{\Gamma\left[(k+2)\alpha\right]}
\end{aligned}
$$

and (D.6d) holds.

Therefore, the solution (1.63a) satisfies the equation (1.62a). □

From (1.63a) for $t = 0$ we have

$$
x(0) = x_0 = \bar{E}\bar{E}^D v + \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{A}^D\right)^k \bar{A}^D\bar{B}u^{(k\alpha)}(0). \tag{1.64}
$$

Therefore, for given admissible $u(t)$ the consistent initial conditions should satisfy the equality (1.64). In particular case, for $u(t) = 0$, we have $x_0 = \bar{E}\bar{E}^D v$ and $x_0 \in \mathrm{Im}\left(\bar{E}\bar{E}^D\right)$.

Example 1.5. Consider the equation (1.49a) with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad 0 < \alpha < 1. \tag{1.65}$$

The pencil of (1.65) is regular, since

$$\det\left[Es^{\alpha} - A\right] = \begin{vmatrix} s^{\alpha}+1 & 0 \\ 0 & 2 \end{vmatrix} = 2\left(s^{\alpha}+1\right) \neq 0.$$

for almost all $s \in \mathbb{C}$.

We chose $c = 1$ and the matrices (1.62b) take the form

$$\begin{aligned} \bar{E} &= [Ec - A]^{-1} E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{A} &= [Ec - A]^{-1} A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}, \\ \bar{B} &= [Ec - A]^{-1} B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}. \end{aligned} \tag{1.66}$$

Using (D.2) and (1.66) we obtain

$$\bar{E}^{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{A}^{D} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.67}$$

It is easy to check that the matrices (1.67) satisfy the conditions (D.2) and (D.6).

Using (1.67) and (1.63b) we obtain

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{\left(\bar{E}^{D}\bar{A}\right)^{k} t^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \begin{bmatrix} (-1)^{k} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.68a}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{\left(\bar{E}^{D}\bar{A}\right)^{k} t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \begin{bmatrix} (-1)^{k} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.68b}$$

since

$$\bar{E}^{D}\bar{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

From (1.63) and (1.68) we have the desired solution in the form

$$\begin{aligned} x(t) =& \Phi_0(t)v + \bar{E}^D \int_0^t \Phi(t-\tau)\bar{B}u(\tau)\mathrm{d}\tau \\ &+ \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right) \sum_{k=0}^{q-1} \left(\bar{E}\bar{A}^D\right)^k \bar{A}^D \bar{B} u^{(k\alpha)}(t) \\ =& \sum_{k=0}^{\infty} \left[\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \begin{bmatrix} (-1)^k & 0 \\ 0 & 0 \end{bmatrix} v + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \right. \\ &\left. + \frac{1}{\Gamma[(k+1)\alpha]} \begin{bmatrix} (-0.5)^k \\ 0 \end{bmatrix} \int_0^t (t-\tau)^{(k+1)\alpha-1} u(\tau)\mathrm{d}\tau \right], \end{aligned}$$

since

$$\bar{E}\bar{E}^D - \mathbb{I}_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},\ q=1,\ \text{and}\ \bar{A}^D\bar{B} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

for arbitrary $v \in \mathbb{R}^2$.

1.10 Definition of n-Order Difference

Definition 1.13. [71, 154, 163] A discrete-time function defined by

$$\Delta^n x_i = \Delta^{n-1} x_i - \Delta^{n-1} x_{i-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{i-k}, \tag{1.69}$$

$$i = 1, 2, 3, \ldots; \quad n \in \mathbb{Z}, \quad x_i \in \mathbb{R},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{1.70}$$

is called the n-order (backward) difference of the function x_i.

Definition 1.14. [71, 154, 163] The n-order (backward) difference on the interval $[0, k]$ of the function x_i is defined as follows

$${}_0\Delta_k^n x_i = \sum_{j=0}^{k} (-1)^j \binom{n}{j} x_{i-j}. \tag{1.71}$$

From (1.69) it follows that the n-order difference can be written as a linear combination of the values of discrete-time function in $n+1$ points.

The definitions are valid for n being natural numbers and integers.

Note that (1.70) is also well defined for fractional and real numbers. In general case n can be also a complex number.

Example 1.6. From (1.69) we have for

$$\begin{aligned} n=1: &\quad \Delta x_i = x_i - x_{i-1},\\ n=2: &\quad \Delta^2 x_i = \Delta x_i - \Delta x_{i-1} = x_i - 2x_{i-1} + x_{i-2},\\ n=3: &\quad \Delta^3 x_i = \Delta^2 x_i - \Delta^2 x_{i-1} = x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}. \end{aligned}$$

From (1.71) we obtain for
$n=-1$:

$$ {}_0\Delta_k^{-1} x_k = \sum_{j=0}^{k} (-1)^j \binom{-1}{j} x_{k-j} = x_k + x_{k-1} + \cdots + x_0 = \sum_{j=0}^{k} x_{k-j}, $$

$n=-2$:

$$ {}_0\Delta_k^{-2} x_k = \sum_{j=0}^{k} (-1)^j \binom{-2}{j} x_{k-j} = x_k + \cdots + (k+1)x_0 = \sum_{j=0}^{k} (j+1) x_{k-j}. $$

Definition 1.15. [71, 154, 163] The discrete-time function

$$ \Delta^{\alpha} x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j} = \sum_{j=0}^{k} c_j x_{k-j}, \tag{1.72} $$

where

$$ c_j = (-1)^j \binom{\alpha}{j}, \tag{1.73} $$

$\alpha \in \mathbb{R}$ and

$$ \binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k=0,\\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} \quad k=1,2,3,\ldots \end{cases} \tag{1.74} $$

is called the fractional α-order difference of the function x_k.

Example 1.7. Using (1.74) for $0 < \alpha < 1$ we obtain for

$$\begin{aligned} k=1: &\quad (-1)^1 \binom{\alpha}{1} = -\alpha < 0,\\ k=2: &\quad (-1)^2 \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} < 0,\\ k=3: &\quad (-1)^3 \binom{\alpha}{3} = -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} < 0. \end{aligned}$$

Lemma 1.3. The coefficients (1.73) satisfy the relation

$$c_{j+1} = c_j \frac{j-\alpha}{j+1}.$$

Proof. From (1.73) we have

$$c_{j+1} = (-1)^{j+1}\binom{\alpha}{j+1} = (-1)^{j+1}\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)(\alpha-j)}{j!(j+1)} = c_j \frac{j-\alpha}{j+1}.$$

□

1.11 State Equations of the Discrete-Time Fractional Linear Systems

1.11.1 Fractional Systems without Delays

The state equations of the fractional discrete-time linear system have the form

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad 0 \le \alpha \le 1, \tag{1.75a}$$

$$y_k = Cx_k + Du_k, \quad k \in \mathbb{Z}_+, \tag{1.75b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Substituting the definition of fractional difference (1.72) into (1.75a) we obtain

$$x_{k+1} + \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1} = Ax_k + Bu_k \tag{1.76a}$$

or

$$\begin{aligned} x_{k+1} &= Ax_k + \sum_{j=1}^{k+1}(-1)^{j+1}\binom{\alpha}{j}x_{k-j+1} + Bu_k \\ &= A_\alpha x_k + \sum_{j=2}^{k+1}(-1)^{j+1}\binom{\alpha}{j}x_{k-j+1} + Bu_k, \end{aligned} \tag{1.76b}$$

where

$$A_\alpha = A + \alpha \mathbb{I}_n.$$

Remark 1.6. From (1.76b) it follows that the fractional system is equivalent to the system with increasing number of delays.

In practice, it is assumed that j is bounded by natural number q. In this case the equations (1.75) take the form

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^{q}(-1)^j\binom{\alpha}{j+1}x_{k-j} + Bu_k, \quad k \in \mathbb{Z}_+, \tag{1.77a}$$

$$y_k = Cx_k + Du_k. \tag{1.77b}$$

Remark 1.7. The equations (1.77) describe a discrete-time linear system with q delays.

1.11.2 Fractional Systems with Delays

Consider the fractional discrete-time linear system with h delays

$$\Delta^\alpha x_{k+1} = \sum_{i=0}^{h} \left(A_i x_{k-i} + B_i u_{k-i} \right), \quad k \in \mathbb{Z}_+, \tag{1.78a}$$

$$y_k = Cx_k + Du_k, \tag{1.78b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 0, \dots, h$; $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Substituting the definition of fractional difference (1.72) into (1.78a) we obtain

$$x_{k+1} = \sum_{j=1}^{k+1} (-1)^{j+1} \binom{\alpha}{j} x_{k-j+1} + \sum_{i=0}^{h} \left(A_i x_{k-i} + B_i u_{k-i} \right), \tag{1.79a}$$

$$y_k = Cx_k + Du_k, \quad k \in \mathbb{Z}_+. \tag{1.79b}$$

If i is bounded by the natural number L then from (1.79) we obtain

$$x_{k+1} = \sum_{j=1}^{L+1} (-1)^{j+1} \binom{\alpha}{j} x_{k-j+1} + \sum_{i=0}^{h} \left(A_i x_{k-i} + B_i u_{k-i} \right),$$

$$y_k = Cx_k + Du_k, \quad k \in \mathbb{Z}_+ \ .$$

1.12 Solution of the State Equations of the Fractional Discrete-Time Linear System

1.12.1 Fractional Systems with Delays

The state equations of the fractional discrete-time linear system with h delays have the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^{j} \binom{\alpha}{j} x_{k-j+1} = \sum_{r=0}^{h} \left(A_r x_{k-r} + B_r u_{k-r} \right), \quad k \in \mathbb{Z}_+, \tag{1.81a}$$

$$y_k = Cx_k + Du_k, \quad 0 \le \alpha \le 1, \tag{1.81b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A_r \in \mathbb{R}^{n\times n}$, $B_r \in \mathbb{R}^{n\times m}$, $r = 0, 1, \dots, h$; $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, h is the number of delays.

Applying the $\mathcal{Z}$-transform method (see Appendix A.4) we shall derive the solution of the state equation (1.81a) of the fractional system.

Theorem 1.17. *The solution of the equation* (1.81a) *has the form*

$$\begin{aligned}x_k =&\Phi_k x_0 + \sum_{r=0}^{h}\sum_{i=0}^{k-r-1}\Phi_{k-r-1-i}B_r u_i + \sum_{j=1}^{k+1}\sum_{l=-1}^{-j+1}(-1)^{j+1}\binom{\alpha}{j}\Phi_{k-l-j}x_l\\ &+\sum_{r=0}^{h}\sum_{l=-1}^{-r}\Phi_{k-r-l-1}A_r x_l + \sum_{r=0}^{h}\sum_{l=-1}^{-r}\Phi_{k-r-l-1}B_r u_l,\end{aligned} \tag{1.82}$$

where

$$x_k \neq 0, \quad u_k \neq 0, \quad k = 0, -1, \dots, -h$$

are initial conditions and the matrices Φ_k *are determined by the equation*

$$\begin{aligned}\Phi_{k+1} &= \Phi_k\left(A_0 + \alpha I_n\right) + \sum_{i=2}^{k+1}(-1)^{i+1}\binom{\alpha}{i}\Phi_{k-i+1} + \sum_{i=1}^{k}\Phi_{k-i}A_i,\\ \Phi_0 &= \mathbb{I}_n\end{aligned}$$

for $k = 0, 1, \dots$.

Proof. Let $X(z)$ be the $\mathcal{Z}$-transform of the discrete-time function x_i defined by (A.16). Applying the $\mathcal{Z}$-transform (Appendix A.4) to the equation (1.81a) we obtain

$$\mathcal{Z}[x_{k+1}]+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}\mathcal{Z}[x_{k-j+1}] = \sum_{r=0}^{h}A_r\mathcal{Z}[x_{k-r}]+\sum_{r=0}^{h}B_r\mathcal{Z}[u_{k-r}]. \tag{1.84}$$

Using (A.17) to (1.84) we get

$$\begin{aligned}zX(z) - zx_0 &+ \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j+1}\left[X(z) + \sum_{l=-1}^{-j+1}x_l z^{-l}\right]\\ &=\sum_{r=0}^{h}A_r z^{-r}\left[X(z) + \sum_{l=-1}^{-r}x_l z^{-l}\right] + \sum_{r=0}^{h}B_r z^{-r}\left[U(z) + \sum_{l=-1}^{-r}u_l z^{-l}\right],\end{aligned} \tag{1.85}$$

where $U(z) = \mathcal{Z}[u_k]$.

Multiplying (1.85) by z^{-1} and solving with respect to $X(z)$ we obtain

$$X(z) = \left[\sum_{j=0}^{k+1}(-1)^j \binom{\alpha}{j} z^{-j}\mathbb{I}_n - \sum_{r=0}^{h} A_r z^{-r-1}\right]^{-1} \times \left\{x_0 + \sum_{j=1}^{k+1}(-1)^{j+1}\binom{\alpha}{j} z^{-j} \sum_{l=-1}^{-j+1} x_l z^{-l} + \sum_{r=0}^{h} A_r z^{-r-1} \sum_{l=-1}^{-r} x_l z^{-l} + \sum_{r=0}^{h} B_r z^{-r-1}\left[U(z) + \sum_{l=-1}^{-r} u_l z^{-l}\right]\right\}. \tag{1.86}$$

Substitution of the expansion

$$\left[\sum_{j=0}^{k+1}(-1)^j \binom{\alpha}{j} z^{-j}\mathbb{I}_n - \sum_{r=0}^{h} A_r z^{-r-1}\right]^{-1} = \sum_{k=0}^{\infty} \Phi_k z^{-k} \tag{1.87}$$

into (1.86) yields

$$X(z) = \sum_{k=0}^{\infty} \Phi_k z^{-k} x_0 + \sum_{k=0}^{\infty}\sum_{r=0}^{h} \Phi_k z^{-k-r-1} B_r U(z) + \sum_{k=0}^{\infty} \Phi_k z^{-k}\left[\sum_{j=1}^{k+1}\sum_{l=-1}^{-j+1}(-1)^{j+1}\binom{\alpha}{j} x_l z^{-j-l} + \sum_{r=0}^{h}\sum_{l=-1}^{-r} A_r x_l z^{-r-l-1} + \sum_{r=0}^{h}\sum_{l=-1}^{-r} B_r u_l z^{-r-l-1}\right]. \tag{1.88}$$

Applying the inverse $\mathcal{Z}$-transform and the convolution theorem (Appendix A.1) to (1.88) we obtain the desired solution (1.82).

From definition of the inverse matrix we have

$$\left[\sum_{j=0}^{k+1}(-1)^j \binom{\alpha}{j} z^{-j}\mathbb{I}_n - \sum_{r=0}^{h} A_r z^{-r-1}\right]\left[\sum_{k=0}^{\infty} \Phi_k z^{-k}\right] = \mathbb{I}_n. \tag{1.89}$$

Comparison of the coefficients at the same powers of z^{-k}, $k = 0, 1, \ldots$; in (1.89) yields

$$z^0: \quad \Phi_0 \cdot \mathbb{I}_n = \mathbb{I}_n,$$
$$z^{-1}: -A_0 + \Phi_1 - \alpha\mathbb{I}_n = 0 \quad \Rightarrow \quad \Phi_1 = A_0 + \alpha\mathbb{I}_n,$$
$$z^{-2}: \Phi_2 - \Phi_1(A_0 + \alpha\mathbb{I}_n) + \cdots = 0$$
$$\Rightarrow \Phi_2 = \Phi_1(A_0 + \alpha\mathbb{I}_n) - \Phi_0\left[\mathbb{I}_n\binom{\alpha}{2} - A_1\right],$$
$$\vdots$$

and in general case the equation (1.83). □

1.12.2 Fractional Systems with Delays in State Vector

In this subsection we present the method for solution of fractional order discrete-time systems with h delays in state vector given by Busłowicz in [12].

Consider the discrete-time linear system with h delays described by the state equation

$$\Delta^\alpha x_{i+1} = A_0 x_i + \sum_{k=1}^{h} A_k x_{i-k} + Bu_i \tag{1.90}$$

with the initial conditions

$$x_{-k} \in \mathbb{R}^n \quad \text{for} \quad k = 0, 1, \ldots, h; \tag{1.91}$$

where h is a positive number (number of delays), $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are the state and input vectors, respectively, $A_k \in \mathbb{R}^{n\times n}$ $(k = 0, 1, \ldots, h)$, $B \in \mathbb{R}^{n\times m}$ and $\Delta^\alpha x_i$ is the fractional difference of order $\alpha \in \mathbb{R}$ of the discrete-time function x_i given by Definition 1.15.

Substitution of (1.72) for $i + 1$ in (1.90) gives the equation

$$x_{i+1} = F_0 x_i + \sum_{k=1}^{h} A_k x_{i-k} + \sum_{k=1}^{i} c_k(\alpha) x_{i-k} + Bu_i, \tag{1.92}$$

where

$$F_0 = A_0 + \alpha\mathbb{I}_n,$$

and

$$c_k(\alpha) = (-1)^k \binom{\alpha}{k+1} \quad \text{for} \quad k = 1, 2, \ldots . \tag{1.93}$$

The coefficients (1.93) can be computed by the following algorithm [13]

$$c_{k+1}(\alpha) = c_k(\alpha)\frac{k - \alpha + 1}{k + 2} \quad \text{for} \quad k = 1, 2, \ldots$$

with $c_1(\alpha) = 0.5\alpha(1 - \alpha)$.

The equation (1.92) describes the discrete-time linear system with increasing number of delays.

Taking $\mathcal{Z}$-transform to both sides of (1.92) with initial conditions (1.91) leads to

$$\begin{aligned} zX(z) - zx_0 =& F_0 X(z) + \sum_{k=1}^{h} A_k z^{-k} \left[X(z) + \sum_{r=-k}^{-1} x_r z^{-r} \right] \\ &+ \sum_{k=1}^{i} c_k(\alpha) z^{-k} X(z) + BU(z), \end{aligned}$$

where $X(z) = \mathcal{Z}[x_i]$, $U(z) = \mathcal{Z}[u_i]$.

The above equation can be written in the form

$$\Delta(z)X(z) = zx_0 + \sum_{k=1}^{h} A_k z^{-k} \sum_{r=-k}^{-1} x_r z^{-r} + BU(z), \tag{1.94}$$

where

$$\Delta(z) = z\mathbb{I}_n - F_0 - \sum_{k=1}^{h} A_k z^{-k} - \sum_{k=1}^{i} \mathbb{I}_n c_k(\alpha) z^{-k} \tag{1.95}$$

is the characteristic matrix.

Solving the equation (1.94) for $X(z)$ we obtain

$$\begin{aligned} X(z) =& \Delta^{-1}(z) z x_0 + \Delta^{-1}(z) \sum_{k=1}^{h} A_k z^{-k} \sum_{r=-k}^{-1} x_r z^{-r} + \Delta^{-1}(z) BU(z) \\ =& \left[\Delta^{-1}(z) z\right] x_0 + \left[\Delta^{-1}(z) z\right] \sum_{k=1}^{h} A_k \sum_{r=0}^{k-1} x_{r-k} z^{-r-1} \\ &+ \left[\Delta^{-1}(z) z\right] B z^{-1} U(z). \end{aligned} \tag{1.96}$$

Taking the inverse $\mathcal{Z}$-transform to (1.96) we obtain the solution of the equation (1.92) (and (1.90)) in the form

$$x_i = \Phi_i x_0 + \sum_{k=1}^{h} \sum_{r=0}^{k-1} \Phi_{i-r-1} A_k x_{r-k} + \sum_{k=0}^{i-1} \Phi_{i-k-1} B u_k, \tag{1.97}$$

where

$$\Phi_i = \mathcal{Z}^{-1}\left[z\Delta^{-1}(z)\right] \tag{1.98}$$

is the state-transition matrix for the equation (1.92) (and (1.90)).

From (1.98) and (1.95) it follows that the state-transition matrix Φ_i satisfies the equation

$$\Phi_{i+1} = F_0\Phi_i + \sum_{k=1}^{h} A_k\Phi_{i-k} + \sum_{k=1}^{i} c_k(\alpha)\Phi_{i-k} \tag{1.99}$$

with the initial conditions

$$\Phi_0 = \mathbb{I}_n, \quad \Phi_i = 0 \quad \text{for} \quad i < 0. \tag{1.100}$$

Lemma 1.4. The state-transition matrix Φ_i also satisfies the equation

$$\Phi_{i+1} = \Phi_i F_0 + \sum_{k=1}^{h} \Phi_{i-k}A_k + \sum_{k=1}^{i} c_k(\alpha)\Phi_{i-k} \tag{1.101}$$

with initial conditions (1.100).

Proof. Consider the equation

$$y_{i+1} = F_0^T y_i + \sum_{k=1}^{h} A_k^T y_{i-k} + \sum_{k=1}^{i} c_k(\alpha) y_{i-k}, \tag{1.102}$$

where $y_i \in \mathbb{R}^n$.

The state-transition matrix Ψ_i for the equation (1.102) can be computed from the formula

$$\Psi_i = \mathcal{Z}^{-1}\left[z\Delta_1^{-1}(z)\right], \tag{1.103}$$

where

$$\Delta_1(z) = z\mathbb{I}_n - F_0^T - \sum_{k=1}^{h} A_k^T z^{-k} - \sum_{k=1}^{i} \mathbb{I}_n c_k(\alpha) z^{-k}. \tag{1.104}$$

Hence, the state-transition matrix (1.103) satisfies the equation

$$\Psi_{i+1} = F_0^T\Psi_i + \sum_{k=1}^{h} A_k^T\Psi_{i-k} + \sum_{k=1}^{i} c_k(\alpha)\Psi_{i-k} \tag{1.105}$$

with the initial conditions $\Psi_0 = \mathbb{I}_n$, $\Psi_i = 0$ for $i < 0$.

From (1.95) and (1.104) it follows that $\Delta(z) = \Delta_1^T(z)$. Hence

$$\Phi_i = \mathcal{Z}^{-1}\left[z\Delta^{-1}(z)\right] = \mathcal{Z}^{-1}\left[(\Delta_1^{-1}(z))^T z\right] = \left\{\mathcal{Z}^{-1}\left[\Delta_1^{-1}(z)z\right]\right\}^T = \Psi_i^T$$

and from (1.105) one obtains

$$\Phi_{i+1}^T = F_0^T\Phi_i^T + \sum_{k=1}^{h} A_k^T\Phi_{i-k}^T + \sum_{k=1}^{i} c_k(\alpha)\Phi_{i-k}^T,$$

which shows that Φ_i satisfies the equation (1.101). □

From the above considerations we have the following theorem.

Theorem 1.18. *The solution of the fractional system* (1.90) *with initial conditions* (1.91) *has the form* (1.97), *where the state-transition matrices can be computed from the recursive formula* (1.99) *or* (1.101).

1.12.3 Fractional Systems without Delays

Substituting in (1.82) $h = 0$ we obtain the following theorem.

Theorem 1.19. *The solution of the equation* (1.76) *has the form*

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i, \tag{1.106}$$

where the matrices Φ_k are determined by the equation

$$\Phi_{k+1} = \Phi_k \left(A + \alpha \mathbb{I}_n\right) + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1}, \quad \Phi_0 = \mathbb{I}_n. \tag{1.107}$$

Theorem 1.20. *Let*

$$\det \left[\sum_{j=0}^{k+1} (-1)^j \binom{\alpha}{j} \mathbb{I}_n z^{-j} - A z^{-1} \right] = \sum_{i=0}^{M} a_{M-i} z^{-i} \tag{1.108}$$

be the characteristic polynomial of the fractional system (1.76) *for $k = L$. The matrices $\Phi_1, \ldots, \Phi_M$ satisfy the equation*

$$\sum_{i=0}^{M} a_i \Phi_i = 0. \tag{1.109}$$

Proof. From definition of the adjoint matrix and (1.108) we have

$$\mathrm{Adj} \left[\sum_{j=0}^{L+1} (-1)^j \binom{\alpha}{j} \mathbb{I}_n z^{-j} - A z^{-1} \right] = \left(\sum_{i=0}^{\infty} \Phi_i z^{-i} \right) \left(\sum_{i=0}^{M} a_{M-i} z^{-i} \right), \tag{1.110}$$

where Adj$F$ denotes the adjoint matrix of F.

Comparing the coefficients of the same powers of z^{-1} in (1.110), we obtain (1.109), since the degree of the matrix

$$\mathrm{Adj} \left[\sum_{j=0}^{L+1} (-1)^j \binom{\alpha}{j} \mathbb{I}_n z^{-j} - A z^{-1} \right]$$

is less than M. □

Theorem 1.20 is an extension of the well-known Cayley-Hamilton theorem for fractional discrete-time linear systems.

Remark 1.8. The degree M of the characteristic polynomial (1.108) depends on k and it increases to infinity for $k \to \infty$. Usually, it is assumed that k is bounded by natural number L. If $k = L$ then $M = N(L+1)$.

1.13 Positive Fractional Linear Systems

In this section the necessary and sufficient conditions for the positivity of the fractional α-order $(0 < \alpha < 1)$ discrete-time linear system

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \tag{1.111a}$$

$$y_k = Cx_k + Du_k \tag{1.111b}$$

will be established, where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 1.16. The system (1.111) is called (internally) positive fractional system if $x_k \in \mathbb{R}^n_+$, $y_k \in \mathbb{R}^p_+$ for every initial conditions $x_0 \in \mathbb{R}^n_+$ and all inputs $u_k \in \mathbb{R}^m_+$, $k \in \mathbb{Z}_+$.

Lemma 1.5. If $0 < \alpha < 1$, then

$$-c_i = (-1)^{i+1} \binom{\alpha}{i} > 0, \quad i = 1, 2, \dots \ .$$

Proof. The proof will be accomplished by induction. The hypothesis is true for $i = 1$, since

$$(-1)^{1+1} \binom{\alpha}{1} = \alpha > 0.$$

Assuming that $(-1)^{k+1} \binom{\alpha}{k} > 0$ for $k \geq 1$ we shall show that the hypothesis is valid for $k+1$. From (1.70) we have

$$\begin{aligned}(-1)^{k+2} \binom{\alpha}{k+1} &= (-1)^{k+2} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!(k+1)} \\ &= (-1)^{k+1} \binom{\alpha}{k} \frac{k-\alpha}{k+1} > 0.\end{aligned}$$

Therefore, the hypothesis is true for $k+1$. This completes the proof. □

Remark 1.9. In a similar way it can be shown that for $1 < \alpha < 2$

$$(-1)^{i+1}\binom{\alpha}{i} < 0, \quad i = 2, 3, \dots \ .$$

Lemma 1.6. Let $0 < \alpha < 1$ and

$$A + \alpha \mathbb{I}_n \in \mathbb{R}_+^{n\times n},$$

then

$$\Phi_k \in \mathbb{R}_+^{n\times n}, \quad k = 1, 2, \dots \ .$$

Proof. The proof follows immediately from (1.107). □

Theorem 1.21. *The fractional system* (1.111) *is (internally) positive if and only if*

$$A_\alpha = A + \alpha \mathbb{I}_n \in \mathbb{R}_+^{n\times n} \quad \text{and} \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{1.112}$$

Proof. Sufficiency. Sufficiency follows from Lemma 1.6 and equation (1.106). From (1.106) it follows that if $\Phi_k \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $x_0 \in \mathbb{R}_+^n$ then $x_k \in \mathbb{R}_+^n$, $k \in \mathbb{Z}_+$. Similarly from (1.111b) we have $y_k \in \mathbb{R}_+^p$ if the conditions (1.112) are satisfied.
Necessity. Let $u_k = 0$ for $k \in \mathbb{Z}_+$. For positive system, from (1.111) for $k = 0$, we have $x_1 = [A + \alpha\mathbb{I}_n]x_0 = A_\alpha x_0 = A_{\alpha 1} \in \mathbb{R}_+^n$, and $y_0 = Cx_0 \in \mathbb{R}_+^p$. Therefore, $A_\alpha \in \mathbb{R}_+^{n\times n}$ and $C \in \mathbb{R}_+^{p\times n}$, since $x_0 \in \mathbb{R}_+^n$ and by Definition 1.16 it is arbitrary. Assuming $x_0 = 0$ from (1.111) for $k = 0$ we obtain $x_1 = Bu_0 \in \mathbb{R}_+^n$ and $y_0 = Du_0 \in \mathbb{R}_+^p$, and this implies $B \in \mathbb{R}_+^{n\times m}$ and $D \in \mathbb{R}_+^{p\times m}$, since $u_0 \in \mathbb{R}_+^m$ and it is arbitrary. □

Definition 1.17. [71] The fractional discrete-time linear system (1.79) with h
delays is called (internally) positive if $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$ for any initial conditions $x_k \in \mathbb{R}_+^n$, $k = 0, -1, \dots, -h$ and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

Theorem 1.22. *The fractional discrete-time linear system* (1.79) *with h delays is (internally) positive for $0 < \alpha < 1$ if and only if*

$$A_k + c_{k+1}\mathbb{I}_n \in \mathbb{R}_+^{n\times n}, \quad c_k = (-1)^{k+1}\binom{\alpha}{k}, \quad B_k \in \mathbb{R}_+^{n\times m}, \quad k = 1, \dots, h;$$
$$C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}.$$

Proof. The proof is similar to the proof of Theorem 1.21. □

1.14 Externally Positive Fractional Systems

Definition 1.18. [71] The fractional discrete-time linear system (1.111) is called externally positive if for any inputs $u_k \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$ and $x_0 = 0$ we have $y_k \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$.

Definition 1.19. [71]The output of the single-input single-output (SISO) linear system for the unit impulse

$$u_i = \begin{cases} 1 & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i > 0 \end{cases}$$

and zero initial conditions is called the impulse response of the system.

In a similar way we define the matrix of impulse responses g_k of the multi-input multi-output (MIMO) linear systems.

Theorem 1.23. *The fractional discrete-time linear system* (1.111) *is externally positive if and only if*

$$g_k \in \mathbb{R}_+^{p\times m}, \quad k \in \mathbb{Z}_+ \tag{1.113}$$

and the matrix of impulse responses is given by

$$g_k = \begin{cases} D & \text{for} \quad k = 0, \\ C\Phi_{k-1}B & \text{for} \quad k = 1, 2, \dots \end{cases} .$$

Proof. Sufficiency. The output of the system (1.111) with zero initial conditions and any input $u_i \in \mathbb{R}_+^m$ is given by

$$y_k = \sum_{i=0}^{k} g_{k-i}u_i, \quad k \in \mathbb{Z}_+. \tag{1.114}$$

If (1.113) holds and $u_i \in \mathbb{R}_+^m$, then from (1.114) we have $y_k \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$. *Necessity.* Necessity follows immediately from Definition 1.19. □

Remark 1.10. Every (internally) positive linear system is always externally positive. This follows from Definitions 1.16 and 1.18.

Example 1.8. Consider the fractional system (1.75a) for $0 < \alpha < 1$ with the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (n = 2).$$

The system is positive, since

$$A + \alpha\mathbb{I}_n = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}.$$

Using (1.107) for $k = 0, 1, \dots$ we obtain

$$\Phi_1 = (A + \alpha\mathbb{I}_n)\Phi_0 = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.115a}$$

$$\Phi_2 = (A + \alpha \mathbb{I}_n)\Phi_1 - \binom{\alpha}{2}\Phi_0 = \begin{bmatrix} \frac{\alpha^2+5\alpha+2}{2} & 0 \\ 0 & \frac{\alpha(1-\alpha)}{2} \end{bmatrix}, \tag{1.115b}$$

$$\Phi_3 = (A + \alpha \mathbb{I}_n)\Phi_2 - \binom{\alpha}{2}\Phi_1 + \binom{\alpha}{3}\Phi_0$$

$$= \begin{bmatrix} \frac{3(\alpha^2+5\alpha+2)(\alpha+1)-\alpha(\alpha-1)(2\alpha+5)}{6} & 0 \\ 0 & \frac{\alpha(1-\alpha)(2-\alpha)}{6} \end{bmatrix}, \tag{1.115c}$$

$$\vdots$$

From (1.106) and (1.107) we have

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i,$$

where Φ_k is defined by (1.115).

1.15 Fractional Different Orders Discrete-Time Linear Systems

Consider the fractional different orders discrete-time linear system

$$\Delta^\alpha x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k), \quad k \in \mathbb{Z}_+, \tag{1.116a}$$

$$\Delta^\beta x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k), \tag{1.116b}$$

where $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^m$ are state and input vectors, respectively and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m}$, $i, j = 1, 2$.

The fractional difference of α order is defined by

$$\Delta^\alpha x(k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j) = \sum_{j=0}^{k} c_\alpha(j)x(k-j), \tag{1.117a}$$

$$c_\alpha(j) = (-1)^j \binom{j}{\alpha} = (-1)^j \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, \tag{1.117b}$$

$$c_\alpha(0) = 1 \; . \tag{1.117c}$$

Using (1.117) we can write the equation (1.116) in the form

$$
\begin{aligned}
x_1(k+1) &= A_{1\alpha}x_1(k) + A_{12}x_2(k) \\
&\quad - \sum_{j=2}^{k+1} c_\alpha(j)x_1(k-j+1) + B_1u(k), \qquad (1.118a) \\
x_2(k+1) &= A_{21}x_1(k) + A_{2\beta}x_2(k) \\
&\quad - \sum_{j=2}^{k+1} c_\beta(j)x_2(k-j+1) + B_2u(k), \qquad (1.118b)
\end{aligned}
$$

where $A_{1\alpha} = A_{11} + \alpha\mathbb{I}_{n_1}$, $A_{2\beta} = A_{22} + \beta\mathbb{I}_{n_2}$.

Applying the $\mathcal{Z}$-transform to (1.118) we obtain

$$
\begin{bmatrix}
\mathbb{I}_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j)\mathbb{I}_{n_1}z^{-j+1} & -A_{12} \\
-A_{21} & \mathbb{I}_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} c_\beta(j)\mathbb{I}_{n_2}z^{-j+1}
\end{bmatrix}
\times \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} zx_{10} \\ zx_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z), \qquad (1.119)
$$

where $X_i(z) = \mathcal{Z}[x_i(k)] = \sum_{k=0}^{\infty} x_i(k)z^{-k}$, $i = 1, 2$; $U(z) = \mathcal{Z}[u(k)]$ and $x_{10} = x_1(0)$, $x_{20} = x_2(0)$.

From (1.119) we have

$$
\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}
= \begin{bmatrix}
\mathbb{I}_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j)\mathbb{I}_{n_1}z^{-j+1} & -A_{12} \\
-A_{21} & \mathbb{I}_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} c_\beta(j)\mathbb{I}_{n_2}z^{-j+1}
\end{bmatrix}^{-1}
\times \left\{ \begin{bmatrix} zx_{10} \\ zx_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z) \right\}.
$$

Let

$$\begin{bmatrix} \mathbb{I}_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j)\mathbb{I}_{n_1}z^{-j+1} & -A_{12} \\ -A_{21} & \mathbb{I}_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} c_\beta(j)\mathbb{I}_{n_2}z^{-j+1} \end{bmatrix}^{-1}$$

$$= \sum_{j=0}^{\infty} \Phi_j z^{-(j+1)}, \tag{1.120}$$

where the matrices Φ_k are defined by

$$\Phi_i = \begin{cases} \mathbb{I}_n \quad (n = n_1 + n_2) & \text{for} \quad i = 0, \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \cdots - D_{i-1}\Phi_0 & \text{for} \quad i = 1, 2, \ldots, k; \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \cdots - D_k\Phi_{i-k-1} & \text{for} \quad i = k+1, k+2, \ldots \ . \end{cases} \tag{1.121}$$

From definition of inverse matrix we have

$$\left[\mathbb{I}_n z - A - D_1 z^{-1} - D_2 z^{-2} - \cdots - D_k z^{-k}\right]\left[\Phi_0 z^{-1} + \Phi_1 z^{-2} + \cdots\right] = \mathbb{I}_n, \tag{1.122}$$

where

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix}, \quad D_k = \begin{bmatrix} c_\alpha(k+1)\mathbb{I}_{n_1} & 0 \\ 0 & c_\beta(k+1)\mathbb{I}_{n_2} \end{bmatrix}. \tag{1.123}$$

Comparison of the coefficient at the same power of z^{-1} in (1.122) we obtain

$$\Phi_0 = \mathbb{I}_n, \quad \Phi_1 = A\Phi_0, \quad \Phi_2 = A\Phi_1 - D_1\Phi_0,$$
$$\Phi_3 = A\Phi_2 - D_1\Phi_1 - D_2\Phi_0, \ldots$$

which can be written in the form (1.121).

Substitution of (1.120) into (1.119) yields

$$\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \sum_{j=0}^{\infty} \Phi_j z^{-j} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=0}^{\infty} \Phi_j z^{-(j+1)} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z). \tag{1.124}$$

Applying the inverse $\mathcal{Z}$-transform and the convolution theorem to (1.124) we obtain

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Phi_k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i. \tag{1.125}$$

Therefore, the following theorem has been proved.

Theorem 1.24. *The solution to the fractional equation* (1.116) *with initial conditions* $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ *is given by* (1.125)*, where* Φ_k *is defined by* (1.121)*.*

1.16 Positive Fractional Different Orders Discrete-Time Linear Systems

Consider the fractional different orders discrete-time linear systems described by the equation (1.116) and

$$y(k) = C\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Du(k), \tag{1.126}$$

where $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ are the state, input and output vectors and $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 1.20. [71] The fractional system (1.116), (1.126) is called positive if $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$, $y(k) \in \mathbb{R}_+^p$ for any initial conditions $x_{10} \in \mathbb{R}_+^{n_1}$, $x_{20} \in \mathbb{R}_+^{n_2}$ and all inputs $u(k) \in \mathbb{R}_+^m$ for $k \in \mathbb{Z}_+$.

Theorem 1.25. *The fractional discrete-time linear system* (1.116), (1.126) *with* $0 < \alpha < 1$, $0 < \beta < 1$ *is positive if and only if*

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \mathbb{R}_+^{n\times n}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}.$$

Proof. Necessity. Let $e_i^{n_j}$ be i-th column of the $n_j \times n_j$ identity matrix, $j = 1, 2$. From (1.118) for $k = 0$, $u(0) = 0$, $x_{20} = 0$ and $x_{10} = e_i^{n_1}$ we have $x_1(1) = A_{1\alpha}e_i^{n_1} \in \mathbb{R}_+^{n_1}$ and $x_2(1) = A_{21}e_i^{n_1} \in \mathbb{R}_+^{n_1}$. This implies the nonnegativity of i-th ($i = 1, \dots, n$) columns of the matrices $A_{1\alpha}$ and A_{21}. Similarly for $k = 0$, $u(0) = 0$, $x_{10} = 0$ and $x_{20} = e_i^{n_2}$ we have $x_1(1) = A_{12}e_i^{n_2} \in \mathbb{R}_+^{n_2}$ and $x_2(1) = A_{2\beta}e_i^{n_2} \in \mathbb{R}_+^{n_2}$. To show that $B_1 \in \mathbb{R}_+^{n_1\times m}$ and $B_2 \in \mathbb{R}_+^{n_2\times m}$ we assume in (1.118) for $k = 0$, $x_1(0) = 0$, $x_2(0) = 0$ and $u(0) = e_i^m$ and we obtain $x_1(0) = B_1e_i^m \in \mathbb{R}_+^{n_1}$ and $x_2(0) = B_2e_i^m \in \mathbb{R}_+^{n_2}$. In a similar way we prove $C \in \mathbb{R}_+^{p\times n}$ and $D \in \mathbb{R}_+^{p\times m}$.
Sufficiency. In Lemma 1.5 was shown that if $0 < \alpha < 1$ and $0 < \beta < 1$ then $c_\alpha(j) < 0$ and $c_\beta(j) < 0$ for $j = 2, \dots, k+1$. From (1.123) if follows that $D_i \in \mathbb{R}_+^n$ for $i = 1, \dots, k$ and from (1.121) we have $\Phi_i \in \mathbb{R}_+^{n\times n}$ for $i = 0, 1 \dots$, since $A \in \mathbb{R}_+^{n\times n}$. From (1.125) we have $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$, $k \in \mathbb{Z}_+$, since $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n\times m}$ and $u(i) \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$. Finally from (1.126) we have $y(k) \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$, since $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$ and $u(k) \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$. □

These considerations can be extended to discrete-time linear systems consisiting of n subsystems with different fractional orders [69].

1.17 Descriptor Fractional Discrete-Time Linear Systems

Following [76], let us consider the descriptor (singular) fractional discrete-time linear system

$$E\Delta^{\alpha}x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+, \quad 0 < \alpha < 1, \tag{1.127}$$

where $\Delta^{\alpha}x_i$ is the α-order fractional difference of the discrete-time function x_i defined by (1.72), $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the input vector and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

It is assumed that $\det E = 0$, but the pencil (E, A) is regular, i.e.

$$\det[Ez - A] \neq 0 \quad \text{for some} \quad z \in \mathbb{C}.$$

Substitution of (1.72) into (1.127) yields

$$Ex_{i+1} = Fx_i - \sum_{k=1}^{i} Ec_{k+1}x_{i-k} + Bu_i, \quad i \in \mathbb{Z}_+, \tag{1.128a}$$

where

$$F = A - Ec_1. \tag{1.128b}$$

Assuming that

$$\det[Ec - F] \neq 0 \quad \text{for some} \quad c \in \mathbb{R}$$

and premultiplying (1.128a) by $[Ec - F]^{-1}$, we obtain

$$\bar{E}x_{i+1} = \bar{F}x_i - \sum_{k=1}^{i} \bar{E}c_{k+1}x_{i-k} + \bar{B}u_i, \tag{1.129a}$$

where

$$\bar{E} = [Ec - F]^{-1}E, \quad \bar{F} = [Ec - F]^{-1}F, \quad \bar{B} = [Ec - F]^{-1}B. \tag{1.129b}$$

Note that the equations (1.127), (1.128a) and (1.129a) have the same solution x_i.

1.17.1 Solution to the State Equation

We will present the solution to the state equation (1.128a) (and (1.127)) by the use of the Drazin inverses of the matrices $\bar{E}$ and $\bar{F}$ (see Appendix D).

Theorem 1.26. *The solution to the state equation* (1.128a) *with an admissible initial condition x_0 is given by*

$$x_i = \left(\bar{E}^D\bar{F}\right)^i \bar{E}^D\bar{E}x_0 + \sum_{k=0}^{i-1}\bar{E}^D\left(\bar{E}^D\bar{F}\right)^{i-k-1}\left[\bar{B}u_k - \sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\right]$$
$$+ \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{F}^D\right)^k \bar{F}^D\bar{B}u_{i+k}, \tag{1.130}$$

where q *is the index of* $\bar{E}$ *defined by the equality* (D.1)*.*

Proof. Using (1.130) and taking into account (D.2) and (D.6), we obtain

$$\begin{aligned}
\bar{E}x_{i+1} =& \bar{E}\left(\bar{E}^D\bar{F}\right)^{i+1}\bar{E}^D\bar{E}x_0 \\
&+ \sum_{k=0}^{i}\bar{E}\bar{E}^D\left(\bar{E}^D\bar{F}\right)^{i-k}\left[\bar{B}u_k - \sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\right] \\
&+ \bar{E}\left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{F}^D\right)^k\bar{F}^D\bar{B}u_{i+k+1} \\
=& \bar{F}\left(\bar{E}^D\bar{F}\right)^i\bar{E}^D\bar{E}x_0 + \sum_{k=0}^{i-1}\left(\bar{E}^D\bar{F}\right)^{i-k}\left[\bar{B}u_k - \sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\right] \\
&+ \bar{B}u_i - \sum_{j=1}^{i}\bar{E}c_{j+1}x_{i-j} + \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=1}^{q-1}\left(\bar{E}\bar{F}^D\right)^k\bar{B}u_{i+k}
\end{aligned}$$

and

$$\begin{aligned}
\bar{F}x_i =& \bar{F}\left(\bar{E}^D\bar{F}\right)^i\bar{E}^D\bar{E}x_0 + \sum_{k=0}^{i-1}\bar{F}\bar{E}^D\left(\bar{E}^D\bar{F}\right)^{i-k-1}\left[\bar{B}u_k - \sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\right] \\
&+ \bar{F}\left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{F}^D\right)^k\bar{F}^D\bar{B}u_{i+k} \\
=& \bar{F}\left(\bar{E}^D\bar{F}\right)^i\bar{E}^D\bar{E}x_0 + \sum_{k=0}^{i-1}\left(\bar{E}^D\bar{F}\right)^{i-k}\left[\bar{B}u_k - \sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\right] \\
&+ \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right)\sum_{k=0}^{q-1}\left(\bar{E}\bar{F}^D\right)^k\bar{B}u_{i+k}.
\end{aligned}$$

Hence

$$\bar{E}x_{i+1} - \bar{F}x_i + \sum_{k=1}^{i}\bar{E}c_{k+1}x_{i-k} = \bar{B}u_i.$$

Thus, the solution (1.130) satisfies the equation (1.129a). □

From (1.130), for $i = 0$ we have

$$x_0 = \bar{E}^D \bar{E} x_0 + \left(\bar{E}\bar{E}^D - \mathbb{I}_n\right) \sum_{k=0}^{q-1} \left(\bar{E}\bar{F}^D\right)^k \bar{F}^D \bar{B} u_k. \tag{1.131}$$

The set of admissible initial conditions x_0 for given input u_i is given by (1.131).

In a particular case, for $u_i = 0$, $i \in \mathbb{Z}_+$, we have $x_0 = \bar{E}^D \bar{E} x_0$. Thus, the equation $E\Delta^\alpha x_{i+1} = Ax_i$ has a unique solution if and only if $x_0 \in \operatorname{Im}\left(\bar{E}^D \bar{E}\right)$.

Example 1.9. Find the solution x_i to the system (1.127) with $\alpha = 0.5$ and the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{1.132}$$

and admissible initial conditions for given input u_i, $i \in \mathbb{Z}_+$.

The pencil of (1.132) is regular, since

$$\det[Ez - A] = \begin{vmatrix} z & 0 \\ -1 & 2 \end{vmatrix} = 2z$$

for almost all $z \in \mathbb{C}$.

From (1.128b) we have

$$F = A - Ec_1 = A + E\alpha = \begin{bmatrix} \alpha & 0 \\ 1 & -2 \end{bmatrix}, \quad q = 1.$$

For $c = 1$ the matrices (1.129b) have the forms

$$\begin{aligned} \bar{E} =& [Ec - F]^{-1} E = \begin{bmatrix} 1-\alpha & 0 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2(1-\alpha)} \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \\ \bar{F} =& [Ec - F]^{-1} F = \begin{bmatrix} 1-\alpha & 0 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha & 0 \\ 1 & -2 \end{bmatrix} = \frac{1}{2(1-\alpha)} \begin{bmatrix} 2\alpha & 0 \\ 1 & -2(1-\alpha) \end{bmatrix} \\ =& \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \\ \bar{B} =& [Ec - F]^{-1} B = \begin{bmatrix} 1-\alpha & 0 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2(1-\alpha)} \begin{bmatrix} 2 \\ 3-2\alpha \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned} \tag{1.133}$$

Using (D.6) and (1.133), we obtain

$$\bar{E} = T \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad T = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

and

$$\bar{E}^D = T \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix}.$$

Note that

$$\det \bar{F} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1 \neq 0$$

and

$$\bar{F}^D = \bar{F}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Taking into account that

$$\bar{E}^D \bar{F} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix},$$

$$\bar{E}\bar{E}^D = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

and using (1.130) we obtain

$$x_i = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix}^i \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i + \sum_{k=0}^{i-1} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix}^{i-k} \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} u_k - \sum_{j=1}^{k} \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} c_{j+1} x_{k-j} \right\}, \tag{1.134}$$

where the coefficients c_j are defined by (1.73) for $\alpha = 0.5$.

From (1.134), for $i = 0$ we have

$$x_0 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_0. \tag{1.135}$$

Hence, for given u_0, the admissible initial condition x_0 should satisfy (1.135).

Chapter 2
Positive Fractional Electrical Circuits

2.1 Fractional Electrical Circuits

Let the current $i_C(t)$ in a supercondensator (shortly condensator) with the capacity C be the α–order derivative of its charge $q(t)$ [59, 71]

$$i_C(t) = \frac{\mathrm{d}^\alpha q(t)}{\mathrm{d}t^\alpha}, \quad 0 < \alpha < 1, \tag{2.1a}$$

where $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ is the α-order derivative defined by (1.9).

Using $q(t) = Cu_C(t)$ we obtain

$$i_C(t) = C\frac{\mathrm{d}^\alpha u_C(t)}{\mathrm{d}t^\alpha}, \tag{2.1b}$$

where $u_C(t)$ is the voltage on the condensator.

Similarly, let the voltage $u_L(t)$ on the coil (inductor) with the inductance L be the β–order derivative of its magnetic flux $\Psi(t)$ [59, 71]

$$u_L(t) = \frac{\mathrm{d}^\beta \Psi(t)}{\mathrm{d}t^\beta}, \quad 0 < \alpha < 1, \tag{2.2a}$$

where $\frac{\mathrm{d}^\beta}{\mathrm{d}t^\alpha}$ is the β-order derivative defined by (1.9).

Taking into account that $\Psi(t) = Li_L(t)$ we obtain

$$u_L(t) = L\frac{\mathrm{d}^\beta i_L(t)}{\mathrm{d}t^\beta}, \tag{2.2b}$$

where $i_L(t)$ is the current in the coil (inductor).

Consider an electrical circuit composed of resistors, n capacitors and m voltage sources. Using the equation (2.1b) and Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

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$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. The components of the state vector $x(t)$ and the input vector $u(t)$ are the voltage on the condensators and source voltages, respectively.

Consider an electrical circuit composed of resistors, n coils and m sources. Similarly, using the equation (2.2b) and Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{\mathrm{d}^\beta x(t)}{\mathrm{d}t^\beta} = Ax(t) + Bu(t), \quad 0 < \beta < 1, \tag{2.4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. In this case the components of the state vector $x(t)$ are the currents in the coils.

Now let us consider electrical circuit composed of resistances, capacitances, coils and voltage (current) sources. As the state variables (the components of the state vector $x(t)$) we choose the voltages on the capacitors and the currents in the coils. Using the equations (2.1b), (2.2b) and Kirchhoff's laws we may write for the fractional linear circuits in the transient states the state equation

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha x_C}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta x_L}{\mathrm{d}t^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad 0 < \alpha < 1,\ 0 < \beta < 1, \tag{2.5}$$

where the components $x_C = x_C(t) \in \mathbb{R}^{n_1}$ are voltages on the condensators, the components $x_L = x_L(t) \in \mathbb{R}^{n_2}$ are currents in the coils, the components of $u = u(t) \in \mathbb{R}^m$ are the voltage or current sources and

$$A_{ij} \in \mathbb{R}^{n_i\times n_j},\ B_i \in \mathbb{R}^{n_i\times m}; \quad i,j = 1,2.$$

The initial conditions for (2.5) have the form

$$x_0 = \begin{bmatrix} x_{C0} \\ x_{L0} \end{bmatrix} = \begin{bmatrix} x_C(0) \\ x_L(0) \end{bmatrix}. \tag{2.6}$$

Theorem 2.1. *The solution of the equation* (2.5) *for* $0 < \alpha < 1$, $0 < \beta < 1$ *with initial conditions* (2.6) *has the form*

$$x(t) = \Phi_0 x_0 + \int_0^t \left[\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{01}\right] u(\tau)\mathrm{d}\tau, \tag{2.7}$$

where

$$x(t) = \begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix}, \quad B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \tag{2.8a}$$

$$\varPhi_0(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}, \tag{2.8b}$$

$$\varPhi_1(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]}, \tag{2.8c}$$

$$\varPhi_2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]}, \tag{2.8d}$$

$$T_{kl} = \begin{cases} \mathbb{I}_n & \text{for } k=l=0, \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k=1,\ l=0, \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k=0,\ l=1, \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} & \text{for } k+l>0. \end{cases} \tag{2.9}$$

Proof. Using the Laplace transforms ($\mathcal{L}$)

$$X_C(s) = \mathcal{L}[x_C(t)], \quad X_L(s) = \mathcal{L}[x_C(t)], \quad \text{and} \quad U(s) = \mathcal{L}[u(t)]$$

we may write the equation (2.5) for $0 < \alpha < 1$, $0 < \beta < 1$ in the form

$$\begin{bmatrix} s^\alpha X_C(s) \\ s^\beta X_L(s) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_C(s) \\ X_L(s) \end{bmatrix} + \begin{bmatrix} s^{\alpha-1}x_{C0} \\ s^{\beta-1}x_{L0} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s), \tag{2.10}$$

since by (1.10) for $0 < \alpha < 1$, $0 < \beta < 1$

$$\mathcal{L}\left[\frac{\mathrm{d}^\alpha x_C}{\mathrm{d}t^\alpha}\right] = s^\alpha X_C(s) - s^{\alpha-1}x_{C0},$$

$$\mathcal{L}\left[\frac{\mathrm{d}^\beta x_L}{\mathrm{d}t^\beta}\right] = s^\beta X_L(s) - s^{\beta-1}x_{L0}.$$

From (2.10) we have

$$\begin{bmatrix} X_C(s) \\ X_L(s) \end{bmatrix} = \begin{bmatrix} \mathbb{I}_{n_1}s^\alpha - A_{11} & -A_{12} \\ -A_{21} & \mathbb{I}_{n_2}s^\beta - A_{22} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} s^{\alpha-1}x_{C0} \\ s^{\beta-1}x_{L0} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s) \right\}. \tag{2.11}$$

Using (2.9) it is easy to verify that

$$\begin{bmatrix} \mathbb{I}_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}s^{-\beta} & \mathbb{I}_{n_2} - A_{22}s^{-\beta} \end{bmatrix} \left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl}s^{-(k\alpha+l\beta)} \right] = \begin{bmatrix} \mathbb{I}_{n_1} & 0 \\ 0 & \mathbb{I}_{n_2} \end{bmatrix}, \tag{2.12}$$

where the matrices T_{kl} are defined by (2.9).

Using (2.12) we obtain

$$\begin{aligned}
&\begin{bmatrix} \mathbb{I}_{n_1}s^{\alpha} - A_{11} & -A_{12} \\ -A_{21} & \mathbb{I}_{n_2}s^{\beta} - A_{22} \end{bmatrix}^{-1} \\
&= \left\{ \begin{bmatrix} \mathbb{I}_{n_1}s^{\alpha} & 0 \\ 0 & \mathbb{I}_{n_2}s^{\beta} \end{bmatrix} \begin{bmatrix} \mathbb{I}_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}s^{-\beta} & \mathbb{I}_{n_2} - A_{22}s^{-\beta} \end{bmatrix} \right\}^{-1} \\
&= \begin{bmatrix} \mathbb{I}_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}s^{-\beta} & \mathbb{I}_{n_2} - A_{22}s^{-\beta} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{I}_{n_1}s^{-\alpha} & 0 \\ 0 & \mathbb{I}_{n_2}s^{-\beta} \end{bmatrix} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl}s^{-(k\alpha+l\beta)} \begin{bmatrix} \mathbb{I}_{n_1}s^{-\alpha} & 0 \\ 0 & \mathbb{I}_{n_2}s^{-\beta} \end{bmatrix}.
\end{aligned} \tag{2.13}$$

Substitution of (2.13) into (2.11)

$$\begin{bmatrix} X_C(s) \\ X_L(s) \end{bmatrix} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl}$$

$$\times \left[x_0 s^{-(k\alpha+l\beta+1)} + \left(B_{10}s^{-[(k+1)\alpha+l\beta]} + B_{01}s^{-[k\alpha+(l+1)\beta]} \right) U(s) \right]. \tag{2.14}$$

Applying the inverse Laplace transform ($\mathcal{L}^{-1}$) and the convolution theorem to (2.14) we obtain

$$\begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} X_C(s) \\ X_L(s) \end{bmatrix} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl}$$

$$\times \mathcal{L}^{-1}\left[x_0 s^{-(k\alpha+l\beta+1)} + \left(B_{10}s^{-[(k+1)\alpha+l\beta]} + B_{01}s^{-[k\alpha+(l+1)\beta]} \right) U(s) \right]$$

and using (A.3) we obtain the desired solution (2.7). □

2.2 Positive Fractional Electrical Circuits

Definition 2.1. The fractional electrical circuit (2.3) (or (2.4), (2.5)) is called the (internally) positive fractional circuit if the state vector $x(t) \in \mathbb{R}_+^n$, $t \geq 0$ for any initial conditions $x(0) = x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 2.2. *The fractional electrical circuit* (2.3) *(or* (2.4)*) is positive if and only if*

$$A \in M_n, \quad B \in \mathbb{R}_+^{n\times m}.$$

Proof follows immediately from Theorem 1.9.

From Theorem 2.2 applied to the fractional circuit (2.5) it follows that the electrical circuit is positive if and only if

$$A_{kk} \in M_{n_k};\ k = 1,2;\ A_{12} \in \mathbb{R}_+^{n_1 \times n_2},\ A_{21} \in \mathbb{R}_+^{n_2 \times n_1},$$
$$B_1 \in \mathbb{R}_+^{n_1 \times m},\ B_2 \in \mathbb{R}_+^{n_2 \times m}.$$

Theorem 2.3. *The fractional electrical circuit is not positive if each its branch contains resistors, condensator and voltage sources.*

The proof of this theorem follows from the examples given in subsections mentioned below.

2.2.1 Fractional R, C, e Type Electrical Circuits

Example 2.1. Consider the fractional electrical circuit shown in Figure 2.1 with given conductances G_k, $k = 1, 2, 3$; capacitances C_1, C_2, C_3 and source voltages e_1, e_2, e_3.

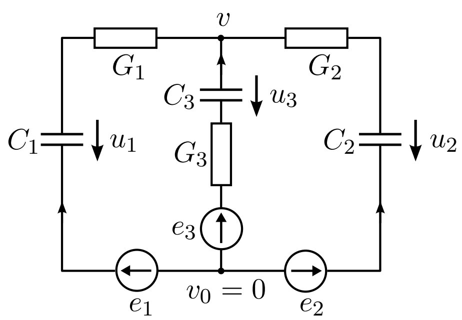

Fig. 2.1 Electrical circuit of Example 2.1

Using Kirchhoff's laws we may write the equations

$$\begin{aligned} C_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} &= G_1(e_1 - u_1 - v), \\ C_2 \frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} &= G_2(e_2 - u_2 - v), \\ C_3 \frac{\mathrm{d}^\alpha u_3}{\mathrm{d}t^\alpha} &= G_3(e_3 - u_3 - v) \end{aligned} \tag{2.15a}$$

and

$$G_3(e_3 - u_3 - v) + G_1(e_1 - u_1 - v) + G_2(e_2 - u_2 - v) = 0. \tag{2.15b}$$

From (2.15b) we have

$$v = \frac{G_1(e_1 - u_1) + G_2(e_2 - u_2) + G_3(e_3 - u_3)}{G_1 + G_2 + G_3}. \tag{2.16}$$

Substitution of (2.16) into (2.15a) yields

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = A_3\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} + B_3\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}, \tag{2.17a}$$

where

$$A_3 = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}, \quad B_3 = \begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix} \tag{2.17b}$$

and

$$\begin{aligned}
a_{11} &= -\frac{G_1(G_2+G_3)}{C_1G}, & a_{12} &= \frac{G_1G_2}{C_1G}, & a_{13} &= \frac{G_1G_3}{C_1G},\\
a_{21} &= \frac{G_1G_2}{C_2G}, & a_{22} &= -\frac{G_2(G_1+G_3)}{C_2G}, & a_{23} &= \frac{G_2G_3}{C_2G},\\
a_{31} &= \frac{G_1G_3}{C_3G}, & a_{32} &= \frac{G_2G_3}{C_3G}, & a_{33} &= -\frac{G_3(G_1+G_2)}{C_3G},
\end{aligned} \tag{2.18a}$$

$$\begin{aligned}
b_{11} &= \frac{G_1(G_2+G_3)}{C_1G}, & b_{12} &= -\frac{G_1G_2}{C_1G}, & b_{13} &= -\frac{G_1G_3}{C_1G},\\
b_{21} &= -\frac{G_1G_2}{C_2G}, & b_{22} &= \frac{G_2(G_1+G_3)}{C_2G}, & b_{23} &= -\frac{G_2G_3}{C_2G},\\
b_{31} &= -\frac{G_1G_3}{C_3G}, & b_{32} &= -\frac{G_2G_3}{C_3G}, & b_{33} &= \frac{G_3(G_1+G_2)}{C_3G},
\end{aligned} \tag{2.18b}$$

where

$$G = G_1 + G_2 + G_3. \tag{2.18c}$$

Remark 2.1. From (2.18) we have

$$\mathbb{1}_3^T\left(\text{diag}\begin{bmatrix}C_1 & C_2 & C_3\end{bmatrix}\right)A_3 = 0 \text{ and } \mathbb{1}_3^T\left(\text{diag}\begin{bmatrix}C_1 & C_2 & C_3\end{bmatrix}\right)B_3 = 0 \tag{2.19}$$

where $\mathbb{1}_3^T = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$.

The equations (2.19) follow from the first Kirchhoff's law

$$C_1\frac{d^\alpha u_1}{dt^\alpha} + C_2\frac{d^\alpha u_2}{dt^\alpha} + C_3\frac{d^\alpha u_3}{dt^\alpha} = 0.$$

From (2.17) and (2.18) it follows that the fractional electrical circuit shown in Figure 2.1 is not positive for any nonzero its parameters C_k, G_k, $k = 1, 2, 3$ and source voltages e_k, $k = 1, 2, 3$.

The fractional electrical circuit is positive if $G_3 = 0$ and $e_1 = e_2 = 0$. In this case we have

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix}u_1\\u_2\end{bmatrix} = A_2\begin{bmatrix}u_1\\u_2\end{bmatrix} + B_2e_3,$$

where

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -\frac{G_1 G_2}{C_1 G} & \frac{G_1 G_2}{C_1 G} \\ \frac{G_1 G_2}{C_2 G} & -\frac{G_1 G_2}{C_2 G} \end{bmatrix} \in M_2,$$

$$B_2 = \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}_+^2.$$

In general case for the fractional electrical circuits with n branches we have the following theorem.

Theorem 2.4. *The fractional electrical circuit is not positive for almost all values of its resistances, capacitances and source voltages if each its branch contains resistor, capacitor and voltage source.*

Proof. From the first Kirchhoff's law we have (Remark 2.1)

$$\mathbb{1}_n^T \left(\text{diag}\left[C_1 \cdots C_n\right]\right) B = 0, \tag{2.21}$$

where C_i is the capacitance of ith $(i = 1, \ldots, n)$ branch and $\mathbb{1}_n^T = \left[1 \cdots 1\right]$.

The equality (2.21) implies that some entries of the matrix B are negative. Therefore, the fractional electrical circuit is not positive. □

Consider the fractional electrical circuit shown in Figure 2.2 with given conductances G_k, $k = 0, 1, \ldots, n$; capacitances C_j, $j = 1, \ldots, n$ and source voltage e.

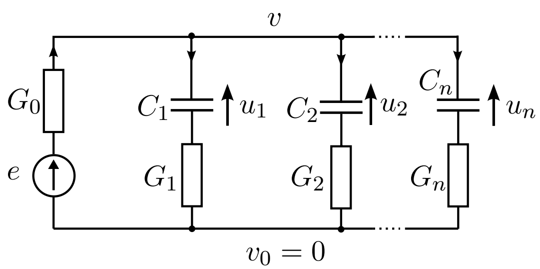

Fig. 2.2 Electrical circuit of R, C, e type

Using (2.1b) and Kirchhoff's laws we may write the equations

$$C_k \frac{\mathrm{d}^\alpha u_k}{\mathrm{d}t^\alpha} = G_k(v - u_k), \quad k = 1, \ldots, n \tag{2.22}$$

and

$$G_0(e - v) = \sum_{j=1}^{n} G_j(v - u_j). \tag{2.23}$$

From (2.23) we have

$$v = \frac{1}{G}\left(G_0 e + \sum_{j=1}^{n} G_j u_j\right), \quad G = \sum_{i}^{n} G_i. \tag{2.24}$$

Substitution of (2.24) into (2.22) yields

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + Be, \tag{2.25}$$

where

$$A = \begin{bmatrix} -\frac{G_1 G - G_1^2}{C_1 G} & \frac{G_1 G_2}{C_1 G} & \dots & \frac{G_1 G_n}{C_1 G} \\ \frac{G_2 G_1}{C_2 G} & -\frac{G_2 G - G_2^2}{C_2 G} & \dots & \frac{G_2 G_n}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_n G_1}{C_n G} & \frac{G_n G_2}{C_n G} & \dots & -\frac{G_n G - G_n^2}{C_n G} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{G_o G_1}{C_1 G} \\ \frac{G_0 G_2}{C_2 G} \\ \vdots \\ \frac{G_0 G_n}{C_n G} \end{bmatrix}. \tag{2.26}$$

From (2.26) it follows that $A \in M_n$ and $B \in \mathbb{R}_+^n$. Therefore, the following theorem has been proved.

Theorem 2.5. *The fractional electrical circuit shown on Figure 2.2 is positive for any values of the conductances G_k, $k = 0, 1, \dots, n$; capacitances C_j, $j = 1, \dots, n$ and source voltage e.*

Example 2.2. Consider the fractional electrical circuit shown in Figure 2.3 with given conductances G_k, $k = 1, \dots, 6$; capacitances C_1, C_2 and source voltages e_2, e_4.

Using Kirchhoff's laws we may write the equations

$$\begin{aligned} C_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} &= G_1(v_1 - u_1), \\ C_2 \frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} &= G_2(e_2 - u_2 - v_3) \end{aligned} \tag{2.27a}$$

and

$$\begin{aligned} &G_1 u_1 - (G_1 + G_3 + G_6)v_1 + G_6 v_2 + G_3 v_3 = 0, \\ &G_6 v_1 - (G_4 + G_5 + G_6)v_2 + G_4 v_3 + G_4 e_4 = 0, \\ &-G_2 u_2 + G_3 v_1 + G_4 v_2 - (G_2 + G_3 + G_4)v_3 + G_2 e_2 - G_4 e_4 = 0. \end{aligned} \tag{2.27b}$$

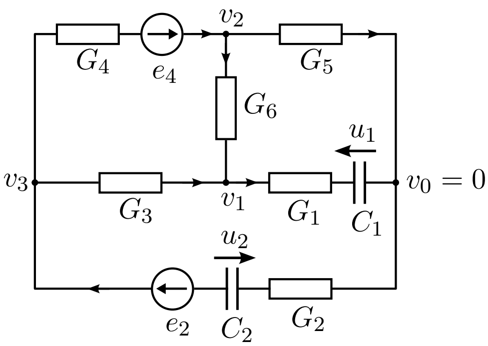

Fig. 2.3 Electrical circuit of Example 2.2

The equations (2.27) can be rewritten in the form

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}-\frac{G_1}{C_1} & 0\\ 0 & -\frac{G_2}{C_2}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}\frac{G_1}{C_1} & 0 & 0\\ 0 & 0 & -\frac{G_2}{C_2}\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}0 & 0\\ \frac{G_2}{C_2} & 0\end{bmatrix}\begin{bmatrix}e_2\\e_4\end{bmatrix}\tag{2.28a}$$

and

$$G\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=-\begin{bmatrix}G_1 & 0\\ 0 & 0\\ 0 & -G_2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & G_4\\ G_2 & -G_4\end{bmatrix}\begin{bmatrix}e_2\\e_4\end{bmatrix},\tag{2.28b}$$

where the matrix

$$G=\begin{bmatrix}-(G_1+G_3+G_2) & G_6 & G_3\\ G_6 & -(G_4+G_5+G_6) & G_4\\ G_3 & G_4 & -(G_2+G_3+G_4)\end{bmatrix}$$

is a Metzler matrix and $-G^{-1}\in\mathbb{R}_+^{3\times 3}$.

From (2.28b) we have

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=-G^{-1}\begin{bmatrix}G_1 & 0\\ 0 & 0\\ 0 & -G_2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}-G^{-1}\begin{bmatrix}0 & 0\\ 0 & G_4\\ G_2 & -G_4\end{bmatrix}\begin{bmatrix}e_2\\e_4\end{bmatrix}.\tag{2.29}$$

Substituting (2.29) into (2.28a) we obtain

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix}u_1\\u_2\end{bmatrix}=A\begin{bmatrix}u_1\\u_2\end{bmatrix}+B\begin{bmatrix}e_2\\e_4\end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{G_1}{C_1} & 0 \\ 0 & -\frac{G_2}{C_2} \end{bmatrix} - \begin{bmatrix} \frac{G_1}{C_1} & 0 & 0 \\ 0 & 0 & -\frac{G_2}{C_2} \end{bmatrix} G^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & 0 \\ 0 & -G_2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ \frac{G_2}{C_2} & 0 \end{bmatrix} - \begin{bmatrix} \frac{G_1}{C_1} & 0 & 0 \\ 0 & 0 & -\frac{G_2}{C_2} \end{bmatrix} G^{-1} \begin{bmatrix} 0 & 0 \\ 0 & G_4 \\ G_2 & -G_4 \end{bmatrix}.$$

The fractional electrical circuit is positive if and only if the matrix A is the Metzler matrix and the matrix B has nonnegative entries.

In general case, let us consider the fractional electrical circuit composed of q conductances G_k, $k = 1, \ldots, q$; r capacitances C_i, $i = 1, \ldots, r$ and m source voltages e_j, $j = 1, \ldots, m$. Let n be the number of linearly independent nodes of the electrical circuit and $n > r$.

Using Kirchhoff's laws we may write the equation

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A_r \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + A_n \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + B_m \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}, \tag{2.30}$$

where u_i is the voltage on the i-th ($i = 1, \ldots, r$) capacitor, v_j is the voltage of the j-th node ($j = 1, \ldots, n$), $A_r \in \mathbb{R}^{r \times r}$ is the diagonal Metzler matrix, $A_n \in \mathbb{R}^{r \times n}$ and $B_m \in \mathbb{R}^{r \times m}$.

Using the node method we obtain

$$G \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -F \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - H \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}, \tag{2.31}$$

where $G \in \mathbb{R}^{n \times n}$ is a Metzler matrix, $F \in \mathbb{R}^{n \times r}$ and $H \in \mathbb{R}^{n \times m}$.

Taking into account that $-G^{-1} \in \mathbb{R}_+^{n \times n}$ from (2.31) we obtain

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -G^{-1}F \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - G^{-1}H \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}. \tag{2.32}$$

Substitution of (2.32) into (2.30) yields

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + B \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}, \tag{2.33}$$

where

$$A = A_r - A_n G^{-1} F, \quad B = B_m - A_n G^{-1} H.$$

The electrical circuit described by the equation (2.33) is positive if and only if the matrix A is a Metzler matrix and the matrix B has non-negative entries.

Therefore, the following theorem has been proved.

Theorem 2.6. *The linear electrical circuit composed of q resistors (conductances), r capacitors and m source voltages is positive if and only if $r < n$ and*

$$A = A_r - A_n G^{-1} F \in M_r, \quad B = B_m - A_n G^{-1} H \in \mathbb{R}_+^{r \times m}.$$

2.2.2 Fractional R, L, e Type Electrical Circuits

Consider the electrical circuit shown on Figure 2.4 with given resistances R_1, R_2, R_3; inductances L_1, L_2, L_3 and source voltages e_1, e_2.

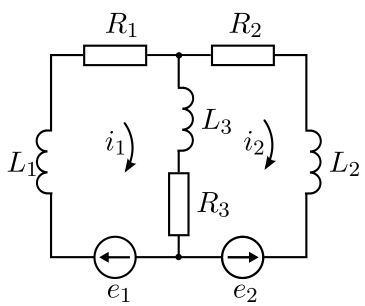

Fig. 2.4 Electrical circuit of R, L, e type

Using (2.2b) and the mesh method for the electrical circuit we obtain the following equations

$$\begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.34a}$$

where

$$\begin{aligned} R_{11} &= R_1 + R_3, \quad R_{12} = R_{21} = R_3, \quad R_{22} = R_2 + R_3, \\ L_{11} &= L_1 + L_3, \quad L_{12} = L_{21} = L_3, \quad L_{22} = L_2 + L_3. \end{aligned} \tag{2.34b}$$

Note that the inverse matrix

$$L^{-1} = \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix}^{-1} = \frac{1}{L_1(L_2 + L_3) + L_2 L_3} \begin{bmatrix} L_{22} & L_{12} \\ L_{21} & L_{11} \end{bmatrix}$$

has positive entries.

From (2.34) we have

$$\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

$$\begin{aligned} A &= L^{-1}\begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix} \\ &= \frac{1}{L_1(L_2+L_3)+L_2L_3}\begin{bmatrix} -L_2(R_1+R_3)-L_3R_1 & L_2R_3-L_3R_2 \\ L_1R_3-L_3R_1 & -L_1(R_2+R_3)-L_3R_2 \end{bmatrix} \end{aligned} \tag{2.35}$$

and $B = L^{-1} \in \mathbb{R}_+^{2\times 2}$.

From (2.35) it follows that $A \in M_2$ if and only if

$$L_2R_3 \geq L_3R_2 \quad \text{and} \quad L_1R_3 \geq L_3R_1. \tag{2.36}$$

Therefore, the fractional electrical circuit is positive if and only if $A \in M_2$, i.e. the condition (2.36) is met.

In general case, let us consider the fractional n-mesh electrical circuit with given resistances R_k, $k = 1, \ldots, q$; inductances $L_1, \ldots, L_r$ for $r \geq n$ and $m \leq n$ mesh source voltages e_{jj}, $j = 1, \ldots, m$. Denote by $i_1, \ldots, i_n$ the mesh currents. In a similar way as for the electrical circuit shown on Figure 2.4 using the mesh method we obtain the equation

$$L\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A'\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix}, \tag{2.37a}$$

where

$$L = \begin{bmatrix} L_{11} & -L_{12} & \cdots & -L_{1n} \\ -L_{21} & L_{22} & \cdots & -L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -L_{n1} & -L_{n2} & \cdots & L_{nn} \end{bmatrix}, \quad A' = \begin{bmatrix} -R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & -R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & -R_{nn} \end{bmatrix}. \tag{2.37b}$$

Note that $-L \in M_n$, $A' \in M_n$ and $L^{-1} \in M_n$.

Premultiplying (2.37a) by L^{-1} we obtain

$$\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + B\begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix}, \tag{2.38a}$$

where

$$A = L^{-1}A', \quad B = L^{-1} \in \mathbb{R}^{n \times n}. \tag{2.38b}$$

Therefore, the electrical circuit (2.38a) is positive if and only if $A = L^{-1}A' \in \mathbb{R}_+^{n \times n}$.

Example 2.3. Consider the fractional electrical circuit shown in Figure 2.5 with given resitances R_k, $k = 1, \ldots, 8$; inductances L_1, L_2 and source voltages e_3, e_4, e_8.

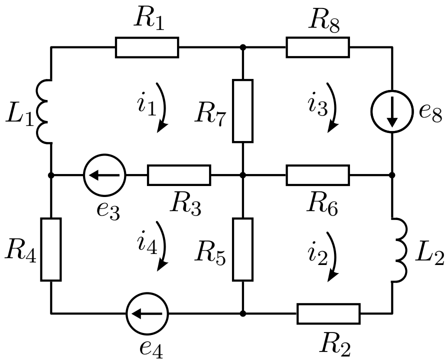

Fig. 2.5 Electrical circuit of Example 2.3

Denote by i_k, $k = 1, \ldots, 4$ the mesh currents and by e_{kk}, $k = 1, \ldots, 4$ the mesh source voltages. Using the mesh method we obtain the following equations

$$\begin{aligned} L_1 \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} &= -R_{11}i_1 + R_{13}i_3 + R_{14}i_4 + e_{11}, \\ L_2 \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} &= -R_{22}i_2 + R_{23}i_3 + R_{24}i_4, \end{aligned} \tag{2.39a}$$

$$\begin{aligned} 0 &= R_{31}i_1 + R_{32}i_2 - R_{33}i_3 + e_{33}, \\ 0 &= R_{41}i_1 + R_{42}i_2 - R_{44}i_4 + e_{44}, \end{aligned} \tag{2.39b}$$

where

$$\begin{aligned} R_{11} &= R_1 + R_3 + R_7, \quad R_{13} = R_{31} = R_7, \quad R_{14} = R_{41} = R_3, \\ R_{22} &= R_2 + R_5 + R_6, \quad R_{23} = R_{32} = R_6, \quad R_{24} = R_{42} = R_5, \\ R_{33} &= R_6 + R_7 + R_8, \quad R_{44} = R_3 + R_4 + R_5, \\ e_{11} &= e_3, \quad e_{33} = e_8, \quad e_{44} = e_4 - e_3. \end{aligned} \tag{2.39c}$$

From (2.39b) we have

$$i_1 = \frac{1}{R_{33}}\left(R_{31}i_1 + R_{32}i_2 + e_{33}\right), \quad i_4 = \frac{1}{R_{44}}\left(R_{41}i_1 + R_{42}i_2 + e_{44}\right). \tag{2.40}$$

Substitution of (2.40) into (2.39a) yields

$$
\begin{aligned}
L_1\frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} &= -R'_{11}i_1 + R'_{12}i_2 + e'_{11},\\
L_2\frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} &= -R'_{21}i_1 - R'_{22}i_2 + e'_{22},
\end{aligned}
\tag{2.41a}
$$

where

$$
\begin{aligned}
R'_{11} &= R_{11} - \frac{R_{13}R_{31}}{R_{33}} - \frac{R_{14}R_{41}}{R_{44}}, \quad R'_{12} = R'_{21} = \frac{R_{31}R_{32}}{R_{33}} + \frac{R_{14}R_{42}}{R_{44}},\\
e'_{11} &= e_{11} + \frac{R_{13}}{R_{33}}e_{33} + \frac{R_{14}}{R_{44}}e_{44}, \qquad R'_{22} = R_{22} - \frac{R_{23}R_{32}}{R_{33}} - \frac{R_{24}R_{42}}{R_{44}},\\
e'_{22} &= \frac{R_{23}}{R_{33}}e_{33} + \frac{R_{24}}{R_{44}}e_{44}.
\end{aligned}
\tag{2.41b}
$$

If the mesh currents i_1, i_2 are chosen as the state variables $x_1 = i_1$, $x_2 = i_2$ then the equations (2.41a) can be written in the form

$$
\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B\begin{bmatrix} e'_{11} \\ e'_{22} \end{bmatrix}, \tag{2.42a}
$$

where

$$
A = \begin{bmatrix} -\frac{R'_{11}}{L_1} & \frac{R'_{12}}{L_1} \\ \frac{R'_{21}}{L_2} & -\frac{R'_{22}}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{2.42b}
$$

From (2.42b) and (2.41b) if followis that A is the Metzler matrix. If we choose

$$
y_1 = R_1 i_1, \quad y_2 = R_8 i_3 - \frac{R_8}{R_{33}}e_{33}
$$

as the output then

$$
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ \frac{R_8R_{31}}{R_{33}} & \frac{R_8R_{32}}{R_{33}} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}
$$

and

$$
C = \begin{bmatrix} R_1 & 0 \\ \frac{R_8R_{31}}{R_{33}} & \frac{R_8R_{32}}{R_{33}} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.43}
$$

The matrices (2.42b) and (2.43) satisfy condition (1.29) and by Theorem 1.9 the fractional electrical circuit is positive.

In general case, let us consider the n-mesh fractional electrical circuit with some given resistances R_k, $k = 1, \dots, q$; inductances $L_1, \dots, L_r$ and m-mesh source voltages e_{jj}, $j = 1, \dots, m$. Let $i_1, \dots, i_n$ be the mesh currents of the fractional electrical circuit.

Using the mesh method in a similar way as for the electrical circuit shown in Figure 2.5 we obtain the equation

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{2.44a}$$

where

$$x_1 = \begin{bmatrix} i_1 \\ \vdots \\ i_r \end{bmatrix}, \quad x_2 = \begin{bmatrix} i_{r+1} \\ \vdots \\ i_n \end{bmatrix}, \quad u = \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix}, \tag{2.44b}$$

$$A_{11} = \begin{bmatrix} -\frac{R_{11}}{L_1} & \frac{R_{12}}{L_1} & \cdots & \frac{R_{1r}}{L_1} \\ \frac{R_{21}}{L_2} & -\frac{R_{22}}{L_2} & \cdots & \frac{R_{2r}}{L_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{R_{r1}}{L_r} & \frac{R_{r2}}{L_r} & \cdots & -\frac{R_{rr}}{L_r} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \frac{R_{1,r+1}}{L_1} & \cdots & \frac{R_{1n}}{L_1} \\ \vdots & \ddots & \vdots \\ \frac{R_{r,r+1}}{L_r} & \cdots & \frac{R_{rn}}{L_r} \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} R_{r+1,1} & \cdots & R_{r+1,r} \\ \vdots & \ddots & \vdots \\ R_{n1} & \cdots & R_{n,r} \end{bmatrix}, A_{22} = \begin{bmatrix} -R_{r+1,r+1} & R_{r+1,r+2} & \cdots & R_{r+1,n} \\ R_{r+2,r+1} & -R_{r+2,r+2} & \cdots & R_{r+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,r+1} & R_{n,r+2} & \cdots & -R_{n,n} \end{bmatrix}, \tag{2.44c}$$

$$R_{ij} = R_{ji} = \begin{cases} > 0 \text{ for } i = j, \\ \geq 0 \text{ for } i \neq j. \end{cases} \tag{2.44d}$$

It is well-known [47] that $(-A_{22})^{-1} \in \mathbb{R}_+^{(n-r)\times(n-r)}$ and from (2.44a) we have

$$x_2 = (-A_{22})^{-1}(A_{21}x_1 + B_2 u) \in \mathbb{R}_+^{n-r} \tag{2.45}$$

for $x_1 \in \mathbb{R}_+^r$ and $u \in \mathbb{R}_+^m$.

Substituting (2.45) into

$$\frac{\mathrm{d}^\beta x_1}{\mathrm{d}t^\beta} = A_{11}x_1 + A_{12}x_2 + B_1 u$$

we obtain

$$\begin{aligned}\frac{d^\beta x_1}{dt^\beta} &= \left[A_{11} + A_{12}(-A_{22})^{-1}A_{21}\right]x_1 + \left[B_1 + A_{12}(-A_{22})^{-1}B_2\right]u \\ &= A'_{11}x_1 + B'_1 u,\end{aligned}$$

where

$$A'_{11} = A_{11} + A_{12}(-A_{22})^{-1}A_{21} \in M_n, \quad B'_1 = B_1 + A_{12}(-A_{22})^{-1}B_2 \in \mathbb{R}_+^{n\times m}. \tag{2.46}$$

In what follows it is assumed that by suitable choice of the outputs of the electrical circuit the matrices C and D have nonnegative entries, i.e.

$$C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{2.47}$$

Therefore, the following theorem has been proved.

Theorem 2.7. *The fractional linear electrical circuit composed of resistors, coils and voltage sources is positive for $r > n$ if its resistances and inductances satisfy the conditions* (2.46) *and* (2.47)*.*

Remark 2.2. In the case $r = n$, if it is possible to choose the n linearly independent meshes so that to each mesh belongs only one coil, the matrix $L = \mathrm{diag}[L_1, \ldots, L_n]$ and the condition (2.38b) is met for any values of the resistances and inductances of the electrical circuit.

Remark 2.3. Note that it is impossible to choose the n linearly independent meshes so that to each mesh belongs only one coil if all branches belonging to the same node contain the coils. In this case we can eliminate one of the branch currents using the fact that the sum of the currents in the coils is equal to zero.

From Theorem 2.7 and Remark 2.2 we have the following important theorem.

Theorem 2.8. *The fractional linear electrical circuit composed of resistors, coils and voltage sources is positive for almost all values of the resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes and the directions of the mesh currents are consistent with the directions of the mesh source voltages.*

2.2.3 Fractional R, L, C Type Electrical Circuits

Consider the fractional electrical circuit shown in Figure 2.6 with given resistance R, inductance L, capacitance C and source voltage e.

Using Kirchhoff's laws we can write the equations

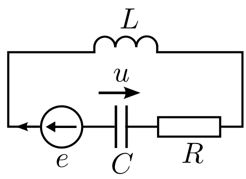

Fig. 2.6 Electrical circuit of R, L, C type

$$i = C\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha},$$
$$e = Ri + L\frac{\mathrm{d}^\beta i}{\mathrm{d}t^\beta} + u,$$

which can be written in the form

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta i}{\mathrm{d}t^\beta} \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be,$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

The matrix A has negative off-diagonal entry $(-1/L)$ and it is not a Metzler matrix for any values of R, L, C. Therefore, the fractional electrical circuit is not positive one for any values of the resistance R, inductance L and capacitance C.

Example 2.4. Consider the fractional electrical circuit shown in Figure 2.7 with given resistances R_1, R_2; inductance L, capacitance C and source voltage e_1, e_2.

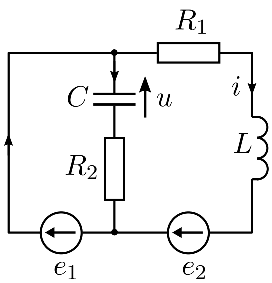

Fig. 2.7 Fractional electrical circuit of Example 2.4

Using Kirchhoff's laws we can write the equations

$$e_1 = R_2 C \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + u,$$
$$e_1 + e_2 = R_1 i + L \frac{\mathrm{d}^\beta i}{\mathrm{d}t^\beta},$$

which can be written in the form

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta i}{\mathrm{d}t^\beta} \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{1}{R_2 C} & 0 \\ 0 & -\frac{R_1}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_2 C} & 0 \\ \frac{1}{L} & \frac{1}{L} \end{bmatrix}. \tag{2.49}$$

From (2.49) it follows that the matrix A is a Metzler matrix and the matrix B has nonnegative entries. Therefore, the R, L, C type fractional electrical circuit is positive for any values of the resistances R_1, R_2; inductance L and capacitance C.

In general case we have the following theorem.

Theorem 2.9. *The fractional electrical circuits of R, L, C type is not positive for almost all values of its resistances, inductances, capacitances and source voltages if at least one its branch contains inductance and capacitance.*

Proof. It is well-known that the linear independent meshes of the electrical circuits can be chosen so that the branch containing the inductance L and capacitance C belongs to the first one. The equation for the first mesh contains the following term

$$e_{11} = L \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} + u_1 + \dots, \tag{2.50}$$

where e_{11} and i_1 are the source voltage and current of the first mesh and u_1 is the voltage on the capacitance C. From (2.50) and

$$i_1 = C \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha}$$

it follows that the matrix A of the electrical circuit has at least one negative off-diagonal entry. Therefore, the matrix A is not a Metzler matrix and the electrical circuit is not positive one. □

Consider the electrical cicuit shown on Figure 2.8 with given resistances R_k, $k = 1, \dots, n$; inductances $L_2, L_4, \dots, L_{n_2}$; capacitances $C_1, C_3, \dots, C_{n_1}$ and source voltages $e_1, e_2, e_4, \dots, e_{n_2}$ $(n = n_1 + n_2)$.

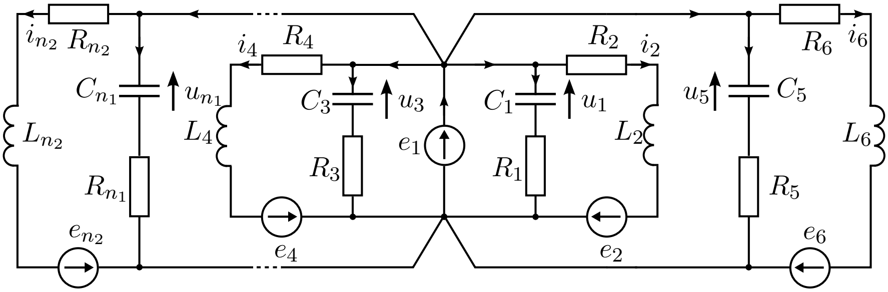

Fig. 2.8 Fractional electrical circuit

Using Kirchhoff's laws we can write the equations

$$e_1 = R_k C_k \frac{\mathrm{d}^\alpha u_k}{\mathrm{d}t^\alpha} + u_k \quad \text{for} \quad k = 1, 3 \ldots, n_1;$$

$$e_1 + e_j = L_j \frac{\mathrm{d}^\beta i_j}{\mathrm{d}t^\beta} + R_j i_j \quad \text{for} \quad j = 2, 4 \ldots, n_2;$$

which can be written in the form

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta i}{\mathrm{d}t^\beta} \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be, \tag{2.52a}$$

where

$$u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n_1} \end{bmatrix}, \quad i = \begin{bmatrix} i_1 \\ i_4 \\ \vdots \\ i_{n_2} \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_4 \\ \vdots \\ e_n \end{bmatrix} \tag{2.52b}$$

and

$$A = \operatorname{diag}\left[-\frac{1}{R_1C_1}, -\frac{1}{R_3C_3}, \dots, -\frac{1}{R_{n_1}}, -\frac{R_2}{L_2}, -\frac{R_4}{L_4}, \dots, -\frac{R_{n_2}}{L_{n_2}}\right] \in M_n, \tag{2.52c}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n\times\left(\frac{n_2}{2}+1\right)}, \quad B_1 = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 & \cdots & 0 \\ \frac{1}{R_3C_3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{R_{n_1}C_{n_1}} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.52d}$$

$$B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & \cdots & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{n_2}} & 0 & 0 & \cdots & \frac{1}{L_{n_2}} \end{bmatrix}.$$

The electrical circuit described by the equation (2.52) is positive for all value of resistances R_k, $k = 1, \dots, n$; inductances L_k, $k = 2, 4, \dots, n_2$ and capacitances C_k, $k = 1, 3, \dots, n_1$.

Therefore, the following theorem has been proved.

Theorem 2.10. *The fractional linear electrical circuit of the structure shown on Figure 2.8 is positive for any values of its resistances, inductances and capacitances.*

Example 2.5. Consider the fractional electrical circuit shown in Figure 2.9 with given resistances R_k, $k = 1, \dots, 5$; inductances L_1, L_2; capacitance C and source voltage e_1.

Using Kirchhoff's laws we can write the equations

$$\begin{aligned} e_1 &= R_1 i_1 + L_1 \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} - R_5 i_2 + (R_3 + R_5) i_3, \\ 0 &= L_2 \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} + u + (R_2 + R_3) i_3 - R_2 i_1 \\ i_2 &= C \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \end{aligned} \tag{2.53a}$$

and

$$(R_2 + R_4) i_1 + (R_4 + R_5) i_2 - (R_2 + R_3 + R_4 + R_5) i_3 = 0. \tag{2.53b}$$

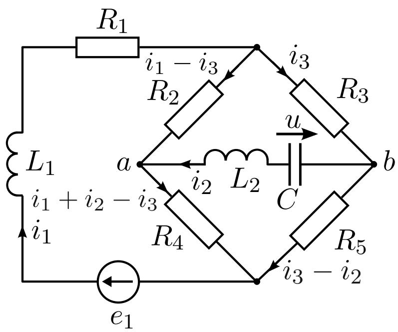

Fig. 2.9 Electrical circuit of Example 2.5

From (2.53b) we have

$$i_3 = \frac{(R_2 + R_4)i_1 + (R_4 + R_5)i_2}{R_2 + R_3 + R_4 + R_5}. \tag{2.54}$$

Substituting (2.54) into (2.53a) we obtain

$$\begin{bmatrix} \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + Be_1, \tag{2.55}$$

where

$$A = \begin{bmatrix} -\frac{R_1}{L_1} - \frac{(R_2 + R_4)(R_3 + R_5)}{L_1(R_2 + R_3 + R_4 + R_5)} & \frac{R_2R_5 - R_3R_4}{L_1(R_2 + R_3 + R_4 + R_5)} & 0 \\ \frac{R_2R_5 - R_3R_4}{L_2(R_2 + R_3 + R_4 + R_5)} & -\frac{(R_2 + R_3)(R_4 + R_5)}{L_2(R_2 + R_3 + R_4 + R_5)} & -\frac{1}{L_2} \\ 0 & \frac{1}{C} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}. \tag{2.56}$$

From (2.56) it follows that the matrix A is not a Metzler matrix if

$$R_2R_5 = R_3R_4. \tag{2.57}$$

If the condition (2.57) is met then the voltage between the points a and b is equal to zero and $u = 0$, $\frac{d^\beta i_2}{dt^\beta} = 0$, $i_2 = 0$. In this case the equation (2.55) takes the form

$$\frac{d^\beta i_1}{dt^\beta} = \left[-\frac{R_1}{L_1} - \frac{(R_2+R_4)(R_3+R_5)}{L_1(R_2+R_3+R_4+R_5)}\right] i_1 + \frac{1}{L_1} e_1. \tag{2.58}$$

The system described by the equation (2.58) is positive. Therefore, we have the following corollary.

Corollary 2.1. If the resistances of the fractional electrical circuit shown in Figure 2.9 satisfy the condition (2.57), then the fractional electrical circuit is positive.

In general case we have what follows.

Corollary 2.2. Nonpositive fractional electrical circuit for some special choice of the parameters (resistances) can be positive one.

Example 2.6. Consider the linear electrical circuit shown on Figure 2.10 with known resistances R_1, R_2, R_3; capacitances C_1, C_2; inductances L_1, L_2 and sources voltages e_1, e_2.

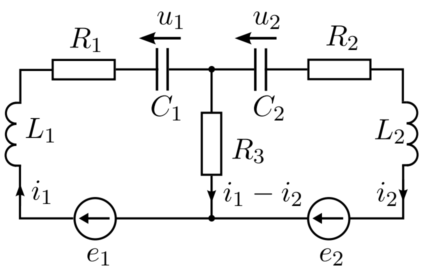

Fig. 2.10 Electrical circuit of Example 2.6

Using relations (2.1b), (2.2b) and Kirchhoff's laws we may write for the circuit the following equations

$$\begin{aligned} i_1 &= C_1 \frac{d^\alpha u_1}{dt^\alpha}, \quad i_2 = C_2 \frac{d^\alpha u_2}{dt^\alpha}, \\ e_1 &= (R_1 + R_3)\, i_1 + L_1 \frac{d^\beta i_1}{dt^\beta} + u_1 - R_3 i_2, \\ e_2 &= (R_2 + R_3)\, i_2 + L_2 \frac{d^\beta i_2}{dt^\beta} + u_2 - R_3 i_1. \end{aligned} \tag{2.59}$$

The equations (2.59) can be written in the form

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.60}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & 0 & -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ 0 & -\frac{1}{L_2} & \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \tag{2.61a}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{2.61b}$$

From (2.61a) it follows that the fractional electrical circuit is not positive, since the matrix A has some off-diagonal entries.

If the fractional linear circuit is not positive but the matrix B has nonnegative entries (see, for example, the circuit shown in Figure 2.10) then using the state-feedback

$$e = K \begin{bmatrix} x_C \\ x_L \end{bmatrix} \tag{2.62}$$

we may usually choose the gain matrix $K \in \mathbb{R}^{m \times n}$, so that the closed-loop system matrix (obtained by substitution of (2.62) into (2.5))

$$A_c = A + BK$$

is a Metzler matrix.

Theorem 2.11. *Let A be not a Metzler matrix but $B \in \mathbb{R}_+^{n \times m}$. Then there exists a gain matrix K such that the closed-loop system matrix $A_c \in M_n$ if and only if*

$$\operatorname{rank}[B, A_c - A] = \operatorname{rank} B. \tag{2.63}$$

Proof. By Kronecker-Cappely theorem the equation

$$BK = A_c - A \tag{2.64}$$

has a solution K for any given B and $A_c - A$ if and only if the condition (2.63) is satisfied. □

Example 2.7. (continuation of Example 2.6). Let

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ \frac{a_1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{a_3}{L_1} \\ 0 & -\frac{a_2}{L_2} & \frac{a_4}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}$$

for $a_k \geq 0$, $k = 1,2,3,4$.

In this case the condition (2.63) takes the form

$$\operatorname{rank}[B, A_c - A] = \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{L_1} & 0 & \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{1}{L_2} & 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} = 2.$$

The equation (2.64) has the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix}$$

and its solution is

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} a_1+1 & 0 & 0 & a_3-R_3 \\ 0 & a_2+1 & a_4-R_3 & 0 \end{bmatrix}. \tag{2.65}$$

The matrix (2.65) has nonnegative entries if $a_k \geq 0$ for $k = 1,2$ and $a_k \geq R_3$ for $k = 3,4$.

2.3 Analysis of the Fractional Electrical Circuits in Transient States

Following [59], let the current in the capacitor be described by (2.1b) and the voltage on the coil be described by (2.2b).

Using the relation (1.10) for (2.1b) ($n = 1$) we obtain

$$I_C(s) = s^\alpha C U_C(s) - s^{\alpha-1} C u_C(0+), \quad 0 < \alpha < 1,$$

where $I_C(s) = \mathcal{L}[i_C(t)]$ and $U_C(s) = \mathcal{L}[u_C(t)]$.

Similarly, using the relation (1.10) for (2.2b) ($n = 1$) we obtain

$$U_L(s) = s^\beta L I_L(s) - s^{\beta-1} L i_L(0+), \quad 0 < \beta < 1,$$

where $U_L(s) = \mathcal{L}[u_L(t)]$ and $I_L(s) = \mathcal{L}[i_L(t)]$.

Impedance of the series connection of the resistance R, capacity C and inductance L described by (2.1b) and (2.2b) will be called the operator inductance.

To simplify the considerations we shall assume that:

1) initial conditions are zero: $i_C(0+) = 0$, $u_L(0+) = 0$,
2) all Laplace transforms $U_C(s)$ and $I_C(s)$ of the capacitors are related by

$$U_C(s) = \frac{1}{s^\alpha C} I_C(s) \tag{2.66a}$$

3) all Laplace transforms $U_L(s)$ and $I_L(s)$ of the coils are related by

$$U_L(s) = s^\beta L I_L(s). \tag{2.66b}$$

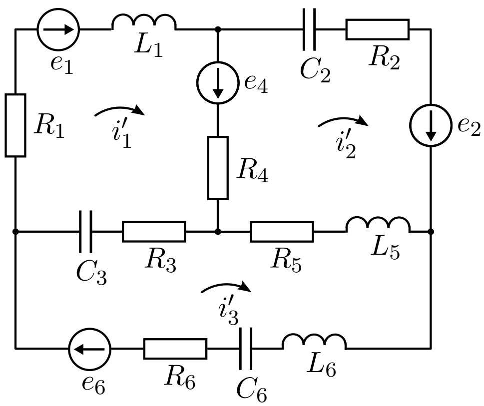

Fig. 2.11 Fractional electrical circuit

First we shall show the essence of the mesh method of the following electrical circuit (Figure 2.11) with given resistances R_k; $k = 1, 2 \ldots, 6$; capacitances C_2, C_3, C_6; inductances L_1, L_5, L_6 and source voltages e_1, e_2, e_4, e_6. Let $I_1'(s)$, $I_2'(s)$, $I_3'(s)$ be the Laplace transforms of the mesh currents $i_1'(t)$, $i_2'(t)$, $i_3'(t)$.

Using Kirchhoff's voltage law and the relations (2.66) for the fractional circuit we obtain the equations

$$\begin{aligned}
E_1(s) + E_4(s) =& \left(R_1 + s^\beta L_1\right) I_1'(s) + R_4\left[I_1'(s) - I_2'(s)\right] \\
&+ \left(R_3 + \frac{1}{s^\alpha C_3}\right)\left[I_1'(s) - I_3'(s)\right] \\
=& Z_{11}(s)I_1'(s) - Z_{12}(s)I_2'(s) - Z_{13}(s)I_3'(s), \\
E_2(s) - E_4(s) =& \left(R_2 + \frac{1}{s^\alpha C_2}\right) I_2'(s) + \left(R_5 + \frac{1}{s^\alpha C_3}\right)\left[I_3'(s) - I_1'(s)\right] \\
&+ R_4\left[I_2'(s) - I_1'(s)\right] \\
=& - Z_{21}(s)I_1'(s) + Z_{22}(s)I_2'(s) - Z_{23}(s)I_3'(s), \\
E_6(s) =& \left(R_6 + s^\beta L_6 + \frac{1}{s^\alpha C_6}\right) I_3'(s) + \left(R_3 + \frac{1}{s^\alpha C_3}\right)\left[I_3'(s) - I_1'(s)\right] \\
&+ \left(R_5 + s^\beta L_5\right)\left[I_3'(s) - I_2'(s)\right] \\
=& -Z_{31}(s)I_1'(s) - Z_{32}(s)I_2'(s) + Z_{33}(s)I_3'(s),
\end{aligned} \tag{2.67}$$

where $E_k(s) = \mathcal{L}[e_k(t)]$, $k = 1, 2, 4, 6$ and

$$\begin{aligned}
&Z_{11}(s) = R_1 + R_5 + R_6 + s^\beta L_1 + \frac{1}{s^\alpha C_3}, \quad Z_{12}(s) = Z_{21}(s) = R_4, \\
&Z_{13}(s) = Z_{31}(s) = R_3 + \frac{1}{s^\alpha C_3}, \quad Z_{22}(s) = R_2 + R_4 + R_5 + s^\beta L_5 + \frac{1}{s^\alpha C_2}, \\
&Z_{23}(s) = Z_{32}(s) = R_5 + s^\beta L_5, \\
&Z_{33}(s) = R_3 + R_5 + R_6 + s^\beta\left(L_5 + L_6\right) + \frac{1}{s^\alpha C_3} + \frac{1}{s^\alpha C_6}.
\end{aligned}$$

Equations (2.67) can be written in the form

$$E'(s) = Z(s)I'(s), \tag{2.68a}$$

where

$$\begin{aligned}
E'(s) &= \begin{bmatrix} E_1(s) + E_4(s) \\ E_2(s) - E_4(s) \\ E_6(s) \end{bmatrix}, \quad I'(s) = \begin{bmatrix} I_1'(s) \\ I_2'(s) \\ I_3'(s) \end{bmatrix}, \\
Z(s) &= \begin{bmatrix} Z_{11}(s) & -Z_{12}(s) & -Z_{13}(s) \\ -Z_{21}(s) & Z_{22}(s) & -Z_{23}(s) \\ -Z_{31}(s) & -Z_{32}(s) & Z_{33}(s) \end{bmatrix}.
\end{aligned} \tag{2.68b}$$

Taking into account that $\det Z(s) \neq 0$ we may find from the equation (2.68a) the vector $I'(s)$

$$I'(s) = Z^{-1}(s)E'(s). \tag{2.69}$$

Applying the inverse Laplace transform $\left(\mathcal{L}^{-1}\right)$ to $I'(s)$ we may find the mesh currents $i'_1(t)$, $i'_2(t)$, $i'_3(t)$ and next from the relations

$$\begin{aligned} i_1(t) &= i'_1(t), & i_2(t) &= i'_2(t), & i_3(t) &= i'_3(t) - i'_1(t), \\ i_4(t) &= i'_1(t) - i'_2(t), & i_5(t) &= i'_2(t) - i'_3(t), & i_6(t) &= i'_3(t) \end{aligned}$$

branch currents $i_k(t)$, $k = 1, \ldots, 6$ in transient state.

Remark 2.4. Choosing as the state variables (the components of the state vector $x(t)$) voltages across the condensators and currents in the coils we may write for the circuit (Figure 2.11) the state equation (2.5) (or (2.3), (2.4)), where the source voltages are the components of $u(t)$ and entries of the matrices A, B depend on the resistances, capacitances and inductances of the circuit. Using the solution (2.7) (or (1.12)) of the equation (2.5) (or (2.3), (2.4)) we may find the transient values of the voltages across the condensators and of the currents of the circuit.

In general case of n-mesh linear electrical circuit we obtain the equation (2.68a), where

$$E'(s) = \begin{bmatrix} E'_1(s) \\ E'_2(s) \\ \vdots \\ E'_n(s) \end{bmatrix}, \quad I'(s) = \begin{bmatrix} I'_1(s) \\ I'_2(s) \\ \vdots \\ I'_n(s) \end{bmatrix},$$

$$Z(s) = \begin{bmatrix} Z_{11}(s) & -Z_{12}(s) & \cdots & -Z_{1n}(s) \\ -Z_{21}(s) & Z_{22}(s) & \cdots & -Z_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ -Z_{n1}(s) & -Z_{n2}(s) & \cdots & Z_{nn}(s) \end{bmatrix}$$

and $E'_k(s)$, $k = 1, 2, \ldots, n$ is the algebraic sum of the Laplace transforms of source voltages in the k-th mesh (with $+$ we take the transform if its direction is consistent with the direction of the mesh current and with $-$ if the direction is opposite), $Z_{kk}(s)$, $k = 1, 2, \ldots, n$ is the sum of the operator impedances of all branches belonging to the k-th mesh and $Z_{kl}(s)$; $k, l = 1, 2, \ldots, n$ is the operator impedance of the branch belonging to the k-th mesh and l-th mesh. $I'_k(s)$, $k = 1, 2, \ldots, n$ is the Laplace transform of the k-th mesh current.

Knowing $E'(s)$ and $Z(s)$ from the equation (2.69) we may find $I'(s)$ and using the inverse Laplace transform we may find the mesh currents $i'(t)$, $k = 1, 2, \ldots, n$ and next the branch currents of the circuit.

2.4 Reciprocity Theorem for Fractional Circuits

Consider the fractional linear circuit composed of resistances, capacitances, inductances and source voltage $e(t)$ in transient state. We choose the linearly independent meshes in such way that the source voltage belongs to the k-th mesh and let $i(t)$ be the current in a branch belonging only to the l-th mesh (Figure 2.12a).

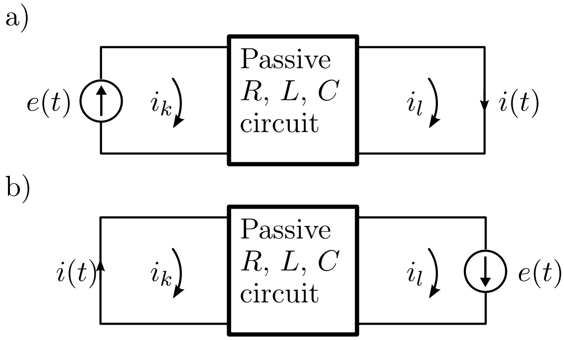

Fig. 2.12 Fractional electrical circuits

Applying to the fractional circuit the mesh method from (2.69) we obtain

$$I_l(s) = \frac{C_{lk}(s)}{\det Z(s)} E(s) \quad \text{for} \quad k, l = 1, 2 \dots, n;$$

where $I_l(s) = \mathcal{L}[i_l(t)]$, $E(s) = \mathcal{L}[e(t)]$, $C_{lk}(s) = (-1)^{l+l} M_{lk}(s)$ and $M_{lk}(s)$ is the minor obtained from the matrix $Z(s)$ by removing its l-th row and its k-th column.

Now we interchange the places of the source voltage $e(t)$ and the observation point of the current $i(t)$ (Figure 2.12b). Applying to the fractional circuit from Figure 2.12b the mesh method from (2.69) we obtain

$$I_k(s) = \frac{C_{kl}(s)}{\det Z(s)} E(s) \quad \text{for} \quad k, l = 1, 2 \dots, n.$$

From symmetry of the matrix $Z(s)$ it follows that $C_{kl}(s) = C_{lk}(s)$ and this implies $I_k(s) = I_l(s)$. Therefore, the following reciprocity theorem for fractional electrical circuits has been proved.

Theorem 2.12. *The ratio of a single source voltage at one point to observed branch current at another one in any linear fractional electrical circuit in transient state is invariant with respect to an interchange of the points of excitation and observation.*

Example 2.8. Consider a fractional linear electrical circuit shown in Figure 2.13 with given resistances R_1, R_2, R_3; capacity C, inductance L and source voltage $e(t)$. The source voltage belongs to the first mesh and $i(t)$ is the current equal to the second mesh current $i_2'(t)$.

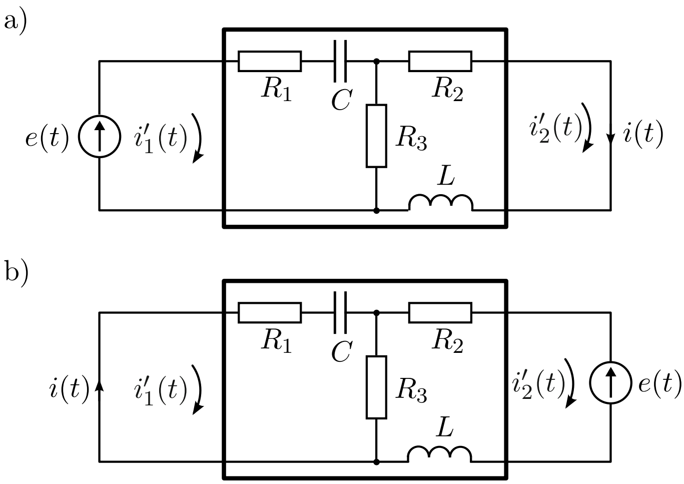

Fig. 2.13 Fractional electrical circuits of Example 2.8

Equation (2.68a) for the fractional circuit (Figure 2.13a) has the form

$$\begin{bmatrix} E(s) \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + R_3 + \dfrac{1}{s^\alpha C} & -R_3 \\ -R_3 & R_2 + R_3 + s^\beta L \end{bmatrix} \begin{bmatrix} I_1'(s) \\ I_2'(s) \end{bmatrix}. \tag{2.70}$$

From (2.70) we have

$$I_2'(s) = \frac{s^\alpha R_3 C}{\Delta(s)} E(s), \tag{2.71}$$

where

$$\Delta(s) = s^\alpha s^\beta LC\,(R_1 + R_3) + s^\alpha C\,[R_2\,(R_1 + R_3) + R_1 R_3] + s^\beta L + R_2 + R_3.$$

Similarly, equation (2.68a) for the fractional electrical circuit (Figure 2.13b) has the form

$$\begin{bmatrix} 0 \\ E(s) \end{bmatrix} = \begin{bmatrix} R_1 + R_3 + \dfrac{1}{s^\alpha C} & -R_3 \\ -R_3 & R_2 + R_3 + s^\beta L \end{bmatrix} \begin{bmatrix} I_1'(s) \\ I_2'(s) \end{bmatrix}. \tag{2.72}$$

From (2.72) we have

$$I_1'(s) = \frac{s^\alpha R_3 C}{\Delta(s)} E(s). \tag{2.73}$$

From comparison of (2.71) and (2.73) we obtain $I_2'(s) = I_1'(s)$.

2.5 Equivalent Voltage Source Theorem and Equivalent Current Source Theorem

Consider fractional linear electrical circuit composed of resistances, capacitances, inductances and source voltages. The circuit can be divided in active part A and passive part B. Both parts are connected in the way shown in Figure 2.14a.

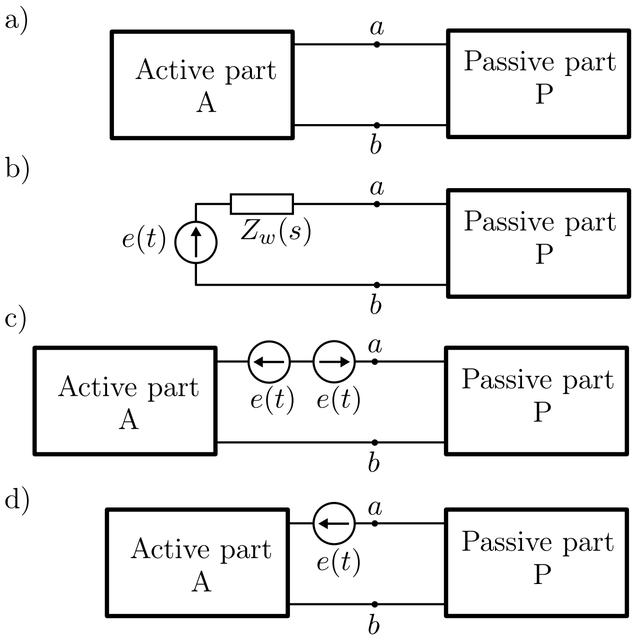

Fig. 2.14 Connection of active part A and passive part B

After disconnecting the passive part P:

1. we register the voltage $u_0(t)$ between the points a-b in transient state,
2. we register the current $i_z(t)$ in transient state when the points a-b are short circuit,
3. we calculate the equivalent operator impedance $Z_w(s)$ of the active part A when all source voltages are zero.

We shall show that the active part A is equivalent to (ideal) voltage source $e(t) = u_0(t)$ connected in series with the operator impedance $Z_w(s)$. To do this we switch on two voltage sources $e(t)$ with opposite directions (Figure 2.14c). By assumption the fractional electrical circuit is linear and we may use the superposition principle. Currents and voltages in transient state in the circuit shown in Figure 2.14c are the sum of the suitable currents and voltages in the circuits shown in Figure 2.14a and Figure 2.14b. The voltage between the points a-b on Figure 2.14d is equal zero and all currents and voltages in the passive part P are equal zero. The voltages and currents in part P shown in Figure 2.14a and Figure 2.14b are the same. This completes the proof of the following theorem.

Theorem 2.13. *Active part of any fractional linear electrical circuit in transient state is equivalent to voltage source $e(t)$ connected in series with the operator impedance $Z_w(s)$ (Figure 2.14b).*

Example 2.9. Consider the fractional electrical circuit shown in Figure 2.15. We divide this circuit into the active part A and the passive part P.

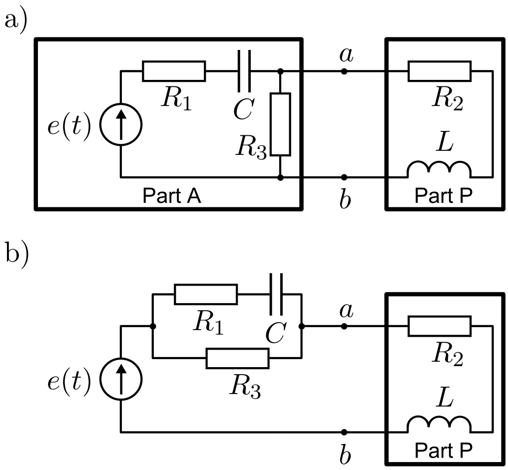

Fig. 2.15 Active and passive parts of Example 2.9

The active part is equivalent to the voltage source $e_z(t)$ connected in series with the operator impedance $Z_w(s)$, where $e_z(t)$ is equal to the voltage on the resistance R_3 when the passive part P is disconnected and the operator impedance

$$Z_w(s) = \frac{R_3\left(R_1 + \dfrac{1}{s^\alpha C}\right)}{R_1 + R_3 + \dfrac{1}{s^\alpha C}}.$$

Using the well-known equivalence of the voltage source and current source we obtain the following theorem.

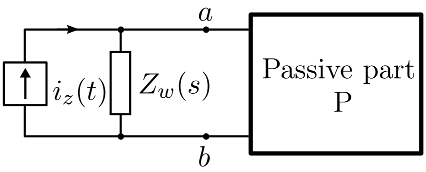

Fig. 2.16 Equivalent current source

Theorem 2.14. *Active part of any fractional linear electrical circuit in transient state is equivalent to current source* $i_z(t)$ *connected in parallel with the operator impredance* $Z_w(s)$ *(Figure 2.16), where* $i_z(t)$ *is the current when the points a-b are short circuit and it is related to the source voltage* $e_z(t)$ *by the equality*

$$\mathcal{L}\left[i_z(t)\right] = \frac{\mathcal{L}\left[e_z(t)\right]}{Z_w(s)}.$$

Theorem 2.14 can be also proved in a similar way as the Theorem 2.13.

Chapter 3
Descriptor Linear Electrical Circuits and Their Properties

3.1 Descriptor Linear Electrical Circuits

Consider the descriptor (singular) linear continuous-time system

$$E\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu, \tag{3.1}$$

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^m$ are the state and input vectors, respectively and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

It is assumed that $\det E = 0$, $\operatorname{rank} B = m$ and the pencil is regular, i.e.

$$\det\left[Es - A\right] \neq 0 \text{ for some } s \in \mathbb{C}. \tag{3.2}$$

<u>Method 1.</u> (Weierstrass-Kronecker decomposition)

It is well-known [44, 45] that if the condition (3.2) is satisfied then there exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} \mathbb{I}_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n_2} \end{bmatrix}, \tag{3.3}$$

where $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with nilpotency index μ ($N^{\mu-1} \neq 0$ and $N^{\mu} = 0$) and $A_1 \in \mathbb{R}^{n_1\times n_1}$, $n_1 + n_2 = n$.

The matrices P and Q can be found by the use of the elementary row (for matrix P) and column (for matrix Q) operations (see Appendix B).

Lemma 3.1. The characteristic polynomial of the descriptor system (3.1) and the characteristic polynomial of the matrix A_1 are related by

$$\det\left[Es - A\right] = k\det\left[\mathbb{I}_{n_1}s - A_1\right],$$

where $k = (-1)^{n_2}\det P^{-1}\det Q^{-1} = (-1)^{n_2}\det(PQ)^{-1}$.

Proof. From (3.2) and (3.3) we have

T. Kaczorek and K. Rogowski, *Fractional Linear Systems and Electrical Circuits*, Studies in Systems, Decision and Control 13, DOI: 10.1007/978-3-319-11361-6_3

$$\begin{aligned}\det\left[Es-A\right] =&\det\left\{P^{-1}\begin{bmatrix}\mathbb{I}_{n_1}s-A_1 & 0\\ 0 & Ns-\mathbb{I}_{n_2}\end{bmatrix}Q^{-1}\right\}\\ =&\det P^{-1}\det Q^{-1}\det\left[\mathbb{I}_{n_1}s-A_1\right]\det\left[Ns-\mathbb{I}_{n_2}\right]\\ =&k\det\left[\mathbb{I}_{n_1}s-A_1\right],\end{aligned}$$

since $\det\left[Ns-\mathbb{I}_{n_2}\right]=(-1)^{n_2}$. □

Method 2. (Shuffle algorithm)

Performing elementary row operations (see Appendix B) on the equation (3.1) or equivalently on the array

$$E \ A \ B$$

we obtain

$$\begin{matrix}\bar{E}_1 & \bar{A}_{11} & \bar{B}_{11}\\ 0 & \bar{A}_{12} & \bar{B}_{12}\end{matrix}$$

and

$$\bar{E}_1\frac{\mathrm{d}x}{\mathrm{d}t}=\bar{A}_{11}x+\bar{B}_{11}u, \tag{3.4a}$$

$$0=\bar{A}_{12}x+\bar{B}_{12}u, \tag{3.4b}$$

where $\bar{E}_1\in\mathbb{R}^{r\times n}$ has full row rank and $r=\mathrm{rank}E$.

Differentiation with respect to time of (3.4b) yields

$$\bar{A}_{12}\frac{\mathrm{d}x}{\mathrm{d}t}=-\bar{B}_{12}\frac{\mathrm{d}u}{\mathrm{d}t}. \tag{3.5}$$

The equations (3.4a) and (3.5) can be written in the form

$$\begin{bmatrix}\bar{E}_1\\ \bar{A}_{12}\end{bmatrix}\frac{\mathrm{d}x}{\mathrm{d}t}=\begin{bmatrix}\bar{A}_{11}\\ 0\end{bmatrix}x+\begin{bmatrix}\bar{B}_{11}\\ 0\end{bmatrix}u+\begin{bmatrix}0\\ -\bar{B}_{12}\end{bmatrix}\frac{\mathrm{d}u}{\mathrm{d}t}. \tag{3.6}$$

If $\det\begin{bmatrix}\bar{E}_1\\ \bar{A}_{12}\end{bmatrix}\neq 0$, then form (3.6) we have

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\hat{A}_1x+\hat{B}_{10}u+\hat{B}_{11}\frac{\mathrm{d}u}{\mathrm{d}t},$$

where

$$\hat{A}_1=\begin{bmatrix}\bar{E}_1\\ \bar{A}_{12}\end{bmatrix}^{-1}\begin{bmatrix}\bar{A}_{11}\\ 0\end{bmatrix},\quad \hat{B}_{10}=\begin{bmatrix}\bar{E}_1\\ \bar{A}_{12}\end{bmatrix}^{-1}\begin{bmatrix}\bar{B}_{11}\\ 0\end{bmatrix},\quad \hat{B}_{11}=\begin{bmatrix}\bar{E}_1\\ \bar{A}_{12}\end{bmatrix}^{-1}\begin{bmatrix}0\\ -\bar{B}_{12}\end{bmatrix}.$$

If $\det\begin{bmatrix}\bar{E}_1\\ \bar{A}_{12}\end{bmatrix}=0$, then performing elementary row operations on the array

$$\begin{matrix} \bar{E}_1 & \bar{A}_{11} & \bar{B}_{11} & 0 \\ \bar{A}_{12} & 0 & 0 & -\bar{B}_{12} \end{matrix}$$

we eliminate the linearly dependent rows in the matrix $\begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix}$ and we obtain

$$\begin{matrix} \bar{E}_2 & \bar{A}_{21} & \bar{B}_{20} & \bar{B}_{21} \\ 0 & \bar{A}_{22} & \bar{B}_{30} & \bar{B}_{31} \end{matrix}$$

and

$$\bar{E}_2 \frac{\mathrm{d}x}{\mathrm{d}t} = \bar{A}_{21} x + \bar{B}_{20} u + \bar{B}_{21} \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{3.7a}$$

$$0 = \bar{A}_{22} x + \bar{B}_{30} u + \bar{B}_{31} \frac{\mathrm{d}u}{\mathrm{d}t}. \tag{3.7b}$$

Differentiation with respect to time of (3.7b) yields

$$\bar{A}_{22} \frac{\mathrm{d}x}{\mathrm{d}t} = -\bar{B}_{30} \frac{\mathrm{d}x}{\mathrm{d}t} - \bar{B}_{31} \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}. \tag{3.8}$$

The equations (3.7a) and (3.8) can be written in the form

$$\begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} \bar{A}_{21} \\ 0 \end{bmatrix} x + \begin{bmatrix} \bar{B}_{20} \\ 0 \end{bmatrix} u + \begin{bmatrix} \bar{B}_{21} \\ -\bar{B}_{30} \end{bmatrix} \frac{\mathrm{d}u}{\mathrm{d}t} + \begin{bmatrix} 0 \\ -\bar{B}_{31} \end{bmatrix} \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}. \tag{3.9}$$

If $\det \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix} \neq 0$, then from (3.9) we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \hat{A}_2 x + \hat{B}_{20} u + \hat{B}_{21} \frac{\mathrm{d}u}{\mathrm{d}t} + \hat{B}_{22} \frac{\mathrm{d}^2 u}{\mathrm{d}t^2},$$

where

$$\hat{A}_2 = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{A}_{21} \\ 0 \end{bmatrix}, \qquad \hat{B}_{20} = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_{20} \\ 0 \end{bmatrix},$$

$$\hat{B}_{21} = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_{21} \\ -\bar{B}_{30} \end{bmatrix}, \quad \hat{B}_{22} = \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\bar{B}_{31} \end{bmatrix}.$$

If $\det \begin{bmatrix} \bar{E}_2 \\ \bar{A}_{22} \end{bmatrix} = 0$, then we repeat the procedure.

It is well-known [50] that if the condition (3.2) is met then after finite number of steps we obtain the standard system equivalent to the desciptor system (3.1).

3.1.1 Regularity of Descriptor Electrical Circuits

We start with simple examples of descriptor linear electrical circuits and next the considerations will be extended to general case.

Example 3.1. Consider electrical circuit shown in Figure 3.1 with given resistance R_1, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2.

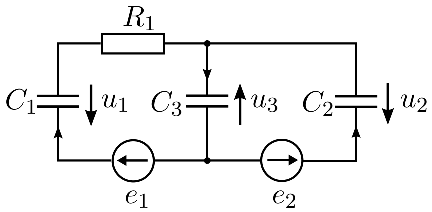

Fig. 3.1 Electrical circuit of Example 3.1

Using Kirchhoff's laws for the electrical circuit we obtain the following equations

$$e_1 = R_1C_1\frac{\mathrm{d}u_1}{\mathrm{d}t} + u_1 + u_3, \tag{3.11a}$$

$$0 = C_1\frac{\mathrm{d}u_1}{\mathrm{d}t} + C_2\frac{\mathrm{d}u_2}{\mathrm{d}t} - C_3\frac{\mathrm{d}u_3}{\mathrm{d}t}, \tag{3.11b}$$

$$e_2 = u_2 + u_3. \tag{3.11c}$$

The equations (3.11) can be written in the form (3.1), where

$$x = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad E = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The assumption (3.2) for the electrcial circuit is satisfied, since

$$\det E = \begin{vmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \tag{3.12a}$$

and

$$\det[Es - A] = \begin{vmatrix} R_1C_1s+1 & 0 & 1 \\ C_1s & C_2s & -C_3s \\ 0 & 1 & 1 \end{vmatrix}$$
$$= R_1C_1(C_2+C_3)s^2 + (C_1+C_2+C_3)s \tag{3.12b}$$

Therefore, the electrcial circuit is a descriptor linear system with regular pencil.

Method 1.

Performing on the matrix

$$Es - A = \begin{bmatrix} R_1C_1s+1 & 0 & 1 \\ C_1s & C_2s & -C_3s \\ 0 & 1 & 1 \end{bmatrix}$$

the following elementary column operations $R\left[3 + 2 \times \frac{C_3}{C_2}\right]$, $R\left[1 + 2 \times -\frac{C_1}{C_2}\right]$, $R\left[1 \times \frac{1}{R_1C_1}\right]$, $R\left[1 \times \frac{1}{C_2}\right]$ we obtain the matrix

$$\begin{bmatrix} s + \dfrac{1}{R_1C_1} & 0 & 1 \\ 0 & s & 0 \\ -\dfrac{1}{R_1C_2} & \dfrac{1}{C_2} & \dfrac{C_2+C_3}{C_2} \end{bmatrix}. \tag{3.13}$$

Performing on the matrix (3.13) the following elementary row operations $L\left[3 \times \frac{C_2}{C_2+C_3}\right]$, $L[1 + 3 \times (-1)]$ and the elementary column operations $R\left[1 + 3 \times \frac{1}{R_1(C_2+C_3)}\right]$, $R\left[2 + 3 \times -\frac{1}{C_2+C_3}\right]$ we obtain the desired matrix

$$\bar{E}s - \bar{A} = P[Es - A]Q = \begin{bmatrix} \mathbb{I}_{n_1}s - A_1 & 0 \\ 0 & Ns - \mathbb{I}_{n_2} \end{bmatrix}, \tag{3.14a}$$

where

$$A_1 = \begin{bmatrix} -\dfrac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)} & \dfrac{1}{C_2+C_3} \\ 0 & 0 \end{bmatrix}, \quad N = [0], \quad n_2 = 1 \tag{3.14b}$$

and

$$P = \begin{bmatrix} 1 & 0 & -\dfrac{C_2}{C_2+C_3} \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{C_2}{C_2+C_3} \end{bmatrix}, \quad Q = \begin{bmatrix} \dfrac{1}{R_1C_1} & 0 & 0 \\ -\dfrac{1}{R_1(C_2+C_3)} & \dfrac{1}{C_2+C_3} & \dfrac{C_3}{C_2} \\ \dfrac{1}{R_1(C_2+C_3)} & -\dfrac{1}{C_2+C_3} & 1 \end{bmatrix}. \quad (3.14c)$$

The matrices (3.14c) can be found by performing the elementary row and column operations on the identity matrix $\mathbb{I}_3$. Performing the elementary row operations $L\left[3 \times \frac{C_2}{C_2+C_3}\right]$ and $L[1 + 3 \times (-1)]$ on the matrix $\mathbb{I}_3$ we obtain the matrix P and the elementary column operations $R\left[3 + 2 \times \frac{C_3}{C_2}\right]$, $R\left[1 + 2 \times -\frac{C_1}{C_2}\right]$, $R\left[1 \times \frac{1}{R_1C_1}\right]$, $R\left[1 \times \frac{1}{C_2}\right]$, $R\left[1 + 3 \times \frac{1}{R_1(C_2+C_3)}\right]$ and $R\left[2 + 3 \times -\frac{1}{C_2+C_3}\right]$ the matrix Q.

<u>Method 2.</u>

In this case the matrix E has already the desired form $\begin{bmatrix} \bar{E}_1 \\ 0 \end{bmatrix}$, where

$$\bar{E}_1 = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \end{bmatrix}$$

and it has full row rank, i.e. rank$\bar{E}_1 = 2$.

Taking into account that

$$\bar{A}_{12} = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}$$

and

$$\det \begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix} = \begin{vmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{vmatrix} = -R_1C_1(C_2+C_3) \neq 0,$$

from (3.6) we obtain

$$\begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \frac{\mathrm{d}u}{\mathrm{d}t}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \hat{A}_1 x + \hat{B}_{10} u + \hat{B}_{11} \frac{\mathrm{d}u}{\mathrm{d}t},$$

where

$$\hat{A}_1 = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{R_1C_1} \\ \frac{1}{R_1(C_2+C_3)} & 0 & \frac{1}{R_1(C_2+C_3)} \\ -\frac{1}{R_1(C_2+C_3)} & 0 & -\frac{1}{R_1(C_2+C_3)} \end{bmatrix}, \tag{3.16}$$

$$\hat{B}_{10} = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1C_1} & 0 \\ -\frac{1}{R_1(C_2+C_3)} & 0 \\ \frac{1}{R_1(C_2+C_3)} & 0 \end{bmatrix},$$

$$\hat{B}_{11} = \begin{bmatrix} R_1C_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{C_3}{C_2+C_3} \\ 0 & \frac{C_2}{C_2+C_3} \end{bmatrix}.$$

Method 3. (Elimination method)

From (3.11c) we have

$$u_3 = e_2 - u_2. \tag{3.17}$$

Substituting (3.17) into (3.11a) and (3.11b) we obtain

$$\begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B_0 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + B_1 \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{3.18a}$$

where

$$A = \begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1(C_2+C_3)} & -\frac{1}{R_1(C_2+C_3)} \end{bmatrix},$$

$$B_0 = \begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1C_1} & -\frac{1}{R_1C_1} \\ -\frac{1}{R_1(C_2+C_3)} & \frac{1}{R_1(C_2+C_3)} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} R_1C_1 & 0 \\ C_1 & C_2+C_3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & C_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{C_3}{C_2+C_3} \end{bmatrix}. \tag{3.18b}$$

The characteristic polynomial of the matrix A has the form

$$\det[\mathbb{I}_2 s - A] = \begin{vmatrix} s + \frac{1}{R_1C_1} & -\frac{1}{R_1C_1} \\ -\frac{1}{R_1(C_2+C_3)} & s + \frac{1}{R_1(C_2+C_3)} \end{vmatrix} = s\left[s + \frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)}\right], \tag{3.19}$$

of the matrix A_1 (given by (3.14b))

$$\det[\mathbb{I}_2 s - A_1] = \begin{vmatrix} s + \frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)} & -\frac{1}{C_2+C_3} \\ 0 & s \end{vmatrix} = s\left[s + \frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)}\right], \tag{3.20}$$

and of the matrix $\hat{A}_1$ (given by (3.16))

$$\det\left[\mathbb{I}_3 s - \hat{A}_1\right] = \begin{vmatrix} s + \frac{1}{R_1C_1} & 0 & \frac{1}{R_1C_1} \\ -\frac{1}{R_1(C_2+C_3)} & s & -\frac{1}{R_1(C_2+C_3)} \\ \frac{1}{R_1(C_2+C_3)} & 0 & s + \frac{1}{R_1(C_2+C_3)} \end{vmatrix} = s\begin{vmatrix} s + \frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1(C_2+C_3)} & s + \frac{1}{R_1(C_2+C_3)} \end{vmatrix} = s^2\left[s + \frac{C_1+C_2+C_3}{R_1C_1(C_2+C_3)}\right]. \tag{3.21}$$

Note that the additional eigenvalue $s = 0$ has been introduced in Method 2 by the differentiation with respect to time of the equation (3.4b).

From (3.12b), (3.19), (3.20) and (3.21) it follows that the spectrum of the electrical circuit is the same for the three different methods and it is equal to

$$\sigma = \left\{ s_1 = 0, s_2 = -\frac{C_1 + C_2 + C_3}{R_1 C_1 (C_2 + C_3)} \right\}.$$

Example 3.2. Consider the descriptor electrical circuit shown in Figure 3.2 with given resistances R_1, R_2, R_3 inductances L_1, L_2, L_3 and source voltages e_1 and e_2.

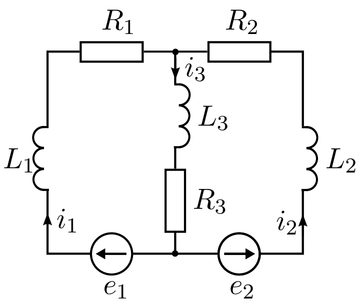

Fig. 3.2 Electrical circuit of Example 3.2

Using Kirchhoff's laws we can write the equations

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt}, \tag{3.22a}$$

$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt}, \tag{3.22b}$$

$$0 = i_1 + i_2 - i_3. \tag{3.22c}$$

The equations (3.22) can be written in the form (3.1), where

$$x = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.23}$$

The assumption (3.2) for the electrical circuit is satisfied, since

$$\det E = \begin{vmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

and

$$\det[Es-A]=\begin{vmatrix} L_1s+R_1 & 0 & L_3s+R_3 \\ 0 & L_2s+R_2 & L_3s+R_3 \\ -1 & -1 & 1 \end{vmatrix}$$
$$=[L_1(L_2+L_3)+L_2L_3]\,s^2$$
$$+[R_1(L_2+L_3)+R_2(L_1+L_3)+R_3(L_1+L_2)]\,s$$
$$+R_1(R_2+R_3)+R_2R_3.$$

Therefore, the electrical circuit shown in Figure 3.2 is a descriptor system with regular pencil.

Method 1.

Performing on the matrix

$$Es-A=\begin{bmatrix} L_1s+R_1 & 0 & L_3s+R_3 \\ 0 & L_2s+R_2 & L_3s+R_3 \\ -1 & -1 & 1 \end{bmatrix} \tag{3.24}$$

the elementary column operations $R\left[3+1\times\left(-\frac{L_3}{L_1}\right)\right]$, $R\left[3+2\times\left(-\frac{L_3}{L_2}\right)\right]$, $R\left[1\times\frac{1}{L_1}\right]$, $R\left[2\times\frac{1}{L_2}\right]$ we obtain

$$\begin{bmatrix} s+\dfrac{R_1}{L_1} & 0 & \dfrac{R_3L_1-R_1L_3}{L_1} \\ 0 & s+\dfrac{R_2}{L_2} & \dfrac{R_3L_2-R_2L_3}{L_2} \\ -\dfrac{1}{L_1} & -\dfrac{1}{L_2} & \Delta \end{bmatrix}, \quad \Delta=\frac{L_2(L_1+L_3)+L_1L_3}{L_1L_2}. \tag{3.25}$$

Next, performing on the matrix (3.25) the elementary row operations $L\left[1+3\times\frac{R_1L_3-R_3L_1}{\Delta L_1}\right]$, $L\left[2+3\times\frac{R_2L_3-R_3L_2}{\Delta L_2}\right]$ we obtain

$$\begin{bmatrix} s+\dfrac{\Delta R_1L_1+R_3L_1-R_1L_3}{\Delta L_1^2} & \dfrac{R_3L_1-R_1L_3}{\Delta L_1L_2} & 0 \\ \dfrac{R_3L_2-R_2L_3}{\Delta L_1L_2} & s+\dfrac{\Delta R_2L_2+R_3L_2-R_2L_3}{\Delta L_2^2} & 0 \\ -\dfrac{1}{L_1} & -\dfrac{1}{L_2} & \Delta \end{bmatrix}$$

and finally $R\left[1+3\times\frac{1}{\Delta L_1}\right]$, $R\left[2+3\times\frac{1}{\Delta L_2}\right]$, $R\left[3\times\frac{1}{\Delta}\right]$ the desired form

$$\bar{E}s-\bar{A}=P[Es-A]Q=\begin{bmatrix} \mathbb{I}_{n_1}s-A_1 & 0 \\ 0 & Ns-\mathbb{I}_{n_2} \end{bmatrix}, \tag{3.26a}$$

where

$$A_1 = \begin{bmatrix} -\dfrac{\Delta R_1 L_1 + R_3 L_1 - R_1 L_3}{\Delta L_1^2} & \dfrac{R_1 L_3 - R_3 L_1}{\Delta L_1 L_2} \\ \dfrac{R_2 L_3 - R_3 L_2}{\Delta L_1 L_2} & -\dfrac{\Delta R_2 L_2 + R_3 L_2 - R_2 L_3}{\Delta L_2^2} \end{bmatrix}, \tag{3.26b}$$

$$N = [0], \quad n_2 = 1$$

and

$$P = \begin{bmatrix} 1 & 0 & \dfrac{R_1 L_3 - R_3 L_1}{\Delta L_1} \\ 0 & 1 & \dfrac{R_2 L_3 - R_3 L_2}{\Delta L_2} \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} \dfrac{\Delta L_1 - L_3}{\Delta L_1^2} & -\dfrac{L_3}{\Delta L_1 L_2} & -\dfrac{L_3}{\Delta L_1} \\ -\dfrac{L_3}{\Delta L_1 L_2} & \dfrac{\Delta L_2 - L_3}{\Delta L_2^2} & -\dfrac{L_3}{\Delta L_2} \\ \dfrac{1}{\Delta L_1} & \dfrac{1}{\Delta L_2} & \dfrac{1}{\Delta} \end{bmatrix}. \tag{3.26c}$$

Performing the elementary row operations on $\mathbb{I}_3$ we obtain the matrix P and the elementary column operations the matrix Q.

Method 2.

The matrix E given by (3.23) has already the desired form $\begin{bmatrix} \bar{E}_1 \\ 0 \end{bmatrix}$, where

$$\bar{E}_1 = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \end{bmatrix}$$

and $\text{rank}\bar{E}_1 = 2$.

Taking into account that

$$\bar{A}_{12} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

and

$$\det \begin{bmatrix} \bar{E}_1 \\ \bar{A}_{12} \end{bmatrix} = \begin{vmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{vmatrix} = -\left[L_1(L_2 + L_3) + L_2 L_3\right] \neq 0$$

from (3.6) we obtain

$$\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \hat{A}_1 x + \hat{B}_{10} u, \tag{3.28a}$$

where

$$\hat{A}_1 = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{\Delta_1} \begin{bmatrix} -R_1(L_2+L_3) & -R_2L_3 & -R_3L_2 \\ R_1L_3 & -R_2(L_1+L_3) & -R_3L_1 \\ -R_1L_2 & -R_2L_1 & -R_3(L_1+L_2) \end{bmatrix}, \tag{3.28b}$$

$$\hat{B}_{10} = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix} L_1+L_3 & -L_3 \\ -L_3 & L_1+L_3 \\ L_2 & L_1 \end{bmatrix},$$

$$\Delta_1 = L_1(L_2+L_3)+L_2L_3.$$

Method 3.

From (3.22c) we have

$$i_3 = i_1 + i_2. \tag{3.29}$$

Substituting (3.29) into (3.22a) and (3.22b) we obtain

$$\begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -(R_1+R_3) & -R_3 \\ -R_3 & -(R_2+R_3) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_0 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{3.30a}$$

where

$$A = \begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix}^{-1} \begin{bmatrix} -(R_1+R_3) & -R_3 \\ -R_3 & -(R_2+R_3) \end{bmatrix}$$

$$= \frac{1}{\Delta_1} \begin{bmatrix} -R_1(L_2+L_3)-R_3L_2 & R_2L_3-R_3L_2 \\ R_1L_3-R_3L_1 & -R_2(L_1+L_3)-R_3L_1 \end{bmatrix}, \tag{3.30b}$$

$$B = \begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix} L_2+L_3 & -L_3 \\ -L_3 & L_1+L_3 \end{bmatrix}, \tag{3.30c}$$

$$\Delta_1 = L_1(L_2+L_3)+L_2L_3.$$

The characteristic polynomial of the matrix (3.30b) has the form

$$\det[\mathbb{I}_2 s - A] = \frac{1}{\Delta_1^2} \begin{vmatrix} \Delta_1 s + R_1(L_2+L_3)+R_3L_2 & R_3L_2-R_2L_3 \\ R_3L_1+R_1L_3 & \Delta_1 s + R_2(L_1+L_3)+R_3L_1 \end{vmatrix}$$

$$= s^2 + \frac{1}{\Delta_1}\left[R_1(L_2+L_3)+R_2(L_1+L_3)+R_3(L_1+L_2)\right]s$$

$$+ \frac{1}{\Delta_1}\left[R_1(R_2+R_3)+R_2R_3\right], \tag{3.31}$$

of the matrix $\hat{A}_1$ (given by (3.28b))

$$\det\left[\mathbb{I}_3 s - \hat{A}_1\right] = \begin{vmatrix} s + \frac{R_1(L_2+L_3)}{\Delta_1} & -\frac{R_2L_3}{\Delta_1} & \frac{R_3L_2}{\Delta_1} \\ -\frac{R_1L_3}{\Delta_1} & s + \frac{R_2(L_1+L_3)}{\Delta_1} & \frac{R_3L_1}{\Delta_1} \\ \frac{R_1L_2}{\Delta_1} & \frac{R_2L_1}{\Delta_1} & s + \frac{R_3(L_1+L_2)}{\Delta_1} \end{vmatrix}$$
$$= s\left\{ s^2 + \frac{1}{\Delta_1}\left[R_1(L_2+L_3) + R_2(L_1+L_3) + R_3(L_1+L_2)\right] s + \frac{1}{\Delta_1}\left[R_1(R_2+R_3) + R_2R_3\right] \right\} \tag{3.32}$$

and of the matrix A_1 (given by (3.26b))

$$\det[\mathbb{I}_2 s - A_1] = \begin{vmatrix} s + \frac{\Delta R_1L_1 + R_3L_1 - R_1L_3}{\Delta L_1^2} & -\frac{R_1L_3 - R_3L_1}{\Delta L_1L_2} \\ -\frac{R_2L_3 - R_3L_2}{\Delta L_1L_2} & s + \frac{\Delta R_2L_2 + R_3L_2 - R_2L_3}{\Delta L_2^2} \end{vmatrix}$$
$$= s^2 + \frac{1}{\Delta_1}\left[R_1(L_2+L_3) + R_2(L_1+L_3) + R_3(L_1+L_2)\right] s + \frac{1}{\Delta_1}\left[R_1(R_2+R_3) + R_2R_3\right]. \tag{3.33}$$

Note that in (3.31) the additional eigenvalue $s = 0$ has been introduced in Method 2 by the differentiation with respect to time of the equation (3.4b).

From (3.24), (3.31), (3.32) and (3.33) it follows that the spectrum of the electrical circuit is the same for the three different methods and it is determined by the zeros of the polynomial (3.24).

Note that the electrical circuit shown in Figure 3.1 contains one mesh consisting of branches with only ideal capacitors and voltage sources and the one shown in Figure 3.2 contains one node with branches with coils. Equations (3.11) and (3.22) consist of two differential equations and one algebraic equation.

In general case we have the following theorem.

Theorem 3.1. *Every electrical circuit is a descriptor system if it contains at least one mesh consisting of only ideal capacitors and voltage sources or at least one node with branches with coils.*

Proof. If the electrical circuit contains at least one mesh consisting of branches with ideal capacitors and voltage sources, then the corresponding rows of the matrix E are zero and the matrix E is singular. Similarly, if the electrical circuit contains at least one node with branches with coils, then the equations written on Kirchhoff's current law for these nodes are algebraic ones and the corresponding rows of E are zero rows and the matrix E is singular. □

Theorem 3.2. *Every descriptor electrical circuit is a linear system with regular pencil.*

Proof. It is well-known [44, 47, 71] that for a descriptor electrical circuit with n branches and q nodes using current Kirchhoff's law we can write $q-1$ algebraic equations and the voltage Kirchhoff's law $n-q+1$ differential equations. The equalities are linearly independent and can be written in the form (3.1). From linear independence of the equations it follows that the condition (3.2) is satisfied and the pencil of the electrical circuit is regular. □

Remark 3.1. The spectrum of descriptor electrical circuits is independent of the method used of their analysis.

3.1.2 Pointwise Completeness of Descriptor Electrical Circuits

Consider the descriptor electrical circuit described by the equation (3.1) for $u(t) = 0$, $t \geq 0$.

Defining the new state vector

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = Q^{-1}x, \quad \bar{x}_1 \in \mathbb{R}^{n_1}, \quad \bar{x}_2 \in \mathbb{R}^{n_2}, \quad n = n_1 + n_2 \tag{3.34}$$

and using (3.1) for $u(t) = 0$, $t \geq 0$ and (3.3) we obtain

$$\frac{\mathrm{d}\bar{x}_1(t)}{\mathrm{d}t} = A_1\bar{x}_1(t)$$

and

$$\bar{x}_1(t) = e^{A_1 t}\bar{x}_{10}, \quad t \geq 0, \tag{3.35}$$

where

$$\bar{x}_{10} = \bar{Q}_1 x_0, \tag{3.36a}$$

$$\begin{bmatrix} \bar{x}_{10} \\ \bar{x}_{20} \end{bmatrix} = Q^{-1}x_0 = \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix} x_0, \quad \bar{Q}_1 \in \mathbb{R}^{n_1 \times n}, \quad \bar{Q}_2 \in \mathbb{R}^{n_2 \times n}. \tag{3.36b}$$

Note that [44]

$$\bar{x}_2(t) = +\sum_{k=0}^{q-2} \delta^{(k)} N^{k+1} \bar{x}_2(0) = 0, \quad t > 0, \tag{3.37}$$

where $\delta^{(k)}$ is the k-th derivative of the Dirac impulse.

From (3.34) for $\bar{x}_2(t_f) = 0$ we have

$$x_f = x_f(t) = Q_1\bar{x}_1(t_f), \tag{3.38}$$

where $Q = \left[Q_1 \; Q_2\right]$, $Q_1 \in \mathbb{R}^{n \times n_1}$, $Q_2 \in \mathbb{R}^{n \times n_2}$.

Definition 3.1. The descriptor electrical circuit (3.1) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \mathbb{R}^n$ there exist initial conditions $x_0 \in \mathbb{R}^n$ satisfying (3.36a) such that $x_f = x(t_f) \in \mathrm{Im} Q_1$.

Theorem 3.3. *The descriptor electrical circuit is pointwise complete for any $t = t_f$ and every $x_f \in \mathbb{R}^n$ satisfying the condition*

$$x_f \in \mathrm{Im} Q_1. \tag{3.39}$$

Proof. Taking into account that for any A_1, $\det\left[e^{A_1 t}\right] \neq 0$ and $\left[e^{A_1 t}\right]^{-1} = e^{-A_1 t}$, from (3.35) and (3.37) for $t = t_f$ we obtain

$$\bar{x}_{10} = e^{-A_1 t} \bar{x}_1(t_f) \quad \text{and} \quad \bar{x}_2(t) = 0.$$

Therefore, from (3.38) it follows that there exist initial conditions $x_0 \in \mathbb{R}^n$ such that $x_f = x(t_f)$ if (3.39) holds. □

Example 3.3. (continuation of Example 3.1)
In this case from (3.14c) we have

$$Q = \left[Q_1 \; Q_2\right], \quad Q_1 = \begin{bmatrix} \frac{1}{R_1 C_1} & 0 \\ -\frac{1}{R_1(C_2 + C_3)} & \frac{1}{C_2 + C_3} \\ \frac{1}{R_1(C_2 + C_3)} & -\frac{1}{C_2 + C_3} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 \\ \frac{C_3}{C_2} \\ 1 \end{bmatrix} \tag{3.40}$$

and

$$x_f \in \mathrm{Im} Q_1 = \begin{bmatrix} \frac{1}{R_1 C_1} a \\ -\frac{1}{R_1(C_2 + C_3)} a + \frac{1}{C_2 + C_3} b \\ \frac{1}{R_1(C_2 + C_3)} a - \frac{1}{C_2 + C_3} b \end{bmatrix} \tag{3.41}$$

for arbitrary a and b.

The eigenvalues of the matrix A_1 (given by (3.14b)) are $s_1 = 0$, $s_2 = -\frac{C_1 + C_2 + C_3}{R_1 C_1 (C_2 + C_3)}$ and using Sylvester formula [44] we obtain

$$e^{A_1 t} = Z_1 + Z_2 e^{s_2 t} = \begin{bmatrix} e^{s_2 t} & \frac{R_1 C_1}{C_1 + C_2 + C_3} \left(1 - e^{s_2 t}\right) \\ 0 & 1 \end{bmatrix},$$

since

$$Z_1 = \frac{A_1 - \mathbb{I}_2 s_2}{s_1 - s_2} = \mathbb{I}_2 - \frac{1}{s_2} A_1 = \begin{bmatrix} 0 & \dfrac{R_1 C_1}{C_1 + C_2 + C_3} \\ 0 & 1 \end{bmatrix},$$

$$Z_2 = \frac{A_1 - \mathbb{I}_2 s_1}{s_2 - s_1} = \frac{1}{s_2} A_1 = \begin{bmatrix} 1 & -\dfrac{R_1 C_1}{C_1 + C_2 + C_3} \\ 0 & 0 \end{bmatrix}$$

Therefore, the descriptor electrical circuit shown in Figure 3.1 is pointwise complete for any $t = t_f$ and every x_f satisfying (3.41).

3.1.3 Pointwise Degeneracy of Descriptor Electrical Circuits

Consider the descriptor electrical circuit described by equation (3.1) for $u(t) = 0$, $t \geq 0$.

Definition 3.2. The descriptor electrical circuit (3.1) is called pointwise degenerated in the direction $v \in \mathbb{R}^n$ for $t = t_f$ if there exists nonzero vector v such that for all initial conditions $x_0 \in \mathrm{Im} Q_1$ the solution of (3.1) satisfies the condition

$$v^T x_f = 0.$$

Theorem 3.4. *The descriptor electrical circuit* (3.1) *is pointwise degenerated in the direction v defined by*

$$v^T Q_1 = 0 \tag{3.43}$$

for any $t_f > 0$ and all initial conditions $\bar{x}_{10} \in \mathrm{Im}\bar{Q}_1$, where Q_1 and $\bar{Q}_1$ are determinated by (3.36b) *and* (3.38)*, respectively.*

Proof. Substitution of (3.35) into (3.38) yields

$$x_f = Q_1 e^{A_1 t_f} \bar{x}_{10}$$

and

$$v^T x_f = v^T Q_1 e^{A_1 t_f} \bar{x}_{10} = 0,$$

since (3.43) holds for all $\bar{x}_{10} = \bar{Q}_1 x_0 \in \mathrm{Im}\bar{Q}_1$. □

Example 3.4. (continuation of Examples 3.1 and 3.3)

From (3.43) and (3.40) we have

$$v^T = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \tag{3.44}$$

since

$$v^T Q_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{R_1 C_1} & 0 \\ -\dfrac{1}{R_1(C_2 + C_3)} & \dfrac{1}{C_2 + C_3} \\ \dfrac{1}{R_1(C_2 + C_3)} & -\dfrac{1}{C_2 + C_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Therefore, the descriptor electrical circuit shown in Figure 3.1 is pointwise degenerated in the direction defined by (3.44) for any $t_f > 0$ and any values of the resistance R_1 and capacitances C_1, C_2, C_3.

3.2 Descriptor Fractional Linear Electrical Circuits

Consider the continuous-time fractional linear system described by the state equation (1.49a).

It is assumed that $\det E = 0$, $\operatorname{rank} B = m$ and the pencil of matrices (E, A) is regular, i.e. condition (1.51) is met.

Example 3.5. Consider the fractional electrical circuit shown in Figure 3.1 with given resistance R, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2.

Using Kirchhoff's laws, for the electrical circuit we can write the equations

$$\begin{aligned} e_1 &= RC_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + u_1 + u_3, \\ 0 &= C_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + C_2 \frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} - C_3 \frac{\mathrm{d}^\alpha u_3}{\mathrm{d}t^\alpha}, \\ e_2 &= u_2 + u_3. \end{aligned} \tag{3.45}$$

The equations (3.45) can be written in the form

$$\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

In this case we have

$$E = \begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.46}$$

Note that the matrix E is singular ($\det E = 0$) but the pencil

$$\begin{aligned} \det\left[Es^\alpha - A\right] &= \begin{vmatrix} RC_1 s^\alpha + 1 & 0 & 1 \\ C_1 s^\alpha & C_2 s^\alpha & -C_3 s^\alpha \\ 0 & 1 & 1 \end{vmatrix} \\ &= \left(RC_1 s^\alpha + 1\right)\left(C_2 + C_3\right) s^\alpha + C_1 s^\alpha \end{aligned} \tag{3.47}$$

is regular.

Therefore, the fractional electrical circuit is a descriptor fractional linear system with regular pencil.

In general case we have the following theorem.

Theorem 3.5. *If the fractional electrical circuit contains at least one mesh consisting of branches with only ideal capacitors and voltage sources, then its matrix E is singular.*

Proof. Note that the row of E corresponding to the mesh is a zero row. This follows from the fact that the equation written with the use of Kirchhoff's voltage law is an algebraic one. □

Example 3.6. Consider the fractional electrical circuit shown in Figure 3.2 with given resistances R_1, R_2, R_3 inductances L_1, L_2, L_3 and source voltages e_1 and e_2.

Using Kirchhoff's laws we can write the equations

$$\begin{aligned} e_1 &= R_1 i_1 + L_1 \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} + R_3 i_3 + L_3 \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta}, \\ e_2 &= R_2 i_2 + L_2 \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} + R_3 i_3 + L_3 \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta}, \\ 0 &= i_1 + i_2 - i_3. \end{aligned} \tag{3.48}$$

Equations (3.48) can be written in the form

$$\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 & -R_2 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

In this case we have

$$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -R_1 & 0 & -R_2 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Note that the matrix E is singular but the pencil

$$\begin{aligned} \det\left[Es^\beta - A\right] &= \begin{vmatrix} L_1 s^\beta + R_1 & 0 & L_3 s^\beta + R_3 \\ 0 & L_2 s^\beta + R_2 & L_3 s^\beta + R_3 \\ -1 & -1 & 1 \end{vmatrix} \\ &= \left[L_1 \left(L_2 + L_3\right) + L_2 L_3\right] s^{2\beta} \\ &\quad + \left[\left(L_2 + L_3\right) R_1 + \left(L_1 + L_3\right) R_2 + \left(L_1 + L_2\right) R_3\right] s^\beta \\ &\quad + R_1 \left(R_2 + R_3\right) + R_2 R_3 \end{aligned}$$

is regular.

Therefore, the fractional electrical circuit is a descriptor linear system with regular pencil.

Theorem 3.6. *If the fractional electrical circuit contains at least one node with branches with coils then its matrix E is singular.*

Proof. Note that the equation written using the current Kirchhoff's current law for this node is an algebraic one and in the matrix E we have zero row. □

In general case we have the following theorem.

Theorem 3.7. *Every fractional electrical circuit is a descriptor (singular) system if it contains at least one mesh consisting of branches with only ideal capacitances and voltage sources or at least one node with branches with coils.*

Proof. By Theorem 3.5 the matrix E of the system is singular if the fractional electrical circuit contains at least one mesh consisting of branches with only ideal capacitors and voltage sources. Similarly, by Theorem 3.6 the matrix E is singular if the fractional electrical circuit contains at least one node with branches with coils. □

Using the solution (1.60) of the equation (1.49a) we may find the voltages on the supercapacitors and currents in the supercoils in transient states of descriptor fractional linear electrical circuits. Knowing the voltages and currents and using (1.61), we may also find any currents and voltages in dscriptor fractional linear electrical circuits.

Example 3.7. (continuation of Example 3.5) Using one of the well-known methods [21, 50, 188] for the pencil (3.47), we can find the matrices

$$P = \begin{bmatrix} \frac{1}{RC_1} & 0 & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{1}{C_2+C_3} & \frac{C_2}{R(C_2+C_3)^2} \\ 0 & 0 & -1 \end{bmatrix}, \tag{3.49a}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{C_3}{C_2+C_3} \\ 0 & -1 & \frac{C_2}{C_2+C_3} \end{bmatrix}, \tag{3.49b}$$

which transform it to the canonical form (1.52) with

$$A_1 = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{RC_1} \\ \frac{1}{R(C_2+C_3)} & -\frac{1}{R(C_2+C_3)} \end{bmatrix},$$
$$N = [0], \quad n_1 = 2, \quad n_2 = 1.$$

Using the matrix B given by (3.46), (3.49) and (1.53c) we obtain

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB = \begin{bmatrix} \frac{1}{RC_1} & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{C_2}{R(C_2+C_3)} \\ 0 & -1 \end{bmatrix}.$$

Using (1.54) we obtain $x_1(t)$ of the fractional circuit for any initial condition $x_{10} \in \mathbb{R}^{n_1}$ and an arbitrary input $u(t)$.

From (1.59) we have

$$x_2(t) = -B_2 u(t),$$

since $N = [0]$.

Taking into account (1.60) and (1.61) we may compute any currents and voltages in singular fractional linear electrical circuit shown in Figure 3.1.

In the same way we may find currents in the supercoils and of the descriptor fractional electrical circuit shown in Figure 3.1.

3.3 Polynomial Approach to Fractional Descriptor Electrical Circuits

First the essensce of the polynomial approach will be shown on the following example.

Example 3.8. Consider the fractional descriptor electrical circuit shown in Figure 3.3 with given resistances R_1, R_1; inductances L_1, L_2 and source current i_z.

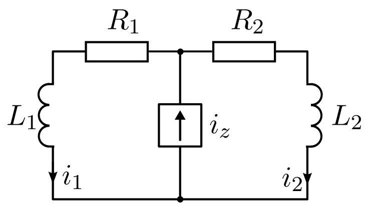

Fig. 3.3 Fractional electrical circuit of Example 3.8

Using Kirchhoff's laws we can write the equations

$$L_1 \frac{d^\beta i_1}{dt^\beta} + R_1 i_1 = L_2 \frac{d^\beta i_2}{dt^\beta} + R_2 i_2, \quad (3.50a)$$

$$i_z = i_1 + i_2. \quad (3.50b)$$

The equations (3.50) can be written in the form (1.49a)

$$E\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + Bi_z, \tag{3.51a}$$

where

$$E = \begin{bmatrix} L_1 & -L_2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & R_2 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.51b}$$

Defining

$$E_1 = \begin{bmatrix} L_1 & -L_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -R_1 & R_2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \end{bmatrix}$$

we can write the equation (3.51) in the form

$$E_1\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A_1\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_1 i_z \tag{3.52a}$$

and

$$0 = A_2\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_2 i_z. \tag{3.52b}$$

The β-order fractional differentiation of (3.52b) yields

$$0 = A_2\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_2\frac{\mathrm{d}^\beta i_z}{\mathrm{d}t^\beta}. \tag{3.53}$$

From (3.52a) and (3.53) we have

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} i_z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix}\frac{\mathrm{d}^\beta i_z}{\mathrm{d}t^\beta}. \tag{3.54}$$

Note that the matrix

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix} \tag{3.55}$$

is nonsingular and premultiplying (3.54) by its inverse we obtain

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \bar{A}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \bar{B}_0 i_z + \bar{B}_1\frac{\mathrm{d}^\beta i_z}{\mathrm{d}t^\beta}, \tag{3.56a}$$

where

$$\bar{A} = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -R_1 & R_2 \\ 0 & 0 \end{bmatrix} = \frac{1}{L_1 + L_2} \begin{bmatrix} -R_1 & R_2 \\ R_1 & -R_2 \end{bmatrix},$$
$$\bar{B}_0 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\bar{B}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{L_1 + L_2} \begin{bmatrix} L_2 \\ L_1 \end{bmatrix}. \tag{3.56b}$$

The standard equation (3.56a) can be also obtained from the equation (3.54) by reducing the matrix (3.55) to the identity matrix $\mathbb{I}_2$ using the elementary row operations (see Appendix B)

$$L\,[1 + 2 \times L_2]\,,\; L\left[1 \times \frac{1}{L_1 + L_2}\right],\; L\,[2 + 1 \times (-1)]\,. \tag{3.57}$$

Performing the elementary row operations (3.57) on the matrix $\begin{bmatrix} 1 & 0 \\ 0 & s^\beta \end{bmatrix}$ we obtain the polynomial matrix

$$L\left(s^\beta\right) = \frac{1}{L_1 + L_2} \begin{bmatrix} 1 & s^\beta L_2 \\ -1 & s^\beta L_1 \end{bmatrix} \tag{3.58}$$

satisfying the equalities

$$\begin{aligned} L\left(s^\beta\right)\left[E s^\beta - A\right] &= \frac{1}{L_1 + L_2} \begin{bmatrix} 1 & s^\beta L_2 \\ -1 & s^\beta L_1 \end{bmatrix} \begin{bmatrix} s^\beta L_1 + R_1 & -s^\beta L_2 - R_2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} s^\beta + \frac{R_1}{L_1 + L_2} & -\frac{R_2}{L_1 + L_2} \\ -\frac{R_1}{L_1 + L_2} & s^\beta + \frac{R_2}{L_1 + L_2} \end{bmatrix} = \left[\mathbb{I}_2 s^\beta - \bar{A}\right], \\ L\left(s^\beta\right) B &= \frac{1}{L_1 + L_2} \begin{bmatrix} 1 & s^\beta L_2 \\ -1 & s^\beta L_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{s^\beta}{L_1 + L_2} \begin{bmatrix} L_2 \\ L_1 \end{bmatrix} = \left[\bar{B}_0 + \bar{B}_1 s^\beta\right]. \end{aligned}$$

Therefore, the reduction of the matrix (3.55) to identity matrix by the use of elementary row operations (3.57) is equivalent to premultiplication of the equation

$$\left[E s^\beta - A\right] X = BU$$

by the polynomial matrix of elementary row operations (3.58), where $X = \mathcal{L}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$, $U = \mathcal{L}\,[i_z]$.

In general case let us consider the continuous-time fractional linear system described by the state equation (1.49a).

It is assumed that $\det E = 0$ but the pencil of matrices (E, A) is regular, i.e. condition (1.51) is met.

Applying to (1.49a) the Laplace transform for zero initial conditions we obtain the equation

$$[Es^\alpha - A]\,X = BU, \tag{3.59}$$

where $X = \mathcal{L}[x(t)]$, $U = \mathcal{L}[u(t)]$.

Theorem 3.8. *There exists a nonsingular polynomial matrix*

$$L(s^\alpha) = L_0 + L_1 s^\alpha + \cdots + L_\mu s^{\alpha\mu}, \tag{3.60}$$

where μ is the nilpotency index of the pair (E, A) (see Appendix C), such that

$$L(s^\alpha)[Es^\alpha - A] = \left[\mathbb{I}_n s^\alpha - \bar{A}\right] \tag{3.61}$$

if and only if the pencil (E, A) is regular, i.e. the condition (1.51) *is met.*

Proof. The matrix $\left[\mathbb{I}_n s^\alpha - \bar{A}\right]$ is nonsingular for every matrix $\bar{A} \in \mathbb{R}^{n\times n}$. From (3.61) and (1.51) it follows that the polynomial matrix (3.60) is nonsingular. Using elementary row operations the singular matrix E can be always reduced to the form $\begin{bmatrix} E_1 \\ 0 \end{bmatrix}$, where E_1 has the full row rank r_1 and L_1 is the matrix of elementary row operations.

Premultiplying (3.59) by L_1 we obtain

$$L_1[Es^\alpha - A]\,X = \begin{bmatrix} E_1 s^\alpha - A_1 \\ -A_2 \end{bmatrix} X = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U, \tag{3.62a}$$

where

$$L_1 E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix},\ L_1 A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ E_1, A_1 \in \mathbb{R}^{r_1\times n},\ L_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ B_1 \in \mathbb{R}^{r_1\times m}. \tag{3.62b}$$

Using (3.62) we can write the equation (1.49a) in the form

$$E_1 \frac{d^\alpha x}{dt^\alpha} = A_1 x + B_1 u, \tag{3.63a}$$

$$0 = A_2 x + B_2 u. \tag{3.63b}$$

The α-order fractional differentiation of (3.63b) yields

$$0 = A_2 \frac{d^\alpha x}{dt^\alpha} + B_2 \frac{d^\alpha u}{dt^\alpha}. \tag{3.64}$$

From (3.63a) and (3.64) we have

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}. \tag{3.65}$$

If the matrix $\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}$ is nonsingular then from (3.65) we have

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \bar{A}_1 x + \bar{B}_{10} u + \bar{B}_{11} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha},$$

where

$$\bar{A}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \quad \bar{B}_{10} = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \bar{B}_{11} = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}.$$

If the matrix $\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}$ is singular then using elementary row operations we may reduce this matrix to the form

$$L_2 \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} = \begin{bmatrix} E_2 \\ 0 \end{bmatrix},$$

where $r_2 = \mathrm{rank} E_2 \geq \mathrm{rank} E_1$ and we repeat the procedure.

It is well known that if the condition (1.51) is satisfied then after μ steps of the procedure we obtain the nonsingular matrix

$$\begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix}. \tag{3.67}$$

Premultiplying the equation

$$\begin{bmatrix} E_\mu \\ -A_{2\mu} \end{bmatrix} \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \begin{bmatrix} A_{2\mu-1} \\ 0 \end{bmatrix} x + \begin{bmatrix} B_{\mu-1,0} \\ 0 \end{bmatrix} u + \begin{bmatrix} B_{\mu-1,1} \\ B_{\mu,1} \end{bmatrix} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + \cdots + \begin{bmatrix} 0 \\ B_{\mu,\mu} \end{bmatrix} \frac{\mathrm{d}^{\mu\alpha} u}{\mathrm{d}t^{\mu\alpha}} \tag{3.68}$$

by the inverse matrix $\begin{bmatrix} E_\mu \\ -A_{2\mu} \end{bmatrix}^{-1}$ we obtain the desired equation

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \bar{A}_\mu x + \bar{B}_0 u + \bar{B}_1 \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + \cdots + \bar{B}_\mu \frac{\mathrm{d}^{\mu\alpha} u}{\mathrm{d}t^{\mu\alpha}}, \tag{3.69a}$$

where

$$\begin{aligned} \bar{A}_\mu &= \begin{bmatrix} E_\mu \\ -A_{2\mu} \end{bmatrix}^{-1} \begin{bmatrix} A_{2\mu-1} \\ 0 \end{bmatrix}, \quad \bar{B}_0 = \begin{bmatrix} E_\mu \\ -A_{2\mu} \end{bmatrix}^{-1} \begin{bmatrix} B_{\mu-1,0} \\ 0 \end{bmatrix}, \\ \bar{B}_1 &= \begin{bmatrix} E_\mu \\ -A_{2\mu} \end{bmatrix}^{-1} \begin{bmatrix} B_{\mu-1,1} \\ B_{\mu,1} \end{bmatrix}, \ldots, \bar{B}_\mu = \begin{bmatrix} E_\mu \\ -A_{2\mu} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_{\mu,\mu} \end{bmatrix}. \end{aligned} \tag{3.69b}$$

The standard equation (3.69a) can be also obtained from the equation (3.68) by reducing the matrix (3.67) to the indentity matrix $\mathbb{I}_n$ using the elementary row operations (Appendix B) and this is equivalent to premultiplication of the equation (3.68) by suitable matrix of elementary row operations

$$L_\mu \begin{bmatrix} E_\mu \\ -A_{2\mu} \end{bmatrix} = \mathbb{I}_n.$$

The desired polynomial matrix of elementary row operations (3.60) is given by

$$L(s^\alpha) = L_\mu \prod_{i=1}^{\mu} \operatorname{diag}\left[\mathbb{I}_{r_i}, \mathbb{I}_{n-r_i} s^\alpha\right].$$

Note that the matrix $\mathbb{I}_{n-r_i} s^\alpha$ corresponds to the fractional differentiation of the algebraic equations. □

The considerations can be easily extended to the linear electrical circuits described by the state equation with different fractional orders.

Example 3.9. Consider the fractional descriptor electrical circuit shown in Figure 3.4 with given resistances R_1, R_2, R_3; inductances L_1, L_2, L_3; capacitance C and source voltages e_1, e_2.

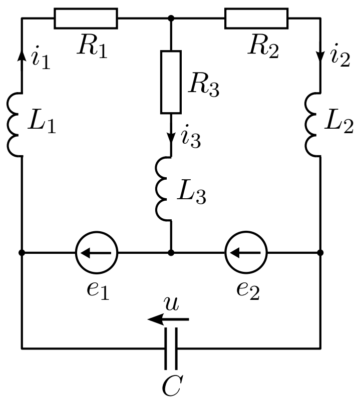

Fig. 3.4 Fractional electrical circuit of Example 3.9

Using Kirchhoff's laws we can write the equations

$$e_1 = L_1 \frac{d^\beta i_1}{dt^\beta} + R_1 i_1 + L_3 \frac{d^\beta i_3}{dt^\beta} + R_3 i_3, \tag{3.70a}$$

$$e_2 = L_2 \frac{d^\beta i_2}{dt^\beta} + R_2 i_2 - L_3 \frac{d^\beta i_3}{dt^\beta} - R_3 i_3, \tag{3.70b}$$

$$i_3 = i_1 - i_2, \tag{3.70c}$$

$$u = e_1 + e_2. \tag{3.70d}$$

The equations (3.70) can be written in the form

$$E\begin{bmatrix}\frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta}\\ \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta}\\ \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta}\\ \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}\end{bmatrix} = A\begin{bmatrix}i_1\\ i_2\\ i_3\\ u\end{bmatrix} + B\begin{bmatrix}e_1\\ e_2\end{bmatrix}, \tag{3.71a}$$

where

$$E = \begin{bmatrix}L_1 & 0 & L_3 & 0\\ 0 & L_2 & -L_3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \quad A = \begin{bmatrix}-R_1 & 0 & -R_3 & 0\\ 0 & -R_2 & R_3 & 0\\ 1 & -1 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}, \quad B = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 1 & 1\end{bmatrix}. \tag{3.71b}$$

The pencil (E, A) is regular, since

$$\begin{aligned}\det\left\{E\begin{bmatrix}\mathbb{I}_3 s^\beta & 0\\ 0 & s^\alpha\end{bmatrix} - A\right\} &= \begin{vmatrix}s^\beta L_1 + R_1 & 0 & s^\beta L_3 + R_3 & 0\\ 0 & s^\beta L_2 + R_2 & -s^\beta L_3 - R_3 & 0\\ -1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1\end{vmatrix}\\ &= (s^\beta L_1 + R_1)\left[s^\beta(L_2 + L_3) + R_3 + R_3\right]\\ &\quad + (s^\beta L_3 + R_3)(s^\beta L_2 + R_2) \neq 0.\end{aligned}$$

Denoting

$$E_1 = \begin{bmatrix}L_1 & 0 & L_3 & 0\\ 0 & L_2 & -L_3 & 0\end{bmatrix}, \quad A_1 = \begin{bmatrix}-R_1 & 0 & -R_3 & 0\\ 0 & -R_2 & R_3 & 0\end{bmatrix},$$
$$A_2 = \begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}, \quad B_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad B_2 = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$$

we can write the equation (3.71a) in the form

$$E_1\begin{bmatrix}\frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta}\\ \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta}\\ \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta}\\ \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}\end{bmatrix} = A_1\begin{bmatrix}i_1\\ i_2\\ i_3\\ u\end{bmatrix} + B_1\begin{bmatrix}e_1\\ e_2\end{bmatrix} \tag{3.72a}$$

and

$$0 = A_2 \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + B_2 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{3.72b}$$

The β-order fractional differentiation of the first equation of (3.72b) and α-order fractional differentiation of the second equation of (3.72b) yield

$$0 = A_2 \begin{bmatrix} \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \end{bmatrix} + B_2 \begin{bmatrix} \frac{\mathrm{d}^\beta e_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\alpha e_2}{\mathrm{d}t^\alpha} \end{bmatrix}. \tag{3.73}$$

From (3.72a) and (3.73) we have

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \end{bmatrix} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}^\beta e_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\alpha e_2}{\mathrm{d}t^\alpha} \end{bmatrix}. \tag{3.74}$$

The matrix

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} = \begin{bmatrix} L_1 & 0 & L_3 & 0 \\ 0 & L_2 & -L_3 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.75}$$

is nonsingular and premultiplying (3.74) by its inverse we obtain

$$\begin{bmatrix} \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \end{bmatrix} = \bar{A} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + \bar{B}_0 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \bar{B}_1 \begin{bmatrix} \frac{\mathrm{d}^\beta e_1}{\mathrm{d}t^\beta} \\ \frac{\mathrm{d}^\alpha e_2}{\mathrm{d}t^\alpha} \end{bmatrix}. \tag{3.76a}$$

where

$$\bar{A} = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{L} \begin{bmatrix} -R_1(L_2 + L_3) & -R_2 L_3 & -R_3 L_2 & 0 \\ -R_1 L_3 & -R_2(L_1 + L_3) & R_3 L_1 & 0 \\ -R_1 L_2 & R_2 L_1 & -R_3(L_1 + L_2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\bar{B}_0 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \frac{1}{L} \begin{bmatrix} L_2 + L_3 & L_3 \\ L_3 & L_1 + L_3 \\ L_2 & -L_1 \\ 0 & 0 \end{bmatrix}, \quad (3.76b)$$
$$\bar{B}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},$$

where $L = L_1(L_2 + L_3) + L_2 L_3$.

The standard equation (3.76a) can be also obtained from the equation (3.74) by reducing the matrix (3.75) to the identity matrix $\mathbb{I}_4$ using elementary row operations

$$L[1 + 3 \times (-L_3)],\ L[2 + 3 \times L_3],\ L\left[1 + 2 \times \frac{L_3}{L_2 + L_3}\right],\ L\left[1 \times \frac{L_2 + L_3}{L}\right],$$
$$L[2 + 1 \times L_3],\ L[3 + 1 \times 1],\ L\left[2 \times \frac{1}{L_2 + L_3}\right],\ L[3 + 2 \times (-1)]. \quad (3.77)$$

Using the elementary row operations (3.77) on the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s^\beta & 0 \\ 0 & 0 & 0 & s^\alpha \end{bmatrix}$$

we obtain the polynomial matrix

$$L\left[s^\alpha, s^\beta\right] = \frac{1}{L} \begin{bmatrix} L_2 + L_3 & L_3 & -L_2 L_3 s^\beta & 0 \\ L_3 & L_1 + L_3 & L_1 L_3 s^\beta & 0 \\ L_2 & -L_1 & L_1 L_2 s^\beta & 0 \\ 0 & 0 & 0 & L s^\alpha \end{bmatrix}$$

satisfying the equation

$$L\left[s^\alpha, s^\beta\right]\left[E\mathrm{diag}\left(s^\beta, s^\beta, s^\beta, s^\alpha\right) - A\right] = \mathrm{diag}\left(s^\beta, s^\beta, s^\beta, s^\alpha\right) - \bar{A}.$$

3.4 Positive Descriptor Fractional Electrical Circuits

Consider the autonomous fractional descriptor continuous-time linear system [75]

$$E\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) = Ax(t), \tag{3.78}$$

where $\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha}$ is the α-order $(0 < \alpha < 1)$ fractional derivative described by (1.9), $x(t) \in \mathbb{R}^n$ is the state vector and $E, A \in \mathbb{R}^{n\times n}$.

It is assumed that $\det E = 0$ but the pencil $(E,\, A)$ is regular, i.e.

$$\det[Es^\alpha - A] \neq 0, \quad \text{for some} \quad s \in \mathbb{C}.$$

Assuming that for some chosen $c \in \mathbb{R}$, $\det\,[Ec - A] \neq 0$ and premultiplying (3.78) by $[Ec - A]^{-1}$ we obtain

$$\bar{E}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) = \bar{A}x(t), \tag{3.79a}$$

where

$$\bar{E} = [Ec - A]^{-1}\,E, \quad \bar{A} = [Ec - A]^{-1}\,A. \tag{3.79b}$$

Note that the equations (3.78) and (3.79a) have the same solution $x(t)$.

From Theorem 1.16 we obtain the following theorem.

Theorem 3.9. *The solution to the equation* (3.78) *is given by*

$$x(t) = \Phi_0(t)\bar{E}\bar{E}^D w, \tag{3.80a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \tag{3.80b}$$

and the vector $w \in \mathbb{R}^n$ is arbitrary.

From (3.80a) we have $x(0) = x_0 = \bar{E}\bar{E}^D w$ and $x_0 \in \mathrm{Im}\bar{E}\bar{E}^D$.

Lemma 3.2. [71] The matrix $\Phi_0(t)$ defined by (3.80b) is nonsingular for any matrix $A \in \mathbb{R}^{n\times n}$ and time $t \geq 0$.

Theorem 3.10. *Let*

$$P = \bar{E}\bar{E}^D, \tag{3.81a}$$

$$Q = \bar{E}^D\bar{A}. \tag{3.81b}$$

Then

$$P^k = P \quad for \quad k = 2, 3, \ldots; \tag{3.82a}$$

$$PQ = QP = Q; \tag{3.82b}$$

$$P\bar{E}^D = \bar{E}^D \tag{3.82c}$$

$$P\Phi_0(t) = \Phi_0(t). \tag{3.82d}$$

Proof. Using (3.81a), (D.2a) and (D.2b) (see Appendix D) we obtain

$$P^2 = \bar{E}\bar{E}^D\bar{E}\bar{E}^D = \bar{E}\bar{E}^D$$

and by induction

$$P^k = P^{k-2}P^2 = P^{k-1} = \cdots = P \quad \text{for} \quad k = 3, 4, \ldots \; .$$

Using (3.81) we have

$$PQ = \bar{E}\bar{E}^D\bar{E}^D\bar{A} = \bar{E}^D\bar{E}\bar{E}^D\bar{A} = \bar{E}^D\bar{A} = Q$$

and

$$QP = \bar{E}^D\bar{A}\bar{E}\bar{E}^D = \bar{E}^D\bar{E}\bar{E}^D\bar{A} = \bar{E}^D\bar{A} = Q.$$

Using (3.81a) and (D.2a) we obtain

$$P\bar{E}^D = \bar{E}\bar{E}^D\bar{E}^D = \bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D.$$

Using (3.80b), (3.81a) and taking into account (D.2a) we have

$$\begin{aligned} P\Phi_0(t) =& \bar{E}\bar{E}^D\sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^k t^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty}\bar{E}\bar{E}^D\bar{E}^D\bar{A}\frac{\left(\bar{E}^D\bar{A}\right)^{k-1} t^{k\alpha}}{\Gamma(k\alpha+1)} \\ =& \sum_{k=0}^{\infty}\frac{\left(\bar{E}^D\bar{A}\right)^k t^{k\alpha}}{\Gamma(k\alpha+1)} = \Phi_0(t). \end{aligned}$$

□

Definition 3.3. [71, 93] The fractional descriptor system (3.78) is called (internally) positive if the state vector $x(t) \in \mathbb{R}_+^n$, $t \geq 0$ for all initial conditions $x_0 \in \mathbb{R}_+^n$.

Theorem 3.11. *[71, 93] The fractional descriptor system* (3.78) *is (internally) positive if and only if*

$$\bar{E}^D\bar{A} \in M_n. \tag{3.83}$$

The proof of this theorem is similar to the proof of Theorem 1.9.

Lemma 3.3. If there exists a nonnegative vector $v \in \mathbb{R}^+$ such that

$$v^T\bar{E} = 0,$$

then

$$v^T \bar{E}^D = 0.$$

Proof. Taking into account that $\bar{E} = VW$ ((D.7) in Appendix D) and

$$\bar{E}^D = V \left[W\bar{E}V\right]^{-1} W$$

we obtain

$$v^T \bar{E}^D = v^T V \left[W\bar{E}V\right]^{-1} W = 0,$$

since $v^T V = 0$. □

Consider the fractional electrical circuits composed of resistors, condensators, coils and voltage (current) sources. It is well-known that such electrical circuits can be described by the equation (1.49a) if as the state variables (components of the state vector $x(t)$) the voltages on the condensators and currents in the coils are chosen [44, 71].

It will be shown that the fractional descriptor electrical circuits are linear systems with regular pencils.

Theorem 3.12. *Every electrical circuit is a descriptor fractional system if it contains at least one mesh consisting of branches with only ideal capacitors and voltage sources or at least one node with branches with coils.*

Proof. If the fractional electrical circuits contains at least one mesh consisting of branches with ideal capacitors and voltage sources, then the rows of the matrix E corresponding to the meshes are zero rows and the matrix E is singular. If the fractional electrical circuit contains at least one node with branches with coils, then the equations written on Kirchhoff's current law for these nodes are algebraic ones and the corresponding rows of E are zero rows and it is singular. □

Theorem 3.13. *Every fractional descriptor electrical circuit is a linear system with regular pencil.*

Proof. It is well-known [44, 71] that for a fractional descriptor electrical circuit with n branches and q nodes using current Kirchhoff's law we can write $q-1$ algebraic equations and using the voltage Kirchhoff's law $n-q+1$ fractional differential equations. The equalities are linearly independent and can be written in the form (1.49a). From linear independence of the equations it follows that the condition (1.51) is satisfied and the pencil of the fractional electrical circuit is regular. □

Theorem 3.14. *The autonomous fractional descriptor linear electrical circuit described by* (3.78) *is positive if and only if the condition* (3.83) *is satisfied.*

The proof follows immediately from Theorem 3.11.

Example 3.10. Consider the fractional descriptor electrical circuit shown in Figure 3.5 with given resistance R, capacitances C_1, C_2 and the source voltage e.

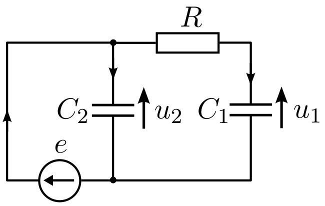

Fig. 3.5 Fractional electrical circuit of Example 3.10

Using Kirchhoff's laws we can write the equations

$$e = RC_1 \frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + u_1, \tag{3.84a}$$

$$e = u_2. \tag{3.84b}$$

The equations (3.84) can be written in the form (1.49a)

$$E \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be, \tag{3.85a}$$

where

$$E = \begin{bmatrix} RC_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.85b}$$

The electrical circuit is regular, since

$$\det [Es^\alpha - A] = \begin{vmatrix} RC_1 s^\alpha + 1 & 0 \\ 0 & 1 \end{vmatrix} = RC_1 s^\alpha + 1 \neq 0.$$

In this case we choose $c = 0$ and using (3.79b) and (3.85b) we obtain

$$\bar{E} = [Ec - A]^{-1} E = [-A]^{-1} E = E = \begin{bmatrix} RC_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{A} = [Ec - A]^{-1} A = [-A]^{-1} A = A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$\bar{E}^D = \begin{bmatrix} \frac{1}{RC_1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}, \quad \bar{E}^D \bar{A} = \begin{bmatrix} -\frac{1}{RC_1} & 0 \\ 0 & 0 \end{bmatrix} \in M_2. \tag{3.86}$$

Therefore, by Theorem 3.11 the fractional descriptor electrical circuit is positive.

Example 3.11. (continuation of Example 3.8) Consider the fractional descriptor electrical circuit shown in Figure 3.3 with state equation matrices given by (3.51b).

The electrical circuit is regular, since

$$\det\left[Es^{\beta}-A\right]=\begin{vmatrix}L_1s^{\beta}+R_1 & -L_2s^{\beta}-R_2\\ 1 & 1\end{vmatrix}=(L_1+L_2)\,s^{\beta}+R_1+R_2\neq 0.$$

In this case we choose $c=0$ and using (3.79b) and (3.51b) we obtain

$$\bar{E}=[Ec-A]^{-1}E=[-A]^{-1}E=\frac{1}{R_1+R_2}\begin{bmatrix}L_1 & -L_2\\ -L_1 & L_2\end{bmatrix}, \qquad (3.87a)$$

$$\bar{A}=[Ec-A]^{-1}A=[-A]^{-1}A=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}. \qquad (3.87b)$$

To compute the Drazin inverse of the matrix (3.87a) we use the procedure given in Appendix D and we obtain

$$\bar{E}=VW,\quad V=\begin{bmatrix}1\\ -1\end{bmatrix},\quad W=\frac{1}{R_1+R_2}\left[L_1\ -L_2\right] \qquad (3.88a)$$

and

$$\begin{aligned}\bar{E}^{D}&=V\left[W\bar{E}V\right]^{-1}W\\ &=\begin{bmatrix}1\\ -1\end{bmatrix}\left[\left(\frac{1}{R_1+R_2}\right)^{2}\left[L_1\ -L_2\right]\begin{bmatrix}L_1 & -L_2\\ -L_1 & L_2\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}\right]^{-1}\\ &\quad\times\frac{1}{R_1+R_2}\left[L_1\ -L_2\right]\\ &=\frac{1}{L_1\left(L_1+L_2\right)}\begin{bmatrix}L_1 & -L_2\\ -L_1 & L_2\end{bmatrix}.\end{aligned} \qquad (3.88b)$$

Taking into account (3.87b) and (3.88) we obtain

$$\begin{aligned}\bar{E}^{D}\bar{A}&=\frac{1}{L_1\left(L_1+L_2\right)}\begin{bmatrix}L_1 & -L_2\\ -L_1 & L_2\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\\ &=\frac{1}{L_1\left(L_1+L_2\right)}\begin{bmatrix}-L_1 & L_2\\ L_1 & -L_2\end{bmatrix}\in M_2\end{aligned}$$

for any values of L_1 and L_2.

Therefore, by Theorem 3.11 the fractional descriptor electrical circuit is positive.

The considerations can be easily extended to fractional descriptor positive electrical circuits composed of resistances, inductances, capacitances and voltage (current) sources.

3.4.1 Pointwise Completeness and Pointwise Degeneracy of Positive Fractional Descriptor Electrical Circuits

Definition 3.4. The positive fractional descriptor electrical circuit described by (3.78) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \mathbb{R}_+^n$ there exists a vector of initial conditions $x_0 \in \mathrm{Im}\left(\bar{E}\bar{E}^D\right) \subset \mathbb{R}_+^n$ such that $x(t_f) = x_f \in \mathbb{R}_+^n$.

Theorem 3.15. *The positive fractional descriptor electrical circuit* (3.78) *is pointwise complete for any* $t = t_f$ *and every final state* $x_f \in \mathbb{R}_+^n$ *belonging to the set*

$$x_f \in \mathrm{Im}\left[\Phi_0(t_f)x_0\right] \subset \mathbb{R}_+^n$$

if and only if $\Phi_0(t_f) \in \mathbb{R}_+^{n\times n}$ *is a monomial matrix.*

Proof. Substituting in (3.80a) $t = t_f$ we obtain

$$x_f = x(t_f) = \Phi_0(t_f)x_0 \tag{3.89}$$

and

$$x_0 = \left[\Phi_0(t_f)\right]^{-1} x_f \in \mathbb{R}_+^n,$$

since the matrix $\Phi_0(t)$ is monomial and $\left[\Phi_0(t_f)\right]^{-1} \in \mathbb{R}_+^{n\times n}$. □

Definition 3.5. The positive fractional descriptor electrical circuit (3.78) is called pointwise degenerated in the direction v for $t = t_f$ if there exists a nonzero vector $v \in \mathbb{R}^n$ such that for all initial conditions $x_0 \in \mathrm{Im}\left(\bar{E}\bar{E}^D\right) \subset \mathbb{R}_+^n$ the condition

$$v^T x_f = 0 \tag{3.90}$$

is satisfied.

Theorem 3.16. *The positive fractional descriptor electrical circuit* (3.78) *is pointwise degenerated in the direction* v *defined by*

$$v^T \bar{E} = 0 \tag{3.91}$$

for any $t_f \geq 0$ *and all initial conditions* $x_0 \in \mathrm{Im}\left(\bar{E}\bar{E}^D\right) \subset \mathbb{R}_+^n$.

Proof. Postmultiplying (3.91) by $\bar{E}^D w$ and using $x_0 = \bar{E}\bar{E}^D w$ and (3.90) we obtain

$$v^T \bar{E}\bar{E}^D w = v^T x_0 = 0.$$

Taking into account (3.80b), (3.89) and (3.91) we obtain

$$v^T x_f = v^T \Phi_0(t_f) x_0 = \sum_{k=0}^{\infty} \frac{v^T \left(\bar{E}^D \bar{A}\right)^k t_f^{k\alpha}}{\Gamma(k\alpha+1)} x_0$$
$$= v^T x_0 + \sum_{k=1}^{\infty} \frac{v^T \bar{E}\bar{E}^D \left(\bar{E}^D \bar{A}\right)^k t_f^{k\alpha}}{\Gamma(k\alpha+1)} = 0,$$

since (D.2b) and (3.91) holds. □

Example 3.12. (continuation of Example 3.10)

In this case the set of admissible (consistent) initial conditions has the form

$$x_0 = \bar{E}\bar{E}^D w = \begin{bmatrix} RC_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{RC_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_{10} \\ 0 \end{bmatrix}, \tag{3.92}$$

where $x_{10} = w_1 > 0$ is arbitrary.

Using (3.89), (3.80b), (3.92) and (3.86)

$$\begin{aligned} x_f = x(t_f) = \Phi_0(t_f) x_0 &= \sum_{k=0}^{\infty} \frac{\left(\bar{E}^D \bar{A}\right)^k t_f^{k\alpha}}{\Gamma(k\alpha+1)} x_0 \\ &= \sum_{k=0}^{\infty} \frac{t_f^{k\alpha}}{\Gamma(k\alpha+1)} \begin{bmatrix} \left(-\frac{1}{RC_1}\right)^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \frac{t_f^{k\alpha}}{\Gamma(k\alpha+1)} \begin{bmatrix} \left(-\frac{1}{RC_1}\right)^k x_{10} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{f1} \\ 0 \end{bmatrix} \in \mathbb{R}_+^2. \end{aligned} \tag{3.93}$$

Therefore, the fractional descriptor electrical circuit is pointwise complete for any t_f and every final state (3.93) for all values of R and C_1.

The electrical circuit is pointwise degenerated in the direction $v^T = \begin{bmatrix} 0 & v_2 \end{bmatrix}$ (v_2 - arbitrary), since from (3.91) we have

$$v^T \bar{E} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} RC_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \text{for } v_1 = 0 \text{ and arbitrary } v_2. \tag{3.94}$$

Using (3.93) and (3.94) we obtain

$$v^T x_f = \begin{bmatrix} 0 & v_2 \end{bmatrix} \begin{bmatrix} x_{f1} \\ 0 \end{bmatrix} = 0.$$

Therefore, the electrical circuit is degenerated in the direction $v^T = \begin{bmatrix} 0 & v_2 \end{bmatrix}$ for all values of R and C_1.

Chapter 4
Stability of Positive Standard Linear Electrical Circuits

4.1 Stability of Positive Electrical Circuits

Consider a linear electrical circuits composed of resistors, coils, capacitors and voltage (current) sources. Using Kirchhoff's laws we may describe the transient states in almost all electrical circuits by state equations [1, 44, 47, 68, 71, 115, 172, 191]

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t), \tag{4.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{4.1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

As the state variables $x_1(t)$,..., $x_n(t)$, (the components of $x(t)$) the currents in the coils and voltages on the condensators are chosen, the components of the input vector $u(t)$ are source voltages or source currents and the components of the output vectors $y(t)$ are currents and voltages of the electrical circuit.

Definition 4.1. The electrical circuit described by the equations (4.1) (shortly electrical circuit (4.1)) is called (internally) positive if for any $x(0) = x_0 \in \mathbb{R}_+^n$ and every $u(t) \in \mathbb{R}_+^m$, $t \geq 0$ we have $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$, $t \geq 0$.

Theorem 4.1. *[30, 47] The electrical circuit* (4.1) *is positive if and only if*

$$A \in M_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}.$$

Definition 4.2. The positive electrical circuit (4.1) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for any} \quad x_0 \in \mathbb{R}_+^n.$$

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Studies in Systems, Decision and Control 13, DOI: 10.1007/978-3-319-11361-6_4

The positive electrical circuit will be called unstable if it is not asymptotically stable.

Theorem 4.2. *[30, 47] The positive electrical circuit* (4.1) *is asymptotically stable if and only if*

$$\mathrm{Re}s_k < 0 \quad \text{for} \quad k = 1, 2, \ldots, n;$$

where s_k*,* $k = 1, 2, \ldots, n$ *are the eigenvalues (not necessarily distinct) of the Metzler matrix* $A = [a_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$*, i.e. the zeros of the polynomial*

$$\begin{aligned} &\det\left[\mathbb{I}_n s - A\right] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0, \\ &a_{n-1} = \mathrm{trace}A = \sum_{i=1}^{n} a_{ii}, \ \ldots, \ a_0 = \det\left[-A\right]. \end{aligned} \tag{4.2}$$

Lemma 4.1. The positive electrical circuit (4.1) is asymptotically unstable if

$$\det A = 0. \tag{4.3}$$

Proof. From (4.2) and $\det A = s_1 s_2 \cdots s_n$ it follows that if (4.3) holds, then at least one eigenvalue of the matrix A is zero. By Theorem 4.2 the positive electrical circuit (4.1) is unstable. □

4.2 Positive Unstable *R*, *L*, *e* Electrical Circuits

In this section, following [73], a class of R, L, e electrical circuits composed of resistors with resistances R_i, $i = 1, 2, \ldots, q_R$; coils with inductances L_j, $j = 1, 2, \ldots, q_L$ and source voltages e_k, $k = 1, 2, \ldots, m_e$ which are unstable for all values of R_i, L_j will be proposed.

Example 4.1. Consider the electrical circuit shown in Figure 4.1 with given resistances R_1, R_2, inductances L_1, L_2 and source voltages e_1, e_2.

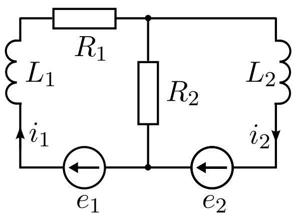

Fig. 4.1 Electrical circuit of Example 4.1

Using Kirchhoff's laws we obtain the following equations

$$\begin{aligned} e_1 &= (R_1 + R_2)\, i_1 - R_2 i_2 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t}, \\ e_2 &= -R_2 i_1 + R_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}. \end{aligned} \tag{4.4}$$

The equations (4.4) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{4.5a}$$

where

$$A = \begin{bmatrix} -\dfrac{R_1 + R_2}{L_1} & \dfrac{R_2}{L_1} \\ \dfrac{R_2}{L_2} & -\dfrac{R_2}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix}.$$

By Theorem 4.1 the electrical circuit is positive for all values of R_1, R_2 and nonzero L_1, L_2, since $A \in M_2$ and $B \in \mathbb{R}_+^{2\times 2}$.

Note that

$$\det A = \begin{vmatrix} -\dfrac{R_1 + R_2}{L_1} & \dfrac{R_2}{L_1} \\ \dfrac{R_2}{L_2} & -\dfrac{R_2}{L_2} \end{vmatrix} = \begin{vmatrix} -\dfrac{R_1}{L_1} & \dfrac{R_2}{L_1} \\ 0 & -\dfrac{R_2}{L_2} \end{vmatrix} = \frac{R_1 R_2}{L_1 L_2}. \tag{4.6}$$

From (4.6) it follows that

$$\det A = 0 \quad \text{if at least one of } R_1, R_2 \text{ is zero.}$$

Therefore, the electrical circuit shown in Figure 4.1 is positive and unstable for $R_1 = 0$ and all values of R_2 and nonzero L_1, L_2 or for $R_2 = 0$ and all values of R_2 and nonzero L_1, L_2.

Note that the positive electrical circuit is unstable if it has at least one mesh containing only inductances and source voltages.

Example 4.2. Consider the electrical circuit shown in Figure 4.2 with given resistances R_1, R_2, R_3, inductances L_1, L_2, L_3 and source voltages e_1, e_2.

Using Kirchhoff's laws we obtain the following equations

$$\begin{aligned} e_1 &= (R_1 + R_2)\, i_1 - R_2 i_2 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t}, \\ 0 &= -R_2 i_2 + (R_2 + R_3)\, i_2 - R_3 i_3 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}, \\ e_2 &= -R_3 i_2 + R_3 i_3 + L_3 \frac{\mathrm{d}i_3}{\mathrm{d}t}. \end{aligned} \tag{4.7}$$

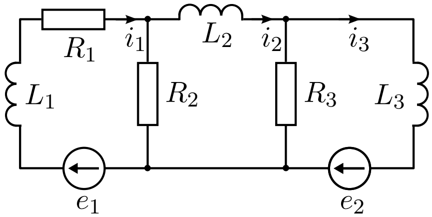

Fig. 4.2 Electrical circuit of Example 4.2

The equations (4.7) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{R_1 + R_2}{L_1} & \frac{R_2}{L_1} & 0 \\ \frac{R_2}{L_2} & -\frac{R_2 + R_3}{L_2} & \frac{R_3}{L_2} \\ 0 & \frac{R_3}{L_3} & -\frac{R_3}{L_3} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{L_3} \end{bmatrix}.$$

By Theorem 4.1 the electrical circuit is positive for all values of R_1, R_2, R_3 and nonzero L_1, L_2, L_3, since $A \in M_3$ and $B \in \mathbb{R}_+^{3\times 2}$.

Note that

$$\det A = \begin{vmatrix} -\frac{R_1 + R_2}{L_1} & \frac{R_2}{L_1} & 0 \\ \frac{R_2}{L_2} & -\frac{R_2 + R_3}{L_2} & \frac{R_3}{L_2} \\ 0 & \frac{R_3}{L_3} & -\frac{R_3}{L_3} \end{vmatrix} = \begin{vmatrix} -\frac{R_1}{L_1} & \frac{R_2}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} & \frac{R_3}{L_2} \\ 0 & 0 & -\frac{R_3}{L_3} \end{vmatrix} = -\frac{R_1 R_2 R_3}{L_1 L_2 L_3}. \tag{4.9}$$

From (4.9) it follows that

$$\det A = 0 \quad \text{if at least one of } R_1, R_2, R_3 \text{ is zero.}$$

Therefore, the electrical circuit shown in Figure 4.2 is positive and unstable for $R_1 = 0$ and all values of R_2, R_3 and nonzero L_1, L_2, L_3 or for $R_2 = 0$ and all values of R_1, R_3, L_1, L_2, L_3 or $R_3 = 0$ and all values of R_1, R_2, L_1, L_2, L_3.

Note that the positive circuit is unstable if it has at least one mesh containing only inductances and source voltages. It is easy to check that if two of the resistances R_1, R_2, R_3 are zero then the matrix A has a double zero eigenvalue ($s_1 = s_2 = 0$).

Example 4.3. Consider the electrical circuit shown in Figure 4.3 with given resistances R_1, R_2, R_3, R_4; inductances L_1, L_2, L_3, L_4 and source voltages e_1, e_2, e_3.

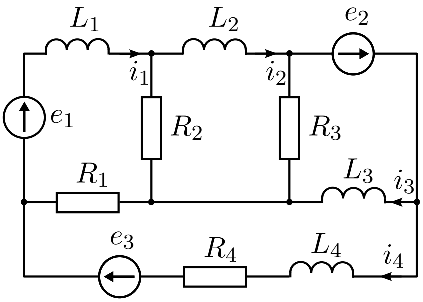

Fig. 4.3 Electrical circuit of Example 4.3

Using Kirchhoff's laws we obtain the following equations

$$\begin{aligned} e_1 &= (R_1 + R_2)\, i_1 - R_2 i_2 - R_1 i_4 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t}, \\ 0 &= -R_2 i_1 + (R_2 + R_3)\, i_2 - R_3 i_3 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}, \\ e_2 &= -R_3 i_2 + R_3 i_3 + L_3 \frac{\mathrm{d}i_3}{\mathrm{d}t}, \\ e_2 + e_3 &= -R_1 i_1 - R_3 i_2 + (R_1 + R_3 + R_4)\, i_4 + L_4 \frac{\mathrm{d}i_4}{\mathrm{d}t}. \end{aligned} \tag{4.10}$$

The equations (4.10) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{R_1+R_2}{L_1} & \frac{R_2}{L_1} & 0 & \frac{R_1}{L_1} \\ \frac{R_2}{L_2} & -\frac{R_2+R_3}{L_2} & \frac{R_3}{L_2} & 0 \\ 0 & \frac{R_3}{L_3} & -\frac{R_3}{L_3} & 0 \\ \frac{R_1}{L_4} & \frac{R_3}{L_4} & 0 & -\frac{R_1+R_3+R_4}{L_4} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{L_3} & 0 \\ 0 & \frac{1}{L_4} & \frac{1}{L_4} \end{bmatrix}.$$

By Theorem 4.1 the electrical circuit is positive for all values of R_1, R_2, R_3, R_4 and nonzero L_1, L_2, L_3, L_4, since $A \in M_4$ and $B \in \mathbb{R}_+^{4\times 3}$.

Note that

$$\det A = \begin{vmatrix} -\frac{R_1+R_2}{L_1} & \frac{R_2}{L_1} & 0 & \frac{R_1}{L_1} \\ \frac{R_2}{L_2} & -\frac{R_2+R_3}{L_2} & \frac{R_3}{L_2} & 0 \\ 0 & \frac{R_3}{L_3} & -\frac{R_3}{L_3} & 0 \\ \frac{R_1}{L_4} & \frac{R_3}{L_4} & 0 & -\frac{R_1+R_3+R_4}{L_4} \end{vmatrix} = \begin{vmatrix} -\frac{R_1}{L_1} & \frac{R_2}{L_1} & 0 & 0 \\ 0 & -\frac{R_2}{L_2} & \frac{R_3}{L_2} & 0 \\ 0 & 0 & -\frac{R_3}{L_3} & 0 \\ \frac{R_1+R_3}{L_4} & \frac{R_3}{L_4} & 0 & -\frac{R_4}{L_4} \end{vmatrix} = \frac{R_1R_2R_3R_4}{L_1L_2L_3L_4}. \tag{4.12}$$

From (4.12) it follows that

$$\det A = 0 \quad \text{if at least one of } R_1, R_2, R_3, R_4 \text{ is zero.}$$

Therefore, the electrical circuit shown in Figure 4.3 is positive and unstable for one zero resistance and for all values of the remaining resistances and all values of nonzero inductances.

The positive circuit is unstable if it has at least one mesh containing only inductances and source voltages.

Thus, in general case we have the following theorem.

Theorem 4.3. *The positive electrical circuit of R, L, e type is unstable if it has at least one mesh containing only coils and voltage sources.*

Proof. To simplify the notation the proof will be accomplished for the positive circuit shown in Figure 4.3 ($n = 4$).

The stability is independent of the inputs and we may assume $e_k = 0$, $k = 1, 2, 3$. If $R_1 = 0$ then for the mesh containing inductances L_1, L_2, L_3 we have

$$L_1 \frac{di_1}{dt} + L_2 \frac{di_2}{dt} + L_3 \frac{di_3}{dt} = 0$$

and this implies linear dependence of rows (and columns) of the matrix A and $\det A = 0$. By Lemma 4.1 the positive electrical circuit is unstable. □

4.3 Positive Unstable G, C, i_s Electrical Circuit

In this subsection, following [73], a class of G, C, i_s electrical circuits composed of resistors with conductances G_i, $i = 1, 2, \ldots, q_G$; condensators with capacitance C_j, $j = 1, 2, \ldots, q_C$ and source currents i_{sk}, $k = 1, 2, \ldots, m_i$ which are positive and unstable for all values of G_i, C_j will be proposed. These considerations are similar (dual) to the considerations in subsection 4.2.

Example 4.4. Consider the electrical circuit shown in Figure 4.4 with given conductances G_1, G_2; capacitances C_1, C_2 and source currents i_{s1}, i_{s2}.

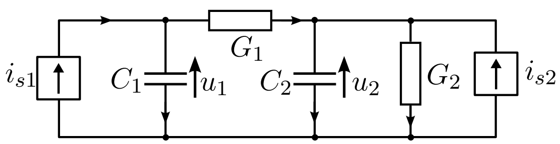

Fig. 4.4 Electrical circuit of Example 4.4

Using Kirchhoff's laws we obtain the following equations

$$\begin{aligned} i_{s1} &= G_1 (u_1 - u_2) + C_1 \frac{du_1}{dt}, \\ i_{s2} &= -G_1 u_1 + (G_1 + G_2) u_2 + C_2 \frac{du_2}{dt}. \end{aligned} \tag{4.13}$$

The equations (4.13) can be written in the form

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B \begin{bmatrix} i_{s1} \\ i_{s2} \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{G_1}{C_1} & \frac{G_1}{C_1} \\ \frac{G_1}{C_2} & -\frac{G_1 + G_2}{C_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{C_1} & 0 \\ 0 & \frac{1}{C_2} \end{bmatrix}.$$

By Theorem 4.1 the electrical circuit is positive for all values of G_1, G_2 and nonzero C_1, C_2, since $A \in M_2$ and $B \in \mathbb{R}_+^{2\times 2}$.

Note that

$$\det A = \begin{vmatrix} -\frac{G_1}{C_1} & \frac{G_1}{C_1} \\ \frac{G_1}{C_2} & -\frac{G_1 + G_2}{C_2} \end{vmatrix} = \begin{vmatrix} -\frac{G_1}{C_1} & 0 \\ \frac{G_1}{C_2} & -\frac{G_2}{C_2} \end{vmatrix} = \frac{G_1 G_2}{C_1 C_2}. \tag{4.15}$$

From (4.15) it follows that

$$\det A = 0 \quad \text{if at least one of } G_1, G_2 \text{ is zero.}$$

Therefore, the electrical circuit shown in Figure 4.4 is positive and unstable for $G_1 = 0$ and all values of G_2 and nonzero C_1, C_2 or for $G_2 = 0$ and all values of G_1, C_1, C_2. If $G_2 = 0$ then positive unstable circuit has one node with branches containing only condensators and current sources.

Example 4.5. Consider the electrical circuit shown in Figure 4.5 with given conductances G_1, G_2, G_3; capacitances C_1, C_2, C_3 and source currents i_{s1}, i_{s2}.

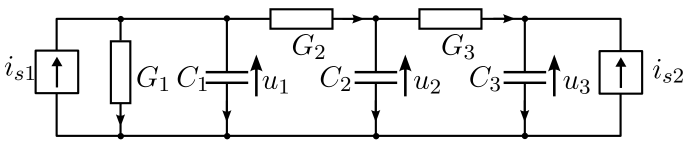

Fig. 4.5 Electrical circuit of Example 4.5

Using Kirchhoff's laws we obtain the following equations

$$\begin{aligned} i_{s1} &= (G_1 + G_2)\, u_1 - G_2 u_2 + C_1 \frac{\mathrm{d}u_1}{\mathrm{d}t}, \\ 0 &= -\, G_2 u_1 + (G_2 + G_3)\, u_2 - G_3 u_3 + C_2 \frac{\mathrm{d}u_2}{\mathrm{d}t}, \\ i_{s2} &= -\, G_3 u_2 + G_3 u_3 + C_3 \frac{\mathrm{d}u_3}{\mathrm{d}t}. \end{aligned} \tag{4.16}$$

The equations (4.16) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + B \begin{bmatrix} i_{s1} \\ i_{s2} \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{G_1+G_2}{C_1} & \frac{G_2}{C_1} & 0 \\ \frac{G_2}{C_2} & -\frac{G_2+G_3}{C_2} & \frac{G_3}{C_2} \\ 0 & \frac{G_3}{C_3} & -\frac{G_3}{C_3} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{C_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C_3} \end{bmatrix}.$$

By Theorem 4.1 the electrical circuit is positive for all values of G_1, G_2, G_3 and nonzero C_1, C_2, C_3, since $A \in M_3$ and $B \in \mathbb{R}_+^{3\times 2}$.

Note that

$$\begin{aligned} \det A &= \begin{vmatrix} -\frac{G_1+G_2}{C_1} & \frac{G_2}{C_1} & 0 \\ \frac{G_2}{C_2} & -\frac{G_2+G_3}{C_2} & \frac{G_3}{C_2} \\ 0 & \frac{G_3}{C_3} & -\frac{G_3}{C_3} \end{vmatrix} \\ &= \begin{vmatrix} -\frac{G_1}{C_1} & \frac{G_2}{C_1} & 0 \\ 0 & -\frac{G_2}{C_2} & \frac{G_3}{C_2} \\ 0 & 0 & -\frac{G_3}{C_3} \end{vmatrix} = -\frac{G_1 G_2 G_3}{C_1 C_2 C_3}. \end{aligned} \tag{4.18}$$

From (4.18) it follows that

$$\det A = 0 \quad \text{if at least one of } G_1, G_2, G_3 \text{ is zero.}$$

Therefore, the electrical circuit shown on Figure 4.5 is positive and unstable for $G_1 = 0$ and all values of G_2, G_3 and nonzero C_1, C_2, C_3 or for $G_2 = 0$ and all values of G_1, G_3, C_1, C_2, C_3 or $G_3 = 0$ and all values of G_1, G_1, C_1, C_2, C_3.

The positive circuit is unstable if $G_1 = 0$. In this case circuit has one node with branches containing only condensators and current sources.

In general case we have the following theorem.

Theorem 4.4. *The positive electrical circuit G, C, i_s type is unstable if it has at least one node with branches containing only condensators and current sources.*

Proof is similar to the proof of Theorem 4.3.

4.4 Positive Unstable *R*, *L*, *C*, *e* Type Electrical Circuits

In this section, following [73], a class of R, L, C, e electrical circuits composed of resistors with resistances R_i, $i = 1, 2, \ldots, q_R$; coils with inductances L_j, $j = 1, 2, \ldots, q_L$; condensators with capacitances C_k, $k = 1, 2, \ldots, q_C$ and source voltages e_l, $l = 1, 2, \ldots, m_e$ which are positive and unstable for all values of parameters will be proposed.

Example 4.6. Consider the electrical circuit shown in Figure 4.6 with given resistance R_1, conductance G_1, inductance L, capacitance C and source voltage e.

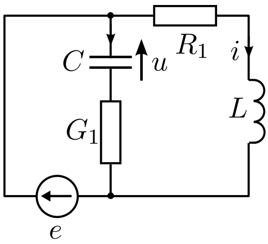

Fig. 4.6 Electrical circuit of Example 4.6

Using Kirchhoff's laws we obtain the following equations

$$
\begin{aligned}
e &= R_1 i + L\frac{\mathrm{d}i}{\mathrm{d}t}, \\
e &= u + \frac{C}{G_1}\frac{\mathrm{d}u}{\mathrm{d}t}.
\end{aligned}
\tag{4.19}
$$

The equations (4.19) can be written in the form

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i \\ u \end{bmatrix} = A\begin{bmatrix} i \\ u \end{bmatrix} + Be,
$$

where

$$
A = \begin{bmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{G_1}{C} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} \\ \frac{G_1}{C} \end{bmatrix}.
$$

The electrical circuit is positive for all values of R_1, G_1 and nonzero L, C, since $A \in M_2$ and $B \in \mathbb{R}_+^{2\times 1}$.

From

$$
\det A = \begin{vmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{G_1}{C} \end{vmatrix} = \frac{R_1 G_1}{LC}
$$

it follows that

$$\det A = 0 \quad \text{if at least one of } R_1, G_1 \text{ are zero.}$$

Therefore, the electrical circuit shown in Figure 4.6 is positive and unstable for $R_1 = 0$ and all values of G_1 and nonzero L, C or for $G_1 = 0$ and all values of R_1 and nonzero L, C.

Note that for $R_1 = 0$ the circuit has one mesh containing only the inductances and source voltages.

Following [68, 74] we will present the following example of positive electrical circuit.

Example 4.7. Consider the electrical circuit shown in Figure 4.7 with given resistances R_k, $k = 1, 2, \ldots, 8$; inductances L_2, L_4, L_6, L_8; capacitances C_1, C_3, C_5, C_7 and source voltages e_1, e_2, e_4, e_6, e_8.

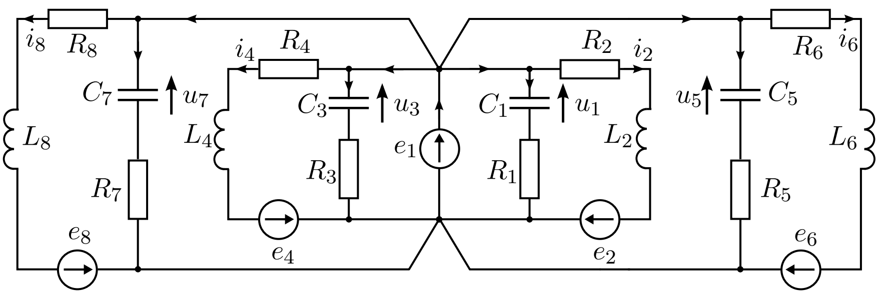

Fig. 4.7 Electrical circuit of Example 4.7

Using Kirchhoff's laws we can write the equations

$$\begin{aligned} e_1 &= u_k + R_k C_k \frac{\mathrm{d}u_k}{\mathrm{d}t} \quad &&\text{for} \quad k = 1, 3, 5, 7; \\ e_1 + e_j &= R_j i_j + L_j \frac{\mathrm{d}i_j}{\mathrm{d}t} \quad &&\text{for} \quad j = 2, 4, 6, 8; \end{aligned}$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be, \tag{4.22a}$$

where

$$u = \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \end{bmatrix}, \quad i = \begin{bmatrix} i_2 \\ i_4 \\ i_6 \\ i_8 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_4 \\ e_6 \\ e_8 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$A = \mathrm{diag}\left[-\frac{1}{R_1 C_1}, -\frac{1}{R_3 C_3}, -\frac{1}{R_5 C_5}, -\frac{1}{R_7 C_7}, -\frac{R_2}{L_2}, -\frac{R_4}{L_4}, -\frac{R_6}{L_6}, -\frac{R_8}{L_8}\right],$$

$$B_1 = \begin{bmatrix} \frac{1}{R_1 C_1} & 0 & 0 & 0 & 0 \\ \frac{1}{R_3 C_3} & 0 & 0 & 0 & 0 \\ \frac{1}{R_5 C_5} & 0 & 0 & 0 & 0 \\ \frac{1}{R_7 C_7} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & 0 & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & 0 & 0 \\ \frac{1}{L_6} & 0 & 0 & \frac{1}{L_6} & 0 \\ \frac{1}{L_8} & 0 & 0 & 0 & \frac{1}{L_8} \end{bmatrix}. \tag{4.22b}$$

From (4.22b) it follows that the matrix A is a Metzler matrix and the matrix B has nonnegative entries. Therefore, the R, L, C type electrical circuit is positive for any values of the resistances, inductances and capacitances.

In general case consider the electrical circuit shown in Figure 2.8 with given resistances $R_2, R_4, \ldots, R_{n_2}$; conductances $G_1, G_3, \ldots, G_{n_1}$; inductances $L_2, L_4, \ldots, L_{n_2}$; capacitances $C_1, C_3, \ldots, C_{n_1}$ and source voltages $e_1, e_2, \ldots, e_{n_2}$.

Using Kirchhoff's laws we can write the equations

$$e_1 = u_k + \frac{C_k}{G_k}\frac{\mathrm{d}u_k}{\mathrm{d}t},$$
$$e_j + e_0 = R_j i_j + L_j \frac{\mathrm{d}i_j}{\mathrm{d}t}$$

for $k = 1, 3, \ldots, n_1$; $j = 2, 4, \ldots, n_2$; which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be, \tag{4.23a}$$

where

$$u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n_1} \end{bmatrix}, \quad i = \begin{bmatrix} i_2 \\ i_4 \\ \vdots \\ i_{n_2} \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n_2} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$A = \mathrm{diag}\left[-\frac{G_1}{C_1}, -\frac{G_3}{C_3}, \dots, -\frac{G_{n_1}}{C_{n_1}}, -\frac{R_2}{L_2}, -\frac{R_4}{L_4}, \dots, -\frac{R_{n_2}}{L_{n_2}}\right],$$

$$B_1 = \begin{bmatrix} \frac{G_1}{C_1} & 0 & \cdots & 0 \\ \frac{G_3}{C_3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_{n_1}}{C_{n_1}} & 0 & \cdots & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & \cdots & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{n_2}} & 0 & 0 & \cdots & \frac{1}{L_{n_2}} \end{bmatrix}. \tag{4.23b}$$

From (4.23) it follows that the electrcal circuit is positive for all values of the resistances and conductances and nonzero inductances and capacitances.

Note that

$$\det A = (-1)^{n_1+n_2} \frac{R_2 R_4 \cdots R_{n_2} G_1 G_3 \cdots G_{n_1}}{L_2 L_4 \cdots L_{n_2} C_1 C_3 \cdots C_{n_1}}$$

and the positive electrical circuit is unstable if at least one of $G_1, G_3, \dots, G_{n_1}$ or one of $R_2, R_4, \dots, L_{n_2}$ is zero for any nonzero values of $C_1, C_3, \dots, C_{n_1}$ and $L_2, L_4, \dots, L_{n_2}$.

From the diagonal form of the matrix A it follows that the multiplicity of its zero eigenvalues is equal to the number of zero conductances and resistances.

If at least one of $R_2, R_4, \dots, L_{n_2}$ is zero then the positive electrical circuit is unstable and it has at least one mesh consisting of branches with only inductances and source voltages.

Therefore, we have the following theorem.

Theorem 4.5. *The positive electrical circuit R, L, C, e type is unstable if it has at least one mesh containing only the inductances and source voltages.*

A similar (dual) theorem can be formulated for positive electrical circuits having one node with branches consisting only of condensators and current sources (Theorem 4.4).

Chapter 5
Reachability, Observability and Reconstructability of Fractional Positive Electrical Circuits and Their Decoupling Zeros

5.1 Decomposition of the Pairs (A, B) and (A, C) of Linear Circuits

Consider the linear continuous-time electrical circuit described by the state equations (4.1).

Definition 5.1. [44, 120] The electrical circuit (4.1) (or the pair (A,B)) is called controllable in the time $0 < t_f < \infty$ if there exists an input $u(t) \in \mathbb{R}^m$, $t \in [0, t_f]$, which steers the state of the system from initial state $x_0 \in \mathbb{R}^n$ to any given final state $x_f \in \mathbb{R}^n$, i.e. $x(t_f) = x_f$.

Theorem 5.1. *[44, 120] The electrical circuit* (4.1) *is controllable in time* $t_f > 0$ *if and only if*

$$\text{rank}C_n = \text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n. \tag{5.1}$$

Definition 5.2. observability!electrical circuit [44] The electrical circuit (4.1) (or the pair (A,C)) is called observable in the time $t_f > 0$ if it is possible to find unique initial state $x_0 \in \mathbb{R}^n$ of the circuit knowing its input $u(t) \in \mathbb{R}^m$ and its output $y(t) \in \mathbb{R}^p$, $t \in [0, t_f]$.

Theorem 5.2. *[44] The electrical circuit* (4.1) *is observable in time* $t_f > 0$ *if and only if*

$$\text{rank}O_n = \text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{5.2}$$

Let the linear electrical circuit (or the pair (A,B)) be uncontrollable and

$$\text{rank}C_n = n_1 < n.$$

Then it is always possible to choose $n_2 = n - n_1$ linearly independent columns

T. Kaczorek and K. Rogowski, *Fractional Linear Systems and Electrical Circuits*,
Studies in Systems, Decision and Control 13, DOI: 10.1007/978-3-319-11361-6_5

$$P_{n_1+1}, \ldots, P_n,$$

which are orthogonal to the linearly independent columns $P_1, \ldots, P_{n_1}$ of the matrix C_n such that the matrix

$$P = \begin{bmatrix} P_1 \cdots P_{n_1} \; P_{n_1+1} \cdots P_n \end{bmatrix} \tag{5.3}$$

is nonsingular, i.e. $\det P \neq 0$.

Using the matrix (5.3) we can reduce the pair (A, B) to the form

$$\bar{A} = P^{-1}AP = \begin{bmatrix} \bar{A}_1 & \bar{A}_{12} \\ 0 & \bar{A}_2 \end{bmatrix}, \quad \bar{B} = P^{-1}B = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, \tag{5.4}$$

where the pair

$$\bar{A}_1 \in \mathbb{R}^{n_1 \times n_1}, \quad \bar{B}_1 \in \mathbb{R}^{n_1 \times m}$$

is controllable and the pair

$$\bar{A}_2 \in \mathbb{R}^{n_2 \times n_2}, \quad \bar{B}_2 = 0 \in \mathbb{R}^{n_2 \times m}$$

is uncontrollable.

It can be easily shown [44, 115] that the transfer matrix of the circuit (4.1) is equal to the transfer matrix of its controllable and observable part.

Let

$$\bar{x}(t) = P^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_1(t) \in \mathbb{R}^{n_1}, \quad x_2(t) \in \mathbb{R}^{n_2}$$

be the new state vector and

$$y(t) = Cx(t) + Du(t) = CPP^{-1}x(t) + Du(t) = \bar{C}_1 x_1(t) + \bar{C}_2 x_2(t) + Du(t),$$

where

$$CP = \begin{bmatrix} \bar{C}_1 \; \bar{C}_2 \end{bmatrix}, \quad \bar{C}_1 \in \mathbb{R}^{p \times n_1}, \quad \bar{C}_2 \in \mathbb{R}^{p \times n_2}.$$

Theorem 5.3. *The transfer matrix of the system* (4.1)

$$T(s) = C \left[\mathbb{I}_n s - A\right]^{-1} B + D \tag{5.5}$$

is equal to the transfer matrix of the controllable part

$$T(s) = \bar{C}_1 \left[\mathbb{I}_{n_1} s - \bar{A}_1\right]^{-1} \bar{B}_1 + D.$$

Proof. Substitution of (5.4) and (5.1) into (5.5) yields

$$
\begin{aligned}
T(s) =& C\left[\mathbb{I}_n s - A\right]^{-1} B + D = CPP^{-1}\left[\mathbb{I}_n s - A\right]^{-1} PP^{-1}B + D \\
=& \bar{C}\left[\mathbb{I}_n s - P^{-1}AP\right]^{-1} \bar{B} + D \\
=& \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} [\mathbb{I}_{n_1} s - \bar{A}_1]^{-1} & * \\ 0 & [\mathbb{I}_{n_2} s - \bar{A}_2]^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} + D \\
=& \bar{C}_1\left[\mathbb{I}_{n_1} s - \bar{A}_1\right]^{-1} \bar{B}_1 + D,
\end{aligned}
$$

where $*$ denotes an unimportant matrix.

Therefore, the transfer matrix (5.5) represents only controllable part of the system. □

Similar (dual) results we have for the pair (A, C).

Let the system (4.1) (or the pair (A, C)) be unobservable and

$$\operatorname{rank} O_n = \hat{n}_1 < n.$$

Then it is always possible to choose $\hat{n}_2 = n - \hat{n}_1$ linearly independent rows

$$Q_{\hat{n}_1+1}, \ldots, Q_n,$$

which are orthogonal to the linearly independent rows $Q_1, \ldots, Q_{\hat{n}_1}$ of the matrix O_n such that the matrix

$$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{\hat{n}_1} \\ Q_{\hat{n}_1+1} \\ \vdots \\ Q_n \end{bmatrix} \tag{5.6}$$

is nonsingular, i.e. $\det Q \neq 0$.

Using the matrix (5.6) we can reduce the pair (A, C) to the form

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix}, \quad \hat{C} = CQ^{-1} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}, \tag{5.7}$$

where the pair

$$\hat{A}_1 \in \mathbb{R}^{\hat{n}_1 \times \hat{n}_1}, \quad \hat{C}_1 \in \mathbb{R}^{p \times \hat{n}_1}$$

is observable and the pair

$$\hat{A}_2 \in \mathbb{R}^{\hat{n}_2 \times \hat{n}_2}, \quad \hat{C}_2 = 0 \in \mathbb{R}^{p \times \hat{n}_2}$$

is unobservable.

Let

$$\hat{x}(t) = Qx(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}, \quad \hat{x}_1(t) \in \mathbb{R}^{\hat{n}_1}, \quad \hat{x}_2(t) \in \mathbb{R}^{\hat{n}_2}$$

be the new state vector and

$$QB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{B}_1 \in \mathbb{R}^{\hat{n}_1 \times m}, \quad \hat{B}_2 \in \mathbb{R}^{\hat{n}_2 \times m}.$$

Theorem 5.4. *The transfer matrix* (5.5) *of the system* (4.1) *is equal to the transfer matrix*

$$T(s) = \hat{C}_1 \left[\mathbb{I}_{\hat{n}_1} s - \hat{A}_1 \right]^{-1} \hat{B}_1 + D.$$

of the observable part.

The proof of this theorem is similar (dual) to the proof of Theorem 5.3.

Let us consider the linear electrical circuits composed of resistors with resistances R, coils with inductances L, capacitors with capacitance C and voltage and current sources. The circuits can be described by the state equations (4.1), where the state variables (the components of the state vector $x(t)$) are the voltages across the capacitors and currents in coils and the inputs are the source voltages (currents). The outputs (the components of the output vector $y(t)$) can be chosen any voltage or current of the electrical circuits.

It is also well-known [44, 115, 172] that the controllability and observability of linear systems are generic properties of the system, i.e. randomly chosen pair (A, B) ((A, C)) is controllable (observable). Therefore, the class of controllable and observable linear electrical circuits is very rich, i.e. any randomly chosen electrical circuit is controllable and observable.

We shall show that the class of uncontrollable and unobservable linear electrical circuits is enough rich. If the pair (A, B) ((A, C)) of the linear electrical circuit is uncontrollable (unobservable), then using the approach presented above we may decompose the pair into controllable and uncontrollable (observable and unobservable) parts.

First the essence of the approach will be shown on the following examples.

Example 5.1. Consider the electrical circuit shown in Figure 5.1 with given resistances R_k, $k = 1, 2, \ldots, 5$; inductance L; capacitances C_1, C_2 and source voltage $e(t) = e$.

As the state variables we choose the voltages u_1, u_2 on the capacitors and the current in the coil. The source voltage $e(t)$ is the input and as the output we choose the voltage u_2, i.e. $y(t) = u_2(t)$.

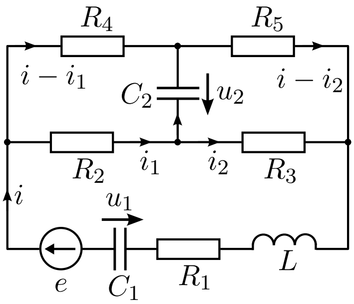

Fig. 5.1 Fractional electrical circuit of Example 5.1

Using Kirchhoff's laws we may write the equations

$$e = R_1 i + L\frac{\mathrm{d}i}{\mathrm{d}t} + u_1 + R_2 i_1 + R_3 i_2, \tag{5.8a}$$

$$i = C_1 \frac{\mathrm{d}u_1}{\mathrm{d}t}, \tag{5.8b}$$

$$i_1 = C_2 \frac{\mathrm{d}u_2}{\mathrm{d}t} + i_2, \tag{5.8c}$$

$$0 = R_2 i_1 + u_2 - R_4(i - i_1), \tag{5.8d}$$

$$0 = R_3 i_2 - u_2 - R_5(i - i_2). \tag{5.8e}$$

From equations (5.8d) and (5.8e) we have

$$i_1 = \frac{R_4 i - u_2}{R_2 + R_4}, \tag{5.9a}$$

$$i_2 = \frac{R_5 i + u_2}{R_3 + R_5}. \tag{5.9b}$$

Substituting (5.9) into (5.8a) and (5.8c) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} + Be,$$

where

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$$

and

$$a_{13} = \frac{1}{C_1}, \quad a_{22} = -\frac{R_2+R_3+R_4+R_5}{C_2(R_2+R_4)(R_3+R_5)}, \quad a_{23} = \frac{R_3R_4-R_2R_5}{C_2(R_2+R_4)(R_3+R_5)},$$
$$a_{31} = -\frac{1}{L}, \quad a_{32} = \frac{R_2R_5-R_3R_4}{L(R_2+R_4)(R_3+R_5)}, \quad b = \frac{1}{L},$$
$$a_{33} = -\frac{R_1(R_2+R_4)(R_3+R_5)+R_2R_4(R_3+R_5)+R_3R_5(R_2+R_4)}{L(R_2+R_4)(R_3+R_5)}.$$

As the output $y(t)$ we choose u_2 and we obtain

$$y(t) = Cx(t) + Du(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad D = [0].$$

We shall show that if the condition

$$R_3R_4 = R_2R_5 \tag{5.11}$$

is satisfied, then the electrical circuit is uncontrollable and unobservable.

Note that if (5.11) holds, then $a_{23} = a_{32} = 0$ and using (5.1) we obtain

$$\text{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & a_{13}b & a_{13}a_{33}b \\ 0 & 0 & 0 \\ b & a_{33}b & (a_{13}a_{31}+a_{33}^2)b \end{bmatrix} = 2 < n = 3.$$

Similarly, using (5.2) for $a_{23} = 0$ we obtain

$$\text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 1 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{22}^2 & 0 \end{bmatrix} = 1 < n = 3. \tag{5.12}$$

Therefore, the electrical circuit is uncontrollable and unobservable if the condition (5.11) is met.

The pair

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$$

is uncontrollable and it can be decomposed into the controllable and uncontrollable parts using the approach presented in above considerations.

In this case $n_1 = 2$, $P_1 = B$, $P_2 = AB$ and we choose

$$P_{n_1+1} = P_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The matrix (5.3) has the form

$$P = \begin{bmatrix} B & AB & P_3 \end{bmatrix} = \begin{bmatrix} 0 & a_{13}b & 0 \\ 0 & 0 & 1 \\ b & a_{33}b & 0 \end{bmatrix} \tag{5.13}$$

and

$$P^{-1} = \begin{bmatrix} -\frac{a_{33}}{a_{13}b} & 0 & \frac{1}{b} \\ \frac{1}{a_{13}b} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.14}$$

Using (5.4) and (5.13), (5.14) we obtain

$$\begin{aligned} \bar{A} =& P^{-1}AP = \begin{bmatrix} -\frac{a_{33}}{a_{13}b} & 0 & \frac{1}{b} \\ \frac{1}{a_{13}b} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \begin{bmatrix} 0 & a_{13}b & 0 \\ 0 & 0 & 1 \\ b & a_{33}b & 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_1 & \bar{A}_{12} \\ 0 & \bar{A}_2 \end{bmatrix}, \\ \bar{B} =& P^{-1}B = \begin{bmatrix} -\frac{a_{33}}{a_{13}b} & 0 & \frac{1}{b} \\ \frac{1}{a_{13}b} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, \end{aligned}$$

where

$$\bar{A}_1 = \begin{bmatrix} 0 & a_{13}a_{31} \\ 1 & a_{33} \end{bmatrix}, \quad \bar{A}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{A}_2 = [a_{22}], \quad \bar{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The pair $(\bar{A}_1, \bar{B}_1)$ is controllable, since

$$\begin{bmatrix} \bar{B}_1 & \bar{A}_1\bar{B}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From (5.12) it follows that the pair

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{5.16}$$

is unobservable and it can be decomposed into the observable and unobservable parts. In this case $\hat{n}_1 = 1$ and we choose $Q_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $Q_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. The matrix (5.6) has the form

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.17}$$

Using (5.7), (5.16) and (5.17) we obtain

$$\begin{aligned} \hat{A} &= QAQ^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix}, \\ \hat{C} &= CQ^{-1} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}, \end{aligned}$$

where

$$\hat{A}_1 = [a_{22}], \quad \hat{A}_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0 & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, \quad \hat{C}_1 = [1].$$

The pair $(\hat{A}_1, \hat{C}_1)$ is observable, since $\hat{C}_1 = [1]$.

If the condition (5.11) is met, then the electrical circuit is uncontrollable and unobservable. Therefore, its transfer matrix is zero, i.e.

$$T(s) = C\left[\mathbb{I}_n s - A\right]^{-1} B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & 0 & -a_{13} \\ 0 & s - a_{22} & 0 \\ -a_{31} & 0 & s - a_{33} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} = 0.$$

Remark 5.1. It can be easily shown that if we choose the state vector of the electrical circuit of the form $x^T(t) = \begin{bmatrix} i & u_1 & u_2 \end{bmatrix}$, then if the condition (5.11) is met, the matrix (5.3) is equal to the identity matrix, i.e. $P = \mathbb{I}_3$.

Example 5.2. Consider the electrical circuit shown in Figure 5.2 with given resistances R_k, $k = 1, 2, \ldots, 6$; inductances L_1, L_2, L_3 and source voltages $e_1(t) = e_1$, $e_2(t) = e_2$.

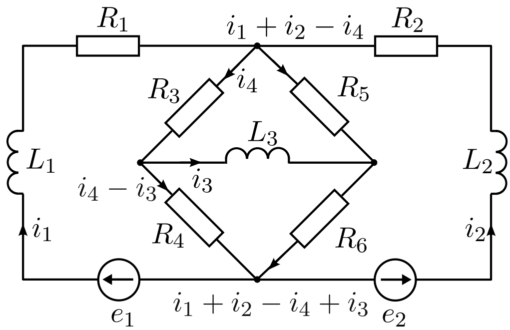

Fig. 5.2 Fractional electrical circuit of Example 5.2

As the state variables we choose the currents i_1, i_2, i_3 in the coils. The source voltages $e_1(t)$, $e_2(t)$ are the inputs and as the output we choose the current $y(t) = i_3(t)$.

Using Kirchhoff's laws we may write the equations

$$e_1 = R_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + R_3 i_4 + R_4(i_4 - i_3), \tag{5.19a}$$

$$e_2 = R_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + R_3 i_4 + R_4(i_4 - i_3), \tag{5.19b}$$

$$L_3 \frac{\mathrm{d}i_3}{\mathrm{d}t} = R_5(i_1 + i_2 - i_4) - R_3 i_4, \tag{5.19c}$$

$$R_3 i_4 + R_4(i_4 - i_3) = R_5(i_1 + i_2 - i_4) + R_6(i_1 + i_2 + i_3 - i_4). \tag{5.19d}$$

From (5.19d) we have

$$i_4 = \frac{(R_5 + R_6)(i_1 + i_2) + (R_4 + R_6)i_3}{R_3 + R_4 + R_5 + R_6}. \tag{5.20}$$

Substituting (5.20) into (5.19a), (5.19b) and (5.19c) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \end{bmatrix}$$

and

$$a_{11} = -\frac{(R_3 + R_4)(R_5 + R_6) + R_1(R_3 + R_4 + R_5 + R_6)}{L_1(R_3 + R_4 + R_5 + R_6)},$$

$$a_{12} = -\frac{(R_3 + R_4)(R_5 + R_6)}{L_1(R_3 + R_4 + R_5 + R_6)}, \quad a_{13} = \frac{R_4 R_5 - R_3 R_6}{L_1(R_3 + R_4 + R_5 + R_6)},$$

$$a_{21} = -\frac{(R_3 + R_4)(R_5 + R_6)}{L_2(R_3 + R_4 + R_5 + R_6)}, \quad a_{23} = \frac{R_4 R_5 - R_3 R_6}{L_2(R_3 + R_4 + R_5 + R_6)},$$

$$a_{22} = -\frac{(R_3 + R_4)(R_5 + R_6) + R_2(R_3 + R_4 + R_5 + R_6)}{L_2(R_3 + R_4 + R_5 + R_6)},$$

$$a_{31} = a_{32} = \frac{R_4 R_5 - R_3 R_6}{L_3(R_3 + R_4 + R_5 + R_6)}, \quad a_{33} = -\frac{(R_3 + R_5)(R_4 + R_6)}{L_3(R_3 + R_4 + R_5 + R_6)},$$

$$b_1 = \frac{1}{L_1}, \quad b_2 = \frac{1}{L_2}.$$

Taking into account that

$$y(t) = Cx(t) + Du(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

we obtain

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = [0].$$

We shall show, that if the condition

$$R_4 R_5 = R_3 R_6 \tag{5.22}$$

is satisfied, then the electrical circuit shown in Figure 5.2 is uncontrollable and unobservable.

Note that if (5.22) holds then $a_{13} = a_{32} = a_{31} = a_{23} = 0$ and using (5.1) we obtain

$$\begin{aligned}
&\operatorname{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} \\
&= \operatorname{rank} \begin{bmatrix} b_1 & 0 & a_{11}b_1 & a_{12}b_2 & (a_{11}^2 + a_{12}a_{21})b_1 & (a_{11}a_{12} + a_{12}a_{22})b_2 \\ 0 & b_2 & a_{21}b_1 & a_{22}b_2 & (a_{21}a_{11} + a_{22}a_{21})b_1 & (a_{21}a_{12} + a_{22}^2)b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= 2 < n = 3.
\end{aligned}$$

Similarly, using (5.2) we obtain

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & a_{33} \\ 0 & 0 & a_{33}^2 \end{bmatrix} = 1 < n = 3.$$

Therefore, the electrical circuit is uncontrollable and unobservable if the condition (5.22) is met.

Note that for $a_{31} = a_{32}$ we have $\bar{A} = A$ and the matrix $P = \mathbb{I}_3$.

In a similar way as in Example 5.1 the matrix Q has the form (5.17) and

$$\begin{aligned}
\hat{A} =& QAQ^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix}, \\
\hat{C} =& CQ^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix},
\end{aligned}$$

where

$$\hat{A}_1 = [a_{33}], \quad \hat{A}_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix}, \quad \hat{C}_1 = [1].$$

The pair $(\hat{A}_1, \hat{C}_1)$ is observable, since $\hat{C}_1 = [1]$.

If the condition (5.22) is met, then the electrical circuit is uncontrollable and unobservable. Therefore, its transfer matrix is zero, i.e.

$$\begin{aligned} T(s) &= C\left[\mathbb{I}_3 s - A\right]^{-1} B \\ &= \begin{bmatrix}0 \; 0 \; 1\end{bmatrix} \begin{bmatrix} s-a_{11} & -a_{12} & 0 \\ -a_{21} & s-a_{22} & 0 \\ 0 & 0 & s-a_{33} \end{bmatrix}^{-1} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix}0 \; 0\end{bmatrix}. \end{aligned}$$

From the above examples we have the following important corollaries.

Corollary 5.1. The linear electrical circuits are uncontrollable and unobservable if and only if their parameters (resistances) satisfy some conditions. For example (5.11) or (5.22).

Corollary 5.2. The uncontrollable pair (A, B) (unobservable pair (A, C)) of the electrical circuit can be decomposed into controllable (observable) and uncontrollable (unobservable) parts by the use of suitable similarity transformations.

Corollary 5.3. The transfer matrices of the uncontrollable and unobservable electrical circuits are zero.

5.2 Reachability of Positive Electrical Circuits

Theorem 5.5. *[1, 44] The solution to the equation* (4.1a) *with initial condition* $x(0) = x_0$ *is given by*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau. \tag{5.24}$$

Definition 5.3. reachability!electrical circuit!positive The positive electrical circuit (4.1) (or the positive pair (A, B)) is called reachable in time t_f if for any given final state $x_f \in \mathbb{R}_+^n$ there exists an input $u(t) \in \mathbb{R}_+^m$, for $t \in [0, t_f]$ that steers the state of the circuit from zero initial state $x(0) = 0$ to the state $x_f \in \mathbb{R}_+^n$, i.e. $x(t_f) = x_f$.

Definition 5.4. A column $a \in \mathbb{R}_+^n$ (row $a^T \in \mathbb{R}_+^{1\times n}$) is called monomial if only one its entry is positive and the remaining entries are zero.

Definition 5.5. A real matrix $A \in \mathbb{R}_+^{n\times n}$ is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 5.6. *The positive electrical circuit* (4.1) *is reachable if the matrix*

$$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} \mathrm{d}\tau, \quad t_f > 0 \tag{5.25}$$

is monomial.

The input

$$u(t) = B^T e^{A^T (t_f - t)} R_f^{-1} x_f \in \mathbb{R}_+^{n \times n} \tag{5.26}$$

for $t \in [0, t_f]$ *steers the state* $x(t)$ *of the circuit from* $x(0) = x_0 = 0$ *to the state* $x(t_f) = x_f$.

Proof. If the matrix R_f given by (5.25) is monomial, then there exists the inverse matrix $R_f^{-1} \in \mathbb{R}_+^{n \times n}$ and the input (5.26) is well defined and nonnegative.

Now we shall show, that the input (5.26) steers the state of the circuit from $x_0 = 0$ to the given final state $x_f \in \mathbb{R}_+^n$ for $t = [0, t_f]$.

Substitution of (5.26) into (5.24) with zero initial condition $x_0 = 0$ yields

$$\begin{aligned} x(t_f) &= \int_0^{t_f} e^{A(t_f - \tau)} B u(\tau) \mathrm{d}\tau = \int_0^{t_f} e^{A(t_f - \tau)} B B^T e^{A^T (t_f - \tau)} \mathrm{d}\tau R_f^{-1} x_f \\ &= \int_0^{t_f} e^{A(\tau)} B B^T e^{A^T (\tau)} \mathrm{d}\tau R_f^{-1} x_f = x_f. \end{aligned}$$

Therefore, the input (5.26) steers the state of the circuit from $x_0 = 0$ to the given final state $x_f \in \mathbb{R}_+^n$ for $t = [0, t_f]$. □

Theorem 5.7. *The positive electrical circuit* (4.1) *is reachable in time* $t \in [0, t_f]$ *if and only if the matrix* $A \in M_n$ *is diagonal and the matrix* $B \in \mathbb{R}_+^{n \times m}$ *is monomial.*

Proof. Sufficiency. It is well known [47] that if $A \in M_n$ is diagonal, then $e^{At} \in \mathbb{R}_+^{n \times n}$ is also diagonal and if $B \in \mathbb{R}_+^{n \times m}$ is monomial, then $BB^T \in \mathbb{R}_+^{n \times n}$ is also monomial.

In this case the matrix

$$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} \mathrm{d}\tau \in \mathbb{R}_+^{n \times n} \tag{5.27}$$

is also monomial and $R_f^{-1} \in \mathbb{R}_+^{n \times n}$.

Necessity. From Cayley-Hamilton theorem [44, 71] we have

$$e^{At} = \sum_{k=0}^{n-1} c_k(t) A^k, \tag{5.28}$$

where $c_k(t)$, $k = 0, 1, \ldots, n-1$ are some nonzero functions of time depending on matrix A.

Substitution of (5.28) into

$$\int_0^{t_f} e^{A(t_f-\tau)} B u(\tau) \mathrm{d}\tau$$

yields

$$x_f = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \begin{bmatrix} v_0(t_f) \\ v_1(t_f) \\ \vdots \\ v_{n-1}(t_f) \end{bmatrix},$$

where

$$v_k(t_f) = \int_0^{t_f} c_k(\tau) u(t_f - \tau) \mathrm{d}\tau, \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$

For given $x_f \in \mathbb{R}_+^n$ it is possible to find nonnegative $v_k(t_f)$ for $k = 0, 1, \ldots, n-1$ if and only if the matrix

$$\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$$

has n linearly independent monomial columns and this takes place only if the matrix $\begin{bmatrix} B, A \end{bmatrix}$ contains n linearly independent columns [47].

Note that for the nonnegative $v_k(t_f)$, $k = 0, 1, \ldots, n-1$ it is possible to find a nonnegative input $u(t) \in \mathbb{R}_+^m$, $t \in [0, t_f]$ only if the matrix $B \in \mathbb{R}_+^{n \times m}$ is monomial and the matrix $A \in M_n$ is diagonal. □

Now let us consider n-meshes electrical circuits with given resistances R_k, $k = 1, \ldots, q$; inductances L_i, $i = 1, \ldots, n$ and m-mesh source voltages e_j, $j = 1, \ldots, m$. It is assumed that to each linearly independent mesh belongs only one inductance and one source voltage. In this case the matrix $A \in M_n$ and the matrix $B \in \mathbb{R}_+^{n \times n}$ is diagonal and the standard electrical circuit is reachable, since $\det B \neq 0$.

Theorem 5.8. *The positive n-meshes electrical circuit with only one inductance in each linearly independent mesh is reachable if*

$$R_{ij} = 0 \quad \text{for} \quad i \neq j, \quad i, j = 1 \ldots, n; \tag{5.29}$$

where R_{ij} is the resistance of the branch belonging the i-th and j-th meshes.

Proof. Note that the matrix $A \in M_n$ is also diagonal if and only if the condition (5.29) is met. By Theorem 5.7 the positive electrical circuit is reachable if and only if the condition (5.29) is satisfied, since in this case $A \in M_n$, $B \in \mathbb{R}_+^{n\times n}$ are both diagonal. □

Example 5.3. Consider the electrical circuit shown in Figure 5.3 with given resistances R_1, R_2, R_3; inductances L_1, L_2 and source voltages e_1, e_2.

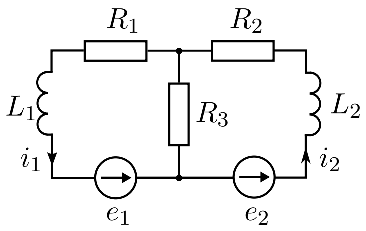

Fig. 5.3 Electrical circuit of Example 5.3

Using Kirchhoff's laws we can write the equations

$$e_1 = R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t},$$
$$e_2 = R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t},$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.30}$$

where

$$A = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{5.31}$$

The electrical circuit is positive, since the matrix A is Metzler matrix and the matrix B has nonnegative entries. Note that the standard pair (5.31) is reachable, since $\det B \neq 0$.

We shall show that the positive electrical circuit is reachable if $R_3 = 0$. In this case

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix} \tag{5.32}$$

and

$$e^{At} = \begin{bmatrix} e^{-\frac{R_1}{L_1}t} & 0 \\ 0 & e^{-\frac{R_2}{L_2}t} \end{bmatrix} \tag{5.33}$$

and from (5.27) we obtain

$$\begin{aligned} R_f &= \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} \mathrm{d}\tau = \int_0^{t_f} \begin{bmatrix} \frac{1}{L_1^2} e^{-\frac{2R_1}{L_1}\tau} & 0 \\ 0 & \frac{1}{L_2^2} e^{-\frac{2R_2}{L_2}\tau} \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} \frac{1}{2L_1R_1}\left(1 - e^{-\frac{2R_1}{L_1}t_f}\right) & 0 \\ 0 & \frac{1}{2L_2R_2}\left(1 - e^{-\frac{2R_2}{L_2}t_f}\right) \end{bmatrix}. \end{aligned} \tag{5.34}$$

The matrix (5.34) is monomial and by Theorem 5.6 the positive electrical circuit is reachable if $R_3 = 0$ and any given R_1, R_2, L_1, L_2.

Example 5.4. Consider the electrical circuit shown in Figure 5.4 with given resistances R, R_1, R_2; inductance L; capacitance C and source volateges e_1, e_2.

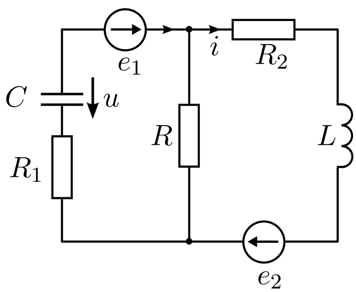

Fig. 5.4 Electrical circuit of Example 5.4

Using Kirchhoff's laws we can write the equations

$$e_1 = R_1 C \frac{\mathrm{d}u}{\mathrm{d}t} + u + R\left(C\frac{\mathrm{d}u}{\mathrm{d}t} - i\right),$$
$$e_2 = R_2 i + L\frac{\mathrm{d}i}{\mathrm{d}t} + R\left(i - C\frac{\mathrm{d}u}{\mathrm{d}t}\right),$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ i \end{bmatrix} = A\begin{bmatrix} u \\ i \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.36a}$$

where

$$A = \begin{bmatrix} -\frac{1}{(R+R_1)C} & \frac{R}{(R+R_1)C} \\ -\frac{R}{(R+R_1)L} & -\frac{R_2(R+R_1)+RR_1}{(R+R_1)L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{(R+R_1)C} & 0 \\ \frac{R}{(R+R_1)L} & \frac{1}{L} \end{bmatrix}. \tag{5.36b}$$

The electrical circuit is not positive, since the matrix A is not a Metzler matrix. The electrical circuit is positive if and only if $R = 0$. In this case from (5.36b) we have

$$A = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_1 C} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}. \tag{5.37}$$

For the matrix A given by (5.37) we obtain

$$e^{At} = \begin{bmatrix} e^{-\frac{1}{R_1 C}t} & 0 \\ 0 & e^{-\frac{R_2}{L}} \end{bmatrix}. \tag{5.38}$$

The positive electrical circuit for $R = 0$ is reachable, since the matrix

$$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} \mathrm{d}\tau = \int_0^{t_f} \begin{bmatrix} \frac{1}{(R_1 C)^2} e^{-\frac{2}{R_1 C}\tau} & 0 \\ 0 & \frac{1}{L^2} e^{-\frac{2R_2}{L}\tau} \end{bmatrix} \mathrm{d}\tau$$
$$= \begin{bmatrix} \frac{1}{2R_1 C}\left(1 - e^{-\frac{2}{R_1 C}t_f}\right) & 0 \\ 0 & \frac{1}{2R_2 L}\left(1 - e^{-\frac{2R_2}{L}t_f}\right) \end{bmatrix}$$

is monomial with positive diagonal entries for all values of R_1, R_2, L, C.

Following [68, 74], consider the electrical circuit shown in Figure 5.5 with given conductances G_k, G'_k, G_{kj}, $k, j = 1 \dots, n$; capacitances C_k, $k = 1, \dots, n$ and source voltages e_k, $k = 1, \dots, n$.

Theorem 5.9. *The electrical circuit shown in Figure 5.5 is positive for all value of the conductances, capacitances and source voltages.*

Proof. Using Kirchhoff's laws and the node method for the electrical circuit we may write the equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = -C^{-1}G' \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + C^{-1}G' \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \tag{5.39a}$$

and

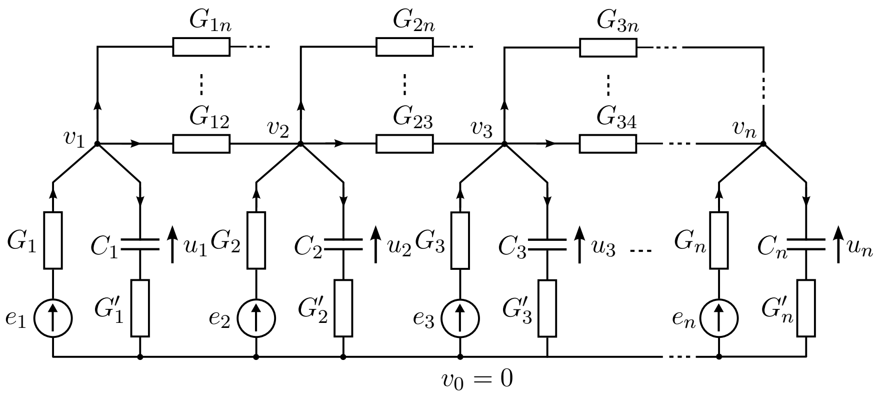

Fig. 5.5 Electrical circuit

$$\bar{G}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -G'\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} - G\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}, \tag{5.39b}$$

where

$$C^{-1} = \operatorname{diag}\left[C_1^{-1}, \dots, C_n^{-1}\right], \quad G' = \operatorname{diag}\left[G'_1, \dots, G'_n\right],$$
$$G = \operatorname{diag}\left[G_1, \dots, G_n\right], \quad \bar{G} = \begin{bmatrix} -G_{11} & G_{12} & \cdots & G_{1n} \\ G_{12} & -G_{22} & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{1n} & G_{2n} & \vdots & -G_{nn} \end{bmatrix} \tag{5.39c}$$

and G_{ii} is the sum of conductances of all branches belonging to the ith node for $i = 1, \dots, n$.

The matrix $\bar{G} \in M_n$ and $-\bar{G} \in \mathbb{R}_+^{n\times n}$. Substituting (5.39b) into (5.39a) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + B\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix},$$

where

$$A = -C^{-1}G'\left[\mathbb{I}_n + \bar{G}^{-1}G'\right] \in M_n$$

and

$$B = -C^{-1}G'\bar{G}^{-1}G \in \mathbb{R}_+^{n\times n},$$

since the matrices C^{-1}, G', G and $-\bar{G}^{-1}$ have nonnegative entries. Therefore, the electrical circuit is positive. □

Note that the standard electrical circuit shown in Figure 5.5 is reachable for all nonzero values of the conductances and capacitances, since $\det B \neq 0$.

Theorem 5.10. *The electrical circuit shown in Figure 5.5 is reachable if and only if*

$$G_{k,j} = 0 \quad \text{for} \quad k \neq j \quad \text{and} \quad k, j = 1, \ldots, n. \tag{5.41}$$

Proof. It is easy to see that the matrices $A \in M_n$ and $B \in \mathbb{R}_+^{n\times n}$ are both diagonal matrices if and only if the condition (5.41) is satisfied. In this case by Theorem 5.7 the electrical circuit is reachable if and only if the conditions (5.41) are met. □

5.3 Observability of Positive Electrical Circuits

Consider a positive electrical circuit described by the state equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t), \tag{5.42a}$$

$$y(t) = Cx(t), \tag{5.42b}$$

where $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ and $A \in M_n$, $C \in \mathbb{R}_+^{p\times n}$.

Definition 5.6. The positive electrical circuit (5.42) is called observable if knowing the output $y(t) \in \mathbb{R}_+^p$ and its derivatives $y^{(k)}(t) = \dfrac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} \in \mathbb{R}_+^p$, $k = 1, 2 \ldots, n-1$, it is possible to find the unique initial value $x_0 = x(0) \in \mathbb{R}_+^n$ of $x(t) \in \mathbb{R}_+^n$.

Theorem 5.11. *The positive electrical circuit* (5.42) *is observable if and only if the matrix $A \in M_n$ is diagonal and the matrix*

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5.43}$$

has n linearly independent monomial rows.

Proof. Substituting of the solution

$$x(t) = e^{At}x_0$$

of the equation (5.42a) into (5.42b) yields

$$y(t) = Ce^{At}x_0. \tag{5.44}$$

From (5.44) we have

$$\begin{bmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} e^{At}x_0. \tag{5.45}$$

It is possible to find from (5.45) $e^{At}x_0 \in \mathbb{R}_+^n$ if and only if the matrix (5.43) has n linearly independent monomial rows. From the equality $e^{At}e^{-At} = \mathbb{I}_n$ it follows that the matrix $e^{At} \in \mathbb{R}_+^n$ for $A \in M_n$ if and only if it is diagonal. Therefore, it is possible to find $x_0 \in \mathbb{R}_+^n$ from the equation (5.45) if and only if the matrix $A \in M_n$ is diagonal and the matrix (5.43) has n linearly independent monomial rows. □

Theorem 5.12. *The positive electrical circuit* (5.42) *is observable if the matrix*

$$O_p = e^{A^T t}C^T C e^{At} \tag{5.46}$$

is monomial.

Proof. Premultiplying (5.44) by $e^{A^T t}C^T$ we obtain

$$e^{A^T t}C^T C e^{At}x_0 = e^{A^T t}C^T y(t). \tag{5.47}$$

If the matrix (5.46) is monomial, then $O_p^{-1} = \left[e^{A^T t}C^T C e^{At}\right]^{-1} \in \mathbb{R}_+^{n\times n}$ and from (5.47) we have

$$x_0 = \left[e^{A^T t}C^T C e^{At}\right]^{-1} e^{A^T t}C^T y(t) \in \mathbb{R}_+^n,$$

since $e^{A^T t}C^T y(t) \in \mathbb{R}_+^n$ for $y(t) \in \mathbb{R}_+^p$. □

Definition 5.7. The positive system (5.42) is called strongly observable on the interval $[0, t_f]$ if knowing $y(t)$ on the interval $[0, t_f]$ it is possible to find (compute) uniquely $x_0 = x(0)$.

Theorem 5.13. *The positive system (electrical circuit)* (5.42) *is strongly observable on the interval* $[0, t_f]$ *if and only if the matrix*

$$W_f = \int_0^{t_f} e^{A^T t}C^T C e^{At}\mathrm{d}t \tag{5.48}$$

is monomial.

Proof. Integrating (5.47) on the interval $[0, t_f]$ we obtain

$$\int_0^{t_f} e^{A^T t}C^T y(t)\mathrm{d}t = \int_0^{t_f} e^{A^T t}C^T C e^{At}\mathrm{d}t x_0. \tag{5.49}$$

Using (5.49) we may find the unique initial value $x_0 = x(0)$ form

$$x_0 = W_f^{-1} \int_0^{t_f} e^{A^T t} C^T y(t) \mathrm{d}t$$

if and only if $W_f^{-1} \in \mathbb{R}_+^{n \times n}$ and this is met if and only if the matrix W_f is monomial. □

Example 5.5. Consider the electrical circuit shown in Figure 5.6 with given resistances R_1, R_2; inductance L; capacitance C and source voltage $e = e(t)$.

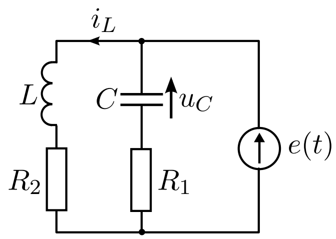

Fig. 5.6 Electrical circuit of Example 5.5

Using Kirhhoff's laws we may write the equations

$$e = u_C + R_1 C \frac{\mathrm{d}u_C}{\mathrm{d}t},$$
$$e = R_2 i_L + L \frac{\mathrm{d}i_L}{\mathrm{d}t},$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_C \\ i_L \end{bmatrix} = A \begin{bmatrix} u_C \\ i_L \end{bmatrix} + Be, \tag{5.50a}$$

where

$$A = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix}. \tag{5.50b}$$

Let

$$y(t) = \begin{bmatrix} u_C \\ i_L \end{bmatrix} = C \begin{bmatrix} u_C \\ i_L \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.51}$$

From (5.50b), (5.51) and Theorem 4.1 if follows that the electrical circuit shown in Figure 5.6 is positive.

We shall show that the positive electrical circuit is strongly observable on the interval $[0, t_f]$.

Using (5.50b), (5.51) and

$$e^{At} = \begin{bmatrix} e^{-\frac{1}{R_1 C}t} & 0 \\ 0 & e^{-\frac{R_2}{L}t} \end{bmatrix},$$

from (5.48) we obtain

$$\begin{aligned} W_f &= \int_0^{t_f} e^{A^T t} C^T C e^{At} \mathrm{d}t = \int_0^{t_f} \begin{bmatrix} e^{-\frac{2}{R_1 C}t} & 0 \\ 0 & e^{-\frac{2R_2}{L}t} \end{bmatrix} \mathrm{d}t \\ &= \begin{bmatrix} -\frac{R_1 C}{2} e^{-\frac{2}{R_1 C}t} & 0 \\ 0 & -\frac{L}{2R_2} e^{-\frac{2R_2}{L}t} \end{bmatrix}_0^{t_f} \\ &= \begin{bmatrix} \frac{R_1 C}{2}\left(1 - e^{-\frac{2}{R_1 C}t_f}\right) & 0 \\ 0 & -\frac{L}{2R_2}\left(1 - e^{-\frac{2R_2}{L}t_f}\right) \end{bmatrix} \end{aligned} \tag{5.52}$$

The matrix (5.52) is monomial and by Theorem 5.13 the positive electrical circuit is strongly observable on the interval $[0, t_f]$ for arbitrary given $t_f > 0$.

5.4 Constructability of Positive Electrical Circuits

Definition 5.8. The positive electrical circutit (5.42) is called constructible if knowing the output $y(t) \in \mathbb{R}_+^p$ and its derivatives $y^{(k)}(t) = \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} \in \mathbb{R}_+^p$, $k = 1, 2 \ldots, n-1$, it is possible to find the state vector $x(t) \in \mathbb{R}_+^n$.

Theorem 5.14. *The positive electrical circutit* (5.42) *is constructible if and only if the matrix* (5.43) *has n linearly independent monomial rows.*

Proof. From (5.42) we have

$$\begin{bmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t). \tag{5.53}$$

It is possible to find from (5.53) the state vector $x(t) \in \mathbb{R}_+^n$ for given $y^{(k)}(t) \in \mathbb{R}_+^p$, $k = 0, 1, 2, \ldots, n-1$ if and only if the matrix (5.43) has n linearly independent monomial rows, since the inverse matrix has nonnegative entries if and only if the matrix is monomial [47]. □

Theorem 5.15. *The positive electrical circuit* (5.42) *is constructible only if the matrix* $\begin{bmatrix} C \\ A \end{bmatrix}$ *has n linearly independent monomial rows.*

Proof. It is easy to show that the matrix (5.43) has n linearly independent rows only if the matrix $\begin{bmatrix} C \\ A \end{bmatrix}$ has n linearly independent monomial rows. □

Remark 5.2. From comparison of Theorem 5.14 and 5.11 if follows that the necessary and suffiecient conditions for the observability are more restrictive than for the constructability.

Theorem 5.16. *If the positive electrical circuit* (5.42) *is observable then it is also constructible.*

Proof follows immediately from comparison of the conditions of Theorems 5.14 and 5.11.

Example 5.6. (continuation of Example 5.3).
Let

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} R_1 i_1(t) \\ R_2 i_2(t) \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}, \quad C = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}. \tag{5.54}$$

The positive circuit described by (5.30) and (5.54) is constructible for all nonzero values of R_1 and R_2, since

$$\det C = \det \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = R_1 R_2 \neq 0.$$

We shall show that the positive circuit is also observable. Using (5.46), (5.32) and (5.54) we obtain

$$\begin{aligned} O_p = e^{A^T t} C^T C e^{At} &= \begin{bmatrix} e^{-\frac{R_1}{L_1}t} & 0 \\ 0 & e^{-\frac{R_2}{L_2}t} \end{bmatrix} \begin{bmatrix} R_1^2 & 0 \\ 0 & R_2^2 \end{bmatrix} \begin{bmatrix} e^{-\frac{R_1}{L_1}t} & 0 \\ 0 & e^{-\frac{R_2}{L_2}t} \end{bmatrix} \\ &= \begin{bmatrix} R_1^2 e^{-\frac{2R_1}{L_1}t} & 0 \\ 0 & R_2^2 e^{-\frac{2R_2}{L_2}t} \end{bmatrix}. \end{aligned} \tag{5.55}$$

Therefore, by Theorem 5.12 the positive circuit is also observable, since the matrix (5.55) is monomial.

Example 5.7. Consider the electrical circuit shown in Figure 5.7 with given conductances G_1, G_1', G_2, G_2', G_{12}; capacitances C_1, C_2 and source voltages e_1, e_2.

Using Kirchhoff's laws we can write the equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.56}$$

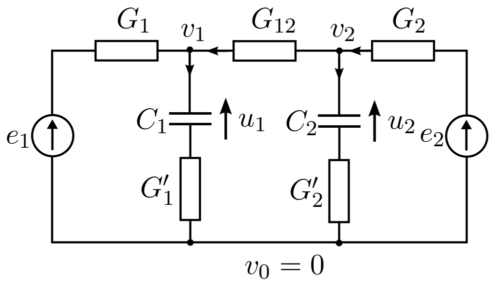

Fig. 5.7 Electrical circuit of Example 5.7

and

$$\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.57a}$$

where

$$G_{11} = G_1 + G_1' + G_{12}, \quad G_{22} = G_2 + G_2' + G_{12}. \tag{5.57b}$$

Taking into account that the matrix

$$\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}$$

is nonsingular and

$$-\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \in \mathbb{R}_+^{2\times 2},$$

from (5.57) we obtain

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right\}. \tag{5.58}$$

Substitution of (5.58) into (5.56) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.59}$$

where

$$A = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} - \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \in M_2, \tag{5.60a}$$

$$B = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}. \tag{5.60b}$$

From (5.60) it follows that A is Metzler matrix and the matrix B has non-negative entries. Therefore, the electrical circuit is positive for all values of the conductances and capacitances.

In this case the matrices (5.60) are diagonal matrices

$$A = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} \frac{1}{G_1 + G_1'} & 0 \\ 0 & \frac{1}{G_2 + G_2'} \end{bmatrix} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} - \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in M_2, \tag{5.61a}$$

$$B = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} \frac{1}{G_1 + G_1'} & 0 \\ 0 & \frac{1}{G_2 + G_2'} \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \in \mathbb{R}_+^{2\times 2} \tag{5.61b}$$

and

$$e^{A\tau} = \begin{bmatrix} e^{a_1\tau} & 0 \\ 0 & e^{a_2\tau} \end{bmatrix}. \tag{5.62}$$

Using (5.62) and (5.61) we obtain

$$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{A^T\tau} \mathrm{d}\tau = \int_0^{t_f} \begin{bmatrix} b_1^2 e^{2a_1\tau} & 0 \\ 0 & b_2^2 e^{2a_2\tau} \end{bmatrix} \mathrm{d}\tau. \tag{5.63}$$

The matrix (5.63) is monomial and by Theorem 5.7 the positive electrical circuit is reachable if $G_{12} = 0$.

Let

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.64}$$

The electrical circuit described by (5.59) and (5.64) is positive. The positive circuit is constructible, since $\det C = 1$.

It is also observable, since by Theorem 5.12 the matrix

$$O_p = e^{A^T t} C^T C e^{At} = \begin{bmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{bmatrix} = \begin{bmatrix} e^{2a_1 t} & 0 \\ 0 & e^{2a_2 t} \end{bmatrix}$$

is monomial.

5.5 Decomposition of the Positive Pair (A, B)

Consider the pair (A, B) with diagonal matrix A of the form

$$A = \mathrm{diag}[a_{11}, a_{22}, \dots, a_{nn}] \in M_n \tag{5.65a}$$

and the matrix B with m linearly independent monomial columns $B_1, B_2, \dots, B_m$

$$B = \left[B_1, B_2, \dots, B_m\right]. \tag{5.65b}$$

By Theorem 5.7 the pair (5.65) is unreachable if $m < n$.

It will be shown that in this case the pair can be decomposed into the reachable pair $(\bar{A}_1, \bar{B}_1)$ and unreachable pair $(\bar{A}_2, \bar{B}_2 = 0)$.

Theorem 5.17. *For unreachable pair* (5.65) $(m < n)$ *there exists a monomial matrix* $P \in \mathbb{R}_+^{n\times n}$ *such that*

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix}, \qquad \bar{B} = PB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, \tag{5.66}$$

where

$$\begin{aligned} \bar{A}_1 &= \mathrm{diag}[\bar{a}_{11}, \bar{a}_{22}, \dots, \bar{a}_{n_1 n_1}] \in M_{n_1}, \\ \bar{A}_2 &= \mathrm{diag}[\bar{a}_{n_1+1, n_1+1}, \dots, \bar{a}_{nn}] \in M_{n_2} \end{aligned}$$

and $\bar{B}_1 \in \mathbb{R}_+^{n_1\times m}$, $n = n_1 + n_2$, *the pair* $(\bar{A}_1, \bar{B}_1)$ *is reachable and the pair* $(\bar{A}_2, \bar{B}_2 = 0)$ *is unreachable.*

Proof. Performing on the matrix B the following elementary row operations:

- interchange the i-th and j-th rows, denoted by $L[i, j]$,
- multiplication of i-th row by positive number c, denoted by $L[i \times c]$,

we may reduce the matrix B to the form $\begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$, where $\bar{B}_1 \in \mathbb{R}_+^{n_1\times m}$ is monomial with positive entries equal to 1.

Performing the same elementary row operations on the identity matrix $\mathbb{I}_n$ we obtain the desired monomial matrix P. It is well-known [47], that $P^{-1} \in \mathbb{R}_+^{n\times n}$ and for diagonal matrix A we have

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix}.$$

□

Example 5.8. Consider the electrical circuit shown in Figure 5.8 with given resistances R_1, R_2, R_3; inductances L_1, L_2, L_3 and source voltages e_1, e_3.

Using Kirchhoff's laws we can write the equations

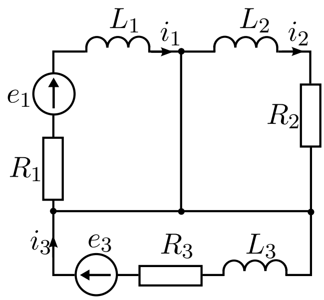

Fig. 5.8 Electrical circuit of Example 5.8

$$L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} = -R_1 i_1 + e_1,$$
$$L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} = -R_2 i_2,$$
$$L_3 \frac{\mathrm{d}i_3}{\mathrm{d}t} = -R_3 i_3 + e_3,$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_3 \end{bmatrix}, \tag{5.67a}$$

where

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \\ 0 & -\frac{R_2}{L_2} & 0 \\ 0 & 0 & -\frac{R_3}{L_3} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{L_3} \end{bmatrix}. \tag{5.67b}$$

By the Theorem 5.7 the positive electrical circuit (or the pair (5.67b)) is unreachable since $n = 3 > m = 2$. The unreachable pair (5.67b) can be decomposed into reachable pair $(\bar{A}_1, \bar{B}_1)$ and unreachable pair $(\bar{A}_2, \bar{B}_2 = 0)$.

In this case the monomial matrix P has the form

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and we obtain

$$\bar{B} = PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{L_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_3} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix},$$

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & 0 \\ 0 & -\frac{R_3}{L_3} & 0 \\ 0 & 0 & -\frac{R_2}{L_2} \end{bmatrix} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix}$$

and

$$\bar{A}_1 = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_3}{L_3} \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} -\frac{R_2}{L_2} \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_3} \end{bmatrix}.$$

The reachable pair $(\bar{A}_1, \bar{B}_1)$ is reachable and the pair $(\bar{A}_2, \bar{B}_2 = 0)$ is unreachable.

5.6 Decomposition of the Positive Pair (A, C)

Let the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}_+^{pn \times n}$$

of the positive unobservable electrical circuit has $n_1 < n$ linearly independent monomial rows. Thus, we can choose n_1 linearly independent rows of the observability matrix O_n as the first n_1 rows q_k $(k = 1, \ldots, n_1)$ of the similarity matrix Q [60, 61]. It is always possible to choose $n - n_1$ linearly independent monomial rows of matrix Q.

The matrix

$$Q^T = \begin{bmatrix} q_1^T & \cdots & q_{n_1}^T & q_{n_1+1}^T & \cdots & q_n^T \end{bmatrix}^T \in \mathbb{R}_+^{n \times n}, \tag{5.68}$$

is monomial and its inverse is equal to the transpose of Q, where every nonzero element is replaced by its inversion. Hence

$$q_k A q_j^T = 0 \quad \text{for} \quad k = 1, 2, \ldots, n_1 \quad \text{and} \quad j = n_1 + 1, \ldots, n. \tag{5.69}$$

Therefore, the following theorem has been proved.

Theorem 5.18. *Let the positive electrical circuit* (5.42) *be unobservable and let there exists the monomial matrix* (5.68). *Then the pair of matrices* (A, C) *of the electrical circuit can be reduced by the use of the matrix* (5.68) *to the form*

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix}, \quad \hat{C} = CQ^{-1} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix},$$
$$\hat{A}_1 \in \mathbb{R}_+^{n_1 \times n_1}, \; \hat{A}_2 \in \mathbb{R}_+^{n_2 \times n_2} \; (n_2 = n - n_1), \; \hat{A}_{21} \in \mathbb{R}_+^{n_2 \times n_1}, \; \hat{C}_1 \in \mathbb{R}_+^{p \times n_1}, \tag{5.70}$$

where the pair $(\hat{A}_1, \hat{C}_1)$ *is observable and the pair of matrices* $(\hat{A}_2, \hat{C}_2 = 0)$ *is unobservable.*

Example 5.9. Consider the positive electrical circuit shown in Figure 5.8 described by the state equation (5.67a) with A given by (5.67b). As the output $y(t)$ we assume

$$y(t) = R_3 i_3 = \begin{bmatrix} 0 & 0 & R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & R_3 \end{bmatrix}. \tag{5.71}$$

In this case the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_3 \\ 0 & 0 & -\frac{R_3^2}{L_3} \\ 0 & 0 & \frac{R_3^3}{L_3^2} \end{bmatrix}$$

has only one monomial linearly independend row $q_1 = C$, i.e. $n_1 = 1$ and the conditions (5.69) are satisfied for $q_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $q_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, since $q_1 A q_j^T = 0$ for $j = 2, 3$.

The matrix (5.68) has the form

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Using (5.70) we obtain

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} 0 & 0 & R_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{R_1}{L_1} & 0 & 0 \\ 0 & -\frac{R_2}{L_2} & 0 \\ 0 & 0 & -\frac{R_3}{L_3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{R_3} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{R_3}{L_3} & 0 & 0 \\ 0 & -\frac{R_1}{L_1} & 0 \\ 0 & 0 & -\frac{R_2}{L_2} \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{bmatrix},$$

$$\hat{C} = CQ^{-1} = \begin{bmatrix} 0 & 0 & R_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{R_3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix},$$

where

$$\hat{A}_1 = \left[-\frac{R_3}{L_3} \right], \quad \hat{A}_2 = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}, \quad \hat{C}_1 = [1].$$

The pair $(\hat{A}_1, \hat{C}_1)$ is observable and the pair of matrices $(\hat{A}_2, \hat{C}_2 = 0)$ is unobservable.

5.7 Decoupling Zeros of the Positive Electrical Circuits

It is well-known [173] that for standard linear systems the input-decoupling zeros are the eigenvalues of the matrix $\bar{A}_2$ of the unreachable (uncontrolable) part $(\bar{A}_2, \bar{B}_2 = 0)$.

Definition 5.9. Let $\bar{A}_2$ be the matrix of unreachable part of the eletrical circuit (4.1). The zeros $s_{i1}, s_{i2}, \dots, s_{1\bar{n}_2}$ of the characteristic polynomial

$$\det[\mathbb{I}_{\bar{n}_2} s - \bar{A}_2] = s^{\bar{n}_2} + \bar{a}_{\bar{n}_2 - 1} s^{\bar{n}_2 - 1} + \dots + \bar{a}_1 s + \bar{a}_0$$

of the matrix $\bar{A}_2$ are called the input-decoupling zeros of the positive system (4.1).

The list of the input-decoupling zeros will be denoted by $Z_i = \{s_{i1}, s_{i2}, \dots, s_{i\bar{n}_2}\}$.

Theorem 5.19. *The state vector $x(t)$ of the positive electrical circuits* (4.1) *is independent of the input-decoupling zeros for any input $u(t)$ and zero initial conditions.*

Proof. Form (4.1) for zero initial conditions $x(0) = 0$ we obtain the Laplace transform of the state equation (4.1a) (see Appendix A)

$$X(s) = \det[\mathbb{I}_n s - A]^{-1} BU(s),$$

where $X(s)$ and $U(s)$ are Laplace transforms of $x(t)$ and $u(t)$, respectively.

Taking into account (5.66) we obtain

$$\begin{aligned} X(s) &= [\mathbb{I}_n s - P^{-1}\bar{A}P]^{-1}P^{-1}BU(s) = P^{-1}[\mathbb{I}_n s - \bar{A}]^{-1}BU(s) \\ &= P^{-1}\begin{bmatrix} \mathbb{I}_{\bar{n}_1}s - \bar{A}_1 & 0 \\ 0 & \mathbb{I}_{\bar{n}_2}s - \bar{A}_2 \end{bmatrix}\begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} U(s) \\ &= P^{-1}\begin{bmatrix} [\mathbb{I}_{\bar{n}_1}s - \bar{A}_1]^{-1} \\ 0 \end{bmatrix} U(s). \end{aligned} \tag{5.72}$$

From (5.72) it follows that $X(s)$ is independent of the matrix $\bar{A}_2$ and of the input-decoupling zeros for any input $u(t)$. □

Example 5.10. (continuation of Example 5.8) In Example 5.8 it was shown that for the unreachable pair $(\bar{A}_2, \bar{B}_2 = 0)$ the matrix $\bar{A}_2$ has the form $\bar{A}_2 = \left[-\frac{R_2}{L_2}\right]$. Therefore, by Definition 5.9 the electrical circuit shown in Figure 5.8 has one input-decoupling zero $s_{i1} = -\frac{R_2}{L_2}$. Note that the input-decoupling zero corresponds to the mesh without the source voltage ($e_2 = 0$).

For standard continuous-time linear systems the output-decoupling zeros are defined as the eigenvalues of the matrix of the unobservable part of the system. In a similar way we will define the output-decoupling zeros of the positive electrical circuits.

Definition 5.10. Let $\hat{A}_2$ be the matrix of unobservable part of the electrical circuit (5.42). The zeros $s_{o1}, s_{o2}, \ldots, s_{o\hat{n}_2}$ of the characteristic polynomial

$$\det[\mathbb{I}_{\hat{n}_2}s - \hat{A}_2] = s^{\hat{n}_2} + \hat{a}_{\hat{n}_2-1}s^{\hat{n}_2-1} + \ldots + \hat{a}_1 s + \hat{a}_0$$

of the matrix $\hat{A}_2$ are called the output-decoupling zeros of the positive electrical circuit (5.42).

The list of the output-decoupling zeros will be denoted by $Z_o = \{s_{o1}, s_{o2}, \ldots, s_{o\hat{n}_2}\}$.

Theorem 5.20. *The output vector* $y(t)$ *of the positive electrical circuit* (5.42) *is independent of the output-decoupling zeros for any input* $\bar{u}(t) = Bu(t)$ *and zero initial conditions.*

Proof is similar to the proof of Theorem 5.19.

Example 5.11. (continuation of Example 5.9) In Example 5.9 it was shown that the matrix $\hat{A}_2$ of the unobservable pair has the form

$$\hat{A}_2 = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}.$$

Therefore, by Definition 5.10 the positive electrical circuit shown in Figure 5.8 has two output-decoupling zeros $s_{o1} = -\frac{R_1}{L_1}, s_{o2} = -\frac{R_2}{L_2}$.

Following the same way as for standard continuous-time linear systems we define the input-output decoupling zeros of the positive systems as follows.

Definition 5.11. Zeros $s_{i0}^{(1)}, s_{i0}^{(2)}, \ldots, s_{i0}^{(k)}$; which are simultaneously the input-decoupling zeros and the output-decoupling zeros of the positive electrical circuit are called the input-output decoupling zeros of the positive electrical circuit, i.e.

$$s_{io}^{(j)} \in Z_i \quad \text{and} \quad s_{io}^{(j)} \in Z_o \quad \text{for} \quad j = 1, 2, \ldots, k; \quad k \le \min(\bar{n}_2, \hat{n}_2).$$

The list of input-output decoupling zeros will be denoted by $Z_{io} = \{z_{io}^{(1)}, \ldots, z_{io}^{(k)}\}$.

Example 5.12. Consider the positive electrical circuit shown in Figure 5.8 with the matrices A, B, C given by (5.67b) and (5.71). In Example 5.10 it was shown that the electrical circuit has one input-decoupling zero $s_{i1} = -\frac{R_2}{L_2}$ and in Example 5.11 that the electrical circuit has two output-decoupling zeros $s_{o1} = -\frac{R_1}{L_1}, s_{o2} = -\frac{R_2}{L_2}$. Therefore, by Definition 5.11 the positive electrical circuit has one input-output decoupling zero $s_{io}^{(1)} = -\frac{R_2}{L_2}$.

5.8 Reachability of Positive Fractional Electrical Circuits

Definition 5.12. The state $x_f \in \mathbb{R}_+^n$ of the fractional circuit described by the state equations (1.11) is called reachable in time t_f if there exists an input $u(t) \in \mathbb{R}_+^n$, $t_f \in [0, t_f]$, which steers the state of the circuit (1.11) from zero intitial state $x_0 \in \mathbb{R}_+^n$ to the final state $x(t_f) = x_f$.

Theorem 5.21. *The positive fractional circuit* (1.11) *is reachable in time* $t_f \in [0, t_f]$ *if and only if the matrix* $A \in M_n$ *is diagonal and the matrix* $B \in \mathbb{R}_+^{n \times n}$ *is monomial.*

Proof. Sufficiency. It is well-known [47, 71] that if $A \in M_n$ is diagonal then $\Phi(t) \in \mathbb{R}_+^{n\times n}$ for $t \geq 0$ is also diagonal and if $B \in \mathbb{R}_+^{n\times n}$ is monomial then $BB^T \in \mathbb{R}_+^{n\times n}$ is also monomial.

In this case the matrix

$$R_f = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)\mathrm{d}\tau \in \mathbb{R}_+^{n\times n} \tag{5.73}$$

is also monomial and $R_f^{-1} \in \mathbb{R}_+^{n\times n}$.

The input

$$u(t) = B^T\Phi^T(t_f - t)R_f^{-1}x_f$$

steers the state of the system (1.11) from $x_0 = 0$ to x_f, since using (1.12) for $x_0 = 0$ and (5.73) we obtain

$$\begin{aligned} x(t_f) &= \int_0^{t_f} \Phi(t_f - \tau)Bu(\tau)\mathrm{d}\tau = \int_0^{t_f} \Phi(t_f - \tau)BB^T\Phi^T(t_f - \tau)\mathrm{d}\tau R_f^{-1}x_f \\ &= \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)\mathrm{d}\tau R_f^{-1}x_f = x_f. \end{aligned}$$

Necessity. Let

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

be the characteristic polynomial of the matrix $A \in M_n$. Then by well-known Cayley-Hamilton theorem [47] we have

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0\mathbb{I}_n = 0. \tag{5.74}$$

Using (5.74) we eliminate from (1.14) A^k for $k = n, n+1, \ldots$ and we obtain

$$\Phi(t) = \sum_{k=0}^{n-1} c_k(t)A^k, \tag{5.75}$$

where $c_k(t)$, $k = 0, 1, \ldots, n-1$ are some nonzero functions of time depending on the matrix A.

Subsitution of (5.75) into

$$\int_0^{t_f} \Phi(t_f - \tau)Bu(\tau)\mathrm{d}\tau$$

yields

$$x_f = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} v_0(t_f) \\ v_1(t_f) \\ \vdots \\ v_{n-1}(t_f) \end{bmatrix},$$

where

$$v_k(t_f) = \int_0^{t_f} c_k(\tau)u(t_f - \tau)\mathrm{d}\tau, \quad k = 0, 1, \ldots, n-1.$$

For given $x_f \in \mathbb{R}_+^n$ it is possible to find nonnegative $v_k(t_f)$ for $k = 0, 1, \ldots, n-1$ if and only if the matrix

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

has n linearly independent monomial columns and this takes place only if the matrix $\begin{bmatrix} B & A \end{bmatrix}$ contains n linearly independent columns [71].

Note that for the nonnegative $v_k(t_f)$ for $k = 0, 1, \ldots, n-1$ it is possible to find a nonnegative input $u(t) \in \mathbb{R}_+^n$, $t \in [0, t_f]$ only if the matrix $B \in \mathbb{R}_+^{n \times n}$ is monomial and the matrix $A \in M_n$ is diagonal. □

Definition 5.13. [105] The fractional positive electrical circuits (2.5) is called reachable in time $[0, t_f]$ if for every given final state $x_f = \begin{bmatrix} x_{Cf} \\ x_{Lf} \end{bmatrix} \in \mathbb{R}_+^n$ there exists an input $u(t) \in \mathbb{R}_+^m$, $t \in [0, t_f]$ which steers the state $\begin{bmatrix} x_C \\ x_L \end{bmatrix}$ of the circuit from zero initial state $x_0 = \begin{bmatrix} x_{C0} \\ x_{L0} \end{bmatrix} = 0$ to final state x_f.

Theorem 5.22. *[105] The fractional positive electrical circuit* (2.5) *is reachable in time* $[0, t_f]$ *if and only if the matrix*

$$R_f = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)\mathrm{d}\tau \in \mathbb{R}_+^{n \times n} \tag{5.76a}$$

is monomial, where

$$\Phi(\tau)B = \Phi_1(\tau)B_{10} + \Phi_2(\tau)B_{01}, \tag{5.76b}$$

where $\Phi_1(\tau)$, $\Phi_2(\tau)$ *and* B_{10}, B_{01} *are defined by* (2.8).

Proof. Note that $R_f^{-1} \in \mathbb{R}_+^{n \times n}$ if and only if the matrix (5.76a) is monomial. The input

$$u(t) = B^T\Phi^T(t_f - \tau)R_f^{-1}x_f \tag{5.77}$$

steers the state of the electrical circuit (2.5) form $x_0 = 0$ to $x_f \in \mathbb{R}_+^m$, since using (2.7) for $x_0 = 0$ and (5.77) we obtain

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau)Bu(\tau) = \int_0^{t_f} \Phi(t_f - \tau)BB^T\Phi^T(t_f - \tau)\mathrm{d}\tau R_f^{-1}x_f$$
$$= \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)\mathrm{d}\tau R_f^{-1}x_f.$$

□

Example 5.13. Consider the electrical circuit shown in Figure 5.3 with given resistances R_1, R_2, R_3; inductances L_1, L_2 and source voltages e_1, e_2. Given the condition, under which the fractional electrical circuit is reachable in time $[0, t_f]$ and find the input $u(t) \in \mathbb{R}_+^2$, $t \geq 0$, which steers the system from $x_0 = 0$ to the final state x_f.

Using Kirchhoff's laws we can write the equations

$$\begin{aligned} e_1 &= R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta}, \\ e_2 &= R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta}, \end{aligned} \quad 0 < \beta < 1,$$

which can be written in the form

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.78}$$

where

$$A = \begin{bmatrix} -\dfrac{R_1 + R_3}{L_1} & \dfrac{R_3}{L_1} \\ \dfrac{R_3}{L_2} & -\dfrac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix}. \tag{5.79}$$

The electrical circuit is positive, since the matrix A is Metzler matrix and the matrix B has nonnegative entries.

Note that the standard pair (5.79) is reachable, since $\det B \neq 0$ but it is not reachable as a positive pair.

We shall show that the positive electrical circuit is reachable if $R_3 = 0$. If $R_3 = 0$, then

$$A = \begin{bmatrix} -\dfrac{R_1}{L_1} & 0 \\ 0 & -\dfrac{R_2}{L_2} \end{bmatrix} \tag{5.80}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\beta - 1}}{\Gamma[(k+1)\beta]} = \sum_{k=0}^{\infty} \frac{t^{(k+1)\beta - 1}}{\Gamma[(k+1)\beta]} \begin{bmatrix} -\dfrac{R_1}{L_1} & 0 \\ 0 & -\dfrac{R_2}{L_2} \end{bmatrix}^k \tag{5.81}$$

and from (5.76) we obtain

$$R_f = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)\mathrm{d}\tau = \int_0^{t_f} \begin{bmatrix} \frac{1}{L_1^2} & 0 \\ 0 & \frac{1}{L_2^2} \end{bmatrix} \Phi^2(\tau)\mathrm{d}\tau = \begin{bmatrix} \frac{1}{L_1^2} & 0 \\ 0 & \frac{1}{L_2^2} \end{bmatrix}$$

$$\times \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{t_f^{(k+l+2)\beta-1}}{[(k+l+2)\beta-1]\Gamma[(k+1)\beta]\Gamma[(l+1)\beta]} \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}^{k+l}. \tag{5.82}$$

The matrix (5.82) is monomial and by Theorem 5.22 the positive fractional electrical circuit is reachable if and only if $R_3 = 0$, since for $R_3 \neq 0$ the matrix R_f is not monomial.

The desired input which steers the state $x(t) = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ from $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the final state $x_f \in \mathbb{R}_+^2$ is given by

$$u(t) = B^T\Phi^T(t_f - t)R_f^{-1}x_f$$

$$= \sum_{k=0}^{\infty} \frac{(t_f - t)^{(k+1)\beta-1}}{\Gamma[(k+1)\beta]} \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}^k R_f^{-1}x_f.$$

Example 5.14. Consider the fractional electrical circuit shown in Figure 5.7 with given conductances G_1, G_1', G_2, G_2', G_{12}; capacitances C_1, C_2 and source voltages e_1, e_2.

Using Kirchhoff's laws we can write the equations

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad 0 < \alpha < 1 \tag{5.83}$$

and

$$\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.84a}$$

where

$$G_{11} = G_1 + G_1' + G_{12}, \quad G_{22} = G_2 + G_2' + G_{12}. \tag{5.84b}$$

Taking into account that the matrix

$$\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}$$

is nonsingular and

$$-\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \in \mathbb{R}_+^{2\times 2},$$

from (5.84) we obtain

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right\}. \tag{5.85}$$

Substitution of (5.85) into (5.83) yields

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

$$A = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} - \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \in M_2, \tag{5.86a}$$

$$B = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}. \tag{5.86b}$$

From (5.86) it follows that A is Metzler matrix and the matrix B has nonnegative entries. Therefore, the fractional electrical circuit is positive for all values of the conductances and capacitances.

We shall show that the fractional positive electrical circuit is reachable in time $[0, t_f]$ if and only if $G_{12} = 0$.

If $G_{12} = 0$, then the matrices (5.86) are diagonal of the form

$$A = \begin{bmatrix} -\frac{G_1 G_1'}{C_1(G_1 + G_1')} & 0 \\ 0 & -\frac{G_2 G_2'}{C_2(G_2 + G_2')} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{G_1 G_1'}{C_1(G_1 + G_1')} & 0 \\ 0 & \frac{G_2 G_2'}{C_2(G_2 + G_2')} \end{bmatrix}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}$$
$$= \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \begin{bmatrix} -\frac{G_1 G_1'}{C_1(G_1+G_1')} & 0 \\ 0 & -\frac{G_2 G_2'}{C_2(G_2+G_2')} \end{bmatrix}^k$$

.

From (5.76) we obtain

$$R_f = \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau) \mathrm{d}\tau = \int_0^{t_f} \begin{bmatrix} \frac{G_1 G_1'}{C_1(G_1+G_1')} & 0 \\ 0 & \frac{G_2 G_2'}{C_2(G_2+G_2')} \end{bmatrix} \Phi^2(\tau) \mathrm{d}\tau.$$

From Theorem 5.22 the fractional positive electrical circuit is reachable if and only if $G_{12} = 0$, since the matrix R_f is monomial only if $G_{12} = 0$.

The desired input which steers the state $x(t) = \begin{bmatrix} u_1(t) \\ i_2(t) \end{bmatrix}$ from $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the final state $x_f \in \mathbb{R}_+^2$ is given by

$$u(t) = B^T \Phi^T (t_f - t) R_f^{-1} x_f$$
$$= -\sum_{k=0}^{\infty} \frac{(t_f - t)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \begin{bmatrix} -\frac{G_1 G_1'}{C_1(G_1+G_1')} & 0 \\ 0 & -\frac{G_2 G_2'}{C_2(G_2+G_2')} \end{bmatrix}^{k+1} R_f^{-1} x_f.$$

5.9 Observability of Positive Fractional Electrical Circuits

Consider the fractional positive electrical circuits described by the equations

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha x_C(t)}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta x_L(t)}{\mathrm{d}t^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix} = Ax(t), \tag{5.87a}$$

$$y(t) = Cx(t), \tag{5.87b}$$

where

$$x(t) = \begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad C \in \mathbb{R}^{p \times n} \tag{5.87c}$$

and $y(t) \in \mathbb{R}^p$ is the output of the electrical circuit.

It is well-known that [67] the fractional electrical circuit (5.87) is positive if and only if

$$A \in M_n \quad \text{and} \quad C \in \mathbb{R}_+^{p\times n}. \tag{5.88}$$

Definition 5.14. The fractional positive electrical circuit (5.87) is called strongly observable in time $[0, t_f]$ if knowing the output $y(t) \in \mathbb{R}_+^p$ for $t \in [0, t_f]$ it is possible to find uniquely the initial value $x_0 = x(0) \in \mathbb{R}_+^n$ of the state vector $x(t) \in \mathbb{R}_+^n$.

Theorem 5.23. *The fractional positive electrical circuit* (5.87) *is strongly observable in time* $[0, t_f]$ *if and only if the matrix*

$$V_f = \int_0^{t_f} \Phi_0^T(t) C^T C \Phi_0(t) \mathrm{d}t \tag{5.89}$$

is monomial, where $\Phi_0(t)$ *is defined by* (2.8).

Proof. From (2.7) for $u(t) = 0$, $t \geq 0$ we have

$$x(t) = \Phi_0(t) x_0. \tag{5.90}$$

Substitution of (5.90) into (5.87b) yields

$$y(t) = C\Phi_0(t) x_0. \tag{5.91}$$

Premultiplying (5.91) by $\Phi_0^T(t) C^T$ and integrating the product in the interval $[0, t_f]$ we obtain

$$\int_0^{t_f} \Phi_0^T(t) C^T y(t) \mathrm{d}t = \int_0^{t_f} \Phi_0^T(t) C^T C \Phi_0(t) \mathrm{d}t x_0 = V_f x_0. \tag{5.92}$$

From (5.92) we can find $x_0 \in \mathbb{R}_+^n$ if and only if the matrix (5.89) is monomial and $V_f^{-1} \in \mathbb{R}_+^{n\times n}$. □

From comparison of Theorem 5.22 and 5.23 we have the following remark.

Remark 5.3. The conditions for strong observability of the fractional positive electrical circuits are dual to the reachability ones. By substituting in the reachability conditions $\Phi(t)$ and B by $\Phi_0^T(t)$ and C^T we obtain the strong observability conditions.

Chapter 6
Standard and Fractional Linear Circuits with Feedbacks

6.1 Linear Dependence on Time of State Variable in Standard Electrical Circuits with State Feedbacks

Let us consider the linear electrical circuit consisting of resistors, capacitors, coils and voltage (current) sources described by the state equation [63]

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Be(t), \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $e(t) \in \mathbb{R}^m$ is the input vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

As state variables $x_1, x_2, \ldots, x_n$ (the components of $x(t)$) usually the currents in coils and voltage across the capacities are chosen. The components of the input vector e are the source voltages or source currents.

Let us assume that the number of state variables n is equal to the number of inputs m, i.e. $n = m$ and $\det B \neq 0$.

Consider the electrical circuit (6.1) with the state-feedback

$$e(t) = Kx(t), \tag{6.2}$$

where $K \in \mathbb{R}^{n\times n}$ is a gain matrix.

Remark 6.1. Note that the electrical circuit with state-feedback (6.2) is equivalent to a linear circuit with controlled voltage or current sources.

Substitution of (6.2) into (6.1) yields

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_c x(t), \tag{6.3}$$

where

$$A_c = A + BK. \tag{6.4}$$

T. Kaczorek and K. Rogowski, *Fractional Linear Systems and Electrical Circuits*,
Studies in Systems, Decision and Control 13, DOI: 10.1007/978-3-319-11361-6_6

We are looking for a gain matrix K, such that the closed-loop system matrix (6.4) has nilpotency index $v = 2$ (see Appendix C).

Theorem 6.1. *Let $m = n$ and $\det B \neq 0$, then the gain matrix K can be chosen so that the state variables in the electrical circuits are linear functions of time for any given initial conditions $x(0) = x_0$.*

Proof. Using Lemmas C.1 and C.2 (given in Appendix C) we choose as the matrix A_c a matrix with nilpotency index $v = 2$.

In this case

$$A_c^k = 0 \quad \text{for } k = 2, 3, \dots. \tag{6.5}$$

If (6.5) holds then the solution of the equation (6.3) has the form

$$x(t) = e^{A_c t} x_0 = \sum_{k=0}^{\infty} \frac{(A_c t)^k}{k!} x_0 = (\mathbb{I}_n + A_c t)\, x_0 \tag{6.6}$$

and the state variables of the closed-loop system are linear functions of time for any initial conditions x_0.

Knowing the matrices A_c, A and B we can find the gain matrix from the equation (6.4), since $\det B \neq 0$.

Hence

$$K = B^{-1}(A_c - A). \tag{6.7}$$

□

Example 6.1. Consider the electrical circuit showin in Figure 6.1 with given resistances R_1, R_2, R_3; inductances L_1, L_2 and voltage sources e_1, e_2.

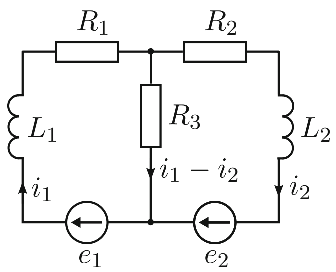

Fig. 6.1 Electrical circuit of Example 6.1

Using Kirchhoff's laws we may write the equations

$$e_1 = (R_1 + R_3)\, i_1 - R_3 i_2 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t}, \tag{6.8a}$$

$$e_2 = (R_2 + R_3)\, i_2 - R_3 i_1 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}. \tag{6.8b}$$

The equations (6.8) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{6.9a}$$

where

$$A = \begin{bmatrix} -\dfrac{R_1+R_3}{L_1} & \dfrac{R_3}{L_1} \\ \dfrac{R_3}{L_2} & -\dfrac{R_2+R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix}. \tag{6.9b}$$

Let

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = K\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}. \tag{6.10}$$

Substitution of (6.10) into (6.9a) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A_c\begin{bmatrix} i_1 \\ i_2 \end{bmatrix},$$

where the matrix A_c is given by (6.4).

We choose the matrix A_c with the nilpotency index $v = 2$ of the form

$$A_c = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \tag{6.11a}$$

or

$$A_c = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \tag{6.11b}$$

where a is any given parameter.

Using (6.7), (6.9b) and (6.11) we obtain

$$K = \begin{bmatrix} R_1+R_3 & aL_1-R_3 \\ -R_3 & R_2+R_3 \end{bmatrix} \quad \text{for (6.11a)}$$

and

$$K = \begin{bmatrix} R_1+R_3 & -R_3 \\ aL_2-R_3 & R_2+R_3 \end{bmatrix} \quad \text{for (6.11b).}$$

From (6.6) we have

$$\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} i_{10}+ai_{20}t \\ i_{20} \end{bmatrix} \quad \text{for (6.11a)}$$

and

$$\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} i_{10} \\ i_{20}+ai_{10}t \end{bmatrix} \quad \text{for (6.11b),}$$

where $i_{10} = i_1(0)$, $i_{20} = i_2(0)$.

Remark 6.2. Note that the electrical circuit considered in Example 6.1 is positive system for all values of resistances R_1, R_2, R_3; nonzero inductances L_1 and L_2, since the matrix A is a Metzler matrix and the matrix B has nonnegative entries.

Example 6.2. Consider the electrical circuit shown in Figure 6.2 with given resistances R_1, R_2; capacitances C_1, C_2; inductance L and voltage sources e_1, e_2, e_3.

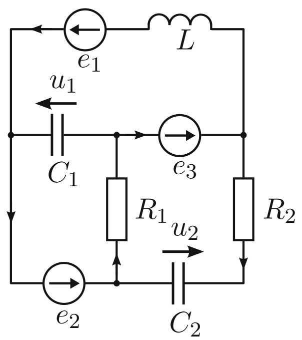

Fig. 6.2 Electrical circuit of Example 6.2

Using Kirchhoff's laws we may write the equations

$$e_1 + e_3 = L\frac{\mathrm{d}i}{\mathrm{d}t} + u_1, \tag{6.14a}$$

$$e_2 = R_1\left(i - C_1\frac{\mathrm{d}u_1}{\mathrm{d}t} + C_2\frac{\mathrm{d}u_2}{\mathrm{d}t}\right) - u_1, \tag{6.14b}$$

$$e_3 = R_2C_2\frac{\mathrm{d}u_2}{\mathrm{d}t} + u_2 + R_1\left(i - C_1\frac{\mathrm{d}u_1}{\mathrm{d}t} + C_2\frac{\mathrm{d}u_2}{\mathrm{d}t}\right). \tag{6.14c}$$

The equations (6.14) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i \\ u_1 \\ u_2 \end{bmatrix} = A\begin{bmatrix} i \\ u_1 \\ u_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{6.15a}$$

where

$$A = \begin{bmatrix} 0 & -\frac{1}{L} & 0 \\ \frac{1}{C_1} & -\frac{R_1+R_2}{R_1R_2C_1} & -\frac{1}{R_2C_1} \\ 0 & -\frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} & 0 & \frac{1}{L} \\ 0 & -\frac{R_1+R_2}{R_1R_2C_1} & \frac{1}{R_2C_1} \\ 0 & -\frac{1}{R_2C_2} & \frac{1}{R_2C_2} \end{bmatrix}. \tag{6.15b}$$

Let

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = K \begin{bmatrix} i \\ u_1 \\ u_2 \end{bmatrix}. \tag{6.16}$$

Substitution of (6.16) into (6.15a) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i \\ u_1 \\ u_2 \end{bmatrix} = A_c \begin{bmatrix} i \\ u_1 \\ u_2 \end{bmatrix},$$

where the matrix A_c is given by (6.4).

We choose the matrix A_c of the form

$$A_c = \begin{bmatrix} 0 & a_1 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix}, \tag{6.17}$$

where a_1, a_2 are some real parameters.

Using (6.7), (6.15b) and (6.17) we obtain

$$\begin{aligned} K &= B^{-1}(A_c - A) \\ &= \begin{bmatrix} \frac{1}{L} & 0 & \frac{1}{L} \\ 0 & -\frac{R_1 + R_2}{R_1 R_2 C_1} & \frac{1}{R_2 C_1} \\ 0 & -\frac{1}{R_2 C_2} & \frac{1}{R_2 C_2} \end{bmatrix}^{-1} \begin{bmatrix} 0 & a_1 + \frac{1}{L} & 0 \\ -\frac{1}{C_1} & \frac{R_1 + R_2}{R_1 R_2 C_1} & \frac{1}{R_2 C_1} \\ 0 & a_2 + \frac{1}{R_2 C_2} & \frac{1}{R_2 C_2} \end{bmatrix} \\ &= \begin{bmatrix} -R_1 & 1 + a_1 L - (R_1 + R_2) C_2 a_2 & -1 \\ R_1 & -1 + R_1 C_2 a_2 & 0 \\ R_1 & (R_1 + R_2) C_2 a_2 & 1 \end{bmatrix}. \end{aligned}$$

From (6.6) we have

$$\begin{bmatrix} i(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 1 & a_1 t & 0 \\ 0 & 1 & 0 \\ 0 & a_2 t & 1 \end{bmatrix} \begin{bmatrix} i_0 \\ u_{10} \\ u_{20} \end{bmatrix} = \begin{bmatrix} i_0 + a_1 u_{10} t \\ u_{10} \\ u_{20} + a_2 u_{10} t \end{bmatrix},$$

where $i_0 = i(0)$, $u_{10} = u_1(0)$, $u_{20} = u_2(0)$.

If $n \neq m$, but for a chosen matrix A_c the following condition is met

$$\operatorname{rank} B = \operatorname{rank}[B, A_c - A], \tag{6.18}$$

then the equation

$$BK = A_c - A \tag{6.19}$$

has a solution K.

Theorem 6.2. *If the condition (6.18) is satisfied, then there exists a gain matrix K, such that the state variables of the electrical circuits are linear functions of time for any given initial conditions x_0.*

Proof. If for a chosen matrix A_c the condition (6.18) is satisfied, then by Kronecker-Capelly theorem the equation (6.19) has a solution K, such that the closed-loop system matrix is equal to A_c. The remaining part of the proof is similar to the proof of Theorem 6.1. □

Example 6.3. Consider the electrical circuit shown in Figure 2.6 with given resistance R, capacity C, inductance L and voltage source e.

The equations

$$i = C\frac{\mathrm{d}u}{\mathrm{d}t},$$
$$e = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + u$$

can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ i \end{bmatrix} = A\begin{bmatrix} u \\ i \end{bmatrix} + Be,$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

If we choose

$$A_c = \begin{bmatrix} 0 & \frac{1}{C} \\ 0 & 0 \end{bmatrix}$$

then the condition (6.18) is satisfied and the equation (6.19) has the form

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{L} & \frac{R}{L} \end{bmatrix}$$

and its solution is

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & R \end{bmatrix}.$$

From (6.6) we have

$$\begin{bmatrix} u(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} 1 & \frac{t}{C} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_0 \\ i_0 \end{bmatrix} = \begin{bmatrix} u_0 + \frac{i_0}{C}t \\ i_0 \end{bmatrix},$$

where $u_0 = u(0)$, $i_0 = i(0)$.

Example 6.4. Consider the electrical circuit shown in Figure 6.3 with given resistances R_1, R_2; capacity C; inductances L_1, L_2 and voltage sources e_1, e_2.

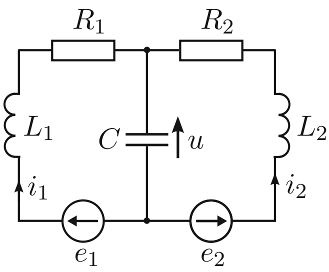

Fig. 6.3 Electrical circuit of Example 6.4

Using Kirchhoff's laws we may write the equations

$$e_1 = R_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + u, \tag{6.22a}$$

$$e_2 = R_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + u, \tag{6.22b}$$

$$C\frac{\mathrm{d}u}{\mathrm{d}t} = i_1 + i_2. \tag{6.22c}$$

The equations (6.22) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix}.$$

If we choose the matrix A_c of the form

$$A_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix},$$

then the condition (6.18) is satisfied and the equation (6.19) takes the form

$$\begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} \frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & \frac{R_2}{L_2} & \frac{1}{L_2} \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.24}$$

The solution of the equation (6.24) is given by

$$K = \begin{bmatrix} R_1 & 0 & 1 \\ 0 & R_2 & 1 \end{bmatrix}.$$

From (6.6) we have

$$\begin{bmatrix} i_1(t) \\ i_2(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{t}{C} & \frac{t}{C} & 1 \end{bmatrix} \begin{bmatrix} i_{10} \\ i_{20} \\ u_0 \end{bmatrix} = \begin{bmatrix} i_{10} \\ i_{20} \\ u_0 + \frac{i_{10}}{C}t + \frac{i_{20}}{C}t \end{bmatrix},$$

where $i_{10} = i_1(0)$, $i_{20} = i_2(0)$, $u_0 = u(0)$.

Remark 6.3. Note that the electrical circuits shown in Figures 6.2 and 6.3 are not positive systems, since the matrices A of these circuits are not Metzler matrices.

Now, consider electrical circuits consisiting of resistors, capacitances, coils and voltage (current) sources described by the equation (6.1) in the case, when the matrix B is singular, i.e. $\det B = 0$.

Theorem 6.3. *The matrix B in the state equation* (6.1) *is singular (*$\det B = 0$*) if one of the following conditions is satisfied:*

1. *to a node are connected branches only with one capacitor and any number branches with coils,*
2. *to a mesh belong branches with only one coil and any number branches with only capacitors.*

Proof. In the first case let us denote the voltage across the condensator with capacity C by u and the currents in the q_1 coils by $i_1, i_2, \ldots, i_{q_1}$. Then using the first Kirchhoff's law we obtain the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sum_{k=1}^{q_1} \frac{i_k}{C}. \tag{6.25}$$

From the equation (6.25) it follows that the row in the matrix B corresponding to $\frac{\mathrm{d}u}{\mathrm{d}t}$ is zero row and $\det B = 0$.

In the second case let us denote the current in the coil with inductance L by i and the voltages across the condensators by $u_1, u_2, \ldots, u_{q_2}$. Then using the second Kirchhoff's law we obtain the equation

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \sum_{k=1}^{q_2} \frac{u_k}{L}. \tag{6.26}$$

From the equation (6.26) it follows that the row in matrix B corresponding to $\frac{\mathrm{d}i}{\mathrm{d}t}$ is zero row and $\det B = 0$. □

The following examples illustrate both cases of the electrical circuits.

Example 6.5. Consider the electrical circuit shown in Figure 6.4 with given resistances R_1, R_2; capacity C; inductances L_1, L_2 and voltage sources e_1, e_2, e_3.

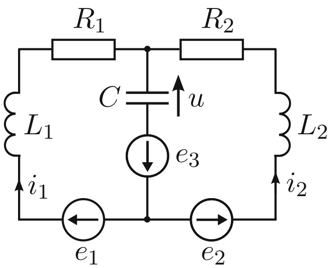

Fig. 6.4 Electrical circuit of Example 6.5

Using Kirchhoff's laws we may write the equations

$$e_1 + e_3 = R_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + u, \tag{6.27a}$$

$$e_2 + e_3 = R_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + u, \tag{6.27b}$$

$$C \frac{\mathrm{d}u}{\mathrm{d}t} = i_1 + i_2. \tag{6.27c}$$

The equations (6.27) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & \frac{1}{L_2} & \frac{1}{L_2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the matrix B is singular with nonnegative entries.

Example 6.6. Consider the electrical circuit shown in Figure 6.5 with given resistances R_1, R_2, R_3; capacities C_1, C_2; inductance L and voltage sources e_1, e_2, e_3.

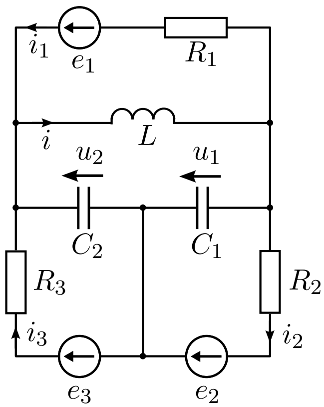

Fig. 6.5 Electrical circuit of Example 6.6

Using Kirchhoff's laws we may write the equations

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = u_1 + u_2, \tag{6.29a}$$

$$e_1 = R_1 i_1 + L\frac{\mathrm{d}i}{\mathrm{d}t}, \tag{6.29b}$$

$$e_2 = R_2\left(i - i_1 + C_1\frac{\mathrm{d}u_1}{\mathrm{d}t}\right) + u_1, \tag{6.29c}$$

$$e_3 = R_3\left(i - i_1 + C_2\frac{\mathrm{d}u_2}{\mathrm{d}t}\right) + u_2. \tag{6.29d}$$

The equations (6.29) can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i \\ u_1 \\ u_2 \end{bmatrix} = A\begin{bmatrix} i \\ u_1 \\ u_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{L} & \frac{1}{L} \\ -\frac{1}{C_1} & -\frac{R_1 + R_2}{R_1 R_2 C_1} & -\frac{1}{R_1 C_1} \\ -\frac{1}{C_2} & -\frac{1}{R_1 C_2} & -\frac{R_1 + R_3}{R_1 R_3 C_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{R_1 C_1} & \frac{1}{R_2 C_1} & 0 \\ \frac{1}{R_1 C_2} & 0 & \frac{1}{R_3 C_2} \end{bmatrix}.$$

Note that the matrix B is singular with nonnegative entries.

In both cases, in similar way as in Example 6.4 we can choose the closed-loop matrices A_c with nilpotency indices $v = 2$ and find gain matrices K, such that the state variables of the closed-loop systems are linear functions of time.

6.2 Zeroing of the State Vector of Standard Circuits by State-Feedbacks

Consider descriptor linear circuit described by the equation (3.1) with regular pencil satisfying (3.2). It is assumed, that $\text{rank}E = r < n$, $\text{rank}B = m$.

To the electrical circuit (3.1) the state-feedback

$$u(t) = Kx(t),$$

where a gain matrix $K \in \mathbb{R}^{m\times n}$ is applied and the equation of closed-loop circuit has the form

$$E\frac{\mathrm{d}x(t)}{\mathrm{d}t} = (A + BK)\,x(t). \tag{6.31}$$

We are looking for a gain matrix K, such that the state vector of the closed-loop circuit satisfies the condition

$$x(t) = 0 \quad \text{for} \quad t > 0 \tag{6.32}$$

for any admissible initial conditions and any values of resistances, inductances and capacitances.

First, similarly as in [48], we shall derive the necessary and sufficient conditions under which the gain matrix exists and satisfies the condition

$$\det\left[Es - (A + BK)\right] = \gamma \neq 0, \tag{6.33}$$

where γ is a real number independent of s.

From the equality

$$\begin{bmatrix} Es - A - BK \end{bmatrix} = \begin{bmatrix} Es - A, B \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ -K \end{bmatrix}$$

and (6.33) it follows that the problem has a solution only if

$$\text{rank}\begin{bmatrix} Es - A, B \end{bmatrix} = n \quad \text{for all finite} \quad s \in \mathbb{C}. \tag{6.34}$$

It is well known [21, 44] that the system (3.1) is completely controllable if and only if the conditions (6.34) and

$$\text{rank}\begin{bmatrix} E, B \end{bmatrix} = n \tag{6.35}$$

are met.

In the solution of the problem the following lemma will be used [19, 132].

Lemma 6.1. If the condition (3.2) is satisfied, then there exist the orthogonal matrices U and V such that

$$U\left[Es - A\right]V = \begin{bmatrix} E_1 s - A_1 & * \\ 0 & E_0 s - A_0 \end{bmatrix}, \quad UB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{6.36a}$$

where $E_1, A_1 \in \mathbb{R}^{n_1 \times n_1}$, $E_0, A_0 \in \mathbb{R}^{n_0 \times n_0}$, $B_1 \in \mathbb{R}^{n_1 \times m}$, the subsystem (E_1,A_1,B_1) is completely controllable, tha pair (E_0,A_0) is regular, E_1 is upper triangular and '$*$' denotes some unimportant matrix. Moreover, the matrices E_1, A_1 and B_1 are of the forms

$$E_1 s - A_1 = \begin{bmatrix} E_{11}s - A_{11} & E_{12}s - A_{12} & \cdots & E_{1,k-1}s - A_{1,k-1} & E_{1k}s - A_{1k} \\ -A_{21} & E_{22}s - A_{22} & \cdots & E_{2,k-1}s - A_{2,k-1} & E_{2k}s - A_{2k} \\ 0 & -A_{32} & \cdots & E_{3,k-1}s - A_{3,k-1} & E_{3k}s - A_{3k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -A_{k,k-1} & E_{kk}s - A_{kk} \end{bmatrix}, \tag{6.36b}$$

$$B_1 = \begin{bmatrix} B_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{6.36c}$$

where $E_{ij}, A_{ij} \in \mathbb{R}^{\bar{n}_i \times \bar{n}_j}$; $i,j = 1,\ldots,k$ and $B_{11} \in \mathbb{R}^{\bar{n}_i \times m}$, $\sum_{i=1}^{n} \bar{n}_i = n_1$ with $B_{11}, A_{21}, \ldots, A_{k,k=1}$ of full row ranks and nonsingular $E_{22}, \ldots, E_{kk}$.

Theorem 6.4. *Let the condition* (3.2) *be satisfied and the matrices* E, A, B *of* (3.1) *be transformed into the forms* (6.36). *A matrix* K *satisfying* (6.33) *exists if and only if*

1. *the subsystem* (E_1, A_1, B_1) *is singular, i.e.*

$$\det E_1 = 0, \tag{6.37a}$$

2. *if* $n_0 > 0$, *then the degree of the polynomial* $\deg\det\left[E_0 s - A_0\right]$ *is zero, i.e.*

$$\deg\det\left[E_0 s - A_0\right] = 0 \quad \textit{for} \quad n_0 > 0. \tag{6.37b}$$

Proof. Necessity. From (6.33) and (6.36a) we have

$$\begin{aligned} \det\left[Es - A - BK\right] =& \det U^{-1} \det V^{-1} \\ & \times \det\left[E_1 s - A_1 - B_1 \bar{K}\right] \det\left[E_0 s - A_0\right] = \alpha, \end{aligned} \tag{6.38}$$

where $\bar{K} = KV \in \mathbb{R}^{m\times n}$ and $\det[E_0 s - A_0] = 1$ if $n_0 = 0$.

From (6.38) it follows that the condition (6.33) holds only if the conditions (6.37) are satisfied.

Sufficiency. Consider first the single-input case ($m = 1$). We have

$$E_1 = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n_1} \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n_1-1} & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix},$$

$$B_1 = b_1 = \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $e_{ii} \neq 0$, $a_{i,i-1} \neq 0$ for $i = 2, \ldots, n_1$ and $b_{11} \neq 0$.

The condition (6.37a) implies $e_{11} = 0$. Premultiplying the matrix $[E_1 s - A_1, b_1]$ by an orthogonal matrix of row operations P_1, it is possible to make the entries $e_{12}, e_{13}, \ldots, e_{1n_1}$ of E_1 zero, since $e_{ii} \neq 0$, $i = 2, \ldots, n_1$.

By this reduction only the entries of the first row of A_1 are modified. We get

$$\bar{E}_1 = P_1 E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix}, \tag{6.39a}$$

$$\bar{A}_1 = P_1 A_1 = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1,n_1-1} & \bar{a}_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix}, \quad \bar{b}_1 = P_1 b_1 = b_1. \tag{6.39b}$$

Let

$$\bar{k}_1 = \frac{1}{b_{11}} \left[-\bar{a}_{11}, -\bar{a}_{12}, \ldots, -\bar{a}_{1,n_1-1}, -\bar{a}_{1,n_1} \right]. \tag{6.40}$$

Using (6.38), (6.39a) and (6.40), we obtain

$$\det\left[\bar{E}_1 s - \bar{A}_1 + \bar{b}_1 \bar{k}_1\right]$$
$$= \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ -a_{21} & e_{22}s - a_{22} & \cdots & e_{2,n_1-1}s - a_{2,n_1-1} & e_{2,n_1}s - a_{2,n_1} \\ 0 & -a_{31} & \cdots & e_{3,n_1-1}s - a_{3,n_1-1} & e_{3,n_1}s - a_{3,n_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -a_{n_1,n_1-1} & e_{n_1,n_1}s - a_{n_1,n_1} \end{vmatrix}$$
$$= a_{21}a_{31}\cdots a_{n_1,n_1-1} = \bar{\gamma},$$

where $\bar{\gamma} = \gamma \det U \det V \det P_1 \det\left[E_0 s - A_0\right]^{-1}$.

The above can be easily extended to multi-input systems, i.e. $m > 1$. In this case the matrix P_1 of orthogonal row operations is chosen, so that all the entries of the first row of $\bar{E}_1 = P_1 E_1$ be zero. By this reduction only the entries of A_{1i}, $i = 1, \ldots, k$ and B_{11} will be modified. The modified matrices will be denoted by $\bar{A}_{1i}$, $i = 1, \ldots, k$ and $\bar{B}_{11}$, respectively.

Let

$$\bar{K} = \bar{B}_1^{-1}\left\{\left[\bar{A}_{11}, \bar{A}_{12}, \ldots, \bar{A}_{1k}\right] + G\right\}. \tag{6.41}$$

The matrix $G \in \mathbb{R}^{m\times n}$ in (6.41) is chosen, so that

$$\bar{E}_1 s - \bar{A}_1 + \bar{B}_1\bar{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{l+1}h \\ \bar{a}_{21} & * & \cdots & * & * \\ 0 & \bar{a}_{32} & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{a}_{l,l-1} & * \end{bmatrix}, \tag{6.42}$$

where '$*$' deontes unimportant entries,

$$h = \frac{\alpha(-1)^{l+1}}{\bar{a}_{21}\bar{a}_{32}\cdots\bar{a}_{l,l-1}c},$$
$$c = \det U^{-1} \det V^{-1} \det P_1^{-1} \det\left[E_0 s - A_0\right].$$

Using (6.38), (6.41) and (6.42), it is easy to verify that

$$\det\left[Es - A - BK\right] = c\det\left[\bar{E}_1 s - \bar{A}_1 + \bar{B}_1\bar{K}\right] = \gamma.$$

□

Remark 6.4. For $m > 1$ there exist many different matrices K satisfying the condition (6.33).

Remark 6.5. If the system order is not high, elementary row and column operations can be used instead of orthogonal operations.

The solution of the problem is based on the following theorem.

Theorem 6.5. *There exists a gain matrix $K \in \mathbb{R}^{m\times n}$, such that* (6.32) *holds if and only if the conditions of Theorem 6.4 are satisfied.*

Proof. By Theorem 6.4 there exists K satisfying (6.33) if and only if the conditions (6.37) are met. In this case, using the Laplace transform from (6.31) we obtain

$$X(s) = [Es - (A + BK)]^{-1} x_0$$

where $X(s) = \mathcal{L}[x(t)]$ is the Laplace transform of $x(t)$ and x_0 is the admissible initial condition.

Taking into account (6.33) we obtain

$$\begin{aligned} X(s) &= \frac{\text{Adj}[Es - (A + BK)]}{\det[Es - (A + BK)]} x_0 = \frac{\text{Adj}[Es - (A + BK)]}{\gamma} x_0 \\ &= (P_0 + P_1 s + \ldots + P_q s^q)\, x_0, \end{aligned} \tag{6.43}$$

where $\text{Adj}[Es - (A + BK)]$ denotes the adjoint matrix and $P_k \in \mathbb{R}^{n\times n}$ for $k = 0, 1, \ldots, q$.

Applying the inverse Laplace transform to (6.43) we obtain

$$x(t) = \sum_{k=0}^{q} P_k x_0 \delta^{(k)}(t) = 0 \quad \text{for} \quad t > 0,$$

where $\delta(t)$ is the Dirac impulse and $\delta^{(k)}(t)$ is its k-th derivative. □

Example 6.7. Consider the electrical circuit shown in Figure 6.6 with given resistance R, capacitances C_1, C_2 and source voltage $e = e(t)$.

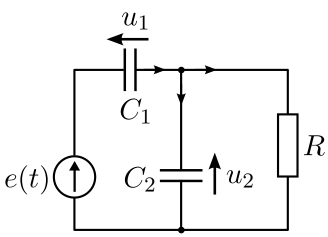

Fig. 6.6 Electrical circuit of Example 6.7

Using Kirchhoff's laws for the electrical circuit we can write the equations

$$\begin{gathered} C_1 \frac{du_1}{dt} - C_2 \frac{du_2}{dt} = \frac{u_2}{R}, \\ u_1 + u_2 = e, \end{gathered}$$

which can be rewritten in the form

$$E \frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be,$$

where

$$E = \begin{bmatrix} C_1 & -C_2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{1}{R} \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The condition (6.34) is satisfied, since

$$\operatorname{rank}[Es - A, B] = \operatorname{rank}\begin{bmatrix} sC_1 & -sC_2 - \frac{1}{R} & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2 \quad \text{for all} \quad s \in \mathbb{C}.$$

For the gain matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ the closed-loop system matrix has the form

$$\left[Es - (A + BK)\right] = \begin{bmatrix} sC_1 & -sC_2 - \frac{1}{R} \\ 1 - k_1 & 1 - k_2 \end{bmatrix}$$

and its determinant is equal to a real number $\gamma \neq 0$

$$\det\left[Es - (A + BK)\right] = s\left[C_1(1 - k_2) + C_2(1 - k_1)\right] + \frac{1 - k_1}{R} = \alpha$$

and

$$k_1 \neq 1 \quad \text{and} \quad k_2 = \frac{C_1 + C_2(1 - k_1)}{C_1}. \tag{6.45}$$

Therefore, for the state feedback matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ with k_1 and k_2 defined by (6.45) we have $u_1(t) = 0$, $u_2(t) = 0$ for $t > 0$.

Example 6.8. Consider the electrical circuit shown in Figure 6.7 with given resistances R_1, R_2; inductances L_1, L_2 and source current $i_s(t) = i_s$.

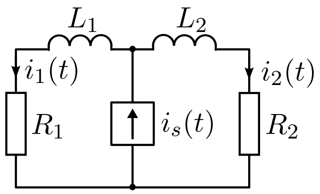

Fig. 6.7 Electrical circuit of Example 6.8

Using Kirchhoff's laws for the electrcial circuit we can write the equations

$$\begin{aligned} 0 &= R_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} - R_2 i_2 - L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \\ i_s &= i_1 + i_2, \end{aligned}$$

which can be rewritten in the form

$$E \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B i_s,$$

where

$$E = \begin{bmatrix} L_1 & -L_2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & R_2 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The condition (6.34) is satisfied, since

$$\text{rank}\,[Es - A, B] = \text{rank} \begin{bmatrix} R_1 + sL_1 & -R_2 - sL_2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2 \quad \text{for all} \quad s \in \mathbb{C}.$$

For the gain matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ the closed-loop system matrix has the form

$$\begin{bmatrix} Es - (A + BK) \end{bmatrix} = \begin{bmatrix} R_1 + sL_1 & -R_2 - sL_2 \\ 1 - k_1 & 1 - k_2 \end{bmatrix}$$

and its determinant is equal to a real number $\gamma \neq 0$

$$\det \begin{bmatrix} Es - (A + BK) \end{bmatrix} = (R_1 + sL_1)(1 - k_2) + (R_2 + sL_2)\,(1 - k_1) = \gamma$$

for k_1, k_2 satisfying the equation

$$\begin{bmatrix} L_2 & L_1 \\ R_2 & R_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} L_1 + L_2 \\ (R_1 + R_2) - \gamma \end{bmatrix}. \tag{6.47}$$

The solution of (6.47) has the form

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} L_2 & L_1 \\ R_2 & R_1 \end{bmatrix}^{-1} \begin{bmatrix} L_1 + L_2 \\ (R_1 + R_2) - \gamma \end{bmatrix}. \tag{6.48}$$

Therefore, for the state feedbacks $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ with k_1 and k_2 given by (6.48) we have $i_1(t) = 0$, $i_2(t) = 0$ for $t > 0$.

Remark 6.6. For the electrical circuit shown in Figure 3.1 the condition (6.34) is not satisfied, since

$$\text{rank}\,[Es - A, B] = \text{rank} \begin{bmatrix} sRC_1 + 1 & 0 & 1 & 1 & 0 \\ sC_1 & sC_2 & -sC_3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = 2 \text{ for all } s \in \mathbb{C}. \tag{6.49}$$

6.3 Zeroing of the State Vector of Standard Circuits by Output-Feedbacks

Following [49], consider descriptor linear circuit described by the equations (3.1) and

$$y(t) = Cx(t) \tag{6.50}$$

where $y = y(t) \in \mathbb{R}^p$ is the output vector and $C \in \mathbb{R}^{p\times m}$. It is assumed that the pencil (E, A) of the circuit is regular, i.e. the condition (3.2) is met.

Without loss of generality of considerations we will assume, that $\text{rank}E = r < n$, $\text{rank}B = m$ and $\text{rank}C = p$.

To the electrical circuit the output feedback

$$u(t) = v(t) - Fy(t),$$

where $v = v(t) \in \mathbb{R}^m$ is a new input vector, $F \in \mathbb{R}^{m\times p}$ is a gain matrix, is applied and the equation of closed-loop circuit has the form

$$E\frac{\mathrm{d}x(t)}{\mathrm{d}t} = (A - BFC)\,x(t) + Bv(t).$$

We are looking for a output-feedback gain matrix $F \in \mathbb{R}^{m\times p}$, such that the state vector of the closed-loop circuit satisfies the condition

$$x(t) = 0 \quad \text{for} \quad t > 0$$

for any admissible initial conditions and any values of resistances, inductances and capacitances.

First we shall derive the necessary and sufficient conditions, under which the gain matrix exists and satisfies the condition

$$\det\left[Es - A + BFC\right] = \gamma, \tag{6.51}$$

where γ is a real number independent of s.

From the equality

$$\left[Es - A + BFC\right] = \left[Es - A \ B\right]\begin{bmatrix}\mathbb{I}_n \\ FC\end{bmatrix} = \left[\mathbb{I}_n \ BF\right]\begin{bmatrix}Es - A \\ C\end{bmatrix}$$

and (6.51) it follows that the problem has a solution only if

$$\text{rank}\left[Es - A \ B\right] = n \quad \text{for all finite} \quad s \in \mathbb{C} \tag{6.52a}$$

and

$$\begin{bmatrix}Es - A \\ C\end{bmatrix} = n \quad \text{for all finite} \quad s \in \mathbb{C}. \tag{6.52b}$$

The problem will be solved by the use of the following two steps procedure.

Step 1. (subproblem 1) Given E, A, B of the circuit (3.1) and a scalar γ. Find a gain matrix $K = FC$ such that

$$\det\left[Es - A + BK\right] = \gamma.$$

Step 2. (subproblem 2) Given C and K depending on some free parameters $k_1, k_2, \ldots$ (found in Step 1). Find desired F satisfying the equation

$$K = FC. \tag{6.53}$$

The solution to the subproblem 1 is derived in subsection 6.2.

Remark 6.7. Note that for $m > 1$ some entries of the matrix $\bar{K}$ in (6.41) can be chosen arbitrarily. Therefore, the matrix $K = \bar{K}V^{-1}$ has a number of free parameters denoted by $k_1, k_2, \dots$.

Taking into account Remark 6.7 we may solve the subproblem 2. The free parameters of the gain matrix K will be chosen, so that the equation (6.53) has a solution F for given C and K.

It is well-known that the equation (6.53) has a solution F if and only if

$$\text{rank}C = \text{rank}\begin{bmatrix} C \\ K \end{bmatrix} \tag{6.54a}$$

or equivalently

$$\text{Im}K^T \subset \text{Im}C^T. \tag{6.54b}$$

The free parameters $k_1, k_2, \dots$ are chosen so that (6.54) holds.

Therefore, the following theorem has been proved.

Theorem 6.6. *Let the conditions* (3.2), (6.52) *and* (6.37) *be satisfied. The problem has a solution, i.e. there exists the output-feedback gain matrix F satisfying* (6.51) *if and only if the free parameters $k_1, k_2, \dots$ of K can be chosen so that the equation* (6.53) *has a solution F for given C and K.*

From the condition (6.54) and (6.40) we have the following corollary.

Corollary 6.1. For $m = 1$ problem has a solution if and only if the row $[\bar{a}_{11}, \bar{a}_{12}, \dots, \bar{a}_{1,n_1-1}, \bar{a}_{1,n_1} - 1]$ is proportional to the matrix C.

Remark 6.8. If the order of system is not high (say $n \leq 5$) the elementary row and column operations instead of the orthogonal operations can be used.

Example 6.9. For the singular system (3.1) and (6.50) with

$$E = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} \tag{6.55}$$

find the gain matrix $F \in \mathbb{R}^{2\times 2}$ such that the condition (6.51) is satisfied for $\gamma = 1$.

In this case the pair (E, A) is regular, since

$$\det[Es - A] = \begin{vmatrix} -1 & 2s+1 & s & -1 \\ 0 & s-1 & -s-2 & 2s \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix}$$
$$=(1-s)^3 + 2(1-s)^2 + (s+2)(1-s) - 4s.$$

The matrices (6.55) have already the desired forms (6.36) with $A_0 = 0$, $B_0 = 0$, $E_1 = E$, $A_1 = A$, $B_1 = B$, $n_1 = n = 4$, $\bar{n}_1 = 2$, $\bar{n}_2 = \bar{n}_3 = 1$, $m = 2$ and

$$E_{11} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$
$$E_{22} = [1], \quad E_{23} = [-1], \quad E_{33} = [1],$$
$$A_{11} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad A_{22} = [1],$$
$$A_{23} = [-1], \quad A_{32} = [2], \quad A_{33} = [1], \quad B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using the elementary row operations (see Appendix B) we obtain

$$P_1 = \begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\left[\bar{E}_1 s - \bar{A}_1, \bar{B}_1\right] = P_1 [Es - A, B] = \begin{bmatrix} -1 & 0 & 5 & -5 & 1 & -2 \\ 0 & s & -1 & 2 & 0 & 1 \\ 0 & 1 & s-1 & 1-s & 0 & 0 \\ 0 & 0 & -2 & s-1 & 0 & 0 \end{bmatrix}.$$

Taking into account that in this case

$$\left[\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{13}\right] = \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$
$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & k_1 & k_2 & k_3 \end{bmatrix}$$

and using (6.41) we obtain

$$K = \bar{K} = \bar{B}_1^{-1}\left\{\left[\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{13}\right] + G\right\} = \begin{bmatrix} 2 & 2k_1 & 2k_2 - 3 & 1 + 2k_3 \\ 0.5 & k_1 & k_2 + 1 & k_3 - 2 \end{bmatrix},$$

where k_1, k_2, k_3 are free parameters.

The free parameters are chosen so that the condition

$$\text{rank}\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = \text{rank}\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \\ 2 & 2k_1 & 2k_2-3 & 1+2k_3 \\ 0.5 & k_1 & k_2+1 & k_3-2 \end{bmatrix} \tag{6.56}$$

is satisfied.

The condition (6.56) is satisfied for $k_1 = 1$, $k_2 = 2$, $k_3 = 0$ and the equation

$$F\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0.5 & 1 & 3 & -2 \end{bmatrix}$$

has the solution

$$F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is easy to check that

$$\det\left[Es - A + BK\right] = \det P_1^{-1}\left[\bar{E}s - \bar{A} + \bar{B}K\right] = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0.5 & s+1 & 2 & 0 \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix} = 1.$$

6.4 Zeroing of the State Vector of Fractional Electrical Circuits by State-Feedbacks

Consider descriptor linear circuit described by the equation (1.49a) with regular pencil satisfying (1.51). It is assumed, that $\text{rank}E = r < n$, $\text{rank}B = m$.

To the fractional electrical circuit the state-feedback

$$u = Kx,$$

where $K \in \mathbb{R}^{m\times n}$ is a gain matrix, is applied and the equation of closed-loop circuit has the form

$$E\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = (A + BK)\,x(t). \tag{6.57}$$

We are looking for a gain matrix K, such that state vector of the closed-loop circuit satisfies the condition

$$x(t) = 0 \text{ for } t > 0 \tag{6.58}$$

for any admissible initial conditions and any values of resistances, inductances and capacitances.

First, similarly as in [48], we shall derive the necessary and sufficient conditions, under which the gain matrix exists and satisfies the condition

$$\det\left[Es^{\alpha} - (A + BK)\right] = \gamma \neq 0, \tag{6.59}$$

where γ is a real number independent of s.

From the equality

$$\left[Es^{\alpha} - A - BK\right] = \left[Es^{\alpha} - A, B\right] \begin{bmatrix} \mathbb{I}_n \\ -K \end{bmatrix}$$

and (6.59) it follows that the problem has a solution only if

$$\operatorname{rank}\left[Es^{\alpha} - A, B\right] = n \quad \text{for all finite} \quad s \in \mathbb{C}. \tag{6.60}$$

It is well known [21, 44] that the system (1.49a) is completely controllable if and only if the conditions (6.60) and

$$\operatorname{rank}\left[E, B\right] = n$$

are met.

The solution of the problem is based on the following theorem.

Theorem 6.7. *There exists a gain matrix $K \in \mathbb{R}^{m\times n}$, such that* (6.58) *holds, where γ is a real number independent of s, if and only if the conditions* (6.37) *are satisfied.*

Proof. By Theorem 6.4 there exists K satisfying (6.59) if and only if the conditions (6.37) are met.

In this case, using the Laplace transform, from (6.57) we obtain

$$X(s) = \left[Es^{\alpha} - (A + BK)\right]^{-1} x_0$$

where $X(s) = \mathcal{L}\left[x(t)\right]$ is the Laplace transform of $x(t)$ and x_0 is the admissible initial condition.

Taking into account (6.59) we obtain

$$\begin{aligned} X(s) &= \frac{\operatorname{Adj}\left[Es^{\alpha} - (A + BK)\right]}{\det\left[Es^{\alpha} - (A + BK)\right]} x_0 = \frac{\operatorname{Adj}\left[Es^{\alpha} - (A + BK)\right]}{\gamma} x_0 \\ &= \left(P_0 + P_1 s^{\alpha} + \ldots + P_q s^{\alpha q}\right) x_0, \end{aligned} \tag{6.61}$$

where $\operatorname{Adj}\left[Es^{\alpha} - (A + BK)\right]$ denotes the adjoint matrix and $P_k \in \mathbb{R}^{n\times n}$ for $k = 0, 1, \ldots, q$.

Applying the inverse Laplace transform to (6.61) we obtain

$$x(t) = \sum_{k=0}^{q} P_k x_0 \delta^{(\alpha k)}(t) = 0 \quad \text{for} \quad t > 0,$$

since $\mathcal{L}\left[\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\delta(t)\right] = s^\alpha$, where $\delta(t)$ is the Dirac impulse and $\delta^{(\alpha k)}(t)$ is its αk-order fractional derivative. □

Example 6.10. Consider the fractional electrical circuit shown in Figure 6.6 with given resistance R, capacitances C_1, C_2 and source voltage $e = e(t)$.

Using Kirchhoff's laws for the electrical circuit we can write the equations

$$C_1\frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} - C_2\frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} = \frac{u_2}{R},$$
$$u_1 + u_2 = e,$$

which can be rewritten in the form

$$E\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be,$$

where

$$E = \begin{bmatrix} C_1 & -C_2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{1}{R} \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The condition (6.60) is satisfied, since

$$\operatorname{rank}\left[Es^\alpha - A, B\right] = \operatorname{rank}\begin{bmatrix} s^\alpha C_1 & -s^\alpha C_2 - \frac{1}{R} & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2 \quad \text{for all} \quad s \in \mathbb{C}.$$

For the gain matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ the closed-loop system matrix has the form

$$\left[Es^\alpha - (A + BK)\right] = \begin{bmatrix} s^\alpha C_1 & -s^\alpha C_2 - \frac{1}{R} \\ 1 - k_1 & 1 - k_2 \end{bmatrix}$$

and its determinant is equal to a real number $\Gamma \neq 0$

$$\det\left[Es^\alpha - (A + BK)\right] = s^\alpha\left[C_1(1 - k_2) + C_2(1 - k_1)\right] + \frac{1 - k_1}{R} = \gamma$$

for

$$k_1 \neq 1 \quad \text{and} \quad k_2 = \frac{C_1 + C_2(1 - k_1)}{C_1}. \tag{6.63}$$

Therefore, for the state feedback matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ with k_1 and k_2 defined by (6.63) we have $u_1(t) = 0$, $u_2(t) = 0$ for $t > 0$.

Example 6.11. Consider the fractional electrical circuit shown in Figure 6.7 with given resistances R_1, R_2; inductances L_1, L_2 and source current $i_s(t) = i_s$.

Using Kirchhoff's laws for the electrcial circuit we can write the equations

$$0 = R_1 i_1 + L_1 \frac{d^\alpha i_1}{dt^\alpha} - R_2 i_2 - L_2 \frac{d^\alpha i_2}{dt^\alpha},$$
$$i_s = i_1 + i_2,$$

which can be rewritten in the form

$$E \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B i_s,$$

where

$$E = \begin{bmatrix} L_1 & -L_2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & R_2 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The condition (6.60) is satisfied, since

$$\operatorname{rank}[Es^\alpha - A, B] = \operatorname{rank} \begin{bmatrix} R_1 + s^\alpha L_1 & -R_2 - s^\alpha L_2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2 \text{ for all } s \in \mathbb{C}.$$

For the gain matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ the closed-loop system matrix has the form

$$\begin{bmatrix} Es^\alpha - (A + BK) \end{bmatrix} = \begin{bmatrix} R_1 + s^\alpha L_1 & -R_2 - s^\alpha L_2 \\ 1 - k_1 & 1 - k_2 \end{bmatrix}$$

and its determinant is equal to a real number $\gamma \neq 0$

$$\det \begin{bmatrix} Es^\alpha - (A + BK) \end{bmatrix} = (R_1 + s^\alpha L_1)(1 - k_2) + (R_2 + s^\alpha L_2)(1 - k_1) = \gamma$$

for k_1, k_2 satisfying the equation

$$\begin{bmatrix} L_2 & L_1 \\ R_2 & R_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} L_1 + L_2 \\ (R_1 + R_2) - a \end{bmatrix}. \tag{6.65}$$

The solution of (6.65) has the form

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} L_2 & L_1 \\ R_2 & R_1 \end{bmatrix}^{-1} \begin{bmatrix} L_1 + L_2 \\ (R_1 + R_2) - a \end{bmatrix}. \tag{6.66}$$

Therefore, for the state feedbacks $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ with k_1 and k_2 given by (6.66), we have $i_1(t) = 0$, $i_2(t) = 0$ for $t > 0$.

Example 6.12. Consider the fractional electrical circuit shown in Figure 6.8 with given resistances R, R_1, R_2; inductances L_1, L_2; capacitances C_1, C_2; source voltage $e = e(t)$ and source current $i_s = i_s(t)$.

Using Kirchhoff's laws for the electrical circuit we can write the equations

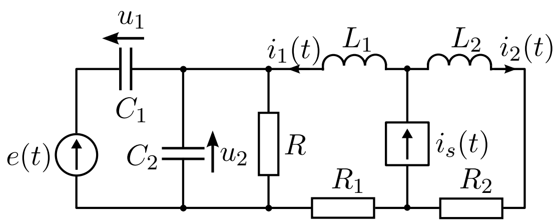

Fig. 6.8 Fractional electrical circuit of Example 6.12

$$C_1 \frac{d^\alpha u_1}{dt^\alpha} - C_2 \frac{d^\alpha u_2}{dt^\alpha} = \frac{u_2}{R} - i_1,$$
$$u_1 + u_2 = e,$$
$$R_1 i_1 + u_2 + L_1 \frac{d^\beta i_1}{dt^\beta} = R_2 i_2 + L_2 \frac{d^\beta i_2}{dt^\beta},$$
$$i_s = i_1 + i_2,$$

which can be written in the form

$$E \begin{bmatrix} \frac{d^\alpha u_1}{dt^\alpha} \\ \frac{d^\alpha u_2}{dt^\alpha} \\ \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e \\ i_s \end{bmatrix}, \quad \begin{matrix} 0 < \alpha < 1, \\ 0 < \beta < 1, \end{matrix}$$

where

$$E = \begin{bmatrix} C_1 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & L_1 & -L_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{1}{R} & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -R_1 & R_2 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The condition (1.51) is satisfied, since

$$\det \left\{ E \begin{bmatrix} \mathbb{I}_2 s^\alpha & 0 \\ 0 & \mathbb{I}_2 s^\beta \end{bmatrix} - A, B \right\}$$
$$= \begin{vmatrix} s^\alpha C_1 & -s^\alpha C_2 - \frac{1}{R} & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & s^\beta L_1 & -s^\beta L_2 - R_2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{vmatrix} = 4$$

for all $s \in \mathbb{C}$.

For the gain matrix

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} \tag{6.68}$$

the closed-loop system matrix has the form

$$\begin{aligned} &\left[E \begin{bmatrix} \mathbb{I}_2 s^\alpha & 0 \\ 0 & \mathbb{I}_2 s^\beta \end{bmatrix} - (A + BK) \right] \\ &= \begin{bmatrix} s^\alpha C_1 & -s^\alpha C_2 - \dfrac{1}{R} & 1 & 0 \\ 1 - k_{11} & 1 - k_{12} & -k_{13} & -k_{14} \\ 0 & 1 & s^\beta L_1 & -s^\beta L_2 - R_2 \\ -k_{21} & -k_{22} & 1 - k_{23} & 1 - k_{24} \end{bmatrix}. \end{aligned} \tag{6.69}$$

Assuming $k_{13} = k_{14} = k_{21} = k_{22} = 0$, we obtain the determinant of the matrix (6.69)

$$\begin{aligned} &\det \left[E \begin{bmatrix} \mathbb{I}_2 s^\alpha & 0 \\ 0 & \mathbb{I}_2 s^\beta \end{bmatrix} - (A + BK) \right] \\ &= \begin{vmatrix} s^\alpha C_1 & -s^\alpha C_2 - \dfrac{1}{R} & 1 & 0 \\ 1 - k_{11} & 1 - k_{12} & 0 & 0 \\ 0 & 1 & s^\beta L_1 & -s^\beta L_2 - R_2 \\ 0 & 0 & 1 - k_{23} & 1 - k_{24} \end{vmatrix} \\ &= \left\{ s^\alpha \left[C_1(1 - k_{12}) + C_2(1 - k_{11}) \right] + \frac{1}{R}(1 - k_{11}) \right\} \\ &\quad \times \left\{ s^\beta \left[L_1(1 - k_{24}) + L_2(1 - k_{23}) \right] + R_1(1 - k_{24}) + R_2(1 - k_{23}) \right\} \\ &\quad + (1 - k_{11})(1 - k_{24}) \\ &= a_1 a_2 + (1 - k_{11})(1 - k_{24}), \end{aligned}$$

where

$$\begin{aligned} &C_1(1 - k_{12}) + C_2(1 - k_{11}) = 0, \quad \frac{1}{R}(1 - k_{11}) = a_1, \\ &L_1(1 - k_{24}) + L_2(1 - k_{23}) = 0, \quad R_1(1 - k_{24}) + R_2(1 - k_{23}) = a_2. \end{aligned} \tag{6.70}$$

From (6.70) we have $k_{11} = 1 - a_1 R$ and

$$\begin{bmatrix} C_1 & 0 & 0 \\ 0 & L_2 & L_1 \\ 0 & R_2 & R_1 \end{bmatrix} \begin{bmatrix} k_{12} \\ k_{23} \\ k_{24} \end{bmatrix} = \begin{bmatrix} C_1 + RC_2 a_1 \\ L_1 + L_2 \\ R_1 + R_2 - a_2 \end{bmatrix}. \tag{6.71}$$

The equation (6.71) has the solution

$$\begin{bmatrix} k_{12} \\ k_{23} \\ k_{24} \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & L_2 & L_1 \\ 0 & R_2 & R_1 \end{bmatrix}^{-1} \begin{bmatrix} C_1 + RC_2 a_1 \\ L_1 + L_2 \\ R_1 + R_2 - a_2 \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{RC_2 a_1}{C_1} \\ 1 + \dfrac{L_1 a_2}{R_1 L_2 - R_2 L_1} \\ 1 - \dfrac{L_2 a_2}{R_1 L_2 - R_2 L_1} \end{bmatrix} \tag{6.72}$$

if

$$\det \begin{bmatrix} C_1 & 0 & 0 \\ 0 & L_2 & L_1 \\ 0 & R_2 & R_1 \end{bmatrix} = C_1(R_1 L_2 - R_2 L_1) \neq 0.$$

Therefore, for the gain matrix (6.68) with $k_{13} = k_{14} = k_{21} = k_{22} = 0$, $k_{11} = 1 - a_1 R$ and k_{12}, k_{23}, k_{24} given by (6.72), we have $u_1(t) = 0$, $u_2(t) = 0$, $i_1(t) = 0$, $i_2(t) = 0$ for $t > 0$.

Remark 6.9. For the fractional electrical circuit shown in Figure 3.1 the condition (1.51) is not satisfied, since

$$\operatorname{rank}[Es^\alpha - A, B] = \operatorname{rank} \begin{bmatrix} s^\alpha RC_1 + 1 & 0 & 1 & 1 & 0 \\ s^\alpha C_1 & s^\alpha C_2 & -s^\alpha C_3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = 2$$

for all $s \in \mathbb{C}$.

Chapter 7
Minimum Energy Control of Electrical Circuits

The minimum energy control problem for standard linear systems was formulated and solved by J. Klamka in [117, 118, 119, 120, 121, 122, 123]. The Klamka's method has been extended to new classes of linear systems [12, 13, 71, 82, 83, 98, 84, 85, 86, 101, 94, 96, 97, 99, 100, 124]

7.1 Minimum Energy Control of Positive Standard Electrical Circuits

Consider the positive electrical circuit described by the equations (4.1) with diagonal $A \in M_n$ and monomial $B \in \mathbb{R}_+^{n\times n}$. If the system is reachable in time $t \in [0, t_f]$, then usually there exists many different inputs $u(t) \in \mathbb{R}_+^n$ that steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Among these inputs we are looking for an input that minimizes the performance index

$$I(u) = \int_0^{t_f} u^T(\tau) Q u(\tau) \mathrm{d}\tau, \tag{7.1}$$

where $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive defined matrix and $Q^{-1} \in \mathbb{R}_+^{n\times n}$.

The minimum energy problem for the positive electrical circuit (4.1) can be stated as follows [101]. Given the reachable matrices $A \in M_n$, $B \in \mathbb{R}_+^{n\times n}$ and $Q \in \mathbb{R}_+^{n\times n}$ of the performance index (7.1), $x_f \in \mathbb{R}_+^n$ and $t_f > 0$, find an input $u(t) \in \mathbb{R}_+^n$ for $t \in [0, t_f]$ that steers the state vector of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (7.1).

To solve the problem we define the matrix

$$W = W(t_f, Q) = \int_0^{t_f} e^{A(t_f-\tau)} B Q^{-1} B^T e^{A^T(t_f-\tau)} \mathrm{d}\tau. \tag{7.2}$$

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Studies in Systems, Decision and Control 13, DOI: 10.1007/978-3-319-11361-6_7

From (7.2) and Theorem 5.7 it follows that the matrix (7.2) is monomial if and only if the positive electrical circuit (4.1) is reachable in time $[0, t_f]$.

In this case we may define the input

$$\hat{u}(t) = Q^{-1}B^T e^{A^T(t_f - t)}W^{-1}x_f \quad \text{for} \quad t \in [0, t_f]. \tag{7.3}$$

Note that the input (7.3) satisfies the condition $u(t) \in \mathbb{R}_+^n$ for $t \in [0, t_f]$ if

$$Q^{-1} \in \mathbb{R}_+^{n\times n} \quad \text{and} \quad W^{-1}x_f \in \mathbb{R}_+^n. \tag{7.4}$$

Theorem 7.1. *Let the positive electrical circuit* (4.1) *be reachable in time* $[0, t_f]$ *and let* $\bar{u}(t) \in \mathbb{R}_+^n$ *for* $t \in [0, t_f]$ *be an input that steers the state of the positive electrical circuit* (4.1) *from* $x_0 = 0$ *to* $x_f \in \mathbb{R}_+^n$. *Then the input* (7.3) *also steers the state of the system from* $x_0 = 0$ *to* $x_f \in \mathbb{R}_+^n$ *and minimizes the performance index* (7.1), *i.e.* $I(\hat{u}) \le I(\bar{u})$.

The minimal value of the performance index (7.1) *is equal to*

$$I(\hat{u}) = x_f^T W^{-1} x_f. \tag{7.5}$$

Proof. If the conditions (7.4) are met, then the input (7.3) is well defined and $\hat{u} \in \mathbb{R}_+^n$ for $t \in [0, t_f]$. We shall show that the input steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$.

Substitution of (7.3) into (5.24) for $t = t_f$ and $x_0 = 0$ yields

$$x(t_f) = \int_0^{t_f} e^{A(t_f-\tau)}B\hat{u}(\tau)\mathrm{d}\tau = \int_0^{t_f} e^{A(t_f-\tau)}BQ^{-1}B^T e^{A^T(t_f-\tau)}\mathrm{d}\tau W^{-1}x_f = x_f,$$

since (7.2) holds.

By the assumption, the inputs $\bar{u}(t)$ and $\hat{u}(t)$, $t \in [0, t_f]$ steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Hence

$$x_f = \int_0^{t_f} e^{A(t_f-\tau)}B\bar{u}(\tau)\mathrm{d}\tau = \int_0^{t_f} e^{A(t_f-\tau)}B\hat{u}(\tau)\mathrm{d}\tau \tag{7.6a}$$

or

$$\int_0^{t_f} e^{A(t_f-\tau)}B\left[\bar{u}(\tau) - \hat{u}(\tau)\right]\mathrm{d}\tau = 0. \tag{7.6b}$$

By transposition of (7.6b) and postmultiplication by $W^{-1}x_f$ we obtain

$$\int_0^{t_f} \left[\bar{u}(\tau) - \hat{u}(\tau)\right]^T B^T e^{A^T(t_f-\tau)}\mathrm{d}\tau W^{-1}x_f = 0. \tag{7.7}$$

Substitution of (7.3) into (7.7) yields

$$\int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T B^T e^{A^T(t_f-\tau)} \mathrm{d}\tau W^{-1} x_f = \int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T Q\hat{u}(\tau)\mathrm{d}\tau = 0. \tag{7.8}$$

Using (7.8) it is easy to verify that

$$\int_0^{t_f} \bar{u}(\tau)^T Q\bar{u}(\tau)\mathrm{d}\tau = \int_0^{t_f} \hat{u}(\tau)^T Q\hat{u}(\tau)\mathrm{d}\tau + \int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T Q[\bar{u}(\tau) - \hat{u}(\tau)]\mathrm{d}\tau. \tag{7.9}$$

From (7.9) it follows that $I(\hat{u}) < I(\bar{u})$, since the second term in the right-hand side of the inequality is nonnegative.

To find the minimal value of the performance index (7.1) we substitute (7.3) into (7.1) and we obtain

$$\begin{aligned} I(\hat{u}) &= \int_0^{t_f} \hat{u}^T(\tau) Q\hat{u}(\tau)\mathrm{d}\tau = x_f^T W^{-1} \int_0^{t_f} e^{A(t_f-\tau)} BQ^{-1} B^T e^{A^T(t_f-\tau)} \mathrm{d}\tau W^{-1} x_f \\ &= x_f^T W^{-1} x_f, \end{aligned}$$

since (7.2) holds. □

From the above considerations we have the following procedure for computation of the optimal inputs that steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (7.1).

Procedure 7.1

Step 1. Knowing $A \in M_n$ compute e^{At}.

Step 2. Using (7.2) compute the matrix W knowing the matrices A, B, Q for given t_f.

Step 3. Using (7.3) compute the optimal input $\hat{u}(t)$.

Step 4. Using (7.5) compute the minimal value of the performance index.

Example 7.1. (continuation of Example 5.3) Consider the positive reachable electrical circuit shown in Figure 5.3 for $R_3 = 0$. Compute the optimal input $u(t) = \left[e_1(t)\ e_2(t)\right]^T$ of the electrical circuit that steers its state vector from $x_0 \left[0\ 0\right]^T$ to the final state $x_f = \left[x_{f1}\ x_{f2}\right]^T$ and minimizes the performance index (7.1) for

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \in \mathbb{R}_+^{2\times 2},\ q_k > 0,\ k = 1, 2. \tag{7.10}$$

Using the Procedure 7.1 we obtain the following.

Step 1. The matrix e^{At} for (5.32) and $R_3 = 0$ is given by (5.33).

Step 2. Using (7.2), (5.33) and (7.10) we obtain

$$\begin{aligned} W &= \int_0^{t_f} e^{A(t_f-\tau)}BQ^{-1}B^T e^{A^T(t_f-\tau)}\mathrm{d}\tau = \int_0^{t_f} e^{A\tau}BQ^{-1}B^T e^{A^T\tau}\mathrm{d}\tau \\ &= \int_0^{t_f} \begin{bmatrix} \frac{1}{q_1 L_1^2} e^{-\frac{2R_1}{L_1}\tau} & 0 \\ 0 & \frac{1}{q_2 L_2^2} e^{-\frac{2R_2}{L_2}\tau} \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} \frac{1}{2q_1 R_1 L_1}\left(1 - e^{-\frac{2R_1}{L_1}t_f}\right) & 0 \\ 0 & \frac{1}{2q_2 R_2 L_2}\left(1 - e^{-\frac{2R_2}{L_2}t_f}\right) \end{bmatrix}. \end{aligned} \tag{7.11}$$

Note that the conditions (7.4) are satisfied for all values of R_1, R_2, L_1, L_2 and $t_f > 0$.

Step 3. The optimal input can be computed using (7.3) and (7.11). We have

$$\hat{u}(t) = Q^{-1}B^T e^{A^T(t_f-t)}W^{-1}x_f = \begin{bmatrix} \frac{2R_1 e^{-\frac{R_1}{L_1}(t_f-t)}}{\left(1 - e^{-\frac{2R_1}{L_1}t_f}\right)} x_{f1} \\ \frac{2R_2 e^{-\frac{R_2}{L_2}(t_f-t)}}{\left(1 - e^{-\frac{2R_2}{L_2}t_f}\right)} x_{f2} \end{bmatrix}$$

Step 4. Using (7.5) and (7.11) we obtain the value of the performance index

$$\begin{aligned} I(\hat{u}) &= x_f^T W^{-1} x_f = \begin{bmatrix} x_{f1} & x_{f2} \end{bmatrix} \begin{bmatrix} \frac{2q_1 R_1 L_1}{\left(1 - e^{-\frac{2R_1}{L_1}t_f}\right)} & 0 \\ 0 & \frac{2q_2 R_2 L_2}{\left(1 - e^{-\frac{2R_2}{L_2}t_f}\right)} \end{bmatrix} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} \\ &= \frac{2q_1 R_1 L_1}{\left(1 - e^{-\frac{2R_1}{L_1}t_f}\right)} x_{f1}^2 + \frac{2q_2 R_2 L_2}{\left(1 - e^{-\frac{2R_2}{L_2}t_f}\right)} x_{f2}^2. \end{aligned}$$

Example 7.2. (continuation of Example 5.4) Consider the positive reachable electrical circuit shown in Figure 5.4 for $R = 0$. Compute the optimal input $u(t) = \left[e_1(t)\ e_2(t)\right]^T$ of the electrical circuit that steers its state vector from $x_0 \left[0\ 0\right]^T$ to the final state $x_f = \left[x_{f1}\ x_{f2}\right]^T$ and minimizes the performance index (7.1) for (7.10).

Using the Procedure 7.1 we obtain the following.

Step 1. The matrix e^{At} for (5.37) is given by (5.38).

Step 2. Using (7.2), (5.38) and (7.10) we obtain

$$\begin{aligned} W &= \int_0^{t_f} e^{A(t_f-\tau)} BQ^{-1}B^T e^{A^T(t_f-\tau)} \mathrm{d}\tau = \int_0^{t_f} e^{A\tau} BQ^{-1}B^T e^{A^T\tau} \mathrm{d}\tau \\ &= \int_0^{t_f} \begin{bmatrix} \frac{1}{q_1(R_1C)^2} e^{-\frac{2}{R_1C}\tau} & 0 \\ 0 & \frac{1}{q_2L^2} e^{-\frac{2R_2}{L}\tau} \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} \frac{1}{2q_1R_1C}\left(1 - e^{-\frac{2}{R_1C}t_f}\right) & 0 \\ 0 & \frac{1}{2q_2R_2L}\left(1 - e^{-\frac{2R_2}{L}t_f}\right) \end{bmatrix}. \end{aligned} \tag{7.12}$$

Step 3. The optimal input can be computed using (7.3) and (7.12). We have

$$\hat{u}(t) = Q^{-1}B^T e^{A^T(t_f-t)} W^{-1} x_f = \begin{bmatrix} \frac{2e^{-\frac{1}{R_1C}(t_f-t)}}{\left(1 - e^{-\frac{2}{R_1C}t_f}\right)} x_{f1} \\ \frac{2R_2 e^{-\frac{R_2}{L}(t_f-t)}}{\left(1 - e^{-\frac{2R_2}{L}t_f}\right)} x_{f2} \end{bmatrix}$$

Step 4. Using (7.5) and (7.12) we obtain the value of the performance index

$$\begin{aligned} I(\hat{u}) &= x_f^T W^{-1} x_f = \left[x_{f1}\ x_{f2}\right] \begin{bmatrix} \frac{2q_1R_1C}{\left(1 - e^{-\frac{2}{R_1C}t_f}\right)} & 0 \\ 0 & \frac{2q_2R_2L}{\left(1 - e^{-\frac{2R_2}{L}t_f}\right)} \end{bmatrix} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} \\ &= \frac{2q_1R_1C}{\left(1 - e^{-\frac{2}{R_1C}t_f}\right)} x_{f1}^2 + \frac{2q_2R_2L}{\left(1 - e^{-\frac{2R_2}{L}t_f}\right)} x_{f2}^2. \end{aligned}$$

7.2 Minimum Energy Control of Fractional Positive Electrical Circuits

Consider the fractional positive electrical circuit (1.11a) with diagonal $A \in M_n$ and monomial $B \in \mathbb{R}$. If the circuit is reachable in time $t \in [0, t_f]$, then usually there exists many different inputs $u(t) \in \mathbb{R}_+^n$, that steers the state of the system form $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Among these inputs we are looking for an input $u(t) \in \mathbb{R}_+^n$, $t \in [0, t_f]$, that minimizes the performance index (7.1).

The minimum energy control problem for the fractional positive electrical circuit (1.11a) can be stated as follows [98]. Given the reachable matrices $A \in M_n$, $B \in \mathbb{R}_+^{n\times n}$, the fractional order α and $Q \in \mathbb{R}_+^{n\times n}$ of the performance index (7.1), $x_f \in \mathbb{R}_+^n$ and $t_f > 0$, find an input $u(t) \in \mathbb{R}_+^n$ for $t \in [0, t_f]$, that steers the state vector of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (7.1).

To solve the problem we define the matrix

$$W(t_f) = \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T (t_f - \tau) \mathrm{d}\tau, \tag{7.13}$$

where $\Phi(t)$ is defined by (1.14).

From (7.13) and Theorem 5.21 it follows that the matrix (7.13) is monomial if and only if the fractional positive electrical circuit (1.11) is reachable in time $[0, t_f]$.

In this case we may define the input

$$\hat{u}(t) = Q^{-1} B^T \Phi^T (t_f - t) W^{-1}(t_f) x_f \quad \text{for} \quad t \in [0, t_f]. \tag{7.14}$$

Note that the input (7.14) satisfies the condition $u(t) \in \mathbb{R}_+^n$ for $t \in [0, t_f]$ if

$$Q^{-1} \in \mathbb{R}_+^{n\times n} \quad \text{and} \quad W^{-1}(t_f) \in \mathbb{R}_+^{n\times n}. \tag{7.15}$$

Theorem 7.2. *Let $\bar{u} \in \mathbb{R}_+^n$ for $t \in [0, t_f]$ be an input that steers the state of the fractional positive electrical circuit* (1.11) *from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Then the input* (7.14) *also steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index* (7.1)*, i.e. $I(\hat{u}) \le I(\bar{u})$.*

The minimal value of the performance index (7.1) *is equal to*

$$I(\hat{u}) = x_f^T W^{-1}(t_f) x_f. \tag{7.16}$$

Proof. If the conditions (7.15) are met then the input (7.14) is well defined and $\hat{u}(t) \in \mathbb{R}_+^n$ for $t \in [0, t_f]$.

We shall show that the input steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Substitution of (7.14) into (1.12) for $t = t_f$ and $x_0 = 0$ yields

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau)B\hat{u}(\tau)\mathrm{d}\tau$$
$$= \int_0^{t_f} \Phi(t_f - \tau)BQ^{-1}B^T\Phi^T(t_f - \tau)\mathrm{d}\tau W^{-1}(t_f)x_f = x_f,$$

since (7.13) holds.

By assumption the inputs $\bar{u}(t)$ and $\hat{u}(t)$ for $t \in [0, t_f]$ steers the state of the system form $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$, i.e.

$$x_f = \int_0^{t_f} \Phi(t_f - \tau)B\bar{u}(\tau)\mathrm{d}\tau = \int_0^{t_f} \Phi(t_f - \tau)B\hat{u}(\tau)\mathrm{d}\tau$$

and

$$\int_0^{t_f} \Phi(t_f - \tau)B\left[\bar{u}(\tau) - \hat{u}(\tau)\right]\mathrm{d}\tau = 0. \tag{7.17}$$

By transposition of (7.17) and postmultiplication by $W^{-1}(t_f)x_f$ we obtain

$$\int_0^{t_f} \left[\bar{u}(\tau) - \hat{u}(\tau)\right]^T B^T\Phi^T(t_f - \tau)\mathrm{d}\tau W^{-1}(t_f)x_f = 0. \tag{7.18}$$

Substitution (7.14) into (7.18) yields

$$\int_0^{t_f} \left[\bar{u}(\tau) - \hat{u}(\tau)\right]^T B^T\Phi^T(t_f - \tau)\mathrm{d}\tau W^{-1}(t_f)x_f = \int_0^{t_f} \left[\bar{u}(\tau) - \hat{u}(\tau)\right]^T Q\hat{u}(\tau)\mathrm{d}\tau = 0. \tag{7.19}$$

Using (7.19) it is easy to verify that

$$\int_0^{t_f} \bar{u}^T(\tau)Q\bar{u}(\tau)\mathrm{d}\tau = \int_0^{t_f} \hat{u}^T(\tau)Q\hat{u}(\tau)\mathrm{d}\tau + \int_0^{t_f} \left[\bar{u}(\tau) - \hat{u}(\tau)\right]^T Q\left[\bar{u}(\tau) - \hat{u}(\tau)\right]\mathrm{d}\tau. \tag{7.20}$$

From (7.20) it follows that $I(\hat{u}) \leq I(\bar{u})$, since the second term in the right-hand side of the inequality is nonnegative.

To find the minimal value of the performance index (7.1) we substitute (7.14) into (7.1) and we obtain

$$I(\hat{u}) = \int_0^{t_f} \hat{u}^T(\tau)Q\hat{u}(\tau)\mathrm{d}\tau = x_f^T W^{-1}(t_f) \int_0^{t_f} \Phi(t_f-\tau)B\Phi^T(t_f-\tau)\mathrm{d}\tau W^{-1}(t_f)x_f$$
$$= x_f^T W^{-1}(t_f)x_f,$$

since (7.13) holds. □

From the above considerations we have the following procedure for computation of the optimal inputs, that steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (7.1).

Procedure 7.2

Step 1. Knowing $A \in M_n$ and using (1.14) compute $\Phi(t)$.

Step 2. Using (7.13) compute the matrix $W(t_f)$ knowing the matrices A, B, Q, fractional order α and some t_f.

Step 3. Using (7.14) obtain the desired optimal input $\hat{u}(t)$ for given $x_f \in \mathbb{R}_+^n$.

Step 4. Using (7.16) compute the minimal value of the performance index.

Example 7.3. (continuation of Example 5.13) The minimum energy control problem of the fractional positive reachable electrical circuit shown in Figure 5.3, where $R_3 = 0$ and described by the state equation (5.78) with matrices (5.79), (5.80) can be stated as follows. Compute the input $\hat{u}(t) \in \mathbb{R}_+^2$ that steers the state of the electrical circuit from zero initial state to $x_f = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and minimizes the performance index (7.1) with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{7.21}$$

Using the Procedure 7.2 we obtain the following.

Step 1. The matrix $\Phi(t)$ has the form (5.81).

Step 2. Using (7.13), (5.81), (5.80) and (7.21) we get

$$W(t_f) = \int_0^{t_f} \Phi(t_f - \tau)BQ^{-1}B^T\Phi^T(t_f - \tau)\mathrm{d}\tau = \int_0^{t_f} \Phi(\tau)BQ^{-1}B^T\Phi^T(\tau)\mathrm{d}\tau$$
$$= \frac{1}{2}\int_0^{t_f} \begin{bmatrix} \frac{1}{L_1^2} & 0 \\ 0 & \frac{1}{L_2^2} \end{bmatrix} \Phi^2(\tau)\mathrm{d}\tau = \frac{1}{2}\begin{bmatrix} \frac{1}{L_1^2} & 0 \\ 0 & \frac{1}{L_2^2} \end{bmatrix}$$
$$\times \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{t_f^{(k+l+2)\beta-1}}{[(k+l+2)\beta-1]\Gamma[(k+1)\beta]\Gamma[(l+1)\beta]} \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}^{k+l}.$$

Step 3. Using (7.14), (7.21), (5.81) and (7.3) we may compute

$$\hat{u}(t) = Q^{-1}B^T\Phi^T(t_f - t)W^{-1}(t_f)x_f$$
$$= \frac{1}{2}\begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \Phi^T(t_f - t)W^{-1}(t_f)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 4. From (7.16) and (7.3) we have the minimal value of the performance index

$$I(\hat{u}) = x_f^T W^{-1}(t_f)x_f = \begin{bmatrix} 1 & 1 \end{bmatrix} W^{-1}(t_f)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

7.3 Minimum Energy Control of Fractional Positive Electrical Circuits with Bounded Inputs

Consider the fractional positive electrical circuit (1.11a) with diagonal $A \in M_n$ and monomial $B \in \mathbb{R}$. If the circuit is reachable in time $t \in [0, t_f]$, then usually there exists many different inputs $u(t) \in \mathbb{R}_+^n$, that steers the state of the system form $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Among these inputs we are looking for an input $u(t) \in \mathbb{R}_+^n$, $t \in [0, t_f]$ satisfying the condition

$$u(t) \le U \in \mathbb{R}_+^n, \quad \text{for} \quad t \in [0, t_f] \tag{7.22}$$

that minimizes the performance index (7.1).

The minimum energy control problem for the fractional positive electrical circuit (1.11a) with bounded inputs can be stated as follows [97]. Given the reachable matrices $A \in M_n$, $B \in \mathbb{R}_+^{n\times n}$, the fractional order α, $U \in \mathbb{R}_+^{n\times n}$ and $Q \in \mathbb{R}_+^{n\times n}$ of the performance index (7.1), $x_f \in \mathbb{R}_+^n$ and $t_f > 0$, find an input $u(t) \in \mathbb{R}_+^n$ for $t \in [0, t_f]$, that steers the state vector of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (7.1).

Theorem 7.3. *Let $\bar{u} \in \mathbb{R}_+^n$ for $t \in [0, t_f]$ be an input satisfying* (7.22)*, that steers the state of the fractional positive electrical circuit* (1.11) *from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Then the input* (7.14) *satisfying* (7.22) *also steers the state of the fractional electrical circuit from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index* (7.1)*, i.e. $I(\hat{u}) \le I(\bar{u})$.*

The minimal value of the performance index (7.1) *is given by* (7.16).

The proof of this theorem is similar to the proof of Theorem 7.2.

To find t_f for which the fractional electrical circuit (1.11a) reaches desired final state $x_f = x(t_f) \in \mathbb{R}_+^n$ with the input $\hat{u}(t)$ for $t \in [0, t_f]$ satisfying the condition (7.22) and minimizing the performance index (7.1), we compute the dervative of (7.14)

$$\frac{\mathrm{d}\hat{u}(t)}{\mathrm{d}t} = Q^{-1}B^T\Psi(t)W^{-1}(t_f)x_f \quad \text{for} \quad t \in [0, t_f],$$

where

$$\Psi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi^T(t_f - t).$$

Knowing $\Psi(t)$ and using the equality

$$\Psi(t)W^{-1}(t_f)x_f = 0 \tag{7.23}$$

we can find $t \in [0, t_f]$, for which $\hat{u}(t)$ reaches its maximal value.

Note that if the fractional electrical circuit is asymptotically stable, then $\lim_{t\to\infty} \Phi(t) = 0$ and $\hat{u}(t)$ reaches its maximal value for $t = t_f$ and if it is unstable then for $t = 0$.

From the above considerations we have the following procedure for computation of the optimal inputs satisfying (7.22), that steers the state of the fractional electrical circuit from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (7.1).

Procedure 7.3

Step 1. Knowing $A \in M_n$ and using (1.14) compute $\Phi(t)$.

Step 2. Using (7.13) compute the matrix $W(t_f)$ for given A, B, Q, α and some t_f.

Step 3. Using (7.14) and (7.23) find t_f, for which $\hat{u}(t)$ satisfying (7.22) reaches its maximal value and the desired $\hat{u}(t)$ for given $U \in \mathbb{R}_+^n$ and $x_f \in \mathbb{R}_+^n$.

Step 4. Using (7.16) compute the minimal value of the performance index.

Example 7.4. Consider the fractional positive reachable electrical circuit shown in Figure 5.3 for $R_1 = R_2 = 1$; $R_3 = 0$, $L_1 = L_2 = 1$ and $\alpha = 0.5$.

Compute the input $\hat{u}(t) \in \mathbb{R}_+^2$ satisfying the condition

$$\hat{u}(t) = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for} \quad t \in [0, t_f],$$

that steers the state of the fractional electrical circuit from zero initial state to the final state $x_f = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and minimizes the performance index (7.1) with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{7.24}$$

Using the Procedure 7.3 we obtain the following.

Step 1. The matrices of the fractional electrical circuit are given by (5.31), (5.32) and we have

$$A = -\mathbb{I}_2, \quad B = \mathbb{I}_2.$$

Knowing $A \in M_n$ and using (1.14) we obtain

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{(k-1)0.5}}{\Gamma[(k+1)0.5]}. \tag{7.25}$$

Step 2. Using (7.13), (7.24), (7.25) and taking into account that $B = \mathbb{I}_2$ we get

$$\begin{aligned} W(t_f) &= \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^T \Phi^T (t_f - \tau) \mathrm{d}\tau \\ &= \int_0^{t_f} \Phi(\tau) B Q^{-1} B^T \Phi^T(\tau) \mathrm{d}\tau = \frac{1}{2} \int_0^{t_f} \Phi^2(\tau) \mathrm{d}\tau \\ &= \frac{1}{2} \int_0^{t_f} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{k+l} \frac{\tau^{(k+l)0.5-1}}{\Gamma[(k+1)0.5]\Gamma[(l+1)0.5]} \mathrm{d}\tau \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{k+l} \frac{t_f^{(k+l)0.5}}{\Gamma[(k+1)0.5]\Gamma[(l+1)0.5](k+l)}. \end{aligned} \tag{7.26}$$

Step 3. Note that the fractional electrical circuit is asymptotically stable. Therefore, the input $\hat{u}(t)$ reaches its maximal value for $t = t_f$.

Using (7.14), (7.24), (7.25) and (7.26) we obtain

$$\hat{u}(t) = Q^{-1}B^T\Phi^T(t_f - t)W^{-1}(t_f)x_f = \frac{1}{2}\left(\sum_{k=0}^{\infty}(-1)^k\frac{(t_f - t)^{(k-1)0.5}}{\Gamma[(k+1)0.5]}\right)$$
$$\times\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{t_f^{(k+l)0.5}}{\Gamma[(k+1)0.5]\Gamma[(l+1)0.5](k+l)}\right)^{-1}\begin{bmatrix}1\\1\end{bmatrix}.$$

Step 4. Using (7.16) and (7.26) we may compute minimal value of the performance index (7.1)

$$I(\hat{u}) = x_f^T W^{-1}(t_f)x_f = \begin{bmatrix}1 & 1\end{bmatrix} W^{-1}(t_f)\begin{bmatrix}1\\1\end{bmatrix}.$$

Chapter 8
Fractional 2D Linear Systems Described by the Standard and Descriptor Roesser Model with Applications

8.1 Fractional Derivatives and Integrals of 2D Functions

We define the fractional-order partial derivative of a 2D continuous function $f(t_1, t_2)$ of two independent real variables $t_1, t_2 > 0$.

Definition 8.1. The α_i-order partial derivative of a 2D continuous function $f(t_1, t_2)$ with respect to t_i is given by the formula

$$ {}^C D_{t_i}^{\alpha_i} f(t_1, t_2) = \frac{\partial^{\alpha_i}}{\partial t_i^{\alpha_i}} f(t_1, t_2) = \frac{1}{\Gamma(N_i - \alpha_i)} \int_0^{t_i} \frac{f_{t_i}^{(N_i)}(\tau)}{(t_i - \tau)^{\alpha_i + 1 - N_i}} \mathrm{d}\tau, \tag{8.1} $$

where $N_i - 1 \le \alpha_i < N_i$, $N_i \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_+$ is the order of fractional partial derivative for $i = 1, 2$, $\Gamma(x)$ is the Euler gamma function defined by Definition 1.1 and

$$ f_{t_i}^{(N_i)}(\tau) = \begin{cases} \dfrac{\partial^{N_1} f(\tau, t_2)}{\partial \tau^{N_1}} & \text{for } i = 1 \\ \dfrac{\partial^{N_2} f(t_1, \tau)}{\partial \tau^{N_2}} & \text{for } i = 2. \end{cases} $$

Definition 8.2. The β_i-order fractional integral of the two-dimensional function $f(t_1, t_2)$ with respect to t_i is defined by

$$ I_{t_i}^{\beta_i} f(t_1, t_2) = \frac{1}{\Gamma(\beta_i)} \int_0^{t_i} (t_i - \tau)^{\beta_i - 1} f_{t_i}(\tau) \mathrm{d}\tau, $$

where $\beta_i > 0$ is the order of integration for $i = 1, 2$ and

$$ f_{t_i}(\tau) = \begin{cases} f(\tau, t_2) & \text{for } i = 1 \\ f(t_1, \tau) & \text{for } i = 2 \end{cases} $$

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Studies in Systems, Decision and Control 13, DOI: 10.1007/978-3-319-11361-6_8

Definition 8.3. An operator defined by

$$D_{t_i}^{\alpha_i} f(t_1,t_2) = \begin{cases} {}^C D_{t_i}^{\alpha_i} f(t_1,t_2) \text{ for } \alpha_i \in \mathbb{R}_+ \\ I_{t_i}^{-\alpha_i} f(t_1,t_2) \text{ for } \alpha_i < 0 \end{cases} \tag{8.2}$$

where $i = 1, 2$ is called a α_i-order ($\alpha_i \in \mathbb{R}$) fractional differintegral of the 2D function $f(t_1, t_2)$ with respect to t_i.

8.2 Descriptor Fractional 2D Roesser Model and Its Solution

Consider the singular (descriptor) fractional-order 2D continuous system described by the state equations [171]

$$E \begin{bmatrix} {}^C D_{t_1}^{\alpha_1} x^h(t_1,t_2) \\ {}^C D_{t_2}^{\alpha_2} x^v(t_1,t_2) \end{bmatrix} = A \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + Bu(t_1,t_2), \tag{8.3a}$$

$$y(t_1,t_2) = C \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + Du(t_1,t_2), \tag{8.3b}$$

where $N_i - 1 < \alpha_i < N_i$, $N_i \in \mathbb{N}$ for $i = 1, 2$; $x^h(t_1, t_2) \in \mathbb{R}^{n_1}$, $x^v(t_1, t_2) \in \mathbb{R}^{n_2}$, $(n = n_1 + n_2)$ are the horizontal and vertical state vectors, respectively, $u(t_1, t_2) \in \mathbb{R}^m$ is the input vector, $y(t_1, t_2) \in \mathbb{R}^p$ is the output vector and the matrices $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

We assume, that in general case, the matrix E of the system (8.3) is singular matrix, i.e. $\det E = 0$.

The boundary conditions for (8.3) are given in the form

$$x^{h(k)}(0,t_2) = \left[\frac{\partial^k x^h(t_1,t_2)}{\partial t_1^k}\right]_{t_1=0} = x_0^{h(k)}(t_2) \tag{8.4a}$$

for $k = 0, 1, \ldots N_1 - 1;\ t_2 \geq 0$,

$$x^{v(l)}(t_1,0) = \left[\frac{\partial^l x^v(t_1,t_2)}{\partial t_2^k}\right]_{t_2=0} = x_0^{v(l)}(t_1), \tag{8.4b}$$

where $l = 0, 1, 2, \ldots, N_2 - 1;\ t_1 \geq 0$ and $x_0^{h(k)}(t_2), x_0^{v(l)}(t_1)$ are given functions.

Theorem 8.1. *[171] The solution to the equation* (8.3a) *with the boundary conditions* (8.4) *is given by*

$$
\begin{aligned}
&\begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} = \sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} T_{ij} \times \\
&\Bigg[E_1 \sum_{k=1}^{N_1} \frac{t_1^{i\alpha_1+k-1}}{\Gamma(i\alpha_1+k)} D_{t_2}^{-(j+1)\alpha_2} x_0^{h(k-1)}(t_2) \\
&+ E_1 \sum_{k=1}^{N_1} \frac{t_1^{i\alpha_1+k-1}}{\Gamma(i\alpha_1+k)} \sum_{l=0}^{M_2-1} \frac{t_2^{(j+1)\alpha_2+l}}{\Gamma[(j+1)\alpha_2+l+1]} \left. \frac{\partial^l x_0^{h(k-1)}(t_2)}{\partial t_2^l} \right|_{t_2=0} \\
&+ E_2 \sum_{l=1}^{N_2} \frac{t_2^{j\alpha_2+l-1}}{\Gamma(j\alpha_2+l)} D_{t_1}^{-(i+1)\alpha_1} x_0^{v(l-1)}(t_1) \\
&+ E_2 \sum_{l=1}^{N_2} \frac{t_2^{j\alpha_2+l-1}}{\Gamma(j\alpha_2+l)} \sum_{k=0}^{M_1-1} \frac{t_1^{(i+1)\alpha_1+k}}{\Gamma[(i+1)\alpha_1+k+1]} \left. \frac{\partial^k x_0^{v(l-1)}(t_1)}{\partial t_1^k} \right|_{t_1=0} \\
&+ BD_{t_1,t_2}^{-(i+1)\alpha_1,-(j+1)\alpha_2} u(t_1,t_2) \\
&+ B \sum_{k=0}^{M_1-1} \frac{t_1^{(i+1)\alpha_1+k}}{\Gamma[(i+1)\alpha_1+k+1]} D_{t_2}^{-(j+1)\alpha_2} \left. \frac{\partial^k u(t_1,t_2)}{\partial {t_1}^k} \right|_{t_1=0} \\
&+ \sum_{l=0}^{M_2-1} \frac{t_2^{(j+1)\alpha_2+l}}{\Gamma[(j+1)\alpha_2+l+1]} D_{t_1}^{-(i+1)\alpha_1} \left. \frac{\partial^l u(t_1,t_2)}{\partial {t_2}^l} \right|_{t_2=0} \\
&+ \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2-1} \frac{t_1^{(i+1)\alpha_1+k} t_2^{(j+1)\alpha_2}}{\Gamma[(i+1)\alpha_1+k+1]\Gamma[(j+1)\alpha_2+l+1]} \frac{\partial^k}{\partial {t_1}^k} \frac{\partial^l}{\partial {t_2}^l} u(t_1,t_2) \Bigg|_{\substack{t_1=0,\\ t_2=0}} \Bigg],
\end{aligned} \tag{8.5}
$$

where

$$E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad E_1 \in \mathbb{R}^{n \times n_1},\ E_2 \in \mathbb{R}^{n \times n_2},$$

$$M_1 - 1 < -(i+1)\alpha_1 \le M_1, \quad M_2 - 1 < -(j+1)\alpha_2 \le M_2; \quad M_1, M_2 \in \mathbb{N}, \tag{8.6}$$

and the transition matrices are defined by the equalities

$$
\begin{cases}
\begin{bmatrix} E_1 & 0 \end{bmatrix} T_{0,-1} + \begin{bmatrix} 0 & E_2 \end{bmatrix} T_{-1,0} - AT_{-1,-1} = \mathbb{I}_n \\
\begin{bmatrix} E_1 & 0 \end{bmatrix} T_{i,j-1} + \begin{bmatrix} 0 & E_2 \end{bmatrix} T_{i-1,j} - AT_{i-1,j-1} = 0 \quad \text{for } i \ge -\mu_1,\ i \ne 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{and } j \ge \mu_2,\ j \ne 0 \\
T_{ij} = 0 \textit{ (zero matrix)} \qquad \textit{for } i < \mu_1 \textit{ and/or } j < \mu_2
\end{cases} \tag{8.7}
$$

Proof. Using (A.12) and (A.13) we obtain the 2D Laplace transform of the state equations (8.3a)

$$\left\{E\begin{bmatrix} p^{\alpha_1}\mathbb{I}_{n_1} & 0 \\ 0 & s^{\alpha_2}\mathbb{I}_{n_2}\end{bmatrix} - A\right\}\begin{bmatrix} X^h(p,s) \\ X^v(p,s)\end{bmatrix} = E\begin{bmatrix} \sum_{k=1}^{N_1} p^{\alpha_1-k}X_{t_1}^{h(k-1)}(0,s) \\ \sum_{l=1}^{N_2} s^{\alpha_2-l}X_{t_2}^{v(l-1)}(p,0)\end{bmatrix} + BU(p,s), \tag{8.8}$$

where $N_i - 1 < \alpha_i < N_i$, $N_i \in \mathbb{N}$ for $i = 1, 2$ and

$$X_{t_1}^{h(k)}(0,s) = \mathcal{L}_{t_2}\left\{ \left.\frac{\partial^k x^h(t_1,t_2)}{\partial t_1{}^k}\right|_{t_1=0} \right\} = \mathcal{L}_{t_2}\left\{x_0^{h(k)}(t_2)\right\},$$
$$X_{t_2}^{v(l)}(p,0) = \mathcal{L}_{t_1}\left\{ \left.\frac{\partial^l x^v(t_1,t_2)}{\partial t_2{}^l}\right|_{t_2=0} \right\} = \mathcal{L}_{t_1}\left\{x_0^{v(l)}(t_1)\right\}$$

for $k = 0, 1, \ldots, N_1 - 1; l = 0, 1, \ldots, N_2 - 1$.

We assume, that the matrix

$$G(p,s) = E\begin{bmatrix} p^{\alpha_1}\mathbb{I}_{n_1} & 0 \\ 0 & s^{\alpha_2}\mathbb{I}_{n_2}\end{bmatrix} - A \tag{8.9}$$

is invertible for some p and s, i.e.

$$\det G(p,s) = \sum_{i=0}^{r_1}\sum_{j=0}^{r_2} d_{ij}p^{i\alpha_1}s^{j\alpha_2} \neq 0 \tag{8.10}$$

for some p and s, where d_{ij} for $i = 0, 1, \ldots, r_1; j = 0, 1, \ldots, r_2$ are real coefficients and depend on the matrices E, A of the system.

If the coefficient (8.10) is met, then the system (8.3) has a solution for $t_1, t_2 \geq 0$ and arbitrary admissible boundary conditions for a given input. Such system (the pair of matrices E, A) is called regular. Note that the system (8.3) is regular for every nonsingular matrix E.

Let the system (8.3a) be regular. Then we may write (8.8) in the form

$$\begin{bmatrix} X^h(p,s) \\ X^v(p,s)\end{bmatrix} = G^{-1}(p,s)\left\{E\begin{bmatrix} \sum_{k=1}^{N_1} p^{\alpha_1-k}X_{t_1}^{h(k-1)}(0,s) \\ \sum_{l=1}^{N_2} s^{\alpha_2-l}X_{t_2}^{v(l-1)}(p,0)\end{bmatrix} + BU(p,s)\right\}. \tag{8.11}$$

The inverse of the polynomial matrix of p, s (8.9) can be expressed in the form of the following sum [43, 128]

$$G^{-1}(p,s) = \sum_{i=-\mu_1}^{\infty}\sum_{j=-\mu_2}^{\infty} T_{ij}p^{-(i+1)\alpha_1}s^{-(j+1)\alpha_2}. \tag{8.12}$$

The pair of natural numbers μ_1 and μ_2, such that $T_{ij} = 0$ for $i < \mu_1$ and/or $j < \mu_2$, is called the index of the pencil of the matrices (E, A). Some method of computation of the transition matrices T_{ij} is given in [128]. It is shown there, that the index of the pair (E, A) is finite number if and only if $d_{r_1,r_2} \neq 0$.

Taking into account the property of the inverse matrices

$$G(p,s)G^{-1}(p,s) = G^{-1}(p,s)G(p,s) = \mathbb{I}_n$$

and using (8.9) and (8.12) we obtain

$$\left\{\begin{bmatrix}E_1 & 0\end{bmatrix} p^{\alpha_1} + \begin{bmatrix}0 & E_2\end{bmatrix} s^{\alpha_2} - A\right\} \left(\sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} T_{ij} p^{-(i+1)\alpha_1} s^{-(j+1)\alpha_2} \right) = \mathbb{I}_n, \tag{8.13}$$

where $E = \begin{bmatrix}E_1 & E_2\end{bmatrix}$, $E_1 \in \mathbb{R}^{n \times n_1}$, $E_2 \in \mathbb{R}^{n \times n_2}$.

Transformation of (8.13) yields

$$\sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} \left\{\begin{bmatrix}E_1 & 0\end{bmatrix} T_{i,j-1} + \begin{bmatrix}0 & E_2\end{bmatrix} T_{i-1,j} - AT_{i-1,j-1}\right\} p^{-i\alpha_1} s^{-j\alpha_2} = \mathbb{I}_n. \tag{8.14}$$

Comparing the coefficients corresponding to the powers of the polynomial variables p and s in (8.14) we obtain (8.7).

Substitution (8.12) to (8.11) yields

$$\begin{aligned}\begin{bmatrix}X^h(p,s)\\ X^v(p,s)\end{bmatrix} = \sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} T_{ij} \Bigg\{ & E_1 \sum_{k=1}^{N_1} p^{-(i\alpha_1+k)} s^{-(j+1)\alpha_2} X_{t_1}^{h(k-1)}(0,s) \\ & + E_2 \sum_{l=1}^{N_2} p^{-(i+1)\alpha_1} s^{-(j\alpha_2+l)} X_{t_2}^{v(l-1)}(p,0) \\ & + Bp^{-(i+1)\alpha_1} s^{-(j+1)\alpha_2} U(p,s) \Bigg\}.\end{aligned} \tag{8.15}$$

Using the 2D inverse Laplace transform to (8.15) we obtain

$$\begin{aligned}\begin{bmatrix}x^h(t_1,t_2)\\ x^v(t_1,t_2)\end{bmatrix} &= \mathcal{L}^{-1}_{t_1,t_2}\left\{\begin{bmatrix}X^h(p,s)\\ X^v(p,s)\end{bmatrix}\right\} \\ &= \sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} T_{ij} \Bigg\{ E_1 \sum_{k=1}^{N_1} \mathcal{L}^{-1}_{t_1,t_2}\left[p^{-(i\alpha_1+k)} s^{-(j+1)\alpha_2} X_{t_1}^{h(k-1)}(0,s)\right] \\ &\qquad + E_2 \sum_{l=1}^{N_2} \mathcal{L}^{-1}_{t_1,t_2}\left[p^{-(i+1)\alpha_1} s^{-(j\alpha_2+l)} X_{t_2}^{v(l-1)}(p,0)\right] \\ &\qquad + B\mathcal{L}^{-1}_{t_1,t_2}\left[p^{-(i+1)\alpha_1} s^{-(j+1)\alpha_2} U(p,s)\right] \Bigg\}.\end{aligned} \tag{8.16}$$

Using (A.12) and (A.14), from (8.16) we obtain the solution of the state equations (8.5). □

Based on the solution (8.5) using (8.3b) we may derive the system response formula $y(t_1, t_2)$ for a given admissible boundary conditions (8.4) and the input $u(t_1, t_2)$ for $t_1, t_2 \geq 0$.

8.3 Fractional-Order Model of the Long Transmission Line

Let us consider the long transmission line with the distributed element model shown in Figure 8.1.

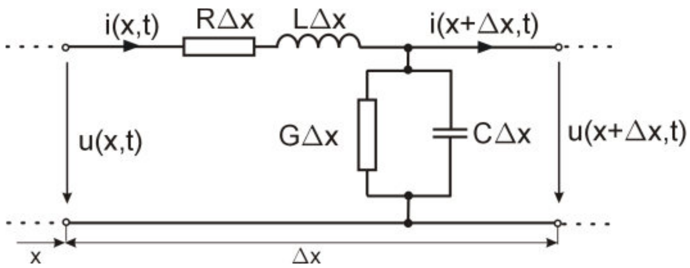

Fig. 8.1 The distributed element model of transmission line

For the circuit shown in Figure 8.1 we may formulate the equations, which describe the current and voltage in this line as a function of time t and space variable x [151, 150, 193]

$$\begin{aligned} -{}^C D_x^\alpha u(x,t) &= Ri(x,t) + L\,{}^C D_t^\beta i(x,t), \\ -{}^C D_x^\alpha i(x,t) &= Gi(x,t) + C\,{}^C D_t^\beta u(x,t), \end{aligned} \tag{8.17}$$

where $u(x,t)$ is the voltage, and $i(x,t)$ is the current at the point x from the beginning of the line for time t; R is distributed resistance, L is distributed inductance, G is distributed conductance, and C is distributed capacitance of the transmission line; $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ are fractional (real) orders with respect to the spatial variable x and time t.

Equations(8.17) may be written in the form of the following matrix equations

$$\begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 1 & C & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} {}^C D_x^{\alpha} x^h(x,t) \\ {}^C D_t^{\beta} x^v(x,t) \end{bmatrix} = \begin{bmatrix} 0 & -R & 0 & 0 \\ -G & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^h(x,t) \\ x^v(x,t) \end{bmatrix}, \tag{8.18a}$$

$$\begin{bmatrix} u(x,t) \\ i(x,t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^h(x,t) \\ x^v(x,t) \end{bmatrix}, \tag{8.18b}$$

where

$$x^h(x,t) = x^v(x,t) = \begin{bmatrix} u(x,t) \\ i(x,t) \end{bmatrix}. \tag{8.18c}$$

The systems (8.18) is an example of singular 2D fractional-order continuous system described by the Roesser model (8.3), where $t_1 = x$ is the spatial variable describing a distance from the beginning of the line, $t_2 = t$ is time variable, $\alpha_1 = \alpha$ is fractional order of partial derivative with respect to spatial variable and $\alpha_2 = \beta$ is the fractional order of partial derivative with respect to time. The matrices of the system (8.3).

$$E = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 1 & C & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -R & 0 & 0 \\ -G & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$
$$B = [0]^{4\times 1}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = [0]_{2\times 1}. \tag{8.19}$$

Note, that the system (8.18) has zero matrices B and D. Therefore it is an autonomous system independent of the input $u(x,t)$.

The boundary conditions (8.4) are the curents and voltages and its integer order derivatives at the beginning of line and for $t = 0$

$$x^{h(k)}(0,t) = \begin{bmatrix} \dfrac{\mathrm{d}^k u(0,t)}{\mathrm{d}x^k} \\ \dfrac{\mathrm{d}^k i(0,t)}{\mathrm{d}x^k} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^k u(x,t)}{\partial x^k} \\ \dfrac{\partial^k i(x,t)}{\partial x^k} \end{bmatrix}_{x=0} = \begin{bmatrix} u_0^{(k)}(t) \\ i_0^{(k)}(t) \end{bmatrix} \tag{8.20a}$$

for $k = 0, 1, \ldots N_1 - 1;\ t_2 \geq 0$,

$$x^{v(l)}(x,0) = \begin{bmatrix} \dfrac{\mathrm{d}^l u(x,0)}{\mathrm{d}t^l} \\ \dfrac{\mathrm{d}^l i(0,t)}{\mathrm{d}t^l} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^l u(x,t)}{\partial t^l} \\ \dfrac{\partial^l i(x,t)}{\partial t^l} \end{bmatrix}_{t=0} = \begin{bmatrix} u_0^{(l)}(x) \\ i_0^{(l)}(x) \end{bmatrix} \tag{8.20b}$$

where $l = 0, 1, 2, \ldots, N_2 - 1;\ t_1 \geq 0$ and $x_0^{h(k)}(t_2)$, $x_0^{v(l)}(t_1)$ are given functions.

Transition matrices for given matrices (8.19) of the system can be computed from (8.7). Hence, we obtain

$$T_{-1,-1} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix}, \tag{8.21a}$$

$$T_{ij} = \left[T_{ij}^1 \; T_{ij}^2\right] = -\begin{bmatrix} 0 & L & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & 0 & L \\ 0 & 0 & C & 0 \end{bmatrix} T_{i-1,j+1} - \begin{bmatrix} 0 & R & 0 & 0 \\ G & 0 & 0 & 0 \\ 0 & 0 & 0 & R \\ 0 & 0 & G & 0 \end{bmatrix} T_{i-1,j} \tag{8.21b}$$

for $i \geq 0$, $j < 0$, where

$$T_{0,-1}^1 = \begin{bmatrix} \mathbb{I}_2 \\ \mathbb{I}_2 \end{bmatrix}, \quad T_{i,j}^1 = [0]_{4\times 2} \quad \text{for} \quad i < 0 \quad \text{and/or} \quad j \geq 0; \tag{8.21c}$$

$$T_{0,-2}^2 = \begin{bmatrix} 0 & -L \\ -C & 0 \\ 0 & -L \\ -C & 0 \end{bmatrix}, \quad T_{i,j}^2 = [0]_{4\times 2} \quad \text{for} \quad i < 0 \quad \text{and/or} \quad j \geq -1. \tag{8.21d}$$

From (8.21) it follows that the indexes of the model $\mu_1 = 1$ and $\mu_2 = \infty$.

Some different method for the computation of the transition matrices of the system is given in [171].

Using Theorem 8.1 we get the solution of the state equation (8.18a) with the matrices (8.19)

$$\begin{aligned} \begin{bmatrix} x^h(x,t) \\ x^v(x,t) \end{bmatrix} &= \sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} T_{ij} \times \\ &\left[E_1 \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_t^{-(j+1)\beta} x_0^h(t) + \sum_{l=0}^{M_2-1} \frac{t^{(j+1)\beta+l}}{\Gamma[(j+1)\beta+l+1]} \frac{\partial^l}{\partial t^l} x_0^h(0) \right) \right. \\ &\left. + E_2 \frac{t^{j\beta}}{\Gamma(j\beta+1)} \left(D_x^{-(i+1)\alpha} x_0^v(x) + \sum_{k=0}^{M_1-1} \frac{x^{(i+1)\alpha+k}}{\Gamma[(i+1)\alpha+k+1]} \frac{\partial^k}{\partial x^k} x_0^v(0) \right) \right], \end{aligned} \tag{8.22}$$

where

$$M_1 - 1 < -(i+1)\alpha \leq M_1, \quad M_2 - 1 < -(j+1)\beta \leq M_2; \quad M_1, M_2 \in \mathbb{N},$$

and

$$\frac{\partial^l}{\partial t^l} x_0^h(0) = \left. \frac{\partial^l}{\partial t^l} x_0^h(t) \right|_{t=0}, \qquad \frac{\partial^k}{\partial x^k} x_0^v(0) = \left. \frac{\partial^k}{\partial x^k} x_0^v(x) \right|_{x=0}.$$

Taking into account (8.18c) and (8.6), from (8.22), we obtain

$$\begin{bmatrix} x^h(x,t) \\ x^v(x,t) \end{bmatrix} = \sum_{i=-1}^{\infty} \sum_{j=-\infty}^{-1} T_{ij}^1 \times$$
$$\left\{ \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_t^{-(j+1)\beta} \begin{bmatrix} u(0,t) \\ i(0,t) \end{bmatrix} + \sum_{l=0}^{M_2-1} \frac{t^{(j+1)\beta+l}}{\Gamma[(j+1)\beta+l+1]} \frac{\partial^l}{\partial t^l} \begin{bmatrix} u(0,0) \\ i(0,0) \end{bmatrix} \right) \right.$$
$$\left. + \frac{t^{j\beta}}{\Gamma(j\beta+1)} \begin{bmatrix} 0 & L \\ C & 0 \end{bmatrix} D_x^{-(i+1)\alpha} \begin{bmatrix} u(x,0) \\ i(x,0) \end{bmatrix} \right\}, \tag{8.23}$$

where

$$M_2 - 1 < -(j+1)\beta \le M_2; \quad M_2 \in \mathbb{N},$$

and

$$\frac{\partial^l}{\partial t^l} \begin{bmatrix} u(0,0) \\ i(0,0) \end{bmatrix} = \frac{\partial^l}{\partial t^l} \begin{bmatrix} u(0,t) \\ i(0,t) \end{bmatrix} \Bigg|_{t=0}$$

Using (8.3b) and (8.19) we get

$$\begin{bmatrix} u(x,t) \\ i(x,t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^h(x,t) \\ x^v(x,t) \end{bmatrix}. \tag{8.24}$$

Equations (8.24) and (8.23) are the solution of fractional-order partial derivative equations (8.17) describing current and voltage in long electrical transmission line for arbitrary distributed parameters R, L, C, G of the line, fractional orders $0 < \alpha < 1$, $0 < \beta < 1$ and admissible boundary conditions (8.20).

8.4 Standard Fractional 2D Roesser Model and Its Solution

Consider the fractional 2D continuous-time system described by the state equations [169]

$$\begin{bmatrix} {}^C D_{t_1}^{\alpha_1} x^h(t_1,t_2) \\ {}^C D_{t_2}^{\alpha_2} x^v(t_1,t_2) \end{bmatrix} = A \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + Bu(t_1,t_2), \tag{8.25a}$$

$$y(t_1,t_2) = C \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + Du(t_1,t_2), \tag{8.25b}$$

where $N_i - 1 < \alpha_i < N_i$, $N_i \in \mathbb{N}$ for $i = 1,2$; $x^h(t_1,t_2) \in \mathbb{R}^{n_1}$, $x^v(t_1,t_2) \in \mathbb{R}^{n_2}$, $(n = n_1 + n_2)$ are the horizontal and vertical state vectors, respectively, $u(t_1,t_2) \in \mathbb{R}^m$ is the input vector, $y(t_1,t_2) \in \mathbb{R}^p$ is the output vector and $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

The boundary conditions for (8.25) are given in the form

$$x^{h(k)}(0,t_2) = \left[\frac{\partial^k x^h(t_1,t_2)}{\partial t_1^k} \right]_{t_1=0} = x_0^{h(k)}(t_2), \tag{8.26a}$$

where $k = 0, 1, \dots N_1 - 1;\ t_2 \geq 0,$

$$x^{v(l)}(t_1, 0) = \left[\frac{\partial^l x^v(t_1, t_2)}{\partial t_2^k}\right]_{t_2=0} = x_0^{v(l)}(t_1), \tag{8.26b}$$

where $l = 0, 1, 2, \dots, N_2-1;\ t_1 \geq 0$ and $x_0^{h(k)}(t_2), x_0^{v(l)}(t_1)$ are given functions.

The model (8.25) is a particular case of singular 2D fractional system described by the Roesser model (8.3), where $E = \mathbb{I}_n$.

Theorem 8.2. *The solution to the equation* (8.25a) *with the boundary conditions* (8.26) *is given by*

$$\begin{aligned}\begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} T_{ij} &\left\{ \sum_{k=1}^{N_1} \frac{t_1^{k+i\alpha_1-1}}{\Gamma(k+i\alpha_1)} I_{t_2}^{j\alpha_2} \begin{bmatrix} x_0^{h(k-1)}(t_2) \\ 0 \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} I_{t_1,t_2}^{(i+1)\alpha_1, j\alpha_2} u(t_1,t_2) \right\} \\ + \sum_{i=0}^{\infty} T_{i0} &\left\{ \sum_{k=1}^{N_1} \frac{t_1^{k+i\alpha_1-1}}{\Gamma(k+i\alpha_1)} \begin{bmatrix} x_0^{h(k-1)}(t_2) \\ 0 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} I_{t_1}^{(i+1)\alpha_1} u(t_1,t_2) \right\} \\ + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} T_{ij} &\left\{ \sum_{l=1}^{N_2} \frac{t_2^{l+j\alpha_2-1}}{\Gamma(l+j\alpha_2)} I_{t_1}^{i\alpha_1} \begin{bmatrix} 0 \\ x_0^{v(l-1)}(t_1) \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} I_{t_1,t_2}^{i\alpha_1,(j+1)\alpha_2} u(t_1,t_2) \right\} \\ + \sum_{j=0}^{\infty} T_{0j} &\left\{ \sum_{l=1}^{N_2} \frac{t_2^{l+j\alpha_2-1}}{\Gamma(l+j\alpha_2)} \begin{bmatrix} 0 \\ x_0^{v(l-1)}(t_1) \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} I_{t_2}^{(j+1)\alpha_2} u(t_1,t_2) \right\}\end{aligned} \tag{8.27a}$$

or

$$\begin{aligned}\begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} &\left\{ \sum_{k=1}^{N_1} \frac{t_1^{k+i\alpha_1-1}}{\Gamma(k+i\alpha_1)} D_{t_2}^{-j\alpha_2} \begin{bmatrix} x_0^{h(k-1)}(t_2) \\ 0 \end{bmatrix} \right. \\ &+ \sum_{l=1}^{N_2} \frac{t_2^{l+j\alpha_2-1}}{\Gamma(l+j\alpha_2)} D_{t_1}^{-i\alpha_1} \begin{bmatrix} 0 \\ x_0^{v(l-1)}(t_1) \end{bmatrix} \\ &\left. + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} D_{t_1,t_2}^{-(i+1)\alpha_1,-j\alpha_2} u(t_1,t_2) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} D_{t_1,t_2}^{-i\alpha_1,-(j+1)\alpha_2} u(t_1,t_2) \right\},\end{aligned} \tag{8.27b}$$

where

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{n_1 \times m},\ B_2 \in \mathbb{R}^{n_2 \times m},$$

and

$$T_{ij} = \begin{cases} \mathbb{I}_n \ (n = n_1 + n_2) & \text{for } i = 0, \ j = 0 \\ T_{10}T_{i-1,j} + T_{01}T_{i,j-1} & \text{for } i + j > 0 \ (i, j \in \mathbb{Z}) \\ 0 & \text{for } i < 0 \text{ and/or } j < 0 \end{cases} \tag{8.28a}$$

where

$$T_{10} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad T_{01} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}. \tag{8.28b}$$

Proof. Using (A.12) and (A.13) we obtain the 2D Laplace transform of the state equation (8.25a)

$$\begin{bmatrix} p^{\alpha_1} X^h(p,s) - \sum_{k=1}^{N_1} p^{\alpha_1 - k} X_{t_1}^{h(k-1)}(0,s) \\ s^{\alpha_2} X^v(p,s) - \sum_{l=1}^{N_2} s^{\alpha_2 - l} X_{t_2}^{v(l-1)}(p,0) \end{bmatrix} = A \begin{bmatrix} X^h(p,s) \\ X^v(p,s) \end{bmatrix} + BU(p,s). \tag{8.29}$$

Premultiplying (8.29) by the matrix

$$M = \text{blockdiag}\left[\mathbb{I}_{n_1} p^{-\alpha_1}, \mathbb{I}_{n_2} s^{-\alpha_2}\right]$$

we obtain

$$\begin{bmatrix} X^h(p,s) \\ X^v(p,s) \end{bmatrix} = G^{-1}(p,s) \left\{ \begin{bmatrix} \sum_{k=1}^{N_1} p^{-k} X_{t_1}^{h(k-1)}(0,s) \\ \sum_{l=1}^{N_2} s^{-l} X_{t_2}^{v(l-1)}(p,0) \end{bmatrix} + \begin{bmatrix} p^{-\alpha_1} B_1 \\ s^{-\alpha_2} B_2 \end{bmatrix} U(p,s) \right\}, \tag{8.30}$$

where

$$G(p,s) = \mathbb{I}_n - MA = \begin{bmatrix} \mathbb{I}_{n_1} - p^{-\alpha_1} A_{11} & -p^{-\alpha_1} A_{12} \\ -s^{-\alpha_2} A_{21} & \mathbb{I}_{n_2} - s^{-\alpha_2} A_{22} \end{bmatrix}. \tag{8.31}$$

Let

$$G^{-1}(p,s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} T_{ij} p^{-i\alpha_1} s^{-j\alpha_2}. \tag{8.32}$$

From

$$G^{-1}(p,s)G(p,s) = G(p,s)G^{-1}(p,s) = \mathbb{I}_n$$

and taking into account (8.31) and (8.32), we get

$$\begin{bmatrix} \mathbb{I}_{n_1} - p^{-\alpha_1} A_{11} & -p^{-\alpha_1} A_{12} \\ -s^{-\alpha_2} A_{21} & \mathbb{I}_{n_2} - s^{-\alpha_2} A_{22} \end{bmatrix} \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} T_{ij} p^{-i\alpha_1} s^{-j\alpha_2} \right) = \begin{bmatrix} \mathbb{I}_{n_1} & 0 \\ 0 & \mathbb{I}_{n_2} \end{bmatrix}. \tag{8.33}$$

Comparing the coefficients at the same powers of p and s in (8.33), we obtain (8.28a).

Substitution (8.32) into (8.30) yields

$$\begin{bmatrix} X^h(p,s) \\ X^v(p,s) \end{bmatrix} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} \left\{ \begin{bmatrix} \sum_{k=1}^{N_1} p^{-k-i\alpha_1} s^{-j\alpha_2} X_{t_1}^{h(k-1)}(0,s) \\ \sum_{l=1}^{N_2} p^{-i\alpha_1} s^{-l-j\alpha_2} X_{t_2}^{v(l-1)}(p,0) \end{bmatrix} + \begin{bmatrix} p^{-(i+1)\alpha_1} s^{-j\alpha_2} B_1 \\ p^{-i\alpha_1} s^{-(j+1)\alpha_2} B_2 \end{bmatrix} U(p,s) \right\} \tag{8.34}$$

Applying the inverse 2D Laplace transtosm to (8.34) and taking into account (A.10) we obtain the formula (8.27a). Using (8.2) and (A.15) we get (8.27b). □

From the solution (8.27) and using (8.25b) we may find output of the system (8.25) $y(t_1, t_2)$, $t_1, t_2 \geq 0$ for arbitrary input $u(t_1, t_2)$, $t_1, t_2 \geq 0$ and arbitrary boundary conditions (8.26).

Example 8.1. Consider the fractional 2D system (8.25) with $\alpha_1 = 0.7$, $\alpha_2 = 0.9$ and matrices

$$\begin{aligned} A &= \begin{bmatrix} -0.9 & 0.7 \\ 0 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = [0]. \end{aligned} \tag{8.35}$$

Find a step response of the system (8.25) with the matrices (8.35), i.e. $y(t_1, t_2)$ for $t_1, t_2 \geq 0$ and the input in the form of 2D step function

$$u(t_1, t_2) = H(t_1, t_2) = \begin{cases} 0 \text{ for } & t_1 < 0 \quad \text{and/or } t_2 < 0 \\ 1 \text{ for } t_1, t_2 \geq 0 \end{cases}$$

and zero boundary conditions

$$x_0^h(t_2) = 0 \text{ for } t_2 \geq 0, \quad x_0^v(t_1) = 0 \text{ for } t_1 \geq 0. \tag{8.36}$$

Note, that

$$y(t_1, t_2) = \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}.$$

The fractional integral (1.6) of 1D step function is given by the formula [163]

$$I_t^\alpha H(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{8.37}$$

for $\alpha > 0$.

Using (8.37) it is easy to show that

$$I_{t_1,t_2}^{\alpha_1,\alpha_2} H(t_1,t_2) = \frac{t_1^{\alpha_1} t_2^{\alpha_2}}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)} \tag{8.38}$$

Substituting (8.35), (8.36), (8.38) into (8.27a) and taking into account that $N_1, N_2 = 1$ we have

$$\begin{aligned}\begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} =& \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} T_{ij} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{t_1^{(i+1)\alpha_1} t_2^{j\alpha_2}}{\Gamma[1+(i+1)\alpha_1]\Gamma(1+j\alpha_2)} \\ &+ \sum_{i=0}^{\infty} T_{i0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{t_1^{(i+1)\alpha_1}}{\Gamma[1+(i+1)\alpha_1]} \\ &+ \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} T_{ij} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{t_1^{i\alpha_1} t_2^{(j+1)\alpha_2}}{\Gamma(1+i\alpha_1)\Gamma[1+(j+1)\alpha_2]} \\ &+ \sum_{j=0}^{\infty} T_{0j} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{t_2^{(j+1)\alpha_2}}{\Gamma[1+(j+1)\alpha_2]},\end{aligned} \tag{8.39}$$

and the transition matrices can be computed from equality (8.28a).

Formula (8.39) defines the step response of the system (8.25) with given matrices (8.35) and known boundary conditions (8.36) of the system.

It is easy to show that the coefficients $\frac{1}{\Gamma(\cdot)}$ in the equation (8.39) strongly decrease for increasing values of i and j. Therefore, in practical applications we may assume that the numbers i and j are bounded by some natural numbers L_1 and L_2.

The state vectors (step response) plots of the system for $L_1, L_2 = 50$ are shown in Figure 8.2 and 8.3.

8.5 Generalization of Cayley-Hamilton Theorem

Theorem 8.3. *Let*

$$\det G(p,s) = \begin{vmatrix} \mathbb{I}_{n_1} - p^{-\alpha_1}A_{11} & -p^{-\alpha_1}A_{12} \\ -s^{-\alpha_2}A_{21} & \mathbb{I}_{n_2} - s^{-\alpha_2}A_{22} \end{vmatrix} = \sum_{k=0}^{n_1}\sum_{l=0}^{n_2} a_{n_1-k,n_2-l} p^{-k\alpha_1} s^{-l\alpha_2} \tag{8.40}$$

be the characteristic polynomial of the system (8.25)*. The transition matrices* T_{ij} *given by* (8.28a) *are defined by the equality*

$$\sum_{k=0}^{n_1}\sum_{l=0}^{n_2} a_{kl} T_{k+m_1,l+m_2} = 0 \tag{8.41}$$

for $m_1, m_2 = 0, 1, \ldots$.

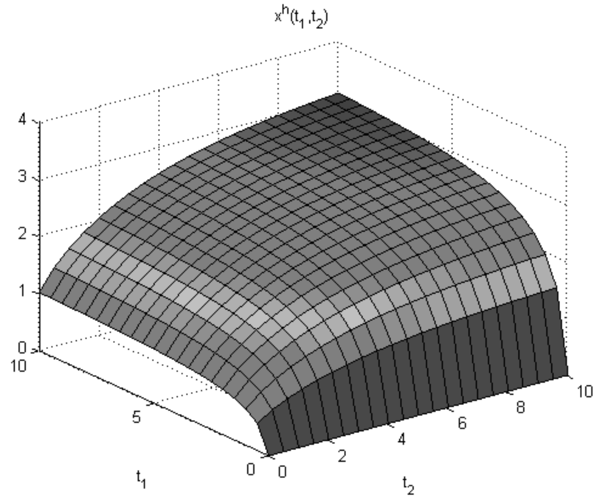

Fig. 8.2 Horizontal state vector $x^h(t_1, t_2)$ of Example 8.1

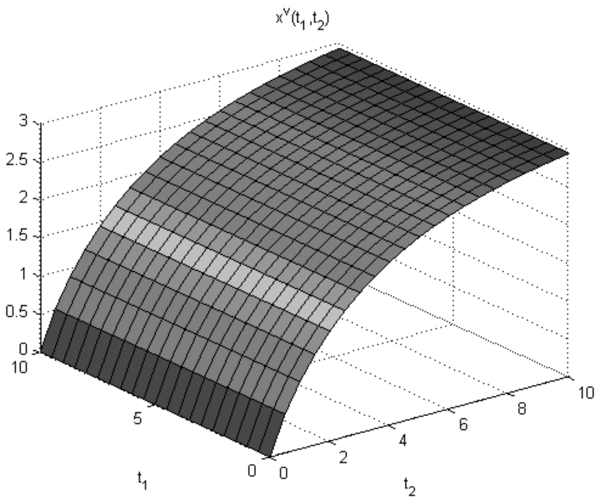

Fig. 8.3 Vertical state vector $x^v(t_1, t_2)$ of Example 8.1

Proof. By definition of inverse matrices, as well as (8.32) and (8.40), we have

$$\begin{aligned}\mathrm{Adj}G(p,s) =& \det G(p,s)G^{-1}(p,s)\\ =&\left(\sum_{k=0}^{n_1}\sum_{l=0}^{n_2} a_{n_1-k,n_2-l}p^{-k\alpha_1}s^{-l\alpha_2}\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij}p^{-i\alpha_1}s^{-j\alpha_2}\right)\\ =&\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\sum_{i=-k}^{\infty}\sum_{j=-l}^{\infty} a_{kl}T_{i+k,j+l}p^{-(i+n_1)\alpha_1}s^{-(j+n_2)\alpha_2},\end{aligned}$$

where $\mathrm{Adj}G(p,s)$ is the adjoint matrix of the polynomial matrix $G(p,s)$.

Comparing coefficiens of the same powers of p and s for $i \geq 0$ and $j \geq 0$, we obtain (8.41), since $\mathrm{Adj}G(p,s)$ is the polynomial matrix of the form

$$\mathrm{Adj}G(p,s) = \sum_{\substack{i=0\\ i,j\neq n_1,n_2}}^{n_1}\sum_{j=0}^{n_2} D_{ij}p^{-i\alpha_2}s^{-j\alpha_2},$$

where $D_{ij} \in \mathbb{R}^{n\times n}$ are some matrices with real entries. □

Theorem 8.3 is a generalization of a well-known Cayley-Hamilton theorem for standard fractional order 2D continuous systems described by the Roesser model.

Appendix A
Laplace Transforms of Continuous-Time Functions and $\mathcal{Z}$-Transforms of Discrete-Time Functions

A.1 Convolutions of Continuous-Time and Discrete-Time Functions and Their Transforms

Definition A.1. The Laplace transform of a continuous-time function $f(t)$ is defined by

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\mathrm{d}t = F(s), \tag{A.1}$$

where $f(t) = 0$ for $t < 0$.

Definition A.2. The continuous-time function defined by

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau)f_2(\tau)\mathrm{d}\tau,$$

is called the convolution of the continuous-time functions $f_1(t)$ and $f_2(t)$.

Theorem A.1. *If*

$$F_1(s) = \mathcal{L}[f_1(t)], \quad F_2(s) = \mathcal{L}[f_2(t)],$$

then

$$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\mathrm{d}\tau\right] = F_1(s)F_2(s). \tag{A.2}$$

Proof. From (A.1) and taking into account that $f_1(t) = 0$, $f_2(t) = 0$ for $t < 0$ we have

$$\begin{aligned}\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\mathrm{d}\tau\right] =&\mathcal{L}\left[\int_0^\infty f_1(t-\tau)f_2(\tau)\mathrm{d}\tau\right]\\ &=\int_0^\infty\left[\int_0^\infty f_1(t-\tau)f_2(\tau)\right]e^{-st}\mathrm{d}\tau\mathrm{d}t\\ &=\int_0^\infty f_1(u)e^{-su}\mathrm{d}u\int_0^\infty f_2(\tau)e^{-s\tau}\mathrm{d}\tau = F_1(s)F_2(s).\end{aligned}$$

□

Definition A.3. The $\mathcal{Z}$-transform of a discrete-time function $f(i)$ is defined by

$$\mathcal{Z}[f(i)] = \sum_{k=0}^{\infty} f(k)z^{-k} = F(z),$$

where $f(i) = 0$ for $i < 0$.

Definition A.4. The discrete-time function defined by

$$f_1(i) * f_2(i) = \sum_{k=0}^{i} f_1(i-k)f_2(k)$$

is called the convolution of the discrete-time functions $f_1(i)$ and $f_2(i)$.

Theorem A.2. *If*

$$F_1(z) = \mathcal{Z}[f_1(i)], \quad F_2(z) = \mathcal{Z}[f_2(i)],$$

then

$$\mathcal{Z}\left[\sum_{k=0}^{i} f_1(i-k)f_2(k)\right] = F_1(z)F_2(z).$$

Proof. The proof is similar to the proof of Theorem A.1. □

A.2 Laplace Transforms of Derivative-Integrals

Theorem A.3. *The Laplace transform of the function t^α has the form*

$$\mathcal{L}\left[t^\alpha\right] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \tag{A.3}$$

for $\alpha \in \mathbb{R}$, $\alpha > -1$.

Proof. Using (A.1) and (1.1) for $\alpha > -1$ we obtain

$$\mathcal{L}\left[t^{\alpha}\right] = \int_{0}^{\infty} t^{\alpha} e^{-st}\mathrm{d}t = \int_{0}^{\infty} \frac{x^{\alpha}}{s^{\alpha+1}} e^{-x}\mathrm{d}x = \frac{1}{s^{\alpha+1}} \int_{0}^{\infty} x^{\alpha} e^{-x}\mathrm{d}x = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

□

Theorem A.4. *The Laplace transform of the first order derivative of the function $f(t)$ has the form*

$$\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right] = sF(s) - f(0). \tag{A.4}$$

Proof. Using (A.1) we have

$$\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right] = \int_{0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} f(t) e^{-st}\mathrm{d}t = \left[e^{-st} f\right]_{0}^{\infty} + s\int_{0}^{\infty} f(t) e^{-st}\mathrm{d}t = sF(s) - f(0^{+}).$$

□

Generalizing (A.4) for n-order derivative we obtain

$$\mathcal{L}\left[\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}} f(t)\right] = s^{n} F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^{+}), \tag{A.5}$$

where $f^{(k)}(0^{+}) = \left.\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right|_{t=0}$.

Theorem A.5. *The Laplace transform of the fractional α-order integral has the form*

$$\mathcal{L}\left[{}_{0}I_{t}^{\alpha} f(t)\right] = \frac{F(s)}{s^{\alpha}}. \tag{A.6}$$

Proof. Using (1.6), (A.3), Theorem A.1 and taking into account that $\alpha > 0$ we obtain

$$\mathcal{L}\left[{}_{0}I_{t}^{\alpha} f(t)\right] = \mathcal{L}\left[\frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau\right] = \frac{1}{\Gamma(\alpha)} \mathcal{L}\left[t^{\alpha-1}\right] \mathcal{L}\left[f(t)\right] = \frac{F(s)}{s^{\alpha}}.$$

□

Theorem A.6. *The Laplace transform of the fractional α-order integral of n-order derivative of the function $f(t)$ has the form*

$$\mathcal{L}\left[{}_{0}I_{t}^{\alpha} f^{(n)}(t)\right] = s^{n-\alpha} F(s) - \sum_{k=1}^{n} s^{n-k-\alpha} f^{(k-1)}(0^{+}).$$

Proof. Using (1.6), (A.3), Theorem A.1 we obtain

$$\begin{aligned}\mathcal{L}\left[{}_0I_t^{\alpha}f^{(n)}(t)\right] &= \mathcal{L}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f^{(n)}(\tau)\mathrm{d}\tau\right] \\ &= \frac{1}{\Gamma(\alpha)}\mathcal{L}\left[t^{\alpha-1}\right]\mathcal{L}\left[f^{(n)}(t)\right] \\ &= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{s^{\alpha}}\left[s^nF(s)-\sum_{k=1}^{n}s^{n-k}f^{(k-1)}(0^+)\right] \\ &= s^{n-\alpha}F(s)-\sum_{k=1}^{n}s^{n-k-\alpha}f^{(k-1)}(0^+).\end{aligned}$$

□

Theorem A.7. *The inverse Laplace transform of the expression* $s^{\alpha}F(s)$ *for* $\alpha > 0$ *is given by*

$$\mathcal{L}^{-1}\left[s^{\alpha}F(s)\right] = {}_0^CD_t^{\alpha}f(t) + \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0^+), \tag{A.7}$$

where $n-1<\alpha<n$, $n\in\mathbb{Z}_+$.

Proof. Let assume $0<\alpha<1$ $(n=1)$, then by Theorem 1.4 and using (A.3) we obtain

$$\begin{aligned}\mathcal{L}^{-1}\left[s^{\alpha}F(s)\right] &= \mathcal{L}^{-1}\left[s^{\alpha}F(s)-s^{\alpha-1}f(0)\right]+\mathcal{L}^{-1}\left[s^{\alpha-1}f(0)\right] \\ &= {}_0^CD_t^{\alpha}f(t)+\mathcal{L}^{-1}\left[\frac{1}{s^{1-\alpha}}\right]f(0) \\ &= {}_0^CD_t^{\alpha}f(t)+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0).\end{aligned} \tag{A.8}$$

Similarly, applying the convolution theorem, for $1<\alpha<2$ $(n=2)$ we get

$$\begin{aligned}\mathcal{L}^{-1}\left[s^{\alpha}F(s)\right] =& \mathcal{L}^{-1}\left[s^{\alpha}F(s)-\sum_{k=1}^{2}s^{\alpha-k}f^{(k-1)}(0^+)\right] \\ &+\mathcal{L}^{-1}\left[\sum_{k=1}^{2}s^{\alpha-k}f^{(k-1)}(0^+)\right] \\ =& {}_0^CD_t^{\alpha}f(t)+\mathcal{L}^{-1}\left[\frac{1}{s^{2-\alpha}}\right]f^{(1)}(0^+)+\mathcal{L}^{-1}\left[s\right]*\mathcal{L}^{-1}\left[\frac{1}{s^{2-\alpha}}\right]f(0) \\ =& {}_0^CD_t^{\alpha}f(t)+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}f^{(1)}(0^+)+\left[\delta^{(1)}(t)\right]*\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right]f(0),\end{aligned} \tag{A.9}$$

since the inverse Laplace transform of the expression s^i is given by the formula [21]

$$\mathcal{L}^{-1}\left[s^i\right] = \delta^{(i)}(t) \quad \text{for} \quad i \in \mathbb{Z}_+,$$

where $\delta^{(i)}(t)$ is i–th distribution derivative of Dirac function and $f(t) * g(t)$ denotes the convolution of functions $f(t)$ and $g(t)$.

Applying the distribution theory [41] for (A.9) and taking into account (1.2) we obtain

$$\begin{aligned}\left[\delta^{(1)}(t)\right] * \left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right] &= \int_0^t \delta^{(1)}(\tau)\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}\mathrm{d}\tau \\ &= -\int_0^t \delta(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}\mathrm{d}\tau \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.\end{aligned} \tag{A.10}$$

Substitution of (A.10) into (A.9) yields

$$\mathcal{L}^{-1}\left[s^\alpha F(s)\right] = {}_0^C D_t^\alpha f(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} f^{(1)}(0^+) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0).$$

Therefore, for arbitrary $n \in \mathbb{N}$, we obtain (A.7). □

Theorem A.7 is a generalization of the formula (4.2.1.3) from the monograph [154] for positive orders $\alpha > 0$.

By Theorems A.3 and A.7 and taking into account (8.2) we obtain the following theorem.

Theorem A.8. *The inverse Laplace transform of the expression $s^\alpha F(s)$ has the form*

$$\mathcal{L}^{-1}[s^\alpha F(s)] = {}_0D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+),$$

where $n-1 < \alpha < n$, $n \in \mathbb{Z}$ and

$${}_0D_t^\alpha f(t) = \begin{cases} {}_0^C D_t^\alpha f(t) \ \textit{for}\ \alpha > 0, \\ {}_0I_t^{-\alpha} f(t) \ \textit{for}\ \alpha < 0. \end{cases}$$

A.3 Laplace Transforms of Two-Dimensional Fractional Differintegrals

Let $F(p, t_2)$ $\big(F(t_1, s)\big)$ be the Laplace transform of a two-dimensional continuous function $f(t_1, t_2)$ with respect to the t_1 $\big(t_2\big)$ given by the formula

$$F(p,t_2)=\mathcal{L}_{t_1}\left[f(t_1,t_2)\right]=\int\limits_0^\infty f(t_1,t_2)e^{-pt_1}\mathrm{d}t_1$$
$$\left(F(t_1,s)=\mathcal{L}_{t_2}\left[f(t_1,t_2)\right]=\int\limits_0^\infty f(t_1,t_2)e^{-st_2}\mathrm{d}t_2\right). \tag{A.11}$$

Definition A.5. The two-dimensional Laplace transform $F(p,s)$ of two-dimensional continuous function $f(t_1,t_2)$ is defined by [51, 154, 163, 171]

$$\begin{aligned}F(p,s)=&\mathcal{L}_{t_1,t_2}\left[f(t_1,t_2)\right]=\mathcal{L}_{t_1}\left\{\mathcal{L}_{t_2}\left[f(t_1,t_2)\right]\right\}=\mathcal{L}_{t_2}\left\{\mathcal{L}_{t_1}\left[f(t_1,t_2)\right]\right\}\\ =&\int\limits_0^\infty\int\limits_0^\infty f(t_1,t_2)e^{-pt_1-st_2}\mathrm{d}t_1\mathrm{d}t_2.\end{aligned} \tag{A.12}$$

From the above considerations the following theorems can be proved.

Theorem A.9. *[171] The Laplace transform of the fractional partial derivative* (8.1) *of the 2D function $f(t_1,t_2)$ with respect to t_1 (t_2) is defined by*

$$\mathcal{L}_{t_1,t_2}\left[{}^C D_{t_1}^{\alpha_1} f(t_1,t_2)\right]=p^{\alpha_1}F(p,s)-\sum_{k=1}^{N_1}p^{\alpha_1-k}F_{t_1}^{(k-1)}(0,s)$$
$$\left(\mathcal{L}_{t_1,t_2}\left[{}^C D_{t_2}^{\alpha_2} f(t_1,t_2)\right]=s^{\alpha_2}F(p,s)-\sum_{l=1}^{N_2}s^{\alpha_2-l}F_{t_2}^{(l-1)}(p,0)\right), \tag{A.13}$$

where

$$F_{t_1}^{(k)}(0,s)=\mathcal{L}_{t_2}\left\{\left.\frac{\partial^k f(t_1,t_2)}{\partial {t_1}^k}\right|_{t_1=0}\right\},$$
$$F_{t_2}^{(l)}(p,0)=\mathcal{L}_{t_1}\left\{\left.\frac{\partial^l f(t_1,t_2)}{\partial {t_2}^l}\right|_{t_2=0}\right\},$$

for $k,l\in\mathbb{Z}_+$.

Theorem A.10. *[171] The inverse 2D Laplace transform of $p^{\alpha_1}s^{\alpha_2}F(p,s)$ is given by*

$$\begin{aligned}\mathcal{L}^{-1}_{t_1,t_2}\left[p^{\alpha_1}s^{\alpha_2}F(p,s)\right] =& D^{\alpha_1,\alpha_2}_{t_1,t_2}f(t_1,t_2)\\ &+\sum_{k=0}^{M_1-1}\frac{t_1^{k-\alpha_1}}{\Gamma(k-\alpha_1+1)}D^{\alpha_2}_{t_2}\left.\frac{\partial^k f(t_1,t_2)}{\partial {t_1}^k}\right|_{t_1=0}\\ &+\sum_{l=0}^{M_2-1}\frac{t_2^{l-\alpha_2}}{\Gamma(l-\alpha_2+1)}D^{\alpha_1}_{t_1}\left.\frac{\partial^l f(t_1,t_2)}{\partial {t_2}^l}\right|_{t_2=0}\\ &+\sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2-1}\frac{t_1^{k-\alpha_1}t_2^{l-\alpha_2}}{\Gamma(k-\alpha_1+1)\Gamma(l-\alpha_2+1)}\left.\frac{\partial^k}{\partial {t_1}^k}\frac{\partial^l}{\partial {t_2}^l}f(t_1,t_2)\right|_{t_1=0,t_2=0}\end{aligned} \tag{A.14}$$

where $M_i-1<\alpha_i<M_i$, $M_i\in\mathbb{Z}$ *for* $i=1,2$ *and*

$$D^{\alpha_1,\alpha_2}_{t_1,t_2}f(t_1,t_2)=D^{\alpha_1}_{t_1}\left[D^{\alpha_2}_{t_2}f(t_1,t_2)\right]=D^{\alpha_2}_{t_2}\left[D^{\alpha_1}_{t_1}f(t_1,t_2)\right]. \tag{A.15}$$

A.4 $\mathcal{Z}$-Transforms of Discrete-Time Functions

Theorem A.11. *If*

$$\mathcal{Z}[x_i]=\sum_{i=0}^{\infty}x_iz^{-i}, \tag{A.16}$$

then

$$\mathcal{Z}[x_{i+1}]=zX(z)-zx_0, \tag{A.17a}$$

$$\mathcal{Z}[x_{i-p}]=z^{-p}X(z)+z^{-p}\sum_{j=-1}^{-p}x_jz^{-j}. \tag{A.17b}$$

Proof. Using (A.16) we obtain

$$\mathcal{Z}[x_{i+1}]=\sum_{i=0}^{\infty}x_{i+1}z^{-i}=\sum_{j=1}^{\infty}x_jz^{-(j-1)}=z\sum_{j=0}^{\infty}x_jz^{-j}-zx_0=zX(z)-zx_0,$$

$$\begin{aligned}\mathcal{Z}[x_{i-p}]&=\sum_{i=0}^{\infty}x_{i-p}z^{-i}=\sum_{j=-p}^{\infty}x_jz^{-(j+p)}=z^{-p}\sum_{j=-p}^{\infty}x_jz^{-j}\\ &=z^{-p}\sum_{j=0}^{\infty}x_jz^{-j}+z^{-p}\sum_{j=-1}^{-p}x_jz^{-j}=z^{-p}X(z)+z^{-p}\sum_{j=-1}^{-p}x_jz^{-j}.\end{aligned}$$

□

Theorem A.12. *Let* $X(z_1,z_2)$ *be the 2D* $\mathcal{Z}$*-transform of the function* x_{ij} *defined by*

$$\mathcal{Z}[x_{ij}]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}x_{ij}z_1^{-i}z_2^{-j}. \tag{A.18}$$

Then

$$\begin{aligned}
\mathcal{Z}[x_{i+1,j+1}] &= z_1 z_2 [X(z_1, z_2) - X(z_1, 0) - X(0, z_2) + x_{00}],\\
\mathcal{Z}[x_{i-k,j+1}] &= z_1^{-k} z_2 [X(z_1, z_2) - X(z_1, 0)],\\
\mathcal{Z}[x_{i+1,j-l}] &= z_1 z_2^{-l} [X(z_1, z_2) - X(0, z_2)],\\
\mathcal{Z}[x_{i-k,j-l}] &= z_1^{-k} z_2^{-l} X(z_1, z_2),\\
\mathcal{Z}[x_{i+1,j}] &= z_1 [X(z_1, z_2) - X(0, z_2)],\\
\mathcal{Z}[x_{i,j+1}] &= z_2 [X(z_1, z_2) - X(z_1, 0)],
\end{aligned}$$

where

$$X(z_1, 0) = \sum_{i=0}^{\infty} x_{i0} z_1^{-i}, \quad X(0, z_2) = \sum_{j=0}^{\infty} x_{0j} z_2^{-j}.$$

Proof. Using (A.18) we obtain

$$\begin{aligned}
\mathcal{Z}[x_{i+1,j+1}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i+1,j+1} z_1^{-i} z_2^{-j} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} x_{kl} z_1^{1-k} z_2^{1-l}\\
&= z_1 z_2 \left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} x_{kl} z_1^{-k} z_2^{-l} - \sum_{k=0}^{\infty} x_{k0} z_1^{-k} - \sum_{l=0}^{\infty} x_{0l} z_2^{-l} + x_{00}\right],\\
\mathcal{Z}[x_{i-k,j+1}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i-k,j+l} z_1^{-i} z_2^{-j} = \sum_{p=1}^{\infty}\sum_{q=-l}^{\infty} x_{pq} z_1^{1-p} z_2^{-(q+l)}\\
&= z_1 z_2^{-l} \left[\sum_{p=1}^{\infty}\sum_{q=-l}^{\infty} x_{pq} z_1^{-p} z_2^{-q}\right]\\
&= z_1 z_2^{-l} \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} x_{pq} z_1^{-p} z_2^{-q} - \sum_{i=0}^{\infty} x_{i0} z_1^{-i}\right],\\
\mathcal{Z}[x_{i+1,j-l}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i+1,j-l} z_1^{-i} z_2^{-j} = \sum_{p=-k}^{\infty}\sum_{q=1}^{\infty} x_{pq} z_1^{-(p+k)} z_2^{1-q}\\
&= z_1^{-k} z_2 \left[\sum_{p=-k}^{\infty}\sum_{q=1}^{\infty} x_{pq} z_1^{-p} z_2^{-q}\right]\\
&= z_1^{-k} z_2 \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} x_{pq} z_1^{-p} z_2^{-q} - \sum_{j=0}^{\infty} x_{0j} z_2^{-j}\right],\\
\mathcal{Z}[x_{i-k,j-l}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i-k,j-l} z_1^{-i} z_2^{-j} = \sum_{p=-k}^{\infty}\sum_{q=-l}^{\infty} x_{pq} z_1^{-(p+k)} z_2^{-(q+l)}\\
&= z_1^{-k} z_2^{-l} \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} x_{pq} z_1^{-p} z_2^{-q}\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{Z}[x_{i+1,j}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i+1,j} z_1^{-i} z_2^{-j} = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} x_{kj} z_1^{1-k} z_2^{-j} \\
&= z_1 \left[\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} x_{kj} z_1^{-k} z_2^{-j}\right] \\
&= z_1 \left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} x_{kj} z_1^{-k} z_2^{-j} - \sum_{j=0}^{\infty} x_{0j} z_2^{-j}\right], \\
\mathcal{Z}[x_{i,j+1}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i,j+1} z_1^{-i} z_2^{-j} = \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} x_{ik} z_1^{-i} z_2^{1-k} \\
&= z_2 \left[\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} x_{ik} z_1^{-i} z_2^{-k}\right] = z_2 \left[\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} x_{ik} z_1^{-i} z_2^{-k} - \sum_{i=0}^{\infty} x_{i0} z_1^{-i}\right].
\end{aligned}$$

□

Appendix B
Elementary Operations on Matrices

Definition B.1. The following operations are called elementary operations on a real matrix $A \in \mathbb{R}^{n \times m}$:

a) multiplication of any i-th row (column) by the number $a \neq 0$,
b) addition to any i-th row (column) of the j-th row (column) multiplied by any number $b \neq 0$,
c) the interchange of any two rows (columns).

In this book the following notation is used.

$L[i \times a]$	multiplication of the i-th row by the number $a \neq 0$,
$R[i \times a]$	multiplication of the i-th column by the number $a \neq 0$,
$L[i + j \times b]$	addition to the i-th row of the j-th row multiplied by the number $b \neq 0$,
$R[i + j \times b]$	addition to the i-th column of the j-th column multiplied by the number $b \neq 0$,
$L[i, j]$	the interchange of the i-th and the j-th rows,
$R[i, j]$	the interchange of the i-th and the j-th columns.

The elementary operations can be extended to polynomial matrices [50].

Appendix C
Nilpotent Matrices

Definition C.1. A real matrix $A \in \mathbb{R}^{n \times n}$ is called nilpotent if there exists a natural number $v \leq n$ such that $A^{v-1} \neq 0$ and $A^v = 0$. The natural number v is called the nilpotency index of the matrix A.

Lemma C.1. Matrices of the form

$$A = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{n,2} & \cdots & a_{n,n-1} & 0 \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{C.1}$$

have the nilpotency index $v = 2$ for any values of the entries $a_{12}, \ldots, a_{1,n-1}$, $a_{n,2}, \ldots, a_{n,n-1}$ and the characteristic polynomials of the form

$$\det[\mathbb{I}_n \lambda - A] = \lambda^n.$$

Proof. Using (C.1) it is easy to check that $A^2 = 0$ and

$$\det[\mathbb{I}_n \lambda - A] = \begin{vmatrix} \lambda & -a_{12} & \cdots & -a_{1,n-1} & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & -a_{n,2} & \cdots & -a_{n,n-1} & \lambda \end{vmatrix} = \lambda^n.$$

□

Lemma C.2. Matrices of the form

$$A = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \tag{C.2}$$

have the nilpotency index $v = 2$ and the chracteristic polynomials of the form

$$\det[\mathbb{I}_{2n}\lambda - A] = \lambda^{2n}$$

for any submatrices $A_{12} \in \mathbb{R}^{n\times n}$.

Proof. Using (C.2) it is easy to verify that $A^2 = 0$ and

$$\det[\mathbb{I}_{2n}\lambda - A] = \begin{vmatrix} \mathbb{I}_n\lambda & -A_{12} \\ 0 & \mathbb{I}_n \end{vmatrix} = \lambda^{2n}.$$

□

From the well-known property of the transposition of the matrix A, $\left(A^k\right)^T = \left(A^T\right)^k$ for $k = 1, 2, \ldots$ we have the following remark.

Remark C.1. The transpose matrix A^T has a nilpotency index v if and only if the matrix A has the same nilpotency index v.

Lemma C.3. A diagonal matrix A with at least one nonzero entry is not the nilpotent matrix.

Proof. This follows immediately from the relation

$$A^k = (\mathrm{diag}\,[a_1, \ldots, a_n])^k = \mathrm{diag}\left[a_1^k, \ldots, a_n^k\right] \neq 0$$

for $k = 1, 2, \ldots$ if at least one from the entries $a_1, \ldots, a_n$ is nonzero. □

Lemma C.4. A nonnegative matrix $A \in \mathbb{R}^{n\times n}$ with at least one nonzero diagonal entry is not nilpotent matrix.

Proof. Let decompose the matrix A as the sum of the diagonal matrix D and the nonnegative matrix B with zero diagonal entries. Let assume that D and B are commutative matrices, i.e. $BD = DB$. Then

$$A^k = (D + B)^k = D^k + BD^{k-1} + \ldots + B^k \quad \text{for } k = 1, 2 \ldots \tag{C.3}$$

If the matrix A has at least one nonzero diagonal entry, then $D \neq 0$ and by Lemma C.3, $D^k \neq 0$ for $k = 1, 2, \ldots$. From (C.3) we have $A^k \neq 0$ for $k = 1, 2, \ldots$, since $D^k \neq 0$ and the remaining entries are nonnegative.
□

Appendix D
Drazin Inverse Matrix

D.1 Definition and Properties of Drazin Inverse Matrix

Definition D.1. [44] The smallest nonnegative integer q satisfying

$$\text{rank}E^q = \text{rank}E^{q+1} \tag{D.1}$$

is called the index of the matrix $E \in \mathbb{R}^{n\times n}$.

Definition D.2. [44] A matrix $E^D \in \mathbb{R}^{n\times n}$ is called the Drazin inverse of $E \in \mathbb{R}^{n\times n}$ if it satisfies the conditions

$$EE^D = E^D E, \tag{D.2a}$$
$$E^D E E^D = E^D, \tag{D.2b}$$
$$E^D E^{q+1} = E^q, \tag{D.2c}$$

where q is the index of E defined by (D.1).

The Drazin inverse E^D of a square matrix E always exists and is unique [18, 44]. If $\det E \neq 0$, then $E^D = E^{-1}$. Some methods for computation of the Drazin inverse are given in [44].

Let us assume that

$$\bar{E} = [Ec - F]^{-1} E, \quad \bar{F} = [Ec - F]^{-1} F, \tag{D.3}$$

where

$$\det [Ec - F] \neq 0 \quad \text{for some } c \in \mathbb{C}. \tag{D.4}$$

Then the matrices $\bar{E}$ and $\bar{F}$ are commuting matrices [44], i.e.

$$\bar{E}\bar{F} = \bar{F}\bar{E} \tag{D.5}$$

for c satisfying the condition (D.4).

Lemma D.1. [18, 44] The matrices $\bar{E}$ and $\bar{F}$ defined by (D.3) satisfy the following equalities

$$\bar{F}^D\bar{E} = \bar{E}\bar{F}^D, \quad \bar{E}^D\bar{F} = \bar{F}\bar{E}^D, \quad \bar{F}^D\bar{E}^D = \bar{E}^D\bar{F}^D, \tag{D.6a}$$

$$\ker\bar{F} \cap \ker\bar{E} = \{0\}, \tag{D.6b}$$

$$\bar{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}, \quad \bar{F} = T\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}T^{-1}, \quad \bar{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}, \tag{D.6c}$$

where $\det T \neq 0$, $J \in \mathbb{R}^{n_1 \times n_1}$ is nonsingular, $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent, $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$, $n_1 + n_2 = n$;

$$\left(\mathbb{I}_n - \bar{E}\bar{E}^D\right)\bar{F}\bar{F}^D = \mathbb{I}_n - \bar{E}\bar{E}^D \quad \text{and} \quad \left(\mathbb{I}_n - \bar{E}\bar{E}^D\right)\left(\bar{E}\bar{F}^D\right)^q = 0. \tag{D.6d}$$

D.2 Procedure for Computation of Drazin Inverse Matrices

To compute the Drazin inverse E^D of the matrix $E \in \mathbb{R}^{n \times n}$ defined by (D.2) the following procedure is recommended.

Procedure C.2.1

Step 1. Find the pair of matrices $V \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times n}$, such that

$$E = VW, \quad \text{rank}V = \text{rank}W = \text{rank}E = r. \tag{D.7}$$

As the r columns (rows) of the matrix V (W) the r linearly independent columns (rows) of the matrix E can be chosen.

Step 2. Compute the nonsingular matrix

$$WEV \in \mathbb{R}^{r \times r}. \tag{D.8}$$

Step 3. The desired Drazin inverse matrix is given by

$$E^D = V\left[WEV\right]^{-1}W. \tag{D.9}$$

Proof. It will be shown that the matrix (D.9) satisfies the three conditions (D.2) of Definition D.2.

Taking into account that $\det WV \neq 0$ and (D.7) we obtain

$$[WEV]^{-1} = [WVWV]^{-1} = [WV]^{-1} [WV]^{-1}. \tag{D.10}$$

Using (D.2a), (D.7) and (D.10) we obtain

$$EE^D = VWV [WEV]^{-1} W = VWV [WV]^{-1} [WV]^{-1} W = V [WV]^{-1} W$$

and

$$E^D E = V [WEV]^{-1} WVW = V [WV]^{-1} [WV]^{-1} WVW = V [WV]^{-1} W.$$

Therefore, the condition (D.2a) is satisfied.

To check the condition (D.2b) we compute

$$\begin{aligned} E^D E E^D =& V [WEV]^{-1} WVWV [WEV]^{-1} W \\ =& V [WVWV]^{-1} WVWV [WEV]^{-1} W \\ =& V [WEV]^{-1} W = E^D. \end{aligned} \tag{D.11}$$

Therefore, the condition (D.2b) is also satisfied.

Using (D.2c) , (D.7), (D.9) and (D.10) we obtain

$$\begin{aligned} E^D E^{q+1} =& V [WEV]^{-1} W (VW)^{q+1} \\ &= V [WV]^{-1} [WV]^{-1} WVW (VW)^q \\ &= V [WV]^{-1} W (VW)^q \\ &= V [WV]^{-1} WVW (VW)^{q-1} \\ &= (VW)^q = E^q, \end{aligned} \tag{D.12}$$

where q is the index of E.

Therefore, the condition (D.2c) is also satisfied. □

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