D

∫

Σ

Yury A. Brychkov

HANDBOOK OF

Special Functions

Derivatives, Integrals, Series and Other Formulas

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

HANDBOOK OF

Special Functions

Derivatives, Integrals, Series and Other Formulas

HANDBOOK OF

Special Functions

Derivatives, Integrals, Series and Other Formulas

Yury A. Brychkov

Computing Center of the Russian Academy of Sciences

Moscow, Russia

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the Taylor & Francis Group, an **informa** business

A CHAPMAN & HALL BOOK

Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-13: 978-1-58488-956-4 (Hardcover)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Contents

Preface

The diversity of problems whose solutions require knowledge of properties of elementary and special functions of mathematical physics has given rise to a large number of handbooks in this field of Calculus. Among these are, above all, the books of the Bateman Manuscript Project, namely, *Higher Transcendental Functions*, Vol. 1–3, and *Tables of Integral Transforms*, Vol. 1–2, by A. Erdelyi (Ed.); *Table of Integrals, Series, and Products* by I.S. Gradshteyn and I.M. Ryzhik; *Handbook of Mathematical Functions* by M. Abramowitz and I. Stegun (Eds.); and the 5-volume handbook *Integrals and Series* by A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev. Due to numerous applications in science and engineering, the theory of special functions is under permanent development, especially in connection with the requirements of modern computer algebra methods.

The present handbook contains mainly new results. Some known formulas are added for the sake of completeness. Special attention is paid to formulas of derivatives of n-th order with respect to the argument and of the first derivatives with respect to the parameters for most elementary and special functions. A considerable part of the book is devoted to formulas of connection and conversion for elementary and special functions, especially hypergeometric and Meijer G functions.

Chapter 1 contains differentiation formulas for various functions. In Chapter 2 limit formulas are given for the special functions that depend on parameters. Chapters 3 to 6 contain formulas of integration and summation for elementary and special functions, new classes of integrals, finite sums and infinite series being considered. In Chapter 7 connection formulas are given for various elementary and special functions. Chapter 8 is devoted to representations of hypergeometric functions and Meijer G functions in terms of other functions for various values of parameters and arguments.

The notations that are standard are listed at the end of the book. In all chapters, unless other restrictions are indicated, k, l, m, n, p, $q = 0, 1, 2, \ldots$

The author hopes that this handbook will be useful to scientists, engineers, postgraduate students and generally to anybody who uses mathematical methods.

Chapter 1

The Derivatives

1.1. Elementary Functions

1.1.1. General formulas

1. $\mathrm{D}^n[f(z^2)] = n! \sum_{k=0}^{[n/2]} \frac{(2z)^{n-2k}}{k!\,(n-2k)!} f^{(n-k)}(z^2).$

2. $\mathrm{D}^n[f(\sqrt{z})] = \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!\,(n-k-1)!} (2\sqrt{z})^{-n-k} f^{(n-k)}(\sqrt{z}) \qquad [n \geq 1].$

3. $\mathrm{D}^n\left[f\left(\frac{1}{z}\right)\right] = (-1)^n (n-1)! \sum_{k=0}^{n-1} \binom{n}{k} \frac{z^{k-2n}}{(n-k-1)!} f^{(n-k)}\left(\frac{1}{z}\right) \qquad [n \geq 1].$

4. $\mathrm{D}^n\left[z^{n-1} f\left(\frac{1}{z}\right)\right] = (-1)^n z^{-n-1} F\left(\frac{1}{z}\right),$ if $\mathrm{D}^n[f(z)] = F(z).$

5. $$\mathrm{D}^n[f_1(z) f_2(z) \ldots f_m(z)] = \sum_{k_1=0}^{n} \binom{n}{k_1} f_1^{(k_1)}(z) \sum_{k_2=0}^{n-k_1} \binom{n-k_1}{k_2} f_2^{(k_2)}(z) \ldots \\ \times \sum_{k_{m-1}=0}^{n-k_1-\ldots-k_{m-2}} \binom{n-k_1-\ldots-k_{m-2}}{k_{m-1}} f_{m-1}^{(k_{m-1})}(z) f_m^{(n-k_1-\ldots-k_{m-1})}(z).$$

6. $$\mathrm{D}^n\left[z^{m+n} f\left(\frac{1}{z}\right) \mathrm{D}^m\left[z^{m-1} g\left(\frac{1}{z}\right)\right]\right] = (-1)^{m+n} z^{-n-1} F\left(\frac{1}{z}\right),$$
if $\mathrm{D}^n[f(z)\,\mathrm{D}^m[g(z)]] = F(z).$

1.1.2. Algebraic functions

1. $\mathrm{D}^n[z^\lambda] = (-1)^n (-\lambda)_n z^{\lambda-n}.$

2. $\mathrm{D}^n[z^\alpha (a-z)^\beta] = n!\, a^n z^{\alpha-n} (a-z)^{\beta-n} P_n^{(\alpha-n,\beta-n)}\left(1 - \frac{2z}{a}\right).$

3. $\mathrm{D}^n[z^\alpha (a-z)^\beta] = n!\, z^\alpha (a-z)^{\beta-n} P_n^{(-\alpha-\beta-1,\beta-n)}\left(1 - \frac{2a}{z}\right).$

4. $\mathrm{D}^n[z^\alpha (a-z)^\beta] = n!\, z^{\alpha-n} (a-z)^\beta P_n^{(\alpha-n,-\alpha-\beta-1)}\left(\frac{a+z}{a-z}\right).$

5. $\mathrm{D}^n[z^\lambda(a+z)^\lambda] = \left(\frac{a}{4}\right)^n n!\,\frac{(-2\lambda)_n}{\left(\frac{1}{2}-\lambda\right)_n} z^{\lambda-n}(a+z)^{\lambda-n} C_n^{\lambda-n+1/2}\left(1+\frac{2z}{a}\right).$

6. $\quad = (-1)^n n!\,(z^2+az)^{\lambda-n/2} C_n^{-\lambda}\left(\frac{2z+a}{2\sqrt{z^2+az}}\right).$

7. $\mathrm{D}^n[z^{-1}(a+z)^{-1}]$

$$= 2(-1)^{n+1} n!\,a^{-1}(a+z)^{-n-1}\left[1-\left(\frac{a+z}{z}\right)^{(n+1)/2} T_{n+1}\left(\frac{a+2z}{2\sqrt{az+z^2}}\right)\right].$$

8. $\mathrm{D}^n[z^{-\lambda-1/2}(a-z)^\lambda]$

$$= 2^{-2n}\,\frac{(2n)!}{\left(\frac{1}{2}-\lambda\right)_n} z^{-\lambda-1/2}(a-z)^{\lambda-n}\, C_{2n}^{\lambda-n+1/2}\left(\sqrt{\frac{a}{z}}\right).$$

9. $\mathrm{D}^n[z^\lambda(a-z)^{n-2\lambda-1}] = (-1)^n n!\,a^{n/2} z^{\lambda-n/2}(a-z)^{-2\lambda-1}\, C_n^{-\lambda}\left(\frac{z+a}{2\sqrt{az}}\right).$

10. $\mathrm{D}^n[z^{n-2\lambda-1}(a+z)^\lambda]$

$$= \left(-\frac{1}{4}\right)^n n!\,\frac{(-2\lambda)_n}{\left(\frac{1}{2}-\lambda\right)_n} z^{n-2\lambda-1}(a+z)^{\lambda-n} C_n^{\lambda-n+1/2}\left(1+\frac{2a}{z}\right).$$

11. $\mathrm{D}^n[z^{-\lambda-3/2}(a-z)^\lambda]$

$$= -\frac{n!}{2}\,\frac{\left(\frac{3}{2}\right)_n}{\left(-\lambda-\frac{1}{2}\right)_{n+1}} a^{-1/2} z^{-\lambda-1}(a-z)^{\lambda-n} C_{2n+1}^{\lambda-n+1/2}\left(\sqrt{\frac{a}{z}}\right).$$

12. $\mathrm{D}^n[z^{n-1/2}(a-z)^\lambda] = n!\,\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{1}{2}-\lambda\right)_n} a^n z^{-1/2}(a-z)^{\lambda-n} C_{2n}^{\lambda-n+1/2}\left(\sqrt{\frac{z}{a}}\right).$

13. $\mathrm{D}^n[z^{n+1/2}(a+z)^\lambda]$

$$= -2^{-2n-1}\,\frac{(2n+1)!}{(\lambda+1)_{n+1}} (z+a)^{\lambda+1/2} C_{2n+1}^{-\lambda-n-1}\left(\sqrt{\frac{z}{z+a}}\right).$$

14. $\mathrm{D}^n[z^{n-1/2}(a+z)^\lambda] = 2^{-2n}\,\frac{(2n)!}{(\lambda+1)_n} z^{-1/2}(a+z)^\lambda C_{2n}^{-\lambda-n}\left(\sqrt{\frac{z}{a+z}}\right).$

15. $\mathrm{D}^n[z^n(a+z)^{-1/2}] = n!\,a^{n/2}(a+z)^{-(n+1)/2} P_n\left(\frac{2a+z}{2\sqrt{a^2+az}}\right).$

16. $\mathrm{D}^n[z^n(a+z)^{n-1/2}] = n!\,a^n(a+z)^{-1/2} P_{2n}\left(\sqrt{1+\frac{z}{a}}\right).$

17. $\mathrm{D}^n[z^n(a+z)^{n+1/2}] = n!\,a^{n+1/2} P_{2n+1}\left(\sqrt{1+\frac{z}{a}}\right).$

18. $\mathrm{D}^n[z^n(a+z)^{-n-1/2}] = n!\,(a+z)^{-n-1/2} P_{2n}\left(\sqrt{\frac{a}{a+z}}\right).$

19. $D^n[z^n(a+z)^{-n-3/2}] = n!\,a^{-1/2}(a+z)^{-n-1}P_{2n+1}\left(\sqrt{\frac{a}{a+z}}\right).$

20. $D^n[z^n(a+z)^n] = n!\,a^n P_n\left(1+\frac{2z}{a}\right).$

21. $D^n[z^{-n-1}(a+z)^n] = (-1)^n n!\,z^{-n-1}P_n\left(1+\frac{2a}{z}\right).$

22. $D^n[z^{n-1/2}(a+z)^{-n-1/2}] = (-1)^n n!\,z^{-1/2}(a+z)^{-n-1/2}P_{2n}\left(\sqrt{\frac{z}{a+z}}\right).$

23. $D^n[z^{n-1/2}(a+z)^n] = n!\,(-a)^n z^{-1/2}P_{2n}\left(\sqrt{\frac{z}{a}}\right).$

24. $D^n[z^{n+1/2}(a+z)^{n+1/2}] = \frac{\left(\frac{3}{2}\right)_n}{n+1}a^n z^{1/2}(a+z)^{1/2}U_n\left(1+\frac{2z}{a}\right).$

25. $\qquad = \frac{\left(\frac{3}{2}\right)_n}{2(n+1)}a^{n+1/2}z^{1/2}U_{2n+1}\left(\sqrt{1+\frac{z}{a}}\right).$

26. $D^n[z^{n-1/2}(a+z)^{n+1/2}] = \left(\frac{1}{2}\right)_n a^{n+1/2}z^{-1/2}T_{2n+1}\left(\sqrt{1+\frac{z}{a}}\right).$

27. $D^n[z^{n+1/2}(a+z)^{n-1/2}] = \left(\frac{1}{2}\right)_n a^n z^{1/2}(a+z)^{-1/2}U_{2n}\left(\sqrt{1+\frac{z}{a}}\right).$

28. $D^n[z^{n+1/2}(a-z)^{n-1/2}] = (-1)^n\left(\frac{1}{2}\right)_n a^{n+1/2}(a-z)^{-1/2}T_{2n+1}\left(\sqrt{\frac{z}{a}}\right).$

29. $D^n[z^{n-1/2}(a+z)^{n-1/2}] = \left(\frac{1}{2}\right)_n a^n z^{-1/2}(a+z)^{-1/2}T_{2n}\left(\sqrt{1+\frac{z}{a}}\right).$

30. $D^n[z^{n-1/2}(a+z)^{-n}] = \left(\frac{1}{2}\right)_n z^{-1/2}(a+z)^{-n}T_{2n}\left(\sqrt{\frac{a}{a+z}}\right).$

31. $D^n[z^{n+1/2}(a+z)^{-n-1}] = \left(\frac{1}{2}\right)_n z^{1/2}(a+z)^{-n-1}U_{2n}\left(\sqrt{\frac{a}{a+z}}\right).$

32. $\qquad = (-1)^n\left(\frac{1}{2}\right)_n (a+z)^{-n-1/2}T_{2n+1}\left(\sqrt{\frac{z}{a+z}}\right).$

33. $D^n[z^{n-1/2}(a+z)^{-n-1}]$

$$= \left(\frac{1}{2}\right)_n (az)^{-1/2}(a+z)^{-n-1/2}T_{2n+1}\left(\sqrt{\frac{a}{a+z}}\right).$$

34. $\qquad = (-1)^n\left(\frac{1}{2}\right)_n z^{-1/2}(a+z)^{-n-1}U_{2n}\left(\sqrt{\frac{z}{a+z}}\right).$

35. $D^n[z^{n+1/2}(a+z)^{-n-2}]$

$$= (-1)^n 2^{-2n}\frac{(2n+1)!}{(n+1)!}z^{1/2}(a+z)^{-n-2}U_n\left(\frac{z-a}{z+a}\right).$$

36. $$= \frac{\left(\frac{3}{2}\right)_n}{2(n+1)} \left(\frac{z}{a}\right)^{1/2} (a+z)^{-n-3/2} U_{2n+1}\left(\sqrt{\frac{a}{a+z}}\right).$$

37. $$\mathrm{D}^n[(a^2-z^2)^\lambda] = (-2a)^n n! \frac{(-\lambda)_n}{(n-2\lambda)_n} (a^2-z^2)^{\lambda-n} C_n^{\lambda-n+1/2}\left(\frac{z}{a}\right).$$

38. $$\mathrm{D}^n[(a^2-z^2)^{-1}] = n!\, a^{-1}(a-z)^{-n-1}\left[1-\left(\frac{z-a}{z+a}\right)^{(n+1)/2} T_{n+1}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right].$$

39. $$\mathrm{D}^{2n}[(a^2-z^2)^{-1}] = (2n)!\, a^{-1}(a^2-z^2)^{-n-1/2} T_{2n+1}\left(\frac{a}{\sqrt{a^2-z^2}}\right).$$

40. $$\mathrm{D}^{2n+1}[(a^2-z^2)^{-1}] = (2n+1)!\, a^{-1} z (a^2-z^2)^{-n-3/2} U_{2n+1}\left(\frac{a}{\sqrt{a^2-z^2}}\right).$$

41. $$\mathrm{D}^n[z^{-1/2}(a+z)^{-1/2}] = (-1)^n n!\, z^{-(n+1)/2}(a+z)^{-(n+1)/2} P_n\left(\frac{2z+a}{2\sqrt{az+z^2}}\right).$$

42. $$\mathrm{D}^n[(a^2+z^2)^{-1/2}] = (-1)^n n!\, (a^2+z^2)^{-(n+1)/2} P_n\left(\frac{z}{\sqrt{a^2+z^2}}\right).$$

43. $$\mathrm{D}^n[(a^2-z^2)^n] = (-2a)^n n!\, P_n\left(\frac{z}{a}\right).$$

44. $$\mathrm{D}^n[(a^2-z^2)^{n-1/2}] = (-2a)^n \left(\frac{1}{2}\right)_n (a^2-z^2)^{-1/2} T_n\left(\frac{z}{a}\right).$$

45. $$\mathrm{D}^n[(az-z^2)^{n-1/2}] = (-a)^n \left(\frac{1}{2}\right)_n (az-z^2)^{-1/2} T_{2n}\left(\sqrt{\frac{z}{a}}\right).$$

46. $$\mathrm{D}^n[(a^2-z^2)^{n+1/2}] = \frac{(-2a)^n}{n+1} \left(\frac{3}{2}\right)_n (a^2-z^2)^{1/2} U_n\left(\frac{z}{a}\right).$$

47. $$\mathrm{D}^n[z^{-n-1}(a^2-z^2)^n] = (-2a)^n n!\, z^{-n-1} P_n\left(\frac{a}{z}\right).$$

48. $$\mathrm{D}^n[z^{n-2\lambda-1}(z^2-a^2)^\lambda] = (2a)^n n! \frac{(-\lambda)_n}{(n-2\lambda)_n} z^{n-2\lambda-1}(z^2-a^2)^{\lambda-n} C_n^{\lambda-n+1/2}\left(\frac{a}{z}\right).$$

49. $$\mathrm{D}^n\left[\frac{1}{\sqrt{z^2+a^2}}\left(z+\sqrt{z^2+a^2}\right)^{1/2}\right] = (-1)^n \frac{\sqrt{2}}{a}\left(\frac{1}{2}\right)_n (z^2+a^2)^{-(2n+1)/4} \sin\left(\frac{2n+1}{2} \operatorname{arccot} \frac{z}{a}\right).$$

1.1.3. The exponential function

1. $$\mathrm{D}^n\left[z^\lambda e^{-az}\right] = n!\, z^{\lambda-n} e^{-az} L_n^{\lambda-n}(az).$$

2. $$\mathrm{D}^n\left[z^\lambda e^{-a/z}\right] = (-1)^n n!\, z^{\lambda-n} e^{-a/z} L_n^{-\lambda-1}\left(\frac{a}{z}\right).$$

3. $\mathrm{D}^n\left[e^{az^2}\right] = (-i)^n a^{n/2} e^{az^2} H_n(i\sqrt{a}\,z).$

4. $\mathrm{D}^n\left[z^{n-1}e^{a/z^2}\right] = i^n a^{n/2} z^{-n-1} e^{a/z^2} H_n\left(\frac{i\sqrt{a}}{z}\right).$

5. $\mathrm{D}^n\left[e^{-az^2}\right] = (-1)^n a^{n/2} e^{-az^2} H_n(\sqrt{a}\,z).$

6. $\mathrm{D}^n\left[z^{n-1}e^{-a/z^2}\right] = a^{n/2} z^{-n-1} e^{-a/z^2} H_n\left(\frac{\sqrt{a}}{z}\right).$

7. $\mathrm{D}^n\left[e^{a\sqrt{z}}\right] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{(1-2n)/4}[I_{n-1/2}(a\sqrt{z}) + I_{1/2-n}(a\sqrt{z})].$

8. $\mathrm{D}^n\left[\frac{1}{\sqrt{z}}\,e^{a\sqrt{z}}\right] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(1+2n)/4}[I_{n+1/2}(a\sqrt{z}) + I_{-n-1/2}(a\sqrt{z})].$

9. $\mathrm{D}^n\left[z^{n-1}e^{a/\sqrt{z}}\right]$

$$= (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(5+2n)/4}\left[I_{n-1/2}\left(\frac{a}{\sqrt{z}}\right) + I_{1/2-n}\left(\frac{a}{\sqrt{z}}\right)\right].$$

10. $\mathrm{D}^n\left[z^{n-1/2}e^{a/\sqrt{z}}\right]$

$$= (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(3+2n)/4}\left[I_{n+1/2}\left(\frac{a}{\sqrt{z}}\right) + I_{-n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right].$$

11. $\mathrm{D}^n\left[e^{-a\sqrt{z}}\right] = (-1)^n \frac{a^{n+1/2}}{2^{n-1/2}\sqrt{\pi}} z^{(1-2n)/4} K_{n-1/2}(a\sqrt{z}).$

12. $\mathrm{D}^n\left[\frac{1}{\sqrt{z}}\,e^{-a\sqrt{z}}\right] = (-1)^n \frac{a^{n+1/2}}{2^{n-1/2}\sqrt{\pi}} z^{-(1+2n)/4} K_{n+1/2}(a\sqrt{z}).$

13. $\mathrm{D}^n\left[z^{n-1}e^{-a/\sqrt{z}}\right] = \frac{a^{n+1/2}}{2^{n-1/2}\sqrt{\pi}} z^{-(2n+5)/4} K_{n-1/2}\left(\frac{a}{\sqrt{z}}\right).$

14. $\mathrm{D}^n\left[z^{-n-1/2}e^{-a/\sqrt{z}}\right] = \frac{a^{n+1/2}}{2^{n-1/2}\sqrt{\pi}} z^{-(3+2n)/4} K_{n+1/2}\left(\frac{a}{\sqrt{z}}\right).$

15. $\mathrm{D}^n\left[e^{(-1)^{j+1}ia\sqrt{z}}\right] = (-1)^{j+1} i\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{(1-2n)/4} H^{(j)}_{1/2-n}(a\sqrt{z})$

$[j = 1,\, 2].$

16. $= (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{(1-2n)/4} H^{(j)}_{n-1/2}(a\sqrt{z}) \quad [j = 1,\, 2].$

17. $\mathrm{D}^n\left[\frac{1}{\sqrt{z}}\,e^{(-1)^{j+1}ia\sqrt{z}}\right]$

$$= (-1)^{j+n+1} i\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(2n+1)/4} H^{(j)}_{n+1/2}(a\sqrt{z}) \quad [j = 1,\, 2].$$

18. $= \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(2n+1)/4} H^{(j)}_{-n-1/2}(a\sqrt{z}) \quad [j = 1,\, 2].$

1.1.4. Hyperbolic functions

1. $D^n[z^\lambda \sinh(az)] = \frac{n!}{2} z^{\lambda-n} \left[e^{az} L_n^{\lambda-n}(-az) - e^{-az} L_n^{\lambda-n}(az)\right].$

2. $D^n[z^\lambda \cosh(az)] = \frac{n!}{2} z^{\lambda-n} \left[e^{az} L_n^{\lambda-n}(-az) + e^{-az} L_n^{\lambda-n}(az)\right].$

3. $D^n[\operatorname{sech}(az)]$

$$= 2(-a)^n \sum_{k=0}^{n} \frac{e^{-kaz} \cos\left[(k+1)\operatorname{arccot} e^{-az}\right]}{(e^{-2az}+1)^{(k+1)/2}} \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^n.$$

4. $D^n[\sinh(az^2)] = \frac{(-i)^n}{2} a^{n/2} e^{az^2} H_n(i\sqrt{a}\,z) - \frac{(-1)^n}{2} a^{n/2} e^{-az^2} H_n(\sqrt{a}\,z).$

5. $D^n[\cosh(az^2)] = \frac{(-i)^n}{2} a^{n/2} e^{az^2} H_n(i\sqrt{a}\,z) + \frac{(-1)^n}{2} a^{n/2} e^{-az^2} H_n(\sqrt{a}\,z).$

6. $D^n[\sinh(a\sqrt{z})] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{(1-2n)/4} I_{1/2-n}(a\sqrt{z}).$

7. $D^n[\cosh(a\sqrt{z})] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{(1-2n)/4} I_{n-1/2}(a\sqrt{z}).$

8. $D^n\left[\frac{\sinh(a\sqrt{z})}{\sqrt{z}}\right] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(1+2n)/4} I_{n+1/2}(a\sqrt{z}).$

9. $D^n\left[\frac{\cosh(a\sqrt{z})}{\sqrt{z}}\right] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(1+2n)/4} I_{-n-1/2}(a\sqrt{z}).$

10. $D^n[\operatorname{sech}\sqrt{z}] = (-1)^n n!\,\pi \left(\frac{4}{\pi^2+4z}\right)^{n+1}$

$$\times\ {}_{2n+4}F_{2n+3}\left(\begin{matrix} 1, \frac{3}{2}, \frac{1}{2}-\frac{i\sqrt{z}}{\pi}, \ldots, \frac{1}{2}-\frac{i\sqrt{z}}{\pi}, \frac{1}{2}+\frac{i\sqrt{z}}{\pi}, \ldots, \frac{1}{2}+\frac{i\sqrt{z}}{\pi} \\ \frac{1}{2}, \frac{3}{2}-\frac{i\sqrt{z}}{\pi}, \ldots, \frac{3}{2}-\frac{i\sqrt{z}}{\pi}, \frac{3}{2}+\frac{i\sqrt{z}}{\pi}, \ldots, \frac{3}{2}+\frac{i\sqrt{z}}{\pi};\ -1 \end{matrix}\right).$$

11. $D^n\left[\frac{\operatorname{csch}\sqrt{z}}{\sqrt{z}}\right] = (-1)^{n+1}\frac{n!}{z^{n+1}} + \frac{2(-1)^n n!}{z^{n+1}}$

$$\times\ {}_{2n+3}F_{2n+2}\left(\begin{matrix} 1, -\frac{i\sqrt{z}}{\pi}, \ldots, -\frac{i\sqrt{z}}{\pi}, \frac{i\sqrt{z}}{\pi}, \ldots, \frac{i\sqrt{z}}{\pi} \\ 1-\frac{i\sqrt{z}}{\pi}, \ldots, 1-\frac{i\sqrt{z}}{\pi}, 1+\frac{i\sqrt{z}}{\pi}, \ldots, 1+\frac{i\sqrt{z}}{\pi};\ -1 \end{matrix}\right).$$

12. $D^n\left[\frac{\tanh\sqrt{z}}{\sqrt{z}}\right] = 2(-1)^n n! \left(\frac{4}{\pi^2+4z}\right)^{n+1}$

$$\times\ {}_{2n+3}F_{2n+2}\left(\begin{matrix} 1, \frac{1}{2}-\frac{i\sqrt{z}}{\pi}, \ldots, \frac{1}{2}-\frac{i\sqrt{z}}{\pi}, \frac{1}{2}+\frac{i\sqrt{z}}{\pi}, \ldots, \frac{1}{2}+\frac{i\sqrt{z}}{\pi} \\ \frac{3}{2}-\frac{i\sqrt{z}}{\pi}, \ldots, \frac{3}{2}-\frac{i\sqrt{z}}{\pi}, \frac{3}{2}+\frac{i\sqrt{z}}{\pi}, \ldots, \frac{3}{2}+\frac{i\sqrt{z}}{\pi};\ 1 \end{matrix}\right).$$

13. $\mathrm{D}^n\left[\dfrac{\coth\sqrt{z}}{\sqrt{z}}\right] = (-1)^{n+1}\dfrac{n!}{z^{n+1}}$

$$+\frac{2(-1)^n n!}{z^{n+1}}\,{}_{2n+3}F_{2n+2}\left(\begin{matrix} 1,\ -\frac{i\sqrt{z}}{\pi},\ \ldots,\ -\frac{i\sqrt{z}}{\pi},\ \frac{i\sqrt{z}}{\pi},\ \ldots,\ \frac{i\sqrt{z}}{\pi} \\ 1-\frac{i\sqrt{z}}{\pi},\ \ldots,\ 1-\frac{i\sqrt{z}}{\pi},\ 1+\frac{i\sqrt{z}}{\pi},\ \ldots,\ 1+\frac{i\sqrt{z}}{\pi};\ 1\end{matrix}\right).$$

14. $\mathrm{D}^n\left[z^{n-1}\sinh\left(\dfrac{a}{\sqrt{z}}\right)\right] = (-1)^n\sqrt{\pi}\left(\dfrac{a}{2}\right)^{n+1/2} z^{-(5+2n)/4} I_{1/2-n}\left(\dfrac{a}{\sqrt{z}}\right).$

15. $\mathrm{D}^n\left[z^{n-1}\cosh\left(\dfrac{a}{\sqrt{z}}\right)\right] = (-1)^n\sqrt{\pi}\left(\dfrac{a}{2}\right)^{n+1/2} z^{-(5+2n)/4} I_{n-1/2}\left(\dfrac{a}{\sqrt{z}}\right).$

16. $\mathrm{D}^n\left[z^{n-1/2}\sinh\left(\dfrac{a}{\sqrt{z}}\right)\right] = (-1)^n\sqrt{\pi}\left(\dfrac{a}{2}\right)^{n+1/2} z^{-(3+2n)/4} I_{n+1/2}\left(\dfrac{a}{\sqrt{z}}\right).$

17. $\mathrm{D}^n\left[z^{n-1/2}\cosh\left(\dfrac{a}{\sqrt{z}}\right)\right] = (-1)^n\sqrt{\pi}\left(\dfrac{a}{2}\right)^{n+1/2} z^{-(3+2n)/4} I_{-n-1/2}\left(\dfrac{a}{\sqrt{z}}\right).$

18. $\mathrm{D}^n[\sinh(a\sqrt[4]{z})]$

$$= \frac{\sqrt{\pi}}{2^{2n+1/2}} a^{n+1/2} z^{(1-6n)/8} \sum_{k=0}^{n-1} \frac{\Gamma(n+k)}{k!\,\Gamma(n-k)} (-a)^{-k} z^{-k/4} I_{k-n+1/2}(a\sqrt[4]{z}) \qquad [n \ge 1].$$

19. $\mathrm{D}^n[\cosh(a\sqrt[4]{z})]$

$$= \frac{\sqrt{\pi}}{2^{2n+1/2}} a^{n+1/2} z^{(1-6n)/8} \sum_{k=0}^{n-1} \frac{\Gamma(n+k)}{k!\,\Gamma(n-k)} (-a)^{-k} z^{-k/4} I_{n-k-1/2}(a\sqrt[4]{z}) \qquad [n \ge 1].$$

20. $\mathrm{D}^n\left[\dfrac{\sinh(a\sqrt[4]{z})}{\sqrt[4]{z}}\right]$

$$= \frac{\sqrt{\pi}}{2^{2n+1/2}} a^{n+1/2} z^{-(6n+1)/8} \sum_{k=0}^{n-1} \frac{\Gamma(n+k)}{k!\,\Gamma(n-k)} (-a)^{-k} z^{-k/4} I_{n-k+1/2}(a\sqrt[4]{z}) \qquad [n \ge 1].$$

21. $\mathrm{D}^n\left[\dfrac{\cosh(a\sqrt[4]{z})}{\sqrt[4]{z}}\right]$

$$= \frac{\sqrt{\pi}}{2^{2n+1/2}} a^{n+1/2} z^{-(6n+1)/8} \sum_{k=0}^{n-1} \frac{\Gamma(n+k)}{k!\,\Gamma(n-k)} (-a)^{-k} z^{-k/4} I_{k-n-1/2}(a\sqrt[4]{z}) \qquad [n \ge 1].$$

1.1.5. Trigonometric functions

1. $\mathrm{D}^n[\sin(az)] = a^n \sin\left(az + \dfrac{n\pi}{2}\right).$

2. $\mathrm{D}^n[\cos(az)] = a^n \cos\left(az + \frac{n\pi}{2}\right).$

3. $$\mathrm{D}^n[\sec(az)] = (-1)^{[(n+1)/2]} a^n \sum_{k=1}^{n} \frac{(-1)^k}{2^k} \binom{n+1}{k+1} \sec^{k+1}(az) \times \sum_{m=0}^{k} \binom{k}{m} (k-2m)^n \cos\left[(k-2m)az - \frac{1-(-1)^n}{4}\pi\right] \quad \text{[[13], (47)].}$$

4. $$= (-1)^{[(n+1)/2]} a^n \sum_{k=1}^{n} \frac{(-1)^k}{2^k} \binom{n+1}{k+1} \sum_{m=0}^{k} \binom{k}{m} (k-2m)^n \sec^{2m+1}(az) \times \sum_{p=0}^{[(k-\gamma)/2]-m} (-1)^p \binom{k-2m}{2p+\gamma} \tan^{2p+\gamma}(az) \quad \left[\gamma = \frac{1-(-1)^n}{2};\ \text{[13], (52)}\right].$$

5. $$= (-1)^{[(n+1)/2]} a^n \sec(az) \sum_{k=0}^{n} \frac{1}{2^k} \sum_{m=0}^{[(k-\gamma)/2]} (-1)^m \binom{k}{2m+\gamma} \times \tan^{2m+\gamma}(az) \sum_{p=0}^{k} (-1)^p \binom{k}{p} (2p+1)^n \quad \left[\gamma = \frac{1-(-1)^n}{2};\ \text{[13], (67)}\right].$$

6. $$= 2(ia)^n \sum_{k=0}^{n} \frac{e^{ikaz} \cos\left[(k+1) \operatorname{arccot} e^{iaz}\right]}{(e^{2iaz}+1)^{(k+1)/2}} \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^n$$

[[13], (75)].

7. $$= (-2)^{n+1} \frac{a^n n!}{(2az+\pi)^{n+1}} + \frac{a^n}{(2\pi)^{n+1}} \left[\psi^{(n)}\left(\frac{3\pi+2az}{4\pi}\right) - \psi^{(n)}\left(\frac{\pi+2az}{4\pi}\right)\right] - (-1)^n \frac{a^n}{(2\pi)^{n+1}} \left[\psi^{(n)}\left(\frac{\pi-2az}{4\pi}\right) - \psi^{(n)}\left(-\frac{\pi+2az}{4\pi}\right)\right].$$

8. $\mathrm{D}^n[\csc(az)]$

$$= (-1)^{[n/2]} a^n \sum_{k=1}^{n} \frac{(-1)^k}{2^k} \binom{n+1}{k+1} \sum_{m=0}^{k} \binom{k}{m} (k-2m)^n \csc^{2m+1}(az) \times \sum_{p=0}^{[(k-\gamma)/2]-m} (-1)^p \binom{k-2m}{2p+\gamma} \cot^{2p+\gamma}(az) \quad \left[\gamma = \frac{1-(-1)^n}{2};\ \text{[13], (120)}\right].$$

9. $$= (-1)^{[n/2]} a^n \csc(az) \sum_{k=1}^{n} \frac{1}{2^k} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (2m+1)^n \times \sum_{p=0}^{[(k-\gamma)/2]} (-1)^p \binom{k}{2p+\gamma} \cot^{2p+\gamma}(az) \quad \left[\gamma = \frac{1-(-1)^n}{2};\ \text{[13], (130)}\right].$$

10. $$= (-1)^{n+1}\frac{n!}{az^{n+1}} + \frac{a^n}{(2\pi)^{n+1}}\left[\psi^{(n)}\left(\frac{\pi+az}{2\pi}\right) - \psi^{(n)}\left(\frac{az}{2\pi}\right)\right] - (-1)^n\frac{a^n}{(2\pi)^{n+1}}\left[\psi^{(n)}\left(\frac{\pi-az}{2\pi}\right) - \psi^{(n)}\left(-\frac{az}{2\pi}\right)\right].$$

11. $$\mathrm{D}^n[\tan(az)] = (-1)^{[n/2]+1}(2a)^n\sum_{k=1}^{n}\frac{\sec^{k+1}(az)}{2^k} \times \sin\left[(k-1)az + \frac{1-(-1)^n}{4}\pi\right]\sum_{m=1}^{k}(-1)^m\binom{k}{m}m^n \quad [[13],\ (13)].$$

12. $$= (-1)z^{[n/2]+1}(2a)^n\sec^2(az)\sum_{k=1}^{n}\frac{1}{2^k}\sum_{m=1}^{k}(-1)^m\binom{k}{m}m^n \times \sum_{p=0}^{[(k+\gamma-2)/2]}(-1)^p\binom{k-1}{2p-\gamma+1}\tan^{2p-\gamma+1}(az) \quad \left[\gamma = \frac{1-(-1)^n}{2};\ [13],\ (24)\right].$$

13. $$= \frac{a^n}{\pi^{n+1}}\left[\psi^{(n)}\left(\frac{1}{2}+\frac{az}{\pi}\right) - (-1)^n\psi^{(n)}\left(\frac{1}{2}-\frac{az}{\pi}\right)\right].$$

14. $$\mathrm{D}^n[\cot(az)] = (-1)^{[(n-1)/2]}(2a)^n\csc^2(az)\sum_{k=0}^{n}\frac{1}{2^k}\sum_{m=1}^{k}(-1)^m\binom{k}{m}m^n \times \sum_{p=0}^{[(k+\gamma-2)/2]}(-1)^p\binom{k-1}{2p-\gamma+1}\cot^{2p-\gamma+1}(az) \quad \left[\gamma = \frac{1-(-1)^n}{2};\ [13],\ (97)\right].$$

15. $$= \frac{a^n}{\pi^{n+1}}\left[(-1)^n\psi^{(n)}\left(-\frac{az}{\pi}\right) - \psi^{(n)}\left(1+\frac{az}{\pi}\right)\right].$$

16. $$\mathrm{D}^n[z^\lambda\sin(az)] = \frac{n!}{2i}z^{\lambda-n}\left[e^{iaz}L_n^{\lambda-n}(-iaz) - e^{-iaz}L_n^{\lambda-n}(iaz)\right].$$

17. $$\mathrm{D}^n[z^\lambda\cos(az)] = \frac{n!}{2}z^{\lambda-n}\left[e^{iaz}L_n^{\lambda-n}(-iaz) + e^{-iaz}L_n^{\lambda-n}(iaz)\right].$$

18. $$\mathrm{D}^n[\sin(az^2)] = \frac{(-1)^n}{2}a^{n/2}e^{(n-2)\pi i/4}\left[i^n e^{iaz^2}H_n\left(e^{3\pi i/4}\sqrt{a}\,z\right) - e^{-iaz^2}H_n\left(e^{\pi i/4}\sqrt{a}\,z\right)\right].$$

19. $$\mathrm{D}^n[\cos(az^2)] = \frac{(-1)^n}{2}a^{n/2}e^{n\pi i/4}\left[i^n e^{iaz^2}H_n\left(e^{3\pi i/4}\sqrt{a}\,z\right) + e^{-iaz^2}H_n\left(e^{\pi i/4}\sqrt{a}\,z\right)\right].$$

20. $$\mathrm{D}^n\left[z^{n-1}\sin\frac{a}{z}\right] = (-a)^n z^{-n-1}\sin\left(\frac{a}{z}+\frac{n\pi}{2}\right).$$

21. $$\mathrm{D}^n\left[z^{n-1}\cos\frac{a}{z}\right] = (-a)^n z^{-n-1}\cos\left(\frac{a}{z}+\frac{n\pi}{2}\right).$$

22. $\mathrm{D}^n[\sin(a\sqrt{z})] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{(1-2n)/4} J_{1/2-n}(a\sqrt{z}).$

23. $\mathrm{D}^n[\cos(a\sqrt{z})] = (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{(1-2n)/4} J_{n-1/2}(a\sqrt{z}).$

24. $\mathrm{D}^n[\sin^m(a\sqrt{z}]$

$$= (-1)^{[m/2]} \frac{\sqrt{\pi}\, a^{n+1/2}}{2^{m+n-1/2} z^{(2n-1)/4}} \sum_{k=0}^{[m/2]-(1+(-1)^m)/2} (-1)^k \binom{m}{k} \times \left\{ \frac{1-(-1)^m}{2} (m-2k)^{n+1/2} J_{1/2-n}((m-2k)a\sqrt{z}) + (-1)^n \frac{1+(-1)^m}{2} (m-2k)^{n+1/2} J_{n-1/2}((m-2k)a\sqrt{z}) \right\} \quad [m \geq 1].$$

25. $\mathrm{D}^n[\cos^m(a\sqrt{z}]$

$$= (-1)^n \frac{\sqrt{\pi}\, a^{n+1/2}}{2^{m+n+1/2} z^{(2n-1)/4}} \sum_{k=0}^{m} \binom{m}{k} (m-2k)^{n+1/2} J_{n-1/2}((m-2k)a\sqrt{z}).$$

26. $\mathrm{D}^n\left[\frac{\sin(a\sqrt{z})}{\sqrt{z}}\right] = (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(1+2n)/4} J_{n+1/2}(a\sqrt{z}).$

27. $\mathrm{D}^n\left[\frac{\cos(a\sqrt{z})}{\sqrt{z}}\right] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(1+2n)/4} J_{-n-1/2}(a\sqrt{z}).$

28. $\mathrm{D}^n[\sec\sqrt{z}] = n!\,\pi\left(\frac{4}{\pi^2-4z}\right)^{n+1}$

$$\times\ {}_{2n+4}F_{2n+3}\left(\begin{matrix} 1,\ \frac{3}{2},\ \frac{1}{2}-\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{1}{2}-\frac{\sqrt{z}}{\pi},\ \frac{1}{2}+\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{1}{2}+\frac{\sqrt{z}}{\pi} \\ \frac{1}{2},\ \frac{3}{2}-\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{3}{2}-\frac{\sqrt{z}}{\pi},\ \frac{3}{2}+\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{3}{2}+\frac{\sqrt{z}}{\pi};\ -1 \end{matrix}\right).$$

29. $\mathrm{D}^n\left[\frac{\csc\sqrt{z}}{\sqrt{z}}\right] = (-1)^{n+1}\frac{n!}{z^{n+1}} + \frac{2(-1)^n n!}{z^{n+1}}$

$$\times\ {}_{2n+3}F_{2n+2}\left(\begin{matrix} 1,\ -\frac{\sqrt{z}}{\pi},\ \ldots,\ -\frac{\sqrt{z}}{\pi},\ \frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{\sqrt{z}}{\pi};\ -1 \\ 1-\frac{\sqrt{z}}{\pi},\ \ldots,\ 1-\frac{\sqrt{z}}{\pi},\ 1+\frac{\sqrt{z}}{\pi},\ \ldots,\ 1+\frac{\sqrt{z}}{\pi} \end{matrix}\right).$$

30. $\mathrm{D}^n\left[\frac{\tan\sqrt{z}}{\sqrt{z}}\right] = 2(n!)\left(\frac{4}{\pi^2-4z}\right)^{n+1}$

$$\times\ {}_{2n+3}F_{2n+2}\left(\begin{matrix} 1,\ \frac{1}{2}-\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{1}{2}-\frac{\sqrt{z}}{\pi},\ \frac{1}{2}+\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{1}{2}+\frac{\sqrt{z}}{\pi} \\ \frac{3}{2}-\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{3}{2}-\frac{\sqrt{z}}{\pi},\ \frac{3}{2}+\frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{3}{2}+\frac{\sqrt{z}}{\pi};\ 1 \end{matrix}\right).$$

31. $\mathrm{D}^n\left[\frac{\cot\sqrt{z}}{\sqrt{z}}\right] = 2(-1)^n n!\, z^{-n-1}$

$$\times\ {}_{2n+3}F_{2n+2}\left(\begin{matrix} 1,\ -\frac{\sqrt{z}}{\pi},\ \ldots,\ -\frac{\sqrt{z}}{\pi},\ \frac{\sqrt{z}}{\pi},\ \ldots,\ \frac{\sqrt{z}}{\pi};\ 1 \\ 1-\frac{\sqrt{z}}{\pi},\ \ldots,\ 1-\frac{\sqrt{z}}{\pi},\ 1+\frac{\sqrt{z}}{\pi},\ \ldots,\ 1+\frac{\sqrt{z}}{\pi} \end{matrix}\right) - (-1)^n\frac{n!}{z^{n+1}}.$$

32. $\mathrm{D}^n\left[z^{n-1}\sin\frac{a}{\sqrt{z}}\right] = (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(2n+5)/4}J_{1/2-n}\left(\frac{a}{\sqrt{z}}\right).$

33. $\mathrm{D}^n\left[z^{n-1/2}\sin\frac{a}{\sqrt{z}}\right] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(2n+3)/4}J_{n+1/2}\left(\frac{a}{\sqrt{z}}\right).$

34. $\mathrm{D}^n\left[z^{n-1}\cos\frac{a}{\sqrt{z}}\right] = \sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(2n+5)/4}J_{n-1/2}\left(\frac{a}{\sqrt{z}}\right).$

35. $\mathrm{D}^n\left[z^{n-1/2}\cos\frac{a}{\sqrt{z}}\right] = (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-(2n+3)/4}J_{-n-1/2}\left(\frac{a}{\sqrt{z}}\right).$

36. $$\mathrm{D}^n[\sin(a\sqrt[4]{z})] = 2^{-2n-1/2}\sqrt{\pi}\,a^{n+1/2}z^{(1-6n)/8} \times\sum_{k=0}^{n-1}(-a)^{-k}\frac{\Gamma(n+k)}{k!\,\Gamma(n-k)}z^{-k/4}J_{k-n+1/2}(a\sqrt[4]{z})\quad [n\ge 1].$$

37. $$\mathrm{D}^n[\cos(a\sqrt[4]{z})] = (-1)^n 2^{-2n-1/2}\sqrt{\pi}\,a^{n+1/2}z^{(1-6n)/8} \times\sum_{k=0}^{n-1}a^{-k}\frac{\Gamma(n+k)}{k!\,\Gamma(n-k)}z^{-k/4}J_{n-k-1/2}(a\sqrt[4]{z})\quad [n\ge 1].$$

38. $$\mathrm{D}^n\left[\frac{\sin(a\sqrt[4]{z})}{\sqrt[4]{z}}\right] = (-1)^n 2^{-2n-1/2}\sqrt{\pi}\,a^{n+1/2}z^{-(1+6n)/8} \times\sum_{k=0}^{n-1}a^{-k}\frac{\Gamma(n+k)}{k!\,\Gamma(n-k)}z^{-k/4}J_{n-k+1/2}(a\sqrt[4]{z})\quad [n\ge 1].$$

39. $$\mathrm{D}^n\left[\frac{\cos(a\sqrt[4]{z})}{\sqrt[4]{z}}\right] = 2^{-2n-1/2}\sqrt{\pi}\,a^{n+1/2}z^{-(1+6n)/8} \times\sum_{k=0}^{n-1}(-1)^k a^{-k}\frac{\Gamma(n+k)}{k!\,\Gamma(n-k)}z^{-k/4}J_{k-n-1/2}(a\sqrt[4]{z})\quad [n\ge 1].$$

40. $\mathrm{D}^{4n}\left[\begin{Bmatrix}\sinh z\sin z\\ \cosh z\cos z\end{Bmatrix}\right] = (-4)^n\begin{Bmatrix}\sinh z\sin z\\ \cosh z\cos z\end{Bmatrix}.$

41. $\mathrm{D}^{4n+1}\left[\begin{Bmatrix}\sinh z\sin z\\ \cosh z\cos z\end{Bmatrix}\right] = (-4)^n(\pm\cosh z\sin z+\sinh z\cos z).$

42. $\mathrm{D}^{4n+2}\left[\begin{Bmatrix}\sinh z\sin z\\ \cosh z\cos z\end{Bmatrix}\right] = \pm(-1)^n 2^{2n+1}\begin{Bmatrix}\cosh z\cos z\\ \sinh z\sin z\end{Bmatrix}.$

43. $\mathrm{D}^{4n+3}\left[\begin{Bmatrix}\sinh z\sin z\\ \cosh z\cos z\end{Bmatrix}\right] = (-1)^{n+1}2^{2n+1}(\cosh z\sin z\mp\sinh z\cos z).$

44. $\mathrm{D}^{4n}\left[\begin{Bmatrix}\sinh z\cos z\\ \cosh z\sin z\end{Bmatrix}\right] = (-4)^n\begin{Bmatrix}\sinh z\cos z\\ \cosh z\sin z\end{Bmatrix}.$

45. $\mathrm{D}^{4n+1}\left[\begin{Bmatrix}\sinh z\cos z\\ \cosh z\sin z\end{Bmatrix}\right] = (-4)^n(\cosh z\cos z\mp\sinh z\sin z).$

46. $$\mathrm{D}^{4n+2}\left[\begin{Bmatrix}\sinh z\cos z\\ \cosh z\sin z\end{Bmatrix}\right] = \mp(-1)^n 2^{2n+1}\begin{Bmatrix}\cosh z\sin z\\ \sinh z\cos z\end{Bmatrix}.$$

47. $$\mathrm{D}^{4n+3}\left[\begin{Bmatrix}\sinh z\cos z\\ \cosh z\sin z\end{Bmatrix}\right] = (-1)^{n+1}2^{2n+1}(\sinh z\sin z \pm \cosh z\cos z).$$

48. $$\mathrm{D}^n\left[\sinh(a\sqrt{z})\begin{Bmatrix}\sin(a\sqrt{z})\\ \cos(a\sqrt{z})\end{Bmatrix}\right] = (\mp 1)^n\sqrt{\pi}\left(\frac{a}{\sqrt{2}}\right)^{n+1/2} z^{(1-2n)/4}$$
$$\times\left[\sin\frac{(6n+3)\pi}{8}\operatorname{ber}_{\pm n\mp 1/2}\left(a\sqrt{2z}\right) + \cos\frac{(6n+3)\pi}{8}\operatorname{bei}_{\pm n\mp 1/2}\left(a\sqrt{2z}\right)\right].$$

49. $$\mathrm{D}^n\left[\cosh(a\sqrt{z})\begin{Bmatrix}\sin(a\sqrt{z})\\ \cos(a\sqrt{z})\end{Bmatrix}\right] = (\pm 1)^{n+1}\sqrt{\pi}\left(\frac{a}{\sqrt{2}}\right)^{n+1/2} z^{(1-2n)/4}$$
$$\times\left[\sin\frac{(6n+3)\pi}{8}\operatorname{bei}_{\mp n\pm 1/2}\left(a\sqrt{2z}\right) - \cos\frac{(6n+3)\pi}{8}\operatorname{ber}_{\mp n\pm 1/2}\left(a\sqrt{2z}\right)\right].$$

50. $$\mathrm{D}^n\left[\frac{1}{\sqrt{z}}\sinh(a\sqrt{z})\begin{Bmatrix}\sin(a\sqrt{z})\\ \cos(a\sqrt{z})\end{Bmatrix}\right] = (\pm 1)^n\sqrt{\pi}\left(\frac{a}{\sqrt{2}}\right)^{n+1/2} z^{-(2n+1)/4}$$
$$\times\left[\sin\frac{(6n+3)\pi}{8}\operatorname{ber}_{\mp n\mp 1/2}\left(a\sqrt{2z}\right) + \cos\frac{(6n+3)\pi}{8}\operatorname{bei}_{\mp n\mp 1/2}\left(a\sqrt{2z}\right)\right].$$

51. $$\mathrm{D}^n\left[\frac{1}{\sqrt{z}}\cosh(a\sqrt{z})\begin{Bmatrix}\sin(a\sqrt{z})\\ \cos(a\sqrt{z})\end{Bmatrix}\right]$$
$$= (\mp 1)^{n+1}\sqrt{\pi}\left(\frac{a}{\sqrt{2}}\right)^{n+1/2} z^{-(2n+1)/4}$$
$$\times\left[\cos\frac{(6n+3)\pi}{8}\operatorname{ber}_{\pm n\pm 1/2}\left(a\sqrt{2z}\right) - \sin\frac{(6n+3)\pi}{8}\operatorname{bei}_{\pm n\pm 1/2}\left(a\sqrt{2z}\right)\right].$$

1.1.6. The logarithmic function

1. $$\mathrm{D}^n\left[\ln\left(\sqrt{z}+\sqrt{z+a}\right)\right]$$
$$= (-1)^{n-1}\frac{(n-1)!}{2}z^{-n/2}(z+a)^{-n/2}P_{n-1}\left(\frac{2z+a}{2\sqrt{z}\sqrt{z+a}}\right) \qquad [n \geq 1].$$

2. $$\mathrm{D}^n\left[z^{n-1}\ln\left(\sqrt{a}+\sqrt{z+a}\right)\right]$$
$$= \frac{(n-1)!}{2z} - \frac{(n-1)!}{2z}\left(\frac{a}{z+a}\right)^{n/2}P_{n-1}\left(\frac{z+2a}{2\sqrt{az+a^2}}\right) \qquad [n \geq 1].$$

3. $$\mathrm{D}^n\left[\ln\left(z+\sqrt{z^2+a^2}\right)\right]$$
$$= (-1)^{n-1}(n-1)!\,(z^2+a^2)^{-n/2}P_{n-1}\left(\frac{z}{\sqrt{z^2+a^2}}\right) \qquad [n \geq 1].$$

4. $$\mathrm{D}^n\left[z^{n-1}\ln\left(a+\sqrt{z^2+a^2}\right)\right]$$
$$= \frac{(n-1)!}{z} - \frac{(n-1)!}{z}a^n(z^2+a^2)^{-n/2}P_{n-1}\left(\frac{a}{\sqrt{z^2+a^2}}\right) \qquad [n \geq 1].$$

5. $$\mathrm{D}^{2n}\left[\ln\frac{a+z}{a-z}\right] = 2(2n-1)!\,z(a^2-z^2)^{-n-1/2}U_{2n-1}\left(\frac{a}{\sqrt{a^2-z^2}}\right) \qquad [n \geq 1].$$

6. $\mathrm{D}^{2n+1}\left[\ln\frac{a+z}{a-z}\right] = 2(2n)!\,(a^2-z^2)^{-n-1/2}T_{2n+1}\left(\frac{a}{\sqrt{a^2-z^2}}\right).$

7. $\mathrm{D}^{n}\left[\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]$
$= (n-1)!a^{2n-1}z^{1/2-n}(a^2-z)^{-n}P_{n-1}^{(1/2-n,\,-n)}\left(1-\frac{2z}{a^2}\right) \quad [n \geq 1].$

8. $\mathrm{D}^{n}\left[z^{n-1}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]$
$= (-1)^n(n-1)!\,az^{n-3/2}(z-a^2)^{-n}P_{n-1}^{(1/2-n,\,-n)}\left(1-\frac{2a^2}{z}\right) \quad [n \geq 1].$

9. $\mathrm{D}^{n}\left[z^{1/2}(a^2-z)^n\,\mathrm{D}^n\left[z^{n-1}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]\right] = 0 \quad [n \geq 1].$

10. $\mathrm{D}^{n}\left[z^{n-1/2}(a^2-z)^n\,\mathrm{D}^n\left[\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]\right] = 0 \quad [n \geq 1].$

11. $\mathrm{D}^{n}\left[z^{-1/2}(a^2-z)^n\,\mathrm{D}^n\left[z^{n-1/2}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]\right]$
$= (2n)!\left(-\frac{a^2}{4}\right)^n z^{-n-1}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}.$

12. $\mathrm{D}^{n}\left[z^{n+1/2}(a^2-z)^n\,\mathrm{D}^n\left[z^{-1/2}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]\right] = \frac{(2n)!}{2^{2n}}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}.$

13. $\mathrm{D}^{n}\left[z^{n+1/2}\,\mathrm{D}^n\left[z^{-1/2}(a^2-z)^{n-1/2}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]\right]$
$= (-1)^n\left(\frac{1}{2}\right)_n^2 a^{2n}(a^2-z)^{-n-1/2}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}.$

14. $\mathrm{D}^{n}\left[z^{n-1/2}\,\mathrm{D}^n\left[(a^2-z)^{n-1/2}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}\right]\right]$
$= (-1)^n\left(\frac{1}{2}\right)_n^2 a^{2n}z^{-1/2}(a^2-z)^{-n-1/2}\ln\frac{a+\sqrt{z}}{a-\sqrt{z}}.$

15. $\mathrm{D}^{n}[z^{n-1}(1-az)^n\,\mathrm{D}^n[\ln(1-az)]] = 0 \quad [n \geq 1].$

16. $\mathrm{D}^{n}[(1-az)^n\,\mathrm{D}^n[z^{n-1}\ln(1-az)]] = 0 \quad [n \geq 1].$

17. $\mathrm{D}^{n}[z^{n+1}(1-az)^{-1}\,\mathrm{D}^n[z^{-1}(1-az)^n\ln(1-az)]]$
$= (-1)^n(n!)^2a^n(1-az)^{-n-1}\ln(1-az).$

18. $\mathrm{D}^{n}[z^{n+1}(1-az)^n\,\mathrm{D}^n[z^{-1}\ln(1-az)]] = (n!)^2a^n\ln(1-az).$

1.1.7. Inverse trigonometric functions

1. $\mathrm{D}^{n}[\arcsin(az)] = (-i)^{n-1}(n-1)!a^n(1-a^2z^2)^{-n/2}P_{n-1}\left(\frac{iaz}{\sqrt{1-a^2z^2}}\right)$
$[n \geq 1].$

2. $\mathrm{D}^n[\arccos(az)]$
$$= (-1)^n i^{n-1}(n-1)!a^n(1-a^2z^2)^{-n/2}P_{n-1}\left(\frac{iaz}{\sqrt{1-a^2z^2}}\right) \quad [n \geq 1].$$

3. $\mathrm{D}^{2n}[\arctan(az)]$
$$= (-1)^n(2n-1)!a^{2n+1}z(1+a^2z^2)^{-n-1/2}U_{2n-1}\left(\frac{1}{\sqrt{1+a^2z^2}}\right) \quad [n \geq 1].$$

4. $\mathrm{D}^{2n+1}[\arctan(az)]$
$$= (-1)^n(2n)!a^{2n+1}(1+a^2z^2)^{-n-1/2}T_{2n+1}\left(\frac{1}{\sqrt{1+a^2z^2}}\right).$$

5. $\mathrm{D}^{2n}[\operatorname{arccot}(az)]$
$$= (-1)^{n+1}(2n-1)!a^{2n+1}z(1+a^2z^2)^{-n-1/2}U_{2n-1}\left(\frac{1}{\sqrt{1+a^2z^2}}\right) \quad [n \geq 1].$$

6. $\mathrm{D}^{2n+1}[\operatorname{arccot}(az)]$
$$= (-1)^{n+1}(2n)!a^{2n+1}(1+a^2z^2)^{-n-1/2}T_{2n+1}\left(\frac{1}{\sqrt{1+a^2z^2}}\right).$$

7. $\mathrm{D}^n[\arcsin(a\sqrt{z})]$
$$= \frac{(-i)^{n-1}}{2}(n-1)!a^n(z-a^2z^2)^{-n/2}P_{n-1}\left(\frac{1-2a^2z}{2a\sqrt{a^2z^2-z}}\right) \quad [n \geq 1].$$

8. $\mathrm{D}^n[\arctan(a\sqrt{z})]$
$$= \frac{(n-1)!}{2}az^{1/2-n}(a^2z+1)^{-n}P_{n-1}^{(1/2-n,\,-n)}(2a^2z+1) \quad [n \geq 1].$$

9. $\mathrm{D}^n\left[z^{n-1}\arcsin\frac{a}{\sqrt{z}}\right]$
$$= -\frac{i^{n-1}}{2}(n-1)!\,a^nz^{-1}(z-a^2)^{-n/2}P_{n-1}\left(\frac{z-2a^2}{2a\sqrt{a^2-z}}\right) \quad [n \geq 1].$$

10. $\mathrm{D}^n\left[z^{n-1}\arctan\frac{a}{\sqrt{z}}\right]$
$$= \frac{(-1)^n}{2}(n-1)!\,az^{n-3/2}(z+a^2)^{-n}P_{n-1}^{(1/2-n,\,-n)}\left(\frac{2a^2}{z}+1\right) \quad [n \geq 1].$$

11. $\mathrm{D}^n\left[z^{n-1/2}(1-a^2z)^{n+1/2}\,\mathrm{D}^n\left[(1-a^2z)^{-1/2}\arcsin(a\sqrt{z})\right]\right]$
$$= \left(\frac{1}{2}\right)_n^2 a^{2n}z^{-1/2}\arcsin(a\sqrt{z}).$$

12. $\mathrm{D}^n\left[z^{n+1/2}(1-a^2z)^{n-1/2}\,\mathrm{D}^n\left[z^{-1/2}\arcsin(a\sqrt{z})\right]\right]$
$$= \left(\frac{1}{2}\right)_n^2 a^{2n}(1-a^2z)^{-1/2}\arcsin(a\sqrt{z}).$$

13. $\mathrm{D}^n\left[z^{-1/2}(1-a^2z)^{n+1/2}\,\mathrm{D}^n\left[z^{n-1/2}(1-a^2z)^{-1/2}\arcsin(a\sqrt{z})\right]\right]$
$$= (-4)^{-n}\,(2n)!\,z^{-n-1}\arcsin(a\sqrt{z}).$$

14. $$\mathrm{D}^n\left[z^{n+1/2}(1-a^2z)^{n+1/2}\,\mathrm{D}^n\left[z^{-1/2}(1-a^2z)^{-1/2}\arcsin(a\sqrt{z})\right]\right] = a^{2n}(n!)^2\arcsin(a\sqrt{z}).$$

15. $$\mathrm{D}^n\left[z^{n+1/2}(1-a^2z)^{-1/2}\,\mathrm{D}^n\left[z^{-1/2}(1-a^2z)^{n-1/2}\arcsin(a\sqrt{z})\right]\right] = (2n)!\left(-\frac{a^2}{4}\right)^n(1-a^2z)^{-n-1}\arcsin(a\sqrt{z}).$$

16. $$\mathrm{D}^n\left[z^{1/2}(1-a^2z)^{n-1/2}\,\mathrm{D}^n[z^{n-1}\arcsin(a\sqrt{z})]\right] = 0 \qquad [n \ge 1].$$

17. $$\mathrm{D}^n\left[z^{n-1/2}(1-a^2z)^{1/2}\,\mathrm{D}^n[(1-a^2z)^{n-1}\arcsin(a\sqrt{z})]\right] = 0 \qquad [n \ge 1].$$

18. $$\mathrm{D}^n\left[z^{n-1/2}(1-a^2z)^{n-1/2}\,\mathrm{D}^n[\arcsin(a\sqrt{z})]\right] = 0 \qquad [n \ge 1].$$

19. $$\mathrm{D}^n\left[z^{1/2}(1+a^2z)^n\,\mathrm{D}^n[z^{n-1}\arctan(a\sqrt{z})]\right] = 0 \qquad [n \ge 1].$$

20. $$\mathrm{D}^n\left[z^{n-1/2}(1+a^2z)^n\,\mathrm{D}^n[\arctan(a\sqrt{z})]\right] = 0 \qquad [n \ge 1].$$

21. $$\mathrm{D}^n\left[z^{-1/2}(1+a^2z)^n\,\mathrm{D}^n\left[z^{n-1/2}\arctan(a\sqrt{z})\right]\right] = (2n)!\left(-\frac{1}{4}\right)^n z^{-n-1}\arctan(a\sqrt{z}).$$

22. $$\mathrm{D}^n\left[z^{n+1/2}(1+a^2z)^n\,\mathrm{D}^n\left[z^{-1/2}\arctan(a\sqrt{z})\right]\right] = (2n)!\left(-\frac{a^2}{4}\right)^n\arctan(a\sqrt{z}).$$

23. $$\mathrm{D}^n\left[z^{n-1/2}\,\mathrm{D}^n\left[(1+a^2z)^{n-1/2}\arctan(a\sqrt{z})\right]\right] = \left(\frac{1}{2}\right)_n^2 a^{2n}z^{-1/2}(1+a^2z)^{-n-1/2}\arctan(a\sqrt{z}).$$

24. $$\mathrm{D}^n\left[z^{n+1/2}\,\mathrm{D}^n\left[z^{-1/2}(1+a^2z)^{n-1/2}\arctan(a\sqrt{z})\right]\right] = \left(\frac{1}{2}\right)_n^2 a^{2n}(1+a^2z)^{-n-1/2}\arctan(a\sqrt{z}).$$

1.2. The Hurwitz Zeta Function $\zeta(\nu, z)$

1.2.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[\zeta(\nu, az)] = (-a)^n(\nu)_n\zeta(\nu+n, az).$

2. $$\mathrm{D}^n\left[z^{n-1}\zeta\left(\nu, \frac{a}{z}\right)\right] = a^n(\nu)_n z^{-n-1}\zeta\left(\nu+n, \frac{a}{z}\right).$$

1.2.2. Derivatives with respect to the parameter

1. $$\frac{\partial}{\partial s}\zeta(s, z)\big|_{s=0} = \ln\frac{\Gamma(z)}{\sqrt{2\pi}}.$$

2. $$\frac{\partial}{\partial s}\zeta\left(s,\ \frac{m}{n}\right)\Big|_{s=-2k+1} = \frac{1}{2k}\left[\psi(2k) - \ln(2n\pi)\right] B_{2k}\left(\frac{m}{n}\right) - \frac{1}{2kn^{2k}}\left[\psi(2k) - \ln(2\pi)\right] B_{2k} - \frac{(-1)^k\pi}{(2n\pi)^{2k}}\sum_{i=1}^{n-1}\sin\frac{2im\pi}{n}\,\psi^{(2k-1)}\left(\frac{i}{n}\right) - \frac{(-1)^k 2(2k-1)!}{(2n\pi)^{2k}} \times \sum_{i=1}^{n-1}\cos\frac{2im\pi}{n}\,\zeta'\left(2k,\ \frac{i}{n}\right) + \frac{1}{n^{2k}}\zeta'(-2k+1) \quad [m < n;\ [59], (5)].$$

3. $$\frac{\partial}{\partial s}\zeta\left(s,\ \frac{1}{2}\right)\Big|_{s=-2n+1} = -\frac{\ln 2}{2^{2n}n}B_{2n} + \left(2^{1-2n} - 1\right)\zeta'(-2n+1) \quad [[59], (17)].$$

4. $$\frac{\partial}{\partial s}\zeta\left(s,\ \frac{3\pm 1}{6}\right)\Big|_{s=-2n+1} = \pm\frac{\sqrt{3}\left(1 - 3^{-2n}\right)\pi}{8\left(1 - 3^{1-2n}\right)n}B_{2n} - \frac{\ln 3}{2^2\, 3^{2n-1}n}B_{2n} \pm \frac{(-1)^n}{2^{2n}3^{2n-1/2}\pi^{2n-1}}\psi^{(2n-1)}\left(\frac{1}{3}\right) + \frac{3^{1-2n} - 1}{2}\zeta'(-2n+1) \quad [[59], (18)].$$

5. $$\frac{\partial}{\partial s}\zeta\left(s,\ \frac{2\pm 1}{4}\right)\Big|_{s=-2n+1} = \pm\frac{\left(1 - 2^{-2n}\right)\pi}{4n}B_{2n} - \frac{\left(1 - 2^{2-2n}\right)\pi}{2^{2n+1}n}B_{2n}\ln 2 \pm \frac{(-1)^k}{2^{6n-1}\pi^{2n-1}}\psi^{(2n-1)}\left(\frac{1}{4}\right) - \frac{1 - 2^{1-2n}}{2^{2n-1}}\zeta'(-2n+1) \quad [[59], (19)].$$

6. $$\frac{\partial}{\partial s}\zeta\left(s,\ \frac{3\pm 2}{6}\right)\Big|_{s=-2n+1} = \pm\frac{\sqrt{3}\left(1 - 3^{-2n}\right)\left(1 + 2^{1-2n}\right)\pi}{8n}B_{2n} + \frac{\left(1 - 3^{1-2n}\right)\pi}{2^{2n+1}n}B_{2n}\ln 2 + \frac{\left(1 - 2^{1-2n}\right)\pi}{2^2 3^{2n-1}n}B_{2n}\ln 3 \pm \frac{(-1)^k\left(2^{2k-1} + 1\right)}{2^{4n-1}3^{2n-1/2}\pi^{2n-1}}\psi^{(2n-1)}\left(\frac{1}{3}\right) + \frac{\left(1 - 2^{1-2n}\right)\left(1 - 3^{1-2n}\right)}{2}\zeta'(-2n+1) \quad [[59], (20)].$$

1.3. The Exponential Integral Ei (z)

1.3.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[\mathrm{Ei}\,(-az)] = (-1)^{n-1}(n-1)!z^{-n}e^{-az}\sum_{k=0}^{n-1}\frac{(az)^k}{k!} \quad [n \geq 1].$$

2. $$= (n-1)!z^{-n}e^{-az}L_{n-1}^{-n}(az) \quad [n \geq 1].$$

3. $\mathrm{D}^n\left[z^{n-1}\,\mathrm{Ei}\left(-\frac{a}{z}\right)\right]=-(n-1)!\,z^{-1}e^{-a/z}\sum_{k=0}^{n-1}\frac{(a/z)^k}{k!}$ $[n\ge 1]$.

4. $\quad=(-1)^n(n-1)!\,z^{-1}e^{-a/z}L_{n-1}^{-n}\left(\frac{a}{z}\right)$ $[n\ge 1]$.

5. $\mathrm{D}^n[e^{az}\,\mathrm{Ei}\,(-az)]=a^ne^{az}\,\mathrm{Ei}\,(-az)+a^n\sum_{k=0}^{n-1}\frac{(-1)^kk!}{(az)^{k+1}}$ $[n\ge 1]$.

6. $\mathrm{D}^n\left[z^{n-1}e^{a/z}\,\mathrm{Ei}\left(-\frac{a}{z}\right)\right]$

$$=(-a)^nz^{-n-1}e^{a/z}\,\mathrm{Ei}\left(-\frac{a}{z}\right)+(-1)^na^{n-1}z^{-n}\sum_{k=0}^{n-1}(-1)^kk!\left(\frac{z}{a}\right)^k\quad[n\ge 1].$$

7. $\mathrm{D}^n[z^{-1}e^{-az}\,\mathrm{D}^n[z^ne^{az}\,\mathrm{Ei}\,(-az)]]=(-1)^n(n!)^2z^{-n-1}\,\mathrm{Ei}\,(-az)$.

8. $\mathrm{D}^n\left[z^{2n+1}e^{-a/z}\,\mathrm{D}^n\left[z^{-1}e^{a/z}\,\mathrm{Ei}\left(-\frac{a}{z}\right)\right]\right]=(-1)^n(n!)^2\,\mathrm{Ei}\left(-\frac{a}{z}\right)$.

9. $\mathrm{D}^n[z^ne^{-az}\,\mathrm{D}^n[e^{az}\,\mathrm{Ei}\,(-az)]]=n!\,a^n\,\mathrm{Ei}\,(-az)$.

10. $\mathrm{D}^n\left[z^ne^{-a/z}\,\mathrm{D}^n\left[z^{n-1}e^{a/z}\,\mathrm{Ei}\left(-\frac{a}{z}\right)\right]\right]=n!\,a^nz^{-n-1}\,\mathrm{Ei}\left(-\frac{a}{z}\right)$.

1.4. The Sine si (z) and Cosine ci (z) Integrals

1.4.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[\mathrm{si}\,(az)]=\frac{(n-1)!}{2i}z^{-n}\left[e^{iaz}L_{n-1}^{-n}(-iaz)-e^{-iaz}L_{n-1}^{-n}(iaz)\right]$ $[n\ge 1]$.

2. $\mathrm{D}^n\left[z^{n-1}\,\mathrm{si}\left(\frac{a}{z}\right)\right]$

$$=(-1)^n\frac{(n-1)!}{2iz}\left[e^{ia/z}L_{n-1}^{-n}\left(-\frac{ia}{z}\right)-e^{-ia/z}L_{n-1}^{-n}\left(\frac{ia}{z}\right)\right]\quad[n\ge 1].$$

3. $\mathrm{D}^n[\mathrm{ci}\,(az)]=\frac{(n-1)!}{2}z^{-n}\left[e^{iaz}L_{n-1}^{-n}(-iaz)+e^{-iaz}L_{n-1}^{-n}(iaz)\right]$ $[n\ge 1]$.

4. $\mathrm{D}^n\left[z^{n-1}\,\mathrm{ci}\left(\frac{a}{z}\right)\right]$

$$=(-1)^n\frac{(n-1)!}{2z}\left[e^{ia/z}L_{n-1}^{-n}\left(-\frac{ia}{z}\right)+e^{-ia/z}L_{n-1}^{-n}\left(\frac{ia}{z}\right)\right]\quad[n\ge 1].$$

5. $\mathrm{D}^n[\sin z\,\mathrm{si}\,(z)-\cos z\,\mathrm{ci}\,(z)]$

$$=(-1)^n\left[\sin\left(z-\frac{n\pi}{2}\right)\mathrm{si}\,(z)-\cos\left(z-\frac{n\pi}{2}\right)\mathrm{ci}\,(z)\right.$$

$$\left.+\frac{1}{z}\sin\frac{n\pi}{2}+\frac{1}{z^{n+2}}\sum_{k=1}^{[n/2]}(n-2k+1)!(-z^2)^k\right].$$

6. $\mathrm{D}^n[\cos z\,\mathrm{si}\,(z)+\sin z\,\mathrm{ci}\,(z)]=(-1)^n\left[\cos\left(z-\frac{n\pi}{2}\right)\mathrm{si}\,(z)\right.$
$\left.+\sin\left(z-\frac{n\pi}{2}\right)\mathrm{ci}\,(z)-\frac{1}{z^{n+1}}\sum_{k=1}^{[n/2]}(n-2k)!\,(-z^2)^k\right].$

1.5. The Error Functions erf (z) and erfc (z)

1.5.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[\mathrm{erf}\,(az)]=(-1)^{n-1}\frac{2a^n}{\sqrt{\pi}}e^{-a^2z^2}H_{n-1}(az)$ $[n\geq 1]$.

2. $\mathrm{D}^n[\mathrm{erf}\,(a\sqrt{z})]=\frac{(n-1)!\,a}{\sqrt{\pi}}z^{1/2-n}e^{-a^2z}L_{n-1}^{1/2-n}(a^2z)$ $[n\geq 1]$.

3. $\mathrm{D}^n[z^{n-1}\,\mathrm{erf}\,(a\sqrt{z})]=\frac{(-1)^{n-1}}{2^{2n-1}\sqrt{\pi}\,z}e^{-a^2z}H_{2n-1}(a\sqrt{z})$ $[n\geq 1]$.

4. $\mathrm{D}^n\left[\frac{1}{\sqrt{z}}\,\mathrm{erf}\,(a\sqrt{z})\right]=\frac{(-1)^n}{\sqrt{\pi}}z^{-n-1/2}\gamma\left(n+\frac{1}{2},\,a^2z\right)$ $[n\geq 1]$.

5. $\mathrm{D}^n\left[z^{n-1}\,\mathrm{erf}\left(\frac{a}{z}\right)\right]=-\frac{2a^n}{\sqrt{\pi}}z^{-n-1}e^{-a^2/z^2}H_{n-1}\left(\frac{a}{z}\right)$ $[n\geq 1]$.

6. $\mathrm{D}^n\left[\mathrm{erf}\left(\frac{a}{\sqrt{z}}\right)\right]=-\frac{z^{-n}}{2^{2n-1}\sqrt{\pi}}e^{-a^2/z}H_{2n-1}\left(\frac{a}{\sqrt{z}}\right)$ $[n\geq 1]$.

7. $\mathrm{D}^n\left[z^{n-1/2}\,\mathrm{erf}\left(\frac{a}{\sqrt{z}}\right)\right]=\frac{1}{\sqrt{\pi z}}\gamma\left(n+\frac{1}{2},\,\frac{a^2}{z}\right)$ $[n\geq 1]$.

8. $\mathrm{D}^n\left[z^{n-1}\,\mathrm{erf}\left(\frac{a}{\sqrt{z}}\right)\right]=(-1)^n\frac{(n-1)!\,a}{\sqrt{\pi}}z^{-1/2}e^{-a^2/z}L_{n-1}^{1/2-n}\left(\frac{a^2}{z}\right)$ $[n\geq 1]$.

9. $\mathrm{D}^n\left[e^{a^2z^2}\,\mathrm{erf}\,(az)\right]=(-ia)^ne^{a^2z^2}H_n(iaz)$
$-\frac{2^{(n+1)/2}}{\sqrt{\pi}}n!\,(-a)^ne^{a^2z^2/2}D_{-n-1}(\sqrt{2}\,az).$

10. $\mathrm{D}^n\left[z^{n-1}e^{a^2/z^2}\,\mathrm{erf}\,(\frac{a}{z})\right]=(ia)^nz^{-n-1}e^{a^2/z^2}H_n\left(\frac{ia}{z}\right)$
$-\frac{2^{(n+1)/2}}{\sqrt{\pi}}n!\,a^nz^{-n-1}e^{a^2/(2z^2)}D_{-n-1}\left(\frac{\sqrt{2}\,a}{z}\right).$

11. $\mathrm{D}^n\left[e^{a^2z}\,\mathrm{erf}\,(a\sqrt{z})\right]=\frac{(-a^2)^n}{\sqrt{\pi}}\left(\frac{1}{2}\right)_n e^{a^2z}\gamma\left(\frac{1}{2}-n,\,a^2z\right).$

12. $\mathrm{D}^n\left[z^{n-1}e^{a^2/z}\operatorname{erf}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{a^{2n}}{\sqrt{\pi}}\left(\frac{1}{2}\right)_n z^{-n-1}e^{a^2/z}\gamma\left(\frac{1}{2}-n,\ \frac{a^2}{z}\right).$

13. $\mathrm{D}^n\left[z^{n-1/2}e^{-a^2z}\,\mathrm{D}^n\left[e^{a^2z}\operatorname{erf}(a\sqrt{z})\right]\right] = \left(\frac{1}{2}\right)_n a^{2n}z^{-1/2}\operatorname{erf}(a\sqrt{z}).$

14. $\mathrm{D}^n\left[z^{n+1/2}e^{-a^2/z}\,\mathrm{D}^n\left[z^{n-1}e^{a^2/z}\operatorname{erf}\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= \left(\frac{1}{2}\right)_n a^{2n}z^{-n-1/2}\operatorname{erf}\left(\frac{a}{\sqrt{z}}\right).$$

15. $\mathrm{D}^n\left[z^{n+1/2}e^{a^2z}\,\mathrm{D}^n\left[z^{-1/2}\operatorname{erf}(a\sqrt{z})\right]\right] = \left(\frac{1}{2}\right)_n (-a^2)^n e^{a^2z}\operatorname{erf}(a\sqrt{z}).$

16. $\mathrm{D}^n\left[z^{n-1/2}e^{a^2/z}\,\mathrm{D}^n\left[z^{n-1/2}\operatorname{erf}\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= \left(\frac{1}{2}\right)_n (-a^2)^n e^{-a^2/z}\operatorname{erf}\left(\frac{a}{\sqrt{z}}\right).$$

17. $\mathrm{D}^n\left[e^{a^2z^2}\operatorname{erfc}(az)\right] = \frac{2^{(n+1)/2}}{\sqrt{\pi}}n!\,(-a)^n e^{a^2z^2/2}D_{-n-1}\left(\sqrt{2}\,az\right).$

18. $\mathrm{D}^n\left[z^{n-1}e^{a^2/z^2}\operatorname{erfc}\left(\frac{a}{z}\right)\right] = \frac{2^{(n+1)/2}}{\sqrt{\pi}}n!\,a^n z^{-n-1}e^{a^2/(2z^2)}D_{-n-1}\left(\frac{\sqrt{2}\,a}{z}\right).$

19. $\mathrm{D}^n\left[z^{n-1/2}e^{a^2z}\operatorname{erfc}(a\sqrt{z})\right] = \frac{2^{1/2-n}}{\sqrt{\pi}}(2n)!\,z^{-1/2}e^{a^2z/2}D_{-2n-1}\left(a\sqrt{2z}\right).$

20. $\mathrm{D}^n\left[z^{-1/2}e^{a^2/z}\operatorname{erfc}\left(\frac{a}{\sqrt{z}}\right)\right]$
$$= (-1)^n\frac{2^{1/2-n}}{\sqrt{\pi}}(2n)!\,z^{-n-1/2}e^{a^2/(2z)}D_{-2n-1}\left(a\sqrt{\frac{2}{z}}\right).$$

21. $\mathrm{D}^n\left[z^{-1/2}e^{-a^2z}\,\mathrm{D}^n\left[z^{n-1/2}e^{a^2z}\operatorname{erfc}(a\sqrt{z})\right]\right]$
$$= (-4)^{-n}(2n)!z^{-n-1}\operatorname{erfc}(a\sqrt{z}).$$

22. $\mathrm{D}^n\left[z^{2n+1/2}e^{-a^2/z}\,\mathrm{D}^n\left[z^{-1/2}e^{a^2/z}\operatorname{erfc}\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= (-4)^{-n}(2n)!\operatorname{erfc}\left(\frac{a}{\sqrt{z}}\right).$$

23. $\mathrm{D}^n\left[z^{1/2}e^{a^2z}\,\mathrm{D}^n[z^{n-1}\operatorname{erfc}(a\sqrt{z})]\right] = 0 \qquad [n \geq 1].$

24. $\mathrm{D}^n\left[z^{n+1/2}e^{-a^2z}\,\mathrm{D}^n\left[z^{-1/2}e^{a^2z}\operatorname{erfc}(a\sqrt{z})\right]\right] = n!\,a^{2n}\operatorname{erfc}(a\sqrt{z}).$

25. $$\mathrm{D}^n\left[z^{n-1/2}e^{-a^2/z}\,\mathrm{D}^n\left[z^{n-1/2}e^{a^2/z}\operatorname{erfc}\left(\frac{a}{\sqrt{z}}\right)\right]\right] = n!\,a^{2n}z^{-n-1}\operatorname{erfc}\left(\frac{a}{\sqrt{z}}\right).$$

1.6. The Fresnel Integrals $S(z)$ and $C(z)$

1.6.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[S(az)] = \sqrt{\frac{a}{2\pi}}\,\frac{(n-1)!}{2i}\,z^{1/2-n}\left[e^{iaz}L_{n-1}^{1/2-n}(-iaz) - e^{-iaz}L_{n-1}^{1/2-n}(iaz)\right] \quad [n \geq 1].$$

2. $$\mathrm{D}^n\left[z^{n-1}S\left(\frac{a}{z}\right)\right] = (-1)^n\sqrt{\frac{a}{2\pi}}\,\frac{(n-1)!}{2i}\,z^{-3/2}\left[e^{ia/z}L_{n-1}^{1/2-n}\left(-\frac{ia}{z}\right) - e^{-ia/z}L_{n-1}^{1/2-n}\left(\frac{ia}{z}\right)\right] \quad [n \geq 1].$$

3. $$\mathrm{D}^n[C(az)] = \sqrt{\frac{a}{2\pi}}\,\frac{(n-1)!}{2}\,z^{1/2-n}\left[e^{iaz}L_{n-1}^{1/2-n}(-iaz) + e^{-iaz}L_{n-1}^{1/2-n}(iaz)\right] \quad [n \geq 1].$$

4. $$\mathrm{D}^n\left[z^{n-1}C\left(\frac{a}{z}\right)\right] = (-1)^n\sqrt{\frac{a}{2\pi}}\,\frac{(n-1)!}{2}\,z^{-3/2}\left[e^{ia/z}L_{n-1}^{1/2-n}\left(-\frac{ia}{z}\right) + e^{-ia/z}L_{n-1}^{1/2-n}\left(\frac{ia}{z}\right)\right] \quad [n \geq 1].$$

1.7. The Generalized Fresnel Integrals $S(z, \nu)$ and $C(z, \nu)$

1.7.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[S(az, \nu)] = -\frac{(n-1)!}{2i}\,a^\nu z^{\nu-n}\left[e^{iaz}L_{n-1}^{\nu-n}(-iaz) - e^{-iaz}L_{n-1}^{\nu-n}(iaz)\right] \quad [n \geq 1].$$

2. $$\mathrm{D}^n\left[z^{n-1}S\left(\frac{a}{z}, \nu\right)\right] = (-1)^{n-1}\frac{(n-1)!}{2i}\,a^\nu z^{-\nu-1}\left[e^{ia/z}L_{n-1}^{\nu-n}\left(-\frac{ia}{z}\right) - e^{-ia/z}L_{n-1}^{\nu-n}\left(\frac{ia}{z}\right)\right] \quad [n \geq 1].$$

3. $$\mathrm{D}^n[C(az, \nu)] = -\frac{(n-1)!}{2}\,a^\nu z^{\nu-n}\left[e^{iaz}L_{n-1}^{\nu-n}(-iaz) + e^{-iaz}L_{n-1}^{\nu-n}(iaz)\right] \quad [n \geq 1].$$

4. $\mathrm{D}^n\left[z^{n-1}C\left(\frac{a}{z}, \nu\right)\right]$
$= (-1)^{n-1}\frac{(n-1)!}{2}a^\nu z^{-\nu-1}\left[e^{ia/z}L_{n-1}^{\nu-n}\left(-\frac{ia}{z}\right) - e^{-ia/z}L_{n-1}^{\nu-n}\left(\frac{ia}{z}\right)\right]$ $[n \geq 1]$.

1.8. The Incomplete Gamma Functions $\gamma(\nu, z)$ and $\Gamma(\nu, z)$

1.8.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[\gamma(\nu, az)] = (n-1)!\,a^\nu z^{\nu-n}e^{-az}L_{n-1}^{\nu-n}(az)$ $[n \geq 1]$.

2. $\mathrm{D}^n\left[z^{n-1}\gamma\left(\nu, \frac{a}{z}\right)\right] = (-1)^n(n-1)!\,a^\nu z^{-\nu-1}e^{-a/z}L_{n-1}^{\nu-n}\left(\frac{a}{z}\right)$ $[n \geq 1]$.

3. $\mathrm{D}^n[z^{-\nu}\gamma(\nu, az)] = (-1)^n z^{-\nu-n}\gamma(\nu+n, az)$.

4. $\mathrm{D}^n\left[z^{n+\nu-1}\gamma\left(\nu, \frac{a}{z}\right)\right] = z^{\nu-1}\gamma\left(\nu+n, \frac{a}{z}\right)$.

5. $\mathrm{D}^n[e^{az}\gamma(\nu, az)] = (1-\nu)_n(-a)^n e^{az}\gamma(\nu-n, az)$.

6. $\mathrm{D}^n\left[z^{n-1}e^{a/z}\gamma\left(\nu, \frac{a}{z}\right)\right] = (1-\nu)_n a^n z^{-n-1}e^{a/z}\gamma\left(\nu-n, \frac{a}{z}\right)$.

7. $\mathrm{D}^n[z^{n-\nu}e^{az}\gamma(\nu, az)] = \frac{n!}{\nu}\,a^\nu\,{}_1F_1\left(\begin{matrix}n+1;\ az\\ \nu+1\end{matrix}\right)$.

8. $\mathrm{D}^n\left[z^{\nu-1}e^{a/z}\gamma\left(\nu, \frac{a}{z}\right)\right] = (-1)^n\frac{n!}{\nu}\,a^\nu z^{-n-1}\,{}_1F_1\left(\begin{matrix}n+1;\ \frac{a}{z}\\ \nu+1\end{matrix}\right)$.

9. $\mathrm{D}^n[z^{\nu+n}e^{-az}\,\mathrm{D}^n[z^{-\nu}e^{az}\gamma(\nu, az)]] = n!\,a^n\gamma(\nu, az)$.

10. $\mathrm{D}^n\left[z^{n-\nu}e^{-a/z}\,\mathrm{D}^n\left[z^{n+\nu-1}e^{a/z}\gamma\left(\nu, \frac{a}{z}\right)\right]\right] = n!\,a^n z^{-n-1}\gamma\left(\nu, \frac{a}{z}\right)$.

11. $\mathrm{D}^n[z^{\nu+n}e^{az}\,\mathrm{D}^n[z^{-\nu}\gamma(\nu, az)]] = (\nu)_n(-a)^n e^{az}\gamma(\nu, az)$.

12. $\mathrm{D}^n\left[z^{n-\nu}e^{a/z}\,\mathrm{D}^n\left[z^{n+\nu-1}\gamma\left(\nu, \frac{a}{z}\right)\right]\right] = (\nu)_n(-a)^n z^{-n-1}e^{a/z}\gamma\left(\nu, \frac{a}{z}\right)$.

13. $\mathrm{D}^n[z^{\nu-1}e^{-az}\,\mathrm{D}^n[z^{n-\nu}e^{az}\gamma(\nu, az)]] = (-1)^n n!\,(1-\nu)_n z^{-n-1}\gamma(\nu, az)$.

14. $\mathrm{D}^n\left[z^{2n-\nu+1}e^{-a/z}\,\mathrm{D}^n\left[z^{\nu-1}e^{a/z}\gamma\left(\nu, \frac{a}{z}\right)\right]\right] = (-1)^n n!\,(1-\nu)_n\gamma\left(\nu, \frac{a}{z}\right)$.

15. $\mathrm{D}^n[z^{n-\nu}e^{-az}\,\mathrm{D}^n[e^{az}\gamma(\nu, az)]] = a^n(1-\nu)_n z^{-\nu}\gamma(\nu, az)$.

16. $\mathrm{D}^n\left[z^{n+\nu}e^{-a/z}\,\mathrm{D}^n\left[z^{n-1}e^{a/z}\gamma\left(\nu, \frac{a}{z}\right)\right]\right] = a^n(1-\nu)_n z^{\nu-n-1}\gamma\left(\nu, \frac{a}{z}\right)$.

17. $\mathrm{D}^n[\Gamma(\nu, az)] = -(n-1)!\,a^\nu z^{\nu-n}e^{-az}L_{n-1}^{\nu-n}(az)$ $[n \geq 1]$.

18. $\mathrm{D}^n\left[z^{n-1}\Gamma\left(\nu,\,\frac{a}{z}\right)\right]=(-1)^{n-1}(n-1)!\,a^{\nu}z^{-\nu-1}e^{-a/z}L_{n-1}^{\nu-n}(az)\quad[n\geq 1].$

19. $\mathrm{D}^n[z^{-\nu}\Gamma(\nu,\,az)]=(-1)^n z^{-\nu-n}\Gamma(\nu+n,\,az).$

20. $\mathrm{D}^n\left[z^{n+\nu-1}\Gamma\left(\nu,\,\frac{a}{z}\right)\right]=z^{\nu-1}\Gamma\left(\nu+n,\,\frac{a}{z}\right).$

21. $\mathrm{D}^n[e^{az}\Gamma(\nu,\,az)]=(1-\nu)_n(-a)^n e^{az}\Gamma(\nu-n,\,az).$

22. $\mathrm{D}^n\left[z^{n-1}e^{a/z}\Gamma\left(\nu,\,\frac{a}{z}\right)\right]=(1-\nu)_n a^n z^{-n-1}e^{a/z}\Gamma\left(\nu-n,\,\frac{a}{z}\right).$

23. $\mathrm{D}^n[z^{n-\nu}e^{az}\Gamma(\nu,\,az)]=n!\,(1-\nu)_n a^{\nu}\Psi\left(\begin{matrix}n+1;\ az\\ \nu+1\end{matrix}\right).$

24. $\mathrm{D}^n\left[z^{\nu-1}e^{a/z}\Gamma\left(\nu,\,\frac{a}{z}\right)\right]=(-1)^n n!\,(1-\nu)_n a^{\nu}z^{-n-1}\Psi\left(\begin{matrix}n+1;\ \frac{a}{z}\\ \nu+1\end{matrix}\right).$

25. $\mathrm{D}^n[\Gamma(1-n,\,az)]=(-1)^n\sqrt{\frac{a}{\pi}}\,z^{1/2-n}e^{-az/2}K_{n-1/2}\left(\frac{az}{2}\right).$

26. $\mathrm{D}^n\left[z^{n-1}\Gamma\left(1-n,\,\frac{a}{z}\right)\right]=\sqrt{\frac{a}{\pi}}\,z^{-3/2}e^{-a/(2z)}K_{n-1/2}\left(\frac{a}{2z}\right).$

1.8.2. Derivatives with respect to the parameter

1. $\dfrac{\partial\gamma(\nu,\,z)}{\partial\nu}=\gamma(\nu,\,z)\ln z-\dfrac{z^{\nu}}{\nu^2}\,{}_2F_2\left(\begin{matrix}\nu,\,\nu;\ -z\\ \nu+1,\,\nu+1\end{matrix}\right).$

1.9. The Parabolic Cylinder Function $D_\nu(z)$

1.9.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[D_\nu(az)]=\left(-\frac{a}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}2^k(-\nu)_k H_{n-k}\left(\frac{az}{2}\right)D_{\nu-k}(az).$

2. $\qquad=\left(-\frac{ia}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}(-2i)^k H_{n-k}\left(\frac{iaz}{2}\right)D_{\nu+k}(az).$

3. $\mathrm{D}^n\left[e^{a^2z^2/4}D_\nu(az)\right]=(-a)^n\,(-\nu)_n e^{a^2z^2/4}D_{\nu-n}(az).$

4. $\mathrm{D}^n\left[e^{-a^2z^2/4}D_\nu(az)\right]=(-a)^n\,e^{-a^2z^2/4}D_{\nu+n}(az).$

5. $\mathrm{D}^n\left[z^{n-1}e^{a^2/(4z^2)}D_\nu\left(\frac{a}{z}\right)\right]=a^n(-\nu)_n z^{-n-1}e^{a^2/(4z^2)}D_{\nu-n}\left(\frac{a}{z}\right).$

6. $\mathrm{D}^n\left[z^{n-1}e^{-a^2/(4z^2)}D_\nu\left(\frac{a}{z}\right)\right]=a^n z^{-n-1}e^{-a^2/(4z^2)}D_{\nu+n}\left(\frac{a}{z}\right).$

7. $\mathrm{D}^n\left[z^{n-\nu/2-1}e^{a^2z/4}D_\nu(a\sqrt{z})\right]=2^{-n}(-\nu)_{2n}\,z^{-\nu/2-1}e^{a^2z/4}D_{\nu-2n}(a\sqrt{z}).$

8. $\mathrm{D}^n\left[z^{n+(\nu-1)/2}e^{-a^2z/4}D_\nu(a\sqrt{z})\right]=(-2)^{-n}\,z^{(\nu-1)/2}e^{-a^2z/4}D_{\nu+2n}(a\sqrt{z}).$

9. $\mathrm{D}^n\left[z^{\nu/2}e^{a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]$
$$=(-2)^{-n}(-\nu)_{2n}\,z^{-n+\nu/2}e^{a^2/(4z)}D_{\nu-2n}\left(\frac{a}{\sqrt{z}}\right).$$

10. $\mathrm{D}^n\left[z^{-(\nu+1)/2}e^{-a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]$
$$=2^{-n}z^{-n-(\nu+1)/2}e^{-a^2/(4z)}D_{\nu+2n}\left(\frac{a}{\sqrt{z}}\right).$$

11. $\mathrm{D}^n\left[z^{-1/2}e^{-a^2z/4}D_{-2n}(a\sqrt{z})\right]=(-2)^{-n}\,z^{-n-1/2}e^{-a^2z/2}.$

12. $\mathrm{D}^n\left[z^{n-1/2}e^{-a^2/(4z)}D_{-2n}\left(\frac{a}{\sqrt{z}}\right)\right]=2^{-n}z^{-1/2}e^{-a^2/(2z)}.$

13. $\mathrm{D}^n\left[z^{-1}e^{-a^2z/4}D_{-2n-1}(a\sqrt{z})\right]=(-1)^n\,2^{-n-1/2}\sqrt{\pi}\,z^{-n-1}\operatorname{erfc}\left(a\sqrt{\frac{z}{2}}\right).$

14. $\mathrm{D}^n\left[z^ne^{-a^2/(4z)}D_{-2n-1}\left(\frac{a}{\sqrt{z}}\right)\right]=2^{-n-1/2}\sqrt{\pi}\operatorname{erfc}\left(\frac{a}{\sqrt{2z}}\right).$

15. $\mathrm{D}^n\left[e^{a^2z/4}D_\nu(a\sqrt{z})\right]$
$$=(-1)^ne^{a^2z/4}\sum_{k=0}^{n-1}\frac{\Gamma(n+k)}{\Gamma(n-k)k!\,(2\sqrt{z})^{n+k}}a^{n-k}(-\nu)_{n-k}D_{\nu-n+k}(a\sqrt{z})$$
$[n\ge 1]$.

16. $\mathrm{D}^n\left[e^{-a^2z/4}D_\nu(a\sqrt{z})\right]$
$$=(-1)^ne^{-a^2z/4}\sum_{k=0}^{n-1}\frac{\Gamma(n+k)}{\Gamma(n-k)k!\,(2\sqrt{z})^{n+k}}a^{n-k}D_{\nu+n-k}(a\sqrt{z})\quad[n\ge 1].$$

17. $\mathrm{D}^n\left[z^{n+1/2}e^{-a^2z/2}\,\mathrm{D}^n\left[z^{-1/2}e^{a^2z/4}D_\nu(a\sqrt{z})\right]\right]$
$$=\left(\frac{1-\nu}{2}\right)_n\left(\frac{a^2}{2}\right)^n e^{-a^2z/4}D_\nu(a\sqrt{z}).$$

18. $\mathrm{D}^n\left[z^{n-1/2}e^{-a^2/(2z)}\,\mathrm{D}^n\left[z^{n-1/2}e^{a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$=\left(\frac{1-\nu}{2}\right)_n\left(\frac{a^2}{2}\right)^n z^{-n-1}e^{-a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right).$$

19. $\mathrm{D}^n\left[z^{n-1/2}e^{-a^2z/2}\,\mathrm{D}^n\left[e^{a^2z/4}D_\nu(a\sqrt{z})\right]\right]$
$$=\left(-\frac{\nu}{2}\right)_n\left(\frac{a^2}{2}\right)^n z^{-1/2}e^{-a^2z/4}D_\nu(a\sqrt{z}).$$

20. $\mathrm{D}^n\left[z^{n+1/2}e^{-a^2/(2z)}\,\mathrm{D}^n\left[z^{n-1}e^{a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= \left(-\frac{\nu}{2}\right)_n\left(\frac{a^2}{2}\right)^n z^{-n-1/2}e^{-a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right).$$

21. $\mathrm{D}^n\left[z^{n-1/2}e^{a^2z/2}\,\mathrm{D}^n\left[e^{-a^2z/4}D_\nu(a\sqrt{z})\right]\right]$
$$= \left(\frac{\nu+1}{2}\right)_n\left(-\frac{a^2}{2}\right)^n z^{-1/2}e^{a^2z/4}D_\nu(a\sqrt{z}).$$

22. $\mathrm{D}^n\left[z^{n+1/2}e^{a^2/(2z)}\,\mathrm{D}^n\left[z^{n-1}e^{-a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= \left(\frac{\nu+1}{2}\right)_n\left(-\frac{a^2}{2}\right)^n z^{-n-1/2}e^{a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right).$$

23. $\mathrm{D}^n\left[z^{n+1/2}e^{a^2z/2}\,\mathrm{D}^n\left[z^{-1/2}e^{-a^2z/4}D_\nu(a\sqrt{z})\right]\right]$
$$= \left(\frac{\nu}{2}+1\right)_n\left(-\frac{a^2}{2}\right)^n e^{a^2z/4}D_\nu(a\sqrt{z}).$$

24. $\mathrm{D}^n\left[z^{n-1/2}e^{a^2/(2z)}\,\mathrm{D}^n\left[z^{n-1/2}e^{-a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= \left(\frac{\nu}{2}+1\right)_n\left(-\frac{a^2}{2}\right)^n z^{-n-1}e^{a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right).$$

25. $\mathrm{D}^n\left[z^{\nu+1/2}e^{-a^2z/2}\,\mathrm{D}^n\left[z^{n-\nu/2-1}e^{a^2z/4}D_\nu(a\sqrt{z})\right]\right]$
$$= (-4)^{-n}(-\nu)_{2n}\,z^{(\nu-1)/2-n}e^{-a^2z/4}D_\nu(a\sqrt{z}).$$

26. $\mathrm{D}^n\left[z^{2n-\nu-1/2}e^{-a^2/(2z)}\,\mathrm{D}^n\left[z^{\nu/2}e^{a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= (-4)^{-n}(-\nu)_{2n}\,z^{-(\nu+1)/2}e^{-a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right).$$

27. $\mathrm{D}^n\left[z^{-\nu-1/2}e^{a^2z/2}\,\mathrm{D}^n\left[z^{n+(\nu-1)/2}e^{-a^2z/4}D_\nu(a\sqrt{z})\right]\right]$
$$= (-4)^{-n}(\nu+1)_{2n}z^{-n-\nu/2-1}e^{a^2z/4}D_\nu(a\sqrt{z}).$$

28. $\mathrm{D}^n\left[z^{2n+\nu+1/2}e^{a^2/(2z)}\,\mathrm{D}^n\left[z^{-(\nu+1)/2}e^{-a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right)\right]\right]$
$$= (-4)^{-n}(\nu+1)_{2n}z^{\nu/2}e^{a^2/(4z)}D_\nu\left(\frac{a}{\sqrt{z}}\right).$$

1.9.2. Derivatives with respect to the order

1. $\left.\frac{\partial D_\nu(z)}{\partial\nu}\right|_{\nu=2n} = 2^{-n-1}e^{-z^2/4}H_{2n}\left(\frac{z}{\sqrt{2}}\right)$
$$\times\left[-\mathbf{C}-\ln 2+\psi\left(\frac{1}{2}-n\right)+\pi\,\mathrm{erfi}\,\left(\frac{z}{\sqrt{2}}\right)-z^2\,{}_2F_2\left(\begin{matrix}1,\,1;\ \frac{z^2}{2}\\ \frac{3}{2},\,2\end{matrix}\right)\right]$$

$$- (-1)^n 2^{n-1} n!\, e^{-z^2/4} \sum_{k=1}^{n} \frac{1}{k} L_{n-k}^{k-1/2}\left(\frac{z^2}{2}\right)$$

$$\times \left[-\sqrt{\frac{\pi}{2}}\, z e^{z^2/2} L_{k-1}^{1/2-k}\left(-\frac{z^2}{2}\right) + \frac{\left(\frac{z^2}{2}\right)^k}{\left(\frac{1}{2}\right)_k} {}_1F_1\left(\begin{matrix} k;\ \frac{z^2}{2} \\ k+\frac{1}{2} \end{matrix}\right)\right]$$

$$- (-1)^n 2^{n-1} \left(\frac{1}{2}\right)_n e^{-z^2/4} \sum_{k=0}^{n} \binom{n}{k} \frac{\left(-\frac{z^2}{2}\right)^k}{\left(\frac{1}{2}\right)_k} \psi\left(\frac{1}{2} - k\right).$$

2. $$\left.\frac{\partial D_\nu(z)}{\partial \nu}\right|_{\nu=2n+1} = 2^{-n-3/2} e^{-z^2/4} H_{2n+1}\left(\frac{z}{\sqrt{2}}\right)$$

$$\times \left[\mathrm{C} + \ln 2 - \psi\left(-\frac{1}{2} - n\right) - \pi \operatorname{erfi}\left(\frac{z}{\sqrt{2}}\right) + z^2 {}_2F_2\left(\begin{matrix} 1, 1;\ \frac{z^2}{2} \\ \frac{3}{2}, 2 \end{matrix}\right)\right]$$

$$+ (-1)^n 2^{-n-2} (2n+1)!\, z^{-1} e^{-z^2/4} \sum_{k=1}^{n} \frac{\left(-2z^2\right)^k}{(2k-1)!\,(n-k+1)!} \psi\left(\frac{1}{2} - k\right)$$

$$+ (-1)^n 2^{n-3/2} n!\, e^{-z^2/4} \sum_{k=1}^{n} \frac{1}{k} \left[L_{n-k}^{k-1/2}\left(\frac{z^2}{2}\right) + 2(n+1) L_{n-k+1}^{k-3/2}\left(\frac{z^2}{2}\right)\right]$$

$$\times \left[\frac{z^{2k-1}}{2^{k-1/2}\left(\frac{1}{2}\right)_k} {}_1F_1\left(\begin{matrix} k;\ \frac{z^2}{2} \\ k+\frac{1}{2} \end{matrix}\right) - \sqrt{\pi}\, e^{z^2/2} L_{k-1}^{1/2-k}\left(-\frac{z^2}{2}\right)\right]$$

$$- (-1)^n 2^{n-1/2} \sqrt{\pi}\, n!\, e^{z^2/4} L_n^{-n-1/2}\left(-\frac{z^2}{2}\right)$$

$$+ (-1)^n \frac{n!}{\left(\frac{3}{2}\right)_n} z^{2n+1} e^{-z^2/4} {}_1F_1\left(\begin{matrix} n+1;\ \frac{z^2}{2} \\ n+\frac{3}{2} \end{matrix}\right) - \frac{z^{2n+1}}{2} e^{-z^2/4} \psi\left(-n - \frac{1}{2}\right).$$

3. $$\left.\frac{\partial}{\partial \nu} \left[D_\nu(z) D_\nu(e^{i\pi/2} z)\right]\right|_{\nu=-1/2} = \frac{e^{3i\pi/4}}{3} z^3 {}_2F_3\left(\begin{matrix} 1, 1;\ \frac{z^4}{16} \\ \frac{5}{4}, \frac{3}{2}, \frac{7}{4} \end{matrix}\right)$$

$$+ \frac{\pi z}{8} \left[2^{3/2} e^{i\pi/4} (\mathrm{C} + 2\ln 2)\, I_{-1/4}\left(\frac{z^2}{4}\right) I_{1/4}\left(\frac{z^2}{4}\right)\right.$$

$$\left. - e^{i\pi/2} (\pi + 2\mathrm{C} + 4\ln 2)\, I_{1/4}^2\left(\frac{z^2}{4}\right) + (\pi - 2\mathrm{C} - 4\ln 2)\, I_{-1/4}^2\left(\frac{z^2}{4}\right)\right]$$

$[z > 0]$.

1.10. The Bessel Function $J_\nu(z)$

1.10.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[J_\nu(az)] = \left(\pm\frac{a}{2}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} J_{\nu \pm 2k \mp n}(az).$$

2. $$= n!\,(-z)^{-n}\sum_{k=0}^{n}\frac{(-az)^{k}}{(n-k)!}(\nu)_{n-k}\sum_{j=0}^{[k/2]}\frac{(2az)^{-j}}{j!\,(k-2j)!}J_{\nu+j-k}(az).$$

3. $$\mathrm{D}^n\left[z^{n-1}J_\nu\left(\frac{a}{z}\right)\right]=\left(\mp\frac{a}{2}\right)^n z^{-n-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{\nu\pm 2k\mp n}\left(\frac{a}{z}\right).$$

4. $$\mathrm{D}^n[J_\nu(a\sqrt{z})]=\left(-\frac{1}{z}\right)^n\sum_{k=0}^{n}(\pm1)^k\binom{n}{k}\left(\mp\frac{\nu}{2}\right)_{n-k}\left(\frac{a\sqrt{z}}{2}\right)^k J_{\nu\pm k}(a\sqrt{z}).$$

5. $$\mathrm{D}^n[z^{\pm\nu/2}J_\nu(a\sqrt{z})]=\left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}J_{\nu\mp n}(a\sqrt{z}).$$

6. $$\mathrm{D}^n[z^{(2n+1)/4}J_{n+1/2}(a\sqrt{z})]=\frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}\sin(a\sqrt{z}).$$

7. $$\mathrm{D}^n[z^{(2n+1)/4}J_{-n-1/2}(a\sqrt{z})]=\frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}\cos(a\sqrt{z}).$$

8. $$\mathrm{D}^n[z^{-(2n+3)/4}J_{n+1/2}(a\sqrt{z})]$$
$$=\sqrt{\pi}\left(\frac{a}{2z}\right)^{n+1/2}J_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)J_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right).$$

9. $$\mathrm{D}^n[z^{\pm\nu/4}J_\nu(a\sqrt[4]{z})]$$
$$=\left(\pm\frac{a}{4}\right)^n z^{(\pm\nu-3n)/4}\sum_{k=0}^{n-1}(\mp a)^{-k}\frac{\Gamma(n+k)}{k!\,\Gamma(n-k)}z^{-k/4}J_{\nu\mp n\pm k}(a\sqrt[4]{z})$$
$[n\ge1]$.

10. $$\mathrm{D}^n\left[z^{n-1}J_\nu\left(\frac{a}{\sqrt{z}}\right)\right]=\frac{1}{z}\sum_{k=0}^{n}(\pm1)^k\binom{n}{k}\left(\mp\frac{\nu}{2}\right)_{n-k}\left(\frac{a}{2\sqrt{z}}\right)^k J_{\nu\pm k}\left(\frac{a}{\sqrt{z}}\right).$$

11. $$\mathrm{D}^n\left[z^{n\pm\nu/2-1}J_\nu\left(\frac{a}{\sqrt{z}}\right)\right]=\left(\pm\frac{a}{2}\right)^n z^{-(n\mp\nu)/2-1}J_{\nu\pm n}\left(\frac{a}{\sqrt{z}}\right).$$

12. $$\mathrm{D}^n\left[z^{(2n-5)/4}J_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right]=\frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}z^{-n-1}\sin\left(\frac{a}{\sqrt{z}}\right).$$

13. $$\mathrm{D}^n\left[z^{(2n-5)/4}J_{-n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right]=\frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}z^{-n-1}\cos\left(\frac{a}{\sqrt{z}}\right).$$

14. $$\mathrm{D}^n\left[z^{(6n-1)/4}J_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right]$$
$$=(-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2}z^{-1/2}J_{n+1/2}\left(\frac{a}{2\sqrt{z}}\right)J_{-n-1/2}\left(\frac{a}{2\sqrt{z}}\right).$$

15. $$\mathrm{D}^n[z^{n-1/2}e^{\pm iaz}J_{n-1/2}(az)]=\frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{n-1}e^{\pm2iaz}\qquad[n\ge1].$$

16. $\mathrm{D}^n\left[z^{-1/2}e^{\pm ia/z}J_{n-1/2}\left(\frac{a}{z}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}}(2a)^{n-1/2}z^{-2n}e^{\pm 2ia/z}$ $[n \geq 1]$.

17. $\mathrm{D}^n[z^{n-1/2}\sin(az)\,J_{n-1/2}(az)] = \frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{n-1}\sin(2az)$.

18. $\mathrm{D}^n[z^{n-1/2}\cos(az)\,J_{n-1/2}(az)] = \frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{n-1}\cos(2az)$ $[n \geq 1]$.

19. $\mathrm{D}^n[z^{n-1/2}\sin(az)\,J_{1/2-n}(az)] = (-1)^{n+1}\frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{n-1}\cos(2az)$ $[n \geq 1]$.

20. $\mathrm{D}^n[z^{n-1/2}\cos(az)\,J_{1/2-n}(az)] = (-1)^n\frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{n-1}\sin(2az)$.

21. $\mathrm{D}^n\left[z^{-1/2}\sin\frac{a}{z}\,J_{n-1/2}\left(\frac{a}{z}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}}(2a)^{n-1/2}z^{-2n}\sin\frac{2a}{z}$.

22. $\mathrm{D}^n\left[z^{-1/2}\cos\frac{a}{z}\,J_{n-1/2}\left(\frac{a}{z}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}}(2a)^{n-1/2}z^{-2n}\cos\frac{2a}{z}$ $[n \geq 1]$.

23. $\mathrm{D}^n[J_\nu^2(a\sqrt{z})] = \left(\frac{a}{2\sqrt{z}}\right)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{\nu+k}(a\sqrt{z})\,J_{\nu-n+k}(a\sqrt{z})$.

24. $\mathrm{D}^n[z^{n-1/2}J_{n-1/2}^2(a\sqrt{z})] = \frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}J_{n-1/2}(2a\sqrt{z})$ $[n \geq 1]$.

25. $\mathrm{D}^n[z^{n-1/2}J_{1/2-n}^2(a\sqrt{z})] = -\frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}J_{n-1/2}(2a\sqrt{z})$ $[n \geq 1]$.

26. $\mathrm{D}^n[z^{n-1/2}J_{n-1/2}(a\sqrt{z})\,J_{1/2-n}(a\sqrt{z})]$

$= \frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}J_{1/2-n}(2a\sqrt{z})$ $[n \geq 1]$.

27. $\mathrm{D}^n[z^nJ_{n-1/2}(a\sqrt{z})\,J_{n+1/2}(a\sqrt{z})] = \frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-1)/4}J_{n+1/2}(2a\sqrt{z})$.

28. $\mathrm{D}^n\left[z^{-1/2}J_{n-1/2}^2\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}}a^{n-1/2}z^{-(6n+1)/4}J_{n-1/2}\left(\frac{2a}{\sqrt{z}}\right)$ $[n \geq 1]$.

29. $\mathrm{D}^n\left[z^{-1/2}J_{1/2-n}^2\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{(-1)^{n+1}}{\sqrt{\pi}}a^{n-1/2}z^{-(6n+1)/4}J_{n-1/2}\left(\frac{2a}{\sqrt{z}}\right)$ $[n \geq 1]$.

30. $\mathrm{D}^n\left[z^{-1/2}J_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)J_{1/2-n}\left(\frac{a}{\sqrt{z}}\right)\right]$

$= \frac{(-1)^n}{\sqrt{\pi}}a^{n-1/2}z^{-(6n+1)/4}J_{1/2-n}\left(\frac{2a}{\sqrt{z}}\right)$.

31. $\mathrm{D}^n\left[z^{-1}J_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)J_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right]$
$$= \frac{(-1)^n}{\sqrt{\pi}}a^{n-1/2}z^{-(6n+3)/4}J_{n+1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

32. $\mathrm{D}^n\left[z^{n-1}J_\nu^2\left(\frac{a}{\sqrt{z}}\right)\right]$
$$= \left(-\frac{a}{2}\right)^n z^{-n/2-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{\nu+k}\left(\frac{a}{\sqrt{z}}\right)J_{\nu-n+k}\left(\frac{a}{\sqrt{z}}\right).$$

1.10.2. Derivatives with respect to the order

1. $$\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=\pm n} = (\pm1)^n\frac{\pi}{2}Y_n(z) \pm (\pm1)^n\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}J_k(z).$$

2. $$\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=1/2} = \sqrt{\frac{2}{\pi z}}\left[\sin z\,\mathrm{ci}\,(2z) - \cos z\,\mathrm{Si}\,(2z)\right] \qquad [[10], 7.9(18)].$$

3. $$\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=-1/2} = \sqrt{\frac{2}{\pi z}}\left[\sin z\,\mathrm{Si}\,(2z) + \cos z\,\mathrm{ci}\,(2z)\right] \qquad [[10], 7.9(19)].$$

4. $$\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=n+1/2} = \mathrm{ci}\,(2z)\,J_{n+1/2}(z) - (-1)^n\,\mathrm{Si}\,(2z)\,J_{-n-1/2}(z)$$
$$+\frac{n!}{2}\left(\frac{2}{z}\right)^n\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^k}{k!\,(n-k)}J_{k+1/2}(z) - \frac{n!\sqrt{\pi z}}{2}\sum_{k=1}^{n}\frac{\left(\frac{2}{z}\right)^k}{(n-k)!\,k}$$
$$\times\sum_{p=0}^{k-1}\frac{z^p}{p!}\left[J_{n-k+1/2}(z)\,J_{p-1/2}(2z) - (-1)^{n-k-p}J_{k-n-1/2}(z)\,J_{1/2-p}(2z)\right].$$

5. $$\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=1/2-n} = \mathrm{ci}\,(2z)\,J_{1/2-n}(z) - (-1)^n\,\mathrm{Si}\,(2z)\,J_{n-1/2}(z)$$
$$-\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(-\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}J_{1/2-k}(z) - \frac{n!\sqrt{\pi}}{2}\sum_{k=1}^{n}\frac{2^k}{(n-k)!\,k}$$
$$\times\sum_{p=0}^{k-1}\frac{z^{p-k+1/2}}{p!}\left[(-1)^k J_{k-n+1/2}(z)\,J_{p-1/2}(2z)\right.$$
$$\left.-(-1)^{n+p}J_{n-k-1/2}(z)\,J_{1/2-p}(2z)\right].$$

6. $$\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=\pm n\pm1/2} = (\mp1)^{n-1}2^{n+1/2}\sqrt{\pi}\,z^{-n-1/2}$$
$$\times\sum_{k=0}^{n-1}\frac{(n-k)!\left(\frac{z}{2}\right)^{2k}}{k!\,(n-2k)!\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k-n+\frac{1}{2}\right)}\left[\left(\psi\left(k+\frac{1}{2}\right)-\psi\left(k-n+\frac{1}{2}\right)\right)\right.$$

$$\times \begin{Bmatrix} \sin z \\ \cos z \end{Bmatrix} \mp \begin{Bmatrix} \sin z \\ \cos z \end{Bmatrix} \operatorname{ci}(2z) + \begin{Bmatrix} \cos z \\ \sin z \end{Bmatrix} \operatorname{Si}(2z)\Bigg]$$

$$+ (\mp 1)^n 2^{n-1/2} \sqrt{\pi}\, z^{-n+1/2} \sum_{k=0}^{n-1} \frac{(n-k-1)!\left(\frac{z}{2}\right)^{2k}}{k!\,(n-2k-1)!\,\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k-n+\frac{1}{2}\right)}$$

$$\times \left[\left(\psi\left(k+\frac{3}{2}\right) - \psi\left(k-n+\frac{1}{2}\right)\right) \begin{Bmatrix} \cos z \\ \sin z \end{Bmatrix}\right.$$

$$\left. - \begin{Bmatrix} \sin z \\ \cos z \end{Bmatrix} \operatorname{Si}(2z) \mp \begin{Bmatrix} \cos z \\ \sin z \end{Bmatrix} \operatorname{ci}(2z)\right] \quad [n \geq 1].$$

1.11. The Bessel Function $Y_\nu(z)$

1.11.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[Y_\nu(az)] = \left(\pm\frac{a}{2}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} Y_{\nu\pm 2k\mp n}(az).$

2. $\qquad = n!\,(-z)^{-n} \sum_{k=0}^{n} \frac{(-az)^k}{(n-k)!} (\nu)_{n-k} \sum_{j=0}^{[k/2]} \frac{(2az)^{-j}}{j!\,(k-2j)!} Y_{\nu+j-k}(az).$

3. $\mathrm{D}^n\left[z^{n-1} Y_\nu\left(\frac{a}{z}\right)\right] = \left(\mp\frac{a}{2}\right)^n z^{-n-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} Y_{\nu\pm 2k\mp n}\left(\frac{a}{z}\right).$

4. $\mathrm{D}^n[z^{\pm\nu/2} Y_\nu(a\sqrt{z})] = \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2} Y_{\nu\mp n}(a\sqrt{z}).$

5. $\mathrm{D}^n[z^{(2n+1)/4} Y_{n+1/2}(a\sqrt{z})] = -\frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \cos(a\sqrt{z}).$

6. $\mathrm{D}^n[z^{(2n+1)/4} Y_{-n-1/2}(a\sqrt{z})] = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \sin(a\sqrt{z}).$

7. $\mathrm{D}^n[z^{-(2n+3)/4} Y_{-n-1/2}(a\sqrt{z})]$

$$= (-1)^{n+1}\sqrt{\pi}\left(\frac{a}{2z}\right)^{n+1/2} Y_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) Y_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right).$$

8. $\mathrm{D}^n[z^{\pm\nu/4} Y_\nu(a\sqrt[4]{z})]$

$$= \left(\pm\frac{a}{4}\right)^n z^{(\pm\nu-3n)/4} \sum_{k=0}^{n-1} (\mp a)^{-k} \frac{\Gamma(n+k)}{k!\,\Gamma(n-k)} z^{-k/4} Y_{\nu\mp n\pm k}(a\sqrt[4]{z}) \quad [n \geq 1].$$

9. $\mathrm{D}^n\left[z^{n\pm\nu/2-1} Y_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2-1} Y_{\nu\pm n}\left(\frac{a}{\sqrt{z}}\right).$

10. $\mathrm{D}^n\left[z^{(2n-5)/4} Y_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{(-1)^{n+1}}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} z^{-n-1} \cos\left(\frac{a}{\sqrt{z}}\right).$

11. $\mathrm{D}^n\left[z^{(2n-5)/4}Y_{-n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} z^{-n-1}\sin\left(\frac{a}{\sqrt{z}}\right).$

12. $\mathrm{D}^n\left[z^{(6n-1)/4}Y_{-n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right]$
$$= -\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2} z^{-1/2}Y_{n+1/2}\left(\frac{a}{2\sqrt{z}}\right)Y_{-n-1/2}\left(\frac{a}{2\sqrt{z}}\right).$$

13. $\mathrm{D}^n[z^{n-1/2}e^{\pm iaz}Y_{1/2-n}(az)] = (-1)^{n+1}\frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{n-1}e^{\pm 2iaz} \quad [n \geq 1].$

14. $\mathrm{D}^n\left[z^{-1/2}e^{\pm ia/z}Y_{1/2-n}\left(\frac{a}{z}\right)\right] = -\frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{-2n}e^{\pm 2ia/z} \quad [n \geq 1].$

15. $\mathrm{D}^n[z^{n-1/2}Y^2_{1/2-n}(a\sqrt{z})] = (-1)^{n+1}\frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}Y_{1/2-n}(2a\sqrt{z})$
$[n \geq 1].$

16. $\mathrm{D}^n[z^{n-1/2}Y_{n-1/2}(a\sqrt{z})\,Y_{1/2-n}(a\sqrt{z})]$
$$= (-1)^{n+1}\frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}Y_{n-1/2}(2a\sqrt{z}) \quad [n \geq 1].$$

17. $\mathrm{D}^n[z^nY_{n-1/2}(a\sqrt{z})\,Y_{n+1/2}(a\sqrt{z})]$
$$= (-1)^{n+1}\frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-1)/4}Y_{-n-1/2}(2a\sqrt{z}).$$

18. $\mathrm{D}^n\left[z^{-1/2}Y^2_{1/2-n}\left(\frac{a}{\sqrt{z}}\right)\right] = -\frac{a^{n-1/2}}{\sqrt{\pi}}z^{-(6n+1)/4}Y_{1/2-n}\left(\frac{2a}{\sqrt{z}}\right) \quad [n \geq 1].$

19. $\mathrm{D}^n\left[z^{-1/2}Y_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)Y_{1/2-n}\left(\frac{a}{\sqrt{z}}\right)\right]$
$$= -\frac{a^{n-1/2}}{\sqrt{\pi}}z^{-(6n+1)/4}Y_{n-1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

20. $\mathrm{D}^n\left[z^{-1}Y_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)Y_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right]$
$$= -\frac{1}{\sqrt{\pi}}a^{n-1/2}z^{-(6n+3)/4}Y_{-n-1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

21. $\mathrm{D}^n[z^{n-1/2}J_{n-1/2}(a\sqrt{z})\,Y_{1/2-n}(a\sqrt{z}]$
$$= (-1)^{n+1}\frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}J_{n-1/2}(2a\sqrt{z}) \quad [n \geq 1].$$

22. $\mathrm{D}^n[z^{n-1/2}J_{1/2-n}(a\sqrt{z})\,Y_{n-1/2}(a\sqrt{z})]$
$$= (-1)^{n+1}\frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}J_{n-1/2}(2a\sqrt{z}) \quad [n \geq 1].$$

23. $$\mathrm{D}^n[z^{n-1/2}J_{n-1/2}(a\sqrt{z})\,Y_{n-1/2}(a\sqrt{z})] = \frac{(-1)^n}{\sqrt{\pi}}\,a^{n-1/2}z^{(2n-3)/4}J_{1/2-n}(2a\sqrt{z}).$$

24. $$\mathrm{D}^n[z^n J_{n-1/2}(a\sqrt{z})\,Y_{-n-1/2}(a\sqrt{z})] = \frac{(-1)^n}{\sqrt{\pi}}\,a^{n-1/2}z^{(2n-1)/4}J_{n+1/2}(2a\sqrt{z}).$$

25. $$\mathrm{D}^n[z^n J_{n+1/2}(a\sqrt{z})\,Y_{1/2-n}(a\sqrt{z})] = \frac{(-1)^{n+1}}{\sqrt{\pi}}\,a^{n-1/2}z^{(2n-1)/4}J_{n+1/2}(2a\sqrt{z}).$$

26. $$\mathrm{D}^n\left[z^{-1/2}J_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)Y_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{a^{n-1/2}}{\sqrt{\pi}}\,z^{-(6n+1)/4}J_{1/2-n}\left(\frac{2a}{\sqrt{z}}\right).$$

27. $$\mathrm{D}^n\left[z^{-1/2}J_{1/2-n}\left(\frac{a}{\sqrt{z}}\right)Y_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = -\frac{a^{n-1/2}}{\sqrt{\pi}}\,z^{-(6n+1)/4}J_{n-1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

28. $$\mathrm{D}^n\left[z^{-1}J_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)Y_{-n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{a^{n-1/2}}{\sqrt{\pi}}\,z^{-(6n+3)/4}J_{n+1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

29. $$\mathrm{D}^n\left[z^{-1}J_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)Y_{1/2-n}\left(\frac{a}{\sqrt{z}}\right)\right] = -\frac{a^{n-1/2}}{\sqrt{\pi}}\,a^{n-1/2}z^{-(6n+3)/4}J_{n+1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

1.11.2. Derivatives with respect to the order

1. $$\left.\frac{\partial Y_\nu(z)}{\partial\nu}\right|_{\nu=\pm n} = -(\pm 1)^n\frac{\pi}{2}\,J_n(z) \pm (\pm 1)^n\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\,Y_k(z).$$

2. $$\left.\frac{\partial Y_\nu(z)}{\partial\nu}\right|_{\nu=\pm n\pm 1/2} = -\pi J_{\pm n\pm 1/2}(z) + (-1)^n\left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=\mp n\mp 1/2}.$$

1.12. The Hankel Functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

1.12.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[z^{\pm\nu/2}H_\nu^{(j)}(a\sqrt{z})] = \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}H_{\nu\mp n}^{(j)}(a\sqrt{z}) \qquad [j=1,\,2].$$

2. $$\mathrm{D}^n\left[z^{n\pm\nu/2-1}H_\nu^{(j)}\left(\frac{a}{\sqrt{z}}\right)\right] = \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2-1}H_{\nu\pm n}^{(j)}\left(\frac{a}{\sqrt{z}}\right) \qquad [j=1,\,2].$$

3. $\mathrm{D}^n[z^{(2n+1)/4}H^{(j)}_{n+1/2}(a\sqrt{z})] = \frac{(-1)^j i}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} e^{(-1)^{j+1}ia\sqrt{z}} \qquad [j=1,\,2].$

4. $\mathrm{D}^n\left[z^{n-1/2}H^{(1)}_{1/2-n}(a\sqrt{z})H^{(2)}_{1/2-n}(a\sqrt{z})\right] = 0 \qquad [n \ge 1].$

1.12.2. Derivatives with respect to the order

1. $\left.\frac{\partial H^{(j)}_\nu(z)}{\partial \nu}\right|_{\nu=n} = (-1)^j \frac{\pi i}{2} H^{(j)}_n(z) + \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)} H^{(j)}_k(z) \qquad [j=1,\,2].$

2. $\left.\frac{\partial H^{(j)}_\nu(z)}{\partial \nu}\right|_{\nu=-n} = (-1)^{j+n} \frac{\pi i}{2} H^{(j)}_n(z) - (-1)^n \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)} H^{(j)}_k(z)$
$[j=1,\,2].$

3. $\left.\frac{\partial H^{(j)}_\nu(z)}{\partial \nu}\right|_{\nu=1/2} = \sqrt{\frac{2}{\pi z}}$
$\times \{e^{(-1)^j iz}[(-1)^{j+1} i\ \mathrm{ci}\,(2z) - \mathrm{Si}\,(2z)] + (-1)^j i\pi \sin z\} \qquad [j=1,\,2].$

4. $\left.\frac{\partial H^{(j)}_\nu(z)}{\partial \nu}\right|_{\nu=-1/2} = \sqrt{\frac{2}{\pi z}}$
$\times \{e^{(-1)^j iz}[\mathrm{ci}\,(2z) + (-1)^{j+1} i\ \mathrm{Si}\,(2z)] + (-1)^j i\pi \cos z\} \qquad [j=1,\,2].$

1.13. The Modified Bessel Function $I_\nu(z)$

1.13.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[I_\nu(az)] = \left(\frac{a}{2}\right)^n \sum_{k=0}^{n}\binom{n}{k} I_{\nu\pm 2k\mp n}(az).$

2. $\quad = n!\,(-z)^{-n}\sum_{k=0}^{n}\frac{(-az)^k}{(n-k)!}(\nu)_{n-k}\sum_{p=0}^{[k/2]}\frac{(2az)^{-j}}{p!\,(k-2p)!} I_{\nu-k+p}(az).$

3. $\mathrm{D}^n\left[z^{n-1}I_\nu\left(\frac{a}{z}\right)\right] = \left(-\frac{a}{2}\right)^n z^{-n-1}\sum_{k=0}^{n}\binom{n}{k} I_{\nu\pm 2k\mp n}\left(\frac{a}{z}\right).$

4. $\mathrm{D}^n[I_\nu(a\sqrt{z})] = \left(-\frac{1}{z}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\mp\frac{\nu}{2}\right)_{n-k}\left(-\frac{a\sqrt{z}}{2}\right)^k I_{\nu\pm k}(a\sqrt{z}).$

5. $\mathrm{D}^n[z^{\pm\nu/2}I_\nu(a\sqrt{z})] = \left(\frac{a}{2}\right)^n z^{(\pm\nu-n)/2} I_{\nu\mp n}(a\sqrt{z}).$

6. $\mathrm{D}^n[z^{(2n+1)/4}I_{n+1/2}(a\sqrt{z})] = \frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}\sinh(a\sqrt{z}).$

7. $\mathrm{D}^n[z^{(2n+1)/4}I_{-n-1/2}(a\sqrt{z})] = \frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}\cosh(a\sqrt{z}).$

8. $\mathrm{D}^n[z^{-(2n+3)/4}I_{n+1/2}(a\sqrt{z})]$

$$= \sqrt{\pi}\left(\frac{a}{2z}\right)^{n+1/2} I_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) I_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right).$$

9. $\mathrm{D}^n\left[z^{\pm\nu/4}I_\nu(a\sqrt[4]{z})\right]$

$$= \left(\frac{a}{4}\right)^n z^{(\pm\nu-3n)/4}\sum_{k=0}^{n-1}(-a)^{-k}\frac{\Gamma(n+k)}{k!\,\Gamma(n-k)}z^{-k/4}I_{\nu\mp n\pm k}(a\sqrt[4]{z}) \quad [n\geq 1].$$

10. $\mathrm{D}^n\left[z^{n-1}I_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{1}{z}\sum_{k=0}^{n}\binom{n}{k}\left(\mp\frac{\nu}{2}\right)_{n-k}\left(-\frac{a}{2\sqrt{z}}\right)^k I_{\nu\pm k}\left(\frac{a}{\sqrt{z}}\right).$

11. $\mathrm{D}^n\left[z^{n\pm\nu/2-1}I_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(-\frac{a}{2}\right)^n z^{-(n\mp\nu)/2-1}I_{\nu\pm n}\left(\frac{a}{\sqrt{z}}\right).$

12. $\mathrm{D}^n\left[z^{(2n-5)/4}I_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}z^{-n-1}\sinh\left(\frac{a}{\sqrt{z}}\right).$

13. $\mathrm{D}^n\left[z^{(2n-5)/4}I_{-n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2}z^{-n-1}\cosh\left(\frac{a}{\sqrt{z}}\right).$

14. $\mathrm{D}^n\left[z^{(6n-1)/4}I_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right]$

$$= (-1)^n\sqrt{\pi}\left(\frac{a}{2}\right)^{n+1/2}z^{-1/2}I_{n+1/2}\left(\frac{a}{2\sqrt{z}}\right) I_{-n-1/2}\left(\frac{a}{2\sqrt{z}}\right).$$

15. $\mathrm{D}^n[z^{n-1/2}e^{\pm az}I_{n-1/2}(az)] = \frac{(2a)^{n-1/2}}{\sqrt{\pi}}z^{n-1}e^{\pm 2az} \quad [n\geq 1].$

16. $\mathrm{D}^n\left[z^{n-1/2}e^{-az}I_{n+1/2}(az)\right] = \frac{(2a)^{-n-1/2}}{\sqrt{\pi}}z^{-n-1}\gamma(2n+1,\,2az).$

17. $\mathrm{D}^n\left[z^{n-1/2}e^{az}I_\nu(az)\right] = (2a)^\nu z^{\nu-1/2}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(2\nu+1)}\,{}_1F_1\left(\begin{matrix}\nu+n+\frac{1}{2}\\ 2\nu+1;\ 2az\end{matrix}\right).$

18. $\mathrm{D}^n\left[z^{-1/2}e^{\pm a/z}I_{n-1/2}\left(\frac{a}{z}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}}(2a)^{n-1/2}z^{-2n}e^{\pm 2a/z} \quad [n\geq 1].$

19. $\mathrm{D}^n\left[z^{-1/2}e^{-a/z}I_{n+1/2}\left(\frac{a}{z}\right)\right] = (-1)^n\frac{(2a)^{-n-1/2}}{\sqrt{\pi}}\gamma\left(2n+1,\,\frac{2a}{z}\right).$

20. $\mathrm{D}^n\left[\frac{1}{\sqrt{z}}e^{a/z}I_\nu\left(\frac{a}{z}\right)\right]$

$$= (-1)^n(2a)^\nu z^{-\nu-n-1/2}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(2\nu+1)}\,{}_1F_1\left(\begin{matrix}\nu+n+\frac{1}{2}\\ 2\nu+1;\ \frac{2a}{z}\end{matrix}\right).$$

21. $\mathrm{D}^n[z^{n-1/2}\sinh(az)\,I_{n-1/2}(az)] = \dfrac{(2a)^{n-1/2}}{\sqrt{\pi}}\,z^{n-1}\sinh(2az).$

22. $\mathrm{D}^n[z^{n-1/2}\cosh(az)\,I_{n-1/2}(az)] = \dfrac{(2a)^{n-1/2}}{\sqrt{\pi}}\,z^{n-1}\cosh(2az) \qquad [n \ge 1].$

23. $\mathrm{D}^n\left[z^{-1/2}\sinh\frac{a}{z}\,I_{n-1/2}\left(\frac{a}{z}\right)\right] = \dfrac{(-1)^n}{\sqrt{\pi}}(2a)^{n-1/2}z^{-2n}\sinh\dfrac{2a}{z}.$

24. $\mathrm{D}^n\left[z^{-1/2}\cosh\frac{a}{z}\,I_{n-1/2}\left(\frac{a}{z}\right)\right] = \dfrac{(-1)^n}{\sqrt{\pi}}(2a)^{n-1/2}z^{-2n}\cosh\dfrac{2a}{z} \qquad [n \ge 1].$

25. $\mathrm{D}^n[I_\nu^2(a\sqrt{z})] = \left(\dfrac{a}{2\sqrt{z}}\right)^n \displaystyle\sum_{k=0}^{n}\binom{n}{k} I_{\nu+k}(a\sqrt{z})\,I_{\nu-n+k}(a\sqrt{z}).$

26. $\mathrm{D}^n[z^{n-1/2}I_{n-1/2}^2(a\sqrt{z})] = \dfrac{a^{n-1/2}}{\sqrt{\pi}}\,z^{(2n-3)/4}I_{n-1/2}(2a\sqrt{z}) \qquad [n \ge 1].$

27. $\mathrm{D}^n[z^{n-1/2}I_{1/2-n}^2(a\sqrt{z})] = \dfrac{a^{n-1/2}}{\sqrt{\pi}}\,z^{(2n-3)/4}I_{n-1/2}(2a\sqrt{z}) \qquad [n \ge 1].$

28. $\mathrm{D}^n[z^{n-1/2}I_{n-1/2}(a\sqrt{z})\,I_{1/2-n}(a\sqrt{z})]$

$$= \frac{a^{n-1/2}}{\sqrt{\pi}}\,z^{(2n-3)/4}I_{1/2-n}(2a\sqrt{z}) \qquad [n \ge 1].$$

29. $\mathrm{D}^n[z^n I_{n-1/2}(a\sqrt{z})\,I_{n+1/2}(a\sqrt{z})] = \dfrac{a^{n-1/2}}{\sqrt{\pi}}\,z^{(2n-1)/4}I_{n+1/2}(2a\sqrt{z}).$

30. $\mathrm{D}^n\left[z^{n-1}I_\nu^2\left(\frac{a}{\sqrt{z}}\right)\right]$

$$= \left(-\frac{a}{2}\right)^n z^{-n/2-1}\sum_{k=0}^{n}\binom{n}{k} I_{\nu+k}\left(\frac{a}{\sqrt{z}}\right) I_{\nu-n+k}\left(\frac{a}{\sqrt{z}}\right).$$

31. $\mathrm{D}^n\left[z^{-1/2}I_{n-1/2}^2\left(\frac{a}{\sqrt{z}}\right)\right] = \dfrac{(-1)^n}{\sqrt{\pi}}\,a^{n-1/2}z^{-(6n+1)/4}I_{n-1/2}\left(\dfrac{2a}{\sqrt{z}}\right)$

$[n \ge 1].$

32. $\mathrm{D}^n\left[z^{-1/2}I_{1/2-n}^2\left(\frac{a}{\sqrt{z}}\right)\right] = \dfrac{(-1)^n}{\sqrt{\pi}}\,a^{n-1/2}z^{-(6n+1)/4}I_{n-1/2}\left(\dfrac{2a}{\sqrt{z}}\right)$

$[n \ge 1].$

33. $\mathrm{D}^n\left[z^{-1/2}I_{n-1/2}\left(\frac{a}{\sqrt{z}}\right) I_{1/2-n}\left(\frac{a}{\sqrt{z}}\right)\right]$

$$= \frac{(-1)^n}{\sqrt{\pi}}\,a^{n-1/2}z^{-(6n+1)/4}I_{1/2-n}\left(\frac{2a}{\sqrt{z}}\right) \qquad [n \ge 1].$$

34. $\mathrm{D}^n\left[z^{-1}I_{n-1/2}\left(\frac{a}{\sqrt{z}}\right) I_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right]$

$$= \frac{(-1)^n}{\sqrt{\pi}}\,a^{n-1/2}z^{-(6n+3)/4}I_{n+1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

35. $\mathrm{D}^n[z^{2\nu+n}e^{-2az}\,\mathrm{D}^n[z^{-\nu}e^{az}I_\nu(az)]] = \left(\nu+\frac{1}{2}\right)_n(2a)^n\,z^\nu e^{-az}I_\nu(az).$

36. $$\mathrm{D}^n\left[z^{n-2\nu}e^{-2a/z}\,\mathrm{D}^n\left[z^{\nu+n-1}e^{a/z}I_\nu\left(\frac{a}{z}\right)\right]\right] = \left(\nu+\frac{1}{2}\right)_n(2a)^n\,z^{-\nu-n-1}e^{-a/z}I_\nu\left(\frac{a}{z}\right).$$

37. $\mathrm{D}^n[z^{n-2\nu}e^{2az}\,\mathrm{D}^n[z^{\nu}e^{-az}I_\nu(az)]] = \left(\frac{1}{2}-\nu\right)_n(-2a)^n\,z^{-\nu}e^{az}I_\nu(az).$

38. $$\mathrm{D}^n\left[z^{n+2\nu}e^{2a/z}\,\mathrm{D}^n\left[z^{n-\nu-1}e^{-a/z}I_\nu\left(\frac{a}{z}\right)\right]\right] = \left(\frac{1}{2}-\nu\right)_n(-2a)^n\,z^{\nu-n-1}e^{a/z}I_\nu\left(\frac{a}{z}\right).$$

39. $\mathrm{D}^n[z^{n-2\nu}e^{-2az}\,\mathrm{D}^n[z^{\nu}e^{az}I_\nu(az)]] = \left(\frac{1}{2}-\nu\right)_n(2a)^n\,z^{-\nu}e^{-az}I_\nu(az).$

40. $$\mathrm{D}^n\left[z^{n+2\nu}e^{-2a/z}\,\mathrm{D}^n\left[z^{n-\nu-1}e^{a/z}I_\nu\left(\frac{a}{z}\right)\right]\right] = \left(\frac{1}{2}-\nu\right)_n(2a)^n\,z^{\nu-n-1}e^{-a/z}I_\nu\left(\frac{a}{z}\right).$$

41. $\mathrm{D}^n[z^{n+2\nu}e^{2az}\,\mathrm{D}^n[z^{-\nu}e^{-az}I_\nu(az)]] = \left(\nu+\frac{1}{2}\right)_n(-2a)^n\,z^{\nu}e^{az}I_\nu(az).$

42. $$\mathrm{D}^n\left[z^{n-2\nu}e^{2a/z}\,\mathrm{D}^n\left[z^{n+\nu-1}e^{-a/z}I_\nu\left(\frac{a}{z}\right)\right]\right] = \left(\nu+\frac{1}{2}\right)_n(-2a)^n\,z^{-\nu-n-1}e^{a/z}I_\nu\left(\frac{a}{z}\right).$$

1.13.2. Derivatives with respect to the order

1. $$\left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=\pm n} = (-1)^{n+1}K_n(z) \pm \frac{n!}{2}\sum_{p=0}^{n-1}\frac{\left(-\frac{z}{2}\right)^{p-n}}{p!\,(n-p)}I_p(z).$$

2. $$\left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=\pm 1/2} = \sqrt{\frac{1}{2\pi z}}\,[e^z\,\mathrm{Ei}\,(-2z) \mp e^{-z}\,\mathrm{Ei}\,(2z)]$$ [[10], 7.8(14)].

3. $$= \sqrt{\frac{2}{\pi z}}\left[\begin{Bmatrix}\sinh z\\ \cosh z\end{Bmatrix}\mathrm{chi}\,(2z) - \begin{Bmatrix}\cosh z\\ \sinh z\end{Bmatrix}\mathrm{shi}\,(2z)\right].$$

4. $$\left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=n+1/2} = \frac{1}{2}\,\mathrm{Ei}\,(-2z)[I_{-n-1/2}(z)+I_{n+1/2}(z)]$$
$$-\frac{(-1)^n}{\pi}\,\mathrm{Ei}\,(2z)\,K_{n+1/2}(z) + (-1)^n\frac{n!}{2}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!\,(n-k)}\left(\frac{z}{2}\right)^{k-n}I_{k+1/2}(z)$$
$$-\frac{n!}{2}\sqrt{\frac{z}{\pi}}\sum_{k=1}^{n}\frac{\left(-\frac{2}{z}\right)^k}{(n-k)!\,k}\sum_{p=0}^{k-1}\frac{(-z)^p}{p!}$$
$$\times\{(-1)^k[I_{n-k+1/2}(z)+I_{k-n-1/2}(z)]K_{p-1/2}(2z) - (-1)^{n-p}K_{n-k+1/2}(z)[I_{p-1/2}(2z)+I_{1/2-p}(2z)]\}.$$

5. $$= \frac{1}{2}[\operatorname{chi}(2z) - \operatorname{shi}(2z)][I_{-n-1/2}(z) + I_{n+1/2}(z)] - \frac{(-1)^n}{\pi} K_{n+1/2}(z)$$
$$\times [\operatorname{chi}(2z) + \operatorname{shi}(2z)] + (-1)^n \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,(n-k)} \left(\frac{z}{2}\right)^{k-n} I_{k+1/2}(z)$$
$$- \frac{n!}{2}\sqrt{\frac{z}{\pi}} \sum_{k=1}^{n} \frac{\left(-\frac{2}{z}\right)^k}{(n-k)!\,k} \sum_{p=0}^{k-1} \frac{(-z)^p}{p!}$$
$$\times \{(-1)^k [I_{n-k+1/2}(z) + I_{k-n-1/2}(z)] K_{p-1/2}(2z) - (-1)^{n-p} K_{n-k+1/2}(z)[I_{p-1/2}(2z) + I_{1/2-p}(2z)]\}.$$

6. $$\left.\frac{\partial I_\nu(z)}{\partial \nu}\right|_{\nu=1/2-n} = \frac{1}{2}[\operatorname{chi}(2z) - \operatorname{shi}(2z)][I_{-n-1/2}(z) + I_{n+1/2}(z)]$$
$$- \frac{(-1)^n}{\pi}[\operatorname{chi}(2z) + \operatorname{shi}(2z)] K_{n-1/2}(z)$$
$$- \frac{n!}{2} \sum_{k=0}^{n-1} \frac{\left(-\frac{z}{2}\right)^{k-n}}{k!\,(n-k)} I_{1/2-k}(z) + \frac{n!}{2}\sqrt{\frac{z}{\pi}} \sum_{k=1}^{n} \frac{\left(\frac{2}{z}\right)^k}{(n-k)!\,k} \sum_{p=0}^{k-1} \frac{z^p}{p!}$$
$$\times \{(-1)^{k-1} [I_{n-k-1/2}(z) + I_{k-n+1/2}(z)] K_{p-1/2}(2z) + (-1)^{n-p} K_{n-k-1/2}(z)[I_{p-1/2}(2z) + I_{1/2-p}(2z)]\}.$$

7. $$\left.\frac{\partial I_\nu(z)}{\partial \nu}\right|_{\nu=\pm n \pm 1/2}$$
$$= 2^{n-1/2}\sqrt{\pi}\, z^{-n-1/2} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k)! \left(\frac{z}{2}\right)^{2k}}{k!\,(n-2k)!\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k-n+\frac{1}{2}\right)}$$
$$\times \left[\mp 2\left(\psi\left(k+\frac{1}{2}\right) - \psi\left(k-n+\frac{1}{2}\right)\right)\begin{Bmatrix}\sinh z\\ \cosh z\end{Bmatrix} + e^z \operatorname{Ei}(-2z) \mp e^{-z}\operatorname{Ei}(2z)\right]$$
$$+ 2^{n-3/2}\sqrt{\pi}\, z^{-n+1/2} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)! \left(\frac{z}{2}\right)^{2k}}{k!\,(n-2k-1)!\,\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k-n+\frac{1}{2}\right)}$$
$$\times \left[\pm 2\left(\psi\left(k+\frac{3}{2}\right) - \psi\left(k-n+\frac{1}{2}\right)\right)\begin{Bmatrix}\cosh z\\ \sinh z\end{Bmatrix} - e^z \operatorname{Ei}(-2z) \mp e^{-z}\operatorname{Ei}(2z)\right].$$

1.14. The Macdonald Function $K_\nu(z)$

1.14.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[K_\nu(az)] = \left(-\frac{a}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} K_{\nu\pm 2k\mp n}(az).$$

2. $$= n!\,(-z)^{-n} \sum_{k=0}^{n} \frac{(az)^k}{(n-k)!} (\nu)_{n-k} \sum_{p=0}^{[k/2]} \frac{(-2az)^{-j}}{p!\,(k-2p)!} K_{\nu-k+p}(az).$$

3. $\mathrm{D}^n\left[z^{n-1}K_\nu\left(\frac{a}{z}\right)\right] = \left(\frac{a}{2}\right)^n z^{-n-1}\sum_{k=0}^{n}\binom{n}{k}K_{\nu\pm 2k\mp n}\left(\frac{a}{z}\right).$

4. $\mathrm{D}^n[z^{\pm\nu/2}K_\nu(a\sqrt{z})] = \left(-\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}K_{\nu\mp n}(a\sqrt{z}).$

5. $\mathrm{D}^n[z^{(2n+1)/4}K_{n+1/2}(a\sqrt{z})] = (-1)^n\frac{\sqrt{\pi}}{2^{n+1/2}}a^{n-1/2}e^{-a\sqrt{z}}.$

6. $\mathrm{D}^n[z^{-(2n+3)/4}K_{n+1/2}(a\sqrt{z})] = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n+1/2} z^{-n-1/2}K^2_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right).$

7. $\mathrm{D}^n[z^{\pm\nu/4}K_\nu(a\sqrt[4]{z})]$

$$= \left(-\frac{a}{4}\right)^n z^{(\pm\nu-3n)/4}\sum_{k=0}^{n-1}a^{-k}\frac{\Gamma(n+k)}{k!\,\Gamma(n-k)}z^{-k/4}K_{\nu\mp n\pm k}(a\sqrt[4]{z}) \quad [n\ge 1].$$

8. $\mathrm{D}^n\left[z^{n\pm\nu/2-1}K_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(\frac{a}{2}\right)^n z^{-(n\mp\nu)/2-1}K_{\nu\pm n}\left(\frac{a}{\sqrt{z}}\right).$

9. $\mathrm{D}^n\left[z^{(6n-1)/4}K_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n+1/2} z^{-1/2}K^2_{n+1/2}\left(\frac{a}{2\sqrt{z}}\right).$

10. $\mathrm{D}^n\left[z^{n-1/2}e^{az}K_{n+1/2}(az)\right] = (-1)^n(2n)!\sqrt{\pi}\,(2a)^{-n-1/2}z^{-n-1}.$

11. $\mathrm{D}^n\left[z^{m+1/2}e^{az}K_{m+1/2}(az)\right]$

$$= (-1)^{m+n}m!\sqrt{\pi}\,(2a)^{n-m-1/2}L_{m-n}^{n-2m-1}(2az) \quad [m\ge n].$$

12. $\mathrm{D}^n\left[z^{n-1/2}e^{az}K_{m+1/2}(az)\right]$

$$= (-1)^m(m+n)!\sqrt{\pi}\,(2a)^{-m-1/2}z^{-m-1}L_{m-n}^{-2m-1}(2az) \quad [m\ge n].$$

13. $\mathrm{D}^n\left[z^{-m-1/2}e^{az}K_{m+1/2}(az)\right]$

$$= (-1)^{m+n}(m+n)!\sqrt{\pi}\,(2a)^{-m-1/2}z^{-2m-n-1}L_m^{-2m-n-1}(2az).$$

14. $\mathrm{D}^n\left[z^{m+1/2}e^{-az}K_{m+1/2}(az)\right]$

$$= (-1)^{m+n}m!\sqrt{\pi}\,(2a)^{n-m-1/2}e^{-2az}L_m^{n-2m-1}(2az).$$

15. $\mathrm{D}^n\left[z^{n-1/2}e^{-az}K_{m+1/2}(az)\right]$

$$= (-1)^m(m+n)!\sqrt{\pi}\,(2a)^{-m-1/2}z^{-m-1}e^{-2az}L_{m+n}^{-2m-1}(2az).$$

16. $\mathrm{D}^n\left[z^{-m-1/2}e^{-az}K_{m+1/2}(az)\right]$

$$= (-1)^m(m+n)!\sqrt{\pi}\,(2a)^{-m-1/2}z^{-2m-n-1}e^{-2az}L_{m+n}^{-2m-n-1}(2az).$$

17. $\mathrm{D}^n\left[z^{-1/2}e^{a/z}K_{m+1/2}\left(\frac{a}{z}\right)\right]$

$$= (-1)^{m+n}(m+n)!\sqrt{\pi}\,(2a)^{-m-1/2}z^{m-n}L_{m-n}^{-2m-1}\left(\frac{2a}{z}\right) \quad [m\ge n].$$

18. $$\mathrm{D}^n\left[z^{m+n-1/2}e^{a/z}K_{m+1/2}\left(\frac{a}{z}\right)\right] = (-1)^m(m+n)!\,\sqrt{\pi}\,(2a)^{-m-1/2}z^{2m}L_m^{-2m-n-1}\left(\frac{2a}{z}\right).$$

19. $$\mathrm{D}^n\left[z^{n-m-3/2}e^{a/z}K_{m+1/2}\left(\frac{a}{z}\right)\right] = (-1)^m m!\,\sqrt{\pi}\,(2a)^{n-m-1/2}z^{-n-1}L_{m-n}^{n-2m-1}\left(\frac{2a}{z}\right) \quad [m \ge n].$$

20. $$\mathrm{D}^n\left[z^{-1/2}e^{-a/z}K_{m+1/2}\left(\frac{a}{z}\right)\right] = (-1)^{m+n}(m+n)!\,\sqrt{\pi}\,(2a)^{-m-1/2}z^{m-n}e^{-2a/z}L_{m+n}^{-2m-1}\left(\frac{2a}{z}\right).$$

21. $$\mathrm{D}^n\left[z^{m+n-1/2}e^{-a/z}K_{m+1/2}\left(\frac{a}{z}\right)\right] = (-1)^{m+n}(m+n)!\,\sqrt{\pi}\,(2a)^{-m-1/2}z^{2m}e^{-2a/z}L_{m+n}^{-2m-n-1}\left(\frac{2a}{z}\right).$$

22. $$\mathrm{D}^n\left[z^{n-m-3/2}e^{-a/z}K_{m+1/2}\left(\frac{a}{z}\right)\right] = (-1)^m m!\,\sqrt{\pi}\,(2a)^{n-m-1/2}z^{-n-1}e^{-2a/z}L_m^{n-2m-1}\left(\frac{2a}{z}\right).$$

23. $$\mathrm{D}^n[K_\nu^2(a\sqrt{z})] = \left(-\frac{a}{2\sqrt{z}}\right)^n\sum_{k=0}^{n}\binom{n}{k}K_{\nu+k}(a\sqrt{z})\,K_{\nu-n+k}(a\sqrt{z}).$$

24. $$\mathrm{D}^n[z^{n-1/2}K_{n-1/2}^2(a\sqrt{z})] = (-1)^n\sqrt{\pi}\,a^{n-1/2}z^{(2n-3)/4}K_{n-1/2}(2a\sqrt{z}).$$

25. $$\mathrm{D}^n[z^nK_{n-1/2}(a\sqrt{z})\,K_{n+1/2}(a\sqrt{z})] = (-1)^n\sqrt{\pi}\,a^{n-1/2}z^{(2n-1)/4}K_{n+1/2}(2a\sqrt{z}).$$

26. $$\mathrm{D}^n\left[z^{n-1}K_\nu^2\left(\frac{a}{\sqrt{z}}\right)\right] = \left(\frac{a}{2}\right)^n z^{-n/2-1}\sum_{k=0}^{n}\binom{n}{k}K_{\nu+k}\left(\frac{a}{\sqrt{z}}\right)K_{\nu-n+k}\left(\frac{a}{\sqrt{z}}\right).$$

27. $$\mathrm{D}^n\left[z^{-1/2}K_{n-1/2}^2\left(\frac{a}{\sqrt{z}}\right)\right] = \sqrt{\pi}\,a^{n-1/2}z^{-(6n+1)/4}K_{n-1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

28. $$\mathrm{D}^n\left[z^{-1}K_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)K_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \sqrt{\pi}\,a^{n-1/2}z^{-(6n+3)/4}K_{n+1/2}\left(\frac{2a}{\sqrt{z}}\right).$$

29. $$\mathrm{D}^n[z^{n-1/2}I_{n-1/2}(a\sqrt{z})\,K_{n-1/2}(a\sqrt{z})] = \frac{a^{n-1/2}}{\sqrt{\pi}}z^{(2n-3)/4}K_{n-1/2}(2a\sqrt{z}) \quad [n \ge 1].$$

30. $$\mathrm{D}^n\left[z^{-1/2} I_{n-1/2}\left(\frac{a}{\sqrt{z}}\right) K_{n-1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{(-1)^n}{\sqrt{\pi}} a^{n-1/2} z^{-(6n+1)/4} K_{n-1/2}\left(\frac{2a}{\sqrt{z}}\right) \quad [n \geq 1].$$

31. $$\mathrm{D}^n[e^{-2az}\,\mathrm{D}^n[z^{n-1/2} e^{az} K_\nu(az)] = (-1)^n\left(\frac{1}{2}-\nu\right)_n \left(\frac{1}{2}+\nu\right)_n z^{-n-1/2} e^{-az} K_\nu(az).$$

32. $$\mathrm{D}^n\left[z^{2n} e^{-2a/z}\,\mathrm{D}^n\left[z^{-1/2} e^{a/z} K_\nu\left(\frac{a}{z}\right)\right]\right] = (-1)^n\left(\frac{1}{2}-\nu\right)_n \left(\frac{1}{2}+\nu\right)_n z^{-1/2} e^{-a/z} K_\nu\left(\frac{a}{z}\right).$$

33. $$\mathrm{D}^n[e^{2az}\,\mathrm{D}^n[z^{n-1/2} e^{-az} K_\nu(az)] = (-1)^n\left(\frac{1}{2}-\nu\right)_n \left(\frac{1}{2}+\nu\right)_n z^{-n-1/2} e^{az} K_\nu(az).$$

34. $$\mathrm{D}^n\left[z^{2n} e^{2a/z}\,\mathrm{D}^n\left[z^{-1/2} e^{-a/z} K_\nu\left(\frac{a}{z}\right)\right]\right] = (-1)^n\left(\frac{1}{2}-\nu\right)_n \left(\frac{1}{2}+\nu\right)_n z^{-1/2} e^{a/z} K_\nu\left(\frac{a}{z}\right).$$

35. $$\mathrm{D}^n[z^{n-2m-1} e^{-2az}\,\mathrm{D}^n[z^{m+1/2} e^{az} K_{m+1/2}(az)] = (-m)_n (2a)^n z^{-m-1/2} e^{-az} K_{m+1/2}(az).$$

36. $$\mathrm{D}^n\left[z^{2m+n+1} e^{-2a/z}\,\mathrm{D}^n\left[z^{n-m-3/2} e^{a/z} K_{m+1/2}\left(\frac{a}{z}\right)\right]\right] = (-m)_n (2a)^n z^{m-n-1/2} e^{-a/z} K_{m+1/2}\left(\frac{a}{z}\right).$$

37. $$\mathrm{D}^n[z^{n+2m+1} e^{2az}\,\mathrm{D}^n[z^{-m-1/2} e^{-az} K_{m+1/2}(az)] = \frac{(m+n)!}{m!}(-2a)^n z^{m+1/2} e^{az} K_{m+1/2}(az).$$

38. $$\mathrm{D}^n\left[z^{n-2m-1} e^{2a/z}\,\mathrm{D}^n\left[z^{n+m-1/2} e^{-a/z} K_{m+1/2}\left(\frac{a}{z}\right)\right]\right] = \frac{(m+n)!}{m!}(-2a)^n z^{-m-n-3/2} e^{a/z} K_{m+1/2}\left(\frac{a}{z}\right).$$

39. $$\mathrm{D}^n[z^{n+2m+1} e^{-2az}\,\mathrm{D}^n[z^{-m-1/2} e^{az} K_{m+1/2}(az)] = \frac{(m+n)!}{m!}(2a)^n z^{m+1/2} e^{-az} K_{m+1/2}(az).$$

40. $$\mathrm{D}^n\left[z^{n-2m-1} e^{-2a/z}\,\mathrm{D}^n\left[z^{n+m-1/2} e^{a/z} K_{m+1/2}\left(\frac{a}{z}\right)\right]\right] = \frac{(m+n)!}{m!}(2a)^n z^{-m-n-3/2} e^{-a/z} K_{m+1/2}\left(\frac{a}{z}\right).$$

41. $$\mathrm{D}^n[e^{-2az}\,\mathrm{D}^n[z^{n-1/2} e^{az} K_{m+1/2}(az)] = \frac{(m+n)!}{(m-n)!} z^{-n-1/2} e^{-az} K_{m+1/2}(az).$$

42. $\mathrm{D}^n\left[z^{2n}e^{-2a/z}\,\mathrm{D}^n\left[z^{-1/2}e^{a/z}K_{m+1/2}\left(\frac{a}{z}\right)\right]\right]$
$$= \frac{(m+n)!}{(m-n)!}z^{-1/2}e^{-a/z}K_{m+1/2}\left(\frac{a}{z}\right).$$

1.14.2. Derivatives with respect to the order

1. $\left.\dfrac{\partial K_\nu(z)}{\partial\nu}\right|_{\nu=0} = 0.$

2. $\left.\dfrac{\partial K_\nu(z)}{\partial\nu}\right|_{\nu=\pm n} = \pm\dfrac{n!}{2}\displaystyle\sum_{p=0}^{n-1}\dfrac{\left(\frac{z}{2}\right)^{p-n}}{p!(n-p)}K_p(z).$

3. $\left.\dfrac{\partial K_\nu(z)}{\partial\nu}\right|_{\nu=\pm 1/2} = \mp\sqrt{\dfrac{\pi}{2z}}\,e^z\,\mathrm{Ei}\,(-2z)$ [[10], 7.8(15)].

4. $\left.\dfrac{\partial K_\nu(z)}{\partial\nu}\right|_{\nu=1/2} = \sqrt{\dfrac{\pi}{2z}}\,e^z\,[\mathrm{shi}\,(2z)-\mathrm{chi}\,(2z)].$

5. $\left.\dfrac{\partial K_\nu(z)}{\partial\nu}\right|_{\nu=n-1/2} = (-1)^n\dfrac{\pi}{2}\,\mathrm{Ei}\,(-2z)[I_{n-1/2}(z)+I_{1/2-n}(z)]$
$$+\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}K_{k-1/2}(z) - (-1)^n\frac{n!}{2}\sqrt{\pi}\sum_{k=1}^{n}\frac{\left(-\frac{2}{z}\right)^k}{(n-k)!\,k}[I_{n-k-1/2}(z)$$
$$+I_{k-n+1/2}(z)]\sum_{p=0}^{k-1}\frac{z^{p+1/2}}{p!}K_{p-1/2}(2z).$$

6. $= (-1)^n\dfrac{\pi}{2}\,[\mathrm{chi}\,(2z)-\mathrm{shi}\,(2z)][I_{n-1/2}(z)+I_{1/2-n}(z)]$
$$+\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}K_{k-1/2}(z) - (-1)^n\frac{n!}{2}\sqrt{\pi}\sum_{k=1}^{n}\frac{\left(-\frac{2}{z}\right)^k}{(n-k)!\,k}[I_{n-k-1/2}(z)$$
$$+I_{k-n+1/2}(z)]\sum_{p=0}^{k-1}\frac{z^{p+1/2}}{p!}K_{p-1/2}(2z).$$

7. $\left.\dfrac{\partial K_\nu(z)}{\partial\nu}\right|_{\nu=n+1/2}$
$$= (-1)^{n+1}2^{n-1/2}\pi^{3/2}z^{-n-1/2}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k)!\left(\frac{z}{2}\right)^{2k}}{k!\,(n-2k)!\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k-n+\frac{1}{2}\right)}$$
$$\times\left[e^{-z}\left(\psi\left(k+\frac{1}{2}\right)-\psi\left(k-n+\frac{1}{2}\right)\right)+e^z\,\mathrm{Ei}\,(-2z)\right]$$
$$+(-1)^{n+1}2^{n-3/2}\pi^{3/2}z^{-n+1/2}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!\left(\frac{z}{2}\right)^{2k}}{k!\,(n-2k-1)!\,\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k-n+\frac{1}{2}\right)}$$
$$\times\left[e^{-z}\left(\psi\left(k+\frac{3}{2}\right)-\psi\left(k-n+\frac{1}{2}\right)\right)-e^z\,\mathrm{Ei}\,(-2z)\right].$$

1.15. The Struve Functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

1.15.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[\mathbf{H}_\nu(az)]$

$$= n!\left(-\frac{2}{z}\right)^n \sum_{k=0}^{[n/2]} \frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(\frac{\nu}{2}\right)_{n-k-p} \left(-\frac{az}{2}\right)^p \mathbf{H}_{\nu-p}(az).$$

2. $$= n!\left(-\frac{2}{z}\right)^n \sum_{k=0}^{[n/2]} \frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(-\frac{\nu}{2}\right)_{n-k-p}$$
$$\times \left[\left(\frac{az}{2}\right)^p \mathbf{H}_{\nu+p}(az) - \frac{1}{\pi}\left(\frac{az}{2}\right)^{\nu+2p-1} \sum_{r=0}^{p-1} \frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\nu+p-r+\frac{1}{2}\right)} \left(\frac{2}{az}\right)^{2r}\right].$$

3. $\mathrm{D}^n[z^{\nu/2}\,\mathbf{H}_\nu(a\sqrt{z})] = \left(\frac{a}{2}\right)^n z^{(\nu-n)/2}\,\mathbf{H}_{\nu-n}(a\sqrt{z}).$

4. $$\mathrm{D}^n[z^{-\nu/2}\,\mathbf{H}_\nu(a\sqrt{z})] = \left(-\frac{a}{2}\right)^n z^{-(n+\nu)/2}\,\mathbf{H}_{\nu+n}(a\sqrt{z})$$
$$- \frac{(-1)^n}{\pi}\left(\frac{a}{2}\right)^{\nu+2n-1} z^{-1/2} \sum_{k=0}^{n-1} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\nu+n-k+\frac{1}{2}\right)} \left(\frac{4}{a^2 z}\right)^k.$$

5. $\mathrm{D}^n[z^{(2n+1)/4}\,\mathbf{H}_{n+1/2}(a\sqrt{z})] = \frac{2}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \sin^2 \frac{a\sqrt{z}}{2}.$

6. $\mathrm{D}^n\left[z^{n-\nu/2-1}\,\mathbf{H}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(-\frac{a}{2}\right)^n z^{-(n+\nu)/2-1}\,\mathbf{H}_{\nu-n}\left(\frac{a}{\sqrt{z}}\right).$

7. $\mathrm{D}^n\left[z^{(2n-5)/4}\,\mathbf{H}_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^n \frac{2}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} z^{-n-1} \sin^2 \frac{a}{2\sqrt{z}}.$

8. $$\mathrm{D}^n\left[z^{n+\nu/2-1}\,\mathbf{H}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(\frac{a}{2}\right)^n z^{(\nu-n)/2-1}\,\mathbf{H}_{\nu+n}\left(\frac{a}{\sqrt{z}}\right)$$
$$- \frac{1}{\pi}\left(\frac{a}{2}\right)^{\nu+2n-1} z^{-n-1/2} \sum_{k=0}^{n-1} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\nu+n-k+\frac{1}{2}\right)} \left(\frac{4z}{a^2}\right)^k.$$

9. $\mathrm{D}^n[\mathbf{L}_\nu(az)]$

$$= n!\left(-\frac{2}{z}\right)^n \sum_{k=0}^{[n/2]} \frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(\frac{\nu}{2}\right)_{n-k-p} \left(-\frac{az}{2}\right)^p \mathbf{L}_{\nu-p}(az).$$

10. $$= n!\left(-\frac{2}{z}\right)^n \sum_{k=0}^{[n/2]} \frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!} \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} \left(-\frac{\nu}{2}\right)_{n-k-p}$$
$$\times \left[\left(\frac{az}{2}\right)^p \mathbf{L}_{\nu+p}(az) + \frac{1}{\pi}\left(\frac{az}{2}\right)^{\nu+2p-1} \sum_{r=0}^{p-1} \frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\nu+p-r+\frac{1}{2}\right)} \left(-\frac{4}{a^2z^2}\right)^r\right].$$

11. $$\mathrm{D}^n[z^{\nu/2}\,\mathbf{L}_\nu(a\sqrt{z})] = \left(\frac{a}{2}\right)^n z^{(\nu-n)/2}\,\mathbf{L}_{\nu-n}(a\sqrt{z}).$$

12. $$\mathrm{D}^n[z^{-\nu/2}\,\mathbf{L}_\nu(a\sqrt{z})] = \left(\frac{a}{2}\right)^n z^{-(n+\nu)/2}\,\mathbf{L}_{\nu+n}(a\sqrt{z})$$
$$+ \frac{1}{\pi}\left(\frac{a}{2}\right)^{\nu+2n-1} z^{-1/2} \sum_{k=0}^{n-1} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\nu+n-k+\frac{1}{2}\right)} \left(-\frac{4}{a^2 z}\right)^k.$$

13. $$\mathrm{D}^n[z^{(2n+1)/4}\,\mathbf{L}_{n+1/2}(a\sqrt{z})] = \frac{2}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \sinh^2\frac{a\sqrt{z}}{2}.$$

14. $$\mathrm{D}^n\left[z^{n-\nu/2-1}\,\mathbf{L}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(-\frac{a}{2}\right)^n z^{-(n+\nu)/2-1}\,\mathbf{L}_{\nu-n}\left(\frac{a}{\sqrt{z}}\right).$$

15. $$\mathrm{D}^n\left[z^{(2n-5)/4}\,\mathbf{L}_{n+1/2}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^n \frac{2}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} z^{-n-1} \sinh^2\frac{a}{2\sqrt{z}}.$$

16. $$\mathrm{D}^n\left[z^{n+\nu/2-1}\,\mathbf{L}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(-\frac{a}{2}\right)^n z^{(\nu-n)/2-1}\,\mathbf{L}_{\nu+n}\left(\frac{a}{\sqrt{z}}\right)$$
$$+ \frac{(-1)^n}{\pi}\left(\frac{a}{2}\right)^{\nu+2n-1} z^{-n-1/2} \sum_{k=0}^{n-1} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\nu+n-k+\frac{1}{2}\right)} \left(-\frac{4z}{a^2}\right)^k.$$

1.15.2. Derivatives with respect to the order

1. $$\left.\frac{\partial\,\mathbf{H}_\nu(z)}{\partial\nu}\right|_{\nu=0} = \frac{1}{2\pi} G_{24}^{32}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{1}{2},\ \frac{1}{2},\ 0,\ 0\end{matrix}\right) - \frac{\pi}{2} J_0(z) \qquad [\operatorname{Re} z \ge 0].$$

2. $$\left.\frac{\partial\,\mathbf{H}_\nu(z)}{\partial\nu}\right|_{\nu=n} = -\frac{\pi}{2} J_n(z) + \frac{2^{n-1}z^{-n}}{\pi} G_{24}^{32}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{1}{2},\ \frac{1}{2},\ n,\ 0\end{matrix}\right)$$
$$+ \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{n-k}} \left(\frac{z}{2}\right)^{n-2k-1} \left[\log\frac{z}{2} - \psi\left(n-k+\frac{1}{2}\right)\right]$$
$$+ \frac{n!}{2}\sum_{k=0}^{n-1} (-1)^k \frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\,\mathbf{H}_{-k}(z) \qquad [\operatorname{Re} z \ge 0].$$

3. $$\left.\frac{\partial\,\mathbf{H}_v(z)}{\partial\nu}\right|_{\nu=-n} = (-1)^{n+1}\frac{\pi}{2}J_n(z) + (-1)^n\frac{2^{n-1}z^{-n}}{\pi}G_{24}^{32}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2},\,\frac{1}{2}\\ \frac{1}{2},\,\frac{1}{2},\,n,\,0\end{matrix}\right) - \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(-\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\,\mathbf{H}_{-k}(z)$$

[Re $z \ge 0$].

4. $$\left.\frac{\partial\,\mathbf{H}_\nu(z)}{\partial\nu}\right|_{\nu=1/2} = \sqrt{\frac{2}{\pi z}} \times \left\{\mathbf{C} + \ln\frac{z}{2} + \sin z[\mathrm{Si}\,(2z) - 2\,\mathrm{Si}\,(z)] + \cos z[\mathrm{ci}\,(2z) - 2\,\mathrm{ci}\,(z)]\right\}$$

[[10], 7.9(22)].

5. $$\left.\frac{\partial\,\mathbf{H}_\nu(z)}{\partial\nu}\right|_{\nu=-1/2} = \sqrt{\frac{2}{\pi z}} \times \{\cos z[\mathrm{Si}\,(2z) - 2\,\mathrm{Si}\,(z)] - \sin z[\mathrm{ci}\,(2z) - 2\,\mathrm{ci}\,(z)]\}$$

[[10], 7.9(23)].

6. $$\left.\frac{\partial\,\mathbf{H}_v(z)}{\partial\nu}\right|_{\nu=n+1/2} = [\mathrm{Si}\,(2z) - 2\,\mathrm{Si}\,(z)]\,J_{n+1/2}(z) + (-1)^n[\mathrm{ci}\,(2z) - 2\,\mathrm{ci}\,(z)]\,J_{-n-1/2}(z) + \ln\frac{z}{2}\,[\mathbf{H}_{n+1/2}(z) - Y_{n+1/2}(z)]$$
$$+ \frac{1}{2\sqrt{\pi}}\left(\frac{2}{z}\right)^{n+1/2}\left(\frac{1}{2}\right)_n\left[3\mathbf{C} + 2\ln 2 + \psi\left(\frac{1}{2} - n\right)\right]$$
$$- \frac{n!}{2}\left(\frac{2}{z}\right)^n\sum_{k=0}^{n-1}\frac{\left(-\frac{z}{2}\right)^k}{k!\,(n-k)}J_{-k-1/2}(z) - \frac{n!}{2\sqrt{\pi}}\left(\frac{2}{z}\right)^{n+1/2}\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k}{k!\,(n-k)}$$
$$- \frac{\left(\frac{z}{2}\right)^{n-1/2}}{\sqrt{\pi}}\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k}{(n-k)!}\left(\frac{2}{z}\right)^{2k}\psi(n-k+1)$$
$$- n!\sqrt{\pi}\left(\frac{2}{z}\right)^{1/2-n}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^k}{k!\,(n-k)}\sum_{p=0}^{n-k-1}\frac{\left(\frac{z}{2}\right)^p}{p!}$$
$$\times\left\{(-1)^{p+1}J_{k+1/2}(z)\left[J_{1/2-p}(z) - 2^{p-1/2}J_{1/2-p}(2z)\right] - (-1)^k J_{-k-1/2}(z)\left[J_{p-1/2}(z) - 2^{p-1/2}J_{p-1/2}(2z)\right]\right\}.$$

7. $$\left.\frac{\partial\,\mathbf{L}_\nu(z)}{\partial\nu}\right|_{\nu=0} = K_0(z) - \frac{1}{\pi^2 z}G_{24}^{42}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}1,\,1\\ \frac{1}{2},\,\frac{1}{2},\,1,\,1\end{matrix}\right)$$

[Re $z \ge 0$].

8. $$= -K_0(z) - \frac{2}{z}G_{46}^{42}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}1,\,1,\,\frac{1}{4},\,\frac{3}{4}\\ \frac{1}{2},\,\frac{1}{2},\,1,\,1,\,\frac{1}{4},\,\frac{3}{4}\end{matrix}\right)$$

[Re $z \ge 0$].

9. $$\left.\frac{\partial\,\mathbf{L}_\nu(z)}{\partial\nu}\right|_{\nu=0} = K_0(z) - \frac{1}{\pi^2 z}G_{24}^{42}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}1,\,1\\ \frac{1}{2},\,\frac{1}{2},\,1,\,1\end{matrix}\right).$$

10. $$= -K_0(z) - \frac{2}{z} G_{46}^{42}\left(\frac{z^2}{4}\,\middle|\, \begin{matrix} 1,\ 1,\ \frac{1}{4},\ \frac{3}{4} \\ \frac{1}{2},\ \frac{1}{2},\ 1,\ 1,\ \frac{1}{4},\ \frac{3}{4} \end{matrix}\right).$$

11. $$\left.\frac{\partial\, \mathbf{L}_\nu(z)}{\partial \nu}\right|_{\nu=n} = (-1)^n K_n(z) + \frac{(-2)^{n-1} z^{-n}}{\pi^2} G_{24}^{42}\left(\frac{z^2}{4}\,\middle|\, \begin{matrix} \frac{1}{2},\ \frac{1}{2} \\ 0,\ \frac{1}{2},\ \frac{1}{2},\ n \end{matrix}\right)$$
$$- \frac{1}{\pi} \sum_{k=0}^{n-1} (-1)^k \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{n-k}} \left(\frac{z}{2}\right)^{n-2k-1} \left[\log\frac{z}{2} - \psi\left(n-k+\frac{1}{2}\right)\right]$$
$$+ \frac{n!}{2} \sum_{k=0}^{n-1} \frac{\left(-\frac{z}{2}\right)^{k-n}}{k!(n-k)} \mathbf{L}_{-k}(z) \quad [\operatorname{Re} z \geq 0].$$

12. $$\left.\frac{\partial\, \mathbf{L}_\nu(z)}{\partial \nu}\right|_{\nu=-n} = (-1)^n K_n(z) + \frac{(-2)^{n-1} z^{-n}}{\pi^2} G_{24}^{42}\left(\frac{z^2}{4}\,\middle|\, \begin{matrix} \frac{1}{2},\ \frac{1}{2} \\ 0,\ \frac{1}{2},\ \frac{1}{2},\ n \end{matrix}\right)$$
$$- \frac{n!}{2} \sum_{k=0}^{n-1} \frac{\left(-\frac{z}{2}\right)^{k-n}}{k!\,(n-k)} \mathbf{L}_{-k}(z) \quad [\operatorname{Re} z \geq 0].$$

13. $$\left.\frac{\partial\, \mathbf{L}_\nu(z)}{\partial \nu}\right|_{\nu=\pm 1/2} = \sqrt{\frac{2^{\pm 1}}{\pi z}} \left\{-\frac{1 \pm 1}{2}\left(\mathbf{C} + \ln\frac{z}{2}\right) \mp e^{-z}[\operatorname{Ei}(2z) - 2\operatorname{Ei}(z)]\right.$$
$$\left. - e^z[\operatorname{Ei}(-2z) - 2\operatorname{Ei}(-z)]\right\} \quad [[10], 7.8(16)].$$

14. $$\left.\frac{\partial\, \mathbf{L}_\nu(z)}{\partial \nu}\right|_{\nu=1/2} = \sqrt{\frac{2}{\pi z}} \left\{\sinh z[\operatorname{shi}(2z) - 2\operatorname{shi}(z)]\right.$$
$$\left. - \cosh z[\operatorname{chi}(2z) - 2\operatorname{chi}(z)] - \ln\frac{z}{2} - \mathbf{C}\right\}.$$

15. $$\left.\frac{\partial\, \mathbf{L}_\nu(z)}{\partial \nu}\right|_{\nu=-1/2} = \sqrt{\frac{2}{\pi z}} \{\cosh z[\operatorname{shi}(2z) - 2\operatorname{shi}(z)]$$
$$- \sinh z[\operatorname{chi}(2z) - 2\operatorname{chi}(z)]\}.$$

16. $$\left.\frac{\partial\, \mathbf{L}_\nu(z)}{\partial \nu}\right|_{\nu=n+1/2} = [\operatorname{shi}(2z) - 2\operatorname{shi}(z)]\, I_{n+1/2}(z)$$
$$- [\operatorname{chi}(2z) - 2\operatorname{chi}(z)]\, I_{-n-1/2}(z) + \ln\frac{z}{2}\, [\mathbf{L}_{n+1/2}(z) - I_{-n-1/2}(z)]$$
$$+ \frac{(-1)^{n+1}}{2\pi} \left(\frac{2}{z}\right)^{n+1/2} \Gamma\left(n+\frac{1}{2}\right) \left[3\mathbf{C} + 2\ln 2 + \psi\left(\frac{1}{2} - n\right)\right]$$
$$+ \frac{n!}{2} \left(-\frac{2}{z}\right)^n \sum_{k=0}^{n-1} \frac{\left(-\frac{z}{2}\right)^k}{k!\,(n-k)} I_{-k-1/2}(z)$$
$$+ (-1)^n \frac{n!}{2\sqrt{\pi}} \left(\frac{2}{z}\right)^{n+1/2} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k}{k!\,(n-k)}$$

$$+\frac{\left(\frac{z}{2}\right)^{n-1/2}}{\sqrt{\pi}}\sum_{k=0}^{n-1}(-1)^k\frac{\left(\frac{1}{2}\right)_k}{(n-k)!}\left(\frac{2}{z}\right)^{2k}\psi(n-k+1)$$

$$+(-1)^n n!\sqrt{\pi}\left(\frac{z}{2}\right)^{1/2-n}\sum_{k=0}^{n-1}\frac{\left(-\frac{z}{2}\right)^k}{k!\,(n-k)}\sum_{p=0}^{n-k-1}\frac{\left(-\frac{z}{2}\right)^p}{p!}$$

$$\times\left\{I_{k+1/2}(z)\left[I_{1/2-p}(z)-2^{p-1/2}I_{1/2-p}(2z)\right]\right.$$

$$\left.-I_{-k-1/2}(z)\left[I_{p-1/2}(z)-2^{p-1/2}I_{p-1/2}(2z)\right]\right\}.$$

17. $$\left.\frac{\partial\,\mathbf{L}_\nu(z)}{\partial\nu}\right|_{\nu=-n-1/2}=\left[2\,\mathrm{chi}\,(z)-\mathrm{chi}\,(2z)\right]I_{n+1/2}(z)$$

$$+\left[\mathrm{shi}\,(2z)-2\,\mathrm{shi}\,(z)\right]I_{-n-1/2}(z)$$

$$-\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(-\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}I_{k+1/2}(z)+\frac{n!}{2}\sqrt{\pi z}\sum_{k=1}^{n}\frac{\left(-\frac{z}{2}\right)^k}{(n-k)!\,k}\sum_{p=0}^{k-1}\frac{(-z)^p}{p!}$$

$$\times\left\{I_{n-k+1/2}(z)\left[2^{1/2-p}I_{p-1/2}(z)-I_{p-1/2}(2z)\right]\right.$$

$$\left.-I_{k-n-1/2}(z)\left[2^{1/2-p}I_{1/2-p}(z)-I_{1/2-p}(2z)\right]\right\}.$$

1.16. The Anger $\mathbf{J}_\nu(z)$ and Weber $\mathbf{E}_\nu(z)$ Functions

1.16.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[\mathbf{J}_\nu(z)]=n!\left(-\frac{z}{2}\right)^{-n}\sum_{k=0}^{[n/2]}\frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!}\sum_{p=0}^{n-k}\binom{n-k}{p}\left(\frac{\nu}{2}\right)_{n-k-p}\left(-\frac{z}{2}\right)^p$$

$$\times\left\{\mathbf{J}_{\nu-p}(z)-(-1)^p\frac{\sin(\nu\pi)}{\pi z}\sum_{r=0}^{p-1}\left(\frac{p-r-\nu+1}{2}\right)_r\left(\frac{2}{z}\right)^r\right\}.$$

2. $$=n!\left(-\frac{z}{2}\right)^{-n}\sum_{k=0}^{[n/2]}\frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!}\sum_{p=0}^{n-k}\binom{n-k}{p}\left(-\frac{\nu}{2}\right)_{n-k-p}\left(\frac{z}{2}\right)^p$$

$$\times\left\{\mathbf{J}_{\nu+p}(z)-(-1)^p\frac{\sin(\nu\pi)}{\pi z}\sum_{r=0}^{p-1}\left(\frac{p-r+\nu+1}{2}\right)_r\left(-\frac{2}{z}\right)^r\right\}.$$

3. $$=\left(\pm\frac{1}{2}\right)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathbf{J}_{\nu\pm2k\mp n}(z).$$

4. $$\mathrm{D}^n[z^{\nu/2}\mathbf{J}_\nu(a\sqrt{z})]=\left(\frac{a}{2}\right)^n z^{(\nu-n)/2}\mathbf{J}_{\nu-n}(a\sqrt{z})$$

$$-\left(-\frac{a}{2}\right)^n\frac{\sin(\nu\pi)}{\pi a}z^{(\nu-n-1)/2}\sum_{k=0}^{n-1}\left(\frac{n-k-\nu+1}{2}\right)_k\left(\frac{2}{a\sqrt{z}}\right)^k.$$

5. $$\mathrm{D}^n[z^{-\nu/2}\mathbf{J}_\nu(a\sqrt{z})] = \left(-\frac{a}{2}\right)^n z^{-(\nu+n)/2}\mathbf{J}_{\nu+n}(a\sqrt{z}) - \left(\frac{a}{2}\right)^n \frac{\sin(\nu\pi)}{\pi a} z^{-(\nu+n+1)/2}\sum_{k=0}^{n-1}\left(\frac{n-k+\nu+1}{2}\right)_k\left(-\frac{2}{a\sqrt{z}}\right)^k.$$

6. $$\mathrm{D}^n\left[z^{n-\nu/2-1}\mathbf{J}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(-\frac{a}{2}\right)^n z^{-(\nu+n)/2-1}\mathbf{J}_{\nu-n}\left(\frac{a}{\sqrt{z}}\right) - \left(\frac{a}{2}\right)^n \frac{\sin(\nu\pi)}{\pi a} z^{-(\nu+n+1)/2}\sum_{k=0}^{n-1}\left(\frac{n-k-\nu+1}{2}\right)_k\left(\frac{2\sqrt{z}}{a}\right)^k.$$

7. $$\mathrm{D}^n\left[z^{n+\nu/2-1}\mathbf{J}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(\frac{a}{2}\right)^n z^{(\nu-n)/2-1}\mathbf{J}_{\nu+n}\left(\frac{a}{\sqrt{z}}\right) - \left(-\frac{a}{2}\right)^n \frac{\sin(\nu\pi)}{\pi a} z^{(\nu-n-1)/2}\sum_{k=0}^{n-1}\left(\frac{n-k+\nu+1}{2}\right)_k\left(-\frac{2\sqrt{z}}{a}\right)^k.$$

8. $$\mathrm{D}^n[\mathbf{E}_\nu(z)] = n!\left(-\frac{z}{2}\right)^{-n}\sum_{k=0}^{[n/2]}\frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!}\sum_{p=0}^{n-k}\binom{n-k}{p}\left(\frac{\nu}{2}\right)_{n-k-p}\left(-\frac{z}{2}\right)^p \times\left\{\mathbf{E}_{\nu-p}(z) + \frac{1}{\pi z}\sum_{r=0}^{p-1}[(-1)^r + (-1)^p\cos(\nu\pi)]\left(\frac{p-r-\nu+1}{2}\right)_r\left(\frac{2}{z}\right)^r\right\}.$$

9. $$= n!\left(-\frac{z}{2}\right)^{-n}\sum_{k=0}^{[n/2]}\frac{\left(-\frac{1}{4}\right)^k}{k!\,(n-2k)!}\sum_{p=0}^{n-k}\binom{n-k}{p}\left(-\frac{\nu}{2}\right)_{n-k-p}\left(\frac{z}{2}\right)^p \times\left\{\mathbf{E}_{\nu+p}(z) + \frac{1}{\pi z}\sum_{r=0}^{p-1}[1 + (-1)^{p+r}\cos(\nu\pi)]\left(\frac{p-r+\nu+1}{2}\right)_r\left(\frac{2}{z}\right)^r\right\}.$$

10. $$= \left(\pm\frac{1}{2}\right)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathbf{E}_{\nu\pm 2k\mp n}(z).$$

11. $$\mathrm{D}^n[z^{\nu/2}\,\mathbf{E}_\nu(a\sqrt{z})] = \left(\frac{a}{2}\right)^n z^{(\nu-n)/2}\,\mathbf{E}_{\nu-n}(a\sqrt{z}) + \frac{1}{\pi a}\left(\frac{a}{2}\right)^n z^{(\nu-n-1)/2} \times\sum_{k=0}^{n-1}[(-1)^k + (-1)^n\cos(\nu\pi)]\left(\frac{n-k-\nu+1}{2}\right)_k\left(\frac{2}{a\sqrt{z}}\right)^k.$$

12. $$\mathrm{D}^n[z^{-\nu/2}\,\mathbf{E}_\nu(a\sqrt{z})] = \left(-\frac{a}{2}\right)^n z^{-(\nu+n)/2}\,\mathbf{E}_{\nu+n}(a\sqrt{z}) + \frac{1}{\pi a}\left(\frac{a}{2}\right)^n z^{-(\nu+n+1)/2} \times\sum_{k=0}^{n-1}[(-1)^n + (-1)^k\cos(\nu\pi)]\left(\frac{n-k+\nu+1}{2}\right)_k\left(\frac{2}{a\sqrt{z}}\right)^k.$$

13. $$\mathrm{D}^n\left[z^{n-\nu/2-1}\,\mathbf{E}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(-\frac{a}{2}\right)^n z^{-(\nu+n)/2-1}\,\mathbf{E}_{\nu-n}\left(\frac{a}{\sqrt{z}}\right) + \left(-\frac{a}{2}\right)^n \frac{z^{-(\nu+n+1)/2}}{\pi a} \times \sum_{k=0}^{n-1}\left[(-1)^k + (-1)^n\cos(\nu\pi)\right]\left(\frac{n-k-\nu+1}{2}\right)_k \left(\frac{2\sqrt{z}}{a}\right)^k.$$

14. $$\mathrm{D}^n\left[z^{n+\nu/2-1}\,\mathbf{E}_\nu\left(\frac{a}{\sqrt{z}}\right)\right] = \left(\frac{a}{2}\right)^n z^{(\nu-n)/2-1}\,\mathbf{E}_{\nu+n}\left(\frac{a}{\sqrt{z}}\right) + \left(-\frac{a}{2}\right)^n \frac{z^{(\nu-n-1)/2}}{\pi a} \times \sum_{k=0}^{n-1}\left[(-1)^n + (-1)^k\cos(\nu\pi)\right]\left(\frac{n-k+\nu+1}{2}\right)_k \left(\frac{2\sqrt{z}}{a}\right)^k.$$

1.16.2. Derivatives with respect to the order

1. $$\left.\frac{\partial \mathbf{J}_\nu(z)}{\partial\nu}\right|_{\nu=n} = \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{2}{z}\right)^{n-k}}{k!\,(n-k)}J_k(z) + \frac{\pi}{2}\,\mathbf{H}_n(z) - \frac{1}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{n-k}}\left(\frac{z}{2}\right)^{n-2k-1} + \frac{(-1)^n}{z}\sum_{k=0}^{n-1}(-1)^k\left(\frac{n-k+1}{2}\right)_k\left(\frac{2}{z}\right)^k.$$

2. $$\left.\frac{\partial \mathbf{J}_\nu(z)}{\partial\nu}\right|_{\nu=-n} = (-1)^{n-1}\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{2}{z}\right)^{n-k}}{k!\,(n-k)}J_k(z) + \frac{\pi}{2}\,\mathbf{H}_{-n}(z) + \frac{(-1)^n}{z}\sum_{k=0}^{n-1}\left(\frac{n-k+1}{2}\right)_k\left(\frac{2}{z}\right)^k.$$

3. $$\left.\frac{\partial \mathbf{E}_\nu(z)}{\partial\nu}\right|_{\nu=n} = \frac{\pi}{2}J_n(z) + \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{2}{z}\right)^{n-k}}{k!\,(n-k)}\left[-\mathbf{H}_k(z) + \frac{1}{\pi}\sum_{r=0}^{k-1}\frac{\left(\frac{1}{2}\right)_r}{\left(\frac{1}{2}\right)_{k-r}}\left(\frac{z}{2}\right)^{k-2r-1}\right] + \frac{1}{2\pi}\sum_{k=0}^{n-1}\left[(-1)^k + (-1)^n\right]\left(\frac{n-k+1}{2}\right)_k\left(-\frac{2}{z}\right)^{k+1}\sum_{r=0}^{k-1}\frac{1}{2r+n-k+1}.$$

4. $$\left.\frac{\partial \mathbf{E}_\nu(z)}{\partial\nu}\right|_{\nu=-n} = \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(-\frac{2}{z}\right)^{n-k}}{k!\,(n-k)}\mathbf{H}_{-k}(z) + (-1)^n\frac{\pi}{2}J_n(z) + \frac{1}{2\pi}\sum_{k=0}^{n-1}\left[(-1)^k + (-1)^n\right]\left(\frac{n-k+1}{2}\right)_k\left(\frac{2}{z}\right)^{k+1}\sum_{r=0}^{k-1}\frac{1}{2r+n-k+1}.$$

1.17. The Kelvin Functions $\mathrm{ber}_\nu(z)$, $\mathrm{bei}_\nu(z)$, $\mathrm{ker}_\nu(z)$ and $\mathrm{kei}_\nu(z)$

1.17.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[z^{\pm\nu/2}\,\mathrm{ber}_\nu(a\sqrt{z})]$
$$= \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}\left[\cos\frac{3n\pi}{4}\,\mathrm{ber}_{\nu\mp n}(a\sqrt{z}) - \sin\frac{3n\pi}{4}\,\mathrm{bei}_{\nu\mp n}(a\sqrt{z})\right].$$

2. $\mathrm{D}^n[z^{\pm\nu/2}\,\mathrm{bei}_\nu(a\sqrt{z})]$
$$= \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}\left[\sin\frac{3n\pi}{4}\,\mathrm{ber}_{\nu\mp n}(a\sqrt{z}) + \cos\frac{3n\pi}{4}\,\mathrm{bei}_{\nu\mp n}(a\sqrt{z})\right].$$

3. $$\mathrm{D}^n[z^{(2n+1)/4}\,\mathrm{ber}_{n+1/2}(a\sqrt{z})] = -\frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \times\left[\sin\frac{3(2n-1)\pi}{8}\sinh\left(a\sqrt{\frac{z}{2}}\right)\cos\left(a\sqrt{\frac{z}{2}}\right) + \cos\frac{3(2n-1)\pi}{8}\cosh\left(a\sqrt{\frac{z}{2}}\right)\sin\left(a\sqrt{\frac{z}{2}}\right)\right].$$

4. $$\mathrm{D}^n[z^{(2n+1)/4}\,\mathrm{bei}_{n+1/2}(a\sqrt{z})] = \frac{1}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \times\left[\cos\frac{3(2n-1)\pi}{8}\sinh\left(a\sqrt{\frac{z}{2}}\right)\cos\left(a\sqrt{\frac{z}{2}}\right) - \sin\frac{3(2n-1)\pi}{8}\cosh\left(a\sqrt{\frac{z}{2}}\right)\sin\left(a\sqrt{\frac{z}{2}}\right)\right].$$

5. $$\mathrm{D}^n[z^{(2n+1)/4}\,\mathrm{ber}_{-n-1/2}(a\sqrt{z})] = \frac{(-1)^{n+1}}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \times\left[\sin\frac{3(2n-1)\pi}{8}\sinh\left(a\sqrt{\frac{z}{2}}\right)\sin\left(a\sqrt{\frac{z}{2}}\right) - \cos\frac{3(2n-1)\pi}{8}\cosh\left(a\sqrt{\frac{z}{2}}\right)\cos\left(a\sqrt{\frac{z}{2}}\right)\right].$$

6. $$\mathrm{D}^n[z^{(2n+1)/4}\,\mathrm{bei}_{-n-1/2}(a\sqrt{z})] = \frac{(-1)^{n}}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \times\left[\cos\frac{3(2n-1)\pi}{8}\sinh\left(a\sqrt{\frac{z}{2}}\right)\sin\left(a\sqrt{\frac{z}{2}}\right) + \sin\frac{3(2n-1)\pi}{8}\cosh\left(a\sqrt{\frac{z}{2}}\right)\cos\left(a\sqrt{\frac{z}{2}}\right)\right].$$

7. $$\mathrm{D}^n\left[z^{-(2n+3)/4}\,\mathrm{ber}_{n+1/2}(a\sqrt{z})\right] = \sqrt{\pi}\left(\frac{a}{2z}\right)^{n+1/2} \times\left\{\cos\frac{3(2n+1)\pi}{8}\left[\mathrm{ber}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\mathrm{ber}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right) - \mathrm{bei}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\mathrm{bei}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right)\right]\right.$$

$$- \sin \frac{3(2n+1)\pi}{8} \left[\mathrm{ber}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) \mathrm{bei}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right) + \mathrm{bei}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) \mathrm{ber}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right) \right] \bigg\}.$$

8. $$\mathrm{D}^n \left[z^{-(2n+3)/4} \, \mathrm{bei}_{n+1/2}(a\sqrt{z}) \right] = \sqrt{\pi} \left(\frac{a}{2z}\right)^{n+1/2} \times \bigg\{ \sin \frac{3(2n+1)\pi}{8} \left[\mathrm{ber}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) \mathrm{ber}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right) - \mathrm{bei}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) \mathrm{bei}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right) \right] + \cos \frac{3(2n+1)\pi}{8} \left[\mathrm{ber}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) \mathrm{bei}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right) + \mathrm{bei}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) \mathrm{ber}_{-n-1/2}\left(\frac{a\sqrt{z}}{2}\right) \right] \bigg\}.$$

9. $$\mathrm{D}^n \left[z^{n-1/2} e^{az/\sqrt{2}} \left(\sin \frac{az}{\sqrt{2}} \, \mathrm{ber}_{n-1/2}(az) + \cos \frac{az}{\sqrt{2}} \, \mathrm{bei}_{n-1/2}(az) \right) \right] = \frac{(2a)^{n-1/2}}{\sqrt{\pi}} z^{n-1} e^{\sqrt{2}\,az} \sin \left[\frac{3(2n-1)\pi}{8} + \sqrt{2}\,az \right] \quad [n \geq 1].$$

10. $$\mathrm{D}^n \left[z^{n-1/2} e^{az/\sqrt{2}} \left(\cos \frac{az}{\sqrt{2}} \, \mathrm{ber}_{n-1/2}(az) - \sin \frac{az}{\sqrt{2}} \, \mathrm{bei}_{n-1/2}(az) \right) \right] = \frac{(2a)^{n-1/2}}{\sqrt{\pi}} z^{n-1} e^{\sqrt{2}\,az} \cos \left[\frac{3(2n-1)\pi}{8} + \sqrt{2}\,az \right] \quad [n \geq 1].$$

11. $$\mathrm{D}^n \left[z^{n-1/2} \left(\sinh \frac{az}{\sqrt{2}} \cos \frac{az}{\sqrt{2}} \, \mathrm{ber}_{n-1/2}(az) - \cosh \frac{az}{\sqrt{2}} \sin \frac{az}{\sqrt{2}} \, \mathrm{bei}_{n-1/2}(az) \right) \right] = \frac{(2a)^{n-1/2}}{\sqrt{\pi}} z^{n-1} \left[\cos \frac{3(2n-1)\pi}{8} \sinh(\sqrt{2}\,az) \cos(\sqrt{2}\,az) - \sin \frac{3(2n-1)\pi}{8} \cosh(\sqrt{2}\,az) \sin(\sqrt{2}\,az) \right].$$

12. $$\mathrm{D}^n \left[z^{n-1/2} \left(\cosh \frac{az}{\sqrt{2}} \sin \frac{az}{\sqrt{2}} \, \mathrm{ber}_{n-1/2}(az) + \sinh \frac{az}{\sqrt{2}} \cos \frac{az}{\sqrt{2}} \, \mathrm{bei}_{n-1/2}(az) \right) \right] = \frac{(2a)^{n-1/2}}{\sqrt{\pi}} z^{n-1} \left[\sin \frac{3(2n-1)\pi}{8} \sinh(\sqrt{2}\,az) \cos(\sqrt{2}\,az) + \cos \frac{3(2n-1)\pi}{8} \cosh(\sqrt{2}\,az) \sin(\sqrt{2}\,az) \right].$$

13. $$\mathrm{D}^n \left[z^{n-1/2} \left(\mathrm{ber}^2_{\pm n \mp 1/2}(a\sqrt{z}) - \mathrm{bei}^2_{\pm n \mp 1/2}(a\sqrt{z}) \right) \right] = \pm \frac{a^{n-1/2}}{\sqrt{\pi}} z^{(2n-3)/4} \times \left[\cos \frac{3(2n-1)\pi}{8} \, \mathrm{ber}_{n-1/2}(2a\sqrt{z}) - \sin \frac{3(2n-1)\pi}{8} \, \mathrm{bei}_{n-1/2}(2a\sqrt{z}) \right] \quad [n \geq 1].$$

14. $$\mathrm{D}^n\left[z^{n-1/2}(\operatorname{ber}_{\pm n\mp 1/2}(a\sqrt{z})\operatorname{bei}_{\pm n\mp 1/2}(a\sqrt{z}))\right] = \pm\frac{a^{n-1/2}}{2\sqrt{\pi}}z^{(2n-3)/4} \times\left[\sin\frac{3(2n-1)\pi}{8}\operatorname{ber}_{n-1/2}(2a\sqrt{z}) + \cos\frac{3(2n-1)\pi}{8}\operatorname{bei}_{n-1/2}(2a\sqrt{z})\right] \quad [n\geq 1].$$

15. $$\mathrm{D}^n[z^{\pm\nu/2}\operatorname{ker}_\nu(a\sqrt{z})] = \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}\left[\cos\frac{3n\pi}{4}\operatorname{ker}_{\nu\mp n}(a\sqrt{z}) - \sin\frac{3n\pi}{4}\operatorname{kei}_{\nu\mp n}(a\sqrt{z})\right].$$

16. $$\mathrm{D}^n[z^{\pm\nu/2}\operatorname{kei}_\nu(a\sqrt{z})] = \left(\pm\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}\left[\sin\frac{3n\pi}{4}\operatorname{ker}_{\nu\mp n}(a\sqrt{z}) + \cos\frac{3n\pi}{4}\operatorname{kei}_{\nu\mp n}(a\sqrt{z})\right].$$

17. $$\mathrm{D}^n[z^{(2n+1)/4}\operatorname{ker}_{n+1/2}(a\sqrt{z})] = (-1)^n\frac{\sqrt{\pi}}{2^{n+1/2}}a^{n-1/2}e^{-a\sqrt{z/2}}\cos\left[a\sqrt{\frac{z}{2}}+\frac{(2n+3)\pi}{8}\right].$$

18. $$\mathrm{D}^n[z^{(2n+1)/4}\operatorname{kei}_{n+1/2}(a\sqrt{z})] = (-1)^{n+1}\frac{\sqrt{\pi}}{2^{n+1/2}}a^{n-1/2}e^{-a\sqrt{z/2}}\sin\left[a\sqrt{\frac{z}{2}}+\frac{(2n+3)\pi}{8}\right].$$

19. $$\mathrm{D}^n\left[z^{-(2n+3)/4}\operatorname{ker}_{n+1/2}(a\sqrt{z})\right] = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2z}\right)^{n+1/2} \times\left\{\cos\frac{3(2n+1)\pi}{8}\left[\operatorname{ker}^2_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) - \operatorname{kei}^2_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\right] - 2\sin\frac{3(2n+1)\pi}{8}\operatorname{ker}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\operatorname{kei}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\right\}.$$

20. $$\mathrm{D}^n\left[z^{-(2n+3)/4}\operatorname{kei}_{n+1/2}(a\sqrt{z})\right] = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2z}\right)^{n+1/2} \times\left\{\sin\frac{3(2n+1)\pi}{8}\left[\operatorname{ker}^2_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right) - \operatorname{kei}^2_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\right] + 2\cos\frac{3(2n+1)\pi}{8}\operatorname{ker}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\operatorname{kei}_{n+1/2}\left(\frac{a\sqrt{z}}{2}\right)\right\}.$$

21. $$\mathrm{D}^n\left[z^{n-1/2}\operatorname{ker}_{n-1/2}(a\sqrt{z})\operatorname{kei}_{n-1/2}(a\sqrt{z})\right] = (-1)^n\frac{\sqrt{\pi}}{2}a^{n-1/2}z^{(2n-3)/4} \times\left[\cos\frac{(2n-1)\pi}{8}\operatorname{kei}_{n-1/2}(2a\sqrt{z}) - \sin\frac{(2n-1)\pi}{8}\operatorname{ker}_{n-1/2}(2a\sqrt{z})\right].$$

22. $$\mathrm{D}^n\left[z^{n-1/2}\left(\operatorname{ker}^2_{n-1/2}(a\sqrt{z}) - \operatorname{kei}^2_{n-1/2}(a\sqrt{z})\right)\right] = (-1)^n\sqrt{\pi}\,a^{n-1/2}z^{(2n-3)/4} \times\left[\cos\frac{(2n-1)\pi}{8}\operatorname{ker}_{n-1/2}(2a\sqrt{z}) + \sin\frac{(2n-1)\pi}{8}\operatorname{kei}_{n-1/2}(2a\sqrt{z})\right].$$

23. $$\mathrm{D}^n\left[z^{n-1/2}\left(\mathrm{ber}_{n-1/2}(a\sqrt{z})\,\mathrm{ker}_{n-1/2}(a\sqrt{z}) - \mathrm{bei}_{n-1/2}(a\sqrt{z})\,\mathrm{kei}_{n-1/2}(a\sqrt{z})\right)\right]$$
$$= \frac{a^{n-1/2}}{\sqrt{\pi}}\,z^{(2n-3)/4}\left[\cos\frac{3(2n-1)\,\pi}{8}\,\mathrm{ker}_{n-1/2}(2a\sqrt{z}) - \sin\frac{3(2n-1)\,\pi}{8}\,\mathrm{kei}_{n-1/2}(2a\sqrt{z})\right].$$

24. $$\mathrm{D}^n\left[z^{n-1/2}\left(\mathrm{ber}_{n-1/2}(a\sqrt{z})\,\mathrm{kei}_{n-1/2}(a\sqrt{z}) + \mathrm{bei}_{n-1/2}(a\sqrt{z})\,\mathrm{ker}_{n-1/2}(a\sqrt{z})\right)\right]$$
$$= \frac{a^{n-1/2}}{\sqrt{\pi}}\,z^{(2n-3)/4}\left[\sin\frac{3(2n-1)\,\pi}{8}\,\mathrm{ker}_{n-1/2}(2a\sqrt{z}) + \cos\frac{3(2n-1)\,\pi}{8}\,\mathrm{kei}_{n-1/2}(2a\sqrt{z})\right].$$

1.17.2. Derivatives with respect to the order

1. $$\left.\frac{\partial\,\mathrm{ber}_\nu(z)}{\partial\nu}\right|_{\nu=n} = -\frac{\pi}{2}\,\mathrm{bei}_n(z) - \mathrm{ker}_n(z) + \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\left[\cos\frac{5(k-n)\pi}{4}\,\mathrm{ber}_k(z) + \sin\frac{5(k-n)\pi}{4}\,\mathrm{bei}_k(z)\right].$$

2. $$\left.\frac{\partial\,\mathrm{bei}_\nu(z)}{\partial\nu}\right|_{\nu=n} = \frac{\pi}{2}\,\mathrm{ber}_n(z) - \mathrm{kei}_n(z) + \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\left[\cos\frac{5(k-n)\pi}{4}\,\mathrm{bei}_k(z) - \sin\frac{5(k-n)\pi}{4}\,\mathrm{ber}_k(z)\right].$$

3. $$\left.\frac{\partial\,\mathrm{ber}_\nu(z)}{\partial\nu}\right|_{\nu=0} = -\frac{\pi}{2}\,\mathrm{bei}_0(z) - \mathrm{ker}_0(z).$$

4. $$\left.\frac{\partial\,\mathrm{bei}_\nu(z)}{\partial\nu}\right|_{\nu=0} = \frac{\pi}{2}\,\mathrm{ber}_0(z) - \mathrm{kei}_0(z).$$

5. $$\left.\frac{\partial\,\mathrm{ker}_\nu(z)}{\partial\nu}\right|_{\nu=n} = \frac{\pi}{2}\,\mathrm{kei}_n(z) + \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\left[\cos\frac{3(k-n)\pi}{4}\,\mathrm{ker}_k(z) - \sin\frac{3(k-n)\pi}{4}\,\mathrm{kei}_k(z)\right].$$

6. $$\left.\frac{\partial\,\mathrm{kei}_\nu(z)}{\partial\nu}\right|_{\nu=n} = -\frac{\pi}{2}\,\mathrm{ker}_n(z) + \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\left[\sin\frac{3(k-n)\pi}{4}\,\mathrm{ker}_k(z) + \cos\frac{3(k-n)\pi}{4}\,\mathrm{kei}_k(z)\right].$$

7. $$\left.\frac{\partial \operatorname{ber}_\nu(z)}{\partial \nu}\right|_{\nu=n+1/2}$$
$$= \frac{1}{6}\operatorname{ber}_{n+1/2}(z)\left[6\mathrm{C} + 6\ln(2z) - z^4\,{}_2F_5\left(\begin{matrix} 1,\,1;\; -\frac{z^4}{16} \\ \frac{5}{4},\,\frac{3}{2},\,\frac{7}{4},\,2,\,2\end{matrix}\right)\right]$$
$$- \frac{1}{4}\operatorname{bei}_{n+1/2}(z)\left[3\pi + 4z^2\,{}_1F_4\left(\begin{matrix} \frac{1}{2};\; -\frac{z^4}{16} \\ \frac{3}{4},\,\frac{5}{4},\,\frac{3}{2},\,\frac{3}{2}\end{matrix}\right)\right]$$
$$+ (-1)^n\sqrt{2}\,z[\operatorname{ber}_{-n-1/2}(z) + \operatorname{bei}_{-n-1/2}(z)]\,{}_1F_4\left(\begin{matrix} \frac{1}{4};\; -\frac{z^4}{16} \\ \frac{1}{2},\,\frac{3}{4},\,\frac{5}{4},\,\frac{5}{4}\end{matrix}\right)$$
$$+ (-1)^n\frac{2\sqrt{2}}{9}z^3[\operatorname{ber}_{-n-1/2}(z) - \operatorname{bei}_{-n-1/2}(z)]\,{}_1F_4\left(\begin{matrix} \frac{3}{4};\; -\frac{z^4}{16} \\ \frac{5}{4},\,\frac{3}{2},\,\frac{7}{4},\,\frac{7}{4}\end{matrix}\right)$$
$$+ \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\left[\cos\frac{3(n-k)\pi}{4}\operatorname{ber}_{k+1/2}(z) + \sin\frac{3(n-k)\pi}{4}\operatorname{bei}_{k+1/2}(z)\right]$$
$$- \frac{n!\sqrt{\pi z}}{2}\sum_{k=1}^{n}\frac{\left(\frac{2}{z}\right)^k}{(n-k)!\,k}\sum_{p=0}^{k-1}\frac{z^p}{p!}\left\{\operatorname{ber}_{n-k+1/2}(z)\right.$$
$$\times\left[\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{p-1/2}(2z) + \sin\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{p-1/2}(2z)\right]$$
$$+ \operatorname{bei}_{n-k+1/2}(z)\left[\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{p-1/2}(2z)\right.$$
$$\left. - \cos\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{p-1/2}(2z)\right]$$
$$- (-1)^{k+n+p}\operatorname{ber}_{k-n-1/2}(z)\left[\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{1/2-p}(2z)\right.$$
$$\left. + \sin\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{1/2-p}(2z)\right]$$
$$- (-1)^{k+n+p}\operatorname{bei}_{k-n-1/2}(z)\left[\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{1/2-p}(2z)\right.$$
$$\left.\left. - \cos\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{1/2-p}(2z)\right]\right\} \qquad [|\arg z| \le \pi/4].$$

8. $$\left.\frac{\partial \operatorname{bei}_\nu(z)}{\partial \nu}\right|_{\nu=n+1/2} = \left.\frac{\partial \operatorname{bei}_\nu(z)}{\partial \nu}\right|_{\nu=n+1/2}$$
$$= \frac{1}{4}\operatorname{ber}_{n+1/2}(z)\left[3\pi + 4z^2\,{}_1F_4\left(\begin{matrix} \frac{1}{2};\; -\frac{z^4}{16} \\ \frac{3}{4},\,\frac{5}{4},\,\frac{3}{2},\,\frac{3}{2}\end{matrix}\right)\right]$$
$$+ \frac{1}{6}\operatorname{bei}_{n+1/2}(z)\left[6\mathrm{C} + 6\ln(2z) - z^4\,{}_2F_5\left(\begin{matrix} 1,\,1;\; -\frac{z^4}{16} \\ \frac{5}{4},\,\frac{3}{2},\,\frac{7}{4},\,2,\,2\end{matrix}\right)\right]$$

$$+ (-1)^n \sqrt{2}\, z[\mathrm{ber}_{-n-1/2}(z) - \mathrm{bei}_{-n-1/2}(z)]\, {}_1F_4\left(\begin{matrix} \frac{1}{4};\ -\frac{z^4}{16} \\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4},\ \frac{5}{4} \end{matrix}\right)$$

$$+ (-1)^n \frac{2\sqrt{2}}{9} z^3 [\mathrm{ber}_{-n-1/2}(z) + \mathrm{bei}_{-n-1/2}(z)]\, {}_1F_4\left(\begin{matrix} \frac{3}{4};\ -\frac{z^4}{16} \\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4},\ \frac{7}{4} \end{matrix}\right)$$

$$- \frac{n!}{2} \sum_{k=0}^{n-1} \frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)} \left[\sin \frac{3(n-k)\pi}{4}\, \mathrm{ber}_{k+1/2}(z) - \cos \frac{3(n-k)\pi}{4}\, \mathrm{bei}_{k+1/2}(z)\right]$$

$$+ \frac{n!\sqrt{\pi z}}{2} \sum_{k=1}^{n} \frac{\left(\frac{2}{z}\right)^k}{(n-k)!\,k} \sum_{p=0}^{k-1} \frac{z^p}{p!} \Big\{\mathrm{ber}_{n-k+1/2}(z)$$

$$\times \left[\sin \frac{3(2k-2p-1)\pi}{8}\, \mathrm{ber}_{p-1/2}(2z) - \cos \frac{3(2k-2p-1)\pi}{8}\, \mathrm{bei}_{p-1/2}(2z)\right]$$

$$- \mathrm{bei}_{n-k+1/2}(z)\Big[\cos \frac{3(2k-2p-1)\pi}{8}\, \mathrm{ber}_{p-1/2}(2z)$$

$$+ \sin \frac{3(2k-2p-1)\pi}{8}\, \mathrm{bei}_{p-1/2}(2z)\Big]$$

$$- (-1)^{k+n+p}\, \mathrm{ber}_{k-n-1/2}(z)\Big[\sin \frac{3(2k-2p-1)\pi}{8}\, \mathrm{ber}_{1/2-p}(2z)$$

$$- \cos \frac{3(2k-2p-1)\pi}{8}\, \mathrm{bei}_{1/2-p}(2z)\Big]$$

$$+ (-1)^{k+n+p}\, \mathrm{bei}_{k-n-1/2}(z)\Big[\cos \frac{3(2k-2p-1)\pi}{8}\, \mathrm{ber}_{1/2-p}(2z)$$

$$+ \sin \frac{3(2k-2p-1)\pi}{8}\, \mathrm{bei}_{1/2-p}(2z)\Big]\Big\} \qquad [|\arg z| \le \pi/4].$$

9. $\left.\dfrac{\partial\, \mathrm{ber}_\nu(z)}{\partial \nu}\right|_{\nu=1/2-n}$

$$= \frac{1}{6}\, \mathrm{ber}_{1/2-n}(z) \left[6\mathrm{C} + 6\ln(2z) - z^4\, {}_2F_5\left(\begin{matrix} 1,\ 1;\ -\frac{z^4}{16} \\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4},\ 2,\ 2 \end{matrix}\right)\right]$$

$$- \frac{1}{4}\, \mathrm{bei}_{1/2-n}(z) \left[3\pi + 4z^2\, {}_1F_4\left(\begin{matrix} \frac{1}{2};\ -\frac{z^4}{16} \\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2},\ \frac{3}{2} \end{matrix}\right)\right]$$

$$+ (-1)^n \sqrt{2}\, z[\mathrm{ber}_{n-1/2}(z) + \mathrm{bei}_{n-1/2}(z)]\, {}_1F_4\left(\begin{matrix} \frac{1}{4};\ -\frac{z^4}{16} \\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4},\ \frac{5}{4} \end{matrix}\right)$$

$$+ (-1)^n \frac{2\sqrt{2}}{9} z^3 [\mathrm{ber}_{n-1/2}(z) - \mathrm{bei}_{n-1/2}(z)]\, {}_1F_4\left(\begin{matrix} \frac{3}{4};\ -\frac{z^4}{16} \\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4},\ \frac{7}{4} \end{matrix}\right)$$

$$- \frac{n!}{2} \sum_{k=0}^{n-1} \frac{\left(-\frac{z}{2}\right)^{k-n}}{k!\,(n-k)} \left[\cos \frac{3(n-k)\pi}{4}\, \mathrm{ber}_{1/2-k}(z) + \sin \frac{3(n-k)\pi}{4}\, \mathrm{bei}_{1/2-k}(z)\right]$$

$$+\frac{n!\sqrt{\pi z}}{2}\sum_{k=1}^{n}\frac{\left(\frac{2}{z}\right)^k}{(n-k)!\,k}\sum_{p=0}^{k-1}\frac{z^p}{p!}\left\{(-1)^{n+p}\operatorname{ber}_{n-k-1/2}(z)\right.$$
$$\times\left[\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{1/2-p}(2z)+\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{1/2-p}(2z)\right]$$
$$+(-1)^{n+p}\operatorname{bei}_{n-k-1/2}(z)\left[\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{1/2-p}(2z)\right.$$
$$\left.-\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{1/2-p}(2z)\right]$$
$$-(-1)^k\operatorname{ber}_{k-n+1/2}(z)\left[\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{p-1/2}(2z)\right.$$
$$\left.+\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{p-1/2}(2z)\right]$$
$$-(-1)^k\operatorname{bei}_{k-n+1/2}(z)\left[\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{p-1/2}(2z)\right.$$
$$\left.\left.-\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{p-1/2}(2z)\right]\right\}\quad[|\arg z|\le\pi/4].$$

10. $$\left.\frac{\partial\operatorname{bei}_\nu(z)}{\partial\nu}\right|_{\nu=1/2-n}=\frac{1}{4}\operatorname{ber}_{1/2-n}(z)\left[3\pi+4z^2\,{}_1F_4\left(\begin{matrix}\frac{1}{2};\ -\frac{z^4}{16}\\ \frac{3}{4},\frac{5}{4},\frac{3}{2},\frac{3}{2}\end{matrix}\right)\right]$$
$$+\frac{1}{6}\operatorname{bei}_{1/2-n}(z)\left[6\,\mathbf{C}+6\ln(2z)-x^4\,{}_2F_5\left(\begin{matrix}1,1;\ -\frac{z^4}{16}\\ \frac{5}{4},\frac{3}{2},\frac{7}{4},2,2\end{matrix}\right)\right]$$
$$-(-1)^n\sqrt{2}\,x[\operatorname{ber}_{n-1/2}(z)-\operatorname{bei}_{n-1/2}(z)]\,{}_1F_4\left(\begin{matrix}\frac{1}{4};\ -\frac{z^4}{16}\\ \frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{5}{4}\end{matrix}\right)$$
$$+(-1)^n\frac{2\sqrt{2}}{9}z^3\,[\operatorname{ber}_{n-1/2}(z)+\operatorname{bei}_{n-1/2}(z)]\,{}_1F_4\left(\begin{matrix}\frac{3}{4};\ -\frac{z^4}{16}\\ \frac{5}{4},\frac{3}{2},\frac{7}{4},\frac{7}{4}\end{matrix}\right)$$
$$+\frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(-\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\left[\sin\frac{3(n-k)\pi}{4}\operatorname{ber}_{1/2-k}(z)-\cos\frac{3(n-k)\pi}{4}\operatorname{bei}_{1/2-k}(z)\right]$$
$$-\frac{n!\sqrt{\pi z}}{2}\sum_{k=1}^{n}\frac{\left(\frac{2}{z}\right)^k}{(n-k)!\,k}\sum_{p=0}^{k-1}\frac{z^p}{p!}\left\{(-1)^{n+p}\operatorname{ber}_{n-k-1/2}(z)\right.$$
$$\times\left[\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{1/2-p}(2z)-\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{1/2-p}(2z)\right]$$
$$-(-1)^{n+p}\operatorname{bei}_{n-k-1/2}(z)\left[\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{1/2-p}(2z)\right.$$
$$\left.+\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{1/2-p}(2z)\right]$$
$$-(-1)^k\operatorname{ber}_{k-n+1/2}(z)\left[\sin\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{p-1/2}(2z)\right.$$
$$\left.-\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{p-1/2}(z)\right]$$

$$+ (-1)^k \operatorname{bei}_{k-n+1/2}(z)\left[\cos\frac{3(2k-2p-1)\pi}{8}\operatorname{ber}_{p-1/2}(2z) + \sin\frac{3(2k-2p-1)\pi}{8}\operatorname{bei}_{p-1/2}(2z)\right]\Bigg\} \quad [|\arg z] \le \pi/4].$$

11. $$\left.\frac{\partial \operatorname{ker}_\nu(z)}{\partial\nu}\right|_{\nu=n-1/2} = \frac{\pi}{2}\operatorname{kei}_{n-1/2}(z) - \frac{\pi^2}{8}\operatorname{ber}_{n-1/2}(z) - \frac{\pi}{2}\ln(2z)\operatorname{bei}_{n-1/2}(z) + (-1)^n\frac{\pi}{2}[\mathbf{C}+\ln(2z)]\operatorname{ber}_{1/2-n}(z) - (-1)^n\frac{\pi^2}{8}\operatorname{bei}_{1/2-n}(z)$$
$$- \frac{\pi z^2}{2}[\operatorname{ber}_{n-1/2}(z) + (-1)^n\operatorname{bei}_{1/2-n}(z)]\,{}_1F_4\left(\begin{matrix}\frac{1}{2};\ -\frac{z^4}{16}\\ \frac{3}{4},\frac{5}{4},\frac{3}{2},\frac{3}{2}\end{matrix}\right)$$
$$+ \frac{\pi z}{\sqrt{2}}[\operatorname{ber}_{n-1/2}(z) + \operatorname{bei}_{n-1/2}(z) + (-1)^n\operatorname{ber}_{1/2-n}(z) - (-1)^n\operatorname{bei}_{1/2-n}(z)]\,{}_1F_4\left(\begin{matrix}\frac{1}{4};\ -\frac{z^4}{16}\\ \frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{5}{4}\end{matrix}\right)$$
$$- \frac{\pi}{18}\Bigg[9\mathbf{C}\operatorname{bei}_{n-1/2}(z) - 2\sqrt{2}\,z^3[\operatorname{ber}_{n-1/2}(z) - \operatorname{bei}_{n-1/2}(z) + (-1)^n\operatorname{ber}_{1/2-n}(z) + (-1)^n\operatorname{bei}_{1/2-n}(z)]\,{}_1F_4\left(\begin{matrix}\frac{3}{4};\ -\frac{z^4}{16}\\ \frac{5}{4},\frac{3}{2},\frac{7}{4},\frac{7}{4}\end{matrix}\right)\Bigg]$$
$$+ \frac{\pi z^4}{12}[\operatorname{bei}_{n-1/2}(z) - (-1)^n\operatorname{ber}_{1/2-n}(z)]\,{}_2F_5\left(\begin{matrix}1,1;\ -\frac{z^4}{16}\\ \frac{5}{4},\frac{3}{2},\frac{7}{4},2,2\end{matrix}\right)$$
$$+ \frac{n!}{2}\sum_{k=0}^{n-1}\frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)}\left[\cos\frac{3(n-k)\pi}{4}\operatorname{ker}_{k-1/2}(z) + \sin\frac{3(n-k)\pi}{4}\operatorname{kei}_{k-1/2}(z)\right]$$
$$- \frac{n!\sqrt{\pi z}}{2}\sum_{k=1}^{n}\frac{\left(-\frac{2}{z}\right)^k}{(n-k)!\,k}\sum_{p=0}^{k-1}\frac{z^p}{p!}\Big\{[\operatorname{ber}_{n-k-1/2}(z) + (-1)^{k+n}\operatorname{bei}_{k-n+1/2}(z)]$$
$$\times\left[\cos\frac{3(2k+6p+3)\pi}{8}\operatorname{ker}_{p-1/2}(2z) - \sin\frac{3(2k+6p+3)\pi}{8}\operatorname{kei}_{p-1/2}(2z)\right] + \left[(-1)^{k+n}\operatorname{ber}_{k-n+1/2}(z) - \operatorname{bei}_{n-k-1/2}(z)\right]$$
$$\times\left[\sin\frac{3(2k+6p+3)\pi}{8}\operatorname{ker}_{p-1/2}(2z) + \cos\frac{3(2k+6p+3)\pi}{8}\operatorname{kei}_{p-1/2}(2z)\right]\Big\} \quad [|\arg z| \le \pi/4].$$

12. $$\left.\frac{\partial \operatorname{kei}_\nu(z)}{\partial\nu}\right|_{\nu=n-1/2} = -\frac{\pi}{2}\operatorname{ker}_{n-1/2}(z)$$
$$+ \frac{\pi}{2}[\mathbf{C} + \ln(2z)][\operatorname{ber}_{n-1/2}(z) + (-1)^n\operatorname{ber}_{1/2-n}(z)] + (-1)^n\frac{\pi^2}{8}\operatorname{ber}_{1/2-n}(z)$$

$$- \frac{\pi^2}{8} \operatorname{bei}_{n-1/2}(z) + \frac{\pi z^2}{2} \left[(-1)^n \operatorname{ber}_{1/2-n}(z) - \operatorname{bei}_{n-1/2}(z)\right] {}_1F_4\left(\begin{matrix} \frac{1}{2};\ -\frac{z^4}{16} \\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2},\ \frac{3}{2} \end{matrix}\right)$$

$$- \frac{\pi z}{\sqrt{2}} \left[\operatorname{ber}_{n-1/2}(z) - \operatorname{bei}_{n-1/2}(z) + (-1)^n \operatorname{ber}_{1/2-n}(z)\right.$$

$$\left. + (-1)^n \operatorname{bei}_{1/2-n}(z)\right] {}_1F_4\left(\begin{matrix} \frac{1}{4};\ -\frac{z^4}{16} \\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4},\ \frac{5}{4} \end{matrix}\right)$$

$$+ \frac{\sqrt{2}\,\pi}{9} z^3 \left[\operatorname{ber}_{n-1/2}(z) + \operatorname{bei}_{n-1/2}(z) - (-1)^n \operatorname{ber}_{1/2-n}(z)\right.$$

$$\left. + (-1)^n \operatorname{bei}_{1/2-n}(z)\right] {}_1F_4\left(\begin{matrix} \frac{3}{4};\ -\frac{z^4}{16} \\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4},\ \frac{7}{4} \end{matrix}\right)$$

$$- \frac{\pi z^4}{12} \left[(-1)^n \operatorname{ber}_{1/2-n}(z) + \operatorname{bei}_{n-1/2}(z)\right] {}_2F_5\left(\begin{matrix} 1,\ 1;\ -\frac{z^4}{16} \\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4},\ 2,\ 2 \end{matrix}\right)$$

$$- \frac{n!}{2} \sum_{k=0}^{n-1} \frac{\left(\frac{z}{2}\right)^{k-n}}{k!\,(n-k)} \left[\sin \frac{3(n-k)\pi}{4} \operatorname{ker}_{k-1/2}(z) - \cos \frac{3(n-k)\pi}{4} \operatorname{kei}_{k-1/2}(z)\right]$$

$$- \frac{n!\,\sqrt{\pi z}}{2} \sum_{k=1}^{n} \frac{\left(\frac{z}{2}\right)^k}{(n-k)!\,k} \sum_{p=0}^{k-1} \frac{z^p}{p!}$$

$$\times \left\{ \left[(-1)^k \operatorname{ber}_{n-k-1/2}(z) + (-1)^n \operatorname{bei}_{k-n+1/2}(z)\right]\right.$$

$$\times \left[\sin \frac{3(2k+6p+3)\pi}{8} \operatorname{ker}_{p-1/2}(2z) + \cos \frac{3(2k+6p+3)\pi}{8} \operatorname{kei}_{p-1/2}(2z)\right]$$

$$+ \left[(-1)^k \operatorname{bei}_{n-k-1/2}(z) - (-1)^n \operatorname{ber}_{k-n+1/2}(z)\right]$$

$$\left. \times \left[\cos \frac{3(2k+6p+3)\pi}{8} \operatorname{ker}_{p-1/2}(2z) - \sin \frac{3(2k+6p+3)\pi}{8} \operatorname{kei}_{p-1/2}(2z)\right] \right\}$$

$[|\arg z| \le \pi/4]$.

13. $\left.\frac{\partial \operatorname{ker}_\nu(z)}{\partial \nu}\right|_{\nu=1/2-n} = (-1)^{n+1} \pi \operatorname{ker}_{n-1/2}(z) + (-1)^{n+1} \left.\frac{\partial \operatorname{kei}_\nu(z)}{\partial \nu}\right|_{\nu=n-1/2}.$

14. $\left.\frac{\partial \operatorname{kei}_\nu(z)}{\partial \nu}\right|_{\nu=1/2-n} = (-1)^{n+1} \pi \operatorname{kei}_{n-1/2}(z) + (-1)^{n} \left.\frac{\partial \operatorname{ker}_\nu(z)}{\partial \nu}\right|_{\nu=n-1/2}.$

1.18. The Legendre Polynomials $P_n(z)$

1.18.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[P_m(az)] = (2n-1)!!\, a^n C_{m-n}^{n+1/2}(az)$ $\quad [m \ge n]$.

2. $\quad = \frac{(m+n)!}{\left(\frac{1}{2}\right)_n (m-n)!} \left(\frac{a}{2}\right)^n (1 - a^2 z^2)^{-n} C_{m+n}^{1/2-n}(az)$ $\quad [m \ge n]$.

3. $$\mathrm{D}^{2n}[z^{-1/2}(1-a^2z)^{n-1/2}P_{2n}(a\sqrt{z})] = (-1)^n\left(\frac{1}{2}\right)_n^2 a^{2n}(z-a^2z^2)^{-n-1/2}P_{2n}\left(\frac{1}{a\sqrt{z}}\right).$$

4. $$\mathrm{D}^{2n}\left[z^n(z-a^2)^{n-1/2}P_{2n}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^n\left(\frac{1}{2}\right)_n^2 a^{2n}(z-a^2)^{-n-1/2}P_{2n}\left(\frac{\sqrt{z}}{a}\right).$$

5. $$\mathrm{D}^{2n+1}[(1-a^2z)^{n-1/2}P_{2n+1}(a\sqrt{z})] = (-1)^n\frac{2n+1}{2}\left(\frac{1}{2}\right)_n^2 a^{2n+1}z^{-n-1/2}(1-a^2z)^{-n-3/2}P_{2n}\left(\frac{1}{a\sqrt{z}}\right).$$

6. $$\mathrm{D}^n\left[z^{-(n+1)/2}(z-a)^{n-1/2}P_n\left(\frac{z+a}{2\sqrt{az}}\right)\right] = \left(\frac{1}{2}\right)_n a^{n/2}z^{-n-1/2}(z-a)^{-1/2}P_{2n}\left(\sqrt{\frac{a}{z}}\right).$$

7. $$\mathrm{D}^n\left[z^{n/2}(a-z)^{n-1/2}P_n\left(\frac{z+a}{2\sqrt{az}}\right)\right] = (-1)^n\left(\frac{1}{2}\right)_n a^{n/2}(a-z)^{-1/2}P_{2n}\left(\sqrt{\frac{z}{a}}\right).$$

8. $$\mathrm{D}^n\left[z^{m/2}(a-z)^{n-m-1}P_m\left(\frac{z+a}{2\sqrt{az}}\right)\right] = \frac{m!}{(m-n)!}a^{n/2}z^{(m-n)/2}(a-z)^{-m-1}P_{m-n}\left(\frac{z+a}{2\sqrt{az}}\right) \quad [m\ge n].$$

9. $$\mathrm{D}^n\left[z^{-(m+1)/2}(a-z)^{m+n}P_m\left(\frac{z+a}{2\sqrt{az}}\right)\right] = (-1)^n\frac{(m+n)!}{m!}a^{n/2}z^{-(m+n+1)/2}(a-z)^mP_{m+n}\left(\frac{z+a}{2\sqrt{az}}\right).$$

10. $$\mathrm{D}^n\left[z^{n/2}(z-a)^nP_n\left(\frac{z+a}{2\sqrt{az}}\right)\right] = n!\,a^{n/2}\left[P_n\left(\sqrt{\frac{z}{a}}\right)\right]^2.$$

11. $$\mathrm{D}^n\left[(z^2-az)^{m/2}P_m\left(\frac{2z-a}{2\sqrt{z^2-az}}\right)\right] = \frac{(-4)^{-n}}{(1/2-m)_n}\frac{(2m)!}{(2m-2n)!}(z^2-az)^{(m-n)/2}P_{m-n}\left(\frac{2z-a}{2\sqrt{z^2-az}}\right) \quad [m\ge n].$$

12. $$\mathrm{D}^n\left[z^{n-m-1}(a-z)^{m/2}P_m\left(\frac{2a-z}{2\sqrt{a^2-az}}\right)\right] = \frac{2^{-2n}a^{n/2}}{\left(\frac{1}{2}-m\right)_n}\frac{(2m)!}{(2m-2n)!}z^{-m-1}(a-z)^{(m-n)/2}P_{m-n}\left(\frac{2a-z}{2\sqrt{a^2-az}}\right) \quad [m\ge n].$$

13. $$\mathrm{D}^n\left[(z^2-a^2)^{-(m+1)/2}P_m\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = (-1)^n\frac{(m+n)!}{m!}(z^2-a^2)^{-(m+n+1)/2}P_{m+n}\left(\frac{z}{\sqrt{z^2-a^2}}\right).$$

14. $$\mathrm{D}^{2n}\left[z^n(a-z)^{-1/2}P_{2n}\left(\sqrt{1-\frac{a}{z}}\right)\right] = \left(\frac{1}{2}\right)_n^2 (a-z)^{-n-1/2}P_{2n}\left(\sqrt{\frac{z}{z-a}}\right).$$

15. $$\mathrm{D}^{2n+1}\left[z^{n+1/2}P_{2n+1}\left(\sqrt{1-\frac{a}{z}}\right)\right] = (-1)^n\frac{2n+1}{2}\left(\frac{1}{2}\right)_n^2 (z-a)^{-n-1/2}P_{2n}\left(\sqrt{\frac{z}{z-a}}\right).$$

16. $$\mathrm{D}^{2n}\left[z^{n-1/2}(az-1)^{-1/2}P_{2n}(\sqrt{1-az})\right] = \left(\frac{1}{2}\right)_n^2 z^{-n-1/2}(az-1)^{-n-1/2}P_{2n}\left(\frac{1}{\sqrt{1-az}}\right).$$

17. $$\mathrm{D}^{2n}\left[z^{n-1/2}(1-az)^n P_{2n}\left(\frac{1}{\sqrt{1-az}}\right)\right] = (-1)^n\left(\frac{1}{2}\right)_n^2 z^{-n-1/2}P_{2n}(\sqrt{1-az}).$$

18. $$\mathrm{D}^{2n}\left[(z^2-a^2)^m P_{2m}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = \frac{(2m)!}{(2m-2n)!}(z^2-a^2)^{m-n}P_{2m-2n}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m \geq n].$$

19. $$\mathrm{D}^{2n+1}\left[(z^2-a^2)^m P_{2m}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = \frac{(2m)!}{(2m-2n-1)!}(z^2-a^2)^{m-n-1/2}P_{2m-2n-1}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m \geq n+1].$$

20. $$\mathrm{D}^{n}\left[z^{-n-1}(a^2-z^2)^n P_{2n}\left(\frac{a}{\sqrt{a^2-z^2}}\right)\right] = (-4)^n\left(\frac{1}{2}\right)_n a^n z^{-2n-1}(a^2-z^2)^{n/2}P_n\left(\frac{a}{\sqrt{a^2-z^2}}\right).$$

21. $$\mathrm{D}^{2n}\left[z^{2n-2m-1}(a^2-z^2)^m P_{2m}\left(\frac{a}{\sqrt{a^2-z^2}}\right)\right] = \frac{(2m)!}{(2m-2n)!}a^{2n}z^{-2m-1}(a^2-z^2)^{m-n}P_{2m-2n}\left(\frac{a}{\sqrt{a^2-z^2}}\right) \quad [m \geq n].$$

22. $$\mathrm{D}^{2n+1}\left[z^{2n-2m}(a^2-z^2)^m P_{2m}\left(\frac{a}{\sqrt{a^2-z^2}}\right)\right] = -\frac{(2m)!}{(2m-2n-1)!}a^{2n+1}z^{-2m-1}(a^2-z^2)^{m-n-1/2}P_{2m-2n-1}\left(\frac{a}{\sqrt{a^2-z^2}}\right) \quad [m \geq n+1].$$

23. $$\mathrm{D}^n[z^{n-1/2}(1-a^2z)^{n+1/2}\,\mathrm{D}^n[(1-a^2z)^{n-m-1}P_{2m}(a\sqrt{z})] = \frac{(2m)!}{(2m-2n)!}\left(\frac{a}{2}\right)^{2n} z^{-1/2}(1-a^2z)^{-m-1/2}P_{2m-2n}(a\sqrt{z}) \quad [m \geq n].$$

24. $$\mathrm{D}^n\Big[(z-a^2)^{n+1/2}\,\mathrm{D}^n\Big[z^m(z-a^2)^{n-m-1}P_{2m}\Big(\frac{a}{\sqrt{z}}\Big)\Big]\Big] = \frac{(2m)!}{(2m-2n)!}\Big(\frac{a}{2}\Big)^{2n} z^{m-n}(z-a^2)^{-m-1/2}P_{2m-2n}\Big(\frac{a}{\sqrt{z}}\Big) \quad [m \geq n].$$

25. $$\mathrm{D}^n\Big[z^n\,\mathrm{D}^n\Big[(a-z)^m P_m\Big(\frac{a+z}{a-z}\Big)\Big]\Big] = \Big[\frac{m!}{(m-n)!}\Big]^2 (a-z)^{m-n}P_{m-n}\Big(\frac{a+z}{a-z}\Big) \quad [m \geq n].$$

26. $$\mathrm{D}^n\Big[z^n\,\mathrm{D}^n\Big[z^{m+n}(a-z)^{-m-1}P_m\Big(\frac{a+z}{a-z}\Big)\Big]\Big] = \Big[\frac{(m+n)!}{m!}\Big]^2 a^n z^m (a-z)^{-m-n-1}P_{m+n}\Big(\frac{a+z}{a-z}\Big).$$

27. $$\mathrm{D}^n\Big[z^n\,\mathrm{D}^n\Big[(a-z)^{-m-1}P_m\Big(\frac{a+z}{a-z}\Big)\Big]\Big] = \Big[\frac{(m+n)!}{m!}\Big]^2 (a-z)^{-m-n-1}P_{m+n}\Big(\frac{a+z}{a-z}\Big).$$

1.19. The Chebyshev Polynomials $T_n(z)$ and $U_n(z)$

1.19.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[T_m(az)] = 2^{n-1}m(n-1)!a^n C^n_{m-n}(az) \quad [m \geq n \geq 1].$$

2. $$\mathrm{D}^n\Big[z^{-1/2}(a-z)^n T_n\Big(\frac{a+z}{a-z}\Big)\Big] = (-1)^n\Big(\frac{1}{2}\Big)_n z^{-n-1/2}(a-z)^n.$$

3. $$\mathrm{D}^{2n}\Big[(a^2-z^2)^m T_m\Big(\frac{a^2+z^2}{a^2-z^2}\Big)\Big] = \frac{(2m)!}{(2m-2n)!}(a^2-z^2)^{m-n}T_{m-n}\Big(\frac{a^2+z^2}{a^2-z^2}\Big) \quad [m \geq n].$$

4. $$\mathrm{D}^{2n}\Big[z^{2n-2m-1}(z^2-a^2)^m T_m\Big(\frac{z^2+a^2}{z^2-a^2}\Big)\Big] = \frac{(2m)!}{(2m-2n)!}a^{2n} \times z^{-2m-1}(z^2-a^2)^{m-n}T_{m-n}\Big(\frac{z^2+a^2}{z^2-a^2}\Big) \quad [m \geq n].$$

5. $$\mathrm{D}^{2n}\Big[(a^2-z^2)^m T_{2m}\Big(\frac{a}{\sqrt{a^2-z^2}}\Big)\Big] = \frac{(2m)!}{(2m-2n)!}(a^2-z^2)^{m-n}T_{2m-2n}\Big(\frac{a}{\sqrt{a^2-z^2}}\Big) \quad [m \geq n].$$

6. $$\mathrm{D}^{2n+1}\Big[(a^2-z^2)^m T_{2m}\Big(\frac{a}{\sqrt{a^2-z^2}}\Big)\Big] = \frac{(2m)!}{(2m-2n-1)!}z(a^2-z^2)^{m-n-1}U_{2m-2n-2}\Big(\frac{a}{\sqrt{a^2-z^2}}\Big) \quad [m \geq n+1].$$

7. $$\mathrm{D}^{2n}\left[(a^2-z^2)^{m+1/2}T_{2m+1}\left(\frac{a}{\sqrt{a^2-z^2}}\right)\right] = \frac{(2m+1)!}{(2m-2n+1)!}(a^2-z^2)^{m-n+1/2}T_{2m-2n+1}\left(\frac{a}{\sqrt{a^2-z^2}}\right) \quad [m\ge n].$$

8. $$\mathrm{D}^{n}\left[(z^2-a^2)^{-m/2}\,T_m\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = -\frac{(m+n-1)!}{(m-1)!}\,(z^2-a^2)^{-(m+n)/2}\,T_{m+n}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m+n\ge 1].$$

9. $$\mathrm{D}^{2n}\left[z^{2n-2m-1}(z^2-a^2)^{m}T_{2m}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = \frac{(2m)!}{(2m-2n)!}a^{2n}z^{-2m-1}(z^2-a^2)^{m-n}T_{2m-2n}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m\ge n].$$

10. $$\mathrm{D}^{2n+1}\left[z^{2n-2m}(z^2-a^2)^{m}T_{2m}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = -\frac{(2m)!}{(2m-2n-1)!}a^{2n+2}z^{-2m-1}(z^2-a^2)^{m-n-1}U_{2m-2n-2}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m\ge n+1].$$

11. $$\mathrm{D}^{2n}\left[z^{2n-2m-2}(z^2-a^2)^{m+1/2}T_{2m+1}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = \frac{(2m+1)!}{(2m-2n+1)!}a^{2n}z^{-2m-2}(z^2-a^2)^{m-n+1/2}T_{2m-2n+1}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m\ge n].$$

12. $$\mathrm{D}^{n}\left[z^{n-1/2}\,\mathrm{D}^{n}\left[(a-z)^{m}T_m\left(\frac{a+z}{a-z}\right)\right]\right] = 2^{-2n}\frac{(2m)!}{(2m-2n)!}z^{-1/2}(a-z)^{m-n}T_{m-n}\left(\frac{a+z}{a-z}\right) \quad [m\ge n].$$

13. $$\mathrm{D}^{n}\left[z^{n+1/2}\,\mathrm{D}^{n}\left[z^{n-m-1}(a-z)^{m}T_m\left(\frac{a+z}{a-z}\right)\right]\right] = \frac{(2m)!}{(2m-2n)!}\left(\frac{a}{4}\right)^{n}z^{-m-1/2}(a-z)^{m-n}T_{m-n}\left(\frac{a+z}{a-z}\right) \quad [m\ge n].$$

14. $$\mathrm{D}^{n}\left[z^{n+1/2}\,\mathrm{D}^{n}\left[z^{-1/2}(a-z)^{-m}T_m\left(\frac{a+z}{a-z}\right)\right]\right] = 2^{-2n}(2m)_{2n}(a-z)^{-m-n}T_{m+n}\left(\frac{a+z}{a-z}\right).$$

15. $$\mathrm{D}^{n}\left[z^{n-1/2}\,\mathrm{D}^{n}\left[z^{m+n-1/2}(a-z)^{-m}T_m\left(\frac{a+z}{a-z}\right)\right]\right] = 2^{-2n}(2m)_{2n}a^{n}z^{m-1}(a-z)^{-m-n}T_{m+n}\left(\frac{a+z}{a-z}\right).$$

16. $$\mathrm{D}^{n}[U_m(az)] = (2a)^{n}n!\,C_{m-n}^{n+1}(az) \quad [m\ge n].$$

17. $$\mathrm{D}^{2n}\left[z(a^2-z^2)^m U_m\left(\frac{a^2+z^2}{a^2-z^2}\right)\right] = \frac{(2m+2)!}{(2m-2n+2)!}\,z(a^2-z^2)^{m-n}U_{m-n}\left(\frac{a^2+z^2}{a^2-z^2}\right) \quad [m\ge n].$$

18. $$\mathrm{D}^{2n}\left[z^{2n-2m-2}(a^2-z^2)^m U_m\left(\frac{a^2+z^2}{a^2-z^2}\right)\right] = \frac{(2m+2)!}{(2m-2n+2)!}\,a^{2n}z^{-2m-2}(a^2-z^2)^{m-n}U_{m-n}\left(\frac{a^2+z^2}{a^2-z^2}\right) \quad [m\ge n].$$

19. $$\mathrm{D}^{2n}\left[z(a^2-z^2)^m U_{2m}\left(\frac{a}{\sqrt{a^2-z^2}}\right)\right] = \frac{(2m+1)!}{(2m-2n+1)!}\,z(a^2-z^2)^{m-n}U_{2m-2n}\left(\frac{a}{\sqrt{a^2-z^2}}\right) \quad [m\ge n].$$

20. $$\mathrm{D}^{2n}\left[z(a^2-z^2)^{m+1/2} U_{2m+1}\left(\frac{a}{\sqrt{a^2-z^2}}\right)\right] = \frac{(2m+2)!}{(2m-2n+2)!}\,z(a^2-z^2)^{m-n+1/2}U_{2m-2n+1}\left(\frac{a}{\sqrt{a^2-z^2}}\right) \quad [m\ge n].$$

21. $$\mathrm{D}^{n}\left[(z^2-a^2)^{-m/2-1}\, U_{m}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = (-1)^n\frac{(m+n)!}{m!}(z^2-a^2)^{-(m+n)/2-1}\,U_{m+n}\left(\frac{z}{\sqrt{z^2-a^2}}\right).$$

22. $$\mathrm{D}^{2n}\left[z^{2n-2m-2}(z^2-a^2)^m U_{2m}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = \frac{(2m+1)!}{(2m-2n+1)!}\,a^{2n}z^{-2m-2}(z^2-a^2)^{m-n}U_{2m-2n}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m\ge n].$$

23. $$\mathrm{D}^{2n}\left[z^{2n-2m-3}(z^2-a^2)^{m+1/2} U_{2m+1}\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right] = \frac{(2m+2)!}{(2m-2n+2)!}\,a^{2n}z^{-2m-3}(z^2-a^2)^{m-n+1/2}U_{2m-2n+1}\left(\frac{z}{\sqrt{z^2-a^2}}\right) \quad [m\ge n].$$

24. $$\mathrm{D}^{n}\left[z^{n-1/2}\,\mathrm{D}^{n}\left[z^{n-m-1}(a-z)^m U_m\left(\frac{a+z}{a-z}\right)\right]\right] = \frac{(2m+2)!}{(2m-2n+2)!}\left(\frac{a}{4}\right)^n z^{-m-3/2}(a-z)^{m-n}U_{m-n}\left(\frac{a+z}{a-z}\right) \quad [m\ge n].$$

25. $$\mathrm{D}^{n}\left[z^{n+1/2}\,\mathrm{D}^{n}\left[(a-z)^m U_m\left(\frac{a+z}{a-z}\right)\right]\right] = 2^{-2n}\frac{(2m+2)!}{(2m-2n+2)!}\,z^{1/2}(a-z)^{m-n}U_{m-n}\left(\frac{a+z}{a-z}\right) \quad [m\ge n].$$

26. $$\mathrm{D}^{n}\left[z^{n+1/2}\,\mathrm{D}^{n}\left[z^{m+n+1/2}(a-z)^{-m-2} U_m\left(\frac{a+z}{a-z}\right)\right]\right] = \frac{(2m+2n+1)!}{(2m+1)!}\left(\frac{a}{4}\right)^n z^{m+1}(a-z)^{-m-n-2}U_{m+n}\left(\frac{a+z}{a-z}\right).$$

27. $$\mathrm{D}^n\left[z^{n-1/2}\,\mathrm{D}^n\left[z^{1/2}(a-z)^{-m-2}U_m\left(\frac{a+z}{a-z}\right)\right]\right] = 2^{-2n}\frac{(2m+2n+1)!}{(2m+1)!}(a-z)^{-m-n-2}U_{m+n}\left(\frac{a+z}{a-z}\right).$$

1.20. The Hermite Polynomials $H_n(z)$

1.20.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[H_m(az)] = \frac{m!}{(m-n)!}(2a)^n H_{m-n}(az) \qquad [m \geq n].$$

2. $$\mathrm{D}^n[z^{-1/2}H_{2m}(a\sqrt{z})] = (-1)^{m+n}2^{2m}m!\left(\frac{1}{2}-m\right)_n z^{-n-1/2}L_m^{-n-1/2}(a^2z).$$

3. $$\mathrm{D}^n[H_{2m+1}(a\sqrt{z})] = (-1)^{m+n}2^{2m+1}m!\left(-\frac{1}{2}-m\right)_n az^{-n+1/2}L_m^{-n+1/2}(a^2z).$$

4. $$\mathrm{D}^n[z^{n-m-1}H_{2m}(a\sqrt{z})] = \frac{(2m)!}{(2m-2n)!}z^{-m-1}H_{2m-2n}(a\sqrt{z}) \qquad [m \geq n].$$

5. $$\mathrm{D}^n[z^{n-m-3/2}H_{2m+1}(a\sqrt{z})] = \frac{(2m+1)!}{(2m-2n+1)!}z^{-m-3/2}H_{2m-2n+1}(a\sqrt{z}) \qquad [m \geq n].$$

6. $$\mathrm{D}^n\left[z^{n-1}H_m\left(\frac{a}{z}\right)\right] = \frac{m!}{(m-n)!}(-2a)^n z^{-n-1}H_{m-n}\left(\frac{a}{z}\right) \qquad [m \geq n].$$

7. $$\mathrm{D}^n\left[z^m H_{2m}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^n\frac{(2m)!}{(2m-2n)!}z^{m-n}H_{2m-2n}\left(\frac{a}{\sqrt{z}}\right) \qquad [m \geq n].$$

8. $$\mathrm{D}^n\left[z^{m+1/2}H_{2m+1}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^n\frac{(2m+1)!}{(2m-2n+1)!}z^{m-n+1/2}H_{2m-2n+1}\left(\frac{a}{\sqrt{z}}\right) \qquad [m \geq n].$$

9. $$\mathrm{D}^n\left[z^{n-1/2}H_{2m}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^m 2^{2m}m!\left(\frac{1}{2}-m\right)_n z^{-1/2}L_m^{-n-1/2}\left(\frac{a^2}{z}\right).$$

10. $$\mathrm{D}^n\left[z^{n-1}H_{2m+1}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^m 2^{2m+1}m!\left(-\frac{1}{2}-m\right)_n az^{-3/2}L_m^{-n+1/2}\left(\frac{a^2}{z}\right).$$

11. $$\mathrm{D}^n\left[e^{-a^2z^2}H_m(az)\right] = (-a)^n e^{-a^2z^2}H_{m+n}(az).$$

12. $$\mathrm{D}^n\left[z^{n-1}e^{-a^2/z^2}H_m\left(\frac{a}{z}\right)\right] = a^n z^{-n-1}e^{-a^2/z^2}H_{m+n}\left(\frac{a}{z}\right).$$

13. $$D^n\left[z^{-1/2}e^{-a^2z}H_{2m}(a\sqrt{z})\right] = (-1)^m 2^{2m}(m+n)!\,z^{-n-1/2}e^{-a^2z}L_{m+n}^{-n-1/2}(a^2z).$$

14. $$D^n\left[e^{-a^2z}H_{2m+1}(a\sqrt{z})\right] = (-1)^m 2^{2m+1}(m+n)!\,az^{-n+1/2}e^{-a^2z}L_{m+n}^{-n+1/2}(a^2z).$$

15. $$D^n\left[z^{m+n-1/2}e^{-a^2z}H_{2m}(a\sqrt{z})\right] = \frac{(-1)^n}{2^{2n}}z^{m-1/2}e^{-a^2z}H_{2m+2n}(a\sqrt{z}).$$

16. $$D^n\left[z^{m+n}e^{-a^2z}H_{2m+1}(a\sqrt{z})\right] = \frac{(-1)^n}{2^{2n}}z^{m}e^{-a^2z}H_{2m+2n+1}(a\sqrt{z}).$$

17. $$D^n\left[z^{n-1/2}e^{-a^2/z}H_{2m}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^{m+n}2^{2m}(m+n)!\,z^{-1/2}e^{-a^2/z}L_{m+n}^{-n-1/2}\left(\frac{a^2}{z}\right).$$

18. $$D^n\left[z^{n-1}e^{-a^2/z}H_{2m+1}\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^{m+n}2^{2m+1}(m+n)!\,az^{-3/2}e^{-a^2/z}L_{m+n}^{-n+1/2}\left(\frac{a^2}{z}\right).$$

19. $$D^n\left[z^{-m-1/2}e^{-a^2/z}H_{2m}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{1}{2^{2n}}z^{-m-n-1/2}e^{-a^2/z}H_{2m+2n}\left(\frac{a}{\sqrt{z}}\right).$$

20. $$D^n\left[z^{-m-1}e^{-a^2/z}H_{2m+1}\left(\frac{a}{\sqrt{z}}\right)\right] = \frac{1}{2^{2n}}z^{-m-n-1}e^{-a^2/z}H_{2m+2n+1}\left(\frac{a}{\sqrt{z}}\right).$$

1.21. The Laguerre Polynomials $L_n^\lambda(z)$

1.21.1. Derivatives with respect to the argument

1. $D\left[L_n^\lambda(z)\right] = \dfrac{z-\lambda-n-1}{z}L_n^\lambda(z) + \dfrac{n+1}{z}L_{n+1}^\lambda(z).$

2. $\qquad = L_n^\lambda(z) - L_n^{\lambda+1}(z).$ [54].

3. $D^n[L_m^\lambda(az)] = (-a)^n L_{m-n}^{\lambda+n}(az)$ $[m \ge n]$.

4. $D^n[z^\lambda L_m^\lambda(az)] = (-1)^n(-\lambda-m)_n z^{\lambda-n}L_m^{\lambda-n}(az).$

5. $D^n[z^{n-m-1}L_m^\lambda(az)] = (-\lambda-m)_n z^{-m-1}L_{m-n}^\lambda(az)$ $[m \ge n]$.

6. $D^n\left[z^{n-1}L_m\left(\dfrac{a}{z}\right)\right] = a^n z^{-n-1}L_{m-n}^n\left(\dfrac{a}{z}\right)$ $[m \ge n]$.

7. $\mathrm{D}^n\left[z^{n-\lambda-1}L_m^{\lambda}\left(\frac{a}{z}\right)\right] = (-\lambda-m)_n z^{-\lambda-1}L_m^{\lambda-n}\left(\frac{a}{z}\right).$

8. $\mathrm{D}^n\left[z^{m}L_m^{\lambda}\left(\frac{a}{z}\right)\right] = (-1)^n(-\lambda-m)_n z^{m-n}L_{m-n}^{\lambda}\left(\frac{a}{z}\right)$ $[m \geq n].$

9. $\mathrm{D}^n[e^{-az}L_m^{\lambda}(az)] = (-a)^n e^{-az}L_m^{\lambda+n}(az).$

10. $\mathrm{D}^n[z^{\lambda}e^{-az}L_m^{\lambda}(az)] = \frac{(m+n)!}{m!}z^{\lambda-n}e^{-az}L_{m+n}^{\lambda-n}(az).$

11. $\mathrm{D}^n[z^{\lambda+m+n}e^{-az}L_m^{\lambda}(az)] = \frac{(m+n)!}{m!}z^{\lambda+m}e^{-az}L_{m+n}^{\lambda}(az).$

12. $\mathrm{D}^n[z^{n-1}e^{-a/z}L_m^{\lambda}(\frac{a}{z})] = a^n e^{-a/z}L_m^{\lambda+n}\left(\frac{a}{z}\right).$

13. $\mathrm{D}^n[z^{n-\lambda-1}e^{-a/z}L_m^{\lambda}\left(\frac{a}{z}\right)] = (-1)^n\frac{(m+n)!}{m!}z^{-\lambda-1}e^{-a/z}L_{m+n}^{\lambda-n}\left(\frac{a}{z}\right).$

14. $$\mathrm{D}^n[z^{-\lambda-m-1}e^{-a/z}L_m^{\lambda}\left(\frac{a}{z}\right)] = (-1)^n\frac{(m+n)!}{m!}z^{-\lambda-m-n-1}e^{-a/z}L_{m+n}^{\lambda}\left(\frac{a}{z}\right).$$

1.21.2. Derivatives with respect to the parameter

1. $\frac{\partial L_n^{\lambda}(z)}{\partial\lambda} = \sum_{k=0}^{n-1}\frac{1}{n-k}L_k^{\lambda}(z)$ [54].

1.22. The Gegenbauer Polynomials $C_n^{\lambda}(z)$

1.22.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[C_m^{\lambda}(az)] = (2a)^n(\lambda)_n C_{m-n}^{\lambda+n}(az)$ $[m \geq n].$

2. $\mathrm{D}^n[z^{\lambda+m+n-1}C_{2m}^{\lambda}(a\sqrt{z})] = (\lambda)_n z^{\lambda+m-1}C_{2m}^{\lambda+n}(a\sqrt{z}).$

3. $\mathrm{D}^n[z^{\lambda+m+n-1/2}C_{2m+1}^{\lambda}(a\sqrt{z})] = (\lambda)_n z^{\lambda+m-1/2}C_{2m+1}^{\lambda+n}(a\sqrt{z}).$

4. $\mathrm{D}^n[z^{n-m-1}C_{2m}^{\lambda}(a\sqrt{z})] = (\lambda)_n z^{-m-1}C_{2m-2n}^{\lambda+n}(a\sqrt{z})$ $[m \geq n].$

5. $\mathrm{D}^n[z^{n-m-3/2}C_{2m+1}^{\lambda}(a\sqrt{z})] = (\lambda)_n z^{-m-3/2}C_{2m-2n+1}^{\lambda+n}(a\sqrt{z})$ $[m \geq n].$

6. $$\mathrm{D}^n[z^{m+n-1/2}(1-a^2z)^{\lambda-1/2}C_{2m}^{\lambda}(a\sqrt{z})] = 2^{-n}\frac{(m+n)!}{m!}\frac{(2m+2n-1)!!}{(2m-1)!!} \times\frac{1}{(1-\lambda)_n}z^{m-1/2}(1-a^2z)^{\lambda-n-1/2}C_{2m+2n}^{\lambda-n}(a\sqrt{z}).$$

7. $$\mathrm{D}^n[z^{m+n}(1-a^2z)^{\lambda-1/2}C_{2m+1}^{\lambda}(a\sqrt{z})] = 2^{-n}\frac{(m+n)!}{m!}\frac{(2m+2n+1)!!}{(2m+1)!!} \times\frac{1}{(1-\lambda)_n}z^{m}(1-a^2z)^{\lambda-n-1/2}C_{2m+2n+1}^{\lambda-n}(a\sqrt{z}).$$

8. $$\mathrm{D}^n\left[z^{n-1}C_m^\lambda\left(\frac{a}{z}\right)\right] = (-2a)^n(\lambda)_n z^{-n-1}C_{m-n}^{\lambda+n}\left(\frac{a}{z}\right) \qquad [m \geq n].$$

9. $$\mathrm{D}^n\left[z^{n-2\lambda}(a-2z)^{\lambda-1/2}C_m^\lambda\left(1-\frac{a}{z}\right)\right]$$
$$= 2^{-n}\frac{(m+n)!}{m!}\frac{(1-2\lambda-m)_n}{(1-\lambda)_n}z^{n-2\lambda}(a-2z)^{\lambda-n-1/2}C_{m+n}^{\lambda-n}\left(1-\frac{a}{z}\right).$$

10. $$\mathrm{D}^n\left[(a-z)^{n-1}C_m^\lambda\left(\frac{a+z}{a-z}\right)\right] = 2^{2n}(\lambda)_n a^n(a-z)^{-n-1}C_{m-n}^{\lambda+n}\left(\frac{a+z}{a-z}\right)] \qquad [m \geq n].$$

11. $$\mathrm{D}^n\left[z^{\lambda-1/2}(a-z)^{n-2\lambda}C_m^\lambda\left(\frac{a+z}{a-z}\right)\right]$$
$$= (-1)^n\frac{(m+n)!}{m!}\frac{(2\lambda)_m\left(\frac{1}{2}-\lambda\right)_n}{(2\lambda-2n)_{m+n}}z^{\lambda-n-1/2}(a-z)^{n-2\lambda}C_{m+n}^{\lambda-n}\left(\frac{a+z}{a-z}\right).$$

12. $$\mathrm{D}^n\left[z^m C_{2m}^\lambda\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^m(\lambda)_n z^{m-n}C_{2m-2n}^{\lambda+n}\left(\frac{a}{\sqrt{z}}\right) \qquad [m \geq n].$$

13. $$\mathrm{D}^n\left[z^{m+1/2}C_{2m+1}^\lambda\left(\frac{a}{\sqrt{z}}\right)\right] = (-1)^n(\lambda)_n z^{m-n+1/2}C_{2m-2n+1}^{\lambda+n}\left(\frac{a}{\sqrt{z}}\right) \qquad [m \geq n].$$

14. $$\mathrm{D}^n\left[z^{-\lambda-m}(z-a^2)^{\lambda-1/2}C_{2m}^\lambda\left(\frac{a}{\sqrt{z}}\right)\right]$$
$$= (-2)^{-n}\frac{(m+n)!}{m!}\frac{(2m+2n-1)!!}{(2m-1)!!}\frac{1}{(1-\lambda)_n}z^{-m-\lambda}$$
$$\times (z-a^2)^{\lambda-n-1/2}C_{2m+2n}^{\lambda-n}\left(\frac{a}{\sqrt{z}}\right).$$

15. $$\mathrm{D}^n\left[z^{-\lambda-m-1/2}(z-a^2)^{\lambda-1/2}C_{2m+1}^\lambda\left(\frac{a}{\sqrt{z}}\right)\right]$$
$$= (-2)^{-n}\frac{(m+n)!}{m!}\frac{(2m+2n+1)!!}{(2m+1)!!}\frac{1}{(1-\lambda)_n}z^{-\lambda-m-1/2}$$
$$\times (a^2-z)^{\lambda-n-1/2}C_{2m+2n+1}^{\lambda-n}\left(\frac{a}{\sqrt{z}}\right).$$

16. $$\mathrm{D}^n\left[z^{m/2}(a-z)^{n-m-1}C_m^\lambda\left(\frac{z+a}{2\sqrt{az}}\right)\right]$$
$$= \frac{(\lambda)_m(1-2\lambda-m)_n}{(1-\lambda-m)_n(\lambda)_{m-n}}a^{n/2}z^{(m-n)/2}(a-z)^{-m-1}C_{m-n}^\lambda\left(\frac{z+a}{2\sqrt{az}}\right) \qquad [m \geq n].$$

17. $$\mathrm{D}^n\left[z^{-\lambda-m/2}(a-z)^{2\lambda+m+n-1}C_m^\lambda\left(\frac{z+a}{2\sqrt{az}}\right)\right]$$
$$= (-1)^n\frac{(m+n)!}{m!}a^{n/2}z^{-\lambda-(m+n)/2}(a-z)^{2\lambda+m-1}C_{m+n}^\lambda\left(\frac{z+a}{2\sqrt{az}}\right).$$

18. $$\mathrm{D}^n\left[\left(z^2-a^2\right)^{-\lambda-m/2}C_m^\lambda\left(\frac{z}{\sqrt{z^2-a^2}}\right)\right]$$
$$= (-1)^n\frac{(m+n)!}{m!}\left(z^2-a^2\right)^{-\lambda-(m+n)/2}C_{m+n}^\lambda\left(\frac{z}{\sqrt{z^2-a^2}}\right).$$

1.22.2. Derivatives with respect to the parameter

1. $$\frac{\partial C_n^\lambda(z)}{\partial\lambda} = [\psi(n+\lambda)-\psi(\lambda)]\,C_n^\lambda(z) + \sum_{k=1}^{[n/2]} \frac{\lambda+n-2k}{k(\lambda+n-k)}\,C_{n-2k}^\lambda(z)$$ [[77], (50)].

2. $$= \left[\psi\left(\lambda+\frac{1}{2}\right)-\psi\left(\lambda+n+\frac{1}{2}\right)-2\psi(2\lambda)+2\psi(2\lambda+2n)\right]C_n^\lambda(z) + 2\sum_{k=0}^{n-1}\frac{\left[1+(-1)^{n-k}\right](k+\lambda)}{(n-k)(2\lambda+k+n)}C_k^\lambda(z)$$ [54].

3. $$= 2[\psi(n+2\lambda)-\psi(2\lambda)]\,C_n^\lambda(z) - \sum_{k=1}^{[n/2]}\frac{(\lambda)_{2k}}{(2\lambda+n)_{2k}}\left(\frac{1}{k}+\frac{2}{2k+2\lambda-1}\right)2^{2k}(z^2-1)^k C_{n-2k}^{\lambda+2k}(z)$$ [[77], (52)].

4. $$\frac{\partial}{\partial\lambda}\left[\frac{C_n^\lambda(z)}{(\lambda)_n}\right] = \frac{1}{(\lambda)_n}\sum_{k=1}^{[n/2]}\frac{\lambda+n-2k}{k(\lambda+n-k)}\,C_{n-2k}^\lambda(z)$$ [[77], (49)].

1.23. The Jacobi Polynomials $P_n^{(\rho,\sigma)}(z)$

1.23.1. Derivatives with respect to the argument

1. $$\mathrm{D}\left[P_n^{(\rho,\sigma)}(z)\right] = \frac{\rho+\sigma+n+1}{1+z}\left[P_n^{(\rho+1,\sigma)}(z)-P_n^{(\rho,\sigma)}(z)\right].$$

2. $$= -\frac{2(\rho+n)}{1-z^2}P_n^{(\rho-1,\sigma)}(z) + \frac{2\rho+n-nz}{1-z^2}P_n^{(\rho,\sigma)}(z).$$

3. $$= \frac{\rho+\sigma+n+1}{1-z}\left[P_n^{(\rho,\sigma)}(z)-P_n^{(\rho,\sigma+1)}(z)\right].$$

4. $$= \frac{2(\sigma+n)}{1-z^2}P_n^{(\rho,\sigma-1)}(z) - \frac{2\sigma+n+nz}{1-z^2}P_n^{(\rho,\sigma)}(z)$$ [54].

5. $$\mathrm{D}^n\left[P_m^{(\rho,\sigma)}(az)\right] = (\rho+\sigma+m+1)_n\left(\frac{a}{2}\right)^n P_{m-n}^{(\rho+n,\sigma+n)}(az)$$ $[m \geq n]$.

6. $$\mathrm{D}^n\left[(1+az)^\sigma P_m^{(\rho,\sigma)}(az)\right] = (-a)^n(-\sigma-m)_n(1+az)^{\sigma-n}P_m^{(\rho+n,\sigma-n)}(az).$$

7. $$\mathrm{D}^n\left[(1+az)^{n-m-1}P_m^{(\rho,\sigma)}(az)\right] = (-a)^n(-\sigma-m)_n(1+az)^{-m-1}P_{m-n}^{(\rho+n,\sigma)}(az)$$ $[m \geq n]$.

8. $$\mathrm{D}^n\left[(1+az)^{\rho+\sigma+m+n}P_m^{(\rho,\sigma)}(az)\right] = a^n(\rho+\sigma+m+1)_n(1+az)^{\rho+\sigma+m}P_m^{(\rho+n,\sigma)}(az).$$

9. $$\mathrm{D}^n\left[(1-az)^\rho(1+az)^\sigma P_m^{(\rho,\sigma)}(az)\right] = (-2a)^n \frac{(m+n)!}{m!}(1-az)^{\rho-n}(1+az)^{\sigma-n} P_{m+n}^{(\rho-n,\sigma-n)}(az).$$

10. $$\mathrm{D}^n\left[(1-az)^{m+n+\rho}(1+az)^\sigma P_m^{(\rho,\sigma)}(az)\right] = (-2a)^n \frac{(m+n)!}{m!}(1-az)^{\rho+m}(1+az)^{\sigma-n} P_{m+n}^{(\rho,\sigma-n)}(az).$$

11. $$\mathrm{D}^n\left[(az-1)^{2\rho+n} P_n^{(\rho,-n-1/2)}(az)\right] = n!\,\frac{(\rho+1)_n}{(2\rho+1)_n}\,a^n(az-1)^{2\rho}\left[C_n^{\rho+1/2}\left(\sqrt{\frac{az+1}{2}}\right)\right]^2.$$

12. $$\mathrm{D}^{2n}\left[z^{2\rho+1} P_m^{(\rho,1/2-m-n)}(1+az^2)\right] = \frac{(\rho+1)_m(-2\rho-1)_{2n}}{(\rho-n+1)_m}\,z^{2\rho-2n+1} P_m^{(\rho-n,1/2-m+n)}(1+az^2).$$

13. $$\mathrm{D}^n\left[z^{n-1} P_m^{(\rho,\sigma)}\left(\frac{a}{z}\right)\right] = (\rho+\sigma+n+1)_n\left(-\frac{a}{2}\right)^n z^{-n-1} P_{m-n}^{(\rho+n,\sigma+n)}\left(\frac{a}{z}\right) \quad [m \geq n].$$

14. $$\mathrm{D}^n[z^{n-\sigma-1}(z+a)^\sigma P_m^{(\rho,\sigma)}\left(\frac{a}{z}\right)] = a^n(-\sigma-m)_n z^{-\sigma-1}(z+a)^{\sigma-n} P_m^{(\rho+n,\sigma-n)}\left(\frac{a}{z}\right).$$

15. $$\mathrm{D}^n\left[z^m(z+a)^{n-m-1} P_m^{(\rho,\sigma)}\left(\frac{a}{z}\right)\right] = a^n(-\sigma-m)_n z^{m-n}(z+a)^{-m-1} P_{m-n}^{(\rho+n,\sigma)}\left(\frac{a}{z}\right) \quad [m \geq n].$$

16. $$\mathrm{D}^n\left[z^{-\rho-\sigma-m-1}(z+a)^{\rho+\sigma+m+n} P_m^{(\rho,\sigma)}\left(\frac{a}{z}\right)\right] = (-a)^n(\rho+\sigma+m+1)_n z^{-\rho-\sigma-m-n-1}(z+a)^{\rho+\sigma+m} P_m^{(\rho+n,\sigma)}\left(\frac{a}{z}\right).$$

17. $$\mathrm{D}^n\left[z^{-\rho-\sigma+n-1}(z-a)^\rho(z+a)^\sigma P_m^{(\rho,\sigma)}\left(\frac{a}{z}\right)\right] = (2a)^n\frac{(m+n)!}{m!} z^{-\rho-\sigma+n-1}(z-a)^{\rho-n}(z+a)^{\sigma-n} P_{m+n}^{(\rho-n,\sigma-n)}\left(\frac{a}{z}\right).$$

18. $$\mathrm{D}^n\left[z^{-\rho-\sigma-m-1}(z-a)^{m+n+\rho}(z+a)^\sigma P_m^{(\rho,\sigma)}\left(\frac{a}{z}\right)\right] = (2a)^n\frac{(m+n)!}{m!} z^{-\rho-\sigma-m-1}(z-a)^{\rho+m}(z+a)^{\sigma-n} P_{m+n}^{(\rho,\sigma-n)}\left(\frac{a}{z}\right).$$

19. $$\mathrm{D}^n\left[z^{n-1} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = \left(\frac{a}{2}\right)^n(\rho+\sigma+m+1)_n z^{-n-1} P_{m-n}^{(\rho+n,\sigma+n)}\left(1-\frac{a}{z}\right) \quad [m \geq n].$$

20. $$\mathrm{D}^n\left[z^m P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = (-1)^n(-\rho-m)_n z^{m-n} P_{m-n}^{(\rho,\sigma+n)}\left(1-\frac{a}{z}\right) \quad [m \geq n].$$

21. $$\mathrm{D}^n\left[z^{-\rho-\sigma-m-1} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = (-1)^n(\rho+\sigma+m+1)_n z^{-\rho-\sigma-m-n-1} P_m^{(\rho,\sigma+n)}\left(1-\frac{a}{z}\right).$$

22. $$\mathrm{D}^n\left[z^{n-\rho-1} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = (-1)^n(\rho+m-n+1)_n z^{-\rho-1} P_m^{(\rho-n,\sigma+n)}\left(1-\frac{a}{z}\right).$$

23. $$\mathrm{D}^n\left[z^m (a-2z)^{\sigma} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = 2^n(-\sigma-m)_n z^m (a-2z)^{\sigma-n} P_m^{(\rho,\sigma-n)}\left(1-\frac{a}{z}\right).$$

24. $$\mathrm{D}^n\left[z^m (a-2z)^{n-m-1} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = a^n(-\sigma-m)_n z^{m-n} (a-2z)^{-m-1} P_{m-n}^{(\rho+n,\sigma)}\left(1-\frac{a}{z}\right) \quad [m \geq n].$$

25. $$\mathrm{D}^n\left[z^m (a-2z)^{n-m-\rho-1} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = (-2)^n(-\rho-m)_n z^m (a-2z)^{-\rho-m-1} P_m^{(\rho-n,\sigma)}\left(1-\frac{a}{z}\right).$$

26. $$\mathrm{D}^n\left[z^{-\rho-\sigma-m-1} (a-2z)^{\sigma} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = 2^n \frac{(m+n)!}{m!} z^{-\rho-\sigma-m-1} (a-2z)^{\sigma-n} P_{m+n}^{(\rho,\sigma-n)}\left(1-\frac{a}{z}\right).$$

27. $$\mathrm{D}^n\left[z^{n-\rho-\sigma-1} (a-2z)^{\sigma} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = 2^n \frac{(m+n)!}{m!} z^{n-\rho-\sigma-1} (a-2z)^{\sigma-n} P_{m+n}^{(\rho-n,\sigma-n)}\left(1-\frac{a}{z}\right).$$

28. $$\mathrm{D}^n\left[z^{-\rho-\sigma-m-1} (a-2z)^{\sigma+m+n} P_m^{(\rho,\sigma)}\left(1-\frac{a}{z}\right)\right] = 2^n \frac{(m+n)!}{m!} z^{-\rho-\sigma-m-1} (a-2z)^{\sigma+m} P_{m+n}^{(\rho-n,\sigma)}\left(1-\frac{a}{z}\right).$$

29. $$\mathrm{D}^n\left[z^{2\rho+n} (a-z)^n P_n^{(\rho,-\rho-n-1/2)}\left(\frac{a+z}{a-z}\right)\right] = n!\,\frac{(\rho+1)_n}{(2\rho+1)_n} a^n z^{2\rho} \left[C_n^{\rho+1/2}\left(\sqrt{1-\frac{z}{a}}\right)\right]^2.$$

30. $$\mathrm{D}^n\left[z^{-2\rho-n-1} (z-a)^n P_n^{(\rho,-\rho-n-1/2)}\left(\frac{z+a}{z-a}\right)\right] = (-1)^n n!\,\frac{(\rho+1)_n}{(2\rho+1)_n} z^{-2\rho-n-1} \left[C_n^{\rho+1/2}\left(\sqrt{1-\frac{a}{z}}\right)\right]^2.$$

1.23.2. Derivatives with respect to parameters

1. $$\frac{\partial P_n^{(\rho,\sigma)}(z)}{\partial\rho} = [\psi(\rho+\sigma+2n+1)-\psi(\rho+\sigma+n+1)]\,P_n^{(\rho,\sigma)}(z) + \sum_{k=0}^{n-1}\frac{\rho+\sigma+2k+1}{(n-k)\,(\rho+\sigma+k+n+1)}\,\frac{(\sigma+k+1)_{n-k}}{(\rho+\sigma+k+1)_{n-k}}\,P_k^{(\rho,\sigma)}(z) \quad [54].$$

2. $$\frac{\partial P_n^{(\rho,\sigma)}(z)}{\partial\sigma} = [\psi(\rho+\sigma+2n+1)-\psi(\rho+\sigma+n+1)]\,P_n^{(\rho,\sigma)}(z) + \sum_{k=0}^{n-1}(-1)^{n-k}\frac{\rho+\sigma+2k+1}{(n-k)(\rho+\sigma+k+n+1)}\,\frac{(\rho+k+1)_{n-k}}{(\rho+\sigma+k+1)_{n-k}}\,P_k^{(\rho,\sigma)}(z) \quad [54].$$

3. $$\frac{\partial}{\partial\rho}\left[\frac{P_n^{(\rho,\sigma)}(z)}{(\rho+1)_n}\right] = \frac{1}{(\rho+1)_n}\sum_{k=1}^{n}(-1)^{k+1}\frac{(-\sigma-n)_k}{(\rho+n+1)_k}\left(\frac{1}{k}+\frac{1}{\rho+k}\right)\left(\frac{z-1}{2}\right)^k P_{n-k}^{(\rho+2k,\sigma)}(z)$$
[[77], (51)].

4. $$\frac{\partial}{\partial\rho}\left[\frac{P_n^{(\rho,\sigma)}(z)}{(\rho+\sigma+n+1)_n}\right] = \frac{(\sigma+1)_n}{(\rho+\sigma+1)_{2n}}\sum_{k=0}^{n-1}\frac{\rho+\sigma+2k+1}{(n-k)\,(\rho+\sigma+k+n+1)}\,\frac{(\rho+\sigma+1)_k}{(\sigma+1)_k}\,P_k^{(\rho,\sigma)}(z)$$
[[77], (48)].

1.24. The Complete Elliptic Integrals $\mathbf{K}(z)$, $\mathbf{E}(z)$ and $\mathbf{D}(z)$

1.24.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n[z^n(1-a^2z)^n\,\mathrm{D}^n[\mathbf{K}(a\sqrt{z})]] = \left(\frac{1}{2}\right)_n^2 a^{2n}\,\mathbf{K}(a\sqrt{z}).$$

2. $$\mathrm{D}^n\left[(1-a^2z)^n\,\mathrm{D}^n\left[z^{n-1/2}\,\mathbf{K}(a\sqrt{z})\right]\right] = (-1)^n\left(\frac{1}{2}\right)_n^2 z^{-n-1/2}\,\mathbf{K}(a\sqrt{z}).$$

3. $$\mathrm{D}^n\left[z^n\,\mathrm{D}^n\left[(1-a^2z)^{n-1/2}\,\mathbf{K}(a\sqrt{z})\right]\right] = (-1)^n\left(\frac{1}{2}\right)_n^2 a^{2n}(1-a^2z)^{-n-1/2}\,\mathbf{K}(a\sqrt{z}).$$

4. $$\mathrm{D}^n[z^n(1-a^2z)^{n-1}\,\mathrm{D}^n[\mathbf{E}(a\sqrt{z})]] = \left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\frac{a^{2n}}{1-a^2z}\,\mathbf{E}(a\sqrt{z}).$$

5. $$\mathrm{D}^n\left[z^2(1-a^2z)^{n-1}\,\mathrm{D}^n\left[z^{n-3/2}\,\mathbf{E}(a\sqrt{z})\right]\right] = (-1)^n\left(-\frac{1}{2}\right)_n^2\frac{z^{1/2-n}}{1-a^2z}\,\mathbf{E}(a\sqrt{z}).$$

6. $$\mathrm{D}^n\left[z^n(1-a^2z)^{-1}\,\mathrm{D}^n\left[(1-a^2z)^{n-1/2}\,\mathbf{E}\left(a\sqrt{z}\right)\right]\right] = (-1)^n\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n a^{2n}(1-a^2z)^{-n-3/2}\,\mathbf{E}\left(a\sqrt{z}\right).$$

7. $$\mathrm{D}^n\left[z^n(1-a^2z)\,\mathrm{D}^n\left[(1-a^2z)^{n-3/2}\,\mathbf{E}\left(a\sqrt{z}\right)\right]\right] = (-1)^n\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n a^{2n}(1-a^2z)^{-n-1/2}\,\mathbf{E}\left(a\sqrt{z}\right).$$

8. $$\mathrm{D}^n[z^n(1-a^2z)^{n+1}\,\mathrm{D}^n[(1-a^2z)^{-1}\,\mathbf{E}\left(a\sqrt{z}\right)]] = \left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n a^{2n}\,\mathbf{E}\left(a\sqrt{z}\right).$$

9. $$\mathrm{D}^n\left[(1-a^2z)^{n+1}\,\mathrm{D}^n\left[z^{n-1/2}(1-a^2z)^{-1}\,\mathbf{E}\left(a\sqrt{z}\right)\right]\right] = (-1)^n\left(\frac{1}{2}\right)_n^2 z^{-n-1/2}\,\mathbf{E}\left(a\sqrt{z}\right).$$

10. $$\mathrm{D}^n[z^{n+1}(1-a^2z)^n\,\mathrm{D}^n[\mathbf{D}\left(a\sqrt{z}\right)]] = \left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n a^{2n}z\,\mathbf{D}\left(a\sqrt{z}\right).$$

11. $$\mathrm{D}^n[z^{n-1}(1-a^2z)^n\,\mathrm{D}^n[z\,\mathbf{D}\left(a\sqrt{z}\right)]] = \left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n a^{2n}\,\mathbf{D}\left(a\sqrt{z}\right).$$

12. $$\mathrm{D}^n\left[z(1-a^2z)^n\,\mathrm{D}^n\left[z^{n-1/2}\,\mathbf{D}\left(a\sqrt{z}\right)\right]\right] = (-1)^n\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n a^{2n}z^{1/2-n}\,\mathbf{D}\left(a\sqrt{z}\right).$$

13. $$\mathrm{D}^n\left[z^{-1}(1-a^2z)^n\,\mathrm{D}^n\left[z^{n+1/2}\,\mathbf{D}\left(a\sqrt{z}\right)\right]\right] = (-1)^n\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n z^{-n-1/2}\,\mathbf{D}\left(a\sqrt{z}\right).$$

14. $$\mathrm{D}^n\left[z^{n+1}\,\mathrm{D}^n\left[(1-a^2z)^{n-1/2}\,\mathbf{D}\left(a\sqrt{z}\right)\right]\right] = (-1)^n\left(\frac{1}{2}\right)_n^2 a^{2n}z\,(1-a^2z)^{-n-1/2}\,\mathbf{D}\left(a\sqrt{z}\right).$$

1.25. The Legendre Function $P_\nu^\mu(z)$

1.25.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n\left[P_\nu^\mu\left(\frac{z}{a}\right)\right] = n!\,(-2a)^n(a^2-z^2)^{-n}\sum_{k=0}^{n}\frac{(2a)^{-k}}{k!}\,\frac{\left(\frac{\mu}{2}\right)_{n-k}}{(n-k+\mu)_{n-k}} \times(\nu-\mu+1)_k(-\mu-\nu)_k(a^2-z^2)^{k/2}C_{n-k}^{k-n+(1-\mu)/2}\left(\frac{z}{a}\right)P_\nu^{\mu-k}\left(\frac{z}{a}\right).$$

2. $$\mathrm{D}^n\left[P_\nu^\mu\left(\frac{z}{a}\right)\right] = n!\,(-2a)^n(a^2-z^2)^{-n}\sum_{k=0}^{n}\frac{(2a)^{-k}}{k!}\,\frac{\left(-\frac{\mu}{2}\right)_{n-k}}{(n-k-\mu)_{n-k}} \times(a^2-z^2)^{k/2}C_{n-k}^{k-n+(\mu+1)/2}\left(\frac{z}{a}\right)P_\nu^{\mu+k}\left(\frac{z}{a}\right).$$

3. $$\mathrm{D}^n\left[(a^2-z^2)^{\mu/2}P_\nu^\mu\left(\frac{z}{a}\right)\right] = (-1)^n(\nu-\mu+1)_n(-\mu-\nu)_n(a^2-z^2)^{(\mu-n)/2}P_\nu^{\mu-n}\left(\frac{z}{a}\right).$$

4. $$\mathrm{D}^n\left[(a^2-z^2)^{-\mu/2}P_\nu^\mu\left(\frac{z}{a}\right)\right] = (-1)^n(a^2-z^2)^{-(\mu+n)/2}P_\nu^{\mu+n}\left(\frac{z}{a}\right).$$

5. $$\mathrm{D}^n\left[z^{n-(\mu+\nu)/2-1}(a-z)^{\mu/2}P_\nu^\mu\left(\sqrt{\frac{z}{a}}\right)\right] = 2^{-n}a^{n/2}(-\mu-\nu)_{2n}z^{-(\mu+\nu)/2-1}(a-z)^{(\mu-n)/2}P_{\nu-n}^{\mu-n}\left(\sqrt{\frac{z}{a}}\right).$$

6. $$\mathrm{D}^n\left[z^{n+(\mu+\nu-1)/2}(a-z)^{-\mu/2}P_\nu^\mu\left(\sqrt{\frac{z}{a}}\right)\right] = (-2)^{-n}a^{n/2}z^{(\mu+\nu-1)/2}(a-z)^{-(\mu+n)/2}P_{\nu+n}^{\mu+n}\left(\sqrt{\frac{z}{a}}\right).$$

7. $$\mathrm{D}^n\left[z^{n+(\nu-\mu-1)/2}(a-z)^{\mu/2}P_\nu^\mu\left(\sqrt{\frac{z}{a}}\right)\right] = 2^{-n}a^{n/2}(\nu-\mu+1)_{2n}z^{(\nu-\mu-1)/2}(a-z)^{(\mu-n)/2}P_{\nu+n}^{\mu-n}\left(\sqrt{\frac{z}{a}}\right).$$

8. $$\mathrm{D}^n\left[z^{n+(\mu-\nu)/2-1}(a-z)^{-\mu/2}P_\nu^\mu\left(\sqrt{\frac{z}{a}}\right)\right] = (-2)^{-n}a^{n/2}z^{(\mu-\nu)/2-1}(a-z)^{-(\mu+n)/2}P_{\nu-n}^{\mu+n}\left(\sqrt{\frac{z}{a}}\right).$$

9. $$\mathrm{D}^n\left[z^{\nu/2}(z-a)^{\mu/2}P_\nu^\mu\left(\sqrt{\frac{a}{z}}\right)\right] = (-2)^{-n}\,(-\mu-\nu)_{2n}z^{(\nu-n)/2}(z-a)^{(\mu-n)/2}P_{\nu-n}^{\mu-n}\left(\sqrt{\frac{a}{z}}\right).$$

10. $$\mathrm{D}^n\left[z^{\nu/2}(z-a)^{-\mu/2}P_\nu^\mu\left(\sqrt{\frac{a}{z}}\right)\right] = 2^{-n}z^{(\nu-n)/2}(z-a)^{-(\mu+n)/2}P_{\nu-n}^{\mu+n}\left(\sqrt{\frac{a}{z}}\right).$$

11. $$\mathrm{D}^n\left[z^{-(\nu+1)/2}(z-a)^{\mu/2}P_\nu^\mu\left(\sqrt{\frac{a}{z}}\right)\right] = (-2)^{-n}(\nu-\mu+1)_{2n}z^{-(\nu+n+1)/2}(z-a)^{(\mu-n)/2}P_{\nu+n}^{\mu-n}\left(\sqrt{\frac{a}{z}}\right).$$

12. $$\mathrm{D}^n\left[z^{-(\nu+1)/2}(z-a)^{-\mu/2}P_\nu^\mu\left(\sqrt{\frac{a}{z}}\right)\right] = 2^{-n}z^{-(\nu+n+1)/2}(z-a)^{-(\mu+n)/2}P_{\nu+n}^{\mu+n}\left(\sqrt{\frac{a}{z}}\right).$$

1.25.2. Derivatives with respect to parameters

1. $$\mathrm{D}_\nu[P_\nu(z)]\big|_{\nu=n} = -\ln\frac{z+1}{2}\,P_n(z) - n!\sum_{k=1}^{n}2^{k+1}\frac{\left(\frac{1}{2}\right)_k}{(k+n)!\,k}(1-z)^kC_{n-k}^{k+1/2}(z)$$ [[73], (5.9)].

2. $\mathrm{D}_\nu[P_\nu(z)]\big|_{\nu=-n-1} = -\,\mathrm{D}_\nu[P_\nu(z)]\big|_{\nu=n}\,.$

3. $\mathrm{D}_\nu[P_\nu^\mu(z)]\big|_{\nu=n-1/2}$

$$= -\ln z\, P_{n-1/2}^\mu(z) + (-1)^n 2^{n-1}\left(\frac{2\mu-2n+1}{4}\right)_n (1-z^2)^{n/2}$$

$$\times \sum_{k=0}^{n}\binom{n}{k}\frac{z^{2k}}{\left(\frac{2\mu-2n+1}{4}\right)_k}\left[2\ln z + \psi\left(\frac{2\mu+2n+1}{4}\right) - \psi\left(\frac{2\mu-2n+4k+1}{4}\right)\right]$$

$$\times\Bigg[\delta_{k,0}P_{-1/2}^{\mu-n}(z) + (-2z)^{-k}(1-z^2)^{-k/2}$$

$$\times\sum_{p=0}^{k-1}\frac{(k+p-1)!}{p!\,(k-p-1)!}\,(2z)^{-p}\,(1-z^2)^{p/2}\,P_{-1/2}^{\mu+k-n-p}(z)\Bigg].$$

4.
$$= \left[\psi\left(\frac12-\mu-n\right) - \psi\left(\frac12-\mu+n\right) - \ln z\right] P_{n-1/2}^\mu(z)$$

$$+\frac{2^{n-1}}{\left(\frac12-\mu\right)_n\left(\frac12+\mu\right)_n}(1-z^2)^{n/2}$$

$$\times\sum_{k=0}^{n}\binom{n}{k}(-z^2)^k\left[2\left(\frac{2\mu-2n+3}{4}\right)_{n-k}\ln z - (n-k)!\sum_{p=0}^{n-k-1}\frac{\left(\frac{2\mu-2n+3}{4}\right)_p}{p!\,(n-k-p)}\right]$$

$$\times\Bigg[\delta_{k,0}P_{-1/2}^{\mu+n}(z) + (-2z)^{-k}(1-z^2)^{-k/2}\sum_{p=0}^{k-1}\frac{(k+p-1)!}{p!\,(k-p-1)!}\left(\frac12-\mu-n\right)_{k-p}^2$$

$$\times(2z)^{-p}(1-z^2)^{p/2}\,P_{-1/2}^{\mu-k+n+p}(z)\Bigg].$$

5. $\mathrm{D}_\nu[P_\nu^\mu(z)]\big|_{\nu=-n-1/2} = -\,\mathrm{D}_\nu[P_\nu^\mu(z)]\big|_{\nu=n-1/2}\,.$

6. $\mathrm{D}_\mu[P_\nu^\mu(z)]\big|_{\mu=0}$

$$= \left\{\frac{\pi\csc(\nu\pi)}{4(\nu+2)}\left[(2\nu+5)\cos(\nu\pi)+1\right]P_\nu(z) - \left[2\nu+5+\cos(\nu\pi)\right]P_\nu(-z)\right\}$$

$$+\frac{\pi\cot\frac{\nu\pi}{2}}{4(\nu+1)(\nu+2)z}\Big\{[\nu+2-(2\nu+3)z^2][P_{\nu+1}(z)+P_{\nu+1}(-z)]$$

$$+\sqrt{1-z^2}\,[P_{\nu+2}^1(z)+P_{\nu+2}^1(-z)]\Big\}.$$

7. $\mathrm{D}_\mu\left[P_{1/2}^\mu(z)\right]\Big|_{\mu=1/2} = \dfrac{(1-z^2)^{-1/4}}{\sqrt{2\pi}}\Big[-2(\mathrm{C}+\ln 2)\,z - \pi\sqrt{1-z^2}$

$$+\left(z+i\sqrt{1-z^2}\right)\ln\left(1-\frac{iz}{\sqrt{1-z^2}}\right) + \left(z+i\sqrt{1-z^2}\right)\ln\left(1-\frac{iz}{\sqrt{1-z^2}}\right)\Big]$$

$[|\arg(1\pm z)|<\pi].$

1.26. The Kummer Confluent Hypergeometric Function ${}_1F_1(a;\ b;\ z)$

1.26.1. Derivatives with respect to the argument

1. $\mathrm{D}^n\left[{}_1F_1\left(\begin{matrix} a;\ cz \\ b \end{matrix}\right)\right] = c^n \dfrac{(a)_n}{(b)_n}\, {}_1F_1\left(\begin{matrix} a+n;\ cz \\ b+n \end{matrix}\right)$ [[6], 6.4.10].

2. $\mathrm{D}^n\left[z^{a+n-1}\, {}_1F_1\left(\begin{matrix} a;\ cz \\ b \end{matrix}\right)\right] = (a)_n z^{a-1}\, {}_1F_1\left(\begin{matrix} a+n \\ b;\ cz \end{matrix}\right)$ [[6], 6.4.11].

3. $\mathrm{D}^n\left[z^{b-1}\, {}_1F_1\left(\begin{matrix} a;\ cz \\ b \end{matrix}\right)\right] = (-1)^n (1-b)_n z^{b-n-1}\, {}_1F_1\left(\begin{matrix} a;\ cz \\ b-n \end{matrix}\right)$ [[6], 6.4.12].

4. $\mathrm{D}^n\left[e^{-cz}\, {}_1F_1\left(\begin{matrix} a;\ cz \\ b \end{matrix}\right)\right] = (-c)^n \dfrac{(b-a)_n}{(b)_n} e^{-cz}\, {}_1F_1\left(\begin{matrix} a;\ cz \\ b+n \end{matrix}\right)$ [[6], 6.4.13].

5. $\mathrm{D}^n\left[z^{b-1} e^{-cz}\, {}_1F_1\left(\begin{matrix} a;\ cz \\ b \end{matrix}\right)\right] = (-1)^n (1-b)_n z^{b-n-1} e^{-cz}\, {}_1F_1\left(\begin{matrix} a-n;\ cz \\ b-n \end{matrix}\right).$

6. $\mathrm{D}^n\left[z^{b-a+n-1} e^{-cz}\, {}_1F_1\left(\begin{matrix} a;\ cz \\ b \end{matrix}\right)\right] = (b-a)_n z^{b-a-1} e^{-cz}\, {}_1F_1\left(\begin{matrix} a-n \\ b;\ cz \end{matrix}\right)$
[[6], 6.4.14].

7. $\mathrm{D}^n\left[z^{n-1}\, {}_1F_1\left(\begin{matrix} a;\ \frac{c}{z} \\ b \end{matrix}\right)\right] = (-c)^n \dfrac{(a)_n}{(b)_n} z^{-n-1}\, {}_1F_1\left(\begin{matrix} a+n;\ \frac{c}{z} \\ b+n \end{matrix}\right).$

8. $\mathrm{D}^n\left[z^{-a}\, {}_1F_1\left(\begin{matrix} a;\ \frac{c}{z} \\ b \end{matrix}\right)\right] = (-1)^n (a)_n z^{-a-n}\, {}_1F_1\left(\begin{matrix} a+n \\ b;\ \frac{c}{z} \end{matrix}\right).$

9. $\mathrm{D}^n\left[z^{n-b}\, {}_1F_1\left(\begin{matrix} a;\ \frac{c}{z} \\ b \end{matrix}\right)\right] = (1-b)_n z^{-b}\, {}_1F_1\left(\begin{matrix} a;\ \frac{c}{z} \\ b-n \end{matrix}\right).$

10. $\mathrm{D}^n\left[z^{n-1} e^{-c/z}\, {}_1F_1\left(\begin{matrix} a;\ \frac{c}{z} \\ b \end{matrix}\right)\right] = c^n \dfrac{(b-a)_n}{(b)_n} z^{-n-1} e^{-c/z}\, {}_1F_1\left(\begin{matrix} a;\ \frac{c}{z} \\ b+n \end{matrix}\right).$

11. $\mathrm{D}^n\left[z^{a-b} e^{-c/z}\, {}_1F_1\left(\begin{matrix} a;\ \frac{c}{z} \\ b \end{matrix}\right)\right] = (-1)^n (b-a)_n z^{a-b-n} e^{-c/z}\, {}_1F_1\left(\begin{matrix} a-n \\ b;\ \frac{c}{z} \end{matrix}\right).$

12. $\mathrm{D}^{2n+\sigma}\left[{}_1F_1\left(\begin{matrix} \frac{1}{2}-n \\ b;\ z^2 \end{matrix}\right)\right] = (-4)^n \dfrac{\left(\frac{1}{2}\right)_n \left(\sigma+\frac{1}{2}\right)_n}{(b)_{n+\sigma}} z^{\sigma}\, {}_1F_1\left(\begin{matrix} n+\sigma+\frac{1}{2} \\ b+n+\sigma;\ z^2 \end{matrix}\right)$
[$\sigma = 0$ or 1; [78]].

13. $\mathrm{D}^{2n+\sigma}\left[z\, {}_1F_1\left(\begin{matrix} \frac{3}{2}-\sigma-n \\ b;\ z^2 \end{matrix}\right)\right] = (-4)^n \dfrac{\left(\frac{3}{2}\right)_n \left(\sigma-\frac{1}{2}\right)_n}{(b)_n} z^{1-\sigma}\, {}_1F_1\left(\begin{matrix} n+\frac{3}{2} \\ b+n;\ z^2 \end{matrix}\right)$
[$\sigma = 0$ or 1; [78]].

14. $$\mathrm{D}^{2n+\sigma}\left[z^{2b-1}\,{}_1F_1\left(\begin{matrix} b-\sigma-n+\frac{1}{2} \\ b;\ z^2\end{matrix}\right)\right]$$
$$=(-1)^\sigma(1-2b)_{2n+\sigma}z^{2b-\sigma-2n-1}\,{}_1F_1\left(\begin{matrix} b+\frac{1}{2} \\ b-n;\ z^2\end{matrix}\right)\quad[\sigma=0\text{ or }1;\ [78]].$$

15. $$\mathrm{D}^{2n+\sigma}\left[z^{2b-2}\,{}_1F_1\left(\begin{matrix} b-n-\frac{1}{2} \\ b;\ z^2\end{matrix}\right)\right]$$
$$=(-1)^\sigma(2-2b)_{2n+\sigma}z^{2b-\sigma-2n-2}\,{}_1F_1\left(\begin{matrix} b-\frac{1}{2};\ z^2 \\ b-\sigma-n\end{matrix}\right)\quad[\sigma=0\text{ or }1;\ [78]].$$

16. $$\mathrm{D}^{2n+\sigma}\left[{}_1F_1\left(\begin{matrix} a;\ z^2 \\ \frac{1}{2}\end{matrix}\right)\right]=2^{2n+2\sigma}(a)_{n+\sigma}z^{\sigma}\,{}_1F_1\left(\begin{matrix} a+n+\sigma \\ \sigma+\frac{1}{2};\ z^2\end{matrix}\right)$$
$$[\sigma=0\text{ or }1;\ [78]].$$

17. $$\mathrm{D}^{2n+\sigma}\left[z\,{}_1F_1\left(\begin{matrix} a;\ z^2 \\ \frac{3}{2}\end{matrix}\right)\right]=2^{2n}(a)_n z^{1-\sigma}\,{}_1F_1\left(\begin{matrix} a+n;\ z^2 \\ \frac{3}{2}-\sigma\end{matrix}\right)$$
$$[\sigma=0\text{ or }1;\ [78]].$$

18. $$\mathrm{D}^{2n+\sigma}\left[e^{-z^2}\,{}_1F_1\left(\begin{matrix} a+n-\frac{1}{2} \\ a;\ z^2\end{matrix}\right)\right]$$
$$=(-1)^\sigma 2^{2n}\frac{\left(\frac{1}{2}\right)_n\left(\sigma+\frac{1}{2}\right)_n}{(a)_{n+\sigma}}z^\sigma e^{-z^2}\,{}_1F_1\left(\begin{matrix} a-\frac{1}{2};\ z^2 \\ a+n+\sigma\end{matrix}\right)\quad[\sigma=0\text{ or }1;\ [78]].$$

19. $$\mathrm{D}^{2n+\sigma}\left[ze^{-z^2}\,{}_1F_1\left(\begin{matrix} a+n+\sigma-\frac{3}{2} \\ a;\ z^2\end{matrix}\right)\right]$$
$$=2^{2n}\frac{\left(\frac{3}{2}\right)_n\left(\sigma-\frac{1}{2}\right)_n}{(a)_n}z^{1-\sigma}e^{-z^2}\,{}_1F_1\left(\begin{matrix} a-\frac{3}{2} \\ a+n;\ z^2\end{matrix}\right)\quad[\sigma=0\text{ or }1;\ [78]].$$

20. $$\mathrm{D}^{2n+\sigma}\left[z^{2a-1}e^{-z^2}\,{}_1F_1\left(\begin{matrix} n+\sigma-\frac{1}{2} \\ a;\ z^2\end{matrix}\right)\right]$$
$$=(-1)^\sigma(1-2a)_{2n+\sigma}z^{2a-2n-\sigma-1}e^{-z^2}\,{}_1F_1\left(\begin{matrix} -n-\frac{1}{2} \\ a-n;\ z^2\end{matrix}\right)\quad[\sigma=0\text{ or }1;\ [78]].$$

21. $$\mathrm{D}^{2n+\sigma}\left[z^{2a-2}e^{-z^2}\,{}_1F_1\left(\begin{matrix} n+\sigma-\frac{1}{2} \\ a;\ z^2\end{matrix}\right)\right]=(-1)^\sigma(2-2a)_{2n+\sigma}$$
$$\times z^{2a-2n-\sigma-2}e^{-z^2}\,{}_1F_1\left(\begin{matrix} \frac{1}{2}-n-\sigma \\ a-n-\sigma;\ z^2\end{matrix}\right)\quad[\sigma=0\text{ or }1;\ [78]].$$

22. $$\mathrm{D}^{2n+\sigma}\left[e^{-z^2}\,{}_1F_1\left(\begin{matrix}a;\ z^2\\ \frac{1}{2}\end{matrix}\right)\right] = (-4)^{n+\sigma}\left(\frac{1}{2}-a\right)_{n+\sigma} z^{\sigma}\,e^{-z^2}\,{}_1F_1\left(\begin{matrix}a-n;\ z^2\\ \sigma+\frac{1}{2}\end{matrix}\right) \quad [\sigma = 0 \text{ or } 1;\ [78]].$$

23. $$\mathrm{D}^{2n+\sigma}\left[ze^{-z^2}\,{}_1F_1\left(\begin{matrix}a;\ z^2\\ \frac{3}{2}\end{matrix}\right)\right] = (-4)^{n}\left(\frac{3}{2}-a\right)_{n} z^{1-\sigma}\,e^{-z^2}\,{}_1F_1\left(\begin{matrix}a-n-\sigma\\ \frac{3}{2}-\sigma;\ z^2\end{matrix}\right) \quad [\sigma = 0 \text{ or } 1;\ [78]].$$

1.26.2. Derivatives with respect to parameters

1. $$\mathrm{D}_a\left[{}_1F_1\left(\begin{matrix}a;\ z\\ n+1\end{matrix}\right)\right]\bigg|_{a=m+n+1} = \frac{m!\,n!}{(m+n)!}\,[\psi(m+1)-\psi(m+n+1)]\,e^{z}L_m^{-n}(-z)$$
$$+ n!\,z^{-n}\left\{[\mathrm{C}+2\ln z+\mathrm{shi}\,(z)-\mathrm{chi}\,(z)]\,L_{m+n}^{-n}(-z) - \sum_{k=1}^{m+n}\frac{1}{k}\,L_{m+n-k}^{k-n}(-z)\left[2(-1)^k e^{z}+L_{k-1}^{-k}(z)\right]\right\}$$
$$- \frac{m!\,(n!)^2}{(m+n)!}\,z^{-n}e^{z}\sum_{k=0}^{m}\binom{m}{k}\frac{z^k}{k!} \times\left\{[\mathrm{C}+\ln z+\psi(m+1)]\,L_n^{k-n}(-z) - \sum_{p=1}^{n}\frac{(-1)^p}{p}\,L_{n-p}^{k-n+p}(-z)\right\}.$$

2. $$\mathrm{D}_a\left[{}_1F_1\left(\begin{matrix}a;\ z\\ \frac{1}{2}-n\end{matrix}\right)\right]\bigg|_{a=m+1/2} = \frac{e^{z}}{\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_n}\left\{2(-1)^n m!\,z^{n+1}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}\right)_k(-z)^{-k}\right.$$
$$\times\sum_{p=0}^{m}\frac{(-1)^p}{p!}\left(k-\frac{1}{2}\right)_p L_{m-p}^{n+p}(-z)\,{}_2F_2\left(\begin{matrix}1,1;\ -z\\ \frac{3}{2}-k-p,\ 2\end{matrix}\right)$$
$$-(-1)^n(m+n)!\left[\psi\left(m+\frac{1}{2}\right)+\ln 2+\mathrm{C}\right]L_{m+n}^{-n-1/2}(-z)$$
$$\left.-(-1)^{m+n}\sum_{k=0}^{m-1}\frac{1}{k!\,(m-k)}\sum_{p=0}^{k}\binom{k}{p}\left(\frac{1}{2}-m\right)_{k-p}L_n^{p-n-1/2}(-z)\right\}.$$

3. $$\mathrm{D}_a\left[{}_1F_1\left(\begin{matrix}a;\ z\\ \frac{3}{2}-n\end{matrix}\right)\right]\Bigg|_{a=3/2} = \frac{(-z)^{n+1}}{\left(-\frac{3}{2}\right)_{n+1}}e^z\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{3}{2}\right)_k(-z)^{-k}\,{}_2F_2\left(\begin{matrix}1,\ 1;\ -z\\ 2,\ \frac{5}{2}-k\end{matrix}\right).$$

4. $$\mathrm{D}_a\left[{}_1F_1\left(\begin{matrix}n+1;\ z\\ a\end{matrix}\right)\right] = z^{1-a}e^zL_n^{1-a}(-z)\gamma(a-1,\ z)+\sum_{k=1}^{n}\frac{1}{k}L_{n-k}^{k-a+1}(-z)L_{k-1}^{a-k-1}(z)$$
$$-(a-1)e^z\sum_{k=0}^{n}\frac{(-z)^k}{k!\,(a+k-1)^2}L_{n-k}^{k}(-z)\,{}_2F_2\left(\begin{matrix}a+k-1,\ a+k-1\\ a+k,\ a+k;\ -z\end{matrix}\right)$$
$$[a\neq 0,\ \pm1,\ \pm2,\ \ldots].$$

5. $$\mathrm{D}_a\left[{}_1F_1\left(\begin{matrix}a;\ z\\ 2a+b\end{matrix}\right)\right]\Bigg|_{a=(n-b+1)/2} = \left[2\psi(n+1)-\psi\left(\frac{n-b+1}{2}\right)-\ln z\right]$$
$$\times\ {}_1F_1\left(\begin{matrix}\frac{n-b+1}{2}\\ n+1;\ z\end{matrix}\right)+(-1)^{n+1}n!\,z^{-n}\Gamma\left(\frac{1-b-n}{2}\right)\Psi\left(\begin{matrix}\frac{1-b-n}{2}\\ 1-n;\ z\end{matrix}\right)$$
$$+\ \frac{(-1)^n(n!)^2z^{-n}}{\Gamma\left(\frac{n-b+1}{2}\right)}\sum_{k=0}^{n-1}\frac{\Gamma\left(k+\frac{1-n-b}{2}\right)}{(k!)^2(n-k)}(-z)^k\,{}_1F_1\left(\begin{matrix}k+\frac{1-b-n}{2}\\ k+1;\ z\end{matrix}\right)\Bigg].$$

6. $$\mathrm{D}_a\left[{}_1F_1\left(\begin{matrix}a;\ -z\\ 2a+b\end{matrix}\right)\right]\Bigg|_{a=(n-b+1)/2} = \frac{1}{2}\left[2\psi(n+1)-\psi\left(\frac{n-b+1}{2}\right)-\ln z\right]$$
$$\times\ {}_1F_1\left(\begin{matrix}\frac{n-b+1}{2};\ -z\\ n+1\end{matrix}\right)+\frac{n!\,z^{-n}}{\Gamma\left(\frac{n-b+1}{2}\right)}$$
$$\times\left[\pi G_{23}^{21}\left(z\,\middle|\,\begin{matrix}\frac{b+n+1}{2},\ -\frac{1}{2}\\ 0,\ n,\ -\frac{1}{2}\end{matrix}\right)+\Gamma(2a+b)\sum_{k=0}^{n-1}\frac{\Gamma\left(k+\frac{1-n-b}{2}\right)}{(k!)^2(n-k)}z^k\right.$$
$$\left.\times\ {}_1F_1\left(\begin{matrix}k+\frac{1-b-n}{2}\\ k+1;\ -z\end{matrix}\right)\right].$$

1.27. The Tricomi Confluent Hypergeometric Function $\Psi(a;\ b;\ z)$

1.27.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n\left[\Psi\left(\begin{matrix}a;\ cz\\ b\end{matrix}\right)\right] = (-c)^n(a)_n\Psi\left(\begin{matrix}a+n;\ cz\\ b+n\end{matrix}\right) \qquad [[6],\ 6.6.11].$$

2. $$\mathrm{D}^n\left[z^{a+n-1}\Psi\left(\begin{matrix}a;\ cz\\ b\end{matrix}\right)\right] = (a)_n(a-b+1)_nz^{a-1}\Psi\left(\begin{matrix}a+n;\ cz\\ b\end{matrix}\right) \qquad [[6],\ 6.6.13].$$

3. $\mathrm{D}^n\left[z^{b-1}\Psi\left(\begin{matrix}a;\ cz\\ b\end{matrix}\right)\right] = (-1)^n(a-b+1)_n z^{b-n-1}\Psi\left(\begin{matrix}a;\ cz\\ b-n\end{matrix}\right)$ [[6], 6.6.12].

4. $\mathrm{D}^n\left[e^{-cz}\Psi\left(\begin{matrix}a;\ cz\\ b\end{matrix}\right)\right] = (-c)^n e^{-cz}\Psi\left(\begin{matrix}a;\ cz\\ b+n\end{matrix}\right)$ [[6], 6.6.14].

5. $\mathrm{D}^n\left[z^{b-a+n-1}e^{-cz}\Psi\left(\begin{matrix}a;\ cz\\ b\end{matrix}\right)\right] = (-1)^n z^{b-a-1}e^{-cz}\Psi\left(\begin{matrix}a-n;\ cz\\ b\end{matrix}\right)$ [[6], 6.6.15].

6. $\mathrm{D}^n\left[z^{n-1}\Psi\left(\begin{matrix}a;\ \frac{c}{z}\\ b\end{matrix}\right)\right] = c^n(a)_n z^{-n-1}\Psi\left(\begin{matrix}a+n;\ \frac{c}{z}\\ b+n\end{matrix}\right).$

7. $\mathrm{D}^n\left[z^{-a}\Psi\left(\begin{matrix}a;\ \frac{c}{z}\\ b\end{matrix}\right)\right] = (-1)^n(a)_n(a-b+1)_n z^{-a-n}\Psi\left(\begin{matrix}a+n;\ \frac{c}{z}\\ b\end{matrix}\right).$

8. $\mathrm{D}^n\left[z^{n-b}\Psi\left(\begin{matrix}a;\ \frac{c}{z}\\ b\end{matrix}\right)\right] = (a-b+1)_n z^{-b}\Psi\left(\begin{matrix}a;\ \frac{c}{z}\\ b-n\end{matrix}\right).$

9. $\mathrm{D}^n\left[z^{n-1}e^{-c/z}\Psi\left(\begin{matrix}a;\ \frac{c}{z}\\ b\end{matrix}\right)\right] = c^n e^{-c/z}z^{-n-1}\Psi\left(\begin{matrix}a;\ \frac{c}{z}\\ b+n\end{matrix}\right).$

10. $\mathrm{D}^n\left[z^{a-b}e^{-c/z}\Psi\left(\begin{matrix}a;\ \frac{c}{z}\\ b\end{matrix}\right)\right] = z^{a-b-n}e^{-c/z}\Psi\left(\begin{matrix}a-n;\ \frac{c}{z}\\ b\end{matrix}\right).$

1.27.2. Derivatives with respect to parameters

1. $$\mathrm{D}_a\left[\Psi\left(\begin{matrix}a;\ z\\ b+n\end{matrix}\right)\right]\Big|_{a=b} = -\left[\psi(b)+\ln z\right]\Psi\left(\begin{matrix}b;\ z\\ b+n\end{matrix}\right) + \sum_{k=0}^{n-1}\binom{n-1}{k}(b)_k\, z^{-b-k}\psi(b+k) - \frac{1}{\Gamma(b)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left\{(-1)^k\,\frac{\pi\csc(b\pi)}{b+k}\,{}_1F_1\left(\begin{matrix}b+k;\ z\\ b+k+1\end{matrix}\right) + \frac{\Gamma(b+k)}{z^{b+k}}\left[\frac{z}{b+k-1}\,{}_2F_2\left(\begin{matrix}1,\,1;\ z\\ 2,\,2-b-k\end{matrix}\right)+\psi(b+k)-\ln z\right]\right\} \quad [n\ge 1].$$

2. $$\mathrm{D}_a\left[\Psi\left(\begin{matrix}a;\ z\\ b+1\end{matrix}\right)\right]\Big|_{a=b} = -z^{-b}\psi(b) - \Gamma(1-b)(-z)^{-b}\gamma(b,\,-z) + \frac{z^{1-b}}{1-b}\,{}_2F_2\left(\begin{matrix}1,\,1;\ z\\ 2-b,\,2\end{matrix}\right).$$

3. $$\mathrm{D}_a\left[\Psi\left(\begin{matrix}a;\ z\\ n\end{matrix}\right)\right]\Big|_{a=m} = -\left[\psi(m)+\ln z\right]\Psi\left(\begin{matrix}m;\ z\\ n\end{matrix}\right) + \frac{z^{-m}}{(m-1)!}\sum_{k=0}^{n-m-1}\binom{n-m-1}{k}(k+m-1)!\,z^{-k}\psi(k+m) - \frac{(-z)^{-m}}{(m-1)!}\sum_{k=0}^{n-m-1}\binom{n-m-1}{k}(k+m-1)!\,(-z)^{-k}$$

$$\times \left\{ \sum_{p=1}^{k+m-1} \frac{1}{p} L_{p-1}^{-p}(z) L_{k+m-p-1}^{-k-m+p}(-z) - e^z [\operatorname{shi}(z) - \operatorname{chi}(z)] \, L_{k+m-1}^{-k-m}(-z) \right\}$$

$[n > m \geq 1].$

4. $$\mathrm{D}_a \left[\Psi\binom{a;\; z}{n+\frac{1}{2}} \right] \bigg|_{a=-m} = \frac{(-1)^m}{z^n} \left(-\frac{1}{2}\right)_n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\left(\frac{3}{2}-n\right)_k}$$

$$\times \left\{ (k+m)! \, L_{k+m}^{-k-1/2}(z) \right.$$

$$\times \left[2z \, {}_2F_2\left(\begin{matrix} 1, 1;\; z \\ \frac{3}{2}, 2 \end{matrix} \right) - \pi \operatorname{erfi}(\sqrt{z}) - \psi\left(\frac{1}{2} - m\right) - \psi\left(n - k - \frac{1}{2}\right) + 2 \right]$$

$$- (k+m)! \sum_{p=1}^{k+m} \frac{1}{p} L_{k+m-p}^{p-k-1/2}(z)$$

$$\times \left[\sqrt{\pi z} \, e^z L_{p-1}^{1/2-p}(-z) - \frac{z^p}{\left(\frac{1}{2}\right)_p} \, {}_1F_1\left(\begin{matrix} p;\; z \\ p+\frac{1}{2} \end{matrix} \right) \right]$$

$$\left. + (-1)^k \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_m \sum_{p=0}^{k+m} \binom{k+m}{p} \frac{(-z)^p}{\left(\frac{1}{2}-k\right)_p} \psi\left(k - p + \frac{1}{2}\right) \right\}.$$

5. $$\mathrm{D}_b \left[\Psi\binom{a;\; z}{b} \right] \bigg|_{b=a+n+1} = (-1)^n n! \, z^{-a-n}$$

$$\times \sum_{k=0}^{n} \left\{ (-1)^k [\psi(k+1) + \psi(a) + \mathbf{C} - \ln z] \right.$$

$$- \frac{z^{k+1}}{(1-a)_{k+1}(k+1)} \, {}_2F_2\left(\begin{matrix} k+1, k+1;\; z \\ k-a+2, k+2 \end{matrix} \right)$$

$$- \Gamma(-a) z^a \left[(e^z - 1)\delta_{n,0} + L_k^{a-k-1}(-z) \right.$$

$$\left. \left. + \frac{(-1)^k}{k!} \sum_{p=0}^{k} \binom{k}{p} (-a)_{k-p} (-z)^{-a} \gamma(a+p+1, -z) \right] \right\}.$$

6. $$\mathrm{D}_b \left[\Psi\binom{m+1;\; z}{b} \right] \bigg|_{b=n+3/2} = \frac{2^{2m}}{(2m)!} \left[\mathbf{C} + 2\ln 2 + \psi\left(m + \frac{1}{2}\right) \right] \sum_{k=0}^{n} \binom{n}{k} (k$$

$$+ m)! \, (-z)^{-k}$$

$$\times \left[\sqrt{\pi} \, z^{-1/2} e^z L_{k+m}^{-k-1/2}(-z) \operatorname{erfc}(\sqrt{z}) - \sum_{p=1}^{k+m} \frac{1}{p} L_{k+m-p}^{p-k-1/2}(-z) L_{p-1}^{1/2-p}(z) \right]$$

$$+\frac{2^{2m}\sqrt{\pi}}{(2m)!}z^{-1/2}e^{z}\sum_{k=0}^{n}\binom{n}{k}(k+m)!\,(-z)^{-k}$$

$$\times\left\{\frac{4z^{1/2}}{\sqrt{\pi}}\sum_{p=0}^{k+m}\frac{(-z)^{p}}{p!\,(2p+1)^{2}}L_{k+m-p}^{p-k}(-z)\,{}_2F_2\left(\begin{matrix}p+\frac{1}{2},\,p+\frac{1}{2};\;-z\\ p+\frac{3}{2},\,p+\frac{3}{2}\end{matrix}\right)\right.$$

$$\left.-\left[\mathrm{C}+\ln(4z)\right]L_{k+m}^{-k-1/2}(-z)+\sum_{p=1}^{k+m}\frac{(-1)^{p}}{p}L_{k+m-p}^{p-k-1/2}(-z)\right\}.$$

7. $$\mathrm{D}_b\left[\Psi\left(\begin{matrix}m+1\\ b;\;z\end{matrix}\right)\right]\Bigg|_{b=3/2-n}=\frac{2^{2m}\left(\frac{1}{2}\right)_m\left(-\frac{1}{2}\right)_n}{(2m)!\left(m+\frac{1}{2}\right)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(k+m)!}{\left(\frac{3}{2}-n\right)_k}$$

$$\times\left[2+\psi\left(m+n+\frac{1}{2}\right)-\psi\left(n-k-\frac{1}{2}\right)\right]\sum_{p=0}^{k+m}\binom{m}{k+m-p}$$

$$\times\left[\sqrt{\pi}\,z^{-1/2}e^{z}L_{p}^{-p-1/2}(-z)\,\mathrm{erfc}\,(\sqrt{z})-\sum_{r=1}^{p}\frac{1}{r}L_{p-r}^{r-p-1/2}(-z)L_{r-1}^{1/2-r}(z)\right]$$

$$+\frac{2^{2m}m!\left(-\frac{1}{2}\right)_n}{(2m)!\left(m+\frac{1}{2}\right)_n}e^{z}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\left(\frac{3}{2}-n\right)_k}\sum_{p=k}^{k+m}\binom{k+m}{p}\frac{1}{(p-k)!}$$

$$\times\left\{4z^{p}\sum_{r=0}^{p}\binom{p}{r}\frac{(-1)^{r}}{(2r+1)^{2}}\,{}_2F_2\left(\begin{matrix}r+\frac{1}{2},\,r+\frac{1}{2};\;-z\\ r+\frac{3}{2},\,r+\frac{3}{2}\end{matrix}\right)\right.$$

$$-p!\,\sqrt{\pi}\,z^{-1/2}[\mathrm{C}+\ln(4z)]\,L_{p}^{-p-1/2}(-z)$$

$$\left.+(-1)^{p}p!\,\sqrt{\pi}\,z^{-1/2}\sum_{r=0}^{p-1}\frac{(-1)^{r}}{p-r}L_{r}^{-r-1/2}(-z)\right\}.$$

8. $$\mathrm{D}_a\left[\Psi\left(\begin{matrix}a;\;z\\ 2a+n\end{matrix}\right)\right]\Bigg|_{a=m}=-\left[\ln z+\psi(m)\right]\Psi\left(\begin{matrix}m;\;z\\ 2m+n\end{matrix}\right)$$

$$+\frac{z^{-m}}{(m-1)!}\sum_{k=0}^{m+n-1}\binom{m+n-1}{k}(k+m-1)!\,z^{-k}\psi(k+m)$$

$$-\frac{(-z)^{-m}}{(m-1)!}\sum_{k=0}^{m+n-1}\binom{m+n-1}{k}(k+m-1)!\,(-z)^{-k}$$

$$\times\left\{e^{z}[\mathrm{shi}\,(z)-\mathrm{chi}\,(z)]\,L_{k+m-1}^{-k-m}(-z)-\sum_{p=1}^{k+m-1}\frac{1}{p}L_{p-1}^{-p}(z)L_{k+m-p-1}^{p-k-m}(-z)\right\}$$

$$[m\geq 1].$$

9. $\mathrm{D}_a\left[\Psi\left(\begin{matrix} a;\ z \\ 2a+b \end{matrix}\right)\right]\Big|_{a=n+(1-b)/2} = -\ln z\,\Psi\left(\begin{matrix} n+\frac{1-b}{2};\ z \\ 2n+1 \end{matrix}\right)$
$$+\,2^{2n-1} n!\, z^{-n} e^{z} \sum_{k=0}^{n-1} \frac{2^{-2k}}{k!\,(n-k)} G_{23}^{30}\left(z \,\middle|\, \begin{matrix} \frac{1-b}{2},\ k-n+\frac{1}{2} \\ \frac{1}{2},\ 2k-n,\ -n \end{matrix}\right).$$

1.28. The Whittaker Functions $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$

1.28.1. Derivatives with respect to the argument

1. $\mathrm{D}^n[z^{n-\mu-1} e^{\pm az/2} M_{\mu,\nu}(az)]$
$$= (\pm 1)^n \left(\frac{1}{2}-\mu+\nu\right)_n z^{-\mu-1} e^{\pm az/2} M_{\mu\mp n,\,\nu}(az).$$

2. $\mathrm{D}^n[z^{\nu-1/2} e^{\pm az/2} M_{\mu,\nu}(az)]$
$$= (-1)^n (-2\nu)_n a^{n/2} z^{\nu-(n+1)/2} e^{\pm az/2} M_{\mu\mp n/2,\,\nu-n/2}(az).$$

3. $\mathrm{D}^n[z^{-\nu-1/2} e^{\pm az/2} M_{\mu,\nu}(az)]$
$$= (\pm 1)^n \frac{\left(\nu-\mu+\frac{1}{2}\right)_n}{(2\nu+1)_n} a^{n/2} z^{-\nu-(n+1)/2} e^{\pm az/2} M_{\mu\mp n/2,\,\nu+n/2}(az).$$

4. $\mathrm{D}^n[z^{\pm\nu-1/2} e^{\pm az/2} W_{\mu,\nu}(az)]$
$$= (-1)^n \left(\frac{1}{2}-\mu-\nu\right)_n a^{n/2} z^{\pm\nu-(n+1)/2} e^{\pm az/2} W_{\mu-n/2,\,\nu\mp n}(az).$$

5. $\mathrm{D}^n[z^{\pm\nu-1/2} e^{az/2} W_{\mu,\nu}(az)]$
$$= (-1)^n \left(\frac{1}{2}-\mu\mp\nu\right)_n a^{n/2} z^{\pm\nu-(n+1)/2} e^{az/2} W_{\mu-n/2,\,\nu\mp n/2}(az).$$

6. $\mathrm{D}^n[z^{n-\mu-1} e^{az/2} W_{\mu,\nu}(az)]$
$$= \left(\frac{1}{2}-\mu-\nu\right)_n \left(\frac{1}{2}-\mu+\nu\right)_n z^{-\mu-1} e^{az/2} W_{\mu-n,\,\nu}(az).$$

7. $\mathrm{D}^n[z^{\pm\nu-1/2} e^{-az/2} W_{\mu,\nu}(az)]$
$$= (-1)^n a^{n/2} z^{\pm\nu-(n+1)/2} e^{-az/2} W_{\mu+n/2,\,\nu\mp n/2}(az).$$

1.29. The Gauss Hypergeometric Function ${}_2F_1(a,\,b;\ c;\ z)$

1.29.1. Derivatives with respect to the argument

1. $\mathrm{D}^n\left[{}_2F_1\left(\begin{matrix} a,\ b \\ c;\ z \end{matrix}\right)\right] = \frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1\left(\begin{matrix} a+n,\ b+n \\ c+n;\ z \end{matrix}\right)$ [[6], 2.8.20].

2. $\mathrm{D}^n\left[z^{a+n-1}\,{}_2F_1\left(\begin{matrix} a,\ b \\ c;\ z \end{matrix}\right)\right] = (a)_n z^{a-1}\,{}_2F_1\left(\begin{matrix} a+n,\ b \\ c;\ z \end{matrix}\right)$ [[6], 2.8.21].

3. $\mathrm{D}^n\left[z^{c-1}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ z\end{matrix}\right)\right] = (-1)^n(1-c)_n z^{c-n-1}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c-n;\ z\end{matrix}\right)$ [[6], 2.8.22].

4. $\mathrm{D}^n\left[(1-z)^{a+n-1}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ z\end{matrix}\right)\right]$

$$= (-1)^n\frac{(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}\,{}_2F_1\left(\begin{matrix}a+n,\ b\\ c+n;\ z\end{matrix}\right)$$ [[6], 2.8.25].

5. $\mathrm{D}^n\left[(1-z)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ z\end{matrix}\right)\right]$

$$= \frac{(c-a)_n(c-b)_n}{(c)_n}(1-z)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c+n;\ z\end{matrix}\right)$$ [[6], 2.8.24].

6. $\mathrm{D}^n\left[z^{c-1}(1-z)^{b-c+n}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ z\end{matrix}\right)\right]$

$$= (-1)^n(1-c)_n z^{c-n-1}(1-z)^{b-c}\,{}_2F_1\left(\begin{matrix}a-n,\ b\\ c-n;\ z\end{matrix}\right)$$ [[6], 2.8.26].

7. $\mathrm{D}^n\left[z^{c-1}(1-z)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ z\end{matrix}\right)\right]$

$$= (-1)^n(1-c)_n z^{c-n-1}(1-z)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a-n,\ b-n\\ c-n;\ z\end{matrix}\right)$$ [[6], 2.8.27].

8. $\mathrm{D}^n\left[z^{c-a+n-1}(1-z)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ z\end{matrix}\right)\right]$

$$= (c-a)_n z^{c-a-1}(1-z)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a-n,\ b\\ c;\ z\end{matrix}\right)$$ [[6], 2.8.23].

9. $\mathrm{D}^n\left[z^{n-1}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right] = (-1)^n\frac{(a)_n(b)_n}{(c)_n}z^{-n-1}\,{}_2F_1\left(\begin{matrix}a+n,\ b+n\\ c+n;\ \frac{1}{z}\end{matrix}\right).$

10. $\mathrm{D}^n\left[z^{-a}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right] = (-1)^n(a)_n z^{-a-n}\,{}_2F_1\left(\begin{matrix}a+n,\ b\\ c;\ \frac{1}{z}\end{matrix}\right).$

11. $\mathrm{D}^n\left[z^{n-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right] = (1-c)_n z^{-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c-n;\ \frac{1}{z}\end{matrix}\right).$

12. $\mathrm{D}^n\left[z^{-a}(z-1)^{a+n-1}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right]$

$$= \frac{(a)_n(c-b)_n}{(c)_n}z^{-a-n}(z-1)^{a-1}\,{}_2F_1\left(\begin{matrix}a+n,\ b\\ c+n;\ \frac{1}{z}\end{matrix}\right).$$

13. $$\mathrm{D}^n\left[z^{c-a-b+n-1}(z-1)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right] = (-1)^n\frac{(c-a)_n(c-b)_n}{(c)_n}z^{c-a-b-1}(z-1)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c+n;\ \frac{1}{z}\end{matrix}\right).$$

14. $$\mathrm{D}^n\left[z^{-b}(z-1)^{b-c+n}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right] = (1-c)_n z^{-b}(z-1)^{b-c}\,{}_2F_1\left(\begin{matrix}a-n,\ b\\ c-n;\ \frac{1}{z}\end{matrix}\right).$$

15. $$\mathrm{D}^n\left[z^{n-a-b}(z-1)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right] = (1-c)_n z^{n-a-b}(z-1)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a-n,\ b-n\\ c-n;\ \frac{1}{z}\end{matrix}\right).$$

16. $$\mathrm{D}^n\left[z^{-b}(z-1)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c;\ \frac{1}{z}\end{matrix}\right)\right] = (-1)^n(c-a)_n z^{-b}(z-1)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a-n,\ b\\ c;\ \frac{1}{z}\end{matrix}\right).$$

17. $$\mathrm{D}^{2n+\sigma}\left[{}_2F_1\left(\begin{matrix}-n+\frac{1}{2},\ b\\ c;\ z^2\end{matrix}\right)\right] = (-4)^n\left(\frac{1}{2}\right)_n\left(\sigma+\frac{1}{2}\right)_n\frac{(b)_{n+\sigma}}{(c)_{n+\sigma}}z^\sigma\,{}_2F_1\left(\begin{matrix}n+\sigma+\frac{1}{2},\ b+n+\sigma\\ c+n+\sigma;\ z^2\end{matrix}\right)$$
$[\sigma=0$ or 1; [79], (36, 40)].

18. $$\mathrm{D}^{2n+\sigma}\left[z\,{}_2F_1\left(\begin{matrix}-n-\sigma+\frac{3}{2},\ b\\ c;\ z^2\end{matrix}\right)\right] = (-4)^n\left(\frac{3}{2}\right)_n\left(\sigma-\frac{1}{2}\right)_n\frac{(b)_{n+\sigma}}{(c)_{n+\sigma}}z^{1-\sigma}\,{}_2F_1\left(\begin{matrix}n+\frac{3}{2},\ b+n\\ c+n;\ z^2\end{matrix}\right)$$
$[\sigma=0$ or 1; [79], (34, 41)].

19. $$\mathrm{D}^{2n+\sigma}\left[z^{2c-1}\,{}_2F_1\left(\begin{matrix}c-n-\sigma+\frac{1}{2},\ b\\ c;\ z^2\end{matrix}\right)\right] = (-1)^\sigma(1-2c)_{2n+\sigma} \times z^{2c-2n-\sigma-1}\,{}_2F_1\left(\begin{matrix}c+\frac{1}{2},\ b\\ c-n;\ z^2\end{matrix}\right)$$
$[\sigma=0$ or 1; [79], (33, 37)].

20. $$\mathrm{D}^{2n+\sigma}\left[z^{2c-2}\,{}_2F_1\left(\begin{matrix} c-n-\frac{1}{2},\, b \\ c;\ z^2 \end{matrix}\right)\right]$$
$$= (-1)^{\sigma}(2-2c)_{2n+\sigma}\, z^{2c-2n-\sigma-2}\,{}_2F_1\left(\begin{matrix} c-\frac{1}{2},\, b \\ c-n-\sigma;\ z^2 \end{matrix}\right)$$
$[\sigma = 0$ or $1;$ [79], (35, 39)].

21. $$\mathrm{D}^{2n+\sigma}\left[(1-z^2)^{b+n-1/2}\,{}_2F_1\left(\begin{matrix} c+n-\frac{1}{2},\, b \\ c;\ z^2 \end{matrix}\right)\right] = (-4)^n\left(\frac{1}{2}\right)_n\left(\sigma+\frac{1}{2}\right)_n$$
$$\times\ \frac{(c-b)_{n+\sigma}}{(c)_{n+\sigma}}\, z^{\sigma}(1-z^2)^{b-n-\sigma-1/2}\,{}_2F_1\left(\begin{matrix} c-\frac{1}{2},\, b \\ c+n+\sigma;\ z^2 \end{matrix}\right)$$
$[\sigma = 0$ or $1;$ [79], (48, 52)].

22. $$\mathrm{D}^{2n+\sigma}\left[z(1-z^2)^{b+n+\sigma-3/2}\,{}_2F_1\left(\begin{matrix} c+n+\sigma-\frac{3}{2},\, b \\ c;\ z^2 \end{matrix}\right)\right]$$
$$= (-4)^n\left(\frac{3}{2}\right)_n\left(\sigma-\frac{1}{2}\right)_n\frac{(c-b)_n}{(c)_n}\, z^{1-\sigma}(1-z^2)^{b-n-3/2}\,{}_2F_1\left(\begin{matrix} c-\frac{3}{2},\, b \\ c+n;\ z^2 \end{matrix}\right)$$
$[\sigma = 0$ or $1;$ [79], (46, 50)].

23. $$\mathrm{D}^{2n+\sigma}\left[z^{2c-1}(1-z^2)^{b-c+n+\sigma-1/2}\,{}_2F_1\left(\begin{matrix} n+\sigma-\frac{1}{2},\, b \\ c;\ z^2 \end{matrix}\right)\right]$$
$$= (-1)^{\sigma}(1-2c)_{2n+\sigma}\, z^{2c-2n-\sigma-1}(1-z^2)^{b-c-n-1/2}\,{}_2F_1\left(\begin{matrix} -n-\frac{1}{2},\, b-n \\ c-n;\ z^2 \end{matrix}\right)$$
$[\sigma = 0$ or $1;$ [79], (45, 49)].

24. $$\mathrm{D}^{2n+\sigma}\left[z^{2c-2}(1-z^2)^{b-c+n+1/2}\,{}_2F_1\left(\begin{matrix} n+\frac{1}{2},\, b \\ c;\ z^2 \end{matrix}\right)\right] = (-1)^{\sigma}(2-2c)_{2n+\sigma}$$
$$\times\ z^{2c-2n-\sigma-2}(1-z^2)^{b-c-n-\sigma+1/2}\,{}_2F_1\left(\begin{matrix} -n-\sigma+\frac{1}{2},\, b-n-\sigma \\ c-n-\sigma;\ z^2 \end{matrix}\right)$$
$[\sigma = 0$ or $1;$ [79], (47, 51)].

25. $$\mathrm{D}^{2n+\sigma}\left[{}_2F_1\left(\begin{matrix} a,\, b \\ \frac{1}{2};\ z^2 \end{matrix}\right)\right]$$
$$= 2^{2n+2\sigma}(a)_{n+\sigma}(b)_{n+\sigma}\, z^{\sigma}\,{}_2F_1\left(\begin{matrix} a+n+\sigma,\, b+n+\sigma \\ \frac{1}{2}+\sigma;\ z^2 \end{matrix}\right)$$
$[\sigma = 0$ or $1;$ [79], (42, 44)].

26. $$\mathrm{D}^{2n+\sigma}\left[z\,{}_2F_1\left(\begin{matrix}a,\,b\\ \frac{3}{2};\ z^2\end{matrix}\right)\right] = 2^{2n}(a)_n(b)_n\,z^{1-\sigma}\,{}_2F_1\left(\begin{matrix}a+n,\,b+n\\ \frac{3}{2}-\sigma;\ z^2\end{matrix}\right)$$

[$\sigma = 0$ or 1; [79], (41, 44)].

27. $$\mathrm{D}^{2n+\sigma}\left[(1-z^2)^{a+b-1/2}\,{}_2F_1\left(\begin{matrix}a,\,b\\ \frac{1}{2};\ z^2\end{matrix}\right)\right] = 2^{2n+2\sigma}\left(\frac{1}{2}-a\right)_{n+\sigma}$$
$$\times\left(\frac{1}{2}-b\right)_{n+\sigma} z^{\sigma}(1-z^2)^{a+b-2n-\sigma-1/2}\,{}_2F_1\left(\begin{matrix}a-n,\,b-n\\ \frac{1}{2}+\sigma;\ z^2\end{matrix}\right)$$

[$\sigma = 0$ or 1; [79], (54, 55)].

28. $$\mathrm{D}^{2n+\sigma}\left[z\,(1-z^2)^{a+b-1/2}\,{}_2F_1\left(\begin{matrix}a,\,b\\ \frac{3}{2};\ z^2\end{matrix}\right)\right] = 2^{2n}\left(\frac{3}{2}-a\right)_n\left(\frac{3}{2}-b\right)_n$$
$$\times\, z^{1-\sigma}(1-z^2)^{a+b-2n-\sigma-3/2}\,{}_2F_1\left(\begin{matrix}a-n-\sigma,\,b-n-\sigma\\ \frac{3}{2}-\sigma;\ z^2\end{matrix}\right)$$

[$\sigma = 0$ or 1; [79], (53, 56)].

29. $$\mathrm{D}^{2n+\sigma}\left[(1-z^2)^{a+n+\sigma-1}\,{}_2F_1\left(\begin{matrix}a,\,a+\sigma-\frac{1}{2}\\ \frac{1}{2};\ z^2\end{matrix}\right)\right]$$
$$= (-4)^n(a)_{n+\sigma}(1-a-\sigma)_{n+\sigma}\,z^{\sigma}(1-z^2)^{a-n-1}\,{}_2F_1\left(\begin{matrix}a+\sigma,\,a+\sigma-\frac{1}{2}\\ \sigma+\frac{1}{2};\ z^2\end{matrix}\right)$$

[$\sigma = 0$ or 1; [79], (58, 60)].

30. $$\mathrm{D}^{2n+\sigma}\left[z(1-z^2)^{a+n-1}\,{}_2F_1\left(\begin{matrix}a,\,a-\sigma+\frac{1}{2}\\ \frac{3}{2};\ z^2\end{matrix}\right)\right] = (-4)^n\,(a)_n(1-a+\sigma)_n$$
$$\times\, z^{1-\sigma}(1-z^2)^{a-n-\sigma-1}\,{}_2F_1\left(\begin{matrix}a-\sigma,\,a-\sigma+\frac{1}{2}\\ \frac{3}{2}-\sigma;\ z^2\end{matrix}\right)$$
[$\sigma = 0$ or 1; [79], (58, 62)].

31. $$\mathrm{D}^{2n+\sigma}\left[(1-z^2)^{n+\sigma}\,{}_2F_1\left(\begin{matrix}1,\,a\\ \frac{1}{2};\ z^2\end{matrix}\right)\right]$$
$$= (-4)^{n+\sigma}(n+\sigma)!\left(\frac{1}{2}-a\right)_{n+\sigma} z^{\sigma}\,{}_2F_1\left(\begin{matrix}n+\sigma+1,\,a\\ \sigma+\frac{1}{2};\ z^2\end{matrix}\right)$$

[$\sigma = 0$ or 1; [79], (57, 59)].

1.29.2. Derivatives with respect to parameters

1. $$\mathrm{D}_a\left[{}_2F_1\left(\begin{matrix}n+1,\,a;\ z\\ a+b\end{matrix}\right)\right]$$
$$= bz(1-z)^{b-n-1}\sum_{k=0}^{n}\frac{(b+1)_k}{k!\,(a+b+k)^2}\,z^k(1-z)^k P_{n-k}^{(k+1,\,b+k-n-1)}(1-2z)$$
$$\times\ {}_3F_2\left(\begin{matrix}a+b+k,\,a+b+k,\,b+k+1;\ z\\ a+b+k+1,\,a+b+k+1\end{matrix}\right).$$

2. $$\mathrm{D}_b\left[{}_2F_1\left(\begin{matrix}n+1,\,a\\ b;\ z\end{matrix}\right)\right] = -az(1-z)^{-a-1}\sum_{k=0}^{n}\frac{(a+1)_k}{k!\,(b+k)^2}(-z)^k(1-z)^{-2k}$$
$$\times P_{n-k}^{(k+1,\,a+k-n-1)}\left(\frac{1+z}{1-z}\right)\,{}_3F_2\left(\begin{matrix}a+k+1,\,b+k,\,b+k;\ \frac{z}{z-1}\\ b+k+1,\,b+k+1\end{matrix}\right).$$

3. $$\mathrm{D}_b\left[{}_2F_1\left(\begin{matrix}\frac{1}{2},\,1\\ b;\ z\end{matrix}\right)\right]\Bigg|_{b=1/2}$$
$$=\frac{1}{2(z-1)}\left[(1+2\sqrt{z}\,)\ln(1+\sqrt{z}\,)+(1-2\sqrt{z}\,)\ln(1-\sqrt{z}\,)-\ln(1-z)\right].$$

4. $$\mathrm{D}_a\left[{}_2F_1\left(\begin{matrix}a,\,b\\ b+1;\ z\end{matrix}\right)\right]$$
$$= bz^{-b}(1-z)^{1-a}\Bigg[(1-z)^{a-1}\,\mathrm{B}\,(1-a,\,b)[\psi(b-a+1)-\psi(1-a)]$$
$$+\frac{1}{1-a}\ln(1-z)\,{}_2F_1\left(\begin{matrix}1-a,\,1-b\\ 2-a;\ 1-z\end{matrix}\right)-\frac{1}{(1-a)^2}\,{}_3F_2\left(\begin{matrix}1-a,\,1-a,\,1-b\\ 2-a,\,2-a;\ 1-z\end{matrix}\right)\Bigg].$$

5. $$\mathrm{D}_a\left[{}_2F_1\left(\begin{matrix}a,\,a+b;\ -z\\ 2a+c\end{matrix}\right)\right]\Bigg|_{a=(n-c+1)/2}$$
$$=\left[-\ln z+2\psi(n+1)-\psi\left(\frac{n-c+1}{2}\right)-\psi\left(b+\frac{n-c+1}{2}\right)\right]$$
$$\times\ {}_2F_1\left(\begin{matrix}\frac{n-c+1}{2},\,b+\frac{n-c+1}{2}\\ n+1;\ -z\end{matrix}\right)$$
$$+\frac{1}{\Gamma\left(\frac{n-c+1}{2}\right)\Gamma\left(b+\frac{n-c+1}{2}\right)}\Bigg\{(-1)^n\frac{2^{n-b}n!\,(1+z)^{(c-n-1)/2-b}}{\Gamma(n-b+1)}$$
$$\times\Bigg[-\sin\left(\frac{c+n}{2}\pi\right)\Gamma(b)\,\Gamma\left(\frac{1+n-c}{2}\right)\Gamma\left(\frac{1-n-c}{2}\right)(1+z)^b$$
$$\times\ {}_2F_1\left(\begin{matrix}\frac{1-n-c}{2},\,\frac{1+n-c}{2}\\ 1-b;\ \frac{1}{1+z}\end{matrix}\right)$$
$$+\sin\left(\frac{2b-c-n}{2}\pi\right)\Gamma(-b)\,\Gamma\left(b+\frac{1-n-c}{2}\right)\Gamma\left(b+\frac{n-c+1}{2}\right)$$

$$\times\ {}_2F_1\left(\begin{matrix}\frac{n+c+1}{2},\ b+\frac{n-c+1}{2}\\ b+1;\ \frac{1}{1+z}\end{matrix}\right)\Bigg] + (n!)^2 z^{-n}$$

$$\times \sum_{k=0}^{n-1} \frac{z^k \Gamma\left(k+\frac{1-n-c}{2}\right)\Gamma\left(k+b+\frac{1-n-c}{2}\right)}{(k!)^2(n-k)}$$

$$\times\ {}_2F_1\left(\begin{matrix}k+\frac{1-n-c}{2},\ k+b+\frac{1-n-c}{2}\\ k+1;\ -z\end{matrix}\right)\Bigg\}.$$

1.30. The Generalized Hypergeometric Function ${}_pF_q((a_p);\ (b_q);\ z)$

1.30.1. Derivatives with respect to the argument

1. $$\mathrm{D}^n\left[{}_pF_q\left(\begin{matrix}(a_p);\ z\\ (b_q)\end{matrix}\right)\right] = \frac{\prod(a_p)_n}{\prod(b_q)_n}\,{}_pF_q\left(\begin{matrix}(a_p)+n;\ z\\ (b_q)+n\end{matrix}\right).$$

2. $$\mathrm{D}^n\left[z^r\,{}_pF_q\left(\begin{matrix}(a_p);\ z^m\\ (b_q)\end{matrix}\right)\right]$$
$$= (-1)^n(-r)_n\, z^{r-n}\,{}_{p+m}F_{q+m}\left(\begin{matrix}(a_p),\ \Delta(m,\ r+1);\ z^m\\ (b_q),\ \Delta(m,\ r-n+1)\end{matrix}\right)$$
$$[r \neq n-1,\ n-2,\ n-3,\ \ldots].$$

3. $$\mathrm{D}^n\left[z^r\,{}_pF_q\left(\begin{matrix}(a_p);\ z\\ (b_q)\end{matrix}\right)\right]$$
$$= \frac{n!}{(n-r)!}\frac{\prod(a_p)_{n-r}}{\prod(b_q)_{n-r}}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)+n-r,\ n+1;\ z\\ (b_q)+n-r,\ n-r+1\end{matrix}\right)$$
$$+ \sum_{k=0}^{-r-1}\frac{(k-n+r+1)_n}{k!}\frac{\prod(a_p)_k}{\prod(b_q)_k}z^{k-n+r}\quad [r = n-1,\ n-2,\ n-3,\ \ldots].$$

4. $$\mathrm{D}^{2n}\left[{}_pF_q\left(\begin{matrix}(a_p);\ z^2\\ (b_q)\end{matrix}\right)\right] = 2^{2n}\left(\frac{1}{2}\right)_n\frac{\prod(a_p)_n}{\prod(b_q)_n}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)+n,\ n+\frac{1}{2}\\ (b_q)+n,\ \frac{1}{2};\ z^2\end{matrix}\right).$$

5. $$\mathrm{D}^{2n+1}\left[{}_pF_q\left(\begin{matrix}(a_p);\ z^2\\ (b_q)\end{matrix}\right)\right]$$
$$= 2^{2n+1}z\left(\frac{3}{2}\right)_n\frac{\prod(a_p)_{n+1}}{\prod(b_q)_{n+1}}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)+n+1,\ n+\frac{3}{2}\\ (b_q)+n+1,\ \frac{3}{2};\ z^2\end{matrix}\right).$$

1.30.2. Derivatives with respect to parameters

1. $$\mathrm{D}_a^n\left[{}_{p+1}F_q\left(\begin{matrix}a,\,(a_p)\\(b_q);\;z\end{matrix}\right)\right]\Bigg|_{a=0}=(-1)^n n!\sum_{k=n}^{\infty}S_n^k\frac{(-z)^k}{k!}\frac{\prod(a_p)_k}{\prod(b_q)_k}.$$

2. $$\mathrm{D}_a\left[{}_{p+2}F_{q+1}\left(\begin{matrix}-n,\,a,\,(a_p)\\b,\,(b_q);\;z\end{matrix}\right)\right]$$
$$=[\psi(b-a)-\psi(b-a+n)]\;{}_{p+2}F_q\left(\begin{matrix}-n,\,a,\,(a_p)\\b,\,(b_q);\;z\end{matrix}\right)$$
$$-\frac{n!}{(a-b+1)_n}\sum_{k=0}^{n-1}\frac{(-1)^k(2k-n+b-a)(b-a-n)_k}{k!\,(n-k)(k-a+b)}$$
$$\times\sum_{j=0}^{n-k}\binom{n-k}{j}(-z)^j\frac{\prod(a_p)}{\prod(b_q)}\;{}_{p+2}F_{q+1}\left(\begin{matrix}-k,\,a+n-k,\,(a_p)+j\\b,\,(b_q)+j;\;z\end{matrix}\right).$$

3. $$\mathrm{D}_b\left[{}_{p+1}F_{q+1}\left(\begin{matrix}-n,\,(a_p);\;z\\b,\,(b_q)\end{matrix}\right)\right]=[\psi(b)-\psi(b+n)]\;{}_{p+1}F_{q+1}\left(\begin{matrix}-n,\,(a_p)\\b,\,(b_q);\;z\end{matrix}\right)$$
$$+\frac{n!}{(b)_n}\sum_{k=0}^{n-1}\frac{(b)_k}{k!\,(n-k)}\;{}_{p+1}F_{q+1}\left(\begin{matrix}-k,\,(a_p)\\b,\,(b_q);\;z\end{matrix}\right).$$

4. $$\mathrm{D}_a\left[{}_{p+2}F_q\left(\begin{matrix}(a_p),\,a,\,b-a\\(b_q);\;z\end{matrix}\right)\right]\Bigg|_{a=n+b/2}$$
$$=\left[\psi\left(\frac{b}{2}-n\right)-\psi\left(\frac{b}{2}+n\right)\right]\;{}_{p+2}F_q\left(\begin{matrix}(a_p),\,\frac{b}{2}-n,\,\frac{b}{2}+n\\(b_q);\;z\end{matrix}\right)$$
$$+2^{2n-1}n!\frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{b}{2}+n\right)}\sum_{k=0}^{n-1}\frac{2^{-2k}\Gamma\left(\frac{b}{2}+2k-n\right)}{k!\,(n-k)\Gamma\left(\frac{b+1}{2}+k-n\right)}$$
$$\times\;{}_{p+3}F_{q+1}\left(\begin{matrix}(a_p),\,\frac{b+1}{2},\,\frac{b}{2}-n,\,\frac{b}{2}+2k-n\\(b_q),\,\frac{b+1}{2}+k-n;\;z\end{matrix}\right).$$

5. $$\mathrm{D}_a\left[{}_{p+2}F_{q+1}\left(\begin{matrix}-n,\,a,\,(a_p)\\a+b,\,(b_q);\;z\end{matrix}\right)\right]$$
$$=[\psi(a+n)-\psi(a)+\psi(a+b)-\psi(a+b+n)]$$
$$\times\;{}_{p+2}F_{q+1}\left(\begin{matrix}-n,\,a,\,(a_p)\\a+b,\,(b_q);\;z\end{matrix}\right)$$
$$+\frac{n!\,(b)_n}{(1-a)_n(a+b)_n}\sum_{k=0}^{n-1}(2k-n+a)\frac{(a-n)_k(a+b)_k}{k!\,(n-k)(a+k)(1-b-n)_k}$$
$$\times\;{}_{p+2}F_{q+1}\left(\begin{matrix}-k,\,a+k-n,\,(a_p)\\a+b,\,(b_q);\;z\end{matrix}\right).$$

6. $$\mathrm{D}_a\left[{}_3F_2\left(\begin{matrix}\frac{1}{2},\,1,\,1\\ a,\,2;\;z\end{matrix}\right)\right]\Bigg|_{a=1/2} = \frac{1}{2z}\left[4\ln(1-z)+4\sqrt{z}\,\ln\frac{1+\sqrt{z}}{1+\sqrt{z}}-\ln^2\frac{1+\sqrt{z}}{1+\sqrt{z}}\right].$$

7. $$\mathrm{D}_a\left[{}_3F_2\left(\begin{matrix}a,\,a,\,a;\;z\\ a+\frac{1}{2},\,2a\end{matrix}\right)\right]\Bigg|_{a=1/2} = -\frac{4}{\pi^2}\,\mathbf{K}\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right)\times\left[\pi\,\mathbf{K}\left(\sqrt{\frac{1+\sqrt{1-z}}{2}}\right)+2\ln\frac{\sqrt{z}}{8}\,\mathbf{K}\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right)\right].$$

8. $$\mathrm{D}_a\left[{}_3F_2\left(\begin{matrix}a,\,a,\,a;\;-z\\ a+1/2,\,2a\end{matrix}\right)\right]\Bigg|_{a=1/2} = -\frac{8}{\pi^2}\,\mathbf{K}\left(\frac{\sqrt{z}}{\sqrt{z+1}+1}\right)\times\left[\frac{\sqrt{2}\,\pi}{\left(\sqrt{z+1}+1\right)^{1/2}\left(\sqrt{z+1}+\sqrt{z}\right)^{1/2}}\,\mathbf{K}\left(\frac{1}{\sqrt{z+1}+\sqrt{z}}\right)+\frac{2\ln\frac{\sqrt{z}}{8}}{\sqrt{z+1}+1}\,\mathbf{K}\left(\frac{\sqrt{z}}{\sqrt{z+1}+1}\right)\right].$$

9. $$\mathrm{D}_d\left[{}_4F_3\left(\begin{matrix}a+\frac{1}{2},\,a+1,\,a-b+c,\,a+b+c\\ 2a+1,\,2d+1,\,2a-2d+1;\;z\end{matrix}\right)\right]\Bigg|_{d=a} = -\left[\psi(a+b+c)+\psi(b-a-c+1)+2\mathbf{C}+\ln\frac{z}{4}\right]\times{}_4F_3\left(\begin{matrix}a+\frac{1}{2},\,a+1,\,a-b+c,\,a+b+c\\ 2a+1,\,2a+1,\,1;\;z\end{matrix}\right)+2^{2a}\sqrt{\pi}\,\frac{\Gamma(2a+1)\Gamma(b-a-c+1)}{\Gamma(a+b+c)}\times G_{55}^{23}\left(z\,\middle|\,\begin{matrix}\frac{1}{2}-a,\,-a,\,1-a-b-c,\,1-a+b-c,\,-\frac{1}{2}\\ 0,\,0,\,-\frac{1}{2},\,-2a,\,-2a\end{matrix}\right).$$

10. $$\mathrm{D}_a\left[{}_4F_3\left(\begin{matrix}a,\,a+\frac{1}{2},\,a+b-c,\,a+b+c\\ 2a,\,d,\,2a-d+1;\;z\end{matrix}\right)\right]\Bigg|_{a=d/2} = -\left[\ln\frac{z}{4}+\psi\left(c+b+\frac{d}{2}\right)+\psi\left(c-b-\frac{d}{2}+1\right)+2\mathbf{C}\right]\times{}_4F_3\left(\begin{matrix}\frac{d}{2},\,\frac{d+1}{2},\,b-c+\frac{d}{2},\,b+c+\frac{d}{2}\\ 1,\,d,\,d;\;z\end{matrix}\right)+2^{d-1}\sqrt{\pi}\,\frac{\Gamma(d)\Gamma\left(c-b-\frac{d}{2}+1\right)}{\Gamma\left(c+b+\frac{d}{2}\right)}\times G_{55}^{23}\left(z\,\middle|\,\begin{matrix}\frac{1-d}{2},\,\frac{2-2b-2c-d}{2},\,1-\frac{d}{2},\,-\frac{1}{2},\,c-b-\frac{d}{2}+1\\ 0,\,0,\,-\frac{1}{2},\,1-d,\,1-d\end{matrix}\right).$$

11. $$\mathrm{D}_a\left[{}_1F_2\left(\begin{matrix}a;\;z\\ 1,\,1\end{matrix}\right)\right]\Bigg|_{a=1} = K_0(2\sqrt{z})+\left(\frac{1}{2}\ln z+\mathbf{C}\right)I_0(2\sqrt{z}).$$

12. $$\mathrm{D}_a\left[{}_1F_2\left(\begin{matrix}a;\ -z\\ 1,\ 1\end{matrix}\right)\right]\bigg|_{a=1} = -\frac{\pi}{2}\,Y_0(2\sqrt{z}) + \left(\mathbf{C} + \frac{1}{2}\ln z\right) J_0(2\sqrt{z}).$$

13. $$\mathrm{D}_a\left[{}_1F_2\left(\begin{matrix}a;\ z\\ \frac{3}{2},\ \frac{3}{2}\end{matrix}\right)\right]\bigg|_{a=3/2} = \frac{1}{4\sqrt{z}}\,\{\sinh(2\sqrt{z})[2\mathbf{C} + \ln(16z) - 2\,\mathrm{chi}\,(4\sqrt{z}) - 4] + 2\cosh(2\sqrt{z})\,\mathrm{shi}\,(4\sqrt{z})\}.$$

14. $$\mathrm{D}_a\left[{}_1F_2\left(\begin{matrix}n+1;\ z\\ a,\ 2-a\end{matrix}\right)\right]\bigg|_{a=1/2} = -2 + \frac{\left(\frac{1}{2}\right)_n}{2(n!)\,z}$$
$$\times \sum_{k=0}^{n}\binom{n}{k}\frac{z^k}{\left(\frac{1}{2}\right)_k}\left[\pi z^{(1-k)/2}\,\mathbf{L}_k(2\sqrt{z}) + \sum_{p=0}^{k-1}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(k-p+\frac{1}{2}\right)}(-z)^{-p}\right]$$
$$- \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\left(-\frac{1}{2}\right)_{k-p} z^{p/2}$$
$$\times\left[\pi\,\mathbf{L}_{p+1}(2\sqrt{z}) + z^{p/2}\sum_{r=0}^{p-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(p-r+\frac{3}{2}\right)}(-z)^{-r}\right].$$

15. $$\mathrm{D}_a\left[{}_1F_2\left(\begin{matrix}n+1;\ -z\\ a,\ 2-a\end{matrix}\right)\right]\bigg|_{a=1/2} = -2 + \frac{\left(\frac{1}{2}\right)_n}{2\,(n!)\,z}$$
$$\times \sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{\left(\frac{1}{2}\right)_k}\left[\pi z^{(1-k)/2}\,\mathbf{H}_k(2\sqrt{z}) - \sum_{r=0}^{k-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(k-r+\frac{3}{2}\right)}\left(\frac{1}{z}\right)^r\right]$$
$$+ \sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{k}\sum_{p=0}^{k}\binom{k}{p}z^{p/2}\left(-\frac{1}{2}\right)_{k-p}$$
$$\times\left[\pi\,\mathbf{H}_{p+1}(2\sqrt{z}) - z^{p/2}\sum_{r=0}^{p-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(p-r+\frac{3}{2}\right)}\left(\frac{1}{z}\right)^r\right].$$

16. $$\mathrm{D}_a\left[{}_1F_2\left(\begin{matrix}1;\ z\\ a,\ a+\frac{1}{2}\end{matrix}\right)\right]\bigg|_{a=3/2} = -\frac{1}{2z}\,\{3 + 2\sinh(2\sqrt{z})\,\mathrm{shi}\,(2\sqrt{z}) + \cosh(2\sqrt{z})\,[-3 + 2\mathbf{C} + 2\ln(2\sqrt{z}) - 2\,\mathrm{chi}\,(2\sqrt{z})]\}.$$

17. $$\mathrm{D}_a\left[{}_1F_2\left(\begin{matrix}a;\ -z\\ a+b,\ 2a+c\end{matrix}\right)\right]\bigg|_{a=(n-c+1)/2}$$
$$= \left[2\psi(n+1) - \ln z - \psi\left(\frac{n-c+1}{2}\right) + \psi\left(b + \frac{n-c+1}{2}\right)\right]$$
$$\times\ {}_1F_2\left(\begin{matrix}\frac{n-c+1}{2};\ -z\\ b+\frac{n-c+1}{2},\ n+1\end{matrix}\right)$$

$$+ n! \frac{\Gamma\left(b + \frac{n-c+1}{2}\right)}{\Gamma\left(\frac{n-c+1}{2}\right)} \left[\pi G_{24}^{21}\left(z \,\middle|\, \begin{matrix} \frac{c-n+1}{2}, -n-\frac{1}{2} \\ 0, -n, -n-\frac{1}{2}, \frac{c-n+1}{2} - b \end{matrix} \right) \right.$$

$$\left. + n! \sum_{k=0}^{n-1} \frac{z^{k-n}}{(k!)^2 (n-k)} \frac{\Gamma\left(k + \frac{1-n-c}{2}\right)}{\Gamma\left(k+b+\frac{1-n-c}{2}\right)} \, {}_1F_2\left(\begin{matrix} k + \frac{1-n-c}{2};\ -z \\ k+1,\ k+b+\frac{1-n-c}{2} \end{matrix} \right) \right].$$

18. $\mathrm{D}_a \left[{}_1F_2\left(\begin{matrix} a;\ z \\ a+b,\ 2a+c \end{matrix} \right) \right] \Big|_{a=(n-c+1)/2}$

$$= \frac{1}{2} \left[2\psi(n+1) - \ln z - \psi\left(\frac{n-c+1}{2}\right) + \psi\left(b + \frac{n-c+1}{2}\right) \right]$$

$$\times \, {}_1F_2\left(\begin{matrix} \frac{n-c+1}{2};\ z \\ b + \frac{n-c+1}{2},\ n+1 \end{matrix} \right) + (-1)^n n! \, z^{-n/2} \frac{\Gamma\left(b + \frac{n-c+1}{2}\right)}{2\Gamma\left(\frac{n-c+1}{2}\right)}$$

$$\times \left[G_{13}^{21}\left(z \,\middle|\, \begin{matrix} \frac{c+1}{2} \\ -\frac{n}{2}, \frac{n}{2}, \frac{c-2b+1}{2} \end{matrix} \right) \right.$$

$$- n! \sum_{k=0}^{n-1} (-1)^k \frac{z^{k-n/2}}{(k!)^2 (n-k)} \frac{\Gamma\left(k + \frac{1-n-c}{2}\right)}{\Gamma\left(k+b+\frac{1-n-c}{2}\right)}$$

$$\left. \times \, {}_1F_2\left(\begin{matrix} k + \frac{1-n-c}{2};\ z \\ k+1,\ k+b+\frac{1-n-c}{2} \end{matrix} \right) \right].$$

19. $\mathrm{D}_a [{}_0F_2(a, 2a+b;\ z)] |_{a=(1-b)/2}$

$$= \left[\psi\left(\frac{b+1}{2}\right) - \ln z - 2\mathbf{C} \right] {}_0F_2\left(\begin{matrix} z \\ 1, \frac{1-b}{2} \end{matrix} \right) + \frac{\pi}{\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{b+3}{2}\right)}$$

$$\times \left\{ z^{(b+1)/2} \tan \frac{b\pi}{2} \Gamma\left(-\frac{b+1}{2}\right) {}_0F_2\left(\begin{matrix} z \\ \frac{b+3}{2}, \frac{b+3}{2} \end{matrix} \right) \right.$$

$$\left. + \Gamma\left(\frac{b+3}{2}\right) G_{14}^{30}\left(z \,\middle|\, \begin{matrix} -\frac{1}{2} \\ 0, 0, \frac{b+1}{2}, -\frac{1}{2} \end{matrix} \right) \right\}.$$

20. $\mathrm{D}_a \left[{}_2F_2\left(\begin{matrix} a, a;\ z \\ 1, a+1 \end{matrix} \right) \right] \Big|_{a=1}$

$$= \frac{1}{z} \left[e^z - 1 + (2 + e^z)(\mathbf{C} + \ln z) - 2\,\mathrm{Ei}(z) - e^z\,\mathrm{Ei}(-z) \right] \quad [z > 0].$$

21. $\mathrm{D}_a \left[{}_2F_2\left(\begin{matrix} a, a;\ -z \\ 1, a+1 \end{matrix} \right) \right] \Big|_{a=1}$

$$= \frac{1}{z} \left[1 - e^{-z} - (2 + e^{-z})(\mathbf{C} + \ln z) + 2\,\mathrm{Ei}(-z) + e^{-z}\,\mathrm{Ei}(z) \right] \quad [z > 0].$$

22. $$\mathrm{D}_a\left[{}_2F_2\left(\begin{matrix} a,\, a+\frac{1}{2};\; z \\ b,\, b+\frac{1}{2}\end{matrix}\right)\right]\Bigg|_{a=b} = 2ze^z\left[\frac{1}{2b}\,{}_2F_2\left(\begin{matrix}1,\,1;\; -z\\ 2,\, b+1\end{matrix}\right)+\frac{1}{2b+1}\,{}_2F_2\left(\begin{matrix}1,\,1;\; -z\\ 2,\, b+\frac{3}{2}\end{matrix}\right)\right].$$

23. $$\mathrm{D}_a\left[{}_2F_2\left(\begin{matrix} a,\, a+b;\; z \\ a+\frac{b+1}{2},\, 2a+c\end{matrix}\right)\right]\Bigg|_{a=(n-c+1)/2}$$
$$= \left[-\ln z-\psi\left(\frac{n-c+1}{2}\right)-\psi\left(\frac{n-c+1}{2}+b\right)\right.$$
$$\left.+\psi\left(\frac{n+b-c}{2}+1\right)+2\psi(n+1)\right]\,{}_2F_2\left(\begin{matrix}\frac{n-c+1}{2},\, \frac{n-c+1}{2}+b;\; z\\ \frac{n+b-c}{2}+1,\, n+1\end{matrix}\right)$$
$$-(-1)^n\,\frac{n!\,\Gamma\left(\frac{n+b-c}{2}+1\right)z^{-n/2}}{\Gamma\left(\frac{n-c+1}{2}\right)\Gamma\left(\frac{n-c+1}{2}+b\right)}\left[G_{23}^{22}\left(z\,\middle|\,\begin{matrix}\frac{c+1}{2},\, \frac{c+1}{2}-b\\ -\frac{n}{2},\, \frac{n}{2},\, \frac{c-b}{2}\end{matrix}\right)\right.$$
$$-n!\,z^{-n/2}\sum_{k=0}^{n-1}\frac{(-z)^k\Gamma\left(k+\frac{1-c-n}{2}\right)\Gamma\left(k+b+\frac{1-c-n}{2}\right)}{(k!)^2(n-k)\Gamma\left(k+1+\frac{b-c-n}{2}\right)}$$
$$\left.\times\,{}_2F_2\left(\begin{matrix}k+\frac{1-c-n}{2},\, k+b+\frac{1-c-n}{2};\; z\\ k+1,\, k+1+\frac{b-n-c}{2}\end{matrix}\right)\right].$$

24. $$\mathrm{D}_a\left[{}_2F_2\left(\begin{matrix} a,\, a+b;\; -z \\ a+\frac{b+1}{2},\, 2a+c\end{matrix}\right)\right]\Bigg|_{a=(n-c+1)/2}$$
$$= \left[-\ln z-\psi\left(\frac{n-c+1}{2}\right)-\psi\left(\frac{n-c+1}{2}+b\right)\right.$$
$$\left.+\psi\left(\frac{n+b-c}{2}+1\right)+2\psi(n+1)\right]\times\,{}_2F_2\left(\begin{matrix}\frac{n-c+1}{2},\, \frac{n-c+1}{2}+b;\; -z\\ \frac{n+b-c}{2}+1,\, n+1\end{matrix}\right)$$
$$+\frac{n!\,\Gamma\left(\frac{n+b-c}{2}+1\right)z^{-n/2}}{\Gamma\left(\frac{n-c+1}{2}\right)\Gamma\left(\frac{n-c+1}{2}+b\right)}\left[\pi G_{34}^{22}\left(z\,\middle|\,\begin{matrix}\frac{c+1}{2},\, \frac{c+1}{2}-b,\, -\frac{n+1}{2}\\ -\frac{n}{2},\, \frac{n}{2},\, -\frac{n+1}{2},\, \frac{c-b}{2}\end{matrix}\right)\right.$$
$$+n!\,z^{-n/2}\sum_{k=0}^{n-1}\frac{z^k\Gamma\left(k+\frac{1-c-n}{2}\right)\Gamma\left(k+b+\frac{1-c-n}{2}\right)}{(k!)^2(n-k)\Gamma\left(k+1+\frac{b-c-n}{2}\right)}$$
$$\left.\times\,{}_2F_2\left(\begin{matrix}k+\frac{1-c-n}{2},\, k+b+\frac{1-c-n}{2};\; -z\\ k+1,\, k+1+\frac{b-n-c}{2}\end{matrix}\right)\right].$$

25. $$\mathrm{D}_a\left[{}_2F_2\left(\begin{matrix}a,\,a;\;z\\ a+1,\,2a+b\end{matrix}\right)\right]\Bigg|_{a=(n-b+1)/2}$$
$$=\left[\frac{2}{n-b+1}-\ln z-\psi\left(\frac{n-b+1}{2}\right)+2\psi(n+1)\right]$$
$$\times\,{}_2F_2\left(\begin{matrix}\frac{n-b+1}{2},\,\frac{n-b+1}{2};\;z\\ \frac{n-b+3}{2},\,n+1\end{matrix}\right)$$
$$-(-1)^n\,\frac{n!\,(n-b+1)}{\Gamma\left(\frac{n-b+1}{2}\right)}z^{-n/2}\left[\frac{1}{2}G_{23}^{22}\left(z\,\middle|\,\begin{matrix}\frac{b+1}{2},\,\frac{b+1}{2}\\ -\frac{n}{2},\,\frac{n}{2},\,\frac{b-1}{2}\end{matrix}\right)\right.$$
$$-n!\,z^{-n/2}\sum_{k=0}^{n-1}\frac{(-z)^k\Gamma\left(k+\frac{1-n-b}{2}\right)}{(k!)^2(n-k)\,(2k-n-b+1)}$$
$$\left.\times\,{}_2F_2\left(\begin{matrix}k+\frac{1-b-n}{2},\,k+\frac{1-n-b}{2};\;z\\ k+1,\,k+\frac{3-n-b}{2}\end{matrix}\right)\right].$$

26. $$\mathrm{D}_a\left[{}_2F_2\left(\begin{matrix}a,\,a;\;-z\\ a+1,\,2a+b\end{matrix}\right)\right]\Bigg|_{a=(n-b+1)/2}$$
$$=\left[\frac{2}{n-b+1}-\ln z-\psi\left(\frac{n-b+1}{2}\right)+2\psi(n+1)\right]$$
$$\times\,{}_2F_2\left(\begin{matrix}\frac{n-b+1}{2},\,\frac{n-b+1}{2};\;-z\\ \frac{n-b+3}{2},\,n+1\end{matrix}\right)$$
$$+\frac{n!\,(n-b+1)z^{-n/2}}{\Gamma\left(\frac{n-b+1}{2}\right)}\left[\frac{\pi}{2}\,G_{34}^{22}\left(z\,\middle|\,\begin{matrix}\frac{b+1}{2},\,\frac{b+1}{2},\,-\frac{n+1}{2}\\ -\frac{n}{2},\,\frac{n}{2},\,\frac{b-1}{2},\,-\frac{n+1}{2}\end{matrix}\right)\right.$$
$$+n!\,z^{-n/2}\sum_{k=0}^{n-1}\frac{z^k\Gamma\left(k+\frac{1-b-n}{2}\right)}{(k!)^2(n-k)\,(2k-n-b+1)}$$
$$\left.\times\,{}_2F_2\left(\begin{matrix}k+\frac{1-b-n}{2},\,k+\frac{1-b-n}{2};\;-z\\ k+1,\,k+\frac{3-b-n}{2}\end{matrix}\right)\right].$$

27. $$\mathrm{D}_a\left[{}_2F_2\left(\begin{matrix}a,\,a+\frac{1}{2};\;z\\ a+1,\,2a+b\end{matrix}\right)\right]\Bigg|_{a=(n-b+1)/2}$$
$$=\left[\frac{2}{n-b+1}-\ln z-\psi\left(\frac{n-b}{2}+1\right)+2\psi(n+1)\right]$$
$$\times\,{}_2F_2\left(\begin{matrix}\frac{n-b+1}{2},\,\frac{n-b}{2}+1;\;z\\ \frac{n-b+3}{2},\,n+1\end{matrix}\right)$$
$$-\frac{(-1)^n n!\,(n-b+1)z^{-n/2}}{2\Gamma\left(\frac{n-b}{2}+1\right)}\left[G_{23}^{22}\left(z\,\middle|\,\begin{matrix}\frac{b}{2},\,\frac{b+1}{2}\\ -\frac{n}{2},\,\frac{n}{2},\,\frac{b-1}{2}\end{matrix}\right)\right.$$

$$- 2(n!)\, z^{-n/2} \sum_{k=0}^{n-1} \frac{(-z)^k \Gamma\left(k - \frac{n+b}{2} + 1\right)}{(k!)^2 (n-k)\,(2k-n-b+1)}$$

$$\left. \times\ {}_2F_2\left(\begin{matrix} k + \frac{1-b-n}{2},\ k - \frac{n+b}{2} + 1;\ z \\ k+1,\ k + \frac{3-n-b}{2} \end{matrix} \right) \right].$$

28. $$\mathrm{D}_a \left[{}_2F_3\left(\begin{matrix} a,\ a+\frac{1}{2};\ z \\ a+1,\ a+1,\ 2a+b \end{matrix} \right) \right] \Bigg|_{a=(n-b+1)/2}$$

$$= \left[\frac{4}{n-b+1} - \ln z + \psi\left(\frac{n-b+1}{2}\right) - \psi\left(\frac{n-b}{2}+1\right) + 2\psi(n+1) \right]$$

$$\times\ {}_2F_3\left(\begin{matrix} \frac{n-b+1}{2},\ \frac{n-b}{2}+1;\ z \\ \frac{n-b+3}{2},\ \frac{n-b+3}{2},\ n+1 \end{matrix} \right)$$

$$- (-1)^n n!\,(n-b+1) \frac{\Gamma\left(\frac{n-b+3}{2}\right)}{\Gamma\left(\frac{n-b}{2}+1\right)} z^{-n/2} \left[\frac{1}{2} G_{24}^{22}\left(z \,\middle|\, \begin{matrix} \frac{b}{2},\ \frac{b+1}{2} \\ -\frac{n}{2},\ \frac{n}{2},\ \frac{b-1}{2},\ \frac{b-1}{2} \end{matrix} \right) \right.$$

$$- n! \sum_{k=0}^{n-1} \frac{(-1)^k z^{k-n/2}}{(k!)^2 (n-k)\,(2k-n-b+1)} \frac{\Gamma\left(k - \frac{n+b}{2} + 1\right)}{\Gamma\left(k + \frac{3-n-b}{2}\right)}$$

$$\left. \times\ {}_2F_3\left(\begin{matrix} k + \frac{1-b-n}{2},\ k - \frac{n+b}{2} + 1;\ z \\ k+1,\ k + \frac{3-n-b}{2},\ k + \frac{3-n-b}{2} \end{matrix} \right) \right].$$

29. $$\mathrm{D}_a \left[{}_2F_3\left(\begin{matrix} a,\ a+\frac{1}{2};\ z \\ a+1,\ a+1,\ 2a+b \end{matrix} \right) \right] \Bigg|_{a=(n-b+1)/2} = \frac{4}{n-b+1} - \ln z$$

$$+ \psi\left(\frac{n-b+1}{2}\right) - \psi\left(\frac{n-b}{2}+1\right)$$

$$+ 2\psi(n+1)\ {}_2F_3\left(\begin{matrix} \frac{n-b+1}{2},\ \frac{n-b}{2}+1;\ z \\ \frac{n-b+3}{2},\ \frac{n-b+3}{2},\ n+1 \end{matrix} \right)$$

$$- (-1)^n n!\,(n-b+1) \frac{\Gamma\left(\frac{n-b+3}{2}\right)}{2\Gamma\left(\frac{n-b}{2}+1\right)} z^{-n/2} \left[\frac{1}{2} G_{24}^{22}\left(z \,\middle|\, \begin{matrix} \frac{b}{2},\ \frac{b+1}{2} \\ -\frac{n}{2},\ \frac{n}{2},\ \frac{b-1}{2},\ \frac{b-1}{2} \end{matrix} \right) \right.$$

$$- v - n! \sum_{k=0}^{n-1} \frac{(-1)^k z^{k-n/2}}{(k!)^2 (n-k)\,(2k-n-b+1)} \frac{\Gamma\left(k - \frac{n+b}{2} + 1\right)}{\Gamma\left(k + \frac{3-n-b}{2}\right)}$$

$$\left. \times\ {}_2F_3\left(\begin{matrix} k + \frac{1-b-n}{2},\ k - \frac{n+b}{2} + 1;\ z \\ k+1,\ k + \frac{3-n-b}{2},\ k + \frac{3-n-b}{2} \end{matrix} \right) \right].$$

30. $$\mathrm{D}_a\left[{}_2F_3\left(\begin{matrix}a,\, a+\frac{1}{2};\; z\\ a+b,\, a+b+\frac{1}{2},\, 2a+c\end{matrix}\right)\right]\Bigg|_{a=(n-c+1)/2}$$
$$= [-\ln z + 2\psi(n+2b-c+1) - 2\psi(n-c+1) + 2\psi(n+1)]$$
$$\times\ {}_2F_3\left(\begin{matrix}\frac{n-c+1}{2},\, \frac{n-c}{2}+1;\; z\\ \frac{n-c+1}{2}+b,\, \frac{n-c}{2}+b+1,\, n+1\end{matrix}\right) + (-1)^n n!\, z^{-n/2}\,\frac{\Gamma(2b-c+n+1)}{\Gamma(n-c+1)}$$
$$\times\left[-2^{-2b} G_{24}^{22}\left(z\,\middle|\,\begin{matrix}\frac{c}{2},\, \frac{c+1}{2}\\ -\frac{n}{2},\, \frac{n}{2},\, \frac{c-2b}{2},\, \frac{c-2b+1}{2}\end{matrix}\right)\right.$$
$$+\, n!\sum_{k=0}^{n-1}\frac{(-1)^k z^{k-n/2}}{(k!)^2(n-k)}\,\frac{\Gamma(2k-n-c+1)}{\Gamma(2k-n+2b-c+1)}$$
$$\left.\times\ {}_2F_3\left(\begin{matrix}k+\frac{1-c-n}{2},\, k-\frac{n+c}{2}+1;\; z\\ k+1,\, k+\frac{1-n-c}{2}+b,\, k-\frac{n+c}{2}+b+1\end{matrix}\right)\right].$$

31. $$\mathrm{D}_a\left[{}_2F_3\left(\begin{matrix}a,\, a+\frac{1}{2};\; -z\\ a+b,\, a+b+\frac{1}{2},\, 2a+c\end{matrix}\right)\right]\Bigg|_{a=(n-c+1)/2}$$
$$= [-\ln z - 2\psi(n-c+1) + 2\psi(n+2b-c+1) + 2\psi(n+1)]$$
$$\times\ {}_2F_3\left(\begin{matrix}\frac{n-c+1}{2},\, \frac{n-c}{2}+1;\; -z\\ \frac{n-c+1}{2}+b,\, \frac{n-c}{2}+b+1,\, n+1\end{matrix}\right) + n!\, z^{-n/2}\,\frac{\Gamma(n+2b-c+1)}{\Gamma(n-c+1)}$$
$$\times\left[2^{-2b}\pi\, G_{35}^{22}\left(z\,\middle|\,\begin{matrix}\frac{c}{2},\, \frac{c+1}{2},\, -\frac{n+1}{2}\\ -\frac{n}{2},\, \frac{n}{2},\, \frac{c-2b}{2},\, \frac{c-2b+1}{2},\, -\frac{n+1}{2}\end{matrix}\right)\right.$$
$$+\, n!\, z^{-n/2}\sum_{k=0}^{n-1}\frac{z^k}{(k!)^2(n-k)}\,\frac{\Gamma(2k-n-c+1)}{\Gamma(2k-n+2b-c+1)}$$
$$\left.\times\ {}_2F_3\left(\begin{matrix}k+\frac{1-c-n}{2},\, k-\frac{n+c}{2}+1;\; -z\\ k+1,\, k+\frac{1-n-c}{2}+b,\, k-\frac{n+c}{2}+b+1\end{matrix}\right)\right].$$

Chapter 2

Limits

2.1. Special Functions

2.1.1. The Bessel functions $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$

1. $$\lim_{\nu\to\infty} \nu^{\nu+1/2}\left(\frac{2}{ez}\right)^{\nu} \left\{\begin{matrix} J_\nu(z) \\ I_\nu(z) \end{matrix}\right\} = \frac{1}{\sqrt{2\pi}}.$$

2. $$\lim_{\nu\to\infty} \nu^{\nu+1/2}\left(\frac{2}{ez}\right)^{\nu} \left\{\begin{matrix} J_\nu(z\sqrt{\nu}) \\ I_\nu(z\sqrt{\nu}) \end{matrix}\right\} = \frac{1}{\sqrt{2\pi}}\, e^{\mp z^2/4}.$$

3. $$\lim_{n\to\infty} (-1)^n\, n\Big[\sin\Big(n^2 z + \frac{n\pi}{2}\Big) J_{n+1/2}(n^2 z) + \cos\Big(n^2 z - \frac{n\pi}{2}\Big) J_{-n-1/2}(n^2 z)\Big] = \sqrt{\frac{2}{\pi z}}\cos\frac{1}{2z}.$$

4. $$\lim_{n\to\infty} \left(\frac{ez}{2n}\right)^{n} \left[\sin z\, J_{n+1/2}(z) - \cos z\, Y_{n+1/2}(z)\right] = \frac{2}{\sqrt{\pi z}}\cos z.$$

5. $$\lim_{n\to\infty} \left(\frac{ez}{2n}\right)^{n} \left[\cos z\, J_{n+1/2}(z) - \sin z\, Y_{n+1/2}(z)\right] = \frac{2}{\sqrt{\pi z}}\sin z.$$

6. $$\lim_{n\to\infty} n\left[J_{n+1/2}^2(nz) + Y_{n+1/2}^2(nz)\right] = \frac{2\operatorname{sgn}(z)}{\pi\sqrt{z^2-1}} \qquad [z^2 > 1].$$

7. $$\lim_{\nu\to\infty} \left(\frac{z}{2\nu}\right)^{\nu-1/2} e^{\nu} K_\nu(z) = \pm\sqrt{\frac{\pi}{z}} \qquad \left[\left\{\begin{matrix} |\arg z| < \pi \\ z < 0 \end{matrix}\right\}\right].$$

2.1.2. The Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

1. $$\lim_{\nu\to\infty} \nu^{(\nu+1)/2}\left(\frac{2}{ez}\right)^{\nu} \mathbf{H}_\nu(z\sqrt{\nu}) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/4}\operatorname{erfi}\left(\frac{z}{2}\right).$$

2. $$\lim_{\nu\to\infty} \nu^{(\nu+1)/2}\left(\frac{2}{ez}\right)^{\nu} \mathbf{L}_\nu(z\sqrt{\nu}) = \frac{1}{\sqrt{2\pi}}\, e^{z^2/4}\operatorname{erf}\left(\frac{z}{2}\right).$$

2.1.3. The Kelvin functions $\operatorname{ber}_\nu(z)$, $\operatorname{bei}_\nu(z)$, $\operatorname{ker}_\nu(z)$ and $\operatorname{kei}_\nu(z)$

1. $$\lim_{\nu\to\infty} \left(\frac{2}{ez}\right)^{\nu} \nu^{\nu+1/2}\left[\cos\frac{3\nu\pi}{4}\operatorname{ber}_\nu(z) + \sin\frac{3\nu\pi}{4}\operatorname{bei}_\nu(z)\right] = \frac{1}{\sqrt{2\pi}}.$$

2. $$\lim_{\nu\to\infty}\left(\frac{z}{2\nu}\right)^{\nu-1/2} e^{\nu}\left[\sin\frac{(6\nu-1)\pi}{8}\operatorname{ker}_{\nu}(z)+\cos\frac{(6\nu-1)\pi}{8}\operatorname{kei}_{\nu}(z)\right] = -\sqrt{\frac{\pi}{z}}\sin\frac{\pi}{8}.$$

2.1.4. The Legendre polynomials $P_n(z)$

1. $$\lim_{n\to\infty} n^{1/2} z^{n/2} P_n\left(\frac{z+1}{2\sqrt{z}}\right) = \frac{1}{\sqrt{\pi(1-z)}} \qquad [|z|<1].$$

2. $$\lim_{n\to\infty} n^{1/2-n}(2z)^{-n} P_n(1+nz) = \frac{1}{\sqrt{\pi}} e^{1/z}.$$

3. $$\lim_{n\to\infty} n^{(1-n)/2}(2z)^{-n} P_n(\sqrt{n}\,z) = \frac{1}{\sqrt{\pi}} e^{-z^{-2}/4}.$$

4. $$\lim_{n\to\infty} (-1)^n n^{1/2} P_{2n}\left(\frac{z}{n}\right) = \frac{\cos(2z)}{\sqrt{\pi}}.$$

5. $$\lim_{n\to\infty} (-1)^n n^{1/2} P_{2n+1}\left(\frac{z}{n}\right) = \frac{\sin(2z)}{\sqrt{\pi}}.$$

6. $$\lim_{n\to\infty} P_n\left(1+\frac{z}{n^2}\right) = I_0\left(\sqrt{2z}\right).$$

7. $$\lim_{n\to\infty} P_n\left(\sqrt{1+\frac{z^2}{n^2}}\right) = I_0(z).$$

8. $$\lim_{n\to\infty} P_n\left(\frac{n}{\sqrt{n^2+z^2}}\right) = J_0(z).$$

9. $$\lim_{n\to\infty} P_n\left(\frac{n+z}{\sqrt{n(n+2z)}}\right) = I_0(z).$$

2.1.5. The Chebyshev polynomials $T_n(z)$ and $U_n(z)$

1. $$\lim_{n\to\infty}\left(z-\sqrt{z^2-1}\right)^n T_n(z) = \frac{1}{2} \qquad [\operatorname{Re} z > 1].$$

2. $$\lim_{n\to\infty} (2zn)^{-n} T_n(1+nz) = \frac{1}{2} e^{1/z}.$$

3. $$\lim_{n\to\infty} (-1)^n T_{2n}\left(\frac{z}{n}\right) = \cos(2z).$$

4. $$\lim_{n\to\infty} (-1)^n T_{2n+1}\left(\frac{z}{n}\right) = \sin(2z).$$

5. $$\lim_{n\to\infty} T_n\left(1+\frac{z}{n^2}\right) = \cosh\sqrt{2z}.$$

6. $$\lim_{n\to\infty} T_n\left(\frac{n^2+z^2}{n^2-z^2}\right) = \cosh(2z).$$

7. $\lim\limits_{n\to\infty} (2z)^{-n} n^{-n/2} T_n(\sqrt{n}\, z) = \dfrac{1}{2} e^{-z^{-2}/4}.$

8. $\lim\limits_{n\to\infty} T_n\left(\sqrt{1+\dfrac{z^2}{n^2}}\right) = \cosh z.$

9. $\lim\limits_{n\to\infty} T_n\left(\dfrac{n}{\sqrt{n^2+z^2}}\right) = \cos z.$

10. $\lim\limits_{n\to\infty} \left(z-\sqrt{z^2-1}\right)^n U_n(z) = \left(2-2z^2+2z\sqrt{z^2-1}\right)^{-1}$ $[\operatorname{Re} z > 1].$

11. $\lim\limits_{n\to\infty} (2zn)^{-n} U_n(1+nz) = e^{1/z}.$

12. $\lim\limits_{n\to\infty} (-1)^n U_{2n}\left(\dfrac{z}{n}\right) = \cos(2z).$

13. $\lim\limits_{n\to\infty} (-1)^n U_{2n+1}\left(\dfrac{z}{n}\right) = \sin(2z).$

14. $\lim\limits_{n\to\infty} \dfrac{1}{n} U_n\left(1+\dfrac{z}{n^2}\right) = \dfrac{\sinh\sqrt{2z}}{\sqrt{2z}}.$

15. $\lim\limits_{n\to\infty} \dfrac{1}{n} U_n\left(\dfrac{n^2+z^2}{n^2-z^2}\right) = \dfrac{\sinh(2z)}{2z}.$

16. $\lim\limits_{n\to\infty} (2z)^{-n} n^{-n/2} U_n(\sqrt{n}\, z) = e^{-z^{-2}/4}.$

17. $\lim\limits_{n\to\infty} \dfrac{1}{n} U_n\left(\sqrt{1+\dfrac{z^2}{n^2}}\right) = \dfrac{\sinh z}{z}.$

18. $\lim\limits_{n\to\infty} \dfrac{1}{n} U_n\left(\dfrac{n}{\sqrt{n^2+z^2}}\right) = \dfrac{\sin z}{z}.$

2.1.6. The Hermite polynomials $H_n(z)$

1. $\lim\limits_{n\to\infty} \dfrac{1}{(2nz)^n} H_n(nz) = e^{-z^{-2}/4}.$

2. $\lim\limits_{n\to\infty} \left(-\dfrac{e}{4n}\right)^n H_{2n}\left(\dfrac{z}{\sqrt{n}}\right) = 2^{1/2}\cos(2z).$

3. $\lim\limits_{n\to\infty} \dfrac{(-e)^n}{(4n)^{n+1/2}} H_{2n+1}\left(\dfrac{z}{\sqrt{n}}\right) = 2^{1/2}\sin(2z).$

2.1.7. The Laguerre polynomials $L_n^\lambda(z)$

1. $\lim\limits_{n\to\infty} n^{-\lambda} L_n^\lambda\left(\dfrac{z}{n}\right) = z^{-\lambda/2} J_\lambda(2\sqrt{z})$ [82].

2. $\lim\limits_{\lambda\to\infty} \lambda^{-n/2} L_n^\lambda(\lambda-\sqrt{\lambda}\, z) = \dfrac{2^{-n/2}}{n!} H_n\left(\dfrac{z}{\sqrt{2}}\right)$ [75].

3. $\lim\limits_{s\to\infty} s^{-n/2} L_n^{\lambda+s}\left(s\frac{\sqrt{s}-z}{\sqrt{s}-t}\right) = \frac{2^{-n/2}}{n!} H_n\left(\frac{z-t}{\sqrt{2}}\right)$ [[80], (3)].

2.1.8. The Gegenbauer polynomials $C_n^\lambda(z)$

1. $\lim\limits_{\lambda\to 0} \lambda^{-1} C_n^\lambda(z) = \frac{2}{n} T_n(z).$

2. $\lim\limits_{\lambda\to\infty} \lambda^{-n} C_n^\lambda(z) = \frac{(-2z)^n}{n!}.$

3. $\lim\limits_{n\to\infty} n^{1-\lambda} z^{n/2} C_n^\lambda\left(\frac{z+1}{2\sqrt{z}}\right) = \frac{(1-z)^{-\lambda}}{\Gamma(\lambda)}$ $[|z|<1]$.

4. $\lim\limits_{\lambda\to\infty} \lambda^{-n/2} C_n^\lambda\left(\frac{z}{\sqrt{\lambda}}\right) = \frac{1}{n!} H_n(z).$

5. $\lim\limits_{n\to\infty} (-2z)^{-n}\, n^{\lambda-n} C_n^{\lambda-n}(nz) = (2z)^{-\lambda}\, I_{-\lambda}\left(\frac{1}{z}\right).$

6. $\lim\limits_{n\to\infty} (-2z)^{-n} n^{\lambda-n} C_n^\lambda(1+nz) = (2z)^{-\lambda} e^{1/z} I_{-\lambda}\left(\frac{1}{z}\right).$

7. $\lim\limits_{n\to\infty} (-1)^n n^{1-\lambda} C_{2n}^\lambda\left(\frac{z}{n}\right) = \frac{\cos(2z)}{\Gamma(\lambda)}.$

8. $\lim\limits_{n\to\infty} (-1)^n n^{1-\lambda} C_{2n+1}^\lambda\left(\frac{z}{n}\right) = \frac{\sin(2z)}{\Gamma(\lambda)}.$

9. $$\lim_{n\to\infty} (-1)^n n^{1/2} \lambda^{-n\lambda} (\lambda-1)^{n\lambda-n} C_{2n}^{n\lambda-n+\mu}\left(\frac{z}{n}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{\lambda}{\lambda-1}\right)^{\mu-1/2} \cos\left(2\sqrt{\lambda}\, z\right) \quad [\lambda\notin[0,\,1]].$$

10. $$\lim_{n\to\infty} (-1)^n n^{1/2} \lambda^{-n\lambda} (\lambda-1)^{n\lambda-n} C_{2n+1}^{n\lambda-n+\mu}\left(\frac{z}{n}\right) = \frac{1}{\sqrt{2\pi}} \frac{\lambda^\mu}{(\lambda-1)^{\mu-1/2}} \sin\left(2\sqrt{\lambda}\, z\right) \quad [\lambda\notin[0,\,1]].$$

11. $\lim\limits_{n\to\infty} \left(\frac{n}{2}\right)^\lambda C_n^{\lambda-n/2}\left(\frac{z}{\sqrt{n}}\right) = \frac{1}{\Gamma(1-\lambda)}\, {}_1F_1\left(\begin{matrix}\lambda;\ -\frac{z^2}{2}\\ \frac{1}{2}\end{matrix}\right).$

12. $\lim\limits_{n\to\infty} n^{1-2\lambda} C_n^\lambda\left(\frac{n}{\sqrt{n^2+z^2}}\right) = \frac{\sqrt{\pi}}{\Gamma(\lambda)} (2z)^{1/2-\lambda} J_{\lambda-1/2}(z).$

2.1.9. The Jacobi polynomials $P_n^{(\rho,\sigma)}(z)$

1. $\lim\limits_{\rho\to\infty} \rho^{-n} P_n^{(\rho,\,\sigma-\rho-n)}(\rho z) = \left(-\frac{z}{2}\right)^n L_n^{-\sigma-n-1}\left(\frac{2}{z}\right).$

2. $\lim\limits_{\sigma\to\infty} P_n^{(\rho,\,\sigma+a)}\left(1+\frac{z}{\sigma}\right) = L_n^\rho\left(-\frac{z}{2}\right).$

3. $\lim\limits_{\sigma\to\infty} P_n^{(\rho,\,-\sigma+a)}\left(\dfrac{\sigma+z}{\sigma-z}\right) = L_n^{\rho}(z).$

4. $\lim\limits_{\rho\to\infty} \rho^{-n/2} P_n^{(\rho,\,\rho)}\left(\dfrac{z}{\sqrt{\rho}}\right) = \dfrac{2^{-n}}{n!} H_n(z).$

5. $\lim\limits_{n\to\infty} n^{-\rho} P_n^{(\rho,\,\sigma-n)}\left(1+\dfrac{z}{n}\right) = \dfrac{1}{\Gamma(\rho+1)}\, {}_1F_1\left(\begin{matrix}\rho+\sigma+1\\ \rho+1;\ \frac{z}{2}\end{matrix}\right).$

6. $\lim\limits_{n\to\infty} n^{-\rho} P_n^{(\rho,\,-n-\rho/2-1/2)}\left(1+\dfrac{z}{n}\right) = \dfrac{\sqrt{\pi}}{\Gamma\left(\frac{\rho+1}{2}\right)} \left(\dfrac{2}{z}\right)^{\rho/2} e^{z/4} I_{\rho/2}\left(\dfrac{z}{4}\right).$

7. $\lim\limits_{n\to\infty} n^{-1/2} P_n^{(1/2,\,-n-1)}\left(1+\dfrac{z^2}{n}\right) = \dfrac{\sqrt{2}}{z}\,\operatorname{erfi}\left(\dfrac{z}{\sqrt{2}}\right).$

8. $\lim\limits_{n\to\infty} n^{-1/2} P_n^{(1/2,\,-n-1/2)}\left(1+\dfrac{z^2}{n}\right) = \dfrac{\sqrt{2}}{z} e^{z^2/2}\operatorname{erf}\left(\dfrac{z}{\sqrt{2}}\right).$

9. $\lim\limits_{n\to\infty} n^{-\rho} P_n^{(\rho,\,-n-\rho)}\left(1+\dfrac{z}{n}\right) = \dfrac{\left(\frac{2}{z}\right)^{\rho}}{\Gamma(\rho)} e^{z/2}\gamma\left(\rho,\ \dfrac{z}{2}\right).$

10. $\lim\limits_{n\to\infty} n^{-\rho} P_n^{(\rho,\,-n-m-\rho-1)}\left(1+\dfrac{z}{n}\right) = \dfrac{m!}{\Gamma(\rho+m+1)} L_m^{\rho}\left(\dfrac{z}{2}\right).$

11. $\lim\limits_{n\to\infty} n^{-\rho} P_n^{(\rho,\,n\sigma-\rho-1)}\left(1+\dfrac{z}{n^2}\right) = \left(\dfrac{2}{\sigma z+z}\right)^{\rho/2} I_{\rho}\left(\sqrt{2(\sigma+1)z}\right).$

2.1.10. Hypergeometric functions

1. $\lim\limits_{s\to\infty} s^{-a}\, {}_1F_1\left(\begin{matrix}a;\ s^2-sz\\ b+s^2\end{matrix}\right) = e^{z^2/4} D_{-a}(z)$ [33].

2. $\lim\limits_{s\to\infty} s^{-a/2}\, {}_2F_1\left(\begin{matrix}a,\ b+s\\ c+\frac{s}{2};\ \frac{1}{2}-\frac{z}{2\sqrt{s}}\end{matrix}\right) = e^{z^2/4} D_{-a}(z)$ [[75], (6)].

Chapter 3

Indefinite Integrals

3.1. Elementary Functions

3.1.1. The logarithmic function

1. $$\int \frac{\ln x \ln^n(ax+b)}{ax+b}\,dx = -\frac{1}{(n+1)a}\ln\left(-\frac{a}{b}\right)\ln^{n+1}(ax+b) - \frac{n!}{a}\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!}\ln^{n-k}(ax+b)\,\mathrm{Li}_{k+2}\left(\frac{ax}{b}+1\right).$$

2. $$\int \frac{\ln^2 x \ln(ax+b)}{(cx+d)^{n+1}}\,dx = \frac{a}{nc}\int \frac{\ln^2 x}{(ax+b)(cx+d)^n}\,dx + \frac{2}{nc}\int \frac{\ln x \ln(ax+b)}{x(cx+d)^n}\,dx - \frac{1}{nc}\frac{\ln^2 x \ln(ax+b)}{(cx+d)^n} \quad [n \geq 1].$$

3. $$\int \frac{\ln x \ln^n(ax+b)}{(ax+b)^{m+1}}\,dx = \frac{n}{m}\int \frac{\ln x \ln^{n-1}(ax+b)}{(ax+b)^{m+1}}\,dx + \frac{1}{ma}\int \frac{\ln x \ln^n(ax+b)}{x(ax+b)^m}\,dx - \frac{1}{ma}\frac{\ln x \ln^n(ax+b)}{(ax+b)^m} \quad [n \geq 1].$$

4. $$\int \frac{\ln x \ln(ax+b)\ln(cx+d)}{x^{n+1}}\,dx = \frac{c}{n}\int \frac{\ln x \ln(ax+b)}{x^n(cx+d)}\,dx + \frac{a}{n}\int \frac{\ln x \ln(cx+d)}{x^n(ax+b)}\,dx + \frac{1}{n}\int \frac{\ln(ax+b)\ln(cx+d)}{x^{n+1}}\,dx - \frac{\ln x}{nx^n}\ln(ax+b)\ln(cx+d) \quad [n \geq 1].$$

5. $$\int \frac{\ln(ax+b)\ln(cx+d)}{x^{n+1}}\,dx = \frac{c}{n}\int \frac{\ln(ax+b)}{x^n(cx+d)}\,dx + \frac{a}{n}\int \frac{\ln(cx+d)}{x^n(ax+b)}\,dx - \frac{1}{nx^n}\ln(ax+b)\ln(cx+d) \quad [n \geq 1].$$

6. $$\int \frac{\ln^2 x \ln^2(ax+b)}{(ax+b)^{n+1}}\,dx = \frac{2}{n}\int \frac{\ln^2 x \ln(ax+b)}{(ax+b)^{n+1}}\,dx + \frac{2}{na}\int \frac{\ln x \ln^2(ax+b)}{x(ax+b)^n}\,dx - \frac{1}{na(ax+b)^n}\ln^2 x \ln^2(ax+b) \quad [n \geq 1].$$

7. $$\int \ln x \ln^n(ax+b)\,dx = \frac{ax+b}{a}\ln x \ln^n(ax+b) - n\int \ln x \ln^{n-1}(ax+b)\,dx - \frac{1}{a}\int \frac{ax+b}{x}\ln^n(ax+b)\,dx.$$

3.2. Special Functions

3.2.1. The Bessel functions $J_\nu(x)$, $Y_\nu(x)$, $I_\nu(x)$ and $K_\nu(x)$

Notation: $Z_\nu(x)$, $\overline{Z}(x) = J_\nu(x)$ or $Y_\nu(x)$.

1. $$\int x^{\nu+2n+1} J_\nu(x)\,dx = (-2)^n n!\, x^{\nu+n+1}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\left(\frac{x}{2}\right)^k J_{\nu+n-k+1}(x).$$

2. $$\int x^n J_\nu(x)\,dx = \sum_{k=0}^{n} 2^k\left(\frac{\nu-n+1}{2}\right)_k x^{n-k} J_{\nu+k+1}(x) + 2^n\left(\frac{\nu-n+1}{2}\right)_n \int J_{\nu+n}(x)\,dx.$$

3. $$\int x^{\nu+1}\ln x\, J_0(x)\, J_\nu(x)\,dx = \frac{x^{\nu+1}}{4(\nu+1)^2}\{2J_0(x)[(\nu+1)\,J_{\nu+1}(x) + x(\nu\ln x + \ln x - 1)\,J_\nu(x)] + x[\pi(\nu+1)[J_\nu(x)\,Y_0(x) + J_{\nu+1}(x)\,Y_1(x)] + 2(\nu\ln x + \ln x - 1)\,J_1(x)\,J_{\nu+1}(x)]\} - \frac{\sqrt{\pi}}{4}G_{35}^{22}\left(x^2\,\middle|\,\begin{matrix}\frac{\nu}{2}+1,\ \frac{\nu+3}{2},\ \nu+\frac{1}{2}\\ \nu+1,\ \nu+1,\ 0,\ 1,\ \nu+\frac{1}{2}\end{matrix}\right).$$

4. $$\int \sin x \ln x\, J_0(x)dx = (x\sin x\ln x - \cos x - 2x\sin x)\,J_0(x) + x\cos x(2-\ln x)\,J_1(x).$$

5. $$\int \cos x\ln x\, J_0(x)dx = (\sin x - 2x\cos x + x\cos x\ln x)\,J_0(x) + x\sin x(\ln x - 2)\,J_1(x).$$

6. $$\int \frac{1}{x\,J_\nu^2(x)} f\left(\frac{J_{-\nu}(x)}{J_\nu(x)}\right)dx = -\frac{\pi}{2}\csc(\nu\pi) F\left(\frac{J_{-\nu}(x)}{J_\nu(x)}\right) \qquad [f(x) = F'(x)].$$

7. $$\int \frac{1}{x\,J_\nu^2(x)} f\left(\frac{Y_\nu(x)}{J_\nu(x)}\right)dx = \frac{\pi}{2} F\left(\frac{Y_\nu(x)}{J_\nu(x)}\right) \qquad [f(x) = F'(x)].$$

8. $$\int \frac{[J_\nu(x)]^{n-1}}{x\,[J_{-\nu}(x)]^{n+1}}\,dx = \frac{\pi}{2n\sin(\nu\pi)}\left[\frac{J_\nu(x)}{J_{-\nu}(x)}\right]^n.$$

9. $$\int \frac{[J_\nu(x)]^{n-1}}{x\,[Y_\nu(x)]^{n+1}}\,dx = -\frac{\pi}{2n}\left[\frac{J_\nu(x)}{Y_\nu(x)}\right]^n.$$

10. $$\int x^{n\nu} Z_\nu^{n-1}(x) Z_{\nu-1}(x)\,dx = \frac{1}{n}x^{n\nu} Z_\nu^n(x).$$

11. $\displaystyle\int x^{-n\nu} Z_\nu^{n-1}(x) Z_{\nu+1}(x)\, dx = -\frac{1}{n} x^{-n\nu} Z_\nu^n(x).$

12. $\displaystyle\int \frac{Z_{\nu-1}(x)}{Z_\nu(x)}\, dx = \ln Z_\nu(x) + \nu \ln x.$

13. $\displaystyle\int \frac{Z_{\nu+1}(x)}{Z_\nu(x)}\, dx = -\ln Z_\nu(x) + \nu \ln x.$

14. $\displaystyle\int x^{2n\nu-1} [Z_\nu(x)]^2 \frac{[Z_{\nu+1}(x)]^{n-1}}{[Z_{\nu-1}(x)]^{n+1}}\, dx = \frac{x^{2n\nu}}{2n\nu} \left[\frac{Z_{\nu+1}(x)}{Z_{\nu-1}(x)}\right]^n.$

15. $\displaystyle\int \frac{1}{x} \left[\frac{Z_0(x)}{Z_1(x)}\right]^2 dx = \frac{Z_0(x)}{x\, Z_1(x)} - \ln x.$

16. $\displaystyle\int \frac{1}{x} \frac{[Z_\nu(x)]^2}{Z_{\nu-1}(x) Z_{\nu+1}(x)}\, dx = \frac{1}{2\nu} \ln \frac{Z_{\nu+1}(x)}{Z_{\nu-1}(x)} + \ln x.$

17. $\displaystyle\int \frac{[Z_\nu(x)]^{n-1}}{x\, [\overline{Z}_\nu(x)]^{n+1}}\, dx = \frac{1}{nx\, [Z_\nu'(x)\overline{Z}_\nu(x) - Z_\nu(x)\overline{Z}_\nu'(x)]} \left(\frac{Z_\nu(x)}{\overline{Z}_\nu(x)}\right)^n.$

18. $\displaystyle\int \frac{1}{x\, J_\nu^2(x)} f\left(\frac{H_\nu^{(1)}(x)}{J_\nu(x)}\right) dx = -\frac{\pi i}{2} \csc(\nu\pi) F\left(\frac{H_\nu^{(1)}(x)}{J_\nu(x)}\right) \qquad [f(x) = F'(x)].$

19. $\displaystyle\int \frac{1}{x\, J_\nu^2(x)} f\left(\frac{H_\nu^{(2)}(x)}{J_\nu(x)}\right) dx = \frac{\pi i}{2} \csc(\nu\pi) F\left(\frac{H_\nu^{(2)}(x)}{J_\nu(x)}\right) \qquad [f(x) = F'(x)].$

20. $\displaystyle\int \frac{1}{x[H_\nu^{(1)}(x)]^2} f\left(\frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)}\right) dx = \frac{\pi i}{4} F\left(\frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)}\right) \qquad [f(x) = F'(x)].$

21. $$\int x^\mu I_\nu(x)\, dx = \sum_{k=0}^{n-1} 2^k \left(\frac{\nu-\mu+1}{2}\right)_k z^{\mu-k} I_{\nu+k+1}(x) + 2^n \left(\frac{\nu-\mu+1}{2}\right)_n \int x^{\mu-n} I_{\nu+n}(x)\, dx.$$

22. $$\int x^\mu I_\nu(x)\, dx = \frac{(-2)^{-n}}{\left(\frac{1+\mu-\nu}{2}\right)_n} \times \left[\int x^{\mu+n} I_{\nu-n}(x)\, dx - \sum_{k=1}^{n} 2^{n-k} \left(\frac{\nu-\mu+1}{2} - n\right)_{n-k} z^{\mu+k} I_{\nu-k+1}(x)\right].$$

23. $$\int x \ln x\, I_0^2(x)\, dx = \frac{x^2}{2} \ln x [I_0^2(x) - I_1^2(x)] - \frac{x^2}{2} \left[I_0^2(x) - I_1^2(x) - \frac{1}{x} I_0(x) I_1(x)\right].$$

24. $\displaystyle\int \frac{1}{x\, I_\nu^2(x)} f\left[\frac{I_{-\nu}(x)}{I_\nu(x)}\right] dx = -\frac{\pi}{2} \csc(\nu\pi) F\left[\frac{I_{-\nu}(x)}{I_\nu(x)}\right] \qquad [f(x) = F'(x)].$

25. $\displaystyle\int x^{n\nu} I_\nu^{n-1}(x)\, I_{\nu-1}(x)\, dx = \frac{1}{n} x^{n\nu} I_\nu^n(x).$

26. $\displaystyle\int x^{-n\nu} I_\nu^{n-1}(x)\, I_{\nu+1}(x)\, dx = \frac{1}{n} x^{-n\nu} I_\nu^n(x).$

27. $\displaystyle\int \frac{I_{\nu+1}(x)}{I_\nu(x)}\, dx = \ln I_\nu(x) - \nu \ln x.$

28. $\displaystyle\int \frac{I_{\nu-1}(x)}{I_\nu(x)}\, dx = \ln I_\nu(x) + \nu \ln x.$

29. $\displaystyle\int e^{\pm x} \ln x\, K_0(x)\, dx = x e^{\pm x}(\ln x - 2)[K_0(x) \pm K_1(x)] \pm e^{\pm x} K_0(x).$

30. $\displaystyle\int \frac{1}{x I_\nu^2(x)} f\left[\frac{K_\nu(x)}{I_\nu(x)}\right] dx = -F\left[\frac{K_\nu(x)}{I_\nu(x)}\right] \qquad [f(x) = F'(x)].$

31. $\displaystyle\int \frac{[I_\nu(x)]^{n-1}}{x\,[K_\nu(x)]^{n+1}}\, dx = \frac{1}{nx\,[I_\nu'(x)\, K_\nu(x) - I_\nu(x)\, K_\nu'(x)]} \left(\frac{I_\nu(x)}{K_\nu(x)}\right)^n.$

32. $\displaystyle\int \frac{K_{\nu-1}(x)}{K_\nu(x)}\, dx = -\ln K_\nu(x) - \nu \ln x.$

33. $\displaystyle\int \frac{K_{\nu+1}(x)}{K_\nu(x)}\, dx = -\ln K_\nu(x) + \nu \ln x.$

34. $\displaystyle\int x^{n\nu} K_\nu^{n-1}(x)\, K_{\nu-1}(x)\, dx = -\frac{1}{n} x^{n\nu} K_\nu^n(x).$

35. $\displaystyle\int x^{-n\nu} K_\nu^{n-1}(x)\, K_{\nu+1}(x)\, dx = -\frac{1}{n} x^{-n\nu} K_\nu^n(x).$

36. $\displaystyle\int K_n(x)\, K_{n+1}(x)\, dx = \frac{(-1)^{n+1}}{2} K_0^2(x) + (-1)^{n+1} \sum_{j=1}^{n} (-1)^j K_j^2(x).$

37. $\displaystyle\int x^{-p} K_\mu(x)\, K_\nu(x)\, dx = I(p, \mu, n),$

$$I(p, \mu, \nu) = -\frac{1}{\mu+\nu+p-1}\big[I(p-1, \mu-1, \nu) + I(p-1, \mu, \nu-1) + x^{1-p} K_\mu(x)\, K_\nu(x)\big] \quad [\mu+\nu+p-1 \neq 0].$$

38. $$I(p, \mu, \nu) = \frac{1}{\mu+\nu-p+1}\big[I(p-1, \mu+1, \nu) + I(p-1, \mu, \nu+1) + x^{1-p} K_\mu(x)\, K_\nu(x)\big] \quad [\mu+\nu-p+1 \neq 0].$$

39. $\displaystyle I(1, \mu, \nu) = -\frac{x}{\mu^2-\nu^2}[K_\mu(x)\, K_\nu'(x) + K_\mu'(x)\, K_\nu(x)].$

40. $$= -\frac{1}{\mu+\nu} K_\mu(x)\, K_\nu(x) - \frac{x}{\mu^2-\nu^2}\left[K_{\mu-1}(x)\, K_\nu(x) - K_\mu(x)\, K_{\nu-1}(x)\right],$$

$$I(1,\, m,\, m) = -\frac{1}{2m}\left[(-1)^m K_0^2(x) + 2\sum_{j=1}^{m-1}(-1)^{j+m} K_j^2(x) + K_m^2(x)\right] \quad [m \ge 1].$$

3.2.2. The Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

1. $$\int x^\mu\, \mathbf{L}_\nu(x)\, dx = \sum_{k=0}^{n-1} 2^k \left(\frac{1-\mu-\nu}{2}\right)_k z^{\mu-k}\, \mathbf{L}_{\nu-k-1}(x) - \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{2^{2k-\nu+1}\left(\frac{1-\mu-\nu}{2}\right)_k z^{\mu+\nu-2k}}{(\mu+\nu-2k)\Gamma\left(\nu-k+\frac{1}{2}\right)} + 2^n \left(\frac{1-\mu-\nu}{2}\right)_n \int x^{\mu-n}\, \mathbf{L}_{\nu-n}(x)\, dx.$$

2. $$\int x^{2n-\nu+1}\, \mathbf{L}_\nu(x)\, dx = n!\, z^{2n-\nu+1} \sum_{k=0}^{n-1} \frac{\left(-\frac{2}{z}\right)^k}{(n-k)!}\, \mathbf{L}_{\nu-k-1}(x) - \frac{n!\, z^{2n+1}}{2^{\nu-1}\sqrt{\pi}} \sum_{k=0}^{n-1}(-1)^k \frac{\left(\frac{2}{z}\right)^{2k}}{(n-k)!\,(2n-2k+1)\Gamma\left(\nu-k+\frac{1}{2}\right)} + (-2)^n n! \left[z^{n-\nu+1}\, \mathbf{L}_{\nu-n-1}(x) - \frac{2^{n-\nu+1} z}{\sqrt{\pi}\,\Gamma\left(\nu-n+\frac{1}{2}\right)}\right].$$

3. $$\int x \ln x\, \mathbf{H}_0(x)\, dx = x \ln x\, \mathbf{H}_1(x) + \mathbf{H}_0(x) - \frac{2x}{\pi}.$$

4. $$\int x \ln x\, \mathbf{L}_0(x)\, dx = x \ln x\, \mathbf{L}_1(x) - \mathbf{L}_0(x) + \frac{2x}{\pi}.$$

3.2.3. The Airy functions $\mathrm{Ai}\,(z)$ and $\mathrm{Bi}\,(z)$

Notation: $y = a\,\mathrm{Ai}\,(x) + b\,\mathrm{Bi}\,(x)$, a, b are constants.

1. $$\int x^n y\, dx = x^{n-1}y' - (n-1)x^{n-2}y + (n-1)(n-2)\int x^{n-3}y\, dx$$ [[53], (4)].

2. $$\int xy\, dx = y'$$ [[53], (6)].

3. $$\int x^2 y\, dx = xy' - y$$ [[53], (7)].

4. $$\int x^3 y\, dx = x^2 y' - 2xy + 2\int y\, dx$$ [[53], (8)].

5. $\displaystyle\int x^4 y\,dx = x^3 y' - 3x^2 y + 6y'$ [[53], (9)].

6. $\displaystyle\int x^n y^2\,dx = \frac{1}{2n+1}\left[x^{n+1}y^2 - x^n y'^2 + nx^{n-1}yy' - \frac{n}{2}(n-1)x^{n-2}y^2 + \frac{n}{2}(n-1)(n-2)\int x^{n-3}y^2\,dx\right]$ [[53], (11)].

7. $\displaystyle\int y^2\,dx = xy^2 - y'^2$ [[53], (12)].

8. $\displaystyle\int xy^2\,dx = \frac{1}{3}\left(x^2y^2 - xy'^2 + yy'\right)$ [[53], (13)).

9. $\displaystyle\int x^2y^2\,dx = \frac{1}{5}\left(x^3y^2 - x^2y'^2 + 2xyy' - y^2\right)$ [[53], (14)].

10. $$\int x^n y^3\,dx = x^{n-1}y^2y' - \frac{n-1}{3}x^{n-2}y^3 - \frac{2}{3}x^{n-2}y'^3 + \frac{1}{3}(n-1)(n-2)\int x^{n-3}y^3\,dx + \frac{2}{3}(n-2)\int x^{n-3}y'^3\,dx$$ [[53], (24)].

11. $$\int x^n y^3\,dx = x^{n-1}y^2y' - \frac{1}{9}(7n-11)x^{n-2}y^3 - \frac{2}{3}x^{n-2}y'^3 + \frac{2}{3}(n-2)x^{n-3}yy'^2 - \frac{1}{3}(n-2)(n-3)x^{n-4}y^2y' + \frac{1}{9}(n-2)(n-3)(n-4)x^{n-5}y^3 + \frac{10}{9}(n-2)^2\int x^{n-3}y^3\,dx - \frac{1}{9}(n-2)(n-3)(n-4)(n-5)\int x^{n-6}y^3\,dx$$ [[53], (27)].

12. $\displaystyle\int x^2y^3\,dx = xy^2y' - \frac{1}{3}y^3 - \frac{2}{3}y'^3$ [[53], (25)].

13. $\displaystyle\int x^3y^3\,dx = x^2y^2y' - \frac{10}{9}xy^3 - \frac{2}{3}xy'^3 + \frac{2}{3}yy'^2 + \frac{10}{9}\int y^3\,dx$ [[53], (28)].

14. $$\int x^4y^3\,dx = x^3y^2y' - \frac{17}{9}x^2y^3 - \frac{2}{3}x^2y'^3 + \frac{4}{3}xyy'^2 - \frac{2}{3}y^2y' + \frac{40}{9}\int xy^3\,dx$$ [[53], (29)].

15. $$\int x^5y^3\,dx = x^4y^2y' - \frac{8}{3}x^3y^3 - \frac{2}{3}x^3y'^3 + 2x^2yy'^2 + 8xy^2y' - \frac{8}{3}y^3 - \frac{20}{3}y'^3$$ [[53], (30)].

16. $$\int x^n y^4\,dx = \frac{1}{5n+3}\Big[3x^{n+1}y^4 - 6x^n y^2 y'^2 + 2nx^{n-1}y^3y' + 3x^{n-1}y'^4 - \frac{n}{2}(n-1)x^{n-2}y^4 + \frac{n}{2}(n-1)(n-2)\int x^{n-3}y^4\,dx - 3(n-1)\int x^{n-2}y'^4\,dx\Big]$$ [[53], (38)].

17. $$\int x^n y^4\,dx = \frac{1}{8n}\Big[3x^{n+1}y^4 - 6x^n y^2y'^2 + (5n-3)x^{n-1}y^3y' + 3x^{n-1}y'^4 - 3(n-1)x^{n-2}yy'^3 - \frac{1}{4}(n-1)(5n-3)x^{n-2}y^4 + \frac{3}{4}(n-1)(n-2)x^{n-4}y'^4 - \frac{3}{4}(n-1)(n-2)(n-4)\int x^{n-5}y'^4\,dx + \frac{1}{4}(n-1)(n-2)(5n-3)\int x^{n-3}y^4\,dx\Big]$$ [[53], (39)].

18. $$\int xy^4\,dx = \frac{1}{8}(3x^2y^4 - 6xy^2y'^2 + 2xy^3y' + 3y'^4)$$ [[53], (40)].

19. $$\int x^2y^4\,dx = \frac{1}{16}\left(3x^3y^4 - 6x^2y^2y'^2 + 7xy^3y' + 3xy'^4 - 3yy'^3 - \frac{7}{4}y^4\right)$$ [[53], (41)].

20. $$\int x^n y'\,dx = x^n y - n\int x^{n-1}y\,dx$$ [[53], (10)].

21. $$\int yy'\,dx = \frac{1}{2}y^2$$ [[53], (16)].

22. $$\int xyy'\,dx = \frac{1}{2}y'^2$$ [[53], (17)].

23. $$\int x^2yy'\,dx = \frac{1}{3}\left(\frac{1}{2}x^2y^2 + xy'^2 - yy'\right)$$ [[53], (18)].

24. $$\int x^n y^2y'\,dx = \frac{1}{3}x^ny^3 - \frac{n}{3}\int x^{n-1}y^3\,dx$$ [[53], (34)].

25. $$\int x^ny^3y'\,dx = \frac{1}{4}x^ny^4 - \frac{n}{4}\int x^{n-1}y^4\,dx$$ [[53], (10)].

26. $$\int x^ny'^2\,dx = \frac{1}{2n+3}\Big[-x^{n+2}y^2 + x^{n+1}y'^2 + (n+2)x^nyy' - \frac{n}{2}(n+2)x^{n-1}y^2 + \frac{1}{2}(n-1)n(n+2)\int x^{n-2}y^2\,dx\Big]$$ [[53], (20)].

27. $$\int y'^2\,dx = \frac{1}{3}\left(-x^2y^2 + xy'^2 + 2yy'\right)$$ [[53], (21)].

28. $$\int xy'^2\,dx = \frac{1}{5}\left(-x^3y^2 + x^2y'^2 + 3xyy' - \frac{3}{2}y^2\right)$$ [[53], (22)].

29. $\displaystyle\int x^2 y'^2\,dx = \frac{1}{7}\left(-x^4y^2 + x^3y'^2 + 4x^2yy' - 4y'^2\right)$ [[53], (23)].

30. $$\int x^n yy'^2\,dx = \frac{1}{2}x^n y^2 y' - \frac{n}{6}x^{n-1}y^3 - \frac{1}{2}\int x^{n+1}y^3\,dx + \frac{n}{6}(n-1)\int x^{n-2}y^3\,dx$$ [[53], (34)].

31. $\displaystyle\int x^3y^2y'\,dx = \frac{1}{3}x^3y^3 - xy^2y' + \frac{1}{3}y^3 + \frac{2}{3}y'^3$ [[53], (36)].

32. $\displaystyle\int xyy'^2\,dx = \frac{1}{3}y'^3$ [[53], (37)].

33. $$\int x^ny^2y'^2\,dx = \frac{1}{3}x^ny^3y' - \frac{n}{12}x^{n-1}y^4 - \frac{1}{3}\int x^{n+1}y^4\,dx - \frac{n}{12}(n-1)\int x^{n-2}y^4\,dx$$ [[53], (47)].

34. $$\int x^ny'^3\,dx = -\frac{2}{3}x^{n+1}y^3 + x^nyy'^2 - \frac{n}{2}x^{n-1}y^2y' + \frac{n}{6}(n-1)x^{n-2}y^3 + \frac{1}{6}(7n+4)\int x^ny^3\,dx - \frac{n}{6}(n-1)(n-2)\int x^{n-3}y^3\,dx$$ [[53], (26)].

35. $\displaystyle\int y'^3\,dx = -\frac{2}{3}xy^3 + yy'^2 + \frac{2}{3}\int y^3\,dx$ [[53], (31)].

36. $\displaystyle\int xy'^3\,dx = -\frac{2}{3}x^2y^3 + xyy'^2 - \frac{1}{2}y^2y' + \frac{11}{6}\int xy^3\,dx$ [[53], (32)].

37. $\displaystyle\int x^2y'^3\,dx = -\frac{2}{3}x^3y^3 + x^2yy'^2 + 2xy^2y' - \frac{2}{3}y^3 - 2y'^3$ [[53], (33)].

38. $$\int x^nyy'^3\,dx = \frac{1}{2}x^ny^2y'^2 - \frac{n}{6}x^{n-1}y^3y' - \frac{1}{4}x^{n+1}y^4 + \frac{n}{24}(n-1)x^{n-2}y^4 + \frac{5n+3}{12}\int x^ny^4\,dx - \frac{n}{24}(n-1)(n-2)\int x^{n-3}y^4\,dx$$ [[53], (48)].

39. $$\int x^ny'^4\,dx = \frac{1}{3(n+1)}\Big[3x^{n+3}y^4 - 6x^{n+2}y^2y'^2 + 2(n+2)x^{n+1}y^3y' + 3x^{n+1}y'^4 - \frac{1}{2}(n+1)(n+2)x^ny^4 - (5n+13)\int x^{n+2}y^4\,dx + \frac{n}{2}(n+1)(n+2)\int x^{n-1}y^4\,dx$$ [[53], (42)].

40. $$\int y'^4\,dx = \frac{3}{16}x^3y^4 - \frac{3}{8}x^2y^2y'^2 - \frac{9}{16}xy^3y' + \frac{3}{16}xy'^4 + \frac{13}{16}yy'^3 + \frac{9}{64}y^4$$ [[53], (45)].

41. $\displaystyle\int \frac{1}{\mathrm{Ai}^2(x)} f\left(\frac{\mathrm{Bi}\,(x)}{\mathrm{Ai}\,(x)}\right)dx = \pi F\left(\frac{\mathrm{Bi}\,(x)}{\mathrm{Ai}\,(x)}\right)$ $[f(x) = F'(x)]$.

42. $\displaystyle\int \frac{1}{\mathrm{Bi}^2(x)} f\left(\frac{\mathrm{Ai}\,(x)}{\mathrm{Bi}\,(x)}\right) dx = -\pi F\left(\frac{\mathrm{Ai}\,(x)}{\mathrm{Bi}\,(x)}\right) \qquad [f(x) = F'(x)].$

3.2.4. Various functions

1. $\displaystyle\int x^n \zeta(s,\, x)\, dx = -\frac{n!\, x^n}{s-1} \sum_{k=0}^{n} (-1)^k \frac{(2-s)_k}{(n-k)!} x^{-k} \zeta(s-k-1,\, x).$

2. $\displaystyle\int e^x \Gamma^2(\nu,\, x)\, dx = \frac{2}{\nu}\left[x^\nu \Gamma(\nu,\, x) - \Gamma(2\nu,\, x)\right] + e^x \Gamma^2(\nu,\, x).$

3. $\displaystyle\int \frac{x^{\nu-1} e^{-x}}{\gamma^2(\nu,\, x)} f\left[\frac{\Gamma(\nu,\, x)}{\gamma(\nu,\, x)}\right] dx = -\frac{1}{\Gamma(\nu)} F\left[\frac{\Gamma(\nu,\, x)}{\gamma(\nu,\, x)}\right] \qquad [f(x) = F'(x)].$

4. $\displaystyle\int \frac{1}{D_\nu^2(x)} f\left[\frac{D_\nu(-x)}{D_\nu(x)}\right] dx = \frac{\Gamma(-\nu)}{\sqrt{2\pi}} F\left[\frac{D_\nu(-x)}{D_\nu(x)}\right] \qquad [f(x) = F'(x)].$

5. $\displaystyle\int \frac{1}{D_\nu^2(x)} f\left[\frac{D_{-\nu-1}(ix)}{D_\nu(x)}\right] dx = i e^{i\nu\pi/2} F\left[\frac{D_{-\nu-1}(ix)}{D_\nu(x)}\right] \qquad [f(x) = F'(x)].$

6. $$\int P_\mu(x) P_\nu(x)\, dx = -\frac{1}{(\mu-\nu)(\mu+\nu+1)}$$
$$\times \left[\mu P_{\mu-1}(x) P_\nu(x) - \nu P_\mu(x) P_{\nu-1}(x) - (\mu-\nu) z P_\mu(x) P_\nu(x)\right].$$

7. $$\int [P_\nu(x)]^2\, dx = -\frac{1}{2\nu+1}$$
$$\times \left[P_{\nu-1}(x) P_\nu(x) - z[P_\nu(x)]^2 + \nu P_\nu(x) \frac{\partial P_{\nu-1}(x)}{\partial \nu} - \nu P_{\nu-1}(x) \frac{\partial P_\nu(x)}{\partial \nu}\right].$$

8. $\displaystyle\int x \ln x\, \mathbf{K}(x)\, dx = (x^2 \ln x - \ln x - x^2 + 1)\, \mathbf{K}(x) + (\ln x - 2)\, \mathbf{E}(x).$

9. $$\int x \ln x\, \mathbf{E}(x)\, dx = \frac{1}{3}\left(x^2 \ln x - \ln x - \frac{4}{3} x^2 + \frac{4}{3}\right) \mathbf{K}(x)$$
$$+ \frac{1}{3}\left(x^2 \ln x + \ln x - \frac{1}{3} x^2 - \frac{7}{3}\right) \mathbf{E}(x).$$

Chapter 4

Definite Integrals

4.1. Elementary Functions

4.1.1. Algebraic functions

Condition: $a > 0$.

1. $$\int_0^a x^{s-1}(a-x)^{t-1}[1-bx(a-x)]^{\nu}\,dx = a^{s+t-1}\,\mathrm{B}\,(s,\,t)$$
$$\times\ {}_3F_2\left(\begin{matrix} s,\, t,\, -\nu;\ \frac{a^2b}{4} \\ \frac{s+t}{2},\, \frac{s+t+1}{2} \end{matrix}\right) \quad [\operatorname{Re} s,\, \operatorname{Re} t > 0;\ |\arg(4-a^2b)| < \pi].$$

2. $$\int_0^a x^{-\nu-3/2}(a-x)^{-\nu-1/2}[1-bx(a-x)]^{\nu}\,dx$$
$$= \sqrt{\pi}\,\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu)}\left(\frac{4}{a^2}-b\right)^{\nu+1/2} \quad [\operatorname{Re}\nu < -1/2;\ |\arg(4-a^2b)| < \pi].$$

3. $$\int_0^a x^{-1/2}(a-x)^{-1/2}[1-bx(a-x)]^{n}\,dx$$
$$= \pi\left(1-\frac{a^2b}{4}\right)^{n/2} P_n\left(\frac{8-a^2b}{4\sqrt{4-a^2b}}\right) \quad [a > 0].$$

4. $$\int_0^a x^{-1/2}(a-x)^{1/2}[1-bx(a-x)]^{1/2}\,dx = a\,\mathbf{E}\left(\frac{a\sqrt{b}}{2}\right)$$
$$[|\arg(4-a^2b)| < \pi].$$

5. $$\int_0^a x^{-1/2}(a-x)^{-1/2}[1-bx(a-x)]^{1/2}\,dx = 2\,\mathbf{E}\left(\frac{a\sqrt{b}}{2}\right)$$
$$[|\arg(4-a^2b)| < \pi].$$

6. $$\int_0^a x^{1/2}(a-x)^{1/2}[1-bx(a-x)]^{-1/2}dx = \frac{2}{b}\left[\mathbf{K}\left(\frac{a\sqrt{b}}{2}\right) - \mathbf{E}\left(\frac{a\sqrt{b}}{2}\right)\right]$$
$$[|\arg(4-a^2b)| < \pi].$$

7. $$\int_0^a x^{-1/2}(a-x)^{-1/2}[1-bx(a-x)]^{-1/2}dx = 2\,\mathbf{K}\left(\frac{a\sqrt{b}}{2}\right)$$
$$[|\arg(4-a^2b)| < \pi].$$

8. $$\int_0^a x^{-1/4}(a-x)^{-3/4}[1-bx(a-x)]^{-1/2}dx$$
$$= \frac{4}{\sqrt{2+a\sqrt{b}}}\,\mathbf{K}\left(\frac{2^{1/2}a^{1/2}b^{1/4}}{\sqrt{2+a\sqrt{b}}}\right) \quad [|\arg(4-a^2b)| < \pi].$$

9. $$\int_0^a x^{-1/4}(a-x)^{-3/4}[1-bx(a-x)]^{-1}dx$$
$$= \frac{\pi\sqrt{2}}{\sqrt{4-a^2b}}\left[\left(1+\frac{a\sqrt{b}}{2}\right)^{1/2} + \left(1-\frac{a\sqrt{b}}{2}\right)^{1/2}\right] \quad [|\arg(4-a^2b)| < \pi].$$

10. $$\int_0^a x^{-1/2}(a-x)^{-1/2}[1-bx(a-x)]^{-1}dx = \frac{2\pi}{\sqrt{4-a^2b}} \quad [|\arg(4-a^2b)| < \pi].$$

11. $$\int_0^a x^{1/2}(a-x)^{1/2}[1-bx(a-x)]^{-1}dx = \frac{\pi}{b}\left[\left(1-\frac{a^2b}{4}\right)^{-1/2} - 1\right]$$
$$[|\arg(4-a^2b)| < \pi].$$

12. $$\int_0^a x^{1/2}(a-x)^{1/2}[1-bx(a-x)]^{-3/2}dx$$
$$= \frac{8}{b(4-a^2b)}\,\mathbf{E}\left(\frac{a\sqrt{b}}{2}\right) - \frac{2}{b}\,\mathbf{K}\left(\frac{a\sqrt{b}}{2}\right) \quad [|\arg(4-a^2b)| < \pi].$$

13. $$\int_0^a \frac{x^{-1/2}(a-x)^{-1/2}}{1+\sqrt{1-bx(a-x)}}\,dx = 2\,\mathbf{K}\left(\frac{a\sqrt{b}}{2}\right) - 2\,\mathbf{D}\left(\frac{a\sqrt{b}}{2}\right)$$
$$[|\arg(4-a^2b)| < \pi].$$

14. $$\int_0^a \frac{dx}{1+\sqrt{1-bx(a-x)}} = \frac{1}{\sqrt{b}}\ln\frac{2+a\sqrt{b}}{2-a\sqrt{b}} + \frac{2}{ab}\ln\left(1-\frac{a^2b}{4}\right)$$
$$[|\arg(4-a^2b)| < \pi].$$

15. $\displaystyle\int_0^a \frac{x^{1/2}(a-x)^{1/2}}{1+\sqrt{1-bx(a-x)}}\,dx = \frac{\pi}{b} - \frac{2}{b}\,\mathbf{E}\left(\frac{a\sqrt{b}}{2}\right)$ $\quad[|\arg(4-a^2b)|<\pi].$

16. $\displaystyle\int_0^a \frac{x^{-1/2}(a-x)^{1/2}}{1+\sqrt{1-bx(a-x)}}\,dx = a\,\mathbf{K}\left(\frac{a\sqrt{b}}{2}\right) - a\,\mathbf{D}\left(\frac{a\sqrt{b}}{2}\right)$ $\quad[|\arg(4-a^2b)|<\pi].$

17. $$\int_0^a x^s(a-x)^{s+1/2}[1+b\sqrt{x(a-x)}]^\nu\,dx = 2^{-2s-1}\sqrt{\pi}\,a^{2s+3/2}\frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)} \times {}_2F_1\left(\begin{matrix}-\nu,\ 2s+2\\ 2s+\frac{5}{2};\ -\frac{ab}{2}\end{matrix}\right) \quad [\operatorname{Re} s>-1;\ |\arg(2+ab)|<\pi].$$

18. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\left[1+b\sqrt{x(a-x)}\right]^n dx = \sqrt{2}\,\pi\left(1+\frac{ab}{2}\right)^{n/2} P_n\left(\frac{4+ab}{2^{3/2}\sqrt{2+ab}}\right) \quad [|\arg(2+ab)|<\pi].$$

19. $$\int_0^a x^{1/2}\left[1-b\sqrt{x(a-x)}\right]^{1/2} dx = \frac{\sqrt{a}}{4b}\left[\frac{1}{\sqrt{2ab}}\left(1+\frac{3ab}{2}\right)\left(1-\frac{ab}{2}\right)\ln\frac{\sqrt{2}+\sqrt{ab}}{\sqrt{2}-\sqrt{ab}}+\frac{3a}{2}-1\right] \quad [|\arg(2+ab)|<\pi].$$

20. $$\int_0^a x^{-1/2}\left[1-b\sqrt{x(a-x)}\right]^{1/2} dx = \sqrt{a}+\frac{1}{\sqrt{2b}}\left(1-\frac{ab}{2}\right)\ln\frac{\sqrt{2}+\sqrt{ab}}{\sqrt{2}-\sqrt{ab}} \quad [|\arg(2-ab)|<\pi].$$

21. $$\int_0^a x^{-1/4}(a-x)^{1/4}\left[1-b\sqrt{x(a-x)}\right]^{1/2} dx = \frac{\sqrt{2}}{3b}\left[(2-ab)\,\mathbf{K}\left(\sqrt{\frac{ab}{2}}\right)+2(ab-1)\,\mathbf{E}\left(\sqrt{\frac{ab}{2}}\right)\right] \quad [|\arg(2-ab)|<\pi].$$

22. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\left[1-b\sqrt{x(a-x)}\right]^{1/2} dx = 2^{3/2}\,\mathbf{E}\left(\sqrt{\frac{ab}{2}}\right) \quad [|\arg(2-ab)|<\pi].$$

23. $$\int_0^a x^{1/2}\left[1-b\sqrt{x(a-x)}\right]^{-1/2}dx = \frac{1}{\sqrt{2}\,b^{3/2}}\left(1+\frac{ab}{2}\right)\ln\frac{\sqrt{2}+\sqrt{ab}}{\sqrt{2}-\sqrt{ab}}-\frac{\sqrt{a}}{b} \qquad [|\arg(2+ab)|<\pi].$$

24. $$\int_0^a x^{-1/2}\left[1+b\sqrt{x(a-x)}\right]^{-1/2}dx = \sqrt{\frac{8}{b}}\arctan\sqrt{\frac{ab}{2}} \qquad [|\arg(2-ab)|<\pi].$$

25. $$\int_0^a x^{-1/2}\left[1-b\sqrt{x(a-x)}\right]^{-1/2}dx = \sqrt{\frac{2}{b}}\ln\frac{\sqrt{2}+\sqrt{ab}}{\sqrt{2}-\sqrt{ab}} \qquad [|\arg(2+ab)|<\pi].$$

26. $$\int_0^a x^{-1/4}(a-x)^{1/4}\left[1-b\sqrt{x(a-x)}\right]^{-1/2}dx = \sqrt{2}\,a\,\mathbf{D}\left(\sqrt{\frac{ab}{2}}\right) \qquad [|\arg(2+ab)|<\pi].$$

27. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\left[1-b\sqrt{x(a-x)}\right]^{-1/2}dx = 2^{3/2}\,\mathbf{K}\left(\sqrt{\frac{ab}{2}}\right) \qquad [|\arg(2-ab)|<\pi].$$

28. $$\int_0^a x^{1/2}\left[1-b\sqrt{x(a-x)}\right]^{-1}dx = \frac{2\sqrt{a}}{b}\left[\frac{2}{\sqrt{ab(2-ab)}}\arcsin\sqrt{\frac{ab}{2}}-1\right] \qquad [|\arg(2-ab)|<\pi].$$

29. $$\int_0^a x^{-1/2}\left[1-b\sqrt{x(a-x)}\right]^{-1}dx = \frac{4}{\sqrt{b(2-ab)}}\arcsin\sqrt{\frac{ab}{2}} \qquad [|\arg(2-ab)|<\pi].$$

30. $$\int_0^a x^{-1/4}(a-x)^{1/4}\left[1+b\sqrt{x(a-x)}\right]^{-1}dx = \frac{\sqrt{2}\,\pi}{b}\left[1-\left(1+\frac{ab}{2}\right)^{-1/2}\right] \qquad [|\arg(2+ab)|<\pi].$$

31. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\left[1+b\sqrt{x(a-x)}\right]^{-1}dx = \frac{2\pi}{\sqrt{ab+2}} \qquad [|\arg(2+ab)|<\pi].$$

32. $$\int_0^a x^{-1/2}\left[1+b\sqrt{x(a-x)}\right]^{-3/2}dx = \frac{4\sqrt{a}}{ab+2} \qquad [|\arg(2+ab)|<\pi].$$

33. $$\int_0^a x^{-1/4}(a-x)^{1/4}\left[1-b\sqrt{x(a-x)}\right]^{-3/2}dx$$
$$= -\frac{2\sqrt{2}}{b\,(ab-2)}\left[(ab-2)\,\mathbf{K}\left(\sqrt{\frac{ab}{2}}\right)+2\,\mathbf{E}\left(\sqrt{\frac{ab}{2}}\right)\right] \qquad [|\arg(2+ab)|<\pi].$$

34. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\left[1-b\sqrt{x(a-x)}\right]^{-3/2}dx = \frac{2^{5/2}}{2-ab}\,\mathbf{E}\left(\sqrt{\frac{ab}{2}}\right)$$
$$[|\arg(2-ab)|<\pi].$$

35. $$\int_0^a x^{1/2}\left[1-b\sqrt{x(a-x)}\right]^{-2}dx$$
$$= \frac{2\sqrt{a}}{b(2-ab)}+\frac{4(ab-1)}{b^{3/2}(2-ab)^{3/2}}\arcsin\sqrt{\frac{ab}{2}} \qquad [|\arg(2-ab)|<\pi].$$

36. $$\int_0^a x^{-1/2}\left[1-b\sqrt{x(a-x)}\right]^{-2}dx$$
$$= \frac{2\sqrt{a}}{2-ab}+\frac{4}{\sqrt{b}\,(2-ab)^{3/2}}\arcsin\sqrt{\frac{ab}{2}} \qquad [|\arg(2-ab)|<\pi].$$

37. $$\int_0^a x^{-1/4}(a-x)^{1/4}\left[1-b\sqrt{x(a-x)}\right]^{-2}dx = \frac{\pi a}{(2-ab)^{3/2}}$$
$$[|\arg(2-ab)|<\pi].$$

38. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\left[1-b\sqrt{x(a-x)}\right]^{-2}dx = \frac{\pi(4-ab)}{(2-ab)^{3/2}}$$
$$[|\arg(2-ab)|<\pi].$$

39. $$\int_0^a x^{-1/2}\left[1-b\sqrt{x(a-x)}\right]^{-5/2}dx = \frac{4\sqrt{a}\,(6-ab)}{3(2-ab)^2} \qquad [|\arg(2-ab)|<\pi].$$

40. $$\int_{-1}^1 \frac{(1-x^2)^{s-1/2}}{(1+2ax+a^2)^s(1+2bx+b^2)^s}dx = \frac{\sqrt{\pi}\,\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s+1)}\,{}_2F_1\left(\begin{matrix}s,\,2s\\ s+1;\ ab\end{matrix}\right)$$
$$[\operatorname{Re} s>-1/2].$$

4.1.2. The exponential function

Condition: $a > 0$.

1. $$\int_0^\infty x^n e^{-bx^2-cx}\,dx = 2^{-n} i^n b^{-(n+1)/2}\left[\frac{\sqrt{\pi}}{2}\,e^{c^2/(4b)}\,\operatorname{erfc}\left(\frac{c}{2\sqrt{b}}\right) H_n\left(\frac{ic}{2\sqrt{b}}\right) + \sum_{k=1}^{n}\binom{n}{k}(-i)^k H_{n-k}\left(\frac{ic}{2\sqrt{b}}\right) H_{k-1}\left(\frac{c}{2\sqrt{b}}\right)\right] \quad [\operatorname{Re} b > 0].$$

2. $$\int_0^a x^{s-1}(a-x)^{t-1} e^{bx(a-x)}\,dx = a^{s+t-1}\,\mathrm{B}\,(s,\,t)\,{}_2F_2\left(\begin{matrix} s,\ t;\ \frac{a^2 b}{4} \\ \frac{s+t}{2},\ \frac{s+t+1}{2} \end{matrix}\right) \quad [\operatorname{Re} s,\ \operatorname{Re} t > 0].$$

3. $$\int_0^a e^{bx(a-x)}\,dx = \sqrt{\frac{\pi}{b}}\,e^{a^2 b/4}\operatorname{erf}\left(\frac{a\sqrt{b}}{2}\right).$$

4. $$\int_0^a x e^{bx(a-x)}\,dx = \frac{a}{2}\sqrt{\frac{\pi}{b}}\,e^{a^2 b/4}\operatorname{erf}\left(\frac{a\sqrt{b}}{2}\right).$$

5. $$\int_0^a x^2 e^{bx(a-x)}\,dx = \frac{a}{2b}\left[\frac{a^2 b+2}{2a}\sqrt{\frac{\pi}{b}}\,e^{a^2 b/4}\operatorname{erf}\left(\frac{a\sqrt{b}}{2}\right) - 1\right].$$

6. $$\int_0^a x^{-1/2}(a-x)^{-1/2} e^{bx(a-x)}\,dx = \pi e^{a^2 b/8} I_0\left(\frac{a^2 b}{8}\right).$$

7. $$\int_0^a x^{1/2}(a-x)^{-1/2} e^{bx(a-x)}\,dx = \frac{\pi a}{2} e^{a^2 b/8} I_0\left(\frac{a^2 b}{8}\right).$$

8. $$\int_0^a x^{1/2}(a-x)^{1/2} e^{bx(a-x)}\,dx = \frac{\pi a^2}{8} e^{a^2 b/8}\left[I_0\left(\frac{a^2 b}{8}\right) + I_1\left(\frac{a^2 b}{8}\right)\right].$$

9. $$\int_0^a x^{3/2}(a-x)^{-1/2} e^{bx(a-x)}\,dx = \frac{\pi a^2}{8} e^{a^2 b/8}\left[3 I_0\left(\frac{a^2 b}{8}\right) - I_1\left(\frac{a^2 b}{8}\right)\right].$$

10. $$\int_0^\infty x^{-1/2}\exp\left(ax - \frac{x^3}{12}\right) dx = \pi^{3/2}\left[\operatorname{Ai}^2(a) + \operatorname{Bi}^2(a)\right] \quad [[65],\ (2.21)].$$

11. $$\int_0^a x^s (a-x)^{s+1/2} e^{b\sqrt{x(a-x)}}\,dx$$
$$= 2^{-2s-1}\sqrt{\pi}\,a^{2s+3/2}\,\frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)}\,{}_1F_1\left(\begin{matrix}2s+2;\ \frac{ab}{2}\\ 2s+\frac{5}{2}\end{matrix}\right) \qquad [\operatorname{Re} s > -1].$$

12. $$\int_0^a x^{1/2} e^{b\sqrt{x(a-x)}}\,dx = \frac{\sqrt{a}}{b} + (ab-1)\sqrt{\frac{\pi}{2b^3}}\,e^{ab/2}\operatorname{erf}\left(\sqrt{\frac{ab}{2}}\right).$$

13. $$\int_0^a x^{-1/2} e^{b\sqrt{x(a-x)}}\,dx = \sqrt{\frac{2\pi}{b}}\,e^{ab/2}\operatorname{erf}\left(\sqrt{\frac{ab}{2}}\right).$$

14. $$\int_0^a x^{-1/4}(a-x)^{1/4} e^{b\sqrt{x(a-x)}}\,dx = \frac{\pi a}{2^{3/2}}\,e^{ab/4}\left[I_0\left(\frac{ab}{4}\right) + I_1\left(\frac{ab}{4}\right)\right].$$

15. $$\int_0^a x^{-1/4}(a-x)^{-3/4} e^{b\sqrt{x(a-x)}}\,dx = \sqrt{2}\,\pi e^{ab/4} I_0\left(\frac{ab}{4}\right).$$

16. $$\int_{-\infty}^{\infty} e^{i(x^3/3+xz)}\,dx = 2\pi\operatorname{Ai}(z) \qquad [\operatorname{Im} z = 0;\ [66]].$$

17. $$\int_0^{\infty} \frac{x}{(x^2+z^2)(e^{2\pi x}+1)}\,dx = \frac{1}{2}\psi\left(z+\frac{1}{2}\right) - \frac{1}{2}\ln z \qquad [\operatorname{Re} z > 0].$$

18. $$\int_0^{\infty} \frac{x}{(x^2+z^2)^{n+1}(e^{2\pi x}+1)}\,dx = \frac{1}{4nz^{2n}}$$
$$+ \frac{(-1)^n}{2(n!)}\sum_{k=0}^{n-1}(-1)^k\frac{(n+k-1)!}{k!\,(n-k-1)!}(2z)^{-n-k}\psi^{(n-k)}\left(z+\frac{1}{2}\right) \qquad [\operatorname{Re} z > 0].$$

4.1.3. Hyperbolic functions

Condition: $a > 0$.

1. $$\int_0^a x^{s-1}(a-x)^{t-1}\sinh\left(b\sqrt{x(a-x)}\right)dx = a^{s+t} b\,\mathrm{B}\left(s+\frac{1}{2},\ t+\frac{1}{2}\right)$$
$$\times\ {}_2F_3\left(\begin{matrix}s+\frac{1}{2},\ t+\frac{1}{2};\ \frac{a^2b^2}{16}\\ \frac{3}{2},\ \frac{s+t+1}{2},\ \frac{s+t}{2}+1\end{matrix}\right) \qquad [\operatorname{Re} s,\ \operatorname{Re} t > -1/2].$$

2. $\displaystyle\int_0^a \sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx = \frac{\pi a}{2} I_1\Bigl(\frac{ab}{2}\Bigr).$

3. $\displaystyle\int_0^a x^{-1/2}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx$
$$= \sqrt{\frac{\pi}{2b}}\left[e^{ab/2}\operatorname{erf}\left(\sqrt{\frac{ab}{2}}\right) - e^{-ab/2}\operatorname{erfi}\left(\sqrt{\frac{ab}{2}}\right)\right].$$

4. $\displaystyle\int_0^a x^{-1}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx$
$$= \frac{\pi ab}{4}\left\{-\pi I_1\Bigl(\frac{ab}{2}\Bigr)\mathbf{L}_0\Bigl(\frac{ab}{2}\Bigr) + I_0\Bigl(\frac{ab}{2}\Bigr)\Bigl[2+\pi\,\mathbf{L}_1\Bigl(\frac{ab}{2}\Bigr)\Bigr]\right\}.$$

5. $\displaystyle\int_0^a x^{1/2}(a-x)^{1/2}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx = \frac{\pi a}{4b}\Bigl[ab\,\mathbf{L}_0\Bigl(\frac{ab}{2}\Bigr) - 2\,\mathbf{L}_1\Bigl(\frac{ab}{2}\Bigr)\Bigr].$

6. $\displaystyle\int_0^a x^{1/2}(a-x)^{-1/2}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx = \frac{\pi a}{2}\,\mathbf{L}_0\Bigl(\frac{ab}{2}\Bigr).$

7. $\displaystyle\int_0^a x^{-1/2}(a-x)^{-1/2}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx = \pi\,\mathbf{L}_0\Bigl(\frac{ab}{2}\Bigr).$

8. $\displaystyle\int_0^a x^{-1/2}(a-x)^{-1}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx = \frac{\pi}{a^{1/2}}\operatorname{erf}\left(\sqrt{\frac{ab}{2}}\right)\operatorname{erfi}\left(\sqrt{\frac{ab}{2}}\right).$

9. $\displaystyle\int_0^a x^{-3/4}(a-x)^{-3/4}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx = \sqrt{\frac{\pi^3 b}{2}}\,I_{1/4}^2\Bigl(\frac{ab}{4}\Bigr).$

10. $\displaystyle\int_0^a x^{-1}(a-x)^{-1}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx$
$$= \frac{\pi b}{2}\left\{-\pi I_1\Bigl(\frac{ab}{2}\Bigr)\mathbf{L}_0\Bigl(\frac{ab}{2}\Bigr) + I_0\Bigl(\frac{ab}{2}\Bigr)\Bigl[2+\pi\,\mathbf{L}_1\Bigl(\frac{ab}{2}\Bigr)\Bigr]\right\}.$$

11. $\displaystyle\int_0^a x^{-5/4}(a-x)^{-5/4}\sinh\bigl(b\sqrt{x(a-x)}\bigr)\,dx$
$$= \sqrt{\frac{\pi^3 b^3}{2}}\Bigl[I_{-1/4}^2\Bigl(\frac{ab}{4}\Bigr) - I_{3/4}^2\Bigl(\frac{ab}{4}\Bigr)\Bigr].$$

12. $\int\limits_0^a x^s (a-x)^{s+1/2} \sinh\left(b \sqrt[4]{x(a-x)}\right) dx$

$$= 2^{-2s-3/2} \sqrt{\pi}\, a^{2s+2} b \frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)} \, {}_1F_2\left(\begin{matrix} 2s+\frac{5}{2};\ \frac{ab^2}{8} \\ \frac{3}{2},\ 2s+3 \end{matrix}\right) \quad [\operatorname{Re} s > -5/4].$$

13. $\int\limits_0^a x^{-1/4} (a-x)^{1/4} \sinh\left(b \sqrt[4]{x(a-x)}\right) dx = \frac{\pi a}{\sqrt{2}} \mathbf{L}_0\left(b\sqrt{\frac{a}{2}}\right)$

$$- \frac{\pi a^{1/2}}{b} \mathbf{L}_1\left(b\sqrt{\frac{a}{2}}\right).$$

14. $\int\limits_0^a x^{-1/2} \sinh\left(b \sqrt[4]{x(a-x)}\right) dx = \pi a^{1/2} I_1\left(b\sqrt{\frac{a}{2}}\right).$

15. $\int\limits_0^a x^{1/2} \sinh\left(b \sqrt[4]{x(a-x)}\right) dx = \frac{\pi a}{2b}\left[3\sqrt{2}\, I_2\left(b\sqrt{\frac{a}{2}}\right) + \sqrt{a}\, b\, I_3\left(b\sqrt{\frac{a}{2}}\right)\right].$

16. $\int\limits_0^a x^{-1/4} (a-x)^{-3/4} \sinh\left(b \sqrt[4]{x(a-x)}\right) dx = \sqrt{2}\pi\, \mathbf{L}_0\left(b\sqrt{\frac{a}{2}}\right).$

17. $\int\limits_0^a x^{-1/2} (a-x)^{-1} \sinh\left(b \sqrt[4]{x(a-x)}\right) dx = \frac{\pi b}{\sqrt{2}}$

$$\times \left\{2I_0\left(b\sqrt{\frac{a}{2}}\right) + \pi\left[I_0\left(b\sqrt{\frac{a}{2}}\right) \mathbf{L}_1\left(b\sqrt{\frac{a}{2}}\right) - I_1\left(b\sqrt{\frac{a}{2}}\right) \mathbf{L}_0\left(b\sqrt{\frac{a}{2}}\right)\right]\right\}.$$

18. $\int\limits_0^a x^{s-1} (a-x)^{t-1} \cosh\left(b\sqrt{x(a-x)}\right) dx$

$$= a^{s+t-1} \mathrm{B}(s,\, t)\, {}_2F_3\left(\begin{matrix} s,\ t;\ \frac{a^2b^2}{16} \\ \frac{1}{2},\ \frac{s+t}{2},\ \frac{s+t+1}{2} \end{matrix}\right) \quad [\operatorname{Re} s,\, \operatorname{Re} t > 0].$$

19. $\int\limits_0^a \cosh\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{2} \mathbf{L}_{-1}\left(\frac{ab}{2}\right).$

20. $\int\limits_0^a x \cosh\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a^2}{4} \mathbf{L}_{-1}\left(\frac{ab}{2}\right).$

21. $\displaystyle\int_0^a x^2 \cosh\left(b\sqrt{x(a-x)}\right) dx$

$$= \frac{a}{8b^2}\left[2\pi ab\,\mathbf{L}_0\!\left(\frac{ab}{2}\right) + \pi(a^2b^2-8)\,\mathbf{L}_1\!\left(\frac{ab}{2}\right) + 2a^2b^2\right].$$

22. $\displaystyle\int_0^a x^{1/2} \cosh\left(b\sqrt{x(a-x)}\right) dx$

$$= \sqrt{\frac{\pi}{8b^3}}\left[(ab-1)e^{ab/2}\operatorname{erf}\left(\sqrt{\frac{ab}{2}}\right) + (ab+1)e^{-ab/2}\operatorname{erfi}\left(\sqrt{\frac{ab}{2}}\right)\right].$$

23. $\displaystyle\int_0^a x^{-1/2} \cosh\left(b\sqrt{x(a-x)}\right) dx$

$$= \sqrt{\frac{\pi}{2b}}\left[e^{ab/2}\operatorname{erf}\left(\sqrt{\frac{ab}{2}}\right) + e^{-ab/2}\operatorname{erfi}\left(\sqrt{\frac{ab}{2}}\right)\right].$$

24. $\displaystyle\int_0^a x^{1/2}(a-x)^{1/2} \cosh\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{4b}\left[2I_1\!\left(\frac{ab}{2}\right) + ab\,I_2\!\left(\frac{ab}{2}\right)\right].$

25. $\displaystyle\int_0^a x^{1/2}(a-x)^{-1/2} \cosh\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{2}\,I_0\!\left(\frac{ab}{2}\right).$

26. $\displaystyle\int_0^a x^{-1/2}(a-x)^{-1/2} \cosh\left(b\sqrt{x(a-x)}\right) dx = \pi\,I_0\!\left(\frac{ab}{2}\right).$

27. $\displaystyle\int_0^a x^{-1/4}(a-x)^{-1/4} \cosh\left(b\sqrt{x(a-x)}\right) dx$

$$= \sqrt{\frac{\pi^3}{8b}}\,I_{1/4}\!\left(\frac{ab}{4}\right)\left[2\,I_{1/4}\!\left(\frac{ab}{4}\right) + ab\,I_{5/4}\!\left(\frac{ab}{4}\right)\right].$$

28. $\displaystyle\int_0^a x^{-1/4}(a-x)^{-3/4} \cosh\left(b\sqrt{x(a-x)}\right) dx = \sqrt{2}\,\pi\cosh\frac{ab}{4}\,I_0\!\left(\frac{ab}{4}\right).$

29. $\displaystyle\int_0^a x^{-3/4}(a-x)^{-3/4} \cosh\left(b\sqrt{x(a-x)}\right) dx = \sqrt{\frac{\pi^3 b}{2}}\,I_{-1/4}^2\!\left(\frac{ab}{4}\right).$

30. $$\int_0^a x^s(a-x)^{s+1/2}\cosh\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= 2^{-2s-1}\sqrt{\pi}\,a^{2s+3/2}\,\frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)}\,{}_1F_2\left(\begin{matrix}2s+2;\ \frac{ab^2}{8}\\ \frac{1}{2},\ 2s+\frac{5}{2}\end{matrix}\right)\qquad [\operatorname{Re} s>-1].$$

31. $$\int_0^a x^{1/2}\cosh\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= \frac{\sqrt{a}\,\pi}{2b^2}\left[ab^2\,\mathbf{L}_{-1}\left(b\sqrt{\frac{a}{2}}\right)-\sqrt{2a}\,b\,\mathbf{L}_0\left(b\sqrt{\frac{a}{2}}\right)+4\,\mathbf{L}_1\left(b\sqrt{\frac{a}{2}}\right)\right].$$

32. $$\int_0^a x^{-1/2}\cosh\left(b\sqrt[4]{x(a-x)}\right)dx=\pi\sqrt{a}\,\mathbf{L}_{-1}\left(b\sqrt{\frac{a}{2}}\right).$$

33. $$\int_0^a x^{1/4}(a-x)^{-1/4}\cosh\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\frac{\pi\sqrt{a}}{b}\,I_1\left(b\sqrt{\frac{a}{2}}\right)+\frac{\pi a}{\sqrt{2}}\,I_2\left(b\sqrt{\frac{a}{2}}\right).$$

34. $$\int_0^a x^{-1/4}(a-x)^{-3/4}\cosh\left(b\sqrt[4]{x(a-x)}\right)dx=\sqrt{2}\,\pi\,I_0\left(b\sqrt{\frac{a}{2}}\right).$$

35. $$\int_0^\infty \frac{x^{2n}}{e^x-1}\left[2^{2n+1}\sinh(2bx)-\sinh(bx)\right]dx$$
$$=\frac{1}{2}\left[\psi^{(2n)}\left(\frac{1}{2}+b\right)-\psi^{(2n)}\left(\frac{1}{2}-b\right)\right]\qquad [\operatorname{Re} b<1/2].$$

36. $$\int_0^\infty \frac{x^{2n+1}}{e^x-1}\left[2^{2n+1}\cosh(2bx)-\cosh(bx)\right]dx$$
$$=\frac{1}{2}\left[\psi^{(2n+1)}\left(\frac{1}{2}+b\right)+\psi^{(2n+1)}\left(\frac{1}{2}-b\right)\right]\qquad [\operatorname{Re} b<1/2].$$

4.1.4. Trigonometric functions

1. $$\int_0^\infty \frac{\sin(ax)}{x\sqrt{x^2+1}}\,dx=\frac{\pi a}{2}\left[K_0(a)\,\mathbf{L}_{-1}(a)+K_1(a)\,\mathbf{L}_0(a)\right]\qquad [a>0].$$

2. $$\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\,dx=\frac{\pi}{4}.$$

3. $$\int_0^a \sin^{\nu-2}(a-x)\sin(\nu x)\,dx = \frac{1}{\nu-1}\sin^{\nu} a \qquad [a>0;\ \operatorname{Re}\nu>1].$$

4. $$\int_0^a \sin^{\nu-2}(a-x)\cos(\nu x)\,dx = \frac{1}{\nu-1}\sin^{\nu-1} a\cos a \qquad [a>0;\ \operatorname{Re}\nu>1].$$

5. $$\int_0^a (\cos x-\cos a)^{\nu-1}\cos(\nu x)\,dx = 2^{\nu-1}\,\mathrm{B}(\nu,\,\nu)\sin^{2\nu-1} a \qquad [a,\ \operatorname{Re}\nu>0].$$

6. $$\int_0^{\pi/2} \frac{\sin^6(nx)}{\sin^6 x}\,dx = \frac{n\pi}{40}(11n^4+5n^2+4).$$

7. $$\int_0^{\pi/2} \sin^2[(n+1)x]\,\frac{\sin^4(nx)}{\sin^4 x}\,dx = \frac{n^2\pi}{8}(n+1).$$

8. $$\int_0^{\pi/2} \frac{\sin^3[(2n+1)x]}{\sin^3 x}\,dx = \frac{\pi}{2}+\frac{3n\pi}{2}(n+1).$$

9. $$\int_0^{\pi/2} \frac{\sin[(2n+1)x]}{\sin^3 x}\sin^2(nx)\sin^2[(n+1)x]\,dx = \frac{n\pi}{8}(n+1).$$

10. $$\int_0^{\infty} x^{-1/2}\cos\left(\frac{x^3}{12}+ax-\frac{b}{x}+\frac{\pi}{4}\right)dx = 2\pi^{3/2}\operatorname{Ai}\left(a+\sqrt{b}\right)\operatorname{Ai}\left(a-\sqrt{b}\right) \qquad [[65],\ (7)].$$

11. $$\int_0^a x^{s-1}(a-x)^{t-1}\sin\left(b\sqrt{x(a-x)}\right)dx = a^{s+t}b\,\mathrm{B}\left(s+\frac{1}{2},\,t+\frac{1}{2}\right) \times {}_2F_3\left(\begin{matrix} s+\frac{1}{2},\ t+\frac{1}{2};\ -\frac{a^2b^2}{16} \\ \frac{3}{2},\ \frac{s+t+1}{2},\ \frac{s+t}{2}+1\end{matrix}\right) \qquad [a>0;\ \operatorname{Re}s,\operatorname{Re}t>-1/2].$$

12. $$\int_0^a \sin\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi a}{2}J_1\left(\frac{ab}{2}\right) \qquad [a>0].$$

13. $$\int_0^a x^{1/2}\sin\left(b\sqrt{x(a-x)}\right)dx = \sqrt{\frac{\pi}{b^3}}\left[\left(ab\sin\frac{ab}{2}+\cos\frac{ab}{2}\right)C\left(\frac{ab}{2}\right) - \left(ab\cos\frac{ab}{2}-\sin\frac{ab}{2}\right)S\left(\frac{ab}{2}\right)-\sqrt{\frac{ab}{\pi}}\right] \qquad [a>0].$$

14. $$\int_0^a x^{-1/2} \sin\left(b\sqrt{x(a-x)}\right) dx = 2\sqrt{\frac{\pi}{b}}\left[\sin\frac{ab}{2}\,C\left(\frac{ab}{2}\right) - \cos\frac{ab}{2}\,S\left(\frac{ab}{2}\right)\right] \quad [a>0].$$

15. $$\int_0^a x^{-1} \sin\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi ab}{4}\left\{\pi J_1\left(\frac{ab}{2}\right)\mathbf{H}_0\left(\frac{ab}{2}\right) + J_0\left(\frac{ab}{2}\right)\left[2 - \pi\,\mathbf{H}_1\left(\frac{ab}{2}\right)\right]\right\} \quad [a>0].$$

16. $$\int_0^a x^{1/2}(a-x)^{1/2} \sin\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{4b}\left[ab\,\mathbf{H}_0\left(\frac{ab}{2}\right) - 2\,\mathbf{H}_1\left(\frac{ab}{2}\right)\right] \quad [a>0].$$

17. $$\int_0^a x^{1/2}(a-x)^{-1/2} \sin\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{2}\,\mathbf{H}_0\left(\frac{ab}{2}\right) \quad [a>0].$$

18. $$\int_0^a x^{-1/2}(a-x)^{-1/2} \sin\left(b\sqrt{x(a-x)}\right) dx = \pi\,\mathbf{H}_0\left(\frac{ab}{2}\right) \quad [a>0].$$

19. $$\int_0^a x^{-3/4}(a-x)^{-3/4} \sin\left(b\sqrt{x(a-x)}\right) dx = \sqrt{\frac{\pi^3 b}{2}}\,J_{1/4}^2\left(\frac{ab}{4}\right) \quad [a>0].$$

20. $$\int_0^a x^{-1}(a-x)^{-1} \sin\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi b}{2}\left\{\pi J_1\left(\frac{ab}{2}\right)\mathbf{H}_0\left(\frac{ab}{2}\right) + J_0\left(\frac{ab}{2}\right)\left[2 - \pi\,\mathbf{H}_1\left(\frac{ab}{2}\right)\right]\right\} \quad [a>0].$$

21. $$\int_0^a x^{-5/4}(a-x)^{-5/4} \sin\left(b\sqrt{x(a-x)}\right) dx = \sqrt{\frac{\pi^3 b^3}{2}}\left[J_{-1/4}^2\left(\frac{ab}{4}\right) + J_{3/4}^2\left(\frac{ab}{4}\right)\right] \quad [a>0].$$

22. $$\int_0^a x^{s}(a-x)^{s+1/2} \sin\left(b\sqrt[4]{x(a-x)}\right) dx = 2^{-2s-3/2}\sqrt{\pi}\,a^{2s+2}b \times \frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)}\,{}_1F_2\left(\begin{matrix} 2s+\frac{5}{2};\ -\frac{ab^2}{8} \\ \frac{3}{2},\ 2s+3 \end{matrix}\right) \quad [a>0;\ \operatorname{Re} s > -5/4].$$

23. $$\int_0^a x^{1/2} \sin\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{\pi a}{2b}\left[3\sqrt{2}\, J_2\left(b\sqrt{\frac{a}{2}}\right) - \sqrt{a}\, b\, J_3\left(b\sqrt{\frac{a}{2}}\right)\right] \quad [a>0].$$

24. $$\int_0^a x^{-1/2} \sin\left(b\sqrt[4]{x(a-x)}\right) dx = \pi a^{1/2} J_1\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

25. $$\int_0^a x^{-1/4}(a-x)^{1/4} \sin\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{\pi a}{\sqrt{2}}\, \mathbf{H}_0\left(b\sqrt{\frac{a}{2}}\right) - \frac{\pi a^{1/2}}{b}\, \mathbf{H}_1\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

26. $$\int_0^a x^{-1/4}(a-x)^{-3/4} \sin\left(b\sqrt[4]{x(a-x)}\right) dx = \sqrt{2}\,\pi\, \mathbf{H}_0\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

27. $$\int_0^a x^{-1/2}(a-x)^{-1} \sin\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{\pi b}{\sqrt{2}}\left\{2J_0\left(b\sqrt{\frac{a}{2}}\right) + \pi\left[J_1\left(b\sqrt{\frac{a}{2}}\right)\mathbf{H}_0\left(b\sqrt{\frac{a}{2}}\right) - J_0\left(b\sqrt{\frac{a}{2}}\right)\mathbf{H}_1\left(b\sqrt{\frac{a}{2}}\right)\right]\right\} \quad [a>0].$$

28. $$\int_0^a x^{s-1}(a-x)^{t-1} \cos\left(b\sqrt{x(a-x)}\right) dx = a^{s+t-1}\,\mathrm{B}(s,\, t)\, {}_2F_3\left(\begin{matrix} s,\ t;\ -\dfrac{a^2b^2}{16} \\ \dfrac{1}{2},\ \dfrac{s+t}{2},\ \dfrac{s+t+1}{2} \end{matrix}\right) \quad [a,\ \operatorname{Re} s,\ \operatorname{Re} t > 0].$$

29. $$\int_0^a \cos\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{2}\, \mathbf{H}_{-1}\left(\frac{ab}{2}\right) \quad [a>0].$$

30. $$\int_0^a x \cos\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a^2}{4}\, \mathbf{H}_{-1}\left(\frac{ab}{2}\right) \quad [a>0].$$

31. $$\int_0^a x^2 \cos\left(b\sqrt{x(a-x)}\right) dx = \frac{a}{8b^2}\left[2\pi ab\, \mathbf{H}_0\left(\frac{ab}{2}\right) - \pi(a^2b^2+8)\, \mathbf{H}_1\left(\frac{ab}{2}\right) + 2a^2b^2\right] \quad [a>0].$$

32. $$\int_0^a x^{-1/2}\cos\left(b\sqrt{x(a-x)}\right)dx$$
$$= 2\sqrt{\frac{\pi}{b}}\left[\cos\frac{ab}{2}\,C\left(\frac{ab}{2}\right)+\sin\frac{ab}{2}\,S\left(\frac{ab}{2}\right)\right] \quad [a>0].$$

33. $$\int_0^a x^{1/2}(a-x)^{1/2}\cos\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi a}{4b}\left[2J_1\left(\frac{ab}{2}\right)-ab\,J_2\left(\frac{ab}{2}\right)\right]$$
$$[a>0].$$

34. $$\int_0^a x^{1/2}(a-x)^{-1/2}\cos\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi a}{2}\,J_0\left(\frac{ab}{2}\right) \quad [a>0].$$

35. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\cos\left(b\sqrt{x(a-x)}\right)dx = \pi\,J_0\left(\frac{ab}{2}\right) \quad [a>0].$$

36. $$\int_0^a x^{-1/4}(a-x)^{-1/4}\cos\left(b\sqrt{x(a-x)}\right)dx$$
$$=\sqrt{\frac{\pi^3}{8b}}\,J_{1/4}\left(\frac{ab}{4}\right)\left[2J_{1/4}\left(\frac{ab}{4}\right)-ab\,J_{5/4}\left(\frac{ab}{4}\right)\right] \quad [a>0].$$

37. $$\int_0^a x^{-1/4}(a-x)^{-3/4}\cos\left(b\sqrt{x(a-x)}\right)dx = \sqrt{2}\,\pi\cos\frac{ab}{4}\,J_0\left(\frac{ab}{4}\right)$$
$$[a>0].$$

38. $$\int_0^a x^{-3/4}(a-x)^{-3/4}\cos\left(b\sqrt{x(a-x)}\right)dx = \sqrt{\frac{\pi^3 b}{2}}\,J_{-1/4}^2\left(\frac{ab}{4}\right) \quad [a>0].$$

39. $$\int_0^a x^{s}(a-x)^{s+1/2}\cos\left(b\sqrt[4]{x(a-x)}\right)dx = 2^{-2s-1}\sqrt{\pi}\,a^{2s+3/2}$$
$$\times\frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)}\,{}_1F_2\left(\begin{matrix}2s+2;\ -\frac{ab^2}{8}\\ \frac{1}{2},\ 2s+\frac{5}{2}\end{matrix}\right) \quad [a>0;\ \operatorname{Re}s>-1].$$

40. $$\int_0^a x^{1/2}\cos\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{a^{1/2}\pi}{2b^2}$$
$$\times\left[ab^2\,\mathbf{H}_{-1}\left(b\sqrt{\frac{a}{2}}\right)-\sqrt{2a}\,b\,\mathbf{H}_0\left(b\sqrt{\frac{a}{2}}\right)+4\,\mathbf{H}_1\left(b\sqrt{\frac{a}{2}}\right)\right] \quad [a>0].$$

41. $$\int_0^a x^{-1/2}\cos\left(b\sqrt[4]{x(a-x)}\right)dx = \pi\sqrt{a}\,\mathbf{H}_{-1}\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

42. $\displaystyle\int_0^a x^{1/4}(a-x)^{-1/4}\cos\left(b\sqrt[4]{x(a-x)}\right)dx$

$$= \frac{\pi\sqrt{a}}{b}J_1\left(b\sqrt{\frac{a}{2}}\right) - \frac{\pi a}{\sqrt{2}}J_2\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

43. $\displaystyle\int_0^a x^{-1/4}(a-x)^{-3/4}\cos\left(b\sqrt[4]{x(a-x)}\right)dx = \sqrt{2}\,\pi J_0\left(b\sqrt{\frac{a}{2}}\right) \qquad [a>0].$

44. $\displaystyle\int_0^{\pi/2} \cos^\mu x\cos(ax)(1+b\cos^2 x)^\nu dx$

$$= \frac{2^{-\mu-1}\pi\Gamma(\mu+1)}{\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)}\,{}_3F_2\left(\begin{matrix}\frac{\mu+1}{2},\ \frac{\mu}{2}+1,\ -\nu;\ -b\\ \frac{\mu-a}{2}+1,\ \frac{\mu+a}{2}+1\end{matrix}\right)$$

$$[\operatorname{Re}\mu>-1;\ |\arg(1+b)|<\pi].$$

45. $\displaystyle\int_0^{\pi/2} \cos(2nx)(1+a\sin^2 x)^\nu dx$

$$= \frac{2^{-2n-1}}{n!}\pi a^n(-\nu)_n\,{}_2F_1\left(\begin{matrix}n+\frac{1}{2},\ n-\nu;\ -a\\ 2n+1\end{matrix}\right) \quad [|\arg(1+a)|<\pi].$$

46. $\displaystyle\int_0^{\pi} \cos(nx)(1+a\cos^2 x)^\nu dx$

$$= \frac{2^{-n}\pi a^{n/2}\Gamma\left(\frac{n}{2}-\nu\right)}{\Gamma\left(\frac{n}{2}+1\right)\Gamma(-\nu)}\cos\frac{n\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{n+1}{2},\ \frac{n}{2}-\nu\\ n+1;\ -a\end{matrix}\right) \quad [|\arg(1+a)|<\pi].$$

47. $\displaystyle\int_0^{\pi} \sin^\mu x\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}(1+b\sin^2 x)^\nu dx$

$$= \frac{2^{-\mu}\pi\Gamma(\mu+1)}{\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)}\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}{}_3F_2\left(\begin{matrix}\frac{\mu+1}{2},\ \frac{\mu}{2}+1,\ -\nu;\ -b\\ \frac{\mu-a}{2}+1,\ \frac{\mu+a}{2}+1\end{matrix}\right)$$

$$[\operatorname{Re}\mu>-1;\ |\arg(1+b)|<\pi].$$

48. $\displaystyle\int_0^{m\pi} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}(1+b\sin^2 x)^\nu\,dx$

$$= \frac{2\sin(m\pi a/2)}{a}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}{}_3F_2\left(\begin{matrix}-\nu,\ \frac{1}{2},\ 1;\ -b\\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right).$$

49. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \cos(b \sin x)\, dx$$
$$= \frac{2\sin(m\pi a/2)}{a} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_1F_2\left(\begin{matrix} 1;\ -\frac{b^2}{4} \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

50. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin(b \sin x)\, dx$$
$$= \frac{2b\sin(m\pi a/2)}{a} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_2F_3\left(\begin{matrix} \frac{1}{2},\, 1;\ -\frac{b^2}{4} \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

51. $$\int_0^{\pi} x \sin(nx - z\sin x)\, dx$$
$$= \frac{\pi}{2} \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{n-k} \left(\frac{z}{2}\right)^{n-2k-1} - (-1)^n \frac{\pi}{z} \sum_{k=0}^{n-1} \left(\frac{n-k+1}{2}\right)_k \left(-\frac{2}{z}\right)^k$$
$$- \frac{n!\,\pi}{2} \sum_{k=0}^{n-1} \frac{\left(\frac{2}{z}\right)^{n-k}}{k!\,(n-k)} J_k(z) - \frac{\pi^2}{2} \mathbf{H}_n(z).$$

52. $$\int_0^{\pi} x \cos(nx - z\sin x)\, dx = \frac{1}{2z} \sum_{k=0}^{n-1} \left[1 + (-1)^{k+n}\right]$$
$$\times \left(\frac{n-k+1}{2}\right)_k \left(\frac{2}{z}\right)^k \left[\psi\left(\frac{n-k+1}{2}\right) - \psi\left(\frac{n+k+1}{2}\right)\right]$$
$$+ \frac{\pi^2}{2} J_n(z) - \frac{n!}{2} \sum_{k=0}^{n-1} \frac{\left(\frac{2}{z}\right)^{n-k}}{k!\,(n-k)} \left[\pi\, \mathbf{H}_k(z) - \sum_{p=0}^{k-1} \left(\frac{1}{2}\right)_p \left(\frac{1}{2}\right)_{k-p} \left(\frac{z}{2}\right)^{k-2p-1}\right].$$

53. $$\int_0^{\infty} e^{-ax} \sin^m(bx) \sin^n(cx)\, dx = \frac{m!}{b} \left(\frac{i}{2}\right)^{m+n+1}$$
$$\times \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(-\frac{m}{2} + \frac{kc}{b} - \frac{nc}{2b} + \frac{ia}{2b}\right)_{m+1}^{-1} \quad [\operatorname{Re} a > 0].$$

54. $$\int_0^{\infty} e^{-ax} \cos^m(bx) \cos^n(cx)\, dx = 2^{-m-n} m! \sum_{k=0}^{n} \binom{n}{k} \frac{1}{a + imb - 2ikc + inc}$$
$$\times\ {}_2F_1\left(\begin{matrix} -m,\, -\frac{m}{2} + \frac{kc}{b} - \frac{nc}{2b} + \frac{ia}{2b};\ -1 \\ 1 - \frac{m}{2} + \frac{kc}{b} - \frac{nc}{2b} + \frac{ia}{2b} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

55. $$\int_0^\infty e^{-ax}\sin^m(bx)\cos^n(cx)\,dx = 2^{-m-n-1}i^{m+1}\frac{m!}{b} \times \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{m}{2}+\frac{kc}{b}-\frac{nc}{2b}+\frac{ia}{2b}\right)_{m+1}^{-1} \quad [\operatorname{Re} a > 0].$$

56. $$\int_0^{m\pi} e^{-ax}(1+b\sin^2 x)^{\nu}\,dx = \frac{1-e^{-m\pi a}}{a}\,{}_3F_2\left(\begin{matrix}-\nu,\frac{1}{2},1;\ -b\\ 1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right)$$
$$[|\arg(1+b)| < \pi \text{ for } \nu \neq 0,1,2,\ldots].$$

57. $$\int_0^{m\pi} e^{-ax}\sin(b\sin x)\,dx = b\frac{1-(-1)^m e^{-m\pi a}}{a^2+1}\,{}_2F_3\left(\begin{matrix}1;\ -\frac{b^2}{4}\\ \frac{3-ia}{2},\frac{3+ia}{2}\end{matrix}\right).$$

58. $$\int_0^{m\pi} e^{-ax}\cos(b\sin x)\,dx = \frac{1-e^{-m\pi a}}{a}\,{}_1F_2\left(\begin{matrix}1;\ -\frac{b^2}{4}\\ 1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right).$$

59. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x}\sin(b\sin x)\,dx = \frac{b}{a}\left(1-e^{-m\pi a}\right)\,{}_1F_2\left(\begin{matrix}\frac{1}{2},1;\ -\frac{b^2}{4}\\ \frac{3}{2},1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right).$$

60. $$\int_0^{m\pi} e^{-ax+b\sin^2 x}\,dx = \frac{1-e^{-m\pi a}}{a}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1;\ b\\ 1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right).$$

61. $$\int_0^{m\pi} \left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\} e^{b\sin^2 x}\,dx = \frac{2}{a}\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1;\ b\\ 1-\frac{a}{2},1+\frac{a}{2}\end{matrix}\right).$$

62. $$\int_0^\infty e^{-ax}(1+b\sin^2 x)^{\nu}\,x = \frac{1}{a}\,{}_3F_2\left(\begin{matrix}-\nu,\frac{1}{2},1;\ -b\\ 1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right)$$
$$[\operatorname{Re} a > 0;\ |\arg(1+b)| < \pi \text{ for } \nu \neq 0,1,2,\ldots].$$

63. $$\int_0^\infty e^{-ax}\sin(b\sin x)\,dx = \frac{b}{a^2+1}\,{}_1F_2\left(\begin{matrix}1;\ -\frac{b^2}{4}\\ \frac{3-ia}{2},\frac{3+ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

64. $$\int_0^\infty e^{-ax}\cos(b\sin x)\,dx = \frac{1}{a}\,{}_1F_2\left(\begin{matrix}1;\ -\frac{b^2}{4}\\ 1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

65. $$\int_0^\infty \frac{e^{-ax}}{\sin x}\sin(b\sin x)\,dx = \frac{b}{a}\,{}_2F_3\left(\begin{matrix}\frac{1}{2},1;\ -\frac{b^2}{4}\\ \frac{3}{2},1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

66. $$\int_0^\infty e^{-ax+b\sin^2 x}\,dx = \frac{1}{a}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1;\ b\\ 1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right) \qquad [\operatorname{Re} a>0].$$

67. $$\int_0^\pi \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} e^{b\sin^2 x}\,dx = \frac{2^{-\mu}\pi\Gamma(\mu+1)}{\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)}$$
$$\times\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}\,{}_2F_2\left(\begin{matrix}\frac{\mu+1}{2},\frac{\mu}{2}+1;\ b\\ \frac{\mu-a}{2}+1,\ \frac{\mu+a}{2}+1\end{matrix}\right).$$

68. $$\int_0^\infty \cosh^\nu x\, e^{a\operatorname{sech}^2 x}\cos(bx)\,dx = \frac{2^{-\nu-2}}{\Gamma(-\nu)}\Gamma\left(\frac{-ib-\nu}{2}\right)\Gamma\left(\frac{ib-\nu}{2}\right)$$
$$\times\,{}_2F_2\left(\begin{matrix}\frac{-ib-\nu}{2},\frac{ib-\nu}{2}\\ -\frac{\nu}{2},\frac{1-\nu}{2};\ a\end{matrix}\right) \qquad [\operatorname{Re}\nu<0].$$

69. $$\int_0^{m\pi}\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\cosh(b\sin x)\,dx$$
$$= \frac{2\sin(m\pi a/2)}{a}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}\,{}_1F_2\left(\begin{matrix}1;\ \frac{b^2}{4}\\ 1-\frac{a}{2},1+\frac{a}{2}\end{matrix}\right).$$

70. $$\int_0^{m\pi}\frac{1}{\sin x}\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\sinh(b\sin x)\,dx$$
$$= \frac{2b\sin(m\pi a/2)}{a}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}\,{}_2F_3\left(\begin{matrix}\frac{1}{2},1;\ \frac{b^2}{4}\\ \frac{3}{2},1-\frac{a}{2},1+\frac{a}{2}\end{matrix}\right).$$

71. $$\int_0^{m\pi}\frac{1}{\sin x}\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\sinh\left(b\sqrt{\sin x}\right)\sin\left(b\sqrt{\sin x}\right)dx$$
$$= \frac{2b^2\sin(m\pi a/2)}{a}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}\,{}_2F_5\left(\begin{matrix}\frac{1}{2},1;\ -\frac{b^4}{64}\\ \frac{3}{4},\frac{5}{4},\frac{3}{2},1-\frac{a}{2},1+\frac{a}{2}\end{matrix}\right).$$

72. $$\int_0^{m\pi}\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\cosh\left(b\sqrt{\sin x}\right)\cos\left(b\sqrt{\sin x}\right)dx$$
$$= \frac{2\sin(m\pi a/2)}{a}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}\,{}_1F_4\left(\begin{matrix}1;\ -\frac{b^4}{64}\\ \frac{1}{4},\frac{3}{4},1-\frac{a}{2},1+\frac{a}{2}\end{matrix}\right).$$

73. $\displaystyle\int_0^{m\pi} e^{-ax}\cosh(b\sin x)\,dx = \frac{1-e^{-m\pi a}}{a}\,{}_1F_2\left(\begin{matrix}1;\ \frac{b^2}{4}\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$

74. $\displaystyle\int_0^{m\pi} \frac{e^{-ax}}{\sin x}\sinh(b\sin x)\,dx = \frac{b}{a}(1-e^{-m\pi a})\,{}_2F_3\left(\begin{matrix}\frac{1}{2},\ 1;\ \frac{b^2}{4}\\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$

75. $\displaystyle\int_0^{m\pi} \frac{e^{-ax}}{\sin x}\sinh\left(b\sqrt{\sin x}\right)\sin\left(b\sqrt{\sin x}\right)dx$

$$= \frac{b^2}{a}(1-e^{-m\pi a})\,{}_2F_5\left(\begin{matrix}\frac{1}{2},\ 1;\ -\frac{b^4}{64}\\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

76. $\displaystyle\int_0^{m\pi} e^{-ax}\cosh\left(b\sqrt{\sin x}\right)\cos\left(b\sqrt{\sin x}\right)dx$

$$= \frac{1-e^{-m\pi a}}{a}\,{}_1F_4\left(\begin{matrix}1;\ -\frac{b^4}{64}\\ \frac{1}{4},\ \frac{3}{4},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

77. $\displaystyle\int_0^{\infty} e^{-ax}\cos(b\sin x)\,dx = \frac{1}{a}\,{}_1F_2\left(\begin{matrix}1;\ \frac{b^2}{4}\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)$ $[\operatorname{Re} a > 0]$.

78. $\displaystyle\int_0^{\infty} \frac{e^{-ax}}{\sin x}\sinh\left(b\sqrt{\sin x}\right)\sin\left(b\sqrt{\sin x}\right)dx$

$$= \frac{b^2}{a}\,{}_2F_5\left(\begin{matrix}\frac{1}{2},\ 1;\ -\frac{b^4}{64}\\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

79. $\displaystyle\int_0^{\infty} e^{-ax}\cosh\left(b\sqrt{\sin x}\right)\cos\left(b\sqrt{\sin x}\right)dx$

$$= \frac{1}{a}\,{}_1F_4\left(\begin{matrix}1;\ -\frac{b^4}{64}\\ \frac{1}{4},\ \frac{3}{4},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

80. $\displaystyle\int_0^{\infty} \cosh^{\nu} x\,\cos(bx)\cos(c\operatorname{sech} x)\,dx = \frac{2^{-\nu-2}}{\Gamma(-\nu)}\Gamma\left(\frac{-ib-\nu}{2}\right)\Gamma\left(\frac{ib-\nu}{2}\right)$

$$\times\,{}_2F_3\left(\begin{matrix}\frac{-ib-\nu}{2},\ \frac{ib-\nu}{2};\ -\frac{c^2}{4}\\ \frac{1}{2},\ -\frac{\nu}{2},\ \frac{1-\nu}{2}\end{matrix}\right) \quad [\operatorname{Re}\nu < 0].$$

81. $$\int_0^\infty \cosh^\nu x \, \cos(bx) \sin(c \operatorname{sech} x)\, dx = \frac{2^{-\nu-1} c}{\Gamma(1-\nu)} \Gamma\left(\frac{1-ib-\nu}{2}\right) \Gamma\left(\frac{1+ib-\nu}{2}\right) \times {}_2F_3\left(\begin{matrix} \frac{1-ib-\nu}{2}, \frac{1+ib-\nu}{2}; -\frac{c^2}{4} \\ \frac{3}{2}, \frac{1-\nu}{2}, 1-\frac{\nu}{2} \end{matrix}\right) \qquad [\operatorname{Re}\nu < 1].$$

82. $$\int_0^{\pi/2} \cos(b\cos x)\cos(c\sin(2nx))\, dx = \frac{\pi}{2} J_0(b)\, J_0(c).$$

83. $$\int_0^a \sinh(b\sqrt{x}) \sin(b\sqrt{a-x})\, dx = \frac{\pi}{8}(ab)^2.$$

84. $$\int_0^a (a-x)^{-1/2} \sinh(b\sqrt{x}) \cos(b\sqrt{a-x})\, dx = \frac{\pi a b}{2}.$$

85. $$\int_0^a x^{-1/2}(a-x)^{-1/2} \cosh(b\sqrt{x}) \cos(b\sqrt{a-x})\, dx = \pi.$$

86. $$\int_0^{\pi/2} \sin(2nx) \sinh(a\sin x) \sin(a\cos x)\, dx = (-1)^{n+1} \frac{\pi a^{2n}}{4(2n)!} \qquad [n \geq 1].$$

87. $$\int_0^{\pi/2} \sin(2nx) \sinh(a\cos x) \sin(a\sin x)\, dx = \frac{\pi a^{2n}}{4(2n)!} \qquad [n \geq 1].$$

88. $$\int_0^{\pi/2} \sinh^2(a\sin x) \sin^2(a\cos x)\, dx = \frac{\pi}{8}\left[I_0(2a) + J_0(2a)\right] - \frac{\pi}{4}.$$

89. $$\int_0^{\pi/2} \cos^2(a\sin x) \cos^2(a\cos x)\, dx = \frac{\pi}{8}\left[2J_0(2a) + J_0(2\sqrt{2}\,a) + 1\right].$$

90. $$\int_0^{\pi/2} \cosh^2(a\cos x) \sin^2(a\sin x)\, dx = \frac{\pi}{8}\left[I_0(2a) - J_0(2a)\right].$$

91. $$\int_0^{\pi/2} \sinh^2(a\cos x) \cos^2(a\sin x)\, dx = \frac{\pi}{8}\left[I_0(2a) - J_0(2a)\right].$$

92. $\displaystyle\int_0^{\pi/2} \sinh^2(a\sin x)\cos^2(a\cos x)\,dx = \frac{\pi}{8}\,[I_0(2a) - J_0(2a)].$

93. $\displaystyle\int_0^{\pi} \cos^2(a\sin x)\sin^2(a\cos x)\,dx = \frac{\pi}{4}\left[1 - J_0(2\sqrt{2}\,a)\right].$

94. $\displaystyle\int_0^{\pi} e^{2a\cos(nx)} \begin{Bmatrix} \sin(a\sin(nx)) \\ \cos(a\sin(nx)) \end{Bmatrix}^2 dx = \mp\frac{\pi}{2} + \frac{\pi}{2}\,I_0(2a).$

4.1.5. The logarithmic function

1. $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{4-x^2}}\,dx = \frac{1}{32\sqrt{3}}\left[-\zeta\left(2,\,\frac{1}{6}\right) + \zeta\left(2,\,\frac{1}{3}\right) - \zeta\left(2,\,\frac{2}{3}\right) + \zeta\left(2,\,\frac{5}{6}\right)\right].$

2. $\displaystyle\int_0^1 \frac{\ln^2 x}{\sqrt{4-x^2}}\,dx = \frac{7\pi^3}{216}.$

3. $\displaystyle\int_0^1 \frac{\ln^3 x}{\sqrt{4-x^2}}\,dx$

$$= -\frac{\pi}{4}\zeta(3) - \frac{\sqrt{3}}{1024}\left[\zeta\left(4,\,\frac{1}{6}\right) - \zeta\left(4,\,\frac{1}{3}\right) + \zeta\left(4,\,\frac{2}{3}\right) - \zeta\left(4,\,\frac{5}{6}\right)\right].$$

4. $\displaystyle\int_0^1 \frac{x^{1/2}}{x^3+1}\ln(1-x)\,dx = -\frac{2}{9}\mathbf{G} - \frac{\pi}{6}\ln 2.$

5. $\displaystyle\int_0^1 \frac{x}{x^4+1}\ln(1-x)\,dx = \frac{1}{32}\left[\pi\ln 2 - 8\mathbf{G} - 4\pi\ln\left(1+\sqrt{2}\right)\right].$

6. $\displaystyle\int_0^1 \frac{(1-x)^{s-1}}{(1-xz)^{s+1}}\ln(1-x)\,dx = \frac{(z-1)^{-1}}{s^2}\,{}_2F_1\left(\begin{matrix}1,\ s;\ z\\ s+1\end{matrix}\right)$

[Re $s > 0$; $|1-z| < \pi$].

7. $\displaystyle\int_0^1 \frac{1}{1+x}\ln\left(1+x^{2+\sqrt{3}}\right)dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \ln 2\ln\left(1+\sqrt{3}\right)$ [60].

8. $\displaystyle\int_0^1 \frac{1}{1+x}\ln\left(1+x^{3+\sqrt{8}}\right)dx = \frac{\pi^2}{24}(3-\sqrt{32})$

$$+\frac{1}{2}\ln 2\left[\ln 2 + \frac{3}{2}\ln\left(3+\sqrt{8}\right)\right]$$ [60].

9. $$\int_0^1 \frac{1}{1+x} \ln\left(1 + x^{4+\sqrt{15}}\right) dx$$
$$= \frac{\pi^2}{12}\left(2 - \sqrt{15}\right) + \ln \frac{1+\sqrt{5}}{2} \ln\left(2+\sqrt{3}\right) + \ln 2 \ln\left(\sqrt{3}+\sqrt{5}\right) \quad [60].$$

10. $$\int_0^1 \frac{1}{1+x} \ln\left(1 + x^{5+\sqrt{24}}\right) dx = \frac{\pi^2}{24}\left(5 - \sqrt{96}\right) + \frac{1}{2} \ln\left(1+\sqrt{2}\right) \ln\left(2+\sqrt{3}\right)$$
$$+ \frac{1}{2} \ln 2 \left[\ln 2 + \frac{3}{2} \ln\left(5+\sqrt{24}\right)\right] \quad [60].$$

11. $$\int_0^1 \frac{1}{1+x} \ln\left(1 + x^{6+\sqrt{35}}\right) dx = \frac{\pi^2}{12}\left(3 - \sqrt{35}\right) + \ln \frac{1+\sqrt{5}}{2} \ln\left(8+3\sqrt{7}\right)$$
$$+ \ln 2 \ln\left(\sqrt{5}+\sqrt{7}\right) \quad [60].$$

12. $$\int_0^1 \frac{1}{1+x} \ln\left(1 + x^{8+\sqrt{63}}\right) dx = \frac{\pi^2}{12}\left(4 - \sqrt{63}\right) + \ln \frac{5+\sqrt{21}}{2} \ln\left(2+\sqrt{3}\right)$$
$$+ \ln 2 \ln\left(3+\sqrt{7}\right) \quad [60].$$

13. $$\int_0^1 \frac{1}{1+x} \ln\left(1 + x^{11+\sqrt{120}}\right) dx = \frac{\pi^2}{24}\left(11 - \sqrt{480}\right)$$
$$+ \frac{1}{2} \ln\left(1+\sqrt{2}\right) \ln\left(4+\sqrt{15}\right) + \frac{1}{2} \ln\left(2+\sqrt{3}\right) \ln\left(3+\sqrt{10}\right)$$
$$+ \frac{1}{2} \ln \frac{1+\sqrt{5}}{2} \ln\left(5+\sqrt{24}\right) + \frac{1}{2} \ln 2 \left[\ln 2 + \frac{3}{2} \ln\left(11+\sqrt{120}\right)\right] \quad [60].$$

14. $$\int_0^1 \frac{1}{1+x} \ln\left(1 + x^{13+\sqrt{168}}\right) dx = \frac{\pi^2}{24}\left(13 - \sqrt{672}\right)$$
$$+ \frac{1}{2} \ln\left(1+\sqrt{2}\right) \ln\left(5+\sqrt{21}\right) + \frac{1}{4} \ln\left(2+\sqrt{3}\right) \ln\left(15+\sqrt{224}\right)$$
$$+ \frac{1}{4} \ln\left(5+\sqrt{24}\right) \ln\left(8+\sqrt{63}\right) + \frac{1}{2} \ln 2 \left[\ln 2 + \frac{3}{2} \ln\left(13+\sqrt{168}\right)\right] \quad [60].$$

15. $$\int_0^1 \frac{1}{1+x} \ln\left(1 + x^{14+\sqrt{195}}\right) dx = \frac{\pi^2}{12}\left(7 - \sqrt{195}\right)$$
$$+ \ln \frac{1+\sqrt{5}}{2} \ln\left(25+4\sqrt{39}\right) + \ln \frac{3+\sqrt{13}}{2} \ln\left(4+\sqrt{15}\right)$$
$$+ \ln 2 \ln\left(\sqrt{15}+\sqrt{13}\right) \quad [60].$$

16. $$\int_0^a x^{s-1}(a-x)^{t-1}\ln\left(1-bx(a-x)\right)dx$$
$$= -a^{s+t+1}b\,\mathrm{B}\left(s+1,\,t+1\right){}_4F_3\left(\begin{matrix}1,\,1,\,s+1,\,t+1;\ \frac{a^2b}{4}\\ 2,\ \frac{s+t+1}{2},\ \frac{s+t}{2}+1\end{matrix}\right)$$
$$\left[a>0;\ \operatorname{Re}s,\,\operatorname{Re}t>-1;\ |\arg(4-a^2b)|<\pi\right].$$

17. $$\int_0^a \ln\left(1-bx(a-x)\right)dx = \frac{4}{\sqrt{b}}\left(1-\frac{a^2b}{4}\right)^{1/2}\arcsin\left(\frac{a\sqrt{b}}{2}\right)-2a$$
$$[a>0;\ |\arg(4-a^2b)|<\pi].$$

18. $$\int_0^a x^{-1}\ln\left(1-bx(a-x)\right)dx = -2\arcsin^2\frac{a\sqrt{b}}{2}\qquad [a>0;\ |\arg(4-a^2b)|<\pi].$$

19. $$\int_0^a \frac{1}{x(a-x)}\ln\left(1-bx(a-x)\right)dx = -\frac{4}{a}\arcsin^2\frac{a\sqrt{b}}{2}$$
$$[a>0;\ |\arg(4-a^2b)|<\pi].$$

20. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\ln\left(1-bx(a-x)\right)dx = 2\pi\ln\frac{2+\sqrt{4-a^2b}}{4}$$
$$[a>0;\ |\arg(4-a^2b)|<\pi].$$

21. $$\int_0^a x^{-3/2}(a-x)^{-3/2}\ln\left(1-bx(a-x)\right)dx = \frac{4\pi}{a^2}\left(\sqrt{4-a^2b}-2\right)$$
$$[a>0;\ |\arg(4-a^2b)|<\pi].$$

22. $$\int_0^a x^{-1/2}(a-x)^{-3/2}\ln\left(1-bx(a-x)\right)dx = \frac{2\pi}{a}\left(\sqrt{4-a^2b}-2\right)$$
$$[a>0;\ |\arg(4-a^2b)|<\pi].$$

23. $$\int_0^\infty \frac{x}{x^2+a^2}\ln\frac{x^2+2bx+c}{x^2-2bx+c}\,dx = 2\pi\arctan\frac{b}{|a|+\sqrt{c-b^2}}$$
$$[b^2\le c;\ [40],\ (42)].$$

24. $$\int_0^\infty \frac{1}{x}\ln\frac{x^2+2bx+c}{x^2-2bx+c}\,dx = 2\pi\arctan\frac{b}{\sqrt{c}}\qquad [b^2\le c;\ [40],\ (43)].$$

25. $$\int_0^\infty \frac{1}{x} \ln \frac{x^2+2bx+c}{x^2+2dx+c}\,dx = \arccos^2 \frac{d}{\sqrt{c}} - \arccos^2 \frac{b}{\sqrt{c}}$$

$[b^2 \le c;\ b^2 \le d;\ [40], (47)]$.

26. $$\int_0^a x^{s+1/2}(a-x)^s \ln\left(1 - b\sqrt{x(a-x)}\right) dx$$
$$= -2^{-2s-2}\pi^{1/2}a^{2s+5/2}b\,\frac{\Gamma(2s+3)}{\Gamma\left(2s+\frac{7}{2}\right)}\,{}_3F_2\left(\begin{matrix}1,\,1,\,2s+3\\ 2,\,2s+\frac{7}{2};\ \frac{ab}{2}\end{matrix}\right)$$

$[a>0;\ \operatorname{Re} s > -1;\ |\arg(2-ab)| < \pi]$.

27. $$\int_0^a x^{1/2} \ln\left(1 - b\sqrt{x(a-x)}\right) dx = \frac{4}{3b^{3/2}}(ab+1)\sqrt{2-ab}\arcsin\left(\sqrt{\frac{ab}{2}}\right) - \frac{2\sqrt{a}}{9b}(5ab+6)$$

$[a>0;\ |\arg(2-ab)| < \pi]$.

28. $$\int_0^a x^{-1/2} \ln\left(1 - b\sqrt{x(a-x)}\right) dx = \frac{4}{\sqrt{b}}\left[\sqrt{2-ab}\arcsin\left(\sqrt{\frac{ab}{2}}\right) - \sqrt{ab}\right]$$

$[a>0;\ |\arg(2-ab)| < \pi]$.

29. $$\int_0^a x^{-1/2}(a-x)^{-1} \ln\left(1 - b\sqrt{x(a-x)}\right) dx = -\frac{4}{\sqrt{a}}\arcsin^2\sqrt{\frac{ab}{2}}$$

$[a>0;\ |\arg(2-ab)| < \pi]$.

30. $$\int_0^a x^{1/4}(a-x)^{-1/4} \ln\left(1 - b\sqrt{x(a-x)}\right) dx$$
$$= \frac{\pi}{\sqrt{2}\,b}\left[ab\ln\left(1+\sqrt{1-\frac{ab}{2}}\right) + 2\sqrt{1-\frac{ab}{2}} + \frac{ab}{2}(1-2\ln 2) - 2\right]$$

$[a>0;\ |\arg(2-ab)| < \pi]$.

31. $$\int_0^a x^{-1/4}(a-x)^{-3/4} \ln\left(1 - b\sqrt{x(a-x)}\right) dx = 2^{3/2}\pi\ln\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{ab}{2}}\right)$$

$[a>0;\ |\arg(2-ab)| < \pi]$.

32. $$\int_0^a x^{-3/4}(a-x)^{-5/4} \ln\left(1 - b\sqrt{x(a-x)}\right) dx = \frac{4\pi}{a}\left(\sqrt{2-ab} - \sqrt{2}\right)$$

$[a>0;\ |\arg(2-ab)| < \pi]$.

33. $\int_0^a x^{s-1}(a-x)^{t-1} \ln\left(b\sqrt{x(a-x)} + \sqrt{1+b^2x(a-x)}\right) dx$

$$= a^{s+t} b\, \mathrm{B}\left(s+\frac{1}{2}, t+\frac{1}{2}\right) {}_4F_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, s+\frac{1}{2}, t+\frac{1}{2};\ -\frac{a^2b^2}{4} \\ \frac{3}{2}, \frac{s+t+1}{2}, \frac{s+t}{2}+1 \end{matrix}\right)$$

$[a, \operatorname{Re} s, \operatorname{Re} t > 0;\ |\arg(4+a^2b^2)| < \pi]$.

34. $\int_0^a x \ln\left(b\sqrt{x(a-x)} + \sqrt{1+b^2x(a-x)}\right) dx$

$$= \frac{a}{b}\sqrt{1+\frac{a^2b^2}{4}}\left[\mathbf{K}\left(\frac{ab}{\sqrt{4+a^2b^2}}\right) - \mathbf{E}\left(\frac{ab}{\sqrt{4+a^2b^2}}\right)\right]$$

$[a > 0;\ |\arg(4+a^2b^2)| < \pi]$.

35. $\int_0^a x^{-1/2}(a-x)^{-1/2} \ln\left(b\sqrt{x(a-x)} + \sqrt{1+b^2x(a-x)}\right) dx$

$$= i\left[\mathrm{Li}_2\left(-\frac{iab}{2}\right) - \mathrm{Li}_2\left(\frac{iab}{2}\right)\right] \quad [a > 0;\ |\arg(4+a^2b^2)| < \pi].$$

36. $\int_0^a x^{1/2}(a-x)^{-1/2} \ln\left(b\sqrt{x(a-x)} + \sqrt{1+b^2x(a-x)}\right) dx$

$$= \frac{ia}{2}\left[\mathrm{Li}_2\left(-\frac{iab}{2}\right) - \mathrm{Li}_2\left(\frac{iab}{2}\right)\right] \quad [a > 0;\ |\arg(4+a^2b^2)| < \pi].$$

37. $\int_0^a x^{s+1/2}(a-x)^{s} \ln\left(b\sqrt[4]{x(a-x)} + \sqrt{1+b^2\sqrt{x(a-x)}}\right) dx$

$$= 2^{-2s-3/2}\pi^{1/2}a^{2s+2}b\frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)}\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, 2s+\frac{5}{2} \\ \frac{3}{2}, 2s+3;\ -\frac{ab^2}{2} \end{matrix}\right)$$

$[a > 0;\ \operatorname{Re} s > -5/4;\ |\arg(2+ab^2)| < \pi]$.

38. $\int_0^a x^{-1/2} \ln\left(b\sqrt[4]{x(a-x)} + \sqrt{1+b^2\sqrt{x(a-x)}}\right) dx$

$$= \frac{2^{3/2}}{b}\sqrt{1+\frac{ab^2}{2}}\left[\mathbf{K}\left(\frac{b\sqrt{a}}{\sqrt{2+ab^2}}\right) - \mathbf{E}\left(\frac{b\sqrt{a}}{\sqrt{2+ab^2}}\right)\right]$$

$[a > 0;\ |\arg(2+ab^2)| < \pi]$.

39. $\int_0^a x^{-1/4}(a-x)^{-3/4} \ln\left(b\sqrt[4]{x(a-x)} + \sqrt{1+b^2\sqrt{x(a-x)}}\right) dx$

$$= \sqrt{2}\,i\left[\mathrm{Li}_2\left(-ib\sqrt{\frac{a}{2}}\right) - \mathrm{Li}_2\left(ib\sqrt{\frac{a}{2}}\right)\right] \quad [a > 0;\ |\arg(2+ab^2)| < \pi].$$

40. $$\int_0^a \frac{x^{1/2}}{\sqrt{1+b^2\sqrt{x(a-x)}}} \ln\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$= \frac{\pi}{4\sqrt{2}\,b^3}\left[ab^2+(ab^2-2)\ln\left(1+\frac{ab^2}{2}\right)\right] \quad [a>0;\ |\arg(2+ab^2)|<\pi].$$

41. $$\int_0^a \frac{x^{-1/2}}{\sqrt{1+b^2\sqrt{x(a-x)}}} \ln\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$= \frac{\pi}{\sqrt{2}\,b}\ln\left(1+\frac{ab^2}{2}\right) \quad [a>0;\ |\arg(2+ab^2)|<\pi].$$

42. $$\int_0^a \frac{x(a-x)^{1/2}}{\sqrt{1+b^2\sqrt{x(a-x)}}} \ln\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$= \frac{\pi}{64\sqrt{2}\,b^5}\left[ab^2(7ab^2-12)+2(3a^2b^4-4ab^2+12)\ln\left(1+\frac{ab^2}{2}\right)\right]$$
$$[a>0;\ |\arg(2+ab^2)|<\pi].$$

43. $$\int_0^a \frac{x^{-1/2}}{\sqrt{1+b^2\sqrt{x(a-x)}}} \ln\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$= \frac{\pi}{\sqrt{2}\,b}\ln\left(1+\frac{ab^2}{2}\right) \quad [a>0;\ |\arg(2+ab^2)|<\pi].$$

44. $$\int_0^a \frac{x^{-1/2}(a-x)^{-1}}{\sqrt{1+b^2\sqrt{x(a-x)}}} \ln\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$= \frac{2\pi}{\sqrt{a}}\arctan\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0;\ |\arg(2+ab^2)|<\pi].$$

45. $$\int_0^1 \frac{x(1-x^2)^{s-1}}{(1-ax^2)^{s+1/2}}\ln\frac{1+x}{1-x}\,dx$$
$$= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(s)}{s\,\Gamma\left(s+\frac{1}{2}\right)}(1-a)^{-1/2}\,{}_2F_1\left(\begin{matrix}1,\ s\\ s+1;\ a\end{matrix}\right) \quad [\operatorname{Re} s>0;\ |1-a|<\pi].$$

46. $$\int_0^a x^{s-1}(1+bx)^{\nu}\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx$$
$$= \frac{1}{2}\sqrt{\pi}\,a^s\,\frac{\Gamma(s)}{s\,\Gamma\left(s+\frac{1}{2}\right)}\,{}_3F_2\left(\begin{matrix}-\nu,\ s,\ s;\ -ab\\ s+\frac{1}{2},\ s+1\end{matrix}\right)$$
$$[a,\ \operatorname{Re} s>0;\ \operatorname{Re}\nu>-1 \text{ for } b<0;\ |\arg(1+ab)|<\pi].$$

47. $$\int_0^a \frac{1}{1+bx} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx = \frac{1}{b} \ln^2\left(\sqrt{ab}+\sqrt{ab+1}\right)$$

$[a>0;\ |\arg(1+ab)|<\pi]$.

48. $$\int_0^a \frac{1}{1-bx} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx = \frac{1}{b} \arcsin^2 \sqrt{ab} \qquad [a>0;\ |\arg(1-ab)|<\pi].$$

49. $$\int_0^a \frac{1}{(1+bx)^2} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx = \sqrt{\frac{a}{b(ab+1)}} \ln\left(\sqrt{ab}+\sqrt{ab+1}\right)$$

$[a>0;\ |\arg(1+ab)|<\pi]$.

50. $$\int_0^a \frac{1}{(1-bx)^2} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx = \sqrt{\frac{a}{b(1-ab)}} \arcsin \sqrt{ab}$$

$[a>0;\ |\arg(1-ab)|<\pi]$.

51. $$\int_0^a \frac{1}{\sqrt{1+bx}} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx = 2\sqrt{\frac{a}{b}} \arctan\left(\sqrt{ab}\right) - \frac{1}{b} \ln(1+ab)$$

$[a>0;\ |\arg(1+ab)|<\pi]$.

52. $$\int_0^a \frac{1}{\sqrt{1-bx}} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx = \sqrt{\frac{a}{b}} \ln \frac{1+\sqrt{ab}}{1-\sqrt{ab}} + \frac{1}{b} \ln(1-ab)$$

$[a>0;\ |\arg(1-ab)|<\pi]$.

53. $$\int_0^a \frac{x}{\sqrt{1+bx}} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx$$
$$= \frac{1}{3b^2}\left[\sqrt{ab}\,(ab-3)\arctan\sqrt{ab} + 2\ln(1+ab) + ab\right]$$

$[a>0;\ |\arg(1+ab)|<\pi]$.

54. $$\int_0^a \frac{x}{\sqrt{1-bx}} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx$$
$$= \frac{1}{6b^2}\left[\sqrt{ab}\,(ab+3)\ln\frac{1+\sqrt{ab}}{1-\sqrt{ab}} + 4\ln(1-ab) - 2ab\right]$$

$[a>0;\ |\arg(1-ab)|<\pi]$.

55. $$\int_0^a \frac{\sqrt{x}}{1+bx} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, dx$$
$$= \frac{\pi\sqrt{a}}{b} - \frac{2\pi}{b^{3/2}} \ln \frac{\sqrt{(ab+1)^{1/2}-1}+\sqrt{(ab+1)^{1/2}+1}}{\sqrt{2}}$$

$[a>0;\ |\arg(1+ab)|<\pi]$.

56. $\int_0^a \frac{\sqrt{x}}{1-bx}\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = -\frac{\pi\sqrt{a}}{b} + \frac{2\pi}{b^{3/2}}\arcsin\sqrt{\frac{1-(1-ab)^{1/2}}{2}}$

$[a>0;\ |\arg(1-ab)|<\pi].$

57. $\int_0^a \frac{x^{-1/2}}{1+bx}\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = \frac{\pi}{\sqrt{b}}\ln\left(\sqrt{ab}+\sqrt{ab+1}\right)$

$[a>0;\ |\arg(1+ab)|<\pi].$

58. $\int_0^a \frac{x^{-1/2}}{1-bx}\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = \frac{\pi}{\sqrt{b}}\arcsin\sqrt{ab}\quad [a>0;\ |\arg(1-ab)|<\pi].$

59. $\int_0^a \frac{x^{-1/2}}{(1+bx)^2}\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx$

$$= \frac{\pi}{2}\left[\sqrt{\frac{a}{ab+1}} + \frac{1}{\sqrt{b}}\ln\left(\sqrt{ab}+\sqrt{ab+1}\right)\right]$$

$[a>0;\ |\arg(1+ab)|<\pi].$

60. $\int_0^a \frac{x^{-1/2}}{(1-bx)^2}\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = \frac{\pi}{2}\sqrt{\frac{a}{1-ab}} + \frac{\pi}{2\sqrt{b}}\arcsin\sqrt{ab}$

$[a>0;\ |\arg(1-ab)|<\pi].$

61. $\int_0^a x^{s-1}(a-x)^{t-1}\ln\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\,dx = 2a^{s+t}b\,\mathrm{B}\left(s+\frac{1}{2},\,t+\frac{1}{2}\right)$

$\times\ {}_4F_3\left(\begin{matrix}\frac{1}{2},\,1,\,s+\frac{1}{2},\,t+\frac{1}{2};\ \frac{a^2b^2}{4}\\ \frac{3}{2},\,\frac{s+t+1}{2},\,\frac{s+t}{2}+1\end{matrix}\right)\quad [a,\,\operatorname{Re}s,\,\operatorname{Re}t>-1;\ |\arg(4-a^2b^2)|<\pi].$

62. $\int_0^a \ln\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\,dx = \frac{\pi a^2 b}{2+\sqrt{4-a^2b^2}}\qquad [a>0;\ |\arg(4-a^2b^2)|<\pi].$

63. $\int_0^a x\ln\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\,dx = \frac{\pi a^3 b}{2}\left(2+\sqrt{4-a^2b^2}\right)^{-1}$

$[a>0;\ |\arg(4-a^2b^2)|<\pi].$

64. $\int_0^a x^2\ln\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\,dx = \frac{\pi}{12b^3}\left[9a^2b^2-4(a^2b^2-1)\sqrt{4-a^2b^2}-8\right]$

$[a>0;\ |\arg(4-a^2b^2)|<\pi].$

65. $$\int_0^a x^{-1} \ln \frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\,dx = 2\pi \arcsin \frac{ab}{2} \qquad [a>0;\ |\arg(4-a^2b^2)|<\pi].$$

66. $$\int_0^a \frac{1}{x(a-x)} \ln \frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\,dx = \frac{4\pi}{a} \arcsin \frac{ab}{2}$$
$$[a>0;\ |\arg(4-a^2b^2)|<\pi].$$

67. $$\int_a^1 \frac{x}{\sqrt{x^2-a^2}} \ln \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\,dx = \frac{\pi}{2}(1-a) \qquad [0\le a\le 1].$$

68. $$\int_0^a x^{s+1/2}(a-x)^s \ln \frac{1+b\sqrt[4]{x(a-x)}}{1-b\sqrt[4]{x(a-x))}}\,dx$$
$$= 2^{-2s-1/2}\pi^{1/2}a^{2s+2}b\,\frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\ 1,\ 2s+\frac{5}{2}\\ \frac{3}{2},\ 2s+3;\ \frac{ab^2}{2}\end{matrix}\right)$$
$$\left[a>0;\ \operatorname{Re} s>-1;\ |\arg(2-ab^2)|<\pi\right].$$

69. $$\int_0^a x^{-1/2} \ln \frac{1+b\sqrt[4]{x(a-x)}}{1-b\sqrt[4]{x(a-x)}}\,dx = \frac{2\pi}{b}\left(\sqrt{2}-\sqrt{2-ab^2}\right)$$
$$[a>0;\ |\arg(2-ab^2)|<\pi].$$

70. $$\int_0^a x^{1/2} \ln \frac{1+b\sqrt[4]{x(a-x)}}{1-b\sqrt[4]{x(a-x)}}\,dx$$
$$= \frac{\pi}{3\sqrt{2}\,b^3}\left[3ab^2-2(ab^2+1)\sqrt{4-2ab^2}+4\right] \qquad [a>0;\ |\arg(2-ab^2)|<\pi].$$

71. $$\int_0^a x^{-1/2}(a-x)^{-1} \ln \frac{1+b\sqrt[4]{x(a-x)}}{1-b\sqrt[4]{x(a-x)}}\,dx = \frac{4\pi}{\sqrt{a}} \arcsin\left(b\sqrt{\frac{a}{2}}\right)$$
$$[a>0;\ |\arg(2-ab^2)|<\pi].$$

72. $$\int_0^\infty \frac{e^{-ax}}{[x(x^2+z^2)(\sqrt{x^2+z^2}+x)]^{1/2}} \ln\left(\sqrt{x^2+z^2}+x\right)dx$$
$$= \frac{\pi}{2^{5/2}z}\left\{\pi \sin\frac{az}{2} J_0\left(\frac{az}{2}\right) - \cos\frac{az}{2} Y_0\left(\frac{az}{2}\right)\right.$$
$$+ 4\cos\frac{az}{2} \ln z\, J_0\left(\frac{az}{2}\right) + 4\sin\frac{az}{2} \ln z\, Y_0\left(\frac{az}{2}\right)$$
$$+ 2J_0\left(\frac{az}{2}\right)\left[\cos\frac{az}{2}\operatorname{ci}(az) + \sin\frac{az}{2}\operatorname{Si}(az)\right]$$
$$\left. - 2Y_0\left(\frac{az}{2}\right)\left[\sin\frac{az}{2}\operatorname{ci}(pz) - \cos\frac{az}{2}\operatorname{Si}(az)\right]\right\} \qquad [\operatorname{Re} a,\ \operatorname{Re} z>0].$$

73. $$\int_0^a x\sinh(bx)\ln\frac{a+\sqrt{a^2-x^2}}{x}\,dx = \frac{\pi^2 a}{4b}\left[I_1(ab)\,\mathbf{L}_0(ab) - I_0(ab)\,\mathbf{L}_1(ab)\right] \qquad [a>0].$$

74. $$\int_0^\infty \left(\frac{1}{4}\operatorname{csch}^2\frac{x}{2} - \frac{1}{x^2}\right)\ln x\,dx = \frac{\mathbf{C}}{2} - \frac{1}{2}\ln(2\pi).$$

75. $$\int_0^a \cosh(bx)\ln\frac{a+\sqrt{a^2-x^2}}{x}\,dx$$
$$= \frac{\pi a}{4}\left[\pi I_0(ab)\,\mathbf{L}_1(ab) - \pi I_1(ab)\,\mathbf{L}_0(ab) + 2I_0(ab)\right] \qquad [a>0].$$

76. $$\int_0^a x^2\cosh(bx)\ln\frac{a+\sqrt{a^2-x^2}}{x}\,dx$$
$$= \frac{\pi a}{2b^2}\left[\pi I_0(ab)\,\mathbf{L}_1(ab) - \pi I_1(ab)\mathbf{L}_0(ab) + ab\,I_1(ab)\right] \qquad [a>0].$$

77. $$\int_0^a \cos bx\ln(a^2-x^2)\,dx$$
$$= \frac{1}{b}\left[\sin(ab)\left(\ln\frac{2a}{b} - \mathbf{C}\right) - \cos(ab)\operatorname{Si}(2ab) + \sin(ab)\operatorname{ci}(2ab)\right] \qquad [a>0].$$

78. $$\int_0^a \frac{\cos(bx)}{\sqrt{a^2-x^2}}\ln(a^2-x^2)\,dx = \frac{\pi^2}{4}Y_0(ab) - \frac{\pi}{2}\left(\ln\frac{2b}{a} + \mathbf{C}\right)J_0(ab) \qquad [a>0].$$

79. $$\int_0^\infty \frac{\cos(ax)}{x^2+z^2}\ln(x^2+z^2)\,dx$$
$$= \frac{\pi}{2z}e^{az}\left[e^{-2az}\left(\ln\frac{2z}{a} - \mathbf{C}\right) - \operatorname{shi}(2az) + \operatorname{chi}(2az)\right] \qquad [a,\ \operatorname{Re} z>0].$$

80. $$\int_0^\infty \frac{\sin(ax)}{\sqrt{x^2+z^2}}\ln(x^2+z^2)\,dx = \frac{\pi}{2}\left(\ln\frac{z}{2a} - \mathbf{C}\right)\left[I_0(az) - \mathbf{L}_0(az)\right]$$
$$+ \frac{1}{4\pi}G_{24}^{32}\left(\frac{a^2z^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\end{matrix}\right) \qquad [a,\ \operatorname{Re} z>0].$$

81. $$\int_0^\infty \frac{\cos(ax)}{\sqrt{x^2+z^2}}\ln(x^2+z^2)\,dx = \left(\ln\frac{z}{2a} - \mathbf{C}\right)K_0(az) \qquad [a,\ \operatorname{Re} z>0].$$

82. $$\int_a^\infty \frac{\sin bx}{\sqrt{x^2-a^2}} \ln\left(x^2-a^2\right) dx = -\frac{\pi^2}{4} Y_0(ab) - \frac{\pi}{2}\left(\ln\frac{2b}{a} + \mathbf{C}\right) J_0(ab)$$

$[a,\, b > 0].$

83. $$\int_a^\infty \frac{\cos bx}{\sqrt{x^2-a^2}} \ln\left(x^2-a^2\right) dx = -\frac{\pi^2}{4} J_0(ab) + \frac{\pi}{2}\left(\ln\frac{2b}{a} + \mathbf{C}\right) Y_0(ab)$$

$[a,\, b > 0].$

84. $$\int_0^a \sqrt{x\,(a-x)} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} \ln\left(ax - x^2\right) dx = \frac{\pi}{4b^2} \begin{Bmatrix} \sin(ab/2) \\ \cos(ab/2) \end{Bmatrix}$$
$$\times \left\{4J_0\left(\frac{ab}{2}\right) + ab\left[\pi Y_1\left(\frac{ab}{2}\right) - 2\left(\ln\frac{4b}{a} + \mathbf{C} - 2\right) J_1\left(\frac{ab}{2}\right)\right]\right\} \quad [a,\, b > 0].$$

85. $$\int_0^a \frac{1}{\sqrt{x\,(a-x)}} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} \ln\left(ax - x^2\right) dx$$
$$= \frac{\pi}{2} \begin{Bmatrix} \sin(ab/2) \\ \cos(ab/2) \end{Bmatrix} \left[\pi Y_0\left(\frac{ab}{2}\right) - 2\left(\ln\frac{4b}{a} + \mathbf{C}\right) J_0\left(\frac{ab}{2}\right)\right] \quad [a,\, b > 0].$$

86. $$\int_0^\infty \frac{\left(\sqrt{x^2+z^2}+x\right)^{1/2}}{\sqrt{x(x^2+z^2)}} \sin(ax) \ln\left(\sqrt{x^2+z^2}+x\right) dx$$
$$= \frac{\pi}{2^{3/2}} e^{-az/2} K_0\left(\frac{az}{2}\right) + \frac{\pi}{2^{1/2}} e^{-az/2} \ln z\, I_0\left(\frac{az}{2}\right)$$
$$- \frac{\pi}{2^{3/2}} e^{az/2} \operatorname{Ei}(-az)\, I_0\left(\frac{az}{2}\right) \quad [\operatorname{Re} z > 0].$$

87. $$\int_0^\infty \frac{\left(\sqrt{x^2+z^2}+x\right)^{-1/2}}{\sqrt{x(x^2+z^2)}} \cos(ax) \ln\left(\sqrt{x^2+z^2}+x\right) dx$$
$$= \frac{\pi}{2^{3/2} a^{1/2} z} e^{-az/2} K_0\left(\frac{az}{2}\right) + \frac{\pi}{2^{1/2} z} e^{-az/2} \ln z\, I_0\left(\frac{az}{2}\right)$$
$$+ \frac{\pi}{2^{3/2} z} e^{az/2} \operatorname{Ei}(-az)\, I_0\left(\frac{az}{2}\right) \quad [\operatorname{Re} z > 0].$$

88. $$\int_0^{\pi/2} \frac{\cos(2nx)}{\sin x} \ln\frac{1+a\sin x}{1-a\sin x}\, dx$$
$$= (-1)^n \frac{2^{-2n}\pi a^{2n+1}}{2n+1}\, {}_3F_2\left(\begin{matrix} n+\frac{1}{2},\, n+\frac{1}{2},\, n+1 \\ n+\frac{3}{2},\, 2n+1;\; a^2 \end{matrix}\right) \quad [|\arg(1-a^2)| < \pi].$$

89. $$\int_0^{\pi/2} \cos^\nu x \cos(ax) \ln\frac{1+b\cos x}{1-b\cos x}\,dx = \frac{2^{-\nu-1}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)}$$
$$\times\ {}_4F_3\left(\begin{matrix}\frac{1}{2},\ 1,\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2};\ b^2\\ \frac{3}{2},\ \frac{\nu-a+3}{2},\ \frac{\nu+a+3}{2}\end{matrix}\right) \quad [\operatorname{Re}\nu > -2;\ |\arg(1-b^2)| < \pi].$$

90. $$\int_0^{\pi} \sin^\nu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} \ln\frac{1+b\sin x}{1-b\sin x}\,dx = \frac{2^{-\nu}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)}$$
$$\times \begin{Bmatrix}\sin(\pi a/2)\\ \cos(\pi a/2)\end{Bmatrix} {}_4F_3\left(\begin{matrix}\frac{1}{2},\ 1,\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2};\ b^2\\ \frac{3}{2},\ \frac{\nu-a+3}{2},\ \frac{\nu+a+3}{2}\end{matrix}\right) \quad [\operatorname{Re}\nu > -2;\ |\arg(1-b^2)| < \pi].$$

91. $$\int_0^{m\pi} \frac{1}{\sin^2 x} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} \ln(1+b\sin^2 x)\,dx = \frac{2b\sin(m\pi a/2)}{a}$$
$$\times \begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix} {}_4F_3\left(\begin{matrix}\frac{1}{2},\ 1,\ 1,\ 1;\ -b\\ 2,\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right) \quad [|\arg(1+b)| < \pi].$$

92. $$\int_0^{\pi/2} \cos^\nu x \cos(ax) \ln(1+b\cos^2 x)\,dx = \frac{2^{-\nu-3}\pi b\,\Gamma(\nu+3)}{\Gamma\left(\frac{\nu-a}{2}+2\right)\Gamma\left(\frac{\nu+a}{2}+2\right)}$$
$$\times\ {}_4F_3\left(\begin{matrix}1,\ 1,\ \frac{\nu+3}{2},\ \frac{\nu}{2}+2;\ -b\\ 2,\ \frac{\nu-a}{2}+2,\ \frac{\nu+a}{2}+2\end{matrix}\right) \quad [\operatorname{Re}\nu > -3;\ |\arg(1+b)| < \pi].$$

93. $$\int_0^{\pi/2} \frac{\cos(ax)}{\sqrt{1+b^2\cos^2 x}} \ln\left(b\cos x + \sqrt{1+b^2\cos^2 x}\right)dx$$
$$= \frac{\cos(a\pi/2)}{1-a^2}\ {}_3F_2\left(\begin{matrix}1,\ 1,\ 1;\ -b^2\\ \frac{3-a}{2},\ \frac{3+a}{2}\end{matrix}\right) \quad [|\arg(1+b^2)| < \pi].$$

94. $$\int_0^{\pi} \sin^\nu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} \ln(1+b\sin^2 x)\,dx = \frac{2^{-\nu-2}\pi b\,\Gamma(\nu+3)}{\Gamma\left(\frac{\nu-a}{2}+2\right)\Gamma\left(\frac{\nu+a}{2}+2\right)}$$
$$\times \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} {}_4F_3\left(\begin{matrix}1,\ 1,\ \frac{\nu+3}{2},\ \frac{\nu}{2}+2;\ -b\\ 2,\ \frac{\nu-a}{2}+2,\ \frac{\nu+a}{2}+2\end{matrix}\right) \quad [\operatorname{Re}\nu > -1;\ |\arg(1+b)| < \pi].$$

95. $$\int_0^{\pi} \frac{\sin^\nu x}{\sqrt{1+b^2\sin^2 x}} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \ln\left(b\sin x + \sqrt{1+b^2\sin^2 x}\right) dx$$
$$= \frac{2^{-\nu-1}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} 1,\ 1,\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2};\ -b^2 \\ \frac{3}{2},\ \frac{\nu-a+3}{2},\ \frac{\nu+a+3}{2} \end{matrix}\right)$$
$[\operatorname{Re}\nu > -1;\ |\arg(1+b^2)| < \pi]$.

96. $$\int_0^{\pi} \sin^\nu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \ln\left(b\sin x + \sqrt{1+b^2\sin^2 x}\right) dx$$
$$= \frac{2^{-\nu-1}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} \frac{1}{2},\ \frac{1}{2},\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2};\ -b^2 \\ \frac{3}{2},\ \frac{\nu-a+3}{2},\ \frac{\nu+a+3}{2} \end{matrix}\right)$$
$[\operatorname{Re}\nu > -2;\ |\arg(1+b^2)| < \pi]$.

97. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \ln\frac{1+b\sin x}{1-b\sin x}\, dx = \frac{4b\sin(m\pi a/2)}{a}$$
$$\times \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} \frac{1}{2},\ \frac{1}{2},\ 1,\ 1;\ b^2 \\ \frac{3}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right) \quad [|\arg(1-b^2)| < \pi].$$

98. $$\int_0^{a} x^{s-1}\sin(bx)\ln\frac{a+\sqrt{a^2-x^2}}{x}\, dx = \frac{\pi^{1/2}a^{s+1}b\,\Gamma\left(\frac{s+1}{2}\right)}{2s(s+1)\Gamma\left(\frac{s}{2}\right)}$$
$$\times {}_2F_3\left(\begin{matrix} \frac{s+1}{2},\ \frac{s+1}{2};\ -\frac{a^2b^2}{4} \\ \frac{3}{2},\ \frac{s}{2}+1,\ \frac{s+3}{2} \end{matrix}\right) \quad [a > 0;\ \operatorname{Re} s > -1].$$

99. $$\int_0^{a} x^{s-1}\cos(bx)\ln\frac{a+\sqrt{a^2-x^2}}{x}\, dx$$
$$= \frac{\pi^{1/2}a^{s}\Gamma\left(\frac{s}{2}\right)}{2s\,\Gamma\left(\frac{s+1}{2}\right)} {}_2F_3\left(\begin{matrix} \frac{s}{2},\ \frac{s}{2};\ -\frac{a^2b^2}{4} \\ \frac{1}{2},\ \frac{s+1}{2},\ \frac{s}{2}+1 \end{matrix}\right) \quad [a,\ \operatorname{Re} s > 0].$$

100. $$\int_0^{a} x\sin(bx)\ln\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\, dx$$
$$= \frac{\pi^2 a}{4b}\left[J_1(ab)\,\mathbf{H}_0(ab) - J_0(ab)\,\mathbf{H}_1(ab)\right] \quad [a > 0].$$

101. $$\int_0^{a} \cos(bx)\ln\frac{a+\sqrt{a^2-x^2}}{x}\, dx$$
$$= \frac{\pi a}{4}\left[\pi J_1(ab)\,\mathbf{H}_0(ab) - \pi J_0(ab)\,\mathbf{H}_1(ab) + 2J_0(ab)\right] \quad [a > 0].$$

102. $$\int_0^a x^2 \cos(bx) \ln \frac{a+\sqrt{a^2-x^2}}{x}\, dx$$
$$= \frac{\pi a}{2b^2}\left[\pi J_0(ab)\,\mathbf{H}_1(ab) - \pi J_1(ab)\,\mathbf{H}_0(ab) + ab\, J_1(ab)\right] \quad [a>0].$$

103. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin^2 x} \ln(1+b\sin^2 x)\, dx = \frac{b}{a}(1-e^{-m\pi a})$$
$$\times\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1,\, 1;\; -b \\ 2,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a>0;\ |\arg(1+b)|<\pi].$$

104. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} \ln \frac{1+b\sin x}{1-b\sin x}\, dx = \frac{2b}{a}\,(1-e^{-m\pi a})\; {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1,\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$$[|\arg(1+b^2)|<\pi].$$

105. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} \ln(1+b\sin x)\, dx = \frac{b}{a}\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1,\, 1;\; b^2 \\ \frac{3}{2},\, \frac{1-ia}{2},\, \frac{1+ia}{2} \end{matrix}\right)$$
$$-\frac{b^2}{2(a^2+1)}\, {}_4F_3\left(\begin{matrix} 1,\, 1,\, 1,\, \frac{3}{2};\; b^2 \\ 2,\, \frac{3-ia}{2},\, \frac{3+ia}{2} \end{matrix}\right) \quad \left[\operatorname{Re} a>0;\ |\arg(1-b^2)|<\pi\right].$$

106. $$\int_0^{\infty} e^{-ax} \ln \frac{1+b\sin x}{1-b\sin x}\, dx = \frac{2b}{a^2+1}\, {}_3F_2\left(\begin{matrix} \frac{1}{2},\, 1,\, 1;\; b^2 \\ \frac{3-ia}{2},\, \frac{3+ia}{2} \end{matrix}\right)$$
$$\left[\operatorname{Re} a>0;\ |\arg(1-b^2)|<\pi\right].$$

107. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} \ln \frac{1+b\sin x}{1-b\sin x}\, dx = \frac{2b}{a}\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1,\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$$\left[\operatorname{Re} a>0;\ |\arg(1-b^2)|<\pi\right].$$

108. $$\int_0^{\infty} \frac{e^{-ax}}{\sin^2 x} \ln(1+b\sin^2 x)\, dx = \frac{b}{a}\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1,\, 1;\; -b \\ 2,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$$[\operatorname{Re} a>0;\ |\arg(1+b)|<\pi].$$

109. $$\int_0^{\infty} \frac{dx}{(x+z)^{n+1}(\ln^2 x+\pi^2)} = \frac{(-1)^n}{n!}\, \mathrm{D}_z^n\left[\frac{1}{\ln z}\right] - \frac{1}{(z-1)^{n+1}}$$
$$[|\arg z|<\pi;\ z\neq 1].$$

110. $$\int_0^1 \left[\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)^2+\frac{\pi^2x^2}{4}\right]^{-1} dx = \frac{4}{5} \qquad [40].$$

111. $$\int_0^1 x^2\left[\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)^2+\frac{\pi^2x^2}{4}\right]^{-1} dx = \frac{36}{175} \qquad [40].$$

112. $$\int_0^1 x^4\left[\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)^2+\frac{\pi^2x^2}{4}\right]^{-1} dx = \frac{92}{875} \qquad [40].$$

113. $$\int_0^1 \frac{1}{a^2x^2+1}\left[\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)^2+\frac{\pi^2x^2}{4}\right]^{-1} dx = \frac{\arctan a}{a-\arctan a}-\frac{3}{a^2}$$

$[\operatorname{Re} a > 0;\ [40]].$

114. $$\int_0^1 x^6\left[\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)^2+\frac{\pi^2x^2}{4}\right]^{-1} dx = \frac{22548}{336875} \qquad [40].$$

115. $$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\ln\left(1+2ax+a^2\right)\ln\left(1+2bx+b^2\right) dx = 2\pi\operatorname{Li}_2(ab).$$

116. $$\int_0^a x^{s-1}(a-x)^{t-1}\ln^2\left(b\sqrt{x(a-x)}+\sqrt{1+b^2x(a-x)}\right) dx$$
$$= a^{s+t+1}b^2\,\mathrm{B}\,(s+1,\,t+1)\;{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,s+1,\,t+1;\ -\frac{a^2b^2}{4}\\ \frac{3}{2},\,2,\,\frac{s+t}{2}+1,\,\frac{s+t+3}{2}\end{matrix}\right)$$

$\left[a>0;\ \operatorname{Re} s,\ \operatorname{Re} t > -1;\ |\arg(4+a^2b^2)|<\pi\right].$

117. $$\int_0^a x^{1/2}(a-x)^{1/2}\ln^2\left(b\sqrt{x(a-x)}+\sqrt{1+b^2x(a-x)}\right) dx$$
$$= \frac{\pi}{16b^2}\left[\left(4+a^2b^2\right)\ln\left(1+\frac{a^2b^2}{4}\right)-a^2b^2\operatorname{Li}_2\left(-\frac{a^2b^2}{4}\right)-a^2b^2\right]$$

$[a>0;\ |\arg(4+a^2b^2)|<\pi].$

118. $$\int_0^a x^{1/2}(a-x)^{-1/2}\ln^2\left(b\sqrt{x(a-x)}+\sqrt{1+b^2x(a-x)}\right) dx$$
$$= -\frac{\pi a}{4}\operatorname{Li}_2\left(-\frac{a^2b^2}{4}\right) \qquad [a>0;\ |\arg(4+a^2b^2)|<\pi].$$

119. $$\int_0^a x^{1/2}(a-x)^{-3/2}\ln^2\left(b\sqrt{x(a-x)}+\sqrt{1+b^2x(a-x)}\right)dx$$
$$=\frac{\pi}{2}\left[4ab\arctan\frac{ab}{2}-4\ln\left(1+\frac{a^2b^2}{4}\right)-\mathrm{Li}_2\left(-\frac{a^2b^2}{4}\right)\right]$$
$[a>0;\ |\arg(4+a^2b^2)|<\pi].$

120. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\ln^2\left(b\sqrt{x(a-x)}+\sqrt{1+b^2x(a-x)}\right)dx$$
$$=-\frac{\pi}{2}\,\mathrm{Li}_2\left(-\frac{a^2b^2}{4}\right)$$ $[a>0;\ |\arg(4+a^2b^2)|<\pi].$

121. $$\int_0^a x^{-1/2}(a-x)^{-3/2}\ln^2\left(b\sqrt{x(a-x)}+\sqrt{1+b^2x(a-x)}\right)dx$$
$$=\frac{2\pi}{a}\left[ab\arctan\frac{ab}{2}-\ln\left(1+\frac{a^2b^2}{4}\right)\right]$$ $[a>0;\ |\arg(4+a^2b^2)|<\pi].$

122. $$\int_0^a x^{-3/2}(a-x)^{-3/2}\ln^2\left(b\sqrt{x(a-x)}+\sqrt{1+b^2x(a-x)}\right)dx$$
$$=\frac{4\pi}{a^2}\left[ab\arctan\frac{ab}{2}-\ln\left(1+\frac{a^2b^2}{4}\right)\right]$$ $[a>0;\ |\arg(4+a^2b^2)|<\pi].$

123. $$\int_0^a x^s(a-x)^{s+1/2}\ln^2\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$=\frac{2^{-2s-2}\sqrt{\pi}\,a^{2s+5/2}b^2\Gamma(2s+3)}{\Gamma\left(2s+\frac{7}{2}\right)}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,2s+3;\ -\frac{ab^2}{2}\\ \frac{3}{2},\,2,\,2s+\frac{7}{2}\end{matrix}\right)$$
$[a>0;\ \operatorname{Re}s>-3/2;\ |\arg(2+ab^2)|<\pi].$

124. $$\int_0^a x^{-1/4}(a-x)^{-3/4}\ln^2\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$=-\frac{\pi}{\sqrt{2}}\,\mathrm{Li}_2\left(-\frac{ab^2}{2}\right)$$ $[a>0;\ |\arg(2+ab^2)|<\pi].$

125. $$\int_0^a x^{1/4}(a-x)^{-1/4}\ln^2\left(b\sqrt[4]{x(a-x)}+\sqrt{1+b^2\sqrt{x(a-x)}}\right)dx$$
$$=-\frac{\pi}{4\sqrt{2}\,b^2}\left[ab^2-(2+ab^2)\ln\left(1+\frac{ab^2}{2}\right)+ab^2\,\mathrm{Li}_2\left(-\frac{ab^2}{2}\right)\right]$$
$[a>0;\ |\arg(2+ab^2)|<\pi].$

126. $$\int_0^1 x^{-1}\ln x \ln\frac{a+x}{a-x}\,dx = \mathrm{Li}_3\left(-\frac{1}{a}\right) - \mathrm{Li}_3\left(\frac{1}{a}\right) \qquad [a>1].$$

127. $$\int_0^a x^{-1/2}\ln(1-bx)\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = 2\pi\sqrt{a}\left(\ln\frac{1+\sqrt{1-ab}}{2}-1\right) + \frac{4\pi}{\sqrt{b}}\arcsin\sqrt{\frac{1-(1-ab)^{1/2}}{2}} \qquad [a>0;\ |\arg(1-ab)|<\pi].$$

128. $$\int_0^a \ln(1+bx)\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = -3a + 2\sqrt{\frac{a(1+ab)}{b}}\ln\left(\sqrt{ab}+\sqrt{1+ab}\right) + \frac{1}{b}\ln^2\left(\sqrt{ab}+\sqrt{1+ab}\right) \qquad [a>0;\ |\arg(1+ab)|<\pi].$$

129. $$\int_0^1 \frac{1}{x}\ln(1-x)\ln\frac{1+\sqrt{1-x}}{\sqrt{x}}\,dx = \frac{1}{4}\left[7\zeta(3) - 2\pi^2\ln 2\right].$$

130. $$\int_0^a x^{-3/2}\ln(1+bx)\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = \frac{2\pi}{\sqrt{a}}\left[1-\sqrt{ab+1}+\sqrt{ab}\ln\left(\sqrt{ab}+\sqrt{ab+1}\right)\right] \qquad [a>0;\ |\arg(1+ab)|<\pi].$$

131. $$\int_0^a x^{-3/2}\ln(1-bx)\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,dx = \frac{2\pi}{\sqrt{a}}\left(1-\sqrt{1-ab}-\sqrt{ab}\arcsin\sqrt{ab}\right) \qquad [a>0;\ |\arg(1-ab)|<\pi].$$

132. $$\int_0^a x^{s-1}\ln\frac{a+\sqrt{a^2-x^2}}{x}\ln\frac{1+bx}{1-bx}\,dx = \frac{2\pi^{1/2}a^{s+1}b\,\Gamma\left(\frac{s+1}{2}\right)}{s^2(s+1)\Gamma\left(\frac{s}{2}\right)} \times \left[(s+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\ 1,\ \frac{s+1}{2};\ a^2b^2\\ \frac{3}{2},\ \frac{s}{2}+1\end{matrix}\right) - {}_3F_2\left(\begin{matrix}1,\ \frac{s+1}{2},\ \frac{s+1}{2};\ a^2b^2\\ \frac{s}{2}+1,\ \frac{s+3}{2}\end{matrix}\right)\right] \qquad [a>0;\ \mathrm{Re}\,s>-1;\ |\arg(1-a^2b^2)|<\pi].$$

133. $$\int_0^a x \ln \frac{a+\sqrt{a^2-x^2}}{x} \ln \frac{1+bx}{1-bx}\, dx$$
$$= \frac{\pi}{2b^2}\left[\arcsin(ab) - 4\arcsin\sqrt{\frac{1-\left(1-a^2b^2\right)^{1/2}}{2}} + ab\left(2-\sqrt{1-a^2b^2}\right)\right]$$
$[a > 0;\ |\arg(1-a^2b^2)| < \pi]$.

134. $$\int_0^1 \frac{1}{x} \ln \frac{1+\sqrt{1-x^2}}{x} \ln \frac{1+x}{1-x}\, dx = \frac{\pi^2}{2} \ln 2.$$

135. $$\int_0^a x^{s-1} \ln\left(bx+\sqrt{1+b^2x^2}\right) \ln \frac{a+\sqrt{a^2-x^2}}{x}\, dx = \frac{\pi^{1/2} a^{s+1} b\, \Gamma\left(\frac{s+1}{2}\right)}{s^2(s+1)\Gamma\left(\frac{s}{2}\right)}$$
$$\times \left[(s+1)\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{s+1}{2} \\ \frac{3}{2}, \frac{s}{2}+1;\ -a^2b^2 \end{matrix}\right) - {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{s+1}{2}, \frac{s+1}{2} \\ \frac{s}{2}+1, \frac{s+3}{2};\ -a^2b^2 \end{matrix}\right)\right]$$
$[a > 0;\ \operatorname{Re} s > -1/2;\ |\arg(1+a^2b^2)| < \pi]$.

136. $$\int_0^a \ln\left(bx+\sqrt{b^2x^2+1}\right) \ln \frac{a+\sqrt{a^2-x^2}}{x}\, dx$$
$$= \frac{ia}{2b}\left[\operatorname{Li}_2(-iab) - \operatorname{Li}_2(iab)\right] + \frac{1}{2b} \ln(1+a^2b^2) - a \arctan(ab)$$
$[a > 0;\ |\arg(1+a^2b^2)| < \pi]$.

137. $$\int_0^1 \ln\left(x+\sqrt{1+x^2}\right) \ln \frac{1+\sqrt{1-x^2}}{x}\, dx = \mathbf{G} - \frac{\pi}{4} + \frac{1}{2} \ln 2.$$

138. $$\int_0^a x^2 \ln\left(bx+\sqrt{b^2x^2+1}\right) \ln \frac{a+\sqrt{a^2-x^2}}{x}\, dx$$
$$= \frac{ia^3}{12}\left[\operatorname{Li}_2(-iab) - \operatorname{Li}_2(iab)\right]$$
$$+ \frac{1}{36b^3}\left[ab(a^2b^2+9)\arctan(ab) - 4\ln(1+a^2b^2) - 5a^2b^2\right]$$
$[a > 0;\ |\arg(1+a^2b^2)| < \pi]$.

139. $$\int_0^1 x^2 \ln\left(x+\sqrt{x^2+1}\right) \ln \frac{1+\sqrt{1-x^2}}{x}\, dx$$
$$= \frac{1}{72}(12\mathbf{G} + 5\pi - 8\ln 2 - 10).$$

140. $$\int_0^a \frac{x^{s-1}}{\sqrt{1+b^2x^2}} \ln\left(bx+\sqrt{1+b^2x^2}\right) \ln \frac{a+\sqrt{a^2-x^2}}{x}\,dx$$
$$= \frac{\pi^{1/2} a^{s+1} b\,\Gamma\left(\frac{s+1}{2}\right)}{2(s+1)\Gamma\left(\frac{s}{2}+1\right)}\, {}_4F_3\left(\begin{matrix}1,\,1,\,\frac{s+1}{2},\,\frac{s+1}{2};\,-a^2b^2\\ \frac{3}{2},\,\frac{s}{2}+1,\,\frac{s+3}{2}\end{matrix}\right)$$
$$[a>0;\ \operatorname{Re} s>-1;\ |\arg(1+a^2b^2)|<\pi].$$

141. $$\int_0^a \frac{x}{\sqrt{b^2x^2+1}} \ln\left(bx+\sqrt{b^2x^2+1}\right) \ln \frac{a+\sqrt{a^2-x^2}}{x}\,dx$$
$$= \frac{\pi}{2b^2}\arctan(ab) + \frac{\pi a}{4b}[\ln(a^2b^2+1)-2] \quad [a>0;\ |\arg(1+a^2b^2)|<\pi].$$

142. $$\int_0^a \frac{x^{-1}}{\sqrt{b^2x^2+1}} \ln\left(bx+\sqrt{b^2x^2+1}\right) \ln \frac{a+\sqrt{a^2-x^2}}{x}\,dx$$
$$= \frac{\pi i}{8}[\operatorname{Li}_2(-a^2b^2) - 4\operatorname{Li}_2(iab)] \quad [a>0;\ |\arg(1+a^2b^2)|<\pi].$$

143. $$\int_0^1 \frac{1}{x\sqrt{x^2+1}} \ln\left(x+\sqrt{x^2+1}\right) \ln \frac{1+\sqrt{1-x^2}}{x}\,dx = \frac{\pi \mathbf{G}}{2}.$$

144. $$\int_0^1 x^{s-1} \ln\left(ax+\sqrt{1+a^2x^2}\right) \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,dx = \frac{2\pi^{1/2} a\,\Gamma\left(\frac{s+1}{2}\right)}{s^2(s+1)\Gamma\left(\frac{s}{2}\right)}$$
$$\times\left[(s+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,\frac{s+1}{2}\\ \frac{3}{2},\,\frac{s}{2}+1;\,-a^2\end{matrix}\right) - {}_3F_2\left(\begin{matrix}\frac{1}{2},\,\frac{s+1}{2},\,\frac{s+1}{2}\\ \frac{s}{2}+1,\,\frac{s+3}{2};\,-a^2\end{matrix}\right)\right]$$
$$\left[|\arg(1+a^2)|<\pi;\ \operatorname{Re} s>0\right].$$

145. $$\int_0^1 \ln\left(ax+\sqrt{a^2x^2+1}\right) \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,dx = i[\operatorname{Li}_2(-ia) - \operatorname{Li}_2(ia)]$$
$$-2\arctan a + \frac{1}{a}\ln(a^2+1) \quad \left[|\arg(1+a^2)|<\pi\right].$$

146. $$\int_0^1 x^{s-1} \ln\frac{a+x}{a-x} \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,dx = \frac{4\pi^{1/2}\Gamma\left(\frac{s+1}{2}\right)}{as^2(s+1)\Gamma\left(\frac{s}{2}\right)}$$
$$\times\left[(s+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\,1,\,\frac{s+1}{2}\\ \frac{3}{2},\,\frac{s}{2}+1;\,a^{-2}\end{matrix}\right) - {}_3F_2\left(\begin{matrix}1,\,\frac{s+1}{2},\,\frac{s+1}{2}\\ \frac{s}{2}+1,\,\frac{s+3}{2};\,a^{-2}\end{matrix}\right)\right]$$
$$\left[|\arg(1+a^{-2})|<\pi;\ \operatorname{Re} s>0\right].$$

147. $$\int_0^1 \ln\frac{1+x}{1-x}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,dx = 8\mathbf{G} - \frac{\pi^2}{2}.$$

148. $$\int_0^1 x\ln\frac{1+x}{1-x}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,dx = 2\pi - \frac{\pi^2}{2}.$$

149. $$\int_0^1 \frac{1}{x}\ln\frac{1+x}{1-x}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,dx = \pi^2\ln 2.$$

150. $$\int_0^1 \frac{1}{\sqrt{x}}\ln\left(\sqrt{x}+\sqrt{x+1}\right)\ln\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\,dx = 4\mathbf{G} - \pi + 2\ln 2.$$

151. $$\int_0^a x^{s-1}\ln\frac{a+\sqrt{a^2-x^2}}{x}\ln^2\left(bx+\sqrt{1+b^2x^2}\right)dx$$
$$= \frac{\pi^{1/2}a^{s+2}b^2\Gamma\left(\frac{s}{2}+1\right)}{2(s+2)\Gamma\left(\frac{s+3}{2}\right)}\,{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,\frac{s}{2}+1,\,\frac{s}{2}+1\\ \frac{3}{2},\,2,\,\frac{s+3}{2},\,\frac{s}{2}+2;\; -a^2b^2\end{matrix}\right)$$
$$[a>0;\ \operatorname{Re}s>-2;\ |\arg(1+a^2b^2)|<\pi].$$

152. $$\int_0^a \ln^2\left(bx+\sqrt{b^2x^2+1}\right)\ln\frac{a+\sqrt{a^2-x^2}}{x}\,dx = \frac{\pi a}{4}[4-2\ln\left(a^2b^2+1\right)$$
$$-\operatorname{Li}_2(-a^2b^2)] - \frac{\pi}{b}\arctan(ab) \quad [a>0;\ |\arg(1+a^2b^2)|<\pi].$$

153. $$\int_0^1 \ln^2\left(x+\sqrt{x^2+1}\right)\ln\frac{1+\sqrt{1-x^2}}{x}\,dx$$
$$= \frac{\pi}{48}\left(48-12\pi+\pi^2-24\ln 2\right) \quad [a>0].$$

154. $$\int_0^1 x^2\ln^2\left(bx+\sqrt{b^2x^2+1}\right)\ln\frac{a+\sqrt{a^2-x^2}}{x}\,dx$$
$$= \frac{\pi}{216b^3}\left[48\arctan(ab) + 3ab(a^2b^2+9)\ln\left(1+a^2b^2\right) - 9a^3b^3\operatorname{Li}_2(-a^2b^2)\right.$$
$$\left. - ab(11a^2b^2+48)\right] \quad [a>0;\ |\arg(1+a^2b^2)|<\pi].$$

155. $$\int_0^1 x^2\ln^2\left(x+\sqrt{x^2+1}\right)\ln\frac{1+\sqrt{1-x^2}}{x}\,dx$$
$$= \frac{\pi}{864}\left(48\pi+3\pi^2+120\ln 2-236\right).$$

156. $$\int_0^a x^{-2}\ln^2\left(bx+\sqrt{b^2x^2+1}\right)\ln\frac{a+\sqrt{a^2-x^2}}{x}\,dx$$
$$=\frac{i\pi b}{2}[\mathrm{Li}_2(-iab)-\mathrm{Li}_2(iab)]-\pi b\arctan(ab)+\frac{\pi}{2a}\ln(a^2b^2+1)$$
$$[a>0;\ |\arg(1+a^2b^2)|<\pi].$$

157. $$\int_0^a x^{-2}\ln^2\left(x+\sqrt{x^2+1}\right)\ln\frac{1+\sqrt{1-x^2}}{x}\,dx=\frac{\pi}{4}(4\mathrm{G}-\pi+2\ln 2).$$

158. $$\int_0^1 x^a(1-x)^{-1/2}(x+3)^{(a-1)/2}\ln^n(x^3+3x^2)\,dx$$
$$=\frac{\sqrt{\pi}}{3}\,\mathrm{D}_a^n\left[2^{a+n}\frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)}\right]\qquad[\operatorname{Re}a>0].$$

159. $$\int_0^1 \frac{x^a(1-x)^{-a/2-3/4}}{\sqrt{2+(\sqrt{2}-1)x}}\ln^n\frac{x^2}{1-x}\,dx$$
$$=-\mathrm{D}_a^n\left[2^{(a+1)/2+n}\cos\frac{(1-2a)\pi}{4}\,\frac{\Gamma(a+1)\,\Gamma\left(\frac{2a+5}{4}\right)\Gamma\left(-a-\frac{1}{2}\right)}{\Gamma\left(\frac{a+3}{4}\right)\Gamma\left(\frac{a}{4}+1\right)}\right]$$
$$[-1<\operatorname{Re}a<1/2].$$

160. $$\int_0^1 x^a(2-x)^{-a-1}\ln^n\frac{x}{2-x}\,dx=\frac{1}{2}\,\mathrm{D}_a^n\left[\psi\left(\frac{a}{2}+1\right)-\psi\left(\frac{a+1}{2}\right)\right]$$
$$[-1<\operatorname{Re}a<0].$$

161. $$\int_0^1 x^a(1-x)^{-3a/2}(3-x)^{a/2}\ln^n\frac{x^2(3-x)}{(1-x)^3}\,dx$$
$$=\mathrm{D}_a^n\left[\frac{2^{n-a}3^{3a/2}\Gamma(a+1)\,\Gamma\left(1-\frac{3a}{2}\right)}{\Gamma\left(1-\frac{a}{2}\right)}\right]\qquad[\operatorname{Re}a>-1].$$

162. $$\int_0^1 x^a(1-x)^{-3a-3/2}(4-x)^{-1/2}\ln^n\frac{x}{(1-x)^3}\,dx$$
$$=-\mathrm{D}_a^n\left[\frac{\cos(a\pi)\Gamma(2a+2)\,\Gamma\left(-3a-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}-a\right)}\right]\qquad[-1<\operatorname{Re}a<-1/6].$$

163. $$\int_0^1 x^{-1/2}(1-x)^{2a+1}(4-x)^a \ln^n \frac{(1-x)^2}{4-x}\,dx = \frac{2\sqrt{\pi}}{3}\,\mathrm{D}_a^n\left[\frac{2^{2a}\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}\right] \quad [\operatorname{Re} a > -1].$$

164. $$\int_0^1 x^a(1-x)^{-2a-5/3}(9-x)^{2a+1} \ln^n \frac{x\,(9-x)^2}{(1-x)^2}\,dx = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\,\mathrm{D}_a^n\left[3^{3a+1}\Gamma(a+1)\,\Gamma\left(-a-\frac{1}{3}\right)\right] \quad [-1 < \operatorname{Re} a < -1/3].$$

165. $$\int_0^1 x^a(1-x)^{-a/2-2/3}(9-x)^{a/2} \ln^n \frac{x^2(9-x)}{1-x}\,dx = -\frac{2^{n+1}}{\Gamma\left(\frac{2}{3}\right)}\,\mathrm{D}_a^n\left[3^{3a/2}\cos\frac{(2\pi-3a)\,\pi}{6}\Gamma\,(a+1)\,\Gamma\left(-a-\frac{1}{3}\right)\right]$$
$$[-1 < \operatorname{Re} a < -1/3].$$

166. $$\int_0^1 x^a(1-x)^{-a/3-1/2}(x+8)^{a/3} \ln^n \frac{x^3(x+8)}{1-x}\,dx = 3^n\sqrt{\pi}\,\mathrm{D}_a^n\left[\frac{2^{2a}\sec\frac{a\pi}{3}\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}\right] \quad [-1 < \operatorname{Re} a < 3/2].$$

167. $$\int_0^1 x^a(1-x)^{-2a-5/3}(x+8)^{-a-4/3} \ln^n \frac{x}{(1-x)^2(x+8)}\,dx = \frac{1}{\Gamma\left(\frac{5}{3}\right)}\,\mathrm{D}_a^n\left[3^{-3a-4}\Gamma(a+1)\,\Gamma\left(-a-\frac{1}{3}\right)\right] \quad [-1 < \operatorname{Re} a < -1/3].$$

168. $$\int_0^1 x^a(1-x)^{-2a-4/3}(x+8)^{-a-2/3} \ln^n \frac{x}{(1-x)^2(x+8)}\,dx$$
$$= \frac{2}{9\Gamma\left(\frac{2}{3}\right)}\,\mathrm{D}_a^n\left[3^{-3a}\cos(a\pi)\Gamma(2a+1)\,\Gamma\left(-2a-\frac{1}{3}\right)\right] \quad [-1 < \operatorname{Re} a < -1/6].$$

169. $$\int_0^1 \frac{x^{-1/2}(1-x)^{a-3/4}}{\left(2+(\sqrt{2}-1)x\right)^{2a}} \ln^n \frac{1-x}{\left(2+(\sqrt{2}-1)x\right)^2}\,dx = \pi\,\mathrm{D}_a^n\left[2^{-3a}\frac{\Gamma\left(a+\frac{1}{4}\right)}{\Gamma\left(\frac{2a+3}{4}\right)\Gamma\left(\frac{a+1}{2}\right)}\right] \quad [-1 < \operatorname{Re} a < 1/2].$$

170. $$\int_0^1 \frac{x^a(1-x)^{-3a-3/2}}{\sqrt{4-x}} \ln^n \frac{x}{(1-x)^3}\,dx$$
$$= \frac{\sqrt{\pi}}{3}\,\mathrm{D}_a^n\left[2^{2a+1}\frac{\Gamma(a+1)\,\Gamma\left(-\frac{1}{2}-3a\right)}{\Gamma^2\left(\frac{1}{2}-a\right)}\right] \quad [-1 < \operatorname{Re} a < -1/6].$$

171. $$\int_0^1 x^a(1-x)^{-3a-3/2}(x+3)^{2a} \ln^n \frac{x^3+3x^2}{(1-x)^3}\,dx$$
$$= \sqrt{\pi}\,\mathrm{D}_a^n\left[3^{3a}\frac{\Gamma(a+1)\,\Gamma\left(-3a-\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{2}-a\right)}\right] \quad [-1 < \operatorname{Re} a < -1/6].$$

172. $$\int_0^1 x^a(1-x)^{-3(2a+1)/2}(x+3)^{-a-1/2} \ln^n \frac{x}{(1-x)^3(x+3)}\,dx$$
$$= \frac{\Gamma\left(\frac{4}{3}\right)}{\sqrt{\pi}}\,\mathrm{D}_a^n\left[2^{-2-12a}3^{3a+3/2}\cos(2a\pi)\frac{\Gamma(4a+1)\,\Gamma\left(\frac{1}{2}-2a\right)\Gamma\left(-\frac{1}{2}-3a\right)}{\Gamma\left(\frac{1}{3}-2a\right)\Gamma\left(\frac{1}{2}+a\right)}\right]$$
$$[-1 < \operatorname{Re} a < -1/6].$$

173. $$\int_0^1 x^a(1-x)^{3a}(9-8x)^{-a-1/2} \ln^n \frac{x(1-x)^3}{9-8x}\,dx$$
$$= \frac{\sqrt{\pi}}{4}\,\mathrm{D}_a^n\left[\frac{2^{-6a}\Gamma(3a+1)}{\Gamma\left(3a+\frac{3}{2}\right)}\right] \quad [\operatorname{Re} a > -1].$$

174. $$\int_0^1 x^a(1-x)^{(a-2)/3}(9-8x)^{-a-1/2} \ln^n \frac{x^3(1-x)}{(9-8x)^3}\,dx$$
$$= \frac{3^n\sqrt{\pi}}{2}\,\mathrm{D}_a^n\left[\frac{2^{-2a}\Gamma\left(\frac{a+1}{3}\right)}{\Gamma\left(\frac{2a+5}{6}\right)}\right] \quad [\operatorname{Re} a > -1].$$

175. $$\int_0^1 x^a(1-x)^{a/3}(9-8x)^{-a-3/2} \ln^n \frac{x^3(1-x)}{(9-8x)^3}\,dx$$
$$= \frac{\sqrt{\pi}}{12}\,\mathrm{D}_a^n\left[\frac{2^{-2a}\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}\right] \quad [\operatorname{Re} a > -1].$$

176. $\displaystyle\int_0^1 x^a(1-x)^{-(a+b)/2-1}\left(1+\sqrt{1-x}\right)^{2b}\ln^n\left(\frac{x}{\sqrt{1-x}}\right)dx$

$$= \mathrm{D}_a^n\left[2^{2a-2b+1}\,\mathrm{B}\left(\frac{a+1}{2},\,b-a\right)\right] \quad [-1<\operatorname{Re}a<-\operatorname{Re}b].$$

177. $\displaystyle\int_0^1 x^a(1-x)^b(2-x)^{-a-2b-2}\ln^m\frac{x}{2-x}\ln^n\frac{1-x}{(2-x)^2}\,dx$

$$= \mathrm{D}_a^m\,\mathrm{D}_b^n\left[\frac{2^{-2b-2}\Gamma\left(\frac{a+1}{2}\right)\Gamma(b+1)}{\Gamma\left(\frac{a+3}{2}+b\right)}\right] \quad [\operatorname{Re}a,\,\operatorname{Re}b>-1].$$

178. $\displaystyle\int_0^1 x^{a-1}(1-x)^{b-2a}(1+x)^{-b}\ln^m\frac{1-x}{1+x}\ln^n\frac{x}{(1-x)^2}\,dx$

$$= \mathrm{D}_b^m\,\mathrm{D}_a^n\left[\frac{\Gamma(a)\Gamma\left(\frac{b-2a+1}{2}\right)}{2^{2a}\Gamma\left(\frac{b+1}{2}\right)}\right] \quad [0<\operatorname{Re}a<\operatorname{Re}(b+1)/2].$$

179. $\displaystyle\int_0^1 \frac{x^{a-1}}{(x^b+1)^{n+1}}\ln^{m+n}x\ln\ln\frac{1}{x}\,dx$

$$= \frac{(-1)^n}{n!}\,\mathrm{D}_a^m\,\mathrm{D}_b^n\left[\frac{\mathrm{C}+\ln(2b)}{2b}\left\{\psi\left(\frac{a}{2b}\right)-\psi\left(\frac{a+b}{2b}\right)\right\}\right.$$
$$\left.+\frac{1}{2b}\left\{\zeta'\left(1,\,\frac{a}{2b}\right)-\zeta'\left(1,\,\frac{a+b}{2b}\right)\right\}\right] \quad [\operatorname{Re}a>0;\ [A2],\ (21)].$$

4.1.6. Inverse trigonometric functions

1. $\displaystyle\int_0^a (a^2-x^2)^{-1/2}\arcsin(bx)\,dx = \frac{1}{2}[\mathrm{Li}_2(ab)+\mathrm{Li}_2(-ab)]$

$$[|\arg(1-a^2b^2)|<\pi].$$

2. $\displaystyle\int_0^a (a^2-x^2)^{1/2}\arcsin(bx)\,dx$

$$= \frac{a}{4b}\left\{\frac{1-a^2b^2}{2ab}\ln\frac{1+ab}{1-ab}+ab[\mathrm{Li}_2(ab)-\mathrm{Li}_2(-ab)]-1\right\}$$
$$[|\arg(1-a^2b^2)|<\pi].$$

3. $\displaystyle\int_0^a x(a^2-x^2)^{1/2}\arcsin(bx)\,dx$

$$= \frac{a^2}{9b}[2(1-2a^2b^2)\,\mathbf{D}(ab)-(1-3a^2b^2)\,\mathbf{K}(ab)] \quad [|\arg(1-a^2b^2)|<\pi].$$

4. $$\int_0^a x(a^2-x^2)^{-1/2}\arcsin(bx)\,dx = a^2b[\mathbf{K}(ab)-\mathbf{D}(ab)]$$
$[|\arg(1-a^2b^2)|<\pi]$.

5. $$\int_0^1 \frac{(1-x^2)^{1/2}}{x(1-b^2x^2)^{1/2}}\arcsin(bx)\,dx = \frac{\pi}{4}\left[\ln\frac{1+b}{1-b}+\frac{1}{b}\ln(1-b^2)\right]$$
$[|\arg(1-b^2)|<\pi]$.

6. $$\int_0^1 \frac{(1-x^2)^{-1/2}}{x(1-b^2x^2)^{1/2}}\arcsin(bx)\,dx = \frac{\pi}{4}\ln\frac{1+b}{1-b}$$ $[|\arg(1-b^2)|<\pi]$.

7. $$\int_0^1 \frac{x(1-x^2)^{s-1}}{(1-x^2z)^{s+1}}\arcsin x\,dx = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(s+\frac{1}{2}\right)}{s^2\Gamma(s)}(1-z)^{-1}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\ s\\ s+1;\ z\end{matrix}\right)$$
$[\operatorname{Re} s>0;\ |1-z|<\pi]$.

8. $$\int_0^a x^{s-1}(a-x)^{t-1}\arcsin\left(b\sqrt{x(a-x)}\right)dx$$
$$= a^{s+t}b\,\mathrm{B}\left(s+\frac{1}{2},\,t+\frac{1}{2}\right)\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ s+\frac{1}{2},\ t+\frac{1}{2};\ \frac{a^2b^2}{4}\\ \frac{3}{2},\ \frac{s+t+1}{2},\ \frac{s+t}{2}+1\end{matrix}\right)$$
$[a,\ \operatorname{Re} s,\ \operatorname{Re} t>-1/2;\ |\arg(4-a^2b^2)|<\pi]$.

9. $$\int_0^a \arcsin\left(b\sqrt{x(a-x)}\right)dx = \frac{a^2b}{2}\left[\mathbf{K}\left(\frac{ab}{2}\right)-\mathbf{D}\left(\frac{ab}{2}\right)\right]$$
$[a>0;\ |\arg(4-a^2b^2)|<\pi]$.

10. $$\int_0^a x\arcsin\left(b\sqrt{x(a-x)}\right)dx = \frac{a^3b}{4}\left[\mathbf{K}\left(\frac{ab}{2}\right)-\mathbf{D}\left(\frac{ab}{2}\right)\right]$$
$[a>0;\ |\arg(4-a^2b^2)|<\pi]$.

11. $$\int_0^a x^{1/2}(a-x)^{-1/2}\arcsin\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{a}{2}\left[\mathrm{Li}_2\left(\frac{ab}{2}\right)-\mathrm{Li}_2\left(-\frac{ab}{2}\right)\right]$$ $[a>0;\ |\arg(4-a^2b^2)|<\pi]$.

12. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\arcsin\left(b\sqrt{x(a-x)}\right)dx = \mathrm{Li}_2\left(\frac{ab}{2}\right)-\mathrm{Li}_2\left(-\frac{ab}{2}\right)$$
$[a>0;\ |\arg(4-a^2b^2)|<\pi]$.

13. $$\int_0^a \frac{1}{\sqrt{1-b^2x(a-x)}} \arcsin\left(b\sqrt{x(a-x)}\right) dx = -\frac{\pi}{2b} \ln\left(1 - \frac{a^2b^2}{4}\right)$$
$$[a > 0;\ |\arg(4 - a^2b^2)| < \pi].$$

14. $$\int_0^a \frac{x^{-1}}{\sqrt{1-b^2x(a-x)}} \arcsin\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi}{2} \ln \frac{2+ab}{2-ab}$$
$$[a > 0;\ |\arg(4 - a^2b^2)| < \pi].$$

15. $$\int_0^a \frac{x^{-1}(a-x)^{-1}}{\sqrt{1-b^2x(a-x)}} \arcsin\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi}{a} \ln \frac{2+ab}{2-ab}$$
$$[a > 0;\ |\arg(4 - a^2b^2)| < \pi].$$

16. $$\int_{-a}^a \frac{(x+a)^{-1}}{(x^2+a^2)^{1/2}} \arcsin \frac{b(x+a)}{\sqrt{x^2+a^2}}\, dx = \frac{\pi b}{2a}\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2};\ 2b^2 \end{matrix}\right)$$
$$[a > 0;\ |\arg(1 - 2b^2)| < \pi].$$

17. $$\int_{-a}^a \frac{x+a}{(x^2+a^2)^{3/2}} \arcsin \frac{b(x+a)}{\sqrt{x^2+a^2}}\, dx = \frac{2b}{a} \left[\mathbf{K}\left(\sqrt{2}\,b\right) - \mathbf{D}\left(\sqrt{2}\,b\right)\right]$$
$$[a > 0;\ |\arg(1 - 2b^2)| < \pi].$$

18. $$\int_0^a x^{s+1/2}(a-x)^s \arcsin\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= 2^{-2s-3/2}\pi^{1/2}a^{2s+2}b \frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)}\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, 2s+\frac{5}{2} \\ \frac{3}{2}, 2s+3;\ \frac{ab^2}{2} \end{matrix}\right)$$
$$\left[a > 0;\ \operatorname{Re} s > -1;\ |\arg(2 - ab^2)| < \pi\right].$$

19. $$\int_0^a x^{1/2} \arcsin\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= \frac{2^{1/2}}{9b^3} \left[(3a^2b^4 - 4ab^2 - 4)\,\mathbf{K}\left(b\sqrt{\frac{a}{2}}\right) + (5ab^2+4)\,\mathbf{E}\left(b\sqrt{\frac{a}{2}}\right)\right]$$
$$[a > 0;\ |\arg(2 - ab^2)| < \pi].$$

20. $$\int_0^a x^{-1/2} \arcsin\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= \frac{2^{1/2}}{b} \left[(ab^2-2)\,\mathbf{K}\left(b\sqrt{\frac{a}{2}}\right) + 2\,\mathbf{E}\left(b\sqrt{\frac{a}{2}}\right)\right] \quad [a > 0;\ |\arg(2 - ab^2)| < \pi].$$

21. $$\int_0^a x^{-1/4}(a-x)^{1/4}\arcsin\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= \frac{a^{1/2}}{2b}\left\{1-\frac{2-ab^2}{2^{3/2}a^{1/2}b}\ln\frac{\sqrt{2}+\sqrt{a}\,b}{\sqrt{2}-\sqrt{a}\,b}\right.$$
$$\left.+\,b\sqrt{\frac{a}{2}}\left[\operatorname{Li}_2\left(b\sqrt{\frac{a}{2}}\right)-\operatorname{Li}_2\left(-b\sqrt{\frac{a}{2}}\right)\right]\right\}\qquad [a>0;\ |\arg(2-ab^2)|<\pi].$$

22. $$\int_0^a x^{-1/4}(a-x)^{-3/4}\arcsin\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= \sqrt{2}\left[\operatorname{Li}_2\left(b\sqrt{\frac{a}{2}}\right)-\operatorname{Li}_2\left(-b\sqrt{\frac{a}{2}}\right)\right]\qquad [a>0;\ |\arg(2-ab^2)|<\pi].$$

23. $$\int_0^a \frac{x(a-x)^{1/2}}{\sqrt{1-b^2\sqrt{x(a-x)}}}\arcsin\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= -\frac{\pi}{64\sqrt{2}\,b^5}\left[ab^2(7ab^2+12)+2(3a^2b^4+4ab^2+12)\ln\left(1-\frac{ab^2}{2}\right)\right.$$
$$[a>0;\ |\arg(2-ab^2)|<\pi].$$

24. $$\int_0^a \frac{x^{1/2}}{\sqrt{1-b^2\sqrt{x(a-x)}}}\arcsin\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= -\frac{\pi}{2^{5/2}b^3}\left[ab^2+(ab^2+2)\ln\left(1-\frac{ab^2}{2}\right)\right]\qquad [a>0;\ |\arg(2-ab^2)|<\pi].$$

25. $$\int_0^a \frac{x^{-1/2}}{\sqrt{1-b^2\sqrt{x(a-x)}}}\arcsin\left(b\sqrt[4]{x(a-x)}\right)dx = -\frac{\pi}{\sqrt{2}\,b}\ln\left(1-\frac{ab^2}{2}\right)$$
$$[a>0;\ |\arg(2-ab^2)|<\pi].$$

26. $$\int_0^a \frac{x^{-1/2}(a-x)^{-1}}{\sqrt{1-b^2\sqrt{x(a-x)}}}\arcsin\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi}{\sqrt{a}}\ln\frac{\sqrt{2}+\sqrt{a}\,b}{\sqrt{2}-\sqrt{a}\,b}$$
$$[a>0;\ |\arg(2-ab^2)|<\pi].$$

27. $$\int_0^{\pi/2}\cos^\nu x\cos(ax)\arcsin(b\cos x)\,dx = \frac{2^{-\nu-2}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)}$$
$$\times\ {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ 1+\frac{\nu}{2},\ \frac{\nu+3}{2};\ b^2\\ \frac{3}{2},\ \frac{\nu-a+3}{2},\ \frac{\nu+a+3}{2}\end{matrix}\right)\qquad [\operatorname{Re}\nu>-2;\ |\arg(1-b^2)|<\pi].$$

28. $$\int_0^{\pi/2} \frac{\cos(2nx)}{\sin x} \arcsin(a \sin x)\, dx = (-1)^n \frac{2^{-2n-1}\sqrt{\pi}\, a^{2n+1}\Gamma\left(n+\frac{1}{2}\right)}{n!\,(2n+1)} \times {}_3F_2\left(\begin{matrix} n+\frac{1}{2},\, n+\frac{1}{2},\, n+\frac{1}{2} \\ n+\frac{3}{2},\, 2n+1;\; a^2 \end{matrix}\right) \quad [|\arg(1-a^2)| < \pi].$$

29. $$\int_0^{\pi/2} \frac{\cos(2nx)}{\sin x\sqrt{1-a^2\sin^2 x}} \arcsin(a \sin x)\, dx = (-1)^n \frac{n!\,\pi^{3/2} a^{2n+1}}{2^{2n+2}\Gamma\left(n+\frac{3}{2}\right)} \times {}_3F_2\left(\begin{matrix} n+\frac{1}{2},\, n+1,\, n+1 \\ n+\frac{3}{2},\, 2n+1;\; a^2 \end{matrix}\right) \quad [|\arg(1-a^2)| < \pi].$$

30. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \arcsin(b \sin x)\, dx = \frac{2b}{a} \sin\frac{m\pi a}{2} \times \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right) \quad [|\arg(1-b^2)| < \pi].$$

31. $$\int_0^{m\pi} \frac{1}{\sin x\sqrt{1-b^2\sin^2 x}} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \arcsin(b \sin x)\, dx = \frac{2b}{a} \sin\frac{m\pi a}{2} \times \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1,\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right) \quad [|\arg(1-b^2)| < \pi].$$

32. $$\int_0^{\pi} \sin^\nu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \arcsin(b \sin x)\, dx = \frac{2^{-\nu-1}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1+\frac{\nu}{2},\, \frac{\nu+3}{2};\; b^2 \\ \frac{3}{2},\, \frac{\nu-a+3}{2},\, \frac{\nu+a+3}{2} \end{matrix}\right) \quad [\operatorname{Re}\nu > -2;\; |\arg(1-b^2)| < \pi].$$

33. $$\int_0^{\pi} \frac{\cos(nx)}{\cos x} \arcsin(a \cos x)\, dx = \frac{2^{-n}\sqrt{\pi}\, a^{n+1}\Gamma\left(\frac{n+1}{2}\right)}{(n+1)\Gamma\left(\frac{n}{2}+1\right)} \cos^2\frac{n\pi}{2}\, {}_3F_2\left(\begin{matrix} \frac{n+1}{2},\, \frac{n+1}{2},\, \frac{n+1}{2} \\ \frac{n+3}{2},\, n+1;\; a^2 \end{matrix}\right) \quad [|\arg(1-a^2)| < \pi].$$

34. $$\int_0^{\pi} \frac{\sin^\nu x}{\sqrt{1-b^2\sin^2 x}} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \arcsin(b\sin x)\,dx$$
$$= \frac{2^{-\nu-1}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2 \end{Bmatrix} {}_4F_3\left(\begin{matrix} 1,\, 1,\, 1+\frac{\nu}{2},\, \frac{\nu+3}{2};\; b^2 \\ \frac{3}{2},\, \frac{\nu-a+3}{2},\, \frac{\nu+a+3}{2} \end{matrix}\right)$$
$[\operatorname{Re}\nu > -2;\; |\arg(1-b^2)| < \pi]$.

35. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} \arcsin(b\sin x)\,dx = \frac{b}{a}\left(1-e^{-m\pi a}\right) {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$[|\arg(1-b^2)| < \pi]$.

36. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x\sqrt{1-b^2\sin^2 x}} \arcsin(b\sin x)\,dx$$
$$= \frac{b}{a}\left(1-e^{-m\pi a}\right) {}_4F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1,\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [|\arg(1-b^2)| < \pi].$$

37. $$\int_0^{\infty} e^{-ax} \arcsin(b\sin x)\,dx = \frac{b}{a^2+1}\, {}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1;\; b^2 \\ \frac{3-ia}{2},\, \frac{3+ia}{2} \end{matrix}\right)$$
$[\operatorname{Re} a > 0;\; |\arg(1-b^2)| < \pi]$.

38. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} \arcsin(b\sin x)\,dx = \frac{b}{a}\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$[\operatorname{Re} a > 0;\; |\arg(1-b^2)| < \pi]$.

39. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x\sqrt{1-b^2\sin^2 x}} \arcsin(b\sin x)\,dx = \frac{b}{a}\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1,\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$[\operatorname{Re} a > 0;\; |\arg(1-b^2)| < \pi]$.

40. $$\int_0^{1} \ln x \arcsin(ax)\,dx = \frac{1}{a}\left(2-2\sqrt{1-a^2} - a\arcsin a + \ln\frac{1+\sqrt{1-a^2}}{2}\right)$$
$[|\arg(1-a^2)| < \pi]$.

41. $$\int_0^{1} \ln x \arcsin x\,dx = 2 - \frac{\pi}{2} - \ln 2.$$

42. $$\int_0^{1} x\ln x \arcsin x\,dx = \frac{\pi}{8}(\ln 2 - 1).$$

43. $\displaystyle\int_0^1 x^2 \ln x \arcsin x\,dx = \frac{1}{54}(14 - 3\pi - 12\ln 2).$

44. $\displaystyle\int_0^1 \frac{1}{x} \ln x \arcsin x\,dx = -\frac{\pi}{48}(\pi^2 + 12\ln^2 2).$

45. $\displaystyle\int_0^1 x(x^2 - x^4)^\nu \ln^n(x^2 - x^4) \arcsin x\,dx = \frac{\pi^{3/2}}{16}\,\mathrm{D}_\nu^n\left[\frac{2^{-2\nu}\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\right]$

$[\operatorname{Re}\nu > -1].$

46. $\displaystyle\int_0^a x^{s-1} \ln\frac{a+\sqrt{a^2-x^2}}{x} \arcsin(bx)\,dx = \frac{\pi^{1/2} a^{s+1} b\,\Gamma\left(\frac{s+1}{2}\right)}{s(s+1)\Gamma\left(\frac{s}{2}\right)}$

$\displaystyle\times\left[(s+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{s+1}{2};\ a^2b^2\\ \frac{3}{2}, \frac{s}{2}+1\end{matrix}\right) - {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{s+1}{2}, \frac{s+1}{2}\\ \frac{s}{2}+1, \frac{s+3}{2};\ a^2b^2\end{matrix}\right)\right]$

$\left[a > 0;\ \operatorname{Re} s > -1;\ |\arg(1-a^2b^2)| < \pi\right].$

47. $\displaystyle\int_0^a \ln\frac{a+\sqrt{a^2-x^2}}{x} \arcsin(bx)\,dx = \frac{a}{2}[\mathrm{Li}_2(ab) - \mathrm{Li}_2(-ab)] - \frac{a}{2}\ln\frac{1+ab}{1-ab}$

$\displaystyle - \frac{1}{2b}\ln(1-a^2b^2) \quad [a > 0;\ |\arg(1-a^2b^2)| < \pi].$

48. $\displaystyle\int_0^a x^2 \ln\frac{a+\sqrt{a^2-x^2}}{x} \arcsin(bx)\,dx = \frac{a^3}{12}[\mathrm{Li}_2(ab) - \mathrm{Li}_2(-ab)]$

$\displaystyle + \frac{1}{72b^3}\left[ab(a^2b^2-9)\ln\frac{1+ab}{1-ab} - 8\ln(1-a^2b^2) + 10a^2b^2\right]$

$[a > 0;\ |\arg(1-a^2b^2)| < \pi].$

49. $\displaystyle\int_0^1 x^2 \ln\frac{1+\sqrt{1-x^2}}{x} \arcsin x\,dx = \frac{1}{144}(20 + 3\pi^2 - 32\ln 2).$

50. $\displaystyle\int_0^a \frac{x^{-1}}{\sqrt{1-b^2x^2}} \ln\frac{a+\sqrt{a^2-x^2}}{x} \arcsin(bx)\,dx = \frac{\pi}{4}[\mathrm{Li}_2(ab) - \mathrm{Li}_2(-ab)]$

$[a > 0;\ |\arg(1-a^2b^2)| < \pi].$

51. $\displaystyle\int_0^a \frac{x}{\sqrt{1-b^2x^2}} \ln\frac{a+\sqrt{a^2-x^2}}{x} \arcsin(bx)\,dx = \frac{\pi a}{2b} - \frac{\pi}{4b^2}\ln\frac{1+ab}{1-ab}$

$\displaystyle - \frac{\pi a}{4b}\ln(1-a^2b^2) \quad [a > 0;\ |\arg(1-a^2b^2)| < \pi].$

52. $$\int_0^1 \frac{x}{\sqrt{1-x^2}} \ln \frac{1+\sqrt{1-x^2}}{x} \arcsin x \, dx = \frac{\pi}{2}(1-\ln 2).$$

53. $$\int_0^1 \frac{x^{-1}}{\sqrt{1-x^2}} \ln \frac{1+\sqrt{1-x^2}}{x} \arcsin x \, dx = \frac{\pi^3}{16}.$$

54. $$\int_0^1 \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \arcsin x \, dx = \frac{\pi^2}{4} - 2\ln 2.$$

55. $$\int_0^1 x^2 \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \arcsin x \, dx = \frac{1}{72}(20+3\pi^2-32\ln 2).$$

56. $$\int_0^1 \frac{x^{-1}}{\sqrt{1-a^2x^2}} \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \arcsin(ax) \, dx = \frac{\pi}{2}[\mathrm{Li}_2(a) - \mathrm{Li}_2(-a)]$$
$[|\arg(1+a)| < \pi]$.

57. $$\int_0^1 x(a^2-x^2)^{-1/2} \arccos x \, dx = \frac{\pi a}{2} - a\,\mathbf{E}\left(\frac{1}{a}\right) \qquad [|\arg(a^2-1)| < \pi].$$

58. $$\int_0^1 (1-a^2x^2)^{-1/2} \arccos x \, dx = \frac{1}{2a}[\mathrm{Li}_2(a) - \mathrm{Li}_2(-a)] \qquad [|\arg(1-a^2)| < \pi].$$

59. $$\int_0^1 (1-a^2x^2)^{-3/2} \arccos x \, dx = \frac{1}{2a} \ln \frac{1+a}{1-a} \qquad [|\arg(1-a^2)| < \pi].$$

60. $$\int_0^1 x^{s-1} \sinh(ax) \arccos x \, dx = \frac{\pi^{1/2} a \Gamma\left(\frac{s}{2}+1\right)}{(s+1)^2 \Gamma\left(\frac{s+1}{2}\right)} {}_2F_3\left(\begin{matrix} \frac{s+1}{2}, \frac{s}{2}+1;\ \frac{a^2}{4} \\ \frac{3}{2}, \frac{s+3}{2}, \frac{s+3}{2} \end{matrix}\right)$$
$[\mathrm{Re}\, s > -1]$.

61. $$\int_0^1 \sinh(ax) \arccos x \, dx = \frac{\pi}{2a}[I_0(a) - 1].$$

62. $$\int_0^1 x \sinh(ax) \arccos x \, dx = \frac{\pi}{2a^2}[a\,\mathbf{L}_{-1}(a) - \mathbf{L}_0(a)].$$

63. $\int_0^1 x^{s-1}\cosh(ax)\arccos x\,dx = \frac{\pi^{1/2}\Gamma\left(\frac{s+1}{2}\right)}{s^2\Gamma\left(\frac{s}{2}\right)}\,{}_2F_3\left(\begin{matrix}\frac{s}{2},\ \frac{s+1}{2};\ \frac{a^2}{4}\\ \frac{1}{2},\ \frac{s}{2}+1,\ \frac{s}{2}+1\end{matrix}\right)$
[Re $s > 0$].

64. $\int_0^1 \cosh(ax)\arccos x\,dx = \frac{\pi}{2a}\,\mathbf{L}_0(a).$

65. $\int_0^1 x\cosh(ax)\arccos x\,dx = \frac{\pi}{2a^2}\,[1 - I_0(a) + a\,I_1(a)].$

66. $\int_0^1 x^{s-1}\sin(ax)\arccos x\,dx = \frac{\pi^{1/2}a\,\Gamma\left(\frac{s}{2}+1\right)}{(s+1)^2\Gamma\left(\frac{s+1}{2}\right)}\,{}_2F_3\left(\begin{matrix}\frac{s+1}{2},\ \frac{s}{2}+1;\ -\frac{a^2}{4}\\ \frac{3}{2},\ \frac{s+3}{2},\ \frac{s+3}{2}\end{matrix}\right)$
[Re $s > -1$].

67. $\int_0^1 \sin(ax)\arccos x\,dx = \frac{\pi}{2a}[1 - J_0(a)].$

68. $\int_0^1 x\sin(ax)\arccos x\,dx = \frac{\pi}{2a^2}\,[\mathbf{H}_0(a) - a\,\mathbf{H}_{-1}(a)].$

69. $\int_0^1 x^{s-1}\cos(ax)\arccos x\,dx = \frac{\pi^{1/2}\Gamma\left(\frac{s+1}{2}\right)}{s^2\Gamma\left(\frac{s}{2}\right)}\,{}_2F_3\left(\begin{matrix}\frac{s}{2},\ \frac{s+1}{2};\ -\frac{a^2}{4}\\ \frac{1}{2},\ \frac{s}{2}+1,\ \frac{s}{2}+1\end{matrix}\right)$
[Re $s > 0$].

70. $\int_0^1 \cos(ax)\arccos x\,dx = \frac{\pi}{2a}\,\mathbf{H}_0(a).$

71. $\int_0^1 x\cos(ax)\arccos x\,dx = \frac{\pi}{2a^2}\,[J_0(a) + a\,J_1(a) - 1].$

72. $\int_0^1 x^{s-1}\ln(1+ax^2)\arccos x\,dx = \frac{\pi^{1/2}a\,\Gamma\left(\frac{s+3}{2}\right)}{s(s+2)^2\Gamma\left(\frac{s}{2}+1\right)}$

$$\times\left[(s+2)\,{}_3F_2\left(\begin{matrix}1,\ 1,\ \frac{s+3}{2}\\ 2,\ \frac{s}{2}+2;\ -a\end{matrix}\right) - 2\,{}_3F_2\left(\begin{matrix}1,\ \frac{s}{2}+1,\ \frac{s+3}{2}\\ \frac{s}{2}+2,\ \frac{s}{2}+2;\ -a\end{matrix}\right)\right]$$
[Re $s > -2$; $|\arg(1+a)| < \pi$].

73. $\displaystyle\int_0^1 x\ln(1+ax^2)\arccos x\,dx$

$$= \frac{\pi}{4a}\left[1-\sqrt{1+a}+(a+2)\ln\frac{1+\sqrt{1+a}}{2}\right] \qquad [|\arg(1+a)|<\pi].$$

74. $\displaystyle\int_0^1 x\ln(1-x^2)\arccos x\,dx = \frac{\pi}{4}(\ln 2-1).$

75. $\displaystyle\int_0^1 \frac{1}{x}\ln(1+ax^2)\arccos x\,dx$

$$= \frac{\pi}{4}\left[\ln^2\frac{1+\sqrt{1+a}}{2} - 2\operatorname{Li}_2\left(\frac{1-\sqrt{1+a}}{2}\right)\right] \qquad [|\arg(1+a)|<\pi].$$

76. $\displaystyle\int_0^1 \frac{1}{x}\ln(1-x^2)\arccos x\,dx = \frac{\pi}{24}\left(12\ln^2 2-\pi^2\right).$

77. $\displaystyle\int_0^1 \frac{1}{x^2}\ln(1-x^2)\arccos x\,dx = \frac{\pi^2}{4} - 4\mathbf{G}.$

78. $\displaystyle\int_0^1 x^{s-1}\ln\frac{a+x}{a-x}\arccos x\,dx = \frac{\pi^{1/2}\Gamma\left(\frac{s}{2}\right)}{2a(s+1)\Gamma\left(\frac{s+3}{2}\right)}$

$$\times\left[(s+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\ 1,\ \frac{s}{2}+1\\ \frac{3}{2},\ \frac{s+3}{2};\ a^{-2}\end{matrix}\right) - {}_3F_2\left(\begin{matrix}1,\ \frac{s+1}{2},\ \frac{s}{2}+1\\ \frac{s+3}{2},\ \frac{s+3}{2};\ a^{-2}\end{matrix}\right)\right]$$

$$\left[\operatorname{Re}s>-1;\ |\arg(a^2-1)|<\pi\right].$$

79. $\displaystyle\int_0^1 \ln\frac{a+x}{a-x}\arccos x\,dx = \pi\left(a-\sqrt{a^2-1}+a\ln\frac{a-\sqrt{a^2-1}}{2a}\right)$

$$\left[|\arg(a^2-1)|<\pi\right].$$

80. $\displaystyle\int_0^1 x^2\arccos x\ln\frac{a+x}{a-x}\,dx$

$$= \frac{\pi}{36}\left[4a^3+9a-4(a^2+2)\sqrt{a^2-1}+12a^3\ln\frac{a+\sqrt{a^2-1}}{2a}\right]$$

$$\left[|\arg(a^2-1)|<\pi\right].$$

81. $$\int_0^1 x^{s-1} \ln\left(ax + \sqrt{1+a^2x^2}\right) \arccos x\, dx = \frac{\pi^{1/2} a\, \Gamma\left(\frac{s}{2}\right)}{4(s+1)\Gamma\left(\frac{s+3}{2}\right)}$$
$$\times \left[(s+1)\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{s}{2}+1 \\ \frac{3}{2}, \frac{s+3}{2}; -a^2 \end{matrix}\right) - {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{s+1}{2}, \frac{s}{2}+1 \\ \frac{s+3}{2}, \frac{s+3}{2}; -a^2 \end{matrix}\right)\right]$$
$$\left[\operatorname{Re} s > -1;\ |\arg(1+a^2)| < \pi\right].$$

82. $$\int_0^1 \ln\left(ax + \sqrt{1+a^2x^2}\right) \arccos x\, dx$$
$$= \frac{\sqrt{1+a^2}}{a}\left[\mathbf{K}\left(\frac{a}{\sqrt{1+a^2}}\right) - 2\,\mathbf{E}\left(\frac{a}{\sqrt{1+a^2}}\right)\right] + \frac{\pi}{2a} \quad [|\arg(1+a^2)| < \pi].$$

83. $$\int_0^1 \ln\left(x + \sqrt{1+x^2}\right) \arccos x\, dx = \frac{\pi}{2} - \frac{2}{\sqrt{\pi}}\Gamma^2\left(\frac{3}{4}\right).$$

84. $$\int_0^1 x \ln\left(ax + \sqrt{1+a^2x^2}\right) \arccos x\, dx$$
$$= \frac{1}{8a}\left\{-2a^2 + ia(a^2+1)\left[\operatorname{Li}_2(-ia) - \operatorname{Li}_2(ia)\right]\right\} \quad [|\arg(1+a^2)| < \pi].$$

85. $$\int_0^1 x \ln\left(x + \sqrt{1+x^2}\right) \arccos x\, dx = \frac{\mathbf{G}}{2} - \frac{1}{4}.$$

86. $$\int_0^1 x^2 \ln\left(ax + \sqrt{1+a^2x^2}\right) \arccos x\, dx = \frac{6a^4 + 5a^2 - 1}{27a^3\sqrt{1+a^2}}\,\mathbf{K}\left(\frac{a}{\sqrt{1+a^2}}\right)$$
$$+ \frac{7}{27a^3}(1-a^2)\sqrt{1+a^2}\ \mathbf{E}\left(\frac{a}{\sqrt{1+a^2}}\right) - \frac{\pi}{9a^3} \quad [|\arg(1+a^2)| < \pi].$$

87. $$\int_0^1 x^2 \ln\left(x + \sqrt{1+x^2}\right) \arccos x\, dx = \frac{5}{54\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right) - \frac{\pi}{9}.$$

88. $$\int_0^1 \frac{1}{x} \ln\left(ax + \sqrt{1+a^2x^2}\right) \arccos x\, dx = \frac{a}{8}\Phi\left(-a^2, 3, \frac{1}{2}\right)$$
$$[|\arg(1+a^2)| < \pi].$$

89. $$\int_0^1 \frac{1}{x} \ln\left(x + \sqrt{1+x^2}\right) \arccos x\, dx = \frac{\pi^3}{32}.$$

90. $$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x^2}} \ln\left(ax+\sqrt{1+a^2x^2}\right) \arccos x\, dx = \frac{\pi^{1/2} a\,\Gamma\left(\frac{s}{2}+1\right)}{2(s+1)\Gamma\left(\frac{s+3}{2}\right)} \times {}_4F_3\left(\begin{matrix} 1,\, 1,\, \frac{s+1}{2},\, \frac{s}{2}+1 \\ \frac{3}{2},\, \frac{s+3}{2},\, \frac{s+3}{2};\, -a^2 \end{matrix}\right) \quad \left[\operatorname{Re} s > 0;\ |\arg(1+a^2)| < \pi\right].$$

91. $$\int_0^1 \frac{1}{\sqrt{1+a^2x^2}} \ln\left(ax+\sqrt{1+a^2x^2}\right) \arccos x\, dx = -\frac{\pi}{8a} \operatorname{Li}_2(-a^2) \quad [|\arg(1+a^2)| < \pi].$$

92. $$\int_0^1 \frac{1}{\sqrt{1+x^2}} \ln\left(x+\sqrt{1+x^2}\right) \arccos x\, dx = \frac{\pi^3}{96}.$$

93. $$\int_0^1 \frac{x^2}{\sqrt{1+a^2x^2}} \ln\left(ax+\sqrt{1+a^2x^2}\right) \arccos x\, dx = \frac{\pi}{16a^3}\left[(a^2+1)\ln(a^2+1) + \operatorname{Li}_2(-a^2)\right] \quad [|\arg(1+a^2)| < \pi].$$

94. $$\int_0^1 \frac{x^2}{\sqrt{1+x^2}} \ln\left(x+\sqrt{1+x^2}\right) \arccos x\, dx = \frac{\pi}{192}(24\ln 2 - \pi^2).$$

95. $$\int_0^1 x^{s-1} \ln^2\left(ax+\sqrt{1+a^2x^2}\right) \arccos x\, dx = \frac{\pi^{1/2} a^2 \Gamma\left(\frac{s+3}{2}\right)}{2s(s+2)\Gamma\left(\frac{s}{2}+2\right)} \times \left[(s+2)\,{}_4F_3\left(\begin{matrix} 1,\, 1,\, 1,\, \frac{s+3}{2};\, -a^2 \\ \frac{3}{2},\, 2,\, \frac{s}{2}+2 \end{matrix}\right) - 2\,{}_4F_3\left(\begin{matrix} 1,\, 1,\, \frac{s}{2}+1,\, \frac{s+3}{2};\, -a^2 \\ \frac{3}{2},\, \frac{s}{2}+2,\, \frac{s}{2}+2 \end{matrix}\right)\right] \quad \left[\operatorname{Re} s > 0;\ |\arg(1+a^2)| < \pi\right].$$

96. $$\int_0^1 x \ln^2\left(ax+\sqrt{1+a^2x^2}\right) \arccos x\, dx = -\frac{\pi}{16a^2}\left[a^2 + (a^2+1)\operatorname{Li}_2(-a^2)\right] \quad [|\arg(1+a^2)| < \pi].$$

97. $$\int_0^1 x \ln^2\left(x+\sqrt{1+x^2}\right) \arccos x\, dx = \frac{\pi^3}{96} - \frac{\pi}{16}.$$

98. $$\int_0^1 x^3 \ln^2\left(ax + \sqrt{1+a^2x^2}\right) \arccos x \, dx$$
$$= \frac{\pi}{512a^4}[4a^2 - 15a^4 + 8(a^2+1)^2 \ln(1+a^2) - 12(a^4-1)\operatorname{Li}_2(-a^2)]$$
$$[|\arg(1+a^2)| < \pi].$$

99. $$\int_0^1 x^3 \ln^2\left(x + \sqrt{1+x^2}\right) \arccos x \, dx = \frac{\pi}{512}(32\ln 2 - 11).$$

100. $$\int_0^1 \frac{1}{x} \ln^2\left(ax + \sqrt{1+a^2x^2}\right) \arccos x \, dx = -\frac{\pi}{8}\operatorname{Li}_3(-a^2)$$
$$[|\arg(1+a^2)| < \pi].$$

101. $$\int_0^1 \frac{1}{x} \ln^2\left(x + \sqrt{x^2+1}\right) \arccos x \, dx = \frac{3\pi}{32}\zeta(3).$$

102. $$\int_0^1 x^{s-1} \arcsin(ax) \arccos x \, dx$$
$$= \frac{\pi^{1/2} a\,\Gamma\left(\frac{s}{2}\right)}{4(s+1)\Gamma\left(\frac{s+3}{2}\right)}\left[(s+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{s}{2}+1\\ \frac{3}{2}, \frac{s+3}{2};\ a^2\end{matrix}\right) - {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{s+1}{2}, \frac{s}{2}+1\\ \frac{s+3}{2}, \frac{s+3}{2};\ a^2\end{matrix}\right)\right]$$
$$\left[\operatorname{Re} s > 0;\ |\arg(1-a^2)| < \pi\right].$$

103. $$\int_0^1 \arcsin(ax) \arccos x \, dx = \frac{1}{2a}[2(a^2-1)\,\mathbf{K}(a) + 4\,\mathbf{E}(a) - \pi]$$
$$[|\arg(1-a^2)| < \pi].$$

104. $$\int_0^1 \arcsin(x) \arccos x \, dx = 2 - \frac{\pi}{2}.$$

105. $$\int_0^1 x \arcsin(ax) \arccos x \, dx = \frac{a}{16}\left[\Phi\left(a^2, 2, \frac{1}{2}\right) - \Phi\left(a^2, 2, \frac{3}{2}\right)\right]$$
$$[|\arg(1-a^2)| < \pi].$$

106. $$\int_0^1 x \arcsin x \arccos x \, dx = \frac{1}{4}.$$

107. $$\int_0^1 x^2 \arcsin(ax) \arccos x\, dx = \frac{1}{27a^3}\left[(6a^4 - 5a^2 - 1)\,\mathbf{K}(a) + 7(a^2+1)\,\mathbf{E}(a) - 3\pi\right]$$ $[|\arg(1-a^2)| < \pi]$.

108. $$\int_0^1 x^2 \arcsin x \arccos x\, dx = \frac{14}{27} - \frac{\pi}{9}.$$

109. $$\int_0^1 \frac{1}{x} \arcsin(ax) \arccos x\, dx = \frac{a}{8}\Phi\left(a^2, 3, \frac{1}{2}\right)$$ $[|\arg(1-a^2)| < \pi]$.

110. $$\int_0^1 \frac{1}{x} \arcsin x \arccos x\, dx = \frac{7}{8}\zeta(3).$$

111. $$\int_0^1 \frac{x^{s-1}}{\sqrt{1-a^2x^2}} \arcsin(ax) \arccos x\, dx = \frac{\pi^{1/2} a\,\Gamma\left(\frac{s}{2}+1\right)}{2(s+1)\Gamma\left(\frac{s+3}{2}\right)} \times {}_4F_3\left(\begin{matrix} 1,\ 1,\ \frac{s+1}{2},\ \frac{s}{2}+1 \\ \frac{3}{2},\ \frac{s+3}{2},\ \frac{s+3}{2};\ a^2 \end{matrix}\right)$$ $[\operatorname{Re} s > 0;\ |\arg(1-a^2)| < \pi]$.

112. $$\int_0^1 \frac{1}{\sqrt{1-a^2x^2}} \arcsin(ax) \arccos x\, dx = \frac{\pi}{8a}\operatorname{Li}_2(a^2)$$ $[|\arg(1-a^2)| < \pi]$.

113. $$\int_0^1 \frac{1}{\sqrt{1-x^2}} \arcsin x \arccos x\, dx = \frac{\pi^3}{48}.$$

114. $$\int_0^1 \frac{x^2}{\sqrt{1-a^2x^2}} \arcsin(ax) \arccos x\, dx = \frac{\pi}{16a^3}\left[(1-a^2)\ln(1-a^2) + \operatorname{Li}_2(a^2)\right]$$ $[|\arg(1-a^2)| < \pi]$.

115. $$\int_0^1 \frac{x^2}{\sqrt{1-x^2}} \arcsin x \arccos x\, dx = \frac{\pi^3}{96}.$$

116. $$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \ln(1+2ax+a^2) \arccos^2 x\, dx = -4\pi \operatorname{Li}_3(a)$$ $[|\arg(1-a^2)| < \pi]$.

117. $$\int_0^a x^{-1}(a^2 - x^2)^{1/2} \arctan(bx)\, dx$$
$$= \frac{\pi}{2a}\left[1 - \sqrt{a^2b^2+1} + ab \ln\left(ab + \sqrt{a^2b^2+1}\right)\right]$$
$$[a > 0;\ |\arg(1 + a^2b^2)| < \pi].$$

118. $$\int_0^a x^{-1}(a^2 - x^2)^{-1/2} \arctan(bx)\, dx = \frac{\pi}{2a} \ln\left(ab + \sqrt{a^2b^2+1}\right)$$
$$[a > 0;\ |\arg(1 + a^2b^2)| < \pi].$$

119. $$\int_0^\infty \frac{1}{x(x^2+a^2)} \arctan x\, dx = \frac{\pi}{2a^2} \ln(1+a) \qquad [\operatorname{Re} a > 0;\ |\arg(1+a)| < \pi].$$

120. $$\int_0^1 \frac{1}{1+x^2} \arctan\left(x^{3+\sqrt{8}}\right) dx = \frac{1}{16} \ln 2 \ln\left(3+\sqrt{8}\right) \qquad [60].$$

121. $$\int_0^1 \frac{1}{1+x^2} \arctan\left(x^{5+\sqrt{24}}\right) dx = \frac{1}{8} \ln\left(1+\sqrt{2}\right) \ln\left(2+\sqrt{3}\right)$$
$$- \frac{1}{16} \ln 2 \ln\left(5+\sqrt{24}\right) \qquad [60].$$

122. $$\int_0^1 \frac{1}{1+x^2} \arctan\left(x^{11+\sqrt{120}}\right) dx = -\frac{1}{8} \ln\left(1+\sqrt{2}\right) \ln\left(4+\sqrt{15}\right)$$
$$- \frac{1}{8} \ln\left(2+\sqrt{3}\right) \ln\left(3+\sqrt{10}\right) + + \frac{3}{8} \ln \frac{1+\sqrt{5}}{2} \ln\left(5+\sqrt{24}\right)$$
$$+ \frac{1}{16} \ln 2 \ln\left(11+\sqrt{120}\right) \qquad [60].$$

123. $$\int_0^1 \frac{1}{1+x^2} \arctan\left(x^{13+\sqrt{168}}\right) dx$$
$$= -\frac{3}{8} \ln\left(1+\sqrt{2}\right) \ln \frac{5+\sqrt{21}}{2} - \frac{1}{16} \ln 2 \ln\left(13+\sqrt{168}\right)$$
$$+ \frac{1}{16} \ln\left(2+\sqrt{3}\right) \ln\left(15+\sqrt{224}\right) + \frac{1}{16} \ln\left(5+\sqrt{24}\right) \ln\left(8+\sqrt{63}\right) \qquad [60].$$

124. $$\int_0^\infty \frac{1}{x(x^2+b^2)} \arctan \frac{ax}{x^2+b^2}\, dx = \frac{\pi}{2b^2} \ln \frac{a+\sqrt{a^2+4b^2}}{2b} \qquad [\operatorname{Re} b > 0].$$

125. $$\int_0^\infty \frac{1}{x(a^2x^2+1)} \arctan[(a^2+1)\,x + a^2x^3]\,dx = \frac{\pi}{2} \ln \frac{(1+a)\left(a+\sqrt{4+a^2}\right)}{2a} \quad [\operatorname{Re} a > 0].$$

126. $$\int_0^a x^{s-1}(a-x)^{t-1} \arctan\left(b\sqrt{x(a-x)}\right) dx = a^{s+t} b\, \mathrm{B}\left(s+\frac{1}{2}, t+\frac{1}{2}\right) {}_3F_2\left(\begin{matrix} \frac{1}{2}, 1, s+\frac{1}{2}, t+\frac{1}{2};\ -\frac{a^2b^2}{4} \\ \frac{3}{2}, \frac{s+t+1}{2}, \frac{s+t}{2}+1 \end{matrix}\right)$$
$$\left[a > 0;\ \operatorname{Re} s,\ \operatorname{Re} t > -1/2;\ |\arg(4+a^2b^2)| < \pi\right].$$

127. $$\int_0^a \arctan\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi}{2b}\left(\sqrt{4+a^2b^2}-2\right) \quad [a > 0;\ |\arg(4+a^2b^2)| < \pi].$$

128. $$\int_0^a x \arctan\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{4b}\left(\sqrt{4+a^2b^2}-2\right) \quad [a > 0;\ |\arg(4+a^2b^2)| < \pi].$$

129. $$\int_0^a x^{-1} \arctan\left(b\sqrt{x(a-x)}\right) dx = \pi \ln\left(\frac{ab}{2} + \sqrt{1+\frac{a^2b^2}{4}}\right) \quad [a > 0;\ |\arg(4+a^2b^2)| < \pi].$$

130. $$\int_0^a x^{-1}(a-x)^{-1} \arctan\left(b\sqrt{x(a-x)}\right) dx = \frac{2\pi}{a} \ln\left(\frac{ab}{2} + \sqrt{1+\frac{a^2b^2}{4}}\right) \quad [a > 0;\ |\arg(4+a^2b^2)| < \pi].$$

131. $$\int_0^a x^{s+1/2}(a-x)^s \arctan\left(b\sqrt[4]{x(a-x)}\right) dx = 2^{-2s-3/2}\pi^{1/2}a^{2s+2}b$$
$$\times \frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)}\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, 1, 2s+\frac{5}{2} \\ \frac{3}{2}, 2s+3;\ -\frac{ab^2}{2} \end{matrix}\right) \quad [a > 0;\ \operatorname{Re} s > -1;\ |\arg(2+ab^2)| < \pi].$$

132. $\displaystyle\int_0^a x^{1/2}\arctan\left(b\sqrt[4]{x(a-x)}\right)dx$

$$= \frac{\pi}{6\sqrt{2}\,b^3}\left[2(ab^2-1)\sqrt{2ab^2+4}-3ab^2+4\right]$$

$$[a>0;\ |\arg(2+ab^2)|<\pi].$$

133. $\displaystyle\int_0^a x^{-1/2}\arctan\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi}{b}\left(\sqrt{ab^2+2}-\sqrt{2}\right)$

$$[a>0;\ |\arg(2+ab^2)|<\pi].$$

134. $\displaystyle\int_0^a x^{-1/2}(a-x)^{-1}\arctan\left(b\sqrt[4]{x(a-x)}\right)dx$

$$= \frac{2\pi}{\sqrt{a}}\ln\left(b\sqrt{\frac{a}{2}}+\sqrt{1+\frac{ab^2}{2}}\right) \quad [a>0;\ |\arg(2+ab^2)|<\pi].$$

135. $\displaystyle\int_0^\infty \frac{1}{e^{2\pi x}+1}\arctan x\,dx = \frac{3}{4}\ln 2-\frac{1}{2}.$

136. $\displaystyle\int_0^{\pi/2}\cos^\nu x\cos(ax)\arctan(b\cos x)\,dx = \frac{2^{-\nu-2}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)}$

$$\times\ {}_4F_3\left(\begin{matrix}\frac{1}{2},\ 1,\ 1+\frac{\nu}{2},\ \frac{\nu+3}{2};\ -b^2\\ \frac{3}{2},\ \frac{\nu-a+3}{2},\ \frac{\nu+a+3}{2}\end{matrix}\right) \quad [\operatorname{Re}\nu>-1;\ |\arg(1+b^2)|<\pi].$$

137. $\displaystyle\int_0^{\pi/2}\frac{\cos(2nx)}{\sin x}\arctan(a\sin x)\,dx$

$$= \frac{2^{-2n-1}\pi a^{2n+1}}{2n+1}\,{}_3F_2\left(\begin{matrix}n+\frac{1}{2},\ n+\frac{1}{2},\ n+1\\ n+\frac{3}{2},\ 2n+1;\ -a^2\end{matrix}\right) \quad [|\arg(1+a^2)|<\pi].$$

138. $\displaystyle\int_0^{m\pi}\frac{1}{\sin x}\left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\}\arctan(b\sin x)\,dx$

$$= \frac{2b}{a}\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\}{}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ 1,\ 1;\ -b^2\\ \frac{3}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right) \quad [|\arg(1+b^2)|<\pi].$$

139. $$\int_0^{\pi} \sin^{\nu} x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \arctan(b\sin x)\,dx = \frac{2^{-\nu-1}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)}$$
$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, \frac{\nu}{2}+1,\, \frac{\nu+3}{2};\; -b^2 \\ \frac{3}{2},\, \frac{\nu-a+3}{2},\, \frac{\nu+a+3}{2} \end{matrix}\right)$$
$$[\operatorname{Re}\nu > -1;\; |\arg(1+b^2)| < \pi].$$

140. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} \arctan(b\sin x)\,dx = \frac{b}{a}\left(1-e^{-m\pi a}\right) {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1,\, 1;\; -b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

141. $$\int_0^{\infty} e^{-ax} \arctan(b\sin x)\,dx = \frac{b}{a^2+1}\, {}_3F_2\left(\begin{matrix} \frac{1}{2},\, 1,\, 1;\; -b^2 \\ \frac{3-ia}{2},\, \frac{3+ia}{2} \end{matrix}\right)$$
$$\left[\operatorname{Re} a > 0;\; |\arg(1+b^2)| < \pi\right].$$

142. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} \arctan(b\sin x)\,dx = \frac{b}{a}\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1,\, 1;\; -b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$$\left[\operatorname{Re} a > 0;\; |\arg(1+b^2)| < \pi\right].$$

143. $$\int_0^{\infty} \frac{\sin(z\arctan x)}{(1+x^2)^{z/2}(e^{ax}-1)}\,dx = \frac{\left(\frac{a}{2\pi}\right)^{z-1}}{2}\zeta\left(z,\, \frac{a}{2\pi}\right) - \frac{1}{2(z-1)} - \frac{\pi}{2a}$$
$$[\operatorname{Re} a > 0].$$

144. $$\int_0^{a} x^{s-1} \ln\frac{a+\sqrt{a^2-x^2}}{x} \arctan(bx)\,dx = \frac{\pi^{1/2} a^{s+1} b\,\Gamma\left(\frac{s+1}{2}\right)}{s(s+1)\Gamma\left(\frac{s}{2}\right)}$$
$$\times\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, \frac{s+1}{2},\, \frac{s+1}{2} \\ \frac{3}{2},\, \frac{s}{2}+1,\, \frac{s+3}{2};\; -a^2b^2 \end{matrix}\right) \quad [a > 0;\; \operatorname{Re} s > -1;\; |\arg(1+a^2b^2)| < \pi].$$

145. $$\int_0^{1} x^{s-1} \ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \arctan(ax)\,dx$$
$$= \frac{2\pi^{1/2} a\,\Gamma\left(\frac{s+1}{2}\right)}{s^2(s+1)\Gamma\left(\frac{s}{2}\right)} \left[(s+1)\, {}_3F_2\left(\begin{matrix} \frac{1}{2},\, 1,\, \frac{s+1}{2} \\ \frac{3}{2},\, \frac{s}{2}+1;\; -a^2 \end{matrix}\right) - {}_3F_2\left(\begin{matrix} 1,\, \frac{s+1}{2},\, \frac{s+1}{2} \\ \frac{s}{2},\, \frac{s+3}{2};\; -a^2 \end{matrix}\right)\right]$$
$$[\operatorname{Re} a > 0;\; \operatorname{Re} s > -1].$$

146. $$\int_0^{1} x \ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \arctan(ax)\,dx = \frac{\pi}{2a^3}\left[a^2\left(\sqrt{a^2+1}-2\right)\right.$$
$$\left. + 2a\ln\left(\sqrt{\frac{1}{2}\sqrt{a^2+1}-\frac{1}{2}} + \sqrt{\frac{1}{2}\sqrt{a^2+1}+\frac{1}{2}}\right)\right] \quad [|\arg(1+a^2)| < \pi].$$

147. $$\int_0^\infty \frac{\ln(x^2+1)}{e^{ax}-1}\arctan x\,dx = \frac{a-\pi}{2a}\ln^2\frac{a}{2\pi} - \frac{\pi}{a}\ln\frac{a}{2\pi}\ln\left[\frac{1}{2\pi}\Gamma^2\left(\frac{a}{2\pi}\right)\right] - 1 - \frac{\pi}{a}\frac{\partial^2}{\partial z^2}\zeta\left(z,\,\frac{a}{2\pi}\right)\bigg|_{z=0} \qquad [\operatorname{Re} a > 0].$$

148. $$\int_0^a x^{s-1}(a^2-x^2)^{t-1}\arcsin^2(bx)\,dx = \frac{1}{2}a^{s+2t}b^2\,\mathrm{B}\left(\frac{s}{2}+1,\,t\right)\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,\frac{s}{2}+1;\;a^2b^2\\ \frac{3}{2},\,2,\,\frac{s}{2}+t+1\end{matrix}\right)$$
$$\left[a,\,\operatorname{Re} t>0;\;\operatorname{Re} s>-2;\;|\arg(1-a^2b^2)|<\pi\right].$$

149. $$\int_0^a (a^2-x^2)^{1/2}\arcsin^2(bx)\,dx = \frac{\pi a^2}{8}\left[\mathrm{Li}_2(a^2b^2) - \left(\frac{1}{a^2b^2}-1\right)\ln(1-a^2b^2) - 1\right]$$
$$[a>0;\;|\arg(1-a^2b^2)|<\pi].$$

150. $$\int_0^a (a^2-x^2)^{-1/2}\arcsin^2(bx)\,dx = \frac{\pi}{4}\,\mathrm{Li}_2(a^2b^2)$$
$$[a>0;\;|\arg(1-a^2b^2)|<\pi].$$

151. $$\int_0^a x^{s-1}(a-x)^{t-1}\arcsin^2\left(b\sqrt{x(a-x)}\right)dx = a^{s+t+1}b^2\,\mathrm{B}(s+1,\,t+1)\,{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,s+1,\,t+1;\;\frac{a^2b^2}{4}\\ \frac{3}{2},\,2,\,\frac{s+t}{2}+1,\,\frac{s+t+3}{2}\end{matrix}\right)$$
$$\left[a>0;\;\operatorname{Re} s,\,\operatorname{Re} t>-1;\;|\arg(4-a^2b^2)|<\pi\right].$$

152. $$\int_0^a x^{1/2}(a-x)^{1/2}\arcsin^2\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{16b^2}\left[a^2b^2 + (4-a^2b^2)\ln\left(1-\frac{a^2b^2}{4}\right) + a^2b^2\,\mathrm{Li}_2\left(\frac{a^2b^2}{4}\right)\right]$$
$$[a>0;\;|\arg(4-a^2b^2)|<\pi].$$

153. $$\int_0^a x^{1/2}(a-x)^{-1/2}\arcsin^2\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi a}{4}\,\mathrm{Li}_2\left(\frac{a^2b^2}{4}\right)$$
$$[a>0;\;|\arg(4-a^2b^2)|<\pi].$$

154. $$\int_0^a x^{1/2}(a-x)^{-3/2}\arcsin^2\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{2}\left[2ab\ln\frac{2+ab}{2-ab}+4\ln\left(1-\frac{a^2b^2}{4}\right)-\mathrm{Li}_2\left(\frac{a^2b^2}{4}\right)\right]$$
$[a>0;\ |\arg(4-a^2b^2)|<\pi].$

155. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\arcsin^2\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{2}\,\mathrm{Li}_2\left(\frac{a^2b^2}{4}\right)$$
$[a>0;\ |\arg(4-a^2b^2)|<\pi].$

156. $$\int_0^a x^{-1/2}(a-x)^{-3/2}\arcsin^2\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{a}\left[ab\ln\frac{2+ab}{2-ab}+2\ln\left(1-\frac{a^2b^2}{4}\right)\right]$$
$[a>0;\ |\arg(4-a^2b^2)|<\pi].$

157. $$\int_0^a x^{-3/2}(a-x)^{-3/2}\arcsin^2\left(b\sqrt{x(a-x)}\right)dx = \frac{2\pi}{a^2}\left[ab\ln\frac{2+ab}{2-ab}+2\ln\left(1-\frac{a^2b^2}{4}\right)\right]$$
$[a>0;\ |\arg(4-a^2b^2)|<\pi].$

158. $$\int_0^a x^{s}(a-x)^{s+1/2}\arcsin^2\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{2^{-2s-2}\sqrt{\pi}\,a^{2s+5/2}b^2\Gamma(2s+3)}{\Gamma\left(2s+\frac{7}{2}\right)}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,2s+3;\ \frac{ab^2}{2}\\ \frac{3}{2},\,2,\,2s+\frac{7}{2}\end{matrix}\right)$$
$[a>0;\ \operatorname{Re}s>-3/2;\ |\arg(2-ab^2)|<\pi].$

159. $$\int_0^a x^{1/4}(a-x)^{-1/4}\arcsin^2\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi}{4\sqrt{2}\,b^2}\left[ab^2+(2-ab^2)\ln\left(1-\frac{ab^2}{2}\right)+ab^2\,\mathrm{Li}_2\left(\frac{ab^2}{2}\right)\right]$$
$[a>0;\ |\arg(2-ab^2)|<\pi].$

160. $$\int_0^a x^{-1/4}(a-x)^{-3/4}\arcsin^2\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi}{\sqrt{2}}\,\mathrm{Li}_2\left(\frac{ab^2}{2}\right)$$
$[a>0;\ |\arg(2-ab^2)|<\pi].$

161. $$\int_0^a x^{-5/4}(a-x)^{-3/4}\arcsin^2\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{2\pi b}{\sqrt{a}}\ln\frac{\sqrt{2}+\sqrt{a}\,b}{\sqrt{2}-\sqrt{a}\,b} + \frac{2^{3/2}\pi}{a}\ln\left(1-\frac{ab^2}{2}\right) \quad [a>0;\ |\arg(2-ab^2)|<\pi].$$

162. $$\int_{-a}^{a}\frac{1}{(x+a)^2}\arcsin^2\frac{b(x+a)}{\sqrt{x^2+a^2}}\,dx = \frac{\pi b^2}{2a}\left[\frac{1}{\sqrt{2}\,b}\ln\frac{1+\sqrt{2}\,b}{1-\sqrt{2}\,b}+\frac{1}{2b^2}\ln\left(1-2b^2\right)\right] \quad [a>0;\ |\arg(1-b^2)|<\pi].$$

163. $$\int_0^a x^{s-1}\ln\frac{a+\sqrt{a^2-x^2}}{x}\arcsin^2(bx)\,dx = \frac{\sqrt{\pi}}{2(s+2)}a^{s+2}\frac{\Gamma\left(\frac{s}{2}+1\right)b^2}{\Gamma\left(\frac{s+3}{2}\right)}\,{}_5F_4\left(\begin{matrix}1,1,1,\frac{s}{2}+1,\frac{s}{2}+1;\ a^2b^2\\ \frac{3}{2},2,\frac{s+3}{2},\frac{s}{2}+2\end{matrix}\right)$$
$$[a>0;\ \operatorname{Re}s>-2;\ |\arg(1-a^2b^2)|<\pi].$$

164. $$\int_0^{m\pi}\frac{e^{-ax}}{\sin^2x}\arcsin^2(b\sin x)\,dx = \frac{b^2}{a}\left(1-e^{-m\pi a}\right)\times{}_5F_4\left(\begin{matrix}\frac{1}{2},1,1,1,1;\ b^2\\ \frac{3}{2},2,1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re}a>0;\ |\arg(1-b^2)|<\pi].$$

165. $$\int_0^{\infty}\frac{e^{-ax}}{\sin^2x}\arcsin^2(b\sin x)\,dx = \frac{b^2}{a}\,{}_5F_4\left(\begin{matrix}\frac{1}{2},1,1,1,1;\ b^2\\ \frac{3}{2},2,1-\frac{ia}{2},1+\frac{ia}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re}a>0;\ |\arg(1-b^2)|<\pi\right].$$

166. $$\int_0^a \ln\frac{a+\sqrt{a^2-x^2}}{x}\arcsin^2(bx)\,dx = \frac{\pi a}{4}\left[\operatorname{Li}_2(a^2b^2)+2\ln\left(1-a^2b^2\right)-4\right] + \frac{\pi}{2b}\ln\frac{1+ab}{1-ab} \quad [a>0;\ |\arg(1-a^2b^2)|<\pi].$$

167. $$\int_0^1 \ln\frac{1+\sqrt{1-x^2}}{x}\arcsin^2x\,dx = \frac{\pi^3}{24}+(\ln 2-1)\pi.$$

168. $$\int_0^a x^2\ln\frac{a+\sqrt{a^2-x^2}}{x}\arcsin^2(bx)\,dx = \frac{\pi a^3}{24}\operatorname{Li}_2(a^2b^2) - \frac{\pi}{216b^3}\left[3ab(a^2b^2-9)\ln\left(1-a^2b^2\right)-24\ln\frac{1+ab}{1-ab}+48ab-11a^3b^3\right]$$
$$[a>0;\ |\arg(1-a^2b^2)|<\pi].$$

169. $$\int_0^1 x^2 \ln \frac{1+\sqrt{1-x^2}}{x} \arcsin^2 x \, dx = \frac{\pi}{432}(3\pi^2 + 96 \ln 2 - 74).$$

170. $$\int_0^a \frac{1}{x^2} \ln \frac{a+\sqrt{a^2-x^2}}{x} \arcsin^2(bx) \, dx$$
$$= \frac{b\pi}{2}\left[\operatorname{Li}_2(ab) - \operatorname{Li}_2(-ab) - \ln \frac{1+ab}{1-ab}\right] - \frac{\pi}{2a} \ln(1-a^2b^2)$$
$[a > 0;\ |\arg(1-a^2b^2)| < \pi]$.

171. $$\int_0^{\pi/2} \cos^\nu x \cos(ax) \arcsin^2(b\cos x) \, dx = \frac{2^{-\nu-3}\pi b \Gamma(\nu+3)}{\Gamma\left(\frac{\nu-a}{2}+2\right)\Gamma\left(\frac{\nu+a}{2}+2\right)}$$
$$\times {}_5F_4\left(\begin{matrix} 1, 1, 1, \frac{\nu+3}{2}, \frac{\nu}{2}+2;\ b \\ \frac{3}{2}, 2, \frac{\nu-a}{2}+2, \frac{\nu+a}{2}+2 \end{matrix}\right) \quad [\operatorname{Re}\nu > 0;\ |\arg(1-b)| < \pi].$$

172. $$\int_0^{\pi} \sin^\nu x \left\{\begin{matrix} \sin(ax) \\ \cos(ax) \end{matrix}\right\} \arcsin^2(b\sin x) \, dx = \frac{2^{-\nu-2}\pi b^2 \Gamma(\nu+3)}{\Gamma\left(\frac{\nu-a}{2}+2\right)\Gamma\left(\frac{\nu+a}{2}+2\right)}$$
$$\times \left\{\begin{matrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{matrix}\right\} {}_5F_4\left(\begin{matrix} 1, 1, 1, \frac{\nu+3}{2}, \frac{\nu}{2}+2;\ b \\ \frac{3}{2}, 2, \frac{\nu-a}{2}+2, \frac{\nu+a}{2}+2 \end{matrix}\right)$$
$[\operatorname{Re}\nu > -1;\ |\arg(1-b)| < \pi]$.

173. $$\int_0^{m\pi} \frac{1}{\sin^2 x} \left\{\begin{matrix} \sin(ax) \\ \cos(ax) \end{matrix}\right\} \arcsin^2(b\sin x) \, dx$$
$$= \frac{2b^2}{a} \sin\frac{m\pi a}{2} \left\{\begin{matrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{matrix}\right\} {}_5F_4\left(\begin{matrix} \frac{1}{2}, 1, 1, 1, 1;\ b^2 \\ \frac{3}{2}, 2, 1-\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right)$$
$[|\arg(1-b^2)| < \pi]$.

174. $$\int_0^1 \frac{1}{x^2} \ln \frac{1+\sqrt{1-x^2}}{x} \arcsin^2(x) \, dx = \frac{\pi^3}{8} - \pi \ln 2.$$

175. $$\int_0^1 x^{s-1} \arcsin^2(ax) \arccos x \, dx = \frac{\pi^{1/2} a^2 \Gamma\left(\frac{s+3}{2}\right)}{2s(s+2)\Gamma\left(\frac{s}{2}+2\right)}$$
$$\times \left[(s+2)\, {}_4F_3\left(\begin{matrix} 1, 1, 1, \frac{s+3}{2} \\ \frac{3}{2}, 2, \frac{s}{2}+2;\ a^2 \end{matrix}\right) - 2\, {}_4F_3\left(\begin{matrix} 1, 1, \frac{s}{2}+1, \frac{s+3}{2} \\ \frac{3}{2}, \frac{s}{2}+2, \frac{s}{2}+2;\ a^2 \end{matrix}\right)\right]$$
$[\operatorname{Re} s > -1;\ |\arg(1-a^2)| < \pi]$.

176. $$\int_0^1 \frac{1}{x} \arccos x \arcsin^2(ax)\, dx = \frac{\pi}{8} \operatorname{Li}_3(a^2) \qquad \left[|\arg(1-a^2)| < \pi\right].$$

177. $$\int_0^1 \frac{1}{x} \arccos x \arcsin^2 x\, dx = \frac{\pi}{8}\zeta(3).$$

178. $$\int_0^1 x \arccos x \arcsin^2(ax)\, dx = \frac{\pi}{16a^2}\left[a^2 + (a^2-1)\operatorname{Li}_2(a^2)\right]$$
$$\left[|\arg(1-a^2)| < \pi\right].$$

179. $$\int_0^1 x^3 \arccos x \arcsin^2(ax)\, dx$$
$$= \frac{\pi}{512a^4}\left[4a^2 + 15a^4 - 8(a^2-1)^2 \ln(1-a^2) + 12(a^4-1)\operatorname{Li}_2(a^2)\right] \mathrm{Li}$$
$$\left[|\arg(1-a^2)| < \pi\right].$$

180. $$\int_0^1 x^3 \arccos x \arcsin^2 x\, dx = \frac{19\pi}{512}.$$

181. $$\int_0^1 x^{s-1} \arccos x \arctan(ax)\, dx = \frac{\pi^{1/2} a\, \Gamma\left(\frac{s}{2}\right)}{4(s+1)\Gamma\left(\frac{s+3}{2}\right)}$$
$$\times \left[(s+1)\, {}_3F_2\left(\begin{matrix}\frac{1}{2}, 1, \frac{s}{2}+1 \\ \frac{3}{2}, \frac{s+3}{2};\ -a^2\end{matrix}\right) - {}_3F_2\left(\begin{matrix}1, \frac{s+1}{2}, \frac{s}{2}+1 \\ \frac{s+3}{2}, \frac{s+3}{2};\ -a^2\end{matrix}\right)\right]$$
$$\left[\operatorname{Re} s > -1;\ |\arg(1+a^2)| < \pi\right].$$

182. $$\int_0^1 \arccos x \arctan(ax)\, dx = \frac{\pi}{2a}\left(\sqrt{1+a^2} - \ln\frac{1+\sqrt{1+a^2}}{2} - 1\right)$$
$$\left[|\arg(1+a^2)| < \pi\right].$$

183. $$\int_0^1 x^2 \arccos x \arctan(ax)\, dx$$
$$= \frac{\pi}{72a^3}(4-9a^2) + \frac{\pi}{18a^3\sqrt{1+a^2}}(2a^4+a^2-1) + \frac{\pi}{6a^3}\ln\frac{1+\sqrt{1+a^2}}{2} \ln$$
$$\left[|\arg(1+a^2)| < \pi\right].$$

184. $\int\limits_0^\infty \left(\frac{\arctan x}{x}\right)^n dx = I_n, \quad I_2 = \pi \ln 2, \quad I_3 = \frac{3\pi}{2}\ln 2 - \frac{\pi^3}{16},$

$$I_4 = 2\pi\left(1 - \frac{\pi^2}{12}\right)\ln 2 - \frac{\pi^3}{12} + \frac{3\pi}{4}\zeta(3), \quad I_5 = \frac{5\pi}{2}\left(1 - \frac{\pi^2}{3}\right)\ln 2 - \frac{5\pi^3}{48} + \frac{\pi^5}{128} + \frac{15\pi}{4}\zeta(3).$$

4.2. The Dilogarithm $\mathrm{Li}_2(z)$

4.2.1. Integrals containing $\mathrm{Li}_2(z)$ and algebraic functions

1. $\int\limits_0^a x^{s-1}(a-x)^{t-1}\,\mathrm{Li}_2(bx(a-x))\,dx$

$$= a^{s+t+1} b\,\mathrm{B}\,(s+1,\, t+1)\, {}_5F_4\left(\begin{matrix}1,\,1,\,1,\,s+1,\,t+1;\; \frac{a^2 b}{4}\\ 2,\,2,\,\frac{s+t}{2}+1,\,\frac{s+t+3}{2}\end{matrix}\right)$$

$\left[a > 0;\ \mathrm{Re}\, s,\, \mathrm{Re}\, t > -1;\ |\arg(4 - a^2 b)| < \pi\right].$

2. $\int\limits_0^a x^{1/2}(a-x)^{-1/2}\,\mathrm{Li}_2(bx(a-x))\,dx = \pi a\,\mathrm{Li}_2\left(\frac{2-\sqrt{4-a^2b}}{4}\right)$

$$- \frac{\pi a}{2}\ln^2\frac{2+\sqrt{4-a^2b}}{4} \quad [a > 0;\ |\arg(4 - a^2 b)| < \pi].$$

3. $\int\limits_0^a x^{-1/2}(a-x)^{-1/2}\,\mathrm{Li}_2(bx(a-x))\,dx = 2\pi\,\mathrm{Li}_2\left(\frac{2-\sqrt{4-a^2b}}{4}\right)$

$$- \pi\ln^2\frac{2+\sqrt{4-a^2b}}{4} \quad [a > 0;\ |\arg(4 - a^2 b)| < \pi].$$

4. $\int\limits_0^a x^{-1/2}(a-x)^{-3/2}\,\mathrm{Li}_2(bx(a-x))\,dx$

$$= \frac{8\pi}{a}\left(1 - \sqrt{1 - \frac{a^2 b}{4}} + \ln\frac{2+\sqrt{4-a^2b}}{4}\right) \quad [a > 0;\ |\arg(4 - a^2 b)| < \pi].$$

5. $\int\limits_0^a x^s(a-x)^{s+1/2}\,\mathrm{Li}_2\left(b\sqrt{x(a-x)}\right)dx = \frac{2^{-2s-2}\sqrt{\pi}\,a^{2s+5/2}\,b\,\Gamma(2s+3)}{\Gamma\left(2s+\frac{7}{2}\right)}$

$$\times\, {}_4F_3\left(\begin{matrix}1,\,1,\,1,\,2s+3\\ 2,\,2,\,2s+\frac{7}{2};\;\frac{ab}{2}\end{matrix}\right) \quad [a > 0;\ \mathrm{Re}\, s > 3/2;\ |\arg(2 - ab)| < \pi].$$

6. $$\int_0^a x^{-1/2}\,\mathrm{Li}_2\left(b\sqrt{x(a-x)}\right)dx = 4a^{1/2}\left(\arcsin^2\sqrt{\frac{ab}{2}}+2\sqrt{\frac{2}{ab}-1}\,\arcsin\sqrt{\frac{ab}{2}}-2\right)$$
$[a>0;\ |\arg(2-ab)|<\pi].$

7. $$\int_0^a x^{-3/4}(a-x)^{-5/4}\,\mathrm{Li}_2\left(b\sqrt{x(a-x)}\right)dx = \frac{2^{7/2}\pi}{a}\left[1-\sqrt{1-\frac{ab}{2}}+\ln\left(1+\sqrt{1-\frac{ab}{2}}\right)-\ln 2\right]$$
$[a>0;\ |\arg(2-ab)|<\pi].$

8. $$\int_0^a x^{-1/4}(a-x)^{-3/4}\,\mathrm{Li}_2\left(b\sqrt{x(a-x)}\right)dx = 2^{1/2}\pi\left[2\,\mathrm{Li}_2\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{ab}{2}}\right)-\ln^2\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{ab}{2}}\right)\right]$$
$[a>0;\ |\arg(2-ab)|<\pi].$

4.2.2. Integrals containing $\mathrm{Li}_2(z)$ and trigonometric functions

1. $$\int_0^{m\pi}\frac{1}{\sin^2 x}\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\mathrm{Li}_2(b\sin^2 x)\,dx = \frac{2b}{a}\sin\frac{m\pi a}{2}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}{}_5F_4\left(\begin{matrix}\frac{1}{2},\,1,\,1,\,1,\,1;\ b\\ 2,\,2,\,1-\frac{a}{2},\,1+\frac{a}{2}\end{matrix}\right)$$
$[a>0;\ |\arg(1-b)|<\pi].$

2. $$\int_0^{\pi}\sin^\nu x\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\mathrm{Li}_2(b\sin^2 x)\,dx = \frac{2^{-\nu-2}\pi b\,\Gamma(\nu+3)}{\Gamma\left(\frac{\nu-a}{2}+2\right)\Gamma\left(\frac{\nu+a}{2}+2\right)}\times\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,\frac{\nu+3}{2},\,\frac{\nu}{2}+2;\ b\\ 2,\,2,\,\frac{\nu-a}{2}+2,\,\frac{\nu+a}{2}+2\end{matrix}\right)$$
$[\mathrm{Re}\,\nu>-1;\ |\arg(1-b)|<\pi].$

3. $$\int_0^{\pi/2}\cos^\nu x\cos(ax)\,\mathrm{Li}_2(b\cos^2 x)\,dx = \frac{2^{-\nu-3}\pi b\,\Gamma(\nu+3)}{\Gamma\left(\frac{\nu-a}{2}+2\right)\Gamma\left(\frac{\nu+a}{2}+2\right)}\,{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,\frac{\nu+3}{2},\,\frac{\nu}{2}+2;\ b\\ 2,\,2,\,\frac{\nu-a}{2}+2,\,\frac{\nu+a}{2}+2\end{matrix}\right)$$
$[\mathrm{Re}\,\nu>-1;\ |\arg(1-b)|<\pi].$

4. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin^2 x} \operatorname{Li}_2(b\sin^2 x)\,dx = \frac{b}{a}(1-e^{-m\pi a})\,{}_5F_4\left(\begin{matrix}\frac{1}{2},\,1,\,1,\,1,\,1;\; b\\ 2,\,2,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right).$$

5. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} \operatorname{Li}_2(b\sin x)\,dx$$
$$= \frac{b}{a}\,{}_5F_4\left(\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2},\,1,\,1;\; b^2\\ \frac{3}{2},\,\frac{3}{2},\,\frac{3-ia}{2},\,\frac{3+ia}{2}\end{matrix}\right) + \frac{b^2}{4(a^2+1)}\,{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,1,\,\frac{3}{2};\; b^2\\ 2,\,2,\,\frac{3-ia}{2},\,\frac{3+ia}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re} a > 0;\; |\arg(1-b^2)| < \pi\right].$$

6. $$\int_0^{\infty} \frac{e^{-ax}}{\sin^2 x} \operatorname{Li}_2(b\sin^2 x)\,dx = \frac{b}{a}\,{}_5F_4\left(\begin{matrix}\frac{1}{2},\,1,\,1,\,1,\,1;\; b\\ 2,\,2,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re} a > 0;\; |\arg(1-b^2)| < \pi\right].$$

4.2.3. Integrals containing $\operatorname{Li}_2(z)$ and the logarithmic function

1. $$\int_0^{a} x^{s-1} \ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} \operatorname{Li}_2(bx)\,dx = \frac{\pi^{1/2} a^{s+1} b\,\Gamma(s)}{2s(s+1)\Gamma\left(s+\frac{3}{2}\right)}$$
$$\times\left[{}_3F_2\left(\begin{matrix}1,\,s+1,\,s+1;\; ab\\ s+\frac{3}{2},\,s+2\end{matrix}\right) - (s+1)\,{}_3F_2\left(\begin{matrix}1,\,1,\,s+1;\; ab\\ 2,\,s+\frac{3}{2}\end{matrix}\right)\right.$$
$$\left.+\, s(s+1)\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,s+1;\; ab\\ 2,\,2,\,s+\frac{3}{2}\end{matrix}\right)\right] \qquad [a > 0;\; \operatorname{Re} s > -2;\; |\arg(1-ab)| < \pi].$$

2. $$\int_0^{a} \ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} \operatorname{Li}_2(bx)\,dx = \left(2a+\frac{1}{b}\right)\arcsin^2(\sqrt{ab})$$
$$+\, 6a\sqrt{\frac{1}{ab}-1}\,\arcsin(\sqrt{ab}) - 7a \qquad [a > 0;\; |\arg(1-ab)| < \pi].$$

3. $$\int_0^{a} x^{-3/2} \ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} \operatorname{Li}_2(bx)\,dx$$
$$= \frac{4\pi}{\sqrt{a}}\left(\sqrt{ab}\arcsin\sqrt{ab} - \ln\frac{1+\sqrt{1-ab}}{2} + 2\sqrt{1-ab} - 2\right)$$
$$[a > 0;\; |\arg(1-ab)| < \pi].$$

4.2.4. Integrals containing $\operatorname{Li}_2(z)$ and inverse trigonometric functions

1. $$\int_0^{1} \frac{1}{x} \arccos\sqrt{x}\, \operatorname{Li}_2(x)\,dx = \frac{\pi}{12}[8\ln^3 2 - 2\pi^2\ln 2 + 12\zeta(3)].$$

2. $$\int_0^1 x^{s-1}\arccos\sqrt{x}\,\mathrm{Li}_2(ax)\,dx = \frac{\pi^{1/2}a\,\Gamma\left(s+\frac{3}{2}\right)}{2s^2(s+1)\Gamma(s+2)} \times\left[{}_3F_2\left(\begin{matrix}1,\ s+1,\ s+\frac{3}{2}\\ s+2,\ s+2;\ a\end{matrix}\right) - (s+1)\,{}_3F_2\left(\begin{matrix}1,\ 1,\ s+\frac{3}{2}\\ 2,\ s+2;\ a\end{matrix}\right) + s(s+1)\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ s+\frac{3}{2}\\ 2,\ 2,\ s+2;\ a\end{matrix}\right)\right] \quad [|\arg(1-a)|<\pi].$$

4.3. The Sine Si (z) and Cosine ci (z) Integrals

4.3.1. Integrals containing Si (z) and algebraic functions

1. $$\int_0^a x^{-1/2}\,\mathrm{Si}\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi\sqrt{a}}{2^{3/2}}\left\{2\sqrt{a}\,J_0\left(b\sqrt{\frac{a}{2}}\right) - 2\sqrt{2}\,J_1\left(b\sqrt{\frac{a}{2}}\right) + \pi\sqrt{a}\,b\left[J_1\left(b\sqrt{\frac{a}{2}}\right)\mathbf{H}_0\left(b\sqrt{\frac{a}{2}}\right) - J_0\left(b\sqrt{\frac{a}{2}}\right)\mathbf{H}_1\left(b\sqrt{\frac{a}{2}}\right)\right]\right\} \quad [a>0].$$

4.3.2. Integrals containing Si (z) and trigonometric functions

1. $$\int_0^a \frac{1}{\sqrt{x(a-x)}}\left[\sin x\,\mathrm{Si}\,(2x)+\cos x\,\mathrm{ci}\,(2x)\right]dx = \frac{\pi^2}{4}\cos\frac{a}{2}\,Y_0\left(\frac{a}{2}\right) + \frac{\pi}{2}\left[\sin\frac{a}{2}\,\mathrm{Si}\,(a)+\cos\frac{a}{2}\,\mathrm{ci}\,(a)\right]J_0\left(\frac{a}{2}\right) \quad [a>0].$$

2. $$\int_0^a \frac{1}{\sqrt{x(a-x)}}\left[\cos x\,\mathrm{Si}\,(2x)-\sin x\,\mathrm{ci}\,(2x)\right]dx = -\frac{\pi^2}{4}\sin\frac{a}{2}\,Y_0\left(\frac{a}{2}\right) + \frac{\pi}{2}\left[\cos\frac{a}{2}\,\mathrm{Si}\,(a)-\sin\frac{a}{2}\,\mathrm{ci}\,(a)\right]J_0\left(\frac{a}{2}\right) \quad [a>0].$$

3. $$\int_0^{\pi/2} \cos^\nu x\cos(ax)\,\mathrm{Si}\,(b\cos x)\,dx = \frac{2^{-\nu-2}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)}\,{}_3F_4\left(\begin{matrix}\frac{1}{2},\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2};\ -\frac{b^2}{4}\\ \frac{3}{2},\ \frac{3}{2},\ \frac{\nu-a+3}{2},\ \frac{\nu+a+3}{2}\end{matrix}\right) \quad [\mathrm{Re}\,\nu>-2].$$

4. $$\int_0^{\pi/2} \frac{\cos(2nx)}{\sin x}\,\mathrm{Si}\,(a\sin x)\,dx = \frac{2^{-2n-1}\pi a^{2n+1}}{(2n+1)!\,(2n+1)}\,{}_2F_3\left(\begin{matrix}n+\frac{1}{2},\ n+\frac{1}{2};\ -\frac{a^2}{4}\\ n+\frac{3}{2},\ n+\frac{3}{2},\ 2n+1\end{matrix}\right).$$

5. $$\int_0^{\pi} \sin^{\nu} x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \operatorname{Si}(b\sin x)\,dx = \frac{2^{-\nu-1}\pi b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_3F_4\left(\begin{matrix} \frac{1}{2},\, \frac{\nu}{2}+1,\, \frac{\nu+3}{2};\, -\frac{b^2}{4} \\ \frac{3}{2},\, \frac{3}{2},\, \frac{\nu-a+3}{2},\, \frac{\nu+a+3}{2} \end{matrix}\right) \quad [\operatorname{Re}\nu > -2].$$

6. $$\int_0^{\pi} \frac{\cos(nx)}{\cos x} \operatorname{Si}(a\cos x)\,dx = \frac{2^{-n}\pi a^{n+1}}{(n+1)!\,(n+1)} \cos\frac{n\pi}{2}\, {}_2F_3\left(\begin{matrix} \frac{n+1}{2},\, \frac{n+1}{2};\, -\frac{a^2}{4} \\ \frac{n+3}{2},\, \frac{n+3}{2},\, n+1 \end{matrix}\right).$$

7. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \operatorname{Si}(b\sin x)\,dx = \frac{2b}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_4\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1;\, -\frac{b^2}{4} \\ \frac{3}{2},\, \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

8. $$\int_0^{\infty} \frac{1}{\sqrt{x}} e^{-ax} [\sin x \operatorname{Si}(2x) + \cos x \operatorname{ci}(2x)]\,dx = -\sqrt{\frac{\pi}{2}}\, \frac{(a+\sqrt{1+a^2})^{1/2}}{(1+a^2)^{1/2}} \ln\left(a+\sqrt{1+a^2}\right) \quad [\operatorname{Re} a > 0].$$

9. $$\int_0^{\infty} \frac{e^{-ax}}{\sqrt{x}} [\sin x \operatorname{ci}(2x) - \cos x \operatorname{Si}(2x)]\,dx = -\sqrt{\frac{\pi}{2}}\, \frac{(1+a^2)^{-1/2}}{(a+\sqrt{1+a^2})^{1/2}} \ln\left(a+\sqrt{1+a^2}\right) \quad [\operatorname{Re} a > 0].$$

10. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} \operatorname{Si}(b\sin x)\,dx = \frac{b}{a}\left(1-e^{-m\pi a}\right) {}_3F_4\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1;\, -\frac{b^2}{4} \\ \frac{3}{2},\, \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

11. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} \operatorname{Si}(b\sin x)\,dx = \frac{b}{a}\, {}_3F_4\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1;\, -\frac{b^2}{4} \\ \frac{3}{2},\, \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

12. $$\int_0^\infty \cosh^\nu x\,\cos(bx)\,\mathrm{Si}(c\,\mathrm{sech}\,x)\,dx = \frac{2^{-\nu-1}c}{\Gamma(1-\nu)}$$
$$\times\,\Gamma\left(\frac{1-\nu-ib}{2}\right)\Gamma\left(\frac{1-\nu+ib}{2}\right)\,{}_3F_4\left(\begin{matrix}\frac{1}{2},\ \frac{1-\nu-ib}{2},\ \frac{1-\nu+ib}{2};\ -\frac{c^2}{4}\\ \frac{3}{2},\ \frac{3}{2},\ \frac{1-\nu}{2},\ 1-\frac{\nu}{2}\end{matrix}\right)$$
$[\mathrm{Re}\,\nu < 1]$.

4.3.3. Integrals containing Si (z) and the logarithmic function

1. $$\int_0^a x^{s-1}\ln\frac{a+\sqrt{a^2-x^2}}{x}\,\mathrm{Si}(bx)\,dx = \frac{\sqrt{\pi}}{2}\,\frac{a^{s+1}b\,\Gamma\left(\frac{s+1}{2}\right)}{s(s+1)\Gamma\left(\frac{s}{2}+1\right)}$$
$$\times\left[(s+1)\,{}_2F_3\left(\begin{matrix}\frac{1}{2},\ \frac{s+1}{2};\ -\frac{a^2b^2}{4}\\ \frac{3}{2},\ \frac{3}{2},\ \frac{s}{2}+1\end{matrix}\right) - {}_2F_3\left(\begin{matrix}\frac{s+1}{2},\ \frac{s+1}{2};\ -\frac{a^2b^2}{4}\\ \frac{3}{2},\ \frac{s}{2}+1,\ \frac{s+3}{2}\end{matrix}\right)\right]$$
$[a > 0;\ \mathrm{Re}\,s > -1]$.

2. $$\int_0^a x\ln\frac{a+\sqrt{a^2-x^2}}{x}\,\mathrm{Si}(bx)\,dx = \frac{\pi a}{8b}\{2ab[ab\,J_0(ab) - J_1(ab)]$$
$$+\,\pi(a^2b^2-1)\,[J_1(ab)\,\mathbf{H}_0(ab) - J_0(ab)\,\mathbf{H}_1(ab)]\}\quad [a>0].$$

3. $$\int_0^a x^3\ln\frac{a+\sqrt{a^2-x^2}}{x}\,\mathrm{Si}(bx)\,dx$$
$$= \frac{\pi a}{24b^3}\{2ab[ab(a^2b^2+1)\,J_0(ab) - (a^2b^2+5)\,J_1(ab)]\}$$
$$+\,\frac{\pi^2 a}{24b^3}(a^4b^4+9)\,[J_1(ab)\,\mathbf{H}_0(ab) - J_0(ab)\,\mathbf{H}_1(ab)]\quad [a>0].$$

4.3.4. Integrals containing Si (z) and inverse trigonometric functions

1. $$\int_0^1 x^{s-1}\arccos x\,\mathrm{Si}(ax)\,dx$$
$$= \frac{\pi^{1/2}a\,\Gamma\left(\frac{s}{2}+1\right)}{2(s+1)\Gamma\left(\frac{s+3}{2}\right)}\,{}_3F_4\left(\begin{matrix}\frac{1}{2},\ \frac{s+1}{2},\ \frac{s}{2}+1;\ -\frac{a^2}{4}\\ \frac{3}{2},\ \frac{3}{2},\ \frac{s+3}{2},\ \frac{s+3}{2}\end{matrix}\right)\quad [\mathrm{Re}\,s > -1].$$

2. $$\int_0^1 \arccos x\,\mathrm{Si}(ax)\,dx = \frac{\pi}{4a}\{-2 + 2(a^2+1)\,J_0(a) - 2a\,J_1(a)$$
$$+\,\pi a^2[J_1(a)\,\mathbf{H}_0(a) - J_0(a)\,\mathbf{H}_1(a)]\}.$$

4.3.5. Integrals containing products of Si (z) and ci (z)

1. $$\int_0^\infty x^{s-1}[\sin(x)\,\mathrm{ci}(2x)-\cos(x)\,\mathrm{Si}(2x)]^2\,dx$$
$$=\frac{2^{-s-4}}{s}\Gamma(s)\left\{\pi^2 s[3-\cos(\pi s)]\sec\frac{\pi s}{2}+4\pi[1+\cos(\pi s)]\csc\frac{\pi s}{2}\right.$$
$$\left.+4s\cos\frac{\pi s}{2}\left[\psi'\left(\frac{1+s}{2}\right)-\psi'\left(\frac{s}{2}\right)\right]\right\}\quad[-2<\operatorname{Re}s<0].$$

2. $$\int_0^\infty x^{s-1}[\sin(x)\,\mathrm{ci}(2x)-\cos(x)\,\mathrm{Si}(2x)][\cos(x)\,\mathrm{ci}(2x)+\sin(x)\,\mathrm{Si}(2x)]dx$$
$$=2^{-s-3}\Gamma(s)\left\{\frac{\pi^2}{2}[\cos(\pi s)+3]\csc\frac{\pi s}{2}\right.$$
$$\left.+\sin\frac{\pi s}{2}\left[3\psi'\left(\frac{1+s}{2}\right)-4\psi'(s)-\psi'\left(\frac{s}{2}\right)\right]\right\}\quad[-1<\operatorname{Re}s<1].$$

3. $$\int_0^a \mathrm{shi}\,(b\sqrt{x})\,\mathrm{Si}\,(b\sqrt{a-x})\,dx=\frac{\pi(ab)^2}{8}\,{}_2F_5\left(\begin{matrix}\frac{1}{2},\frac{1}{2};\ \frac{a^2b^4}{256}\\ \frac{3}{4},\frac{5}{4},\frac{3}{2},\frac{3}{2},2\end{matrix}\right)\quad[a>0].$$

4.4. The Error Functions erf (z), erfi (z) and erfc (z)

4.4.1. Integrals containing erf (z) and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1}\,\mathrm{erf}\left(b\sqrt{x(a-x)}\right)dx=\frac{2}{\sqrt{\pi}}a^{s+t}b\,\mathrm{B}\left(s+\frac{1}{2},t+\frac{1}{2}\right)$$
$$\times\,{}_3F_3\left(\begin{matrix}\frac{1}{2},s+\frac{1}{2},t+\frac{1}{2};\ -\frac{a^2b^2}{4}\\ \frac{3}{2},\frac{s+t+1}{2},\frac{s+t}{2}+1\end{matrix}\right)\quad[a>0;\ \operatorname{Re}s,\operatorname{Re}t>-1/2].$$

2. $$\int_0^a x^{s+1/2}(a-x)^s\,\mathrm{erf}\left(b\sqrt[4]{x(a-x)}\right)dx=2^{-2s-1/2}a^{2s+2}b\frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)}$$
$$\times\,{}_2F_2\left(\begin{matrix}\frac{1}{2},2s+\frac{5}{2};\ -\frac{ab^2}{2}\\ \frac{3}{2},2s+3\end{matrix}\right)[a>0;\ \operatorname{Re}s>-5/4].$$

3. $$\int_0^a x^{1/2}\,\mathrm{erf}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\sqrt{\frac{\pi}{2}}\frac{a}{3b}e^{-ab^2/4}\left[ab^2I_0\left(\frac{ab^2}{4}\right)+(ab^2+1)\,I_1\left(\frac{ab^2}{4}\right)\right]\quad[a>0].$$

4. $$\int_0^a x^{-1/2}\operatorname{erf}\left(b\sqrt[4]{x(a-x)}\right)dx = \sqrt{\frac{\pi}{2}}\,abe^{-ab^2/4}\left[I_0\left(\frac{ab^2}{4}\right)+I_1\left(\frac{ab^2}{4}\right)\right]$$
$[a>0]$.

4.4.2. Integrals containing erf (z), erfc (z) and the exponential function

1. $$\int_0^a x^{s-1}(a-x)^{t-1}e^{b^2x(a-x)}\operatorname{erf}\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{2}{\sqrt{\pi}}a^{s+t}b\,\mathrm{B}\left(s+\frac{1}{2},\,t+\frac{1}{2}\right)\,{}_3F_3\left(\begin{matrix}1,\, s+\frac{1}{2},\, t+\frac{1}{2};\ \frac{a^2b^2}{4}\\ \frac{3}{2},\, \frac{s+t+1}{2},\, \frac{s+t}{2}+1\end{matrix}\right)$$
$[a>0;\ \operatorname{Re}s,\, \operatorname{Re}t>-1/2]$.

2. $$\int_0^a e^{b^2x(a-x)}\operatorname{erf}\left(b\sqrt{x(a-x)}\right)dx = \frac{\sqrt{\pi}}{b}\left(e^{a^2b^2/4}-1\right)$$ $[a>0]$.

3. $$\int_0^a x^{-1}e^{b^2x(a-x)}\operatorname{erf}\left(b\sqrt{x(a-x)}\right)dx = \pi\operatorname{erfi}\left(\frac{ab}{2}\right)$$ $[a>0]$.

4. $$\int_0^a x^{-1}(a-x)^{-1}e^{b^2x(a-x)}\operatorname{erf}\left(b\sqrt{x(a-x)}\right)dx = \frac{2\pi}{a}\operatorname{erfi}\left(\frac{ab}{2}\right)$$ $[a>0]$.

5. $$\int_0^a x^{s+1/2}(a-x)^s e^{b^2\sqrt{x(a-x)}}\operatorname{erf}\left(b\sqrt[4]{x(a-x)}\right)dx = 2^{-2s-1/2}a^{2s+2}b$$
$$\times\frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)}\,{}_2F_2\left(\begin{matrix}1,\, 2s+\frac{5}{2};\ \frac{ab^2}{2}\\ \frac{3}{2},\, 2s+3\end{matrix}\right)$$ $[a>0;\ \operatorname{Re}s>-5/4]$.

6. $$\int_0^a x^{1/2}e^{b^2\sqrt{x(a-x)}}\operatorname{erf}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= \frac{1}{2b^3}\sqrt{\frac{\pi}{2}}\left[2-ab^2+2(ab^2-1)e^{ab^2/2}\right]$$ $[a>0]$.

7. $$\int_0^a x^{-1/2}e^{b^2\sqrt{x(a-x)}}\operatorname{erf}\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\sqrt{2\pi}}{b}\left(e^{ab^2/2}-1\right)$$ $[a>0]$.

8. $$\int_0^a x^{-1/2}(a-x)^{-1}e^{b^2\sqrt{x(a-x)}}\operatorname{erf}\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{2\pi}{\sqrt{a}}\operatorname{erfi}\left(b\sqrt{\frac{a}{2}}\right) \qquad [a>0].$$

4.4.3. Integrals containing erf (z) and trigonometric functions

1. $$\int_0^\pi \sin(ax)\operatorname{erf}(b\sin x)\,dx = \frac{2b\sin(\pi a)}{\sqrt{\pi}\,(1-a^2)}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},\,1;\;-b^2\\ \frac{3-a}{2},\,\frac{3+a}{2}\end{matrix}\right).$$

2. $$\int_0^\pi \frac{\cos(nx)}{\cos x}\operatorname{erf}(a\cos x)\,dx = \frac{2^{1-n}\sqrt{\pi}\,a^{n+1}}{(n+1)\Gamma\left(\frac{n}{2}+1\right)}\cos\frac{n\pi}{2}\,{}_2F_2\left(\begin{matrix}\frac{n+1}{2},\,\frac{n+1}{2};\;-a^2\\ \frac{n+3}{2},\,n+1\end{matrix}\right).$$

3. $$\int_0^{\pi/2} \frac{\cos(2nx)}{\sin x}\operatorname{erf}(a\sin x)\,dx = \frac{2^{-2n}\sqrt{\pi}\,a^{2n+1}\Gamma\left(n+\frac{1}{2}\right)}{n!\,(2n+1)}\,{}_2F_2\left(\begin{matrix}n+\frac{1}{2},\,n+\frac{1}{2};\;-a^2\\ n+\frac{3}{2},\,2n+1\end{matrix}\right).$$

4. $$\int_0^{\pi/2} \cos^\nu x\cos(ax)\operatorname{erf}(b\cos x)\,dx = \frac{2^{-\nu-1}\sqrt{\pi}\,b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \times {}_3F_3\left(\begin{matrix}\frac{1}{2},\,1+\frac{\nu}{2},\,\frac{\nu+3}{2};\;b^2\\ \frac{3}{2},\,\frac{\nu-a+3}{2},\,\frac{\nu+a+3}{2}\end{matrix}\right) \qquad [\operatorname{Re}\nu>-2].$$

5. $$\int_0^\pi \sin^\nu x\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\operatorname{erf}(b\sin x)\,dx = \frac{2^{-\nu}\sqrt{\pi}\,b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \times \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}{}_3F_3\left(\begin{matrix}\frac{1}{2},\,1+\frac{\nu}{2},\,\frac{\nu+3}{2};\;-b^2\\ \frac{3}{2},\,\frac{\nu-a+3}{2},\,\frac{\nu+a+3}{2}\end{matrix}\right) \qquad [\operatorname{Re}\nu>-2].$$

6. $$\int_0^{\pi/2} \cos^\nu x\cos(ax)e^{b^2\cos^2 x}\operatorname{erf}(b\cos x)\,dx = \frac{2^{-\nu-1}\sqrt{\pi}\,b\,\Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)} \times {}_3F_3\left(\begin{matrix}1,\,1+\frac{\nu}{2},\,\frac{\nu+3}{2};\;b^2\\ \frac{3}{2},\,\frac{\nu-a+3}{2},\,\frac{\nu+a+3}{2}\end{matrix}\right) \qquad [\operatorname{Re}\nu>-2].$$

7. $$\int_0^{\pi} \sin^{\nu} x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} e^{b^2 \sin^2 x} \operatorname{erf}(b \sin x)\, dx = \frac{2^{-\nu}\sqrt{\pi}\, b\, \Gamma(\nu+2)}{\Gamma\left(\frac{\nu-a+3}{2}\right)\Gamma\left(\frac{\nu+a+3}{2}\right)}$$
$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_3F_3\left(\begin{matrix} 1, 1+\frac{\nu}{2}, \frac{\nu+3}{2};\ b^2 \\ \frac{3}{2}, \frac{\nu-a+3}{2}, \frac{\nu+a+3}{2} \end{matrix}\right) \qquad [\operatorname{Re}\nu > -1].$$

8. $$\int_0^{\infty} e^{-ax} \operatorname{erf}(b \sin x)\, dx = \frac{2b}{\sqrt{\pi}\,(a^2+1)}\, {}_2F_2\left(\begin{matrix} \frac{1}{2}, 1;\ -b^2 \\ \frac{3-ia}{2}, \frac{3+ia}{2} \end{matrix}\right) \qquad [\operatorname{Re} a > 0].$$

9. $$\int_0^{\infty} e^{-ax+b^2\sin^2 x} \operatorname{erf}(b \sin x)\, dx = \frac{2b}{\sqrt{\pi}\,(a^2+1)}\, {}_2F_2\left(\begin{matrix} 1, 1;\ b^2 \\ \frac{3-ia}{2}, \frac{3+ia}{2} \end{matrix}\right)$$
$$[\operatorname{Re} a > 0].$$

4.4.4. Integrals containing erf (z) and the logarithmic function

1. $$\int_0^{a} x^{s-1} \ln \frac{a+\sqrt{a^2-x^2}}{x} \operatorname{erf}(bx)\, dx = \frac{2a^{s+1} b\, \Gamma\left(\frac{s+1}{2}\right)}{s(s+1)\Gamma\left(\frac{s}{2}\right)}$$
$$\times\, {}_3F_3\left(\begin{matrix} \frac{1}{2}, \frac{s+1}{2}, \frac{s+1}{2};\ -a^2b^2 \\ \frac{3}{2}, \frac{s}{2}+1, \frac{s+3}{2} \end{matrix}\right) \qquad [a > 0;\ \operatorname{Re} s > -1].$$

4.4.5. Integrals containing erf (z), erfi (z) and inverse trigonometric functions

1. $$\int_0^{1} \arccos x \operatorname{erf}(ax)\, dx$$
$$= \frac{\sqrt{\pi}}{2a}\left\{ e^{-a^2/2}\left[(a^2+1)\, I_0\left(\frac{a^2}{2}\right) + a^2 I_1\left(\frac{a^2}{2}\right)\right] - 1\right\}.$$

2. $$\int_0^{1} x^2 \arccos x \operatorname{erf}(ax)\, dx = \frac{\sqrt{\pi}}{36a^3}$$
$$\times \left[(4a^4+3a^2+6)e^{-a^2/2} I_0\left(\frac{a^2}{2}\right) + a^2(4a^2-1)e^{-a^2/2} I_1\left(\frac{a^2}{2}\right) - 6\right].$$

3. $$\int_0^{1} e^{a^2x^2} \arccos x \operatorname{erf}(ax)\, dx = \frac{\sqrt{\pi}}{4a}[\operatorname{Ei}(a^2) - 2\ln a - \mathbf{C}].$$

4. $$\int_0^1 x^2 e^{a^2x^2} \arccos x \operatorname{erf}(ax)\,dx = \frac{\sqrt{\pi}}{8a^3}\left[2e^{a^2} - a^2 - \operatorname{Ei}(a^2) + 2\ln a + \mathbf{C} - 2\right].$$

5. $$\int_0^1 \arccos x \operatorname{erfi}(ax)\,dx = \frac{\sqrt{\pi}}{2a}\left\{1 + e^{a^2/2}\left[(a^2-1)\,I_0\left(\frac{a^2}{2}\right) - a^2 I_1\left(\frac{a^2}{2}\right)\right]\right\}.$$

6. $$\int_0^1 x^2 \arccos x \operatorname{erfi}(ax)\,dx = \frac{\sqrt{\pi}}{36a^3}\left[(4a^4 - 3a^2 + 6)e^{a^2/2} I_0\left(\frac{a^2}{2}\right) - a^2(4a^2+1)e^{a^2/2} I_1\left(\frac{a^2}{2}\right) - 6\right].$$

7. $$\int_0^1 e^{-a^2x^2} \arccos x \operatorname{erfi}(ax)\,dx = \frac{\sqrt{\pi}}{4a}[\mathbf{C} + 2\ln a - \operatorname{Ei}(-a^2)].$$

8. $$\int_0^1 x^2 e^{-a^2x^2} \arccos x \operatorname{erfi}(ax)\,dx = \frac{\sqrt{\pi}}{8a^3}\left[2e^{-a^2} + a^2 - \operatorname{Ei}(-a^2) + 2\ln a + \mathbf{C} - 2\right].$$

4.4.6. Integrals containing products of $\operatorname{erf}(z)$, $\operatorname{erfc}(z)$ and $\operatorname{erfi}(z)$

1. $$\int_0^\infty x \operatorname{erfi}(ax)\operatorname{erfc}(bx)\,dx = \frac{a^2+b^2}{4\pi a^2 b^2}\ln\frac{b+a}{b-a} - \frac{1}{2\pi ab} \qquad [a < b].$$

2. $$\int_0^\infty \frac{1}{x}\operatorname{erfi}(ax)\operatorname{erfc}(bx)\,dx = \frac{1}{\pi}\left[\operatorname{Li}_2\left(\frac{a}{b}\right) - \operatorname{Li}_2\left(-\frac{a}{b}\right)\right] \qquad [|a| < |b|].$$

3. $$\int_0^\infty \frac{1}{x}\operatorname{erfi}(x)\operatorname{erfc}(x)\,dx = \frac{\pi}{4}.$$

4. $$\int_0^a \operatorname{erf}(b\sqrt{x})\operatorname{erf}(b\sqrt{a-x})\,dx = \frac{1}{b^2}\left(e^{-ab^2} + ab^2 - 1\right) \qquad [a > 0].$$

5. $$\int_0^a \operatorname{erfi}(b\sqrt{x})\operatorname{erfi}(b\sqrt{a-x})\,dx = \frac{1}{b^2}\left(e^{ab^2} - ab^2 - 1\right) \qquad [a > 0].$$

6. $$\int_0^a \operatorname{erfi}(b\sqrt{x})\operatorname{erf}(b\sqrt{a-x})\,dx = a^2b^2$$
$$\times\left[I_0(ab^2) - \frac{1}{ab^2}I_1(ab^2) + \frac{\pi}{2}I_0(ab^2)\,\mathbf{L}_1(ab^2) - \frac{\pi}{2}I_1(ab^2)\,\mathbf{L}_0(ab^2)\right]$$
$[a > 0]$.

7. $$\int_0^a e^{-b^2x}\operatorname{erfi}(b\sqrt{x})\operatorname{erf}(b\sqrt{a-x})\,dx = 2ae^{-ab^2/2}I_1\left(\frac{ab^2}{2}\right)$$ $[a > 0]$.

8. $$\int_0^a x^{s-1}(a-x)^{t-1}\operatorname{erf}\left(b\sqrt[4]{x(a-x)}\right)\operatorname{erfi}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= \frac{4}{\pi}a^{s+t}b^2\,\mathrm{B}\left(s+\frac{1}{2},\, t+\frac{1}{2}\right)\,{}_4F_5\left(\begin{matrix}\frac{1}{2},\,1,\,s+\frac{1}{2},\,t+\frac{1}{2};\ \frac{a^2b^2}{16}\\ \frac{3}{4},\,\frac{3}{2},\,\frac{5}{4},\,\frac{s+t+1}{2},\,\frac{s+t}{2}+1\end{matrix}\right)$$
$[a > 0;\ \operatorname{Re} s, \operatorname{Re} t > -1/2]$.

9. $$\int_0^a e^{b^2x}\operatorname{erf}(b\sqrt{x})\operatorname{erfi}(b\sqrt{a-x})\,dx = 2ae^{ab^2/2}I_1\left(\frac{ab^2}{2}\right)$$ $[a > 0]$.

10. $$\int_0^a e^{2b^2x}\operatorname{erf}(b\sqrt{x})\operatorname{erfi}(b\sqrt{a-x})\,dx = \frac{e^{ab^2}}{b^2}\left[\cosh(ab^2) - 1\right]$$ $[a > 0]$.

4.5. The Fresnel Integrals $S(z)$ and $C(z)$

4.5.1. Integrals containing $S(z)$ and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1}S\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{a^{s+t+1/2}}{3}\sqrt{\frac{2b^3}{\pi}}\,\mathrm{B}\left(s+\frac{3}{4},\, t+\frac{3}{4}\right)\,{}_3F_4\left(\begin{matrix}\frac{3}{4},\,s+\frac{3}{4},\,t+\frac{3}{4};\ -\frac{a^2b^2}{16}\\ \frac{3}{2},\,\frac{7}{4},\,\frac{2s+2t+3}{4},\,\frac{2s+2t+5}{4}\end{matrix}\right)$$
$[a > 0;\ \operatorname{Re} s, \operatorname{Re} t > -3/4]$.

2. $$\int_0^a x^{s+1/2}(a-x)^s\,S\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= \frac{2^{-2s-5/4}}{3}a^{2s+9/4}b^{3/2}\,\frac{\Gamma\left(2s+\frac{11}{4}\right)}{\Gamma\left(2s+\frac{13}{4}\right)}\,{}_2F_3\left(\begin{matrix}\frac{3}{4},\,2s+\frac{11}{4};\ -\frac{ab^2}{8}\\ \frac{3}{2},\,\frac{7}{4},\,2s+\frac{13}{4}\end{matrix}\right)$$
$[a > 0;\ \operatorname{Re} s > -11/8]$.

4.5.2. Integrals containing $S(z)$ and trigonometric functions

1. $$\int_0^{\pi/2} \frac{\cos(2nx)}{\sin^{3/2} x} S(a\sin x)\,dx = \frac{2^{-2n-1/2}\sqrt{\pi}\,a^{2n+3/2}}{(2n+1)!\,(4n+3)}\,{}_2F_3\left(\begin{matrix} n+\frac{1}{2},\ n+\frac{3}{4};\ -\frac{a^2}{4} \\ n+\frac{3}{2},\ n+\frac{7}{4},\ 2n+1 \end{matrix}\right).$$

2. $$\int_0^{\pi} \sin^{\mu} x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} S(b\sin x)\,dx = \frac{2^{-\mu-1}\Gamma\left(\mu+\frac{5}{2}\right)\sqrt{b^3\pi}}{3\Gamma\left(\frac{2\mu-2a+5}{4}\right)\Gamma\left(\frac{2\mu+2a+5}{4}\right)}$$
$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_3F_4\left(\begin{matrix} \frac{3}{4},\ \frac{2\mu+5}{4},\ \frac{2\mu+7}{4};\ -\frac{b^2}{4} \\ \frac{3}{2},\ \frac{7}{4},\ \frac{2\mu-2a+7}{4},\ \frac{2\mu+2a+7}{4} \end{matrix}\right) \qquad [\operatorname{Re}\mu > -5/2].$$

3. $$\int_0^{m\pi} \frac{1}{\sin^{3/2} x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} S(b\sin x)\,dx$$
$$= \frac{2^{3/2}}{3a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \left(\frac{b^3}{\pi}\right)^{1/2} {}_3F_4\left(\begin{matrix} \frac{1}{2},\ \frac{3}{4},\ 1;\ -\frac{b^2}{4} \\ \frac{3}{2},\ \frac{7}{4},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right).$$

4. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin^{3/2} x} S(b\sin x)\,dx$$
$$= \frac{1-e^{-m\pi a}}{3a} \left(\frac{2b^3}{\pi}\right)^{1/2} {}_3F_4\left(\begin{matrix} \frac{1}{2},\ \frac{3}{4},\ 1;\ -\frac{b^2}{4} \\ \frac{3}{2},\ \frac{7}{4},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

5. $$\int_0^{\infty} \frac{e^{-ax}}{\sin^{3/2} x} S(b\sin x)\,dx = \frac{1}{3a}\left(\frac{2b^3}{\pi}\right)^{1/2} {}_3F_4\left(\begin{matrix} \frac{1}{2},\ \frac{3}{4},\ 1;\ -\frac{b^2}{4} \\ \frac{3}{2},\ \frac{7}{4},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$
$$[\operatorname{Re} a > 0].$$

6. $$\int_0^{\infty} \cosh^{\nu} x \cos(bx)\, S(c\operatorname{sech} x)\,dx = \frac{2^{-\nu}c^{3/2}}{3\sqrt{\pi}\,\Gamma\left(\frac{3}{2}-\nu\right)}$$
$$\times \Gamma\left(\frac{3-2\nu-2ib}{4}\right)\Gamma\left(\frac{3-2\nu+2ib}{4}\right) {}_3F_4\left(\begin{matrix} \frac{3}{4},\ \frac{3-2\nu-ib}{4},\ \frac{3-2\nu+2ib}{4} \\ \frac{3}{2},\ \frac{7}{4},\ \frac{3-2\nu}{4},\ \frac{5-2\nu}{4};\ -\frac{c^2}{4} \end{matrix}\right)$$
$$[\operatorname{Re}\nu < 3/2].$$

4.5.3. Integrals containing $S(z)$ and the logarithmic function

1. $$\int_0^a x^{s-1} \ln \frac{a+\sqrt{a^2-x^2}}{x} S(bx)\,dx = \frac{1}{3} a^{s+3/2} \left(\frac{b}{2}\right)^{3/2} \frac{\Gamma\left(\frac{2s+3}{4}\right)}{s(2s+3)\Gamma\left(\frac{2s+5}{4}\right)}$$
$$\times \left[(2s+3)\,{}_2F_3\left(\begin{matrix}\frac{3}{4}, \frac{2s+3}{4}; -\frac{a^2b^2}{4}\\ \frac{3}{2}, \frac{7}{4}, \frac{2s+5}{4}\end{matrix}\right) - {}_2F_3\left(\begin{matrix}\frac{2s+3}{4}, \frac{2s+3}{4}; -\frac{a^2b^2}{4}\\ \frac{3}{2}, \frac{2s+5}{4}, \frac{2s+7}{4}\end{matrix}\right)\right]$$
$[a > 0;\ \operatorname{Re} s > -3/2]$.

4.5.4. Integrals containing $C(z)$ and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1} C\left(b\sqrt{x(a-x)}\right) dx$$
$$= a^{s+t-1/2}\sqrt{\frac{2b}{\pi}}\, \mathrm{B}\left(s+\frac{1}{4}, t+\frac{1}{4}\right) {}_3F_4\left(\begin{matrix}\frac{1}{4}, s+\frac{1}{4}, t+\frac{1}{4}; -\frac{a^2b^2}{16}\\ \frac{1}{2}, \frac{5}{4}, \frac{2s+2t+1}{4}, \frac{2s+2t+3}{4}\end{matrix}\right)$$
$[a > 0;\ \operatorname{Re} s, \operatorname{Re} t > -1/4]$.

2. $$\int_0^a x^{s+1/2}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= 2^{-2s-3/4} a^{2s+7/4} b^{1/2} \frac{\Gamma\left(2s+\frac{9}{4}\right)}{\Gamma\left(2s+\frac{11}{4}\right)} {}_2F_3\left(\begin{matrix}\frac{1}{4}, 2s+\frac{9}{4}; -\frac{ab^2}{8}\\ \frac{1}{2}, \frac{5}{4}, 2s+\frac{11}{4}\end{matrix}\right)$$
$[a > 0;\ \operatorname{Re} s > -9/8]$.

4.5.5. Integrals containing $C(z)$ and trigonometric functions

1. $$\int_0^{\pi/2} \frac{\cos(2nx)}{\sqrt{\sin x}} C(a\sin x)\,dx = \frac{\sqrt{\pi}\left(\frac{a}{2}\right)^{2n+1/2}}{(2n)!\,(4n+1)} {}_1F_2\left(\begin{matrix}n+\frac{1}{4}; -\frac{a^2}{4}\\ n+\frac{5}{4}, 2n+1\end{matrix}\right).$$

2. $$\int_0^{\pi} \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} C(b\sin x)\,dx = \frac{2^{-\mu}\Gamma(\mu+\frac{3}{2})\sqrt{b\pi}}{\Gamma\left(\frac{2\mu-2a+5}{2}\right)\Gamma\left(\frac{2\mu+2a+5}{2}\right)}\sqrt{\frac{b}{\pi}}$$
$$\times \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} {}_3F_4\left(\begin{matrix}\frac{1}{4}, \frac{2\mu+3}{4}, \frac{2\mu+5}{4}; -\frac{b^2}{4}\\ \frac{1}{2}, \frac{5}{4}, \frac{2\mu-2a+5}{4}, \frac{2\mu+2a+5}{4}\end{matrix}\right) \quad [\operatorname{Re}\mu > -3/2].$$

3. $$\int_0^{m\pi} \frac{1}{\sqrt{\sin x}} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} C(b\sin x)\,dx$$
$$= \frac{2^{3/2}}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix} \left(\frac{b}{\pi}\right)^{1/2} {}_2F_3\left(\begin{matrix}\frac{1}{4}, 1; -\frac{b^2}{4}\\ \frac{5}{4}, 1-\frac{a}{2}, 1+\frac{a}{2}\end{matrix}\right).$$

4. $\displaystyle\int_0^{m\pi} \frac{e^{-ax}}{\sqrt{\sin x}} C(b\sin x)\,dx = \frac{1}{a}\sqrt{\frac{2b}{\pi}}\,(1-e^{-m\pi a})\,{}_2F_3\left(\begin{matrix}\frac{1}{4},\,1;\,-\frac{b^2}{4}\\ \frac{5}{4},\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right).$

5. $\displaystyle\int_0^{\infty} \frac{e^{-ax}}{\sqrt{\sin x}} C(b\sin x)\,dx = \frac{1}{a}\left(\frac{2b}{\pi}\right)^{1/2}{}_2F_3\left(\begin{matrix}\frac{1}{4},\,1;\,-\frac{b^2}{4}\\ \frac{5}{4},\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)\qquad [\operatorname{Re} a > 0].$

6. $\displaystyle\int_0^{\infty} \cosh^{\nu} x\,\cos(bx)\,C(c\operatorname{sech} x)\,dx = \frac{2^{-\nu-1}c^{1/2}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}$

$$\times\,\Gamma\left(\frac{1-2\nu-2ib}{4}\right)\Gamma\left(\frac{1-2\nu+2ib}{4}\right){}_3F_4\left(\begin{matrix}\frac{1}{4},\,\frac{1-2\nu-2ib}{4},\,\frac{1-2\nu+2ib}{4}\\ \frac{1}{2},\,\frac{5}{4},\,\frac{1-2\nu}{4},\,\frac{3-2\nu}{4};\,-\frac{c^2}{4}\end{matrix}\right)$$

$[\operatorname{Re}\nu < 1].$

4.5.6. Integrals containing $C(z)$ and the logarithmic function

1. $\displaystyle\int_0^{a} x^{s-1}\ln\frac{a+\sqrt{a^2-x^2}}{x}\,C(bx)\,dx = \sqrt{\frac{b}{8}}\,\frac{a^{s+1/2}\Gamma\left(\frac{2s+1}{4}\right)}{s(2s+1)\Gamma\left(\frac{2s+3}{4}\right)}$

$$\times\left[(2s+1)\,{}_2F_3\left(\begin{matrix}\frac{1}{4},\,\frac{2s+1}{4};\,-\frac{a^2b^2}{4}\\ \frac{1}{2},\,\frac{5}{4},\,\frac{2s+3}{4}\end{matrix}\right) - {}_2F_3\left(\begin{matrix}\frac{2s+1}{4},\,\frac{2s+1}{4};\,-\frac{a^2b^2}{4}\\ \frac{1}{2},\,\frac{2s+3}{4},\,\frac{2s+5}{4}\end{matrix}\right)\right]$$

$[a > 0;\ \operatorname{Re} s > 1/2].$

4.6. The Incomplete Gamma Function $\gamma(\nu, z)$

4.6.1. Integrals containing $\gamma(\nu, z)$ and algebraic functions

1. $\displaystyle\int_0^{a} x^{s-1}(a-x)^{t-1}\gamma(\nu,\,bx(a-x))\,dx = \frac{a^{s+t+2\nu-1}b^{\nu}}{\nu}\,\mathrm{B}\,(s+\nu,\,t+\nu)$

$$\times\,{}_3F_3\left(\begin{matrix}\nu,\,s+\nu,\,t+\nu;\,-\frac{a^2b}{4}\\ \nu+1,\,\frac{s+t}{2}+\nu,\,\frac{s+t+1}{2}+\nu\end{matrix}\right)\qquad [a,\,\operatorname{Re}\nu,\,\operatorname{Re}(s+\nu),\,\operatorname{Re}(t+\nu) > 0].$$

2. $\displaystyle\int_0^{a} x^{s+1/2}(a-x)^{s}\gamma\left(\nu,\,b\sqrt{x(a-x)}\right)dx$

$$= 2^{-2s-\nu-1}a^{2s+\nu+3/2}b^{\nu}\,\frac{\sqrt{\pi}\,\Gamma(2s+\nu+2)}{\nu\Gamma\left(2s+\nu+\frac{5}{2}\right)}\,{}_2F_2\left(\begin{matrix}\nu,\,2s+\nu+2;\,-\frac{ab}{2}\\ \nu+1,\,2s+\nu+\frac{5}{2}\end{matrix}\right)$$

$[a,\,\operatorname{Re}\nu,\,\operatorname{Re}(s+\nu) > 0].$

4.6.2. Integrals containing $\gamma(\nu, z)$ and the exponential function

1. $\displaystyle\int_0^a x^{s-1}(a-x)^{t-1}e^{bx(a-x)}\gamma(\nu,\, bx(a-x))\,dx$

$$= \frac{a^{s+t+2\nu-1}b^{\nu}}{\nu}\,\mathrm{B}\,(s+\nu,\, t+\nu)\,{}_3F_3\left(\begin{matrix}1,\, s+\nu,\, t+\nu;\, \frac{a^2b}{4}\\ \nu+1,\, \frac{s+t}{2}+\nu,\, \frac{s+t+1}{2}+\nu\end{matrix}\right)$$
$[a,\, \mathrm{Re}\,\nu,\, \mathrm{Re}\,(s+\nu),\, \mathrm{Re}\,(t+\nu) > 0].$

2. $\displaystyle\int_0^a e^{bx(a-x)}\gamma(\nu,\, bx(a-x))\,dx = \frac{\sqrt{\pi}\,\Gamma(\nu)}{\Gamma\left(\nu+\frac{1}{2}\right)}b^{-1/2}e^{a^2b/4}\gamma\left(\nu+\frac{1}{2},\, \frac{a^2b}{4}\right)$
$[a,\, \mathrm{Re}\,\nu > 0].$

3. $\displaystyle\int_0^a x^{s+1/2}(a-x)^{s}e^{b\sqrt{x(a-x)}}\gamma\left(\nu,\, b\sqrt{x(a-x)}\right)dx$

$$= 2^{-2s-\nu-1}a^{2s+\nu+3/2}b^{\nu}\frac{\sqrt{\pi}\,\Gamma(2s+\nu+2)}{\nu\Gamma\left(2s+\nu+\frac{5}{2}\right)}\,{}_2F_2\left(\begin{matrix}1,\, 2s+\nu+2;\, \frac{ab}{2}\\ \nu+1,\, 2s+\nu+\frac{5}{2}\end{matrix}\right)$$
$[a,\, \mathrm{Re}\,\nu > 0;\; \mathrm{Re}\,(s+\nu/2) > -1].$

4. $\displaystyle\int_0^a x^{-1/2}e^{b\sqrt{x(a-x)}}\gamma\left(\nu,\, b\sqrt{x(a-x)}\right)dx$

$$= \sqrt{\frac{2\pi}{b}}\,e^{ab/2}\,\frac{\Gamma(\nu)}{\Gamma\left(\nu+\frac{1}{2}\right)}\gamma\left(\nu+\frac{1}{2},\, \frac{ab}{2}\right)\qquad [a>0;\; \mathrm{Re}\,\nu > -1/2].$$

4.6.3. Integrals containing $\gamma(\nu, z)$ and trigonometric functions

1. $\displaystyle\int_0^{m\pi}\left\{\begin{matrix}\sin\,(ax)\\ \cos\,(ax)\end{matrix}\right\}\sin^{-2\nu}x\,\gamma(\nu,\, b\sin^2 x)\,dx$

$$= \frac{2b^{\nu}}{\nu a}\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin\,(m\pi a/2)\\ \cos\,(m\pi a/2)\end{matrix}\right\}{}_3F_3\left(\begin{matrix}\nu,\, \frac{1}{2},\, 1;\, -b\\ \nu+1,\, 1-\frac{a}{2},\, 1+\frac{a}{2}\end{matrix}\right)\qquad [\mathrm{Re}\,\nu > 0].$$

2. $\displaystyle\int_0^{\pi/2}\cos^{\mu}x\cos\,(ax)e^{b\cos^2 x}\gamma(\nu,\, b\cos^2 x)\,dx$

$$= \frac{2^{-\mu-2\nu-1}\pi b^{\nu}\Gamma(\mu+2\nu+1)}{\nu\Gamma\left(\frac{\mu+2\nu-a+2}{2}\right)\Gamma\left(\frac{\mu+2\nu+a+2}{2}\right)}$$
$$\times\,{}_3F_3\left(\begin{matrix}1,\, \frac{1+\mu}{2}+\nu,\, \frac{\mu}{2}+\nu+1;\, b\\ \nu+1,\, \frac{\mu-a}{2}+\nu+1,\, \frac{\mu+a}{2}+\nu+1\end{matrix}\right)\qquad [\mathrm{Re}\,\nu,\, \mathrm{Re}\,(\mu+\nu+1) > 0].$$

3. $$\int_0^{\pi} \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} e^{b\sin^2 x} \gamma(\nu, b\sin^2 x)\, dx$$
$$= \frac{2^{-\mu-2\nu}\pi b^\nu \Gamma(\mu+2\nu+1)}{\nu\Gamma\left(\frac{\mu+2\nu-a}{2}+1\right)\Gamma\left(\frac{\mu+2\nu+a}{2}+1\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix}$$
$$\times\, {}_3F_3\left(\begin{matrix} 1,\, \frac{\mu+1}{2}+\nu,\, \frac{\mu}{2}+\nu+1;\; b \\ \nu+1,\, \frac{\mu-a}{2}+\nu+1,\, \frac{\mu+a}{2}+\nu+1 \end{matrix} \right)$$
$[\operatorname{Re}\nu,\ \operatorname{Re}(\mu+\nu+(3\pm1)/2) > 0].$

4. $$\int_0^{m\pi} \sin^{-2\nu} x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} e^{b\sin^2 x} \gamma(\nu, b\sin^2 x)\, dx$$
$$= \frac{2b^\nu}{\nu a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1;\; b \\ \nu+1,\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix} \right) \quad [\operatorname{Re}\nu > 0].$$

5. $$\int_0^{m\pi} e^{-ax+b\sin^2 x} \sin^{-2\nu} x\, \gamma(\nu, b\sin^2 x)\, dx$$
$$= \left(1-e^{-m\pi a}\right)\frac{b^\nu}{\nu a}\, {}_3F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1;\; b \\ \nu+1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right) \quad [\operatorname{Re}\nu > 0].$$

6. $$\int_0^{m\pi} e^{-ax} \sin^{-2\nu} x\, \gamma(\nu, b\sin^2 x)\, dx$$
$$= \left(1-e^{-m\pi a}\right)\frac{b^\nu}{\nu a}\, {}_3F_3\left(\begin{matrix} \nu,\, \frac{1}{2},\, 1;\; b \\ \nu+1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right) \quad [\operatorname{Re}\nu > 0].$$

7. $$\int_0^{\infty} e^{-ax} \sin^{-2\nu} x\, \gamma(\nu, b\sin^2 x)\, dx = \frac{b^\nu}{\nu a}\, {}_3F_3\left(\begin{matrix} \nu,\, \frac{1}{2},\, 1;\; b \\ \nu+1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right)$$
$[\operatorname{Re} a,\ \operatorname{Re}\nu > 0].$

8. $$\int_0^{\infty} e^{-ax+b\sin^2 x} \sin^{-2\nu} x\, \gamma(\nu, b\sin^2 x)\, dx$$
$$= \frac{b^\nu}{\nu a}\, {}_3F_3\left(\begin{matrix} \frac{1}{2},\, 1,\, 1;\; b \\ \nu+1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right) \quad [\operatorname{Re} a,\ \operatorname{Re}\nu > 0].$$

4.6.4. Integrals containing $\gamma(\nu, z)$ and the logarithmic function

1. $$\int_0^a x^{s-1} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} \gamma(\nu, bx)\,dx = \frac{\pi^{1/2} a^{s+\nu} b^{\nu} \Gamma(s+\nu)}{2\nu(s+\nu)\Gamma\left(s+\nu+\frac{1}{2}\right)} \times {}_3F_3\left(\begin{matrix} \nu,\, s+\nu,\, s+\nu;\; -ab \\ \nu+1,\, s+\nu+\frac{1}{2},\, s+\nu+1 \end{matrix}\right) \quad [a,\, \operatorname{Re}\nu,\, \operatorname{Re}(s+\nu) > 0].$$

2. $$\int_0^a x^{s-1} e^{bx} \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} \gamma(\nu, bx)\,dx = \frac{\pi^{1/2} a^{s+\nu} b^{\nu} \Gamma(s+\nu)}{2\nu(s+\nu)\Gamma\left(s+\nu+\frac{1}{2}\right)} \times {}_3F_3\left(\begin{matrix} 1,\, s+\nu,\, s+\nu;\; ab \\ \nu+1,\, s+\nu+\frac{1}{2},\, s+\nu+1 \end{matrix}\right) \quad [a,\, \operatorname{Re}\nu,\, \operatorname{Re}(s+\nu) > 0].$$

4.6.5. Integrals containing $\gamma(\nu, z)$, erf (z) and erfi (z)

1. $$\int_0^a \operatorname{erf}\left(\sqrt{b(a-x)}\right) \gamma(\nu, bx)\,dx = \frac{b^{-1}\Gamma(\nu)}{2\Gamma\left(\nu+\frac{3}{2}\right)} \left[2(ab)^{\nu+3/2} e^{-ab} + (2ab-2\nu-1)\gamma\left(\nu+\frac{3}{2},\, ab\right)\right] \quad [a,\, \operatorname{Re}\nu > 0].$$

2. $$\int_0^a e^{bx} \operatorname{erfi}\left(\sqrt{b(a-x)}\right) \gamma(\nu, bx)\,dx = \frac{\Gamma(\nu)}{\Gamma\left(\nu+\frac{5}{2}\right)} a^{\nu+3/2} b^{\nu+1/2} {}_1F_1\left(\begin{matrix} \frac{3}{2};\; ab \\ \nu+\frac{5}{2} \end{matrix}\right) \quad [a,\, \operatorname{Re}\nu > 0].$$

3. $$\int_0^a e^{2bx} \operatorname{erfi}\left(\sqrt{b(a-x)}\right) \gamma(\nu, bx)\,dx = \frac{\Gamma(\nu)}{\Gamma\left(\nu+\frac{5}{2}\right)} a^{\nu+3/2} b^{\nu+1/2} e^{ab} {}_1F_2\left(\begin{matrix} 1;\; \frac{a^2b^2}{4} \\ \frac{2\nu+5}{4},\, \frac{2\nu+7}{4} \end{matrix}\right) \quad [a,\, \operatorname{Re}\nu > 0].$$

4.6.6. Integrals containing products of $\gamma(\nu, z)$

1. $$\int_0^a e^{2bx} \gamma(\mu, bx)\gamma(\nu, b(a-x))\,dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu+2)} a^{\mu+\nu+1} b^{\mu+\nu} e^{ab+\mu\pi i} \times {}_1F_2\left(\begin{matrix} 1;\; \frac{a^2b^2}{4} \\ \frac{\mu+\nu}{2}+1,\, \frac{\mu+\nu+3}{2} \end{matrix}\right) \quad [a,\, b,\, \operatorname{Re}\mu,\, \operatorname{Re}\nu > 0].$$

4.7. The Bessel Function $J_\nu(z)$

4.7.1. Integrals containing $J_\nu(z)$ and algebraic functions

1. $$\int_0^\infty \frac{x^{n+2p+1}}{(x^4+ax^2+b)^{m+p+1/2}} J_n(cx)\,dx$$
$$= \frac{(-1)^{m+n+p}}{\left(\frac{1}{2}\right)_m \left(m+\frac{1}{2}\right)_p} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathrm{D}_a^p\, \mathrm{D}_b^m \left[u_+^k u_-^{n-k} I_k(cu_+)\, I_{n-k}(cu_-)\right]$$
$$\left[n < 4m+2p;\ u_\pm = 2^{-1}\left(a \pm 2\sqrt{b}\right)^{1/2}\right].$$

2. $$\int_0^a x^s (a-x)^{s+1/2} J_\nu\left(b\sqrt[4]{x(a-x)}\right) dx = 2^{-2s-3\nu/2-1}\sqrt{\pi}\, a^{2s+(\nu+3)/2} b^\nu$$
$$\times \frac{\Gamma\left(2s+\frac{\nu}{2}+2\right)}{\Gamma(\nu+1)\Gamma\left(2s+\frac{\nu+5}{2}\right)}\, {}_1F_2\left(\begin{matrix} 2s+\frac{\nu}{2}+2;\ -\frac{ab^2}{8} \\ \nu+1,\ 2s+\frac{\nu+5}{2} \end{matrix}\right)$$
$$[a>0;\ \operatorname{Re}(s+\nu/4) > -1].$$

3. $$\int_0^a x^{-\nu/4}(a-x)^{-(\nu+2)/4} J_\nu\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= 2^{(2\nu+3)/4}\sqrt{\pi}\, a^{(1-2\nu)/4} b^{-1/2}\, \mathbf{H}_{\nu-1/2}\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

4. $$\int_0^a x^{\nu/4}(a-x)^{(\nu-2)/4} J_\nu\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= 2^{(3-2\nu)/4}\sqrt{\pi}\, a^{(2\nu+1)/4} b^{-1/2} J_{\nu+1/2}\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0;\ \operatorname{Re}\nu > -2].$$

5. $$\int_0^a x^{-1/4}(a-x)^{-3/4} J_\nu\left(b\sqrt[4]{x(a-x)}\right) dx = 2^{1/2}\pi J_{\nu/2}^2\left(b\sqrt{\frac{a}{8}}\right)$$
$$[a>0;\ \operatorname{Re}\nu > -1].$$

6. $$\int_0^a x^{1/2} J_0\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= \frac{1}{b^3}\left[\sqrt{2}\,(ab^2-2)\sin\left(b\sqrt{\frac{a}{2}}\right) + 2\sqrt{a}\,b\cos\left(b\sqrt{\frac{a}{2}}\right)\right]$$
$$[a>0].$$

7. $$\int_0^a x^{-1/2} J_0\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{2^{3/2}}{b}\sin\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

8. $$\int_0^a x^{-1/4}(a-x)^{1/4} J_0\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{\pi a}{2^{3/2}}\left[J_0^2\left(b\sqrt{\frac{a}{8}}\right) - J_1^2\left(b\sqrt{\frac{a}{8}}\right)\right] \quad [a>0].$$

9. $$\int_0^a x^{-1/4}(a-x)^{1/4} J_1\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{2}{b^2}\left[\sqrt{2}\sin\left(b\sqrt{\frac{a}{2}}\right) - \sqrt{a}\,b\cos\left(b\sqrt{\frac{a}{2}}\right)\right] \quad [a>0].$$

10. $$\int_0^a x^{-3/4}(a-x)^{-1/4} J_1\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{4}{\sqrt{a}\,b}\left[1-\cos\left(b\sqrt{\frac{a}{2}}\right)\right] \quad [a>0].$$

4.7.2. Integrals containing $J_\nu(z)$ and the exponential function

1. $$\int_0^\infty x^{m+(n-1)/2}(x+z)^{(n-1)/2} e^{-ax} J_\nu\left(b\sqrt{x^2+xz}\right) dx$$
$$= (-1)^m 2^n b^{-\nu}\,\mathrm{D}_h^n\,\mathrm{D}_a^m\left[h^{(\nu+n)/2} e^{az/2} I_{(\nu+n)/2}(u_-)\,K_{(\nu+n)/2}(u_+)\right]\Big|_{h=c^2}$$
$$\left[u_\pm = z\left(\sqrt{a^2+h}\pm a\right)/4;\ \operatorname{Re}(2\nu+2m+n)>2;\ \operatorname{Re} a > |\operatorname{Im} b|;\ |\arg z|<\pi\right].$$

4.7.3. Integrals containing $J_\nu(z)$ and trigonometric functions

1. $$\int_0^a x^{m+\nu/2}(a-x)^n \sin\left(b\sqrt{a-x}\right) J_\nu(c\sqrt{x})\,dx = (-1)^{m+n} 2^{m+1/2}\sqrt{\pi}$$
$$\times\, a^{(2m+2\nu+3)/4} c^\nu \sum_{k=0}^m (-1)^k \binom{m}{k}(-m-\nu)_{m-k}\left(-\frac{\sqrt{a}\,c^2}{2}\right)^k$$
$$\times\, \mathrm{D}_b^{2n}\left[b(b^2+c^2)^{-(2m+2k+2\nu+3)/4} J_{m+k+\nu+3/2}\left(\sqrt{a(b^2+c^2)}\right)\right] \quad [a>0].$$

2. $$\int_0^\pi \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} J_\nu(b\sin x)\,dx$$
$$= \frac{2^{-\mu-2\nu}\pi b^\nu\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a}{2}+1\right)\Gamma\left(\frac{\mu+\nu+a}{2}+1\right)}\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}$$
$$\times\, {}_2F_3\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1;\ -\frac{b^2}{4}\\ \nu+1,\ \frac{\mu+\nu-a}{2}+1,\ \frac{\mu+\nu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}(\mu+\nu)>-1].$$

3. $$\int_0^{\pi/2} \cos^\mu x \cos(ax)\, J_\nu(b\cos x)\, dx$$

$$= \frac{2^{-\mu-2\nu-1}\pi b^\nu \Gamma(\mu+\nu+1)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a}{2}+1\right)\Gamma\left(\frac{\mu+\nu+a}{2}+1\right)}$$

$$\times\ {}_2F_3\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1;\ -\frac{b^2}{4}\\ \nu+1,\ \frac{\mu+\nu-a)}{2}+1,\ \frac{\mu+\nu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}(\mu+\nu)>-1].$$

4. $$\int_0^{\pi/2} \cos(2nx)\sin^{-\nu} x\, J_\nu(a\sin x)\, dx$$

$$= \frac{2^{-4n-\nu-1}\pi a^{2n+\nu}}{n!\,\Gamma(n+\nu+1)}\ {}_1F_2\left(\begin{matrix}n+\frac{1}{2};\ -\frac{a^2}{4}\\ n+\nu+1,\ 2n+1\end{matrix}\right).$$

5. $$\int_0^{m\pi} e^{-ax}\sin^{-\nu} x\, J_\nu(b\sin x)\, dx$$

$$= \frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)a}\left(1-e^{-m\pi a}\right)\ {}_2F_3\left(\begin{matrix}\frac{1}{2},\ 1;\ -\frac{b^2}{4}\\ \nu+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

6. $$\int_0^{\infty} e^{-ax}\sin^{-\nu} x\, J_\nu(b\sin x)\, dx = \frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)a}\ {}_2F_3\left(\begin{matrix}\frac{1}{2},\ 1;\ -\frac{b^2}{4}\\ \nu+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)$$

$$[\operatorname{Re} a>0].$$

7. $$\int_0^{\pi} \cos 2mx\,(a^2-b^2\sin^2 x)^{n/2} J_n\left(\sqrt{a^2-b^2\sin^2 x}\right) dx$$

$$= (-1)^{m+n}\frac{\pi}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}b^{2n-2k}\left(a-\sqrt{a^2-b^2}\right)^{2k-n}$$

$$\times J_{m+k}\left(\frac{a-\sqrt{a^2-b^2}}{2}\right) J_{m-n+k}\left(\frac{a+\sqrt{a^2-b^2}}{2}\right) \quad [0<b\le a].$$

8. $$\int_0^{2\pi} \cos mx\,(a^2+b^2+2ab\cos x)^{n/2} J_n\left(\sqrt{a^2+b^2+2ab\cos x}\right) dx$$

$$= (-1)^{m+n}2\pi\sum_{k=0}^{n}(-1)^k\binom{n}{k}a^k b^{n-k} J_{m+k}(a)\, J_{m-n+k}(b).$$

4.7.4. Integrals containing $J_\nu(z)$ and the logarithmic function

1. $$\int_0^1 x\ln x\, J_0(ax)\,dx = \frac{1}{a^2}\,[J_0(a)-1].$$

2. $$\int_0^a \frac{x\ln x}{\sqrt{a^2-x^2}}\, J_0(bx)\,dx = \frac{1}{b}\{\sin(ab)\ln a + \sin(ab)[\mathrm{ci}\,(2ab)-\mathrm{ci}\,(ab)] + \cos(ab)[\mathrm{Si}\,(ab)-\mathrm{Si}\,(2ab)]\} \quad [a>0].$$

3. $$\int_0^a x^{s-1}\ln\frac{a+\sqrt{a^2-x^2}}{x}\, J_\nu(bx)\,dx = \frac{\pi^{1/2}a^{s+\nu}\left(\frac{b}{2}\right)^\nu \Gamma\left(\frac{s+\nu}{2}\right)}{2(s+\nu)\Gamma\left(\frac{s+\nu+1}{2}\right)\Gamma(\nu+1)} \times {}_2F_3\left(\begin{matrix}\frac{s+\nu}{2},\ \frac{s+\nu}{2};\ -\frac{a^2b^2}{4}\\ \frac{s+\nu+1}{2},\ \frac{s+\nu}{2}+1,\ \nu+1\end{matrix}\right) \quad [a,\ \mathrm{Re}\,(s+\nu)>0].$$

4. $$\int_0^a \ln\frac{a+\sqrt{a^2-x^2}}{x}\, J_1(bx)\,dx = \frac{1}{b}[\mathbf{C}+\ln(ab)-\mathrm{ci}\,(ab)] \quad [a>0].$$

5. $$\int_0^a x\ln\frac{a+\sqrt{a^2-x^2}}{x}\, J_0(bx)\,dx = \frac{1}{b^2}[1-\cos(ab)] \quad [a>0].$$

6. $$\int_0^a x^3\ln\frac{a+\sqrt{a^2-x^2}}{x}\, J_0(bx)\,dx = \frac{1}{b^4}[3ab\sin(ab)+(4-a^2b^2)\cos(ab)-4] \quad [a>0].$$

7. $$\int_0^\infty x^{m+n}e^{-px}\ln x\, J_n(ax)\,dx = (-1)^{m+n}(2a)^n\,\mathrm{D}_p^m\,\mathrm{D}_u^n\left[(p^2+u)^{-1/2}\left(\ln\frac{p+\sqrt{p^2+u}}{2(p^2+u)}-\mathbf{C}\right)\right]\Bigg|_{u=a^2} \quad [\mathrm{Re}\,p>|\mathrm{Im}\,a|].$$

4.7.5. Integrals containing $J_\nu(z)$ and inverse trigonometric functions

1. $$\int_0^1 x^{s-1}\arccos x\, J_\nu(ax)\,dx = \frac{\pi^{1/2}\left(\frac{a}{2}\right)^\nu \Gamma\left(\frac{s+\nu+1}{2}\right)}{(s+\nu)^2\Gamma(\nu+1)\,\Gamma\left(\frac{s+\nu}{2}\right)}\,{}_2F_3\left(\begin{matrix}\frac{s+\nu}{2},\ \frac{s+\nu+1}{2};\ -\frac{a^2}{4}\\ \frac{s+\nu}{2}+1,\ \frac{s+\nu}{2}+1,\ \nu+1\end{matrix}\right) \quad [\mathrm{Re}\,(s+\nu)>0].$$

2. $\displaystyle\int_0^1 \arccos x\, J_0(ax)\,dx = \frac{1}{a}\,\mathrm{Si}\,(a).$

3. $\displaystyle\int_0^1 x\arccos x\, J_0(ax)\,dx = \frac{\pi}{2a} J_0\Bigl(\frac{a}{2}\Bigr)\, J_1\Bigl(\frac{a}{2}\Bigr).$

4. $\displaystyle\int_0^1 x^2\arccos x\, J_0(ax)\,dx = \frac{1}{a^3}\,[2\sin a - a\cos a - \mathrm{Si}\,(a)].$

5. $\displaystyle\int_0^1 \arccos x\, J_1(ax)\,dx = \frac{\pi}{2a}\,\Bigl[1 - J_0^2\Bigl(\frac{a}{2}\Bigr)\Bigr].$

6. $\displaystyle\int_0^1 x\arccos x\, J_1(ax)\,dx = \frac{1}{a^2}\,[\mathrm{Si}\,(a) - \sin a].$

7. $\displaystyle\int_0^1 \frac{1}{x}\arccos x\, J_1(ax)\,dx = \mathrm{Si}\,(a) + \frac{1}{a}\,(\cos a - 1).$

4.7.6. Integrals containing $J_\nu(z)$, $\mathrm{Si}\,(z)$ and $\mathrm{ci}\,(z)$

1. $$\int_0^\infty x^{s-1}[\sin(x)\,\mathrm{Si}\,(2x) + \cos(x)\,\mathrm{ci}\,(2x)]\,J_\nu(x)\,dx$$
$$= -\frac{2^{-s-1}\Gamma(s+\nu)}{\pi^{1/2}\Gamma(\nu-s+1)}\Gamma\Bigl(\frac{1}{2}-s\Bigr)\cos\Bigl(\frac{s+\nu}{2}\pi\Bigr)$$
$$\times\Bigl[\psi\Bigl(\frac{1-s-\nu}{2}\Bigr) + \psi\Bigl(\frac{1-s+\nu}{2}\Bigr) - \psi\Bigl(1-\frac{s-\nu}{2}\Bigr) - \psi\Bigl(\frac{s+\nu}{2}\Bigr)\Bigr]$$
$[0 < \mathrm{Re}\,(s+\nu) < 3/2].$

2. $$\int_0^\infty x^{s-1}[\sin(x)\,\mathrm{ci}\,(2x) - \cos(x)\,\mathrm{Si}\,(2x)]\,J_\nu(x)\,dx$$
$$= -\frac{2^{-s-1}\Gamma(s+\nu)}{\pi^{1/2}\Gamma(\nu-s+1)}\Gamma\Bigl(\frac{1}{2}-s\Bigr)\sin\Bigl(\frac{s+\nu}{2}\pi\Bigr)$$
$$\times\Bigl[\psi\Bigl(1-\frac{s+\nu}{2}\Bigr) - \psi\Bigl(\frac{1-s+\nu}{2}\Bigr) + \psi\Bigl(1+\frac{\nu-s}{2}\Bigr) - \psi\Bigl(\frac{1+s+\nu}{2}\Bigr)\Bigr]$$
$[-1/2 < \mathrm{Re}\,(s+\nu) < 3/2].$

4.7.7. Integrals containing products of $J_\nu(z)$

1. $$\int_0^a J_\mu(x)\,J_\nu(a-x)\,dx = \frac{a}{\mu+\nu}\left\{J_{\mu+\nu}(a) - \frac{(a/2)^{\mu+\nu}}{\Gamma(\mu+\nu+1)}\right.$$
$$\times\left[\frac{\cos a}{\mu+\nu+1}\,{}_3F_4\left(\begin{matrix}\frac{2\mu+2\nu+1}{4},\ \frac{\mu+\nu+1}{2},\ \frac{2\mu+2\nu+3}{4};\ -a^2\\ \frac{1}{2},\ \frac{\mu+\nu+3}{2},\ \mu+\nu+\frac{1}{2},\ \mu+\nu+1\end{matrix}\right)\right.$$
$$\left.\left.+\frac{a\sin a}{\mu+\nu+2}\,{}_3F_4\left(\begin{matrix}\frac{2\mu+2\nu+3}{4},\ \frac{\mu+\nu}{2}+1,\ \frac{2\mu+2\nu+5}{4};\ -a^2\\ \frac{3}{2},\ \frac{\mu+\nu}{2}+2,\ \mu+\nu+1,\ \mu+\nu+\frac{3}{2}\end{matrix}\right)\right]\right\}$$
$[a>0;\ \operatorname{Re}\mu,\ \operatorname{Re}\nu>-1]$.

2. $$\int_0^a \frac{1}{x^2}\,J_\mu(x)\,J_\nu(a-x)\,dx = \frac{1}{2\mu}\left[\frac{1}{\mu-1}\,J_{\mu+\nu-1}(a) + \frac{1}{\mu+1}\,J_{\mu+\nu+1}(a)\right]$$
$[a>0;\ \operatorname{Re}\mu>1;\ \operatorname{Re}\nu>-1]$.

3. $$\int_0^a \frac{1}{x(a-x)^2}\,J_\mu(x)\,J_\nu(a-x)\,dx$$
$$= \frac{1}{2\mu\nu a}\left[\frac{\mu+\nu-1}{\nu-1}\,J_{\mu+\nu-1}(a) + \frac{\mu+\nu+1}{\nu+1}\,J_{\mu+\nu+1}(a)\right]$$
$[a,\ \operatorname{Re}\mu>0;\ \operatorname{Re}\nu>1]$.

4. $$\int_0^a \frac{1}{x^2(a-x)^2}\,J_\mu(x)\,J_\nu(a-x)\,dx$$
$$= \frac{1}{2\mu\nu a^2}\left[\frac{(\mu+\nu-1)(\mu+\nu-2)}{(\mu-1)(\nu-1)}\,J_{\mu+\nu-1}(a)\right.$$
$$\left.+\frac{(\mu+\nu+1)(\mu+\nu+2)}{(\mu+1)(\nu+1)}\,J_{\mu+\nu+1}(a)\right]$$ $[a>0;\ \operatorname{Re}\mu,\ \operatorname{Re}\nu>1]$.

5. $$\int_0^a x^{(2n-1)/4}(a-x)^{\nu/2}J_{n-1/2}(b\sqrt{x})\,J_\nu\left(c\sqrt{a-x}\right)dx$$
$$= 2b^{n-1/2}c^\nu\left(\frac{a}{b^2+c^2}\right)^{(2\nu+2n+1)/4} J_{\nu+n+1/2}\left(\sqrt{ab^2+ac^2}\right)$$
$[a>0;\ \operatorname{Re}\nu>-1]$.

6. $$\int_0^a x^{m+\mu/2}(a-x)^{n+\nu/2}J_\mu(b\sqrt{x})\,J_\nu\left(c\sqrt{a-x}\right)dx = (-1)^{m+n}2^{m+n+1}b^\mu c^\nu$$
$$\times\left(\frac{a}{b^2+c^2}\right)^{(\mu+\nu+m+n+1)/2}\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}(-\mu-m)_{m-j}(-\nu-n)_{n-k}$$

$$\times \frac{b^{2j}c^{2k}}{2^{j+k}}\left(\frac{a}{b^2+c^2}\right)^{(j+k)/2} J_{\mu+\nu+j+k+m+n+1}\left(\sqrt{ab^2+ac^2}\right)$$
$$[a>0;\ \operatorname{Re}\mu>-m-1;\ \operatorname{Re}\nu>-n-1].$$

7. $$\int_0^a x^{n-1/2}(a-x)^{\nu/2} J_{n-1/2}(b\sqrt{x})\, J_{1/2-n}(b\sqrt{x})\, J_\nu\left(c\sqrt{a-x}\right) dx$$
$$= (-1)^{n-1}\frac{2^{n+1/2}}{\sqrt{\pi}} c^\nu \left(\frac{a}{4b^2+c^2}\right)^{(2\nu+2n+1)/4} \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{1}{2}\right)_{n-k-1}$$
$$\times \left(\frac{4ab^4}{4b^2+c^2}\right)^{k/2} J_{\nu+n+k+1/2}\left(\sqrt{4ab^2+ac^2}\right) \quad [n\ge 1;\ a>0;\ \operatorname{Re}\nu>-1].$$

8. $$\int_a^b \frac{1}{b-x} J_1(b-x) J_0\left(\sqrt{x^2-a^2}\right) dx = \frac{b-a}{\sqrt{b^2-a^2}} J_1\left(\sqrt{b^2-a^2}\right)$$
$$[b>a>0].$$

9. $$\int_a^b \frac{1}{\sqrt{x^2-a^2}} J_0(b-x) J_1\left(\sqrt{x^2-a^2}\right) dx$$
$$= \frac{1}{a}\left[J_0(b-a) - J_0\left(\sqrt{b^2-a^2}\right)\right] \quad [b>a>0].$$

10. $$\int_0^1 e^{2ax} J_0^2\left(a\sqrt{x-x^2}\right) dx = e^a I_0(a)\left[1+\frac{\pi}{2}\,\mathbf{L}_1(a)\right] - \frac{\pi}{2} e^a I_1(a)\,\mathbf{L}_0(a).$$

11. $$\int_0^1 \frac{x}{\sqrt{1-x^2}} \cosh\left[(a+b)\sqrt{1-x^2}\right] J_0(ax)\, J_0(bx)\, dx$$
$$= I_0(2\sqrt{ab})\left[1+\frac{\pi}{2}\,\mathbf{L}_1(2\sqrt{ab})\right] - \frac{\pi}{2} I_1(2\sqrt{ab}) L_0(2\sqrt{ab}).$$

12. $$\int_0^{\pi/2} \cos(2nx)\sin^{-\mu-\nu} x\, J_\mu(a\sin x)\, J_\nu(a\sin x)\, dx$$
$$= \frac{2^{-4n-\mu-\nu-1}\pi a^{2n+\mu+\nu}\Gamma(2n+\mu+\nu+1)}{n!\,\Gamma(n+\mu+1)\Gamma(n+\nu+1)}$$
$$\times \frac{1}{\Gamma(n+\mu+\nu+1)}\, {}_3F_4\left(\begin{matrix} n+\frac{\mu+\nu+1}{2},\ n+\frac{\mu+\nu}{2}+1,\ n+\frac{1}{2};\ -a^2 \\ n+\mu+1,\ n+\nu+1,\ n+\mu+\nu+1,\ 2n+1\end{matrix}\right).$$

13. $$\int_0^\pi \cos(nx)\cos^{-\mu-\nu}x\,J_\mu(a\cos x)\,J_\nu(a\cos x)\,dx$$
$$= \frac{2^{-2n-\mu-\nu}\pi a^{n+\mu+\nu}\Gamma(n+\mu+\nu+1)}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}+\mu+1\right)\Gamma\left(\frac{n}{2}+\nu+1\right)}$$
$$\times\frac{\cos(n\pi/2)}{\Gamma\left(\frac{n}{2}+\mu+\nu+1\right)}\,{}_3F_4\left(\begin{matrix}\frac{n+\mu+\nu+1}{2},\ \frac{n+\mu+\nu}{2}+1,\ \frac{n+1}{2};\ -a^2\\ \frac{n}{2}+\mu+1,\ \frac{n}{2}+\nu+1,\ \frac{n}{2}+\mu+\nu+1,\ n+1\end{matrix}\right).$$

14. $$\int_0^\pi \sin(ax)\sin^{1-\mu-\nu}x\,J_\mu(b\sin x)\,J_\nu(b\sin x)\,dx$$
$$= \frac{\left(\frac{b}{2}\right)^{\mu+\nu}\sin(\pi a)}{\Gamma(\mu+1)\Gamma(\nu+1)\left(1-a^2\right)}\,{}_4F_5\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1,\ 1,\ \frac{3}{2};\ -b^2\\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ \frac{3-a}{2},\ \frac{3+a}{2}\end{matrix}\right).$$

15. $$\int_0^{m\pi} e^{-ax}\sin^{-\mu-\nu}x\,J_\mu(b\sin x)\,J_\nu(b\sin x)\,dx$$
$$= \frac{\left(\frac{b}{2}\right)^{\mu+\nu}\left(1-e^{-m\pi a}\right)}{\Gamma(\mu+1)\Gamma(\nu+1)a}\,{}_4F_5\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1,\ \frac{1}{2},\ 1;\ -b^2\\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

16. $$\int_0^\infty e^{-ax}\sin^{-\mu-\nu}x\,J_\mu(b\sin x)\,J_\nu(b\sin x)\,dx$$
$$= \frac{\left(\frac{b}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)a}\,{}_4F_5\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1,\ \frac{1}{2},\ 1;\ -b^2\\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)\quad [\operatorname{Re} a>0].$$

17. $$\int_0^a x\ln\frac{a+\sqrt{a^2-x^2}}{x}\,J_0^2(bx)\,dx = \frac{a}{2b}\left[2ab\,J_0(2ab)-J_1(2ab)\right]$$
$$+\frac{\pi a^2}{2}\left[J_1(2ab)\,\mathbf{H}_0(2ab)-J_0(2ab)\,\mathbf{H}_1(2ab)\right]\quad [a>0].$$

18. $$\int_0^a x^3\ln\frac{a+\sqrt{a^2-x^2}}{x}\,J_0^2(bx)\,dx$$
$$= \frac{a}{12b^3}\left[2ab(a^2b^2-2)\,J_0(2ab)-(a^2b^2-4)\,J_1(2ab)\right]$$
$$+\frac{\pi a^2}{48b^2}(4a^2b^2-3)\left[J_1(2ab)\mathbf{H}_0(2ab)-J_0(2ab)\,\mathbf{H}_1(2ab)\right]\quad [a>0].$$

19. $$\int_0^a (a^2-x^2)^{(2n-1)/4} J_{n-1/2}\left(b\sqrt{a^2-x^2}\right) J_\nu(cx)\,dx$$
$$= (-1)^n 2^{n-1/2}\sqrt{\pi}\, b^{n-1/2}$$
$$\times \mathrm{D}_u^n \left[J_{\nu/2}\left(\frac{a}{2}\sqrt{u+c^2}+\frac{a\sqrt{u}}{2}\right) J_{\nu/2}\left(\frac{a}{2}\sqrt{u+c^2}-\frac{a\sqrt{u}}{2}\right)\right]\Big|_{u=b^2}$$
$[a>0;\ \operatorname{Re}\nu>-1]$.

20. $$\int_0^1 x\ln x\, J_0^2(ax)\,dx = -\frac{1}{2}\left[J_0^2(a)+J_1^2(a)-\frac{1}{a}J_0(a)\,J_1(a)\right].$$

21. $$\int_0^1 x\ln x\, J_1^2(ax)\,dx = \frac{1}{2a^2}[1-(a^2+1)\,J_0^2(a)+a\,J_0(a)\,J_1(a)-a^2J_1^2(a)].$$

22. $$\int_0^1 \frac{1}{x^2}\arccos x\, J_1^2(ax)\,dx$$
$$= -\frac{1}{6a}\,[3a-2a(4a^2+1)\,J_0(2a)+(4a^2-1)\,J_1(2a)]$$
$$+\frac{2\pi a^2}{3}[J_1(2a)\,\mathbf{H}_0(2a)-J_0(2a)\,\mathbf{H}_1(2a)].$$

23. $$\int_{-1}^1 J_0\left(a\sqrt{x-1}\right) J_0\left(a\sqrt{x+1}\right) J_0\left(b\sqrt{x-1}\right) J_0\left(b\sqrt{x+1}\right) dx$$
$$= 2\,{}_1F_4\left(\begin{matrix}\frac{1}{2};\ \frac{a^2b^2}{4}\\ 1,\,1,\,1,\,\frac{3}{2}\end{matrix}\right).$$

4.8. The Bessel Function $Y_\nu(z)$

4.8.1. Integrals containing $Y_\nu(z)$ and algebraic functions

1. $$\int_0^a \frac{1}{\sqrt{a^2-x^2}}Y_0(cx)\,dx = \frac{\pi}{2}J_0\left(\frac{ac}{2}\right)Y_0\left(\frac{ac}{2}\right)$$ $[a>0]$.

4.8.2. Integrals containing $Y_\nu(z)$ and $J_\nu(z)$

1. $$\int_0^a \frac{1}{x}J_1(x)\,Y_0(a-x)\,dx = \frac{2}{\pi a}J_0(a)+Y_1(a)$$ $[a>0]$.

2. $$\int_0^a e^{-ax}\left[\frac{2}{\pi bx}J_0(bx)+Y_1(bx)\right]dx$$
$$=\frac{2}{\pi b}\left(1-\frac{a}{\sqrt{a^2+b^2}}\right)\ln\frac{a+\sqrt{a^2+b^2}}{b}\qquad [\operatorname{Re}a>|\operatorname{Im}b|].$$

4.9. The Modified Bessel Function $I_\nu(z)$

4.9.1. Integrals containing $I_\nu(z)$ and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1}I_\nu\left(b\sqrt{x(a-x)}\right)dx$$
$$=\mathrm{B}\left(s+\frac{\nu}{2},\,t+\frac{\nu}{2}\right)\frac{a^{s+t+\nu-1}}{\Gamma(\nu+1)}\left(\frac{b}{2}\right)^\nu {}_2F_3\left(\begin{matrix}s+\frac{\nu}{2},\,t+\frac{\nu}{2};\;\frac{a^2b^2}{16}\\ \nu+1,\,\frac{s+t+\nu}{2},\,\frac{s+t+\nu+1}{2}\end{matrix}\right).$$

2. $$\int_0^a x^{\nu/2}(a-x)^{\nu/2}I_\nu\left(b\sqrt{x(a-x)}\right)dx$$
$$=\sqrt{\pi}\left(\frac{a}{2}\right)^{\nu+1/2}\left(\frac{2}{b}\right)^{1/2}I_{\nu+1/2}\left(\frac{ab}{2}\right)\qquad [a>0;\ \operatorname{Re}\nu>-1].$$

3. $$\int_0^a I_0\left(b\sqrt{x(a-x)}\right)dx=\frac{2}{b}\sinh\left(\frac{ab}{2}\right)\qquad [a>0].$$

4. $$\int_0^a x^{-1/2}(a-x)^{-1/2}I_0\left(b\sqrt{x(a-x)}\right)dx=\pi I_0^2\left(\frac{ab}{4}\right)\qquad [a>0].$$

5. $$\int_0^a x^s(a-x)^{s+1/2}I_\nu\left(b\sqrt[4]{x(a-x)}\right)dx=2^{-2s-3\nu/2-1}\sqrt{\pi}\,a^{2s+(\nu+3)/2}b^\nu$$
$$\times\frac{\Gamma\left(2s+\frac{\nu}{2}+2\right)}{\Gamma(\nu+1)\Gamma\left(2s+\frac{\nu+5}{2}\right)}\,{}_1F_2\left(\begin{matrix}2s+\frac{\nu}{2}+2;\;\frac{ab^2}{8}\\ \nu+1,\,2s+\frac{\nu+5}{2}\end{matrix}\right)\qquad [a>0;\ \operatorname{Re}(s+\nu/4)>-1].$$

6. $$\int_0^a x^{\nu/4}(a-x)^{(\nu-2)/4}I_\nu\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=2^{(3-2\nu)/4}\sqrt{\pi}\,a^{(2\nu+1)/4}b^{-1/2}I_{\nu+1/2}\left(b\sqrt{\frac{a}{2}}\right)\qquad [a>0;\ \operatorname{Re}\nu>-3].$$

7. $$\int_0^a x^{-\nu/4}(a-x)^{-(\nu+2)/4} I_\nu\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= 2^{(2\nu+3)/4}\sqrt{\pi}\, a^{(1-2\nu)/4} b^{-1/2}\, \mathbf{L}_{\nu-1/2}\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0;\ \operatorname{Re}\nu > -2].$$

8. $$\int_0^a x^{-1/4}(a-x)^{-3/4} I_\nu\left(b\sqrt[4]{x(a-x)}\right) dx = 2^{1/2}\pi I_{\nu/2}^2\left(b\sqrt{\frac{a}{8}}\right)$$
$$[a>0;\ \operatorname{Re}\nu > -3/2].$$

9. $$\int_0^a x^{1/2} I_0\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= \frac{1}{b^3}\left[\sqrt{2}\,(ab^2+2)\sinh\left(b\sqrt{\frac{a}{2}}\right) - 2\sqrt{a}\,b\cosh\left(b\sqrt{\frac{a}{2}}\right)\right] \quad [a>0].$$

10. $$\int_0^a x^{-1/2} I_0\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{2^{3/2}}{b}\sinh\left(b\sqrt{\frac{a}{2}}\right) \quad [a>0].$$

11. $$\int_0^a x^{-1/4}(a-x)^{1/4} I_0\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= \frac{\pi a}{2^{3/2}}\left[I_0^2\left(b\sqrt{\frac{a}{8}}\right) + I_1^2\left(b\sqrt{\frac{a}{8}}\right)\right] \quad [a>0].$$

12. $$\int_0^a x^{-1/4}(a-x)^{1/4} I_1\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= \frac{2}{b^2}\left[\sqrt{a}\,b\cosh\left(b\sqrt{\frac{a}{2}}\right) - \sqrt{2}\sinh\left(b\sqrt{\frac{a}{2}}\right)\right] \quad [a>0].$$

13. $$\int_0^a x^{-3/4}(a-x)^{-1/4} I_1\left(b\sqrt[4]{x(a-x)}\right) dx = \frac{4}{\sqrt{a}\,b}\left[\cosh\left(b\sqrt{\frac{a}{2}}\right) - 1\right]$$
$$[a>0].$$

4.9.2. Integrals containing $I_\nu(z)$ and the exponential function

1. $$\int_0^a x^{s-1}(a-x)^{t-1} e^{bx(a-x)} I_\nu(bx(a-x))\, dx$$
$$= \mathrm{B}\,(s+\nu,\, t+\nu)\frac{a^{s+t+2\nu-1}}{\Gamma(\nu+1)}\left(\frac{b}{2}\right)^\nu {}_3F_3\left(\begin{matrix}\nu+\frac{1}{2},\ s+\nu,\ t+\nu;\ \frac{a^2b}{2}\\ 2\nu+1,\ \frac{s+t}{2}+\nu,\ \frac{s+t+1}{2}+\nu\end{matrix}\right)$$
$$[a,\ \operatorname{Re}(s+\nu),\ \operatorname{Re}(t+\nu) > 0].$$

2. $$\int_0^a e^{bx(a-x)} I_0(bx(a-x))\,dx = \sqrt{\frac{\pi}{2b}}\,\operatorname{erfi}\left(a\sqrt{\frac{b}{2}}\right) \qquad [a>0].$$

3. $$\int_0^a x e^{bx(a-x)} I_0(bx(a-x))\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2b}}\,\operatorname{erfi}\left(a\sqrt{\frac{b}{2}}\right) \qquad [a>0].$$

4. $$\int_0^a x^2 e^{bx(a-x)} I_0(bx(a-x))\,dx$$
$$= \frac{3a^2b+1}{8}\sqrt{\frac{\pi}{2b^3}}\,\operatorname{erfi}\left(a\sqrt{\frac{b}{2}}\right) - \frac{a}{8b}e^{a^2b/2} \qquad [a>0].$$

5. $$\int_0^a x^{-1} e^{bx(a-x)} I_\nu(bx(a-x))\,dx$$
$$= \frac{a}{\nu}\sqrt{\frac{\pi b}{8}}\,e^{a^2b/4}\left[I_{\nu-1/2}\left(\frac{a^2b}{4}\right) - I_{\nu+1/2}\left(\frac{a^2b}{4}\right)\right] \qquad [a, \operatorname{Re}\nu>0].$$

6. $$\int_0^a x^{-1} e^{bx(a-x)} I_1(bx(a-x))\,dx = \frac{1}{a^2b}(2e^{a^2b/2} - a^2b - 2) \qquad [a>0].$$

7. $$\int_0^a x^{-1}(a-x)^{-1} e^{bx(a-x)} I_\nu(bx(a-x))\,dx$$
$$= \frac{1}{\nu}\sqrt{\frac{\pi b}{2}}\,e^{a^2b/4}\left[I_{\nu-1/2}\left(\frac{a^2b}{4}\right) - I_{\nu+1/2}\left(\frac{a^2b}{4}\right)\right] \qquad [a, \operatorname{Re}\nu>0].$$

8. $$\int_0^a x^{-1}(a-x)^{-1} e^{bx(a-x)} I_1(bx(a-x))\,dx = \frac{2}{a^3b}(2e^{a^2b/2} - a^2b - 2)$$
$$[a>0].$$

9. $$\int_0^a x^{1/2} e^{b\sqrt{x(a-x)}} I_0\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{a^{1/2}}{4b}\left[e^{ab} - \left(\frac{1}{2} - ab\right)\sqrt{\frac{\pi}{ab}}\,\operatorname{erfi}\left(\sqrt{ab}\right)\right] \qquad [a>0].$$

10. $$\int_0^a x^{-1/2} e^{b\sqrt{x(a-x)}} I_0\left(b\sqrt{x(a-x)}\right)dx = \sqrt{\frac{\pi}{b}}\,\operatorname{erfi}\left(\sqrt{ab}\right) \qquad [a>0].$$

11. $$\int_0^a x^{-1/2} e^{b\sqrt{x(a-x)}} I_1\left(b\sqrt{x(a-x)}\right) dx = \frac{1}{a^{1/2}b}\left[2e^{ab} - \sqrt{\pi ab}\,\operatorname{erfi}\left(\sqrt{ab}\right) - 2\right] \quad [a > 0].$$

12. $$\int_0^a x^{-1}(a-x)^{-1/2} e^{b\sqrt{x(a-x)}} I_1\left(b\sqrt{x(a-x)}\right) dx = \frac{2}{a^{3/2}b}\left(e^{ab} - ab - 1\right) \quad [a > 0].$$

4.9.3. Integrals containing $I_\nu(z)$ and trigonometric functions

1. $$\int_0^{\pi/2} \cos^\mu x \cos(ax)\, I_\nu(b\cos x)\, dx = \frac{2^{-\mu-2\nu-1}\pi b^\nu \Gamma(\mu+\nu+1)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a}{2}+1\right)\Gamma\left(\frac{\mu+\nu+a}{2}+1\right)} \times {}_2F_3\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1;\ \frac{b^2}{4}\\ \nu+1,\ \frac{\mu+\nu-a}{2}+1,\ \frac{\mu+\nu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}(\mu+\nu) > -1].$$

2. $$\int_0^{\pi} \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} I_\nu(b\sin x)\, dx = \frac{2^{-\mu-2\nu}\pi b^\nu \Gamma(\mu+\nu+1)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a}{2}+1\right)\Gamma\left(\frac{\mu+\nu+a}{2}+1\right)} \times \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} {}_2F_3\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1;\ \frac{b^2}{4}\\ \nu+1,\ \frac{\mu+\nu-a}{2}+1,\ \frac{\mu+\nu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}(\mu+\nu) > -1].$$

3. $$\int_0^{m\pi} \sin^{-\nu} x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} I_\nu(b\sin x)\, dx = \frac{2}{a}\sin\frac{m\pi a}{2}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix} \frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_2F_3\left(\begin{matrix}\frac{1}{2},\ 1;\ \frac{b^2}{4}\\ \nu+1,\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right).$$

4. $$\int_0^{m\pi} \sin^{-2\nu} x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} I_\nu(b\sin^2 x)\, dx = \frac{2}{a}\sin\frac{m\pi a}{2}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix} \times \frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_4F_5\left(\begin{matrix}\frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1;\ b\\ \nu+1,\ 1-\frac{a}{4},\ 1-\frac{a}{4},\ 1+\frac{a}{4},\ 1+\frac{a}{4}\end{matrix}\right).$$

5. $$\int_0^{m\pi} e^{-ax}\sin^{-\nu}x\,I_\nu(b\sin x)\,dx$$
$$= (1-e^{-m\pi a})\frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)a}\,{}_2F_3\left(\begin{matrix}\frac{1}{2},\,1;\;\frac{b^2}{4}\\ \nu+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right).$$

6. $$\int_0^{m\pi} e^{-ax+b\sin^2 x}\sin^{-2\nu}x\,I_\nu(b\sin^2 x)\,dx$$
$$= (1-e^{-m\pi a})\frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)a}\,{}_3F_3\left(\begin{matrix}\frac{1}{2},\,1,\,\nu+\frac{1}{2};\;2b\\ 2\nu+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right).$$

7. $$\int_0^{\infty} e^{-ax}\sin^{-\nu}x\,I_\nu(b\sin x)\,dx = \frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)a}\,{}_2F_3\left(\begin{matrix}\frac{1}{2},\,1;\;\frac{b^2}{4}\\ \nu+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)$$
[Re $a>0$].

8. $$\int_0^{\infty} e^{-ax+b\sin^2 x}\sin^{-2\nu}x\,I_\nu(b\sin^2 x)\,dx$$
$$= \frac{\left(\frac{b}{2}\right)^\nu}{\Gamma(\nu+1)a}\,{}_3F_3\left(\begin{matrix}\nu+\frac{1}{2},\,\frac{1}{2},\,1;\;2b\\ 2\nu+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)$$ [Re $a>0$].

9. $$\int_0^{\pi} \sin^\nu x\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\cos(b\sin x)\,I_0(b\sin x)\,dx$$
$$= \frac{2^{-\nu}\pi\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}{}_4F_4\left(\begin{matrix}\frac{1}{4},\,\frac{3}{4},\,\frac{\nu+1}{2},\,1+\frac{\nu}{2};\;b\\ \frac{1}{2},\,\frac{1}{2},\,1+\frac{\nu-a}{2},\,1+\frac{\nu+a}{2}\end{matrix}\right)$$
[Re $\nu>-1$].

10. $$\int_0^{\pi} \sin^\mu x\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\sinh(b\sin x)\,I_\nu(b\sin x)\,dx$$
$$= \frac{2^{-\mu-2\nu-1}\pi b^{\nu+1}\Gamma(\mu+\nu+2)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a+3}{2}\right)\Gamma\left(\frac{\mu+\nu+a+3}{2}\right)}$$
$$\times\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}{}_4F_5\left(\begin{matrix}\frac{2\nu+3}{4},\,\frac{2\nu+5}{4},\,\frac{\mu+\nu}{2}+1,\,\frac{\mu+\nu+3}{2};\;b^2\\ \frac{3}{2},\,\nu+1,\,\nu+\frac{3}{2},\,\frac{\mu+\nu-a+3}{2},\,\frac{\mu+\nu+a+3}{2}\end{matrix}\right)$$
[Re $(\mu+\nu)>-3$].

11. $$\int_0^{\pi} \sin^{\mu} x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \cosh(b \sin x)\, I_{\nu}(b \sin x)\, dx$$
$$= \frac{2^{-\mu-2\nu}\pi b^{\nu}\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a}{2}+1\right)\Gamma\left(\frac{\mu+\nu+a}{2}+1\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix}$$
$$\times\, {}_4F_5\left(\begin{matrix} \frac{2\nu+3}{4}, \frac{2\nu+5}{4}, \frac{\mu+\nu}{2}+1, \frac{\mu+\nu+3}{2};\ b^2 \\ \frac{3}{2}, \nu+1, \nu+\frac{3}{2}, \frac{\mu+\nu-a+3}{2}, \frac{\mu+\nu+a+3}{2} \end{matrix}\right) \quad [\mathrm{Re}\,(\mu+\nu) > -2].$$

12. $$\int_0^{\pi/2} \cos^{\mu} x \cos(ax) \sinh(b\cos x)\, I_{\nu}(b\cos x)\, dx$$
$$= \frac{2^{-\mu-2\nu-2}\pi b^{\nu+1}\Gamma(\mu+\nu+2)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a+3}{2}\right)\Gamma\left(\frac{\mu+\nu+a+3}{2}\right)}$$
$$\times\, {}_4F_5\left(\begin{matrix} \frac{2\nu+3}{4}, \frac{2\nu+5}{4}, \frac{\mu+\nu}{2}+1, \frac{\mu+\nu+3}{2};\ b^2 \\ \frac{3}{2}, \nu+1, \nu+\frac{3}{2}, \frac{\mu+\nu-a+3}{2}, \frac{\mu+\nu+a+3}{2} \end{matrix}\right) \quad [\mathrm{Re}\,(\mu+\nu) > -3].$$

13. $$\int_0^{\pi/2} \cos^{\mu} x \cos(ax) \cosh(b\cos x)\, I_{\nu}(b\cos x)\, dx$$
$$= \frac{2^{-\mu-2\nu-1}\pi b^{\nu}\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+\nu-a}{2}+1\right)\Gamma\left(\frac{\mu+\nu+a}{2}+1\right)}$$
$$\times\, {}_4F_5\left(\begin{matrix} \frac{2\nu+1}{4}, \frac{2\nu+3}{4}, \frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1;\ b^2 \\ \frac{1}{2}, \nu+\frac{1}{2}, \nu+1, \frac{\mu+\nu-a}{2}+1, \frac{\mu+\nu+a}{2}+1 \end{matrix}\right) \quad [\mathrm{Re}\,(\mu+\nu) > -2].$$

14. $$\int_0^{\infty} \cosh^{\mu}(b\cos x)\, \cos(bx)\, I_{\nu}(c\,\mathrm{sech}\, x)\, dx = \frac{2^{-\mu-2}c^{\nu}}{\Gamma(\nu+1)\Gamma(\nu-\mu)}$$
$$\times\, \Gamma\left(\frac{\nu-\mu-ib}{2}\right)\Gamma\left(\frac{\nu-\mu+ib}{2}\right) {}_2F_3\left(\begin{matrix} \frac{\nu-\mu-ib}{2}, \frac{\nu-\mu+ib}{2};\ \frac{c^2}{4} \\ \nu+1, \frac{\nu-\mu}{2}, \frac{\mu-\nu+1}{2} \end{matrix}\right)$$
$$[\mathrm{Re}\,(\nu-\mu) > 0].$$

4.9.4. Integrals containing $I_{\nu}(z)$ and the logarithmic function

1. $$\int_0^{a} x^{s-1} \ln \frac{a+\sqrt{a^2-x^2}}{x}\, I_{\nu}(bx)\, dx = \frac{\pi^{1/2}a^{s+\nu}\left(\frac{b}{2}\right)^{\nu}\Gamma\left(\frac{s+\nu}{2}\right)}{2(s+\nu)\Gamma\left(\frac{s+\nu+1}{2}\right)\Gamma(\nu+1)}$$
$$\times\, {}_2F_3\left(\begin{matrix} \frac{s+\nu}{2}, \frac{s+\nu}{2};\ \frac{a^2b^2}{4} \\ \frac{s+\nu+1}{2}, \frac{s+\nu}{2}+1, \nu+1 \end{matrix}\right) \quad [a,\ \mathrm{Re}\,(s+\nu) > 0].$$

2. $\displaystyle\int_0^a \ln \frac{a+\sqrt{a^2-x^2}}{x}\, I_1(bx)\,dx = \frac{1}{b}\,[\operatorname{chi}(ab) - \ln(ab) - \mathbf{C}] \qquad [a>0].$

3. $\displaystyle\int_0^a x\ln \frac{a+\sqrt{a^2-x^2}}{x}\, I_0(bx)\,dx = \frac{\cosh(ab)-1}{b^2} \qquad [a>0].$

4. $\displaystyle\int_0^a x^3\ln \frac{a+\sqrt{a^2-x^2}}{x}\, I_0(bx)\,dx = \frac{1}{b^4}[(a^2b^2+4)\cosh(ab) - 3ab\sinh(ab) - 4] \qquad [a>0].$

5. $\displaystyle\int_0^1 x\ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\, I_0(ax)\,dx = \frac{4}{a^2}\sinh^2\frac{a}{2}.$

6. $$\int_0^a x^{s-1}e^{bx}\ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, I_\nu(bx)\,dx = \frac{2^{-\nu-1}\pi^{1/2}a^{s+\nu}b^\nu\Gamma(s+\nu)}{(s+\nu)\Gamma\left(s+\nu+\frac{1}{2}\right)\Gamma(\nu+1)} \times {}_3F_3\left(\begin{matrix}\nu+\frac{1}{2},\ s+\nu,\ s+\nu;\ 2ab\\ 2\nu+1,\ s+\nu+\frac{1}{2},\ s+\nu+1\end{matrix}\right) \qquad [a,\ \operatorname{Re}(s+\nu)>0].$$

4.9.5. Integrals containing $I_\nu(z)$ and inverse trigonometric functions

1. $$\int_0^1 x^{s-1}\arccos x\, I_\nu(ax)\,dx = \frac{\pi^{1/2}\left(\frac{a}{2}\right)^\nu\Gamma\left(\frac{s+\nu+1}{2}\right)}{(s+\nu)^2\Gamma(\nu+1)\,\Gamma\left(\frac{s+\nu}{2}\right)}\,{}_2F_3\left(\begin{matrix}\frac{s+\nu}{2},\ \frac{s+\nu+1}{2};\ \frac{a^2}{4}\\ \frac{s+\nu}{2}+1,\ \frac{s+\nu}{2}+1,\ \nu+1\end{matrix}\right) \qquad [\operatorname{Re}(s+\nu)>0].$$

2. $\displaystyle\int_0^1 \arccos x\, I_0(ax)\,dx = \frac{1}{a}\operatorname{shi}(a).$

3. $\displaystyle\int_0^1 x\arccos x\, I_0(ax)\,dx = \frac{\pi}{2a} I_0\left(\frac{a}{2}\right) I_1\left(\frac{a}{2}\right).$

4. $\displaystyle\int_0^1 x^2\arccos x\, I_0(ax)\,dx = \frac{1}{a^3}[a\cosh a - 2\sinh a + \operatorname{shi}(a)].$

5. $\displaystyle\int_0^1 \frac{1}{x}\arccos x\, I_1(ax)\,dx = \operatorname{shi}(a) + \frac{1-\cosh a}{a}.$

6. $$\int_0^1 \arccos x\, I_1(ax)\, dx = \frac{\pi}{2a}\left[I_0^2\left(\frac{a}{2}\right) - 1\right].$$

7. $$\int_0^1 x \arccos x\, I_1(ax)\, dx = \frac{1}{a^2}\left[\sinh a - \operatorname{shi}(a)\right].$$

4.9.6. Integrals containing $I_\nu(z)$ and special functions

1. $$\int_0^\infty x^{s-1} e^x \operatorname{Ei}(-2x)\, I_\nu(x)\, dx$$
$$= -\frac{2^{-s}\pi^{1/2}}{s+\nu}\sec(\nu\pi)\frac{\Gamma(s+\nu)}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1+2\nu)}\, {}_3F_2\left(\begin{matrix}\nu+\frac{1}{2},\, s+\nu,\, s+\nu\\ s+\nu+1,\, 2\nu+1;\, 1\end{matrix}\right)$$
$[\operatorname{Re}(s-\nu) > 0].$

2. $$\int_0^a e^{bx} \operatorname{erf}\left(\sqrt{2b(a-x)}\right) I_0(bx)\, dx$$
$$= \left(\pi a^3 b\right)^{1/2}\left[I_{-1/4}(ab)\, I_{1/4}(ab) - I_{-3/4}(ab)\, I_{3/4}(ab)\right]$$
$[\operatorname{Re} a > 0].$

3. $$\int_0^a e^{-bx} \operatorname{erf}\left(\sqrt{2b(a-x)}\right) I_0(bx)\, dx = \sqrt{\frac{2a}{b\pi}} - \frac{e^{-2ab}}{2b}\operatorname{erfi}\left(\sqrt{2ab}\right).$$

4. $$\int_0^a e^{bx} \operatorname{erfi}\left(\sqrt{2b(a-x)}\right) I_0(bx)\, dx = \frac{e^{2ab}}{2b}\operatorname{erf}\left(\sqrt{2ab}\right) - \sqrt{\frac{2a}{b\pi}}\,.$$

5. $$\int_0^a x^{\mu/2}(a-x)^{\nu/2} J_\mu(b\sqrt{x})\, I_\nu(b\sqrt{a-x})\, dx = \frac{a^{\mu+\nu+1} b^{\mu+\nu}}{2^{\mu+\nu}\Gamma(\mu+\nu+2)}$$
$[a > 0;\ \operatorname{Re}\mu,\, \operatorname{Re}\nu > -1].$

6. $$\int_0^{m\pi} \left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\} \sin^{-\nu} x\, J_\nu\left(b\sqrt{\sin x}\right) I_\nu\left(b\sqrt{\sin x}\right) dx = 2\sin\frac{m\pi a}{2}$$
$$\times \left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\} \frac{\left(\frac{b}{2}\right)^{2\nu}}{\Gamma^2(\nu+1)a}\, {}_2F_5\left(\begin{matrix}\frac{1}{2},\, 1;\, -\frac{b^4}{64}\\ \frac{\nu+1}{2},\, \frac{\nu}{2}+1,\, \nu+1,\, 1-\frac{a}{2},\, 1+\frac{a}{2}\end{matrix}\right).$$

7. $\displaystyle\int_0^{m\pi} e^{-ax}\sin^{-\nu}x\,J_\nu\big(b\sqrt{\sin x}\big)\,I_\nu\big(b\sqrt{\sin x}\big)\,dx$

$$= (1-e^{-m\pi a})\frac{\left(\frac{b}{2}\right)^{2\nu}}{\Gamma^2(\nu+1)a}\,{}_2F_5\left(\begin{matrix}\frac{1}{2},\,1;\;-\frac{b^4}{64}\\ \frac{\nu+1}{2},\,\frac{\nu}{2}+1,\,\nu+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right).$$

8. $\displaystyle\int_0^{\infty} e^{-ax}\sin^{-\nu}x\,J_\nu\big(b\sqrt{\sin x}\big)\,I_\nu\big(b\sqrt{\sin x}\big)\,dx$

$$= \frac{\left(\frac{b}{2}\right)^{2\nu}}{\Gamma^2(\nu+1)a}\,{}_2F_5\left(\begin{matrix}\frac{1}{2},\,1;\;-\frac{b^4}{64}\\ \frac{\nu+1}{2},\,\frac{\nu}{2}+1,\,\nu+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)\qquad [\operatorname{Re}a>0].$$

9. $\displaystyle\int_0^{a} x^{s-1}\ln\frac{a^2+\sqrt{a^4-x^4}}{x^2}\,J_\nu(bx)\,I_\nu(bx)\,dx$

$$= \frac{2^{-2\nu-3}\pi^{1/2}a^{s+2\nu}b^{2\nu}\Gamma\left(\frac{s+2\nu}{4}\right)}{(s+2\nu)\Gamma\left(\frac{s+2\nu+2}{4}\right)\Gamma^2(\nu+1)}$$

$$\times\,{}_2F_5\left(\begin{matrix}\frac{s+2\nu}{4},\,\frac{s+2\nu}{4};\;-\frac{a^4b^4}{64}\\ \frac{s+2\nu+2}{2},\,\frac{s+2\nu}{2}+1,\,\frac{\nu+1}{2},\,\frac{\nu}{2}+1,\,\nu+1\end{matrix}\right)\qquad [\operatorname{Re}(s+2\nu)>0].$$

10. $\displaystyle\int_{-1}^{1} e^{ax}J_0\Big(a\sqrt{1-x^2}\Big)\,J_0\big(b\sqrt{1-x}\big)\,I_0\big(b\sqrt{1+x}\big)\,dx = 2\,{}_1F_3\left(\begin{matrix}\frac{1}{2};\;\frac{ab^2}{2}\\ 1,\,1,\,\frac{3}{2}\end{matrix}\right).$

4.9.7. Integrals containing products of $I_\nu(z)$

1. $\displaystyle\int_0^{a} x^{\mu}(a-x)^{\nu}I_\mu(bx)\,I_\nu(b(a-x))\,dx$

$$= a^{\mu+\nu+1/2}\sqrt{\frac{b}{2\pi}}\,\mathrm{B}\left(\mu+\frac{1}{2},\,\nu+\frac{1}{2}\right)I_{\mu+\nu+1/2}(ab)$$

$$[a>0;\ \operatorname{Re}\mu,\,\operatorname{Re}\nu>-1/2].$$

2. $\displaystyle\int_0^{a} I_0^2\big(b\sqrt{x(a-x)}\big)\,dx = a\,I_0(ab)+\frac{\pi a}{2}[I_0(ab)\,\mathbf{L}_1(ab)-I_1(ab)\,\mathbf{L}_0(ab)]$

$$[a>0].$$

3. $\displaystyle\int_0^{a} x\,I_0^2\big(b\sqrt{x(a-x)}\big)\,dx = \frac{a^2}{2}I_0(ab)+\frac{\pi a^2}{4}[I_0(ab)\,\mathbf{L}_1(ab)-I_1(ab)\,\mathbf{L}_0(ab)]$

$$[a>0].$$

4. $$\int_0^a I_1^2\left(b\sqrt{x(a-x)}\right)dx = \frac{2}{b}I_1(ab) - a\,I_0(ab) - \frac{\pi a}{2}\left[I_0(ab)\,\mathbf{L}_1(ab) - I_1(ab)\,\mathbf{L}_0(ab)\right] \quad [a>0].$$

5. $$\int_0^a x^{-1/2} I_0^2\left(b\sqrt[4]{x(a-x)}\right)dx = 2a^{1/2}I_0(b\sqrt{2a}) + \pi a^{1/2}\left[I_0(b\sqrt{2a})\,\mathbf{L}_1(b\sqrt{2a}) - I_1(b\sqrt{2a})\,\mathbf{L}_0(b\sqrt{2a})\right] \quad [a>0].$$

6. $$\int_0^a x^{-1/2} I_1^2\left(b\sqrt[4]{x(a-x)}\right)dx = -2a^{1/2}I_0(b\sqrt{2a}) + \frac{2^{3/2}}{b}I_1(b\sqrt{2a}) + \pi a^{1/2}\left[I_1(b\sqrt{2a})\,\mathbf{L}_0(b\sqrt{2a}) - I_0(b\sqrt{2a})\,\mathbf{L}_1(b\sqrt{2a})\right] \quad [a>0].$$

7. $$\int_0^a x^{-1/4}(a-x)^{1/4} I_0\left(b\sqrt[4]{x(a-x)}\right) I_1\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi a^{1/2}}{2b}\left[I_1(b\sqrt{2a})\,\mathbf{L}_0(b\sqrt{2a}) - I_0(b\sqrt{2a})\,\mathbf{L}_1(b\sqrt{2a})\right] \quad [a>0].$$

8. $$\int_0^a x^{-3/4}(a-x)^{-1/4} I_0\left(b\sqrt[4]{x(a-x)}\right) I_1\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{2}{a^{1/2}b}\left[I_0(b\sqrt{2a}) - 1\right] \quad [a>0].$$

9. $$\int_0^a x^{-1}(a-x)^{-1/2} I_1^2\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{2^{3/2}}{ab}I_1(b\sqrt{2a}) - \frac{2}{a^{1/2}} \quad [a>0].$$

10. $$\int_0^a x^{\nu}(a-x)^{\nu}e^{2bx}I_\nu(bx)\,I_\nu(b(a-x))\,dx = \left(\frac{a}{2}\right)^{4\nu+1} b^{2\nu}e^{ab}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(2\nu+\frac{3}{2}\right)}\,{}_1F_2\left(\begin{matrix}\nu+\frac{1}{2};\ a^2b^2\\ 2\nu+1,\,2\nu+\frac{3}{2}\end{matrix}\right) \quad [a>0;\ \operatorname{Re}\nu>-1/2].$$

11. $$\int_0^a e^{2bx}I_0(bx)\,I_0(b(a-x))\,dx = ae^{ab}\left[I_0(2ab) + \frac{\pi}{2}I_0(2ab)\,\mathbf{L}_1(2ab) - \frac{\pi}{2}I_1(2ab)\mathbf{L}_0(2ab)\right].$$

12. $$\int_0^1 x\ln x\,I_0^2(ax)\,dx = \frac{1}{2}\left[I_1^2(a) - I_0^2(a) + \frac{1}{a}I_0(a)\,I_1(a)\right].$$

13. $$\int_0^1 x\ln x\, I_1^2(ax)\,dx = \frac{1}{2a^2}[1+(a^2-1)\,I_0^2(a) - a\,I_0(a)\,I_1(a) - a^2 I_1^2(a)].$$

14. $$\int_0^a x\ln\frac{a+\sqrt{a^2-x^2}}{x}\,I_0^2(bx)\,dx = \frac{a}{2b}[2ab\,I_0(2ab) - I_1(2ab)] + \frac{\pi a^2}{2}[I_0(2ab)\,\mathbf{L}_1(2ab) - I_1(2ab)\,\mathbf{L}_0(2ab)] \quad [a>0].$$

15. $$\int_0^a x^3\ln\frac{a+\sqrt{a^2-x^2}}{x}\,I_0^2(bx)\,dx = \frac{a}{12b^3}[2ab(a^2b^2+2)\,I_0(2ab) - (a^2b^2+4)\,I_1(2ab)] + \frac{\pi a^2}{48b^2}(4a^2b^2+3)\,[I_0(2ab)\mathbf{L}_1(2ab) - I_1(2ab)\,\mathbf{L}_0(2ab)] \quad [a>0].$$

16. $$\int_0^a x^{(\nu-1)/2}(a-x)^\nu J_\nu(2b\sqrt{x})\,I_\nu^2(b\sqrt{a-x})\,dx = \frac{\Gamma^2\left(\nu+\frac{1}{2}\right)a^{3\nu+1/2}b^{3\nu}}{\pi^{1/2}\Gamma^2(\nu+1)\Gamma\left(3\nu+\frac{3}{2}\right)}\,{}_2F_5\left(\begin{matrix}\frac{1}{2},\,\nu+\frac{1}{2};\;\frac{a^2b^4}{16}\\ \frac{\nu+1}{2},\,\frac{\nu}{2}+1,\,\frac{6\nu+3}{4},\,\frac{6\nu+5}{4},\,\nu+1\end{matrix}\right) \quad [a>0].$$

17. $$\int_0^a x^\nu(a-x)^\nu J_\nu^2(b\sqrt{x})\,I_\nu^2(b\sqrt{a-x})\,dx = \frac{2^{-6\nu-1}a^{4\nu+1}b^{4\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma^3(\nu+1)\Gamma\left(2\nu+\frac{3}{2}\right)} \times {}_2F_5\left(\begin{matrix}\frac{1}{2},\,\nu+\frac{1}{2};\;\frac{a^2b^4}{16}\\ \frac{\nu+1}{2},\,\frac{\nu}{2}+1,\,\nu+1,\,2\nu+1,\,2\nu+\frac{3}{2}\end{matrix}\right) \quad [a>0].$$

18. $$\int_{-1}^1 J_0\left(a\sqrt{1+x}\right)J_0\left(b\sqrt{1+x}\right)I_0\left(a\sqrt{1-x}\right)I_0\left(b\sqrt{1-x}\right)dx = 2\,{}_1F_4\left(\begin{matrix}\frac{1}{2};\;\frac{a^2b^2}{4}\\ 1,\,1,\,1,\,\frac{3}{2}\end{matrix}\right).$$

19. $$\int_0^a x^{-\nu}(a-x)^{-\nu}J_\nu(b\sqrt{x})\,J_{-\nu}(b\sqrt{x})\,I_\nu(b\sqrt{a-x})\,I_{-\nu}(b\sqrt{a-x})\,dx = \frac{\sin^2(\nu\pi)\Gamma(1-\nu)}{\nu^2\pi^{3/2}\Gamma\left(\frac{3}{2}-\nu\right)}\left(\frac{2}{a}\right)^{2\nu-1}{}_2F_5\left(\begin{matrix}\frac{1}{2},\,\nu+\frac{1}{2};\;\frac{a^2b^4}{16}\\ 1-\nu,\,\frac{3}{2}-\nu,\,\frac{\nu+1}{2},\,\frac{\nu}{2}+1,\,\nu+1\end{matrix}\right) \quad [a>0].$$

4.10. The Macdonald Function $K_\nu(z)$

4.10.1. Integrals containing $K_\nu(z)$, $J_\nu(z)$, $Y_\nu(z)$ and $I_\nu(z)$

1. $$\int_0^a \frac{1}{x} I_1(x)\, K_0(a-x)\,dx = \frac{2}{\pi a} I_0(a) - K_1(a).$$

2. $$\int_0^\infty e^{-ax}\left[\frac{1}{bx} I_0(bx) - K_1(bx)\right] dx = \frac{1}{b}\left(\frac{a}{\sqrt{a^2-b^2}} - 1\right) \ln \frac{a+\sqrt{a^2-b^2}}{b}$$

[$\operatorname{Re} a > |\operatorname{Re} b|$].

3. $$\int_0^\infty x\, J_1(ax)\, I_1(ax)\, Y_0(bx)\, K_0(bx)\, dx = -\frac{1}{2\pi a^2} \ln\left(1 - \frac{a^4}{b^4}\right) \qquad [0 < a < b].$$

4.10.2. Integrals containing products of $K_\nu(z)$

1. $$\int_0^\infty \frac{x^{\nu-2}}{(x+a)^\nu} K_0(x)\, K_\nu(x+a)\, dx$$
$$= 2^{-\nu}\sqrt{\pi}\, \frac{\Gamma^2(\nu-1)}{\Gamma(\nu+\frac{1}{2})} a^{-\nu-1} \left[\nu(\nu-1)\, K_0(a) + (2\nu-1) a\, K_1(a)\right]$$

[$|\arg a| < \pi$, $\operatorname{Re}\nu > 1$].

2. $$\int_0^\infty x\, K_0^3(x)\, dx = \frac{1}{6}\psi'\left(\frac{1}{3}\right) - \frac{\pi^2}{9} \qquad [37].$$

3. $$\int_0^\infty x\, K_0^4(x)\, dx = \frac{7}{8}\zeta(3) \qquad [37].$$

4. $$\int_0^\infty x^3 K_0^4(x)\, dx = -\frac{3}{16} + \frac{7}{32}\zeta(3) \qquad [37].$$

5. $$\int_0^\infty x^5 K_0^4(x)\, dx = -\frac{27}{64} + \frac{49}{128}\zeta(3) \qquad [37].$$

6. $$\int_0^\infty x^7 K_0^4(x)\, dx = -\frac{37}{16} + \frac{63}{32}\zeta(3) \qquad [37].$$

7. $$\int_0^\infty x^3 K_0^2(x)\, K_1^2(x)\, dx = \frac{1}{16} + \frac{7}{32}\zeta(3) \qquad [37].$$

8. $$\int_0^\infty x^2 K_0^3(x)\, K_1(x)\, dx = \frac{7}{16}\zeta(3) \qquad [37].$$

9. $$\int_0^\infty x^4 K_0^3(x)\, K_1(x)\, dx = -\frac{3}{16} + \frac{7}{32}\zeta(3) \qquad [37].$$

10. $$\int_0^\infty x^4 K_0(x)\, K_1^3(x)\, dx = \frac{1}{4} \qquad [37].$$

11. $$\int_0^\infty x^7\, K_1^4(x)\, dx = \frac{201}{64} - \frac{315}{128}\zeta(3) \qquad [37].$$

4.11. The Struve Functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

4.11.1. Integrals containing $\mathbf{H}_\nu(z)$, $\mathbf{L}_\nu(z)$ and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1} \left\{ \begin{matrix} \mathbf{H}_\nu\left(b\sqrt{x(a-x)}\right) \\ \mathbf{L}_\nu\left(b\sqrt{x(a-x)}\right) \end{matrix} \right\} dx$$
$$= 2^{-\nu} a^{s+t+\nu} b^{\nu+1} \frac{\Gamma(s+\frac{\nu+1}{2})\Gamma(t+\frac{\nu+1}{2})}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)\Gamma(s+t+\nu+1)}$$
$$\times\, {}_3F_4\left(\begin{matrix} 1,\, s+\frac{\nu+1}{2},\, t+\frac{\nu+1}{2};\ \mp\frac{a^2b^2}{16} \\ \frac{3}{2},\, \nu+\frac{3}{2},\, \frac{s+t+\nu+1}{2},\, \frac{s+t+\nu}{2}+1 \end{matrix} \right)$$
$[a>0;\ \operatorname{Re}(s+\nu), \operatorname{Re}(s+\nu) > -1]$.

2. $$\int_0^a x^{-\nu/2}(a-x)^{-\nu/2}\, \mathbf{L}_\nu\left(b\sqrt{x(a-x)}\right) dx$$
$$= 2^\nu \sqrt{\pi}\, a^{1/2-\nu} b^{-1/2} I_{\nu-1/2}\left(\frac{ab}{2}\right) - \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{b}{2}\right)^{\nu-1} \qquad [a>0].$$

3. $$\int_0^a \mathbf{L}_0\left(b\sqrt{x(a-x)}\right) dx = \frac{2}{b}\left[\cosh\left(\frac{ab}{2}\right) - 1\right] \qquad [a>0].$$

4. $$\int_0^a x^{s+1/2}(a-x)^s \left\{ \begin{matrix} \mathbf{H}_\nu\left(b\sqrt[4]{x(a-x)}\right) \\ \mathbf{L}_\nu\left(b\sqrt[4]{x(a-x)}\right) \end{matrix} \right\} dx = 2^{-2s-3(\nu+1)/2} a^{2s+\nu/2+2} b^{\nu+1}$$
$$\times \frac{\Gamma\left(2s+\frac{\nu+5}{2}\right)}{\Gamma\left(2s+\frac{\nu}{2}+3\right)\Gamma\left(\nu+\frac{3}{2}\right)}\, {}_2F_3\left(\begin{matrix} 1,\, 2s+\frac{\nu+5}{2};\ \mp\frac{ab^2}{8} \\ \frac{3}{2},\, \nu+\frac{3}{2},\, 2s+\frac{\nu}{2}+3 \end{matrix} \right)$$
$[a>0;\ \operatorname{Re}(4s+\nu) > -5]$.

5. $$\int_0^a x^{-1/2}\,\mathbf{H}_0\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx = \frac{2^{3/2}}{b}\left[1-\cos\left(b\sqrt{\frac{a}{2}}\right)\right] \qquad [a>0].$$

6. $$\int_0^a x^{-1/2}(a-x)^{-1}\,\mathbf{H}_0\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx = \frac{4}{\sqrt{a}}\operatorname{Si}\left(b\sqrt{\frac{a}{2}}\right) \qquad [a>0].$$

7. $$\int_0^a x^{-1/4}(a-x)^{1/4}\,\mathbf{H}_1\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx$$
$$= \frac{2^{3/2}}{b^2}\left[1-\cos\left(b\sqrt{\frac{a}{2}}\right)-\frac{ab^2}{4}\sin\left(b\sqrt{\frac{a}{2}}\right)\right]+\frac{a}{2^{1/2}} \qquad [a>0].$$

8. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\,\mathbf{H}_1\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx = -\frac{4}{\sqrt{a}\,b}\sin\left(b\sqrt{\frac{a}{2}}\right)+2^{3/2} \qquad [a>0].$$

9. $$\int_0^a x^{-3/4}(a-x)^{-5/4}\,\mathbf{H}_1\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx$$
$$= \frac{2}{a^{3/2}b}\left[-2\sin\left(b\sqrt{\frac{a}{2}}\right)+\sqrt{2a}\,b\cos\left(b\sqrt{\frac{a}{2}}\right)+ab^2\operatorname{Si}\left(b\sqrt{\frac{a}{2}}\right)\right] \qquad [a>0].$$

10. $$\int_0^a x^{-1/2}\,\mathbf{L}_0\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx = \frac{2^{3/2}}{b}\left[\cosh\left(b\sqrt{\frac{a}{2}}\right)-1\right] \qquad [a>0].$$

11. $$\int_0^a x^{-1/2}(a-x)^{-1}\,\mathbf{L}_0\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx = \frac{4}{\sqrt{a}}\operatorname{shi}\left(b\sqrt{\frac{a}{2}}\right) \qquad [a>0].$$

12. $$\int_0^a x^{-1/4}(a-x)^{1/4}\,\mathbf{L}_1\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx$$
$$= \frac{2^{3/2}}{b^2}\left[1-\cosh\left(b\sqrt{\frac{a}{2}}\right)+\frac{ab^2}{4}\sinh\left(b\sqrt{\frac{a}{2}}\right)\right]-\frac{a}{2^{1/2}} \qquad [a>0].$$

13. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\,\mathbf{L}_1\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx = \frac{4}{\sqrt{a}\,b}\sinh\left(b\sqrt{\frac{a}{2}}\right)-2^{3/2} \qquad [a>0].$$

14. $$\int_0^a x^{-3/4}(a-x)^{-5/4}\,\mathbf{L}_1\bigl(b\sqrt[4]{x(a-x)}\bigr)\,dx$$
$$= \frac{2}{a^{3/2}b}\left[2\sinh\left(b\sqrt{\frac{a}{2}}\right)-\sqrt{2a}\,b\cosh\left(b\sqrt{\frac{a}{2}}\right)+ab^2\operatorname{shi}\left(b\sqrt{\frac{a}{2}}\right)\right] \qquad [a>0].$$

4.11.2. Integrals containing $\mathbf{H}_\nu(z)$ and hyperbolic functions

1. $\displaystyle\int_0^a \frac{x^{\nu/2}}{\sqrt{a-x}} \sinh\left(b\sqrt{a-x}\right) \mathbf{H}_\nu(b\sqrt{x})\,dx$

$$= \frac{a^{\nu+3/2} b^{\nu+2}}{2^\nu \sqrt{\pi}\,\Gamma\left(\nu+\frac{5}{2}\right)}\, {}_2F_5\left(\begin{matrix} \frac{1}{2},\, 1;\; \frac{a^2b^4}{256} \\ \frac{3}{4},\, \frac{5}{4},\, \frac{3}{2},\, \frac{2\nu+5}{4},\, \frac{2\nu+7}{4} \end{matrix}\right) \quad [a>0;\ \operatorname{Re}\nu > -5/2].$$

4.11.3. Integrals containing $\mathbf{H}_\nu(z)$, $\mathbf{L}_\nu(z)$ and trigonometric functions

1. $\displaystyle\int_0^{\pi/2} \sin^{-\nu-1} x \cos(2nx)\, \mathbf{H}_\nu(a\sin x)\,dx$

$$= \frac{2^{-4n-\nu-2}\pi a^{2n+\nu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)}\, {}_2F_3\left(\begin{matrix} n+\frac{1}{2},\, n+1;\; -\frac{a^2}{4} \\ n+\nu+\frac{3}{2},\, n+\frac{3}{2},\, 2n+1 \end{matrix}\right).$$

2. $\displaystyle\int_0^{\pi} \cos^{-\nu-1} x \cos(nx)\, \mathbf{H}_\nu(a\cos x)\,dx$

$$= \frac{2^{-2n-\nu-1}\pi a^{n+\nu+1}}{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{n+3}{2}+\nu\right)} \cos\frac{n\pi}{2}\, {}_2F_3\left(\begin{matrix} \frac{n+1}{2},\, \frac{n}{2}+1;\; -\frac{a^2}{4} \\ \frac{n+3}{2}+\nu,\, \frac{n+3}{2},\, n+1 \end{matrix}\right).$$

3. $\displaystyle\int_0^{\pi} \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \mathbf{H}_\nu(b\sin x)\,dx$

$$= \frac{2^{-\mu-2\nu-1}\sqrt{\pi}\, b^{\nu+1}\Gamma(\mu+\nu+2)}{\Gamma\left(\frac{\mu+\nu-a+3}{2}\right)\Gamma\left(\frac{\mu+\nu+a+3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}$$

$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_3F_4\left(\begin{matrix} 1,\, \frac{\mu+\nu}{2}+1,\, \frac{\mu+\nu+3}{2};\; -\frac{b^2}{4} \\ \frac{3}{2},\, \frac{\mu+\nu-a+3}{2},\, \frac{\mu+\nu+a+3}{2},\, \nu+\frac{3}{2} \end{matrix}\right)$$

$[\operatorname{Re}(\mu+\nu) > -(5\pm 1)/2]$.

4. $\displaystyle\int_0^{\pi} \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \mathbf{L}_\nu(b\sin x)\,dx$

$$= \frac{2^{-\mu-2\nu-1}\sqrt{\pi}\, b^{\nu+1}\Gamma(\mu+\nu+2)}{\Gamma\left(\frac{\mu+\nu-a+3}{2}\right)\Gamma\left(\frac{\mu+\nu+a+3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}$$

$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_3F_4\left(\begin{matrix} 1,\ \frac{\mu+\nu}{2}+1,\ \frac{\mu+\nu+3}{2};\ \frac{b^2}{4} \\ \frac{3}{2},\ \frac{\mu+\nu-a+3}{2},\ \frac{\mu+\nu+a+3}{2},\ \nu+\frac{3}{2} \end{matrix} \right)$$

$$[\operatorname{Re}(\mu+\nu) > -(5\pm 1)/2].$$

5. $$\int_0^{m\pi} \sin^{-\nu-1} x \sin(ax) \begin{Bmatrix} \mathbf{H}_\nu(b\sin x) \\ \mathbf{L}_\nu(b\sin x) \end{Bmatrix} dx$$

$$= [1-\cos(m\pi a)] \frac{2^{-\nu}\pi^{-1/2}b^{\nu+1}}{\Gamma\left(\nu+\frac{3}{2}\right)a} {}_3F_4\left(\begin{matrix} \frac{1}{2},\ 1,\ 1;\ \mp\frac{b^2}{4} \\ \frac{3}{2},\ \nu+\frac{3}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

6. $$\int_0^{m\pi} \sin^{-\nu-1} x \cos(ax) \begin{Bmatrix} \mathbf{H}_\nu(b\sin x) \\ \mathbf{L}_\nu(b\sin x) \end{Bmatrix} dx$$

$$= \sin(m\pi a) \frac{2^{-\nu}\pi^{-1/2}b^{\nu+1}}{\Gamma\left(\nu+\frac{3}{2}\right)a} {}_3F_4\left(\begin{matrix} \frac{1}{2},\ 1,\ 1;\ \mp\frac{b^2}{4} \\ \frac{3}{2},\ \nu+\frac{3}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

7. $$\int_0^{m\pi} e^{-ax} \sin^{-\nu-1} x \begin{Bmatrix} \mathbf{H}_\nu(b\sin x) \\ \mathbf{L}_\nu(b\sin x) \end{Bmatrix} dx$$

$$= (1-e^{-m\pi a}) \frac{2^{-\nu}\pi^{-1/2}b^{\nu+1}}{\Gamma\left(\nu+\frac{3}{2}\right)a} {}_3F_4\left(\begin{matrix} \frac{1}{2},\ 1,\ 1;\ \mp\frac{b^2}{4} \\ \frac{3}{2},\ \nu+\frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix} \right).$$

8. $$\int_0^{\infty} e^{-ax} \sin^{-\nu-1} x \begin{Bmatrix} \mathbf{H}_\nu(b\sin x) \\ \mathbf{L}_\nu(b\sin x) \end{Bmatrix} dx$$

$$= \frac{2^{-\nu}\pi^{-1/2}b^{\nu+1}}{\Gamma(\nu+3/2)a} {}_3F_4\left(\begin{matrix} \frac{1}{2},\ 1,\ 1;\ \mp\frac{b^2}{4} \\ \frac{3}{2},\ \nu+\frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix} \right) \quad [\operatorname{Re} a > 0].$$

4.11.4. Integrals containing $\mathbf{H}_\nu(z)$, $\mathbf{L}_\nu(z)$ and the logarithmic function

1. $$\int_0^a x^{s-1} \ln\frac{a+\sqrt{a^2-x^2}}{x} \begin{Bmatrix} \mathbf{H}_\nu(bx) \\ \mathbf{L}_\nu(bx) \end{Bmatrix} dx = \frac{a^{s+\nu+1}\left(\frac{b}{2}\right)^{\nu+1}\Gamma\left(\frac{s+\nu+1}{2}\right)}{(s+\nu+1)\Gamma\left(\frac{s+\nu}{2}+1\right)\Gamma\left(\nu+\frac{3}{2}\right)}$$

$$\times {}_3F_4\left(\begin{matrix} 1,\ \frac{s+\nu+1}{2},\ \frac{s+\nu+1}{2};\ \mp\frac{a^2b^2}{4} \\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{s+\nu}{2}+1,\ \frac{s+\nu+3}{2} \end{matrix} \right) \quad [a>0;\ \operatorname{Re}(s+\nu) > -1].$$

2. $$\int_0^a x \ln\frac{a+\sqrt{a^2-x^2}}{x} \mathbf{H}_0(bx)\,dx = \frac{1}{b^2}[ab-\sin(ab)] \qquad [a>0].$$

3. $$\int_0^a x \ln \frac{a+\sqrt{a^2-x^2}}{x} \mathbf{L}_0(bx)\,dx = \frac{1}{b^2}\left[\sinh(ab) - ab\right] \qquad [a>0].$$

4.11.5. Integrals containing $\mathbf{H}_\nu(z)$, $\mathbf{L}_\nu(z)$ and inverse trigonometric functions

1. $$\int_0^1 x^{s-1} \arccos x \left\{ \begin{matrix} \mathbf{H}_\nu(ax) \\ \mathbf{L}_\nu(ax) \end{matrix} \right\} dx$$
$$= \frac{a^{\nu+1}\Gamma\left(\frac{s+\nu}{2}+1\right)}{2^\nu(s+\nu+1)^2\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right)}\, {}_3F_3\left(\begin{matrix} 1,\ \frac{s+\nu+1}{2},\ \frac{s+\nu}{2}+1;\ \mp\frac{a^2}{4} \\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{s+\nu+3}{2},\ \frac{s+\nu+3}{2} \end{matrix} \right)$$
$$[\operatorname{Re}(s+\nu) > -1].$$

2. $$\int_0^1 \arccos x\, \mathbf{H}_0(ax)\,dx = \frac{1}{a}[\mathbf{C} - \operatorname{ci}(a) + \ln a] \qquad [|\arg a| < \pi].$$

3. $$\int_0^1 \frac{1}{x} \arccos x\, \mathbf{H}_1(ax)\,dx = \mathbf{C} - 1 - \operatorname{ci}(a) + \ln a + \frac{1}{a}\sin a \qquad [|\arg a| < \pi].$$

4. $$\int_0^1 \arccos x\, \mathbf{L}_0(ax)\,dx = \frac{1}{a}[\operatorname{chi}(a) - \mathbf{C} - \ln a] \qquad [|\arg a| < \pi].$$

5. $$\int_0^1 \frac{1}{x} \arccos x\, \mathbf{L}_1(ax)\,dx = 1 - \mathbf{C} + \operatorname{chi}(a) - \ln a - \frac{1}{a}\sinh a \qquad [|\arg a| < \pi].$$

6. $$\int_0^a x^{\mu/2}(a-x)^{\nu/2}\, \mathbf{H}_\mu(b\sqrt{x})\mathbf{L}_\nu(b\sqrt{a-x})\,dx$$
$$= \frac{(ab)^{\mu+\nu+2}}{2^{\mu+\nu}\pi\Gamma(\mu+\nu+3)}\, {}_2F_5\left(\begin{matrix} \frac{1}{2},\ 1;\ \frac{a^2b^4}{256} \\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2},\ \mu+\nu+\frac{3}{2},\ \mu+\nu+2 \end{matrix} \right)$$
$$[a>0;\ \operatorname{Re}\mu,\ \operatorname{Re}\nu > -3/2].$$

4.12. The Kelvin Functions $\operatorname{ber}_\nu(z)$, $\operatorname{bei}_\nu(z)$, $\operatorname{ker}_\nu(z)$ and $\operatorname{kei}_\nu(z)$

4.12.1. Integrals containing $\operatorname{ber}_\nu(z)$, $\operatorname{bei}_\nu(z)$, $\operatorname{ker}_\nu(z)$, $\operatorname{kei}_\nu(z)$ and algebraic functions

1. $$\int_0^a \frac{1}{\sqrt{a^2-x^2}} \operatorname{ber}_\nu(bx)\,dx = \frac{\pi}{2}\left[\operatorname{ber}^2_{\nu/2}\left(\frac{ab}{2}\right) - \operatorname{bei}^2_{\nu/2}\left(\frac{ab}{2}\right)\right] \qquad [a>0].$$

2. $\displaystyle\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{bei}_\nu(bx)\,dx = \pi\,\mathrm{ber}_{\nu/2}\left(\frac{ab}{2}\right)\mathrm{bei}_{\nu/2}\left(\frac{ab}{2}\right)$ $[a>0]$.

3. $\displaystyle\int_a^\infty \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{ker}_\nu(bx)\,dx = \frac{1}{2}\left[\mathrm{ker}^2_{\nu/2}\left(\frac{ab}{2}\right)-\mathrm{kei}^2_{\nu/2}\left(\frac{ab}{2}\right)\right]$ $[a>0]$.

4. $\displaystyle\int_a^\infty \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{kei}_\nu(bx)\,dx = \mathrm{ker}_{\nu/2}\left(\frac{ab}{2}\right)\mathrm{kei}_{\nu/2}\left(\frac{ab}{2}\right)$ $[a>0]$.

4.13. The Airy Functions Ai (z) and Bi (z)

4.13.1. Integrals containing products of Ai (z) and Bi (z)

1. $\displaystyle\int_0^\infty x^{s-1}\,\mathrm{Ai}\,(x)\,\mathrm{Bi}\,(x)\,dx = \frac{2}{\pi^{1/2}}12^{-(2s+5)/6}\frac{\Gamma(s)\,\Gamma\left(\frac{1-2s}{6}\right)}{\Gamma\left(\frac{1+s}{3}\right)\Gamma\left(\frac{2-s}{3}\right)}$

$[0<\mathrm{Re}\,s<1/2;$ [65], (2.16)].

2. $\displaystyle\int_0^\infty x^{s-1}\,\mathrm{Ai}\,(x)\,\mathrm{Bi}\,(-x)\,dx = \frac{12^{(s-5)/6}}{\pi^{1/2}}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{6}\right)}{\Gamma\left(\frac{s+4}{6}\right)\Gamma\left(\frac{2-s}{6}\right)}$

$[\mathrm{Re}\,s>0;$ [65], (2.7)].

3. $\displaystyle\int_0^\infty x^{s-1}\,\mathrm{Ai}^2(x)\,dx = \frac{2^{-2(s+1)/3}3^{-(2s+5)/6}\Gamma(s)}{\pi^{1/2}\Gamma\left(\frac{2s+5}{6}\right)}$ $[\mathrm{Re}\,s>0]$.

4. $\displaystyle\int_0^\infty x^{s-1}\,\mathrm{Ai}\left(xe^{\pi i/6}\right)\mathrm{Ai}\left(xe^{-\pi i/6}\right)dx = \frac{2^{(s-8)/3}3^{(s-5)/6}}{\pi^{3/2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{6}\right)$

$[\mathrm{Re}\,s>0;$ [65], (4.5)].

5. $\displaystyle\int_0^\infty x^{s-1}\left[\mathrm{Ai}^2(-x)+\mathrm{Bi}^2(-x)\right]dx = \frac{4}{\pi^{3/2}}12^{-(2s+5)/6}\Gamma\,(s)\,\Gamma\left(\frac{1-2s}{6}\right)$

$[0<\mathrm{Re}\,s<1/2;$ [65], (2.28)].

6. $\displaystyle\int_0^\infty \mathrm{Ai}^3(x)\,dx = \frac{\Gamma^2\left(\frac{1}{3}\right)}{12\pi^2}-\frac{\Gamma\left(\frac{2}{3}\right)}{2^{5/3}\pi^2}\,{}_2F_1\left(\begin{matrix}\frac{1}{6},\ \frac{1}{3}\\ \frac{7}{6};\ \frac{1}{4}\end{matrix}\right)$ [[66], (2.11)].

7. $\displaystyle\int_0^\infty \mathrm{Bi}^3(-x)\,dx = -\frac{\Gamma^2\left(\frac{1}{3}\right)}{2\sqrt{3}\pi^2}+\frac{3^{3/2}\left(\frac{2}{3}\right)}{2^{5/3}\pi^2}\,{}_2F_1\left(\begin{matrix}\frac{1}{6},\ \frac{1}{3}\\ \frac{7}{6};\ \frac{1}{4}\end{matrix}\right)$ [[66], (3.15)].

8. $\displaystyle\int_0^\infty \operatorname{Ai}^2(x)\operatorname{Bi}(x)\,dx = \frac{\Gamma^2\left(\frac{1}{3}\right)}{12\sqrt{3}\,\pi^2} + \frac{\Gamma\left(\frac{2}{3}\right)}{2^{5/3}\sqrt{3}\,\pi^2}\,{}_2F_1\left(\begin{matrix}\frac{1}{6},\ \frac{1}{3}\\ \frac{7}{6};\ \frac{1}{4}\end{matrix}\right)$ [[66], (2.27)].

9. $\displaystyle\int_0^\infty \operatorname{Ai}^2(-x)\operatorname{Bi}(-x)\,dx = \frac{\Gamma^2\left(\frac{1}{3}\right)}{6\sqrt{3}\,\pi^2} - \frac{\Gamma\left(\frac{2}{3}\right)}{2^{5/3}\sqrt{3}\,\pi^2}\,{}_2F_1\left(\begin{matrix}\frac{1}{6},\ \frac{1}{3}\\ \frac{7}{6};\ \frac{1}{4}\end{matrix}\right)$ [[66], (2.34)].

10. $\displaystyle\int_0^\infty \operatorname{Ai}(-x)\operatorname{Bi}^2(-x)\,dx = \frac{\Gamma^2\left(\frac{1}{3}\right)}{6\pi^2} - \frac{\Gamma\left(\frac{2}{3}\right)}{2^{5/3}\pi^2}\,{}_2F_1\left(\begin{matrix}\frac{1}{6},\ \frac{1}{3}\\ \frac{7}{6};\ \frac{1}{4}\end{matrix}\right)$ [[66], (3.7)].

11. $\displaystyle\int_0^\infty \operatorname{Ai}^3(x)\operatorname{Bi}(x)\,dx = \frac{1}{24\pi}$ [[67], (3.7)].

12. $\displaystyle\int_0^\infty \operatorname{Ai}^3(-x)\operatorname{Bi}(-x)\,dx = \frac{1}{12\pi}$ [[67], (2.18)].

13. $\displaystyle\int_0^\infty \operatorname{Ai}(-x)\operatorname{Bi}^3(-x)\,dx = \frac{1}{12\pi}$ [[67], (4.7)].

14. $\displaystyle\int_0^\infty \operatorname{Ai}^4(x)\,dx = \frac{\ln 3}{24\pi^2}$ [57].

15. $\displaystyle\int_0^\infty x\operatorname{Ai}^4(x)\,dx = \frac{3^{-5/6}}{32\pi^3}\Gamma^2\left(\frac{1}{3}\right) - \frac{3^{5/3}}{128\pi^4}\Gamma^4\left(\frac{2}{3}\right)$ [57].

16. $\displaystyle\int_0^\infty x^2\operatorname{Ai}^4(x)\,dx = \frac{7}{1024\sqrt[3]{9}\,\pi^4}\Gamma^4\left(\frac{1}{3}\right) - \frac{3^{5/6}}{128\pi^3}\Gamma^2\left(\frac{2}{3}\right)$ [57].

4.14. The Legendre Polynomials $P_n(z)$

4.14.1. Integrals containing $P_n(z)$ and algebraic functions

1. $\displaystyle\int_a^1 (x-a)^{-1/2}P_n(x)\,dx = \frac{2}{2n+1}\sqrt{1-a}\,U_{2n}\left(\sqrt{\frac{a+1}{2}}\right)$ [Re $a > 0$].

2. $\displaystyle\int_0^a (a^2-x^2)^{-1/2}P_{2n}(bx)\,dx = (-1)^n\frac{\left(\frac{1}{2}\right)_n\pi}{n!2}P_{2n}\left(\sqrt{1-a^2b^2}\right)$ [Re $a > 0$].

3. $$\int_{-1}^{1} \frac{1}{\left(1-2ax+a^2\right)^{3/2}} P_n(x)\, dx = \frac{2a^n}{1-a^2}.$$

4. $$\int_{-1}^{1} \frac{1}{(2a-x)^{n+2}} P_n(x)\, dx = \frac{1}{n+1} \left(\frac{2}{4a^2-1}\right)^{n+1} \qquad \left[\left|\arg(4a^2-1) < \pi\right].$$

5. $$\int_{0}^{1} \frac{1}{\left(x^2+a^2\right)^{n+3/2}} P_{2n}(x)\, dx = \frac{(-1)^n}{2n+1} a^{-2n-2} (a^2+1)^{-n-1/2} \qquad [\operatorname{Re} a > 0].$$

6. $$\int_{0}^{1} \frac{x}{\left(x^2+a^2\right)^{n+5/2}} P_{2n+1}(x)\, dx = \frac{(-1)^n}{2n+3} a^{-2n-2} (a^2+1)^{-n-3/2} \qquad [\operatorname{Re} a > 0].$$

7. $$\int_{0}^{a} (a^2-x^2)^{s-1} (1-b^2x^2)^{-s-n-1/2} P_{2n}(bx)\, dx$$
$$= \frac{(-1)^n}{2} \mathrm{B}\left(n+\frac{1}{2},\, s\right) a^{2s-1} (1-a^2b^2)^{-n-1/2} P_n^{(s-1/2,0)}(1-2a^2b^2)$$
$$[a > 0].$$

8. $$\int_{0}^{a} x(a^2-x^2)^{s-1} (1-b^2x^2)^{-s-n-3/2} P_{2n+1}(bx)\, dx$$
$$= \frac{(-1)^n}{2} \mathrm{B}\left(n+\frac{3}{2},\, s\right) a^{2s+1} b(1-a^2b^2)^{-n-3/2} P_n^{(s+1/2,0)}(1-2a^2b^2)$$
$$[a > 0].$$

9. $$\int_{0}^{a} (1-b^2x^2)^{-n-3/2} P_{2n}(bx)\, dx = \frac{b^{-1}}{2n+1} (1-a^2b^2)^{-n-1/2} P_{2n+1}(ab).$$

10. $$\int_{-1}^{1} \frac{1}{1-x} \left[1-P_n(x)\right] dx = 2\mathbf{C} + 2\psi(n+1).$$

11. $$\int_{-1}^{1} \frac{1}{(1-x)^{3/2}} \left[1-P_n(x)\right] dx = 2^{3/2} n.$$

12. $$\int_{0}^{a} P_n(1+bx(a-x))\, dx = \frac{a}{2n+1} U_{2n}\left(\sqrt{1+\frac{a^2b}{8}}\right).$$

13. $$\int_0^a (a-x)^{s-1}(1+bx)^{-s-n-1}P_{2n}\left(\sqrt{1+bx}\right)dx = \frac{2(2n)!}{(2s)_{2n+1}}a^s(1+ab)^{-n-1}C_{2n}^{s+1/2}\left(\sqrt{1+ab}\right) \quad [a>0].$$

14. $$\int_0^a (a-x)^{-1/2}(1+bx)^{-n-3/2}P_{2n}\left(\sqrt{1+bx}\right)dx = \frac{2a^{1/2}}{2n+1}(1+ab)^{-n-1}U_{2n}\left(\sqrt{1+ab}\right) \quad [a>0].$$

15. $$\int_0^a x^{-1/2}(a-x)^{-1/2}P_{2n}\left(b\sqrt{x(a-x)}\right)dx = (-1)^n\pi\frac{\left(\frac{1}{2}\right)_n}{n!}P_{2n}\left(\sqrt{1-\frac{a^2b^2}{4}}\right) \quad [a>0].$$

16. $$\int_0^a x^{s-1}(a-x)^{s-1/2}P_n\left(1+b\sqrt{x(a-x)}\right)dx = \frac{2^{-2s+1}\sqrt{\pi}\,a^{2s-1/2}\Gamma(2s)}{\Gamma\left(2s+\frac{1}{2}\right)}\,{}_3F_2\left(\begin{matrix}-n,\, n+1,\, 2s\\ 1,\, 2s+\frac{1}{2};\; -\frac{ab}{4}\end{matrix}\right) \quad [a>0;\ \operatorname{Re}\nu>0].$$

17. $$\int_0^a x^{-1/2}P_n\left(1+b\sqrt{x(a-x)}\right)dx = \frac{2a^{1/2}}{2n+1}U_{2n}\left(\sqrt{1+\frac{ab}{4}}\right) \quad [a>0].$$

18. $$\int_0^a x^{-1/4}(a-x)^{-3/4}P_n\left(1+b\sqrt{x(a-x)}\right)dx = \sqrt{2}\pi\left[P_n\left(\sqrt{1+\frac{ab}{4}}\right)\right]^2 \quad [a>0].$$

19. $$\int_0^a P_{2n}\left(\sqrt{1-bx(a-x)}\right)dx = (-1)^n\frac{n!}{\left(\frac{3}{2}\right)_n}ab^{-1/2}\left[a\sqrt{b}\,P_{2n}\left(\frac{a\sqrt{b}}{2}\right)-2P_{2n-1}\left(\frac{a\sqrt{b}}{2}\right)\right] \quad [a>0;\ n\geq 1].$$

20. $$\int_0^a \frac{1}{\sqrt{1-bx(a-x)}}P_{2n+1}\left(\sqrt{1-bx(a-x)}\right)dx = \frac{2(-1)^n n!}{\left(\frac{3}{2}\right)_n b^{1/2}}P_{2n+1}\left(\frac{a\sqrt{b}}{2}\right) \quad [a>0].$$

21. $$\int_0^a x^{-1/2} P_{2n}\left(\sqrt{1-b\sqrt{x(a-x)}}\right) dx$$
$$= (-1)^{n+1} \frac{(n-1)!}{\left(\frac{1}{2}\right)_n} \left(\frac{2}{b}\right)^{1/2} \left[\sqrt{\frac{ab}{2}}\, P_{2n}\left(\sqrt{\frac{ab}{2}}\right) - P_{2n+1}\left(\sqrt{\frac{ab}{2}}\right)\right] \quad [a>0].$$

22. $$\int_0^a \frac{x^{-1/2}}{\sqrt{1-b\sqrt{x(a-x)}}} P_{2n+1}\left(\sqrt{1-b\sqrt{x(a-x)}}\right) dx$$
$$= (-1)^n \frac{n!}{\left(\frac{3}{2}\right)_n} \frac{2^{3/2}}{b^{1/2}} P_{2n+1}\left(\sqrt{\frac{ab}{2}}\right) \quad [a>0].$$

23. $$\int_0^a x^{-1/2}\left[1-b\sqrt{x(a-x)}\right]^n P_n\left(\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right) dx$$
$$= -\frac{2^{1-n} n!}{\left(\frac{3}{2}\right)_n} b^{-1/2}(ab-2)^{n+1/2} P_{2n+1}\left(\sqrt{\frac{ab}{ab-2}}\right) \quad [a>0].$$

24. $$\int_0^a x^{-1/4}(a-x)^{-3/4} P_{2n}\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= (-1)^n \sqrt{2}\,\pi \frac{\left(\frac{1}{2}\right)_n}{n!} P_{2n}\left(\sqrt{1-\frac{ab^2}{2}}\right) \quad [a>0].$$

25. $$\int_0^b x^{-n-3} e^{-a/x^2} P_n\left(\frac{x}{b}\right) dx = \frac{a^{-n/2-1}}{2^{n+1}} e^{-a/b^2} H_n\left(\frac{\sqrt{a}}{b}\right) \quad [\operatorname{Re} a>0].$$

4.14.2. Integrals containing $P_n(z)$ and trigonometric functions

1. $$\int_0^\pi \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_{2n}(b\sin x)\, dx$$
$$= (-1)^n \frac{2^{-\mu}\pi\Gamma(\mu+1)\left(\frac{1}{2}\right)_n}{n!\,\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix}$$
$$\times\, {}_4F_3\left(\begin{matrix} -n,\, n+\frac{1}{2},\, \frac{\mu+1}{2},\, \frac{\mu}{2}+1;\; b^2 \\ \frac{1}{2},\, \frac{\mu-a}{2}+1,\, \frac{\mu+a}{2}+1 \end{matrix}\right) \quad [\operatorname{Re}\mu>-1].$$

2. $$\int_0^\pi \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_{2n+1}(b \sin x)\, dx = (-1)^n \frac{2^{-\mu-1}\pi b\,\Gamma(\mu+2)\left(\frac{3}{2}\right)_n}{n!\,\Gamma\left(\frac{\mu-a+3}{2}\right)\Gamma\left(\frac{\mu+a+3}{2}\right)}$$
$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{\mu}{2}+1,\, \frac{\mu+3}{2};\; b^2 \\ \frac{3}{2},\, \frac{\mu-a+3}{2},\, \frac{\mu+a+3}{2} \end{matrix}\right) \qquad [\operatorname{Re}\mu > -2].$$

3. $$\int_0^\pi \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_n\left(\frac{b}{\sin x}\right) dx = \frac{2^{2n-\mu}\pi b^n \Gamma(\mu-n+1)\left(\frac{1}{2}\right)_n}{n!\,\Gamma\left(\frac{\mu-a-n}{2}+1\right)\Gamma\left(\frac{\mu+a-n}{2}+1\right)}$$
$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{\mu-n+1}{2},\, \frac{\mu-n}{2}+1;\; b^{-2} \\ \frac{1}{2}-n,\, \frac{\mu-a-n}{2}+1,\, \frac{\mu+a-n}{2}+1 \end{matrix}\right) \qquad [\operatorname{Re}\mu > n-1].$$

4. $$\int_0^\pi \sin x \sin(ax) P_{2n}\left(\sqrt{1+b\sin^2 x}\right) dx = \frac{\sin(\pi a)}{1-a^2}\, {}_3F_2\left(\begin{matrix} -n,\, n+\frac{1}{2},\, \frac{3}{2} \\ \frac{3-a}{2},\, \frac{3+a}{2};\; -b \end{matrix}\right).$$

5. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_{2n}(b\sin x)\, dx$$
$$= (-1)^n \frac{2\left(\frac{1}{2}\right)_n}{n!\,a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\, n+\frac{1}{2},\, 1;\; b^2 \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

6. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_n(\cos x)\, dx$$
$$= (-1)^n \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2};\; 1 \\ 1-a,\, 1+a \end{matrix}\right).$$

7. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_{2n+1}(b\sin x)\, dx$$
$$= (-1)^n \frac{2b}{n!\,a} \left(\frac{3}{2}\right)_n \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{1}{2},\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

8. $$\int_0^{m\pi} \frac{1}{\cos x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_{2n+1}(\cos x)\, dx$$
$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{1}{2};\; 1 \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

9. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_n(1 + b\sin^2 x)\,dx$$
$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2};\; -\frac{b}{2} \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

10. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_{2n}\left(\sqrt{1 + b\sin^2 x}\right)dx$$
$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\, n+\frac{1}{2},\, \frac{1}{2};\; -b \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

11. $$\int_0^{m\pi} \frac{1}{\sqrt{1 + b\sin^2 x}} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_{2n+1}\left(\sqrt{1 + b\sin^2 x}\right)dx$$
$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{1}{2};\; -b \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

12. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} (1 + b\sin^2 x)^{n/2} P_n\left(\frac{1}{\sqrt{1 + b\sin^2 x}}\right)dx$$
$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2};\; -b \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

13. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^n x\, P_n\left(\frac{b}{\sin x}\right)dx$$
$$= 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{\left(\frac{1}{2}\right)_n}{n!\,a} (2b)^n\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2},\, 1;\; b^{-2} \\ \frac{1}{2}-n,\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

14. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \cos^n x\, P_n\left(\frac{1}{\cos x}\right)dx$$
$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2};\; 1 \\ 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

15. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^{2n} x\, P_n(\cot^2 x)\,dx$$
$$= \frac{2^{n+1}\left(\frac{1}{2}\right)_n}{n!\,a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} -n,\, -n,\, \frac{1}{2},\, 1;\; 2 \\ -2n,\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

16. $$\int_0^{m\pi} e^{-ax} P_{2n}(b\sin x)\,dx$$
$$= (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!\,a}\,(1-e^{-m\pi a})\;{}_3F_2\left(\begin{matrix}-n,\ n+\frac{1}{2},\ 1;\ b^2\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

17. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} P_{2n+1}(b\sin x)\,dx$$
$$= (-1)^n \frac{\left(\frac{3}{2}\right)_n b}{n!\,a}\,(1-e^{-m\pi a})\;{}_4F_3\left(\begin{matrix}-n,\ n+\frac{3}{2},\ \frac{1}{2},\ 1;\ b^2\\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

18. $$\int_0^{m\pi} \frac{e^{-ax}}{\cos x} P_{2n+1}(\cos x)\,dx = \frac{1}{a}\,(1-e^{-m\pi a})\;{}_3F_2\left(\begin{matrix}-n,\ n+\frac{3}{2},\ \frac{1}{2};\ 1\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

19. $$\int_0^{m\pi} e^{-ax} P_n(1+b\sin^2 x)\,dx = \frac{1}{a}\,(1-e^{-m\pi a})\;{}_3F_2\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2};\ -\frac{b}{2}\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

20. $$\int_0^{m\pi} e^{-ax} P_{2n}\left(\sqrt{1+b\sin^2 x}\right)dx$$
$$= \frac{1}{a}(1-e^{-m\pi a})\;{}_3F_2\left(\begin{matrix}-n,\ n+\frac{1}{2},\ \frac{1}{2};\ -b\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

21. $$\int_0^{m\pi} \frac{e^{-ax}}{\sqrt{1+b\sin^2 x}} P_{2n+1}\left(\sqrt{1+b\sin^2 x}\right)dx$$
$$= \frac{1}{a}\,(1-e^{-m\pi a})\;{}_3F_2\left(\begin{matrix}-n,\ n+\frac{3}{2},\ \frac{1}{2};\ -b\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

22. $$\int_0^{m\pi} e^{-ax}(1+b\sin^2 x)^{n/2} P_n\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right)dx$$
$$= \frac{1}{a}\,(1-e^{-m\pi a})\;{}_3F_2\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2};\ -b\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

23. $$\int_0^{m\pi} e^{-ax}\sin^n x\, P_n\left(\frac{b}{\sin x}\right)dx = \frac{\left(\frac{1}{2}\right)_n}{n!\,a}(2b)^n\left(1-e^{-m\pi a}\right)\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ b^{-2}\\ \frac{1}{2}-n,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

24. $$\int_0^{m\pi} e^{-ax}\cos^n x\, P_n\left(\frac{1}{\cos x}\right)dx = \frac{1}{a}\left(1-e^{-m\pi a}\right)\,{}_3F_2\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2};\ 1\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

25. $$\int_0^{m\pi} e^{-ax}\sin^{2n} x\, P_n(\cot^2 x)\,dx = 2^n\left(1-e^{-m\pi a}\right)\frac{\left(\frac{1}{2}\right)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\ -n,\ \frac{1}{2},\ 1;\ 2\\ -2n,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

26. $$\int_0^{\infty} e^{-ax} P_{2n}(b\sin x)\,dx = (-1)^n\frac{\left(\frac{1}{2}\right)_n}{n!\,a}\,{}_3F_2\left(\begin{matrix}-n,\ n+\frac{1}{2},\ 1;\ b^2\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

27. $$\int_0^{\infty} e^{-ax} P_{2n+1}(b\sin x)\,dx = (-1)^n\frac{\left(\frac{3}{2}\right)_n}{n!}\frac{b}{a^2+1}\,{}_3F_2\left(\begin{matrix}-n,\ n+\frac{3}{2},\ 1;\ b^2\\ \frac{3-ia}{2},\ \frac{3+ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

28. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} P_{2n+1}(b\sin x)\,dx = (-1)^n\frac{\left(\frac{3}{2}\right)_n b}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\ n+\frac{3}{2},\ \frac{1}{2},\ 1;\ b^2\\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

29. $$\int_0^{\infty} \frac{e^{-ax}}{\cos x} P_{2n+1}(\cos x)\,dx = \frac{1}{a}\,{}_3F_2\left(\begin{matrix}-n,\ n+\frac{3}{2},\ \frac{1}{2};\ b\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

30. $$\int_0^{\infty} e^{-ax} P_n(1+b\sin^2 x)\,dx = \frac{1}{a}\,{}_3F_2\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2};\ -\frac{b}{2}\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

31. $$\int_0^{\infty} e^{-ax} P_{2n}\left(\sqrt{1+b\sin^2 x}\right)dx = \frac{1}{a}\,{}_3F_2\left(\begin{matrix}-n,\ n+\frac{1}{2},\ \frac{1}{2};\ -b\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

32. $$\int_0^\infty \frac{e^{-ax}}{\sqrt{1+b\sin^2 x}} P_{2n+1}\left(\sqrt{1+b\sin^2 x}\right) dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{1}{2};\, -b \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

33. $$\int_0^\infty e^{-ax}(1+b\sin^2 x)^{n/2} P_n\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right) dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2};\, -b \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

34. $$\int_0^\infty e^{-ax}\sin^n x\, P_n\left(\frac{b}{\sin x}\right) dx = \frac{\left(\frac{1}{2}\right)_n}{n!\, a}(2b)^n\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2},\, 1;\, b^{-2} \\ \frac{1}{2}-n,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

35. $$\int_0^\infty e^{-ax}\cos^n x\, P_n\left(\frac{1}{\cos x}\right) dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2};\, 1 \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

36. $$\int_0^\infty e^{-ax}\sin^{2n} x\, P_n(\cot^2 x)\, dx = 2^n \frac{\left(\frac{1}{2}\right)_n}{n!\, a}\, {}_4F_3\left(\begin{matrix} -n,\, -n,\, \frac{1}{2},\, 1;\, 2 \\ -2n,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

4.14.3. Integrals containing $P_n(z)$ and the logarithmic function

Condition: $a > 0$.

1. $$\int_{-a}^a (a-x)^{n-1} \ln\left(\frac{x+a}{2a}\right) P_n\left(\frac{x}{a}\right) dx = \frac{\left(-\frac{a}{2}\right)^n}{n^2}\left[\frac{n!}{\left(\frac{1}{2}\right)_n} - 2^{2n}\right].$$

2. $$\int_0^a \ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} P_n(1+bx)\, dx = \frac{1}{n(n+1)b}\left[\frac{(n+1)!}{\left(\frac{1}{2}\right)_{n+1}} P_{n+1}^{(-1/2,-3/2)}(1+ab) - 1\right].$$

3. $$\int_0^a \ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} P_{2n}(\sqrt{1-bx})\, dx = \frac{1}{(n+1)(2n-1)b}\left[1 - \frac{(n+1)!}{\left(\frac{1}{2}\right)_{n+1}} P_{n+1}^{(-1/2,-2)}(1-2ab)\right].$$

4. $$\int_0^a \ln \frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}} \frac{P_{2n+1}(\sqrt{1-bx})}{\sqrt{1-bx}}\,dx$$
$$= \frac{1}{(n+1)(2n+1)b}\left[1 - \frac{(n+1)!}{\left(\frac{1}{2}\right)_{n+1}} P_{n+1}^{(-1/2,\,-1)}(1-2ab)\right].$$

4.14.4. Integrals containing $P_n(z)$, $J_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$

1. $$\int_{-1}^1 e^{ax} J_0\left(a\sqrt{1-x^2}\right) P_n(x)dx = \frac{2a^n}{n!\,(2n+1)}.$$

2. $$\int_{-1}^1 J_0\left(a\sqrt{1-x}\right) I_0\left(b\sqrt{1+x}\right) P_n(x)dx = \frac{2^{1-n}a^n}{(n!)^2\,(2n+1)}.$$

3. $$\int_0^1 x^{-5/2} K_{n+1/2}\left(\frac{a}{x}\right) P_n(x)\,dx = \frac{\sqrt{\pi}}{4}\left(\frac{2}{a}\right)^{3/2} e^{-a} \qquad [\operatorname{Re} a > 0].$$

4. $$\int_0^1 x(1-x^2)^{-7/4} K_{2n+3/2}\left(\frac{a}{\sqrt{1-x^2}}\right) P_{2n+1}(x)\,dx$$
$$= (-1)^n \frac{2^{1/2}a^{-3/2}}{n!}\Gamma\left(n+\frac{3}{2}\right) K_0(a) \qquad [\operatorname{Re} a > 0].$$

5. $$\int_a^\infty K_0(2\sqrt{x})\, P_{2n+1}\left(i\sqrt{\frac{x}{a}-1}\right) dx$$
$$= (-1)^n i \frac{\Gamma\left(n+\frac{3}{2}\right)}{n!} a^{1/4} K_{2n+3/2}(2\sqrt{a}) \qquad [a > 0].$$

6. $$\int_{2a}^\infty K_0(\sqrt{x})\, P_n\left(\frac{x}{a}-1\right) dx = \sqrt{8a}\, K_{2n+1}\left(\sqrt{2a}\right) \qquad [a > 0].$$

4.14.5. Integrals containing products of $P_n(z)$

1. $$\int_1^a P_{2n}(x) P_{2n}\left(\frac{x}{a}\right) dx = \frac{1}{4n+1}\left(a^{2n+1} - a^{-2n}\right).$$

2. $$\int_1^a P_{2n+1}(x) P_{2n+1}\left(\frac{x}{a}\right) dx = \frac{1}{4n+3}\left(a^{2n+2} - a^{-2n-1}\right).$$

3. $$\int_1^a P_n(1-2x)P_n\left(1-\frac{2x}{a}\right)dx = \frac{1}{2n+1}(a^{n+1}-a^{-n}).$$

4. $$\int_a^1 \frac{1}{x}P_n(1-2x)P_n\left(1-\frac{2x}{a}\right)dx = -\ln a \qquad [a>0].$$

5. $$\int_a^1 \frac{1}{x^2}P_n(x)P_n\left(\frac{x}{a}\right)dx = \frac{1}{a}-1.$$

6. $$\int_0^{\pi/2} \cos^\nu x\cos(ax)\left[P_n\left(\sqrt{1+b\cos^2 x}\right)\right]^2 dx$$
$$= \frac{2^{-\nu-1}\pi\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}\, {}_5F_4\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2},\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1\\ 1,\ 1,\ \frac{\nu-a}{2}+1,\ \frac{\nu+a}{2}+1;\ -b\end{matrix}\right)$$
$$[\operatorname{Re}\nu>-1].$$

7. $$\int_0^{\pi} \sin^\nu x\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\left[P_n\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx = \frac{2^{-\nu}\pi\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}$$
$$\times\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}{}_5F_4\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2},\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1\\ 1,\ 1,\ \frac{\nu-a}{2}+1,\ \frac{\nu+a}{2}+1;\ -b\end{matrix}\right) \qquad [\operatorname{Re}\nu>-1].$$

8. $$\int_0^{m\pi} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}[P_n(\cos x)]^2\, dx$$
$$= \frac{2}{a}\sin\frac{m\pi a}{2}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}{}_4F_3\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2},\ \frac{1}{2};\ 1\\ 1,\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right).$$

9. $$\int_0^{m\pi} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\left[P_n\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx$$
$$= \frac{2}{a}\sin\frac{m\pi a}{2}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}{}_4F_3\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2},\ \frac{1}{2};\ -b\\ 1,\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right).$$

10. $$\int_0^{m\pi} e^{-ax}[P_n(\cos x)]^2\, dx = \frac{1}{a}\left(1-e^{-m\pi a}\right){}_4F_3\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2},\ \frac{1}{2};\ 1\\ 1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

11. $$\int_0^{m\pi} e^{-ax}\left[P_n\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx$$
$$= \frac{1}{a}\left(1-e^{-m\pi a}\right) {}_4F_3\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2},\, \frac{1}{2};\ -b \\ 1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

12. $$\int_0^{\infty} e^{-ax}[P_n(\cos x)]^2\, dx = \frac{1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2},\, \frac{1}{2};\ 1 \\ 1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \qquad [\operatorname{Re} a > 0].$$

13. $$\int_0^{\infty} e^{-ax}\left[P_n\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx = \frac{1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2},\, \frac{1}{2};\ -b \\ 1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$
$$[\operatorname{Re} a > 0].$$

14. $$\int_0^1 [P_m(x)]^2[P_n(x)]^2\, dx$$
$$= \frac{1}{(2m+1)(2n+1)}\, {}_9F_8\left(\begin{matrix} -m,\, m+1,\, -n,\, n+1,\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{5}{4};\ 1 \\ \frac{1}{2}-m,\, \frac{3}{2}+m,\, \frac{1}{2}-n,\, \frac{3}{2}+n,\, \frac{1}{4},\, 1,\, 1,\, 1 \end{matrix}\right).$$

4.15. The Chebyshev Polynomials $T_n(z)$

4.15.1. Integrals containing $T_n(z)$ and algebraic functions

1. $$\int_0^1 \frac{(1-x^2)^{-1/2}}{(x^2+a^2)^{n+1}} T_{2n}(x)\, dx = (-1)^n \pi \frac{\left(\frac{1}{2}\right)_n}{2(n!)} a^{-2n-1}(a^2+1)^{-n-1/2}$$
$$[\operatorname{Re} a > 0].$$

2. $$\int_0^1 \frac{x(1-x^2)^{-1/2}}{(x^2+a^2)^{n+2}} T_{2n+1}(x)\, dx = (-1)^n \pi \frac{\left(\frac{3}{2}\right)_n}{4(n+1)!} a^{-2n-1}(a^2+1)^{-n-3/2}$$
$$[\operatorname{Re} a > 0].$$

3. $$\int_0^a (a^2-x^2)^{-1/2}(1-b^2x^2)^{-n-1} T_{2n}(bx)\, dx$$
$$= (-1)^n \frac{\pi}{2}(1-a^2b^2)^{-n-1/2} P_{2n}\left(\sqrt{1-a^2b^2}\right) \qquad [a > 0].$$

4. $$\int_0^a x^{-1/2}(1-bx)^{-n-3/2} T_{2n}\left(\sqrt{1+bx}\right) dx$$
$$= \frac{2a^{1/2}}{2n+1}(1+ab)^{-n-1/2} U_{2n}\left(\sqrt{1+ab}\right) \qquad [a > 0].$$

5. $$\int_0^a x^{-1/2}(a-x)^{-1/2}(1+bx)^{-n-1}T_{2n}\left(\sqrt{1+bx}\right)dx = \pi(1+ab)^{-n-1/2}P_{2n}\left(\sqrt{1+ab}\right) \quad [a>0].$$

6. $$\int_0^a x^{-1/2}(a-x)^{-1/2}T_{2n}\left(b\sqrt{x(a-x)}\right)dx = (-1)^n\frac{\pi}{2}\left[P_n\left(1-\frac{a^2b^2}{2}\right)+P_{n-1}\left(1-\frac{a^2b^2}{2}\right)\right] \quad [n\ge 1;\ a>0].$$

7. $$\int_0^a \frac{x^{-1/2}(a-x)^{-1/2}}{\sqrt{1+bx(a-x)}}T_{2n+1}\left(\sqrt{1+bx(a-x)}\right)dx = \pi P_n\left(1+\frac{a^2b}{2}\right) \quad [a>0].$$

8. $$\int_0^a x^s(a-x)^{s-1/2}T_{2n}\left(b\sqrt[4]{x(a-x)}\right)dx = (-1)^n\frac{\pi^{1/2}a^{2s+1/2}\Gamma(2s+1)}{2^{2s}\Gamma\left(2s+\frac{3}{2}\right)}\,{}_3F_2\left(\begin{matrix}-n,\ n,\ 2s+1\\ \frac{1}{2},\ 2s+\frac{3}{2};\ \frac{ab^2}{2}\end{matrix}\right) \quad [a>0;\ \operatorname{Re}s>-1/2].$$

9. $$\int_0^a x^s(a-x)^{s-1/2}T_{2n+1}\left(b\sqrt[4]{x(a-x)}\right)dx = (-1)^n\frac{(2n+1)\pi^{1/2}a^{2s+1}b\,\Gamma\left(2s+\frac{3}{2}\right)}{2^{2s+1/2}\Gamma(2s+2)}\,{}_3F_2\left(\begin{matrix}-n,\ n+1,\ 2s+\frac{3}{2}\\ \frac{3}{2},\ 2s+2;\ \frac{ab^2}{2}\end{matrix}\right)$$ $$[a>0;\ \operatorname{Re}s>-3/4].$$

10. $$\int_0^a x^s(a-x)^{s-1/2}T_{2n}\left(\sqrt{1+b\sqrt{x(a-x)}}\right)dx = \frac{\pi^{1/2}a^{2s+1/2}\Gamma(2s+1)}{2^{2s}\Gamma\left(2s+\frac{3}{2}\right)}\,{}_3F_2\left(\begin{matrix}-n,\ n,\ 2s+1\\ \frac{1}{2},\ 2s+\frac{3}{2};\ -\frac{ab}{2}\end{matrix}\right) \quad [a>0;\ \operatorname{Re}s>-1/2].$$

11. $$\int_0^a \frac{x^s(a-x)^{s-1/2}}{\sqrt{1+b\sqrt{x(a-x)}}}T_{2n+1}\left(\sqrt{1+b\sqrt{x(a-x)}}\right)dx = \frac{\pi^{1/2}a^{2s+1/2}\Gamma(2s+1)}{2^{2s}\Gamma\left(2s+\frac{3}{2}\right)}\,{}_3F_2\left(\begin{matrix}-n,\ n+1,\ 2s+1\\ \frac{1}{2},\ 2s+\frac{3}{2};\ -\frac{ab}{2}\end{matrix}\right) \quad [a>0;\ \operatorname{Re}s>-1/2].$$

4.15.2. Integrals containing $T_n(z)$ and trigonometric functions

1. $$\int_0^\pi \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n}(b\sin x)\,dx$$
$$= (-1)^n \frac{2^{-\mu}\pi\Gamma(\mu+1)}{\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix}$$
$$\times\, {}_4F_3\left(\begin{matrix} -n,\ n,\ \frac{\mu+1}{2},\ \frac{\mu}{2}+1;\ b^2 \\ \frac{1}{2},\ \frac{\mu-a}{2}+1,\ \frac{\mu+a}{2}+1 \end{matrix}\right) \quad [\operatorname{Re}\mu > -1].$$

2. $$\int_0^\pi \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n+1}(b\sin x)\,dx$$
$$= (-1)^n \frac{2^{-\mu-1}(2n+1)\pi b\Gamma(\mu+2)}{\Gamma\left(\frac{\mu-a+3}{2}\right)\Gamma\left(\frac{\mu+a+3}{2}\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix}$$
$$\times\, {}_4F_3\left(\begin{matrix} -n,\ n+1,\ \frac{\mu}{2}+1,\ \frac{\mu+3}{2};\ b^2 \\ \frac{3}{2},\ \frac{\mu-a+3}{2},\ \frac{\mu+a+3}{2} \end{matrix}\right) \quad [\operatorname{Re}\mu > -2].$$

3. $$\int_0^\pi \sin x \sin(ax) T_{2n}\left(\sqrt{1+b\sin^2 x}\right)dx = \frac{\sin(\pi a)}{1-a^2}\, {}_4F_3\left(\begin{matrix} -n,\ n,\ 1,\ \frac{3}{2};\ -b \\ \frac{1}{2},\ \frac{3-a}{2},\ \frac{3+a}{2} \end{matrix}\right).$$

4. $$\int_0^\pi \frac{\sin x \sin(ax)}{\sqrt{1+b\sin^2 x}} T_{2n+1}\left(\sqrt{1+b\sin^2 x}\right)dx$$
$$= \frac{\sin(\pi a)}{1-a^2}\, {}_4F_3\left(\begin{matrix} -n,\ n+1,\ 1,\ \frac{3}{2};\ -b \\ \frac{1}{2},\ \frac{3-a}{2},\ \frac{3+a}{2} \end{matrix}\right).$$

5. $$\int_0^\pi \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_n\left(\frac{b}{\sin x}\right)dx = \frac{2^{2n-\mu}\pi b^n\Gamma(\mu-n+1)}{\Gamma\left(\frac{\mu-a-n}{2}+1\right)\Gamma\left(\frac{\mu+a-n}{2}+1\right)}$$
$$\times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{\mu-n+1}{2},\ \frac{\mu-n}{2}+1;\ b^{-2} \\ 1-n,\ \frac{\mu-a-n}{2}+1,\ \frac{\mu+a-n}{2}+1 \end{matrix}\right) \quad [\operatorname{Re}\mu > -1].$$

6. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n}(b\sin x)\,dx$$
$$= (-1)^n \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ n,\ 1;\ b^2 \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right).$$

7. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n+1}(b\sin x)\,dx$$

$$= 2(-1)^n \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{2n+1}{a} b\, {}_4F_3\left(\begin{matrix} -n,\ n+1,\ \frac{1}{2},\ 1;\ b^2 \\ \frac{3}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

8. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n}(\cos x)\,dx$$

$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ n,\ 1;\ 1 \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

9. $$\int_0^{m\pi} \frac{1}{\cos x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n+1}(\cos x)\,dx$$

$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ n+1,\ 1;\ 1 \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

10. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_n(1+b\sin^2 x)\,dx$$

$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ n,\ 1;\ -\frac{b}{2} \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

11. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n}\left(\sqrt{1+b\sin^2 x}\right)dx$$

$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ n,\ 1;\ -b \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

12. $$\int_0^{m\pi} \frac{1}{\sqrt{1+b\sin^2 x}} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} T_{2n+1}\left(\sqrt{1+b\sin^2 x}\right)dx$$

$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ n+1,\ 1;\ -b \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

13. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} (1+b\sin^2 x)^{n/2} T_n\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right)dx$$

$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ 1;\ -b \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

14. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^n x \, T_n\!\left(\frac{b}{\sin x}\right) dx$$
$$= \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{2^n b^n}{a} \, {}_4F_3\!\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}, \frac{1}{2}, 1;\ b^{-2} \\ 1-n, 1-\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right) \quad [n \geq 1].$$

15. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \cos^n x \, T_n\!\left(\frac{1}{\cos x}\right) dx$$
$$= \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\!\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}, 1;\ 1 \\ 1-\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right).$$

16. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \cos^{2n} x \, T_{2n}(i\tan x)\, dx$$
$$= (-1)^n \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\!\left(\begin{matrix} -n, \frac{1}{2}-n, 1;\ 1 \\ 1-\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right).$$

17. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^{2n} x \, T_n(\cot^2 x)\, dx$$
$$= \frac{2^n}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_3\!\left(\begin{matrix} -n, \frac{1}{2}-n, \frac{1}{2}, 1;\ 2 \\ 1-2n, 1-\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right) \quad [n \geq 2].$$

18. $$\int_0^{m\pi} e^{-ax} T_{2n}(b\sin x)\, dx = \frac{(-1)^n}{a} \left(1-e^{-m\pi a}\right) {}_3F_2\!\left(\begin{matrix} -n, n, 1;\ b^2 \\ 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

19. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} T_{2n+1}(b\sin x)\, dx$$
$$= (-1)^n (2n+1) \frac{b}{a} \left(1-e^{-m\pi a}\right) {}_4F_3\!\left(\begin{matrix} -n, n+1, \frac{1}{2}, 1;\ b^2 \\ \frac{3}{2}, 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

20. $$\int_0^{m\pi} e^{-ax} T_{2n}(\cos x)\, dx = \frac{1}{a} \left(1-e^{-m\pi a}\right) {}_3F_2\!\left(\begin{matrix} -n, n, 1;\ 1 \\ 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

21. $$\int_0^{m\pi} \frac{e^{-ax}}{\cos x} T_{2n+1}(\cos x)\, dx = \frac{1}{a} \left(1-e^{-m\pi a}\right) {}_3F_2\!\left(\begin{matrix} -n, n+1, 1;\ 1 \\ 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

22. $$\int_0^{m\pi} e^{-ax} T_n(1+b\sin^2 x)\, dx = \frac{1}{a} \left(1-e^{-m\pi a}\right) {}_3F_2\!\left(\begin{matrix} -n, n, 1;\ -\frac{b}{2} \\ 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

23. $$\int_0^{m\pi} e^{-ax} T_{2n}\left(\sqrt{1+b\sin^2 x}\right) dx = \frac{1}{a}(1-e^{-m\pi a})\, {}_3F_2\left(\begin{matrix} -n,\, n,\, 1;\ -b \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

24. $$\int_0^{m\pi} \frac{e^{-ax}}{\sqrt{1+b\sin^2 x}} T_{2n+1}\left(\sqrt{1+b\sin^2 x}\right) dx$$
$$= \frac{1}{a}\,(1-e^{-m\pi a})\, {}_3F_2\left(\begin{matrix} -n,\, n+1,\, 1;\ -b \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

25. $$\int_0^{m\pi} e^{-ax}(1+b\sin^2 x)^{n/2} T_n\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right) dx$$
$$= \frac{1}{a}\,(1-e^{-m\pi a})\, {}_3F_2\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, 1;\ -b \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

26. $$\int_0^{m\pi} e^{-ax}\sin^n x\, T_n\left(\frac{b}{\sin x}\right) dx$$
$$= (1-e^{-m\pi a})(1+\delta_{0,n})\,\frac{2^{n-1}b^n}{a}\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2},\, 1;\ b^{-2} \\ 1-n,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

27. $$\int_0^{m\pi} e^{-ax}\cos^n x\, T_n\left(\frac{1}{\cos x}\right) dx = \frac{1}{a}\,(1-e^{-m\pi a})\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, 1;\ 1 \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

28. $$\int_0^{m\pi} e^{-ax}\sin^{2n} x\, T_n(\cot^2 x)\, dx$$
$$= (1-e^{-m\pi a})(1+\delta_{0,n})\,\frac{2^{n-1}}{a}\, {}_4F_3\left(\begin{matrix} -n,\, \frac{1}{2}-n,\, \frac{1}{2},\, 1;\ 2 \\ 1-2n,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

29. $$\int_0^{\infty} e^{-ax} T_{2n}(b\sin x)\, dx = \frac{(-1)^n}{a}\, {}_3F_2\left(\begin{matrix} -n,\, n,\, 1;\ b^2 \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \qquad [\operatorname{Re} a > 0].$$

30. $$\int_0^{\infty} e^{-ax} T_n(\cos x)\, dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -n,\, n,\, 1;\ 1 \\ 1-ia,\, 1+ia \end{matrix}\right) \qquad [\operatorname{Re} a > 0].$$

31. $$\int_0^{\infty} e^{-ax} T_{2n}(\cos x)\, dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -n,\, n,\, 1;\ 1 \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \qquad [\operatorname{Re} a > 0].$$

32. $$\int_0^\infty e^{-ax} T_{2n+1}(b \sin x)\, dx = (-1)^n \frac{(2n+1)b}{a^2+1}\, {}_3F_2\left(\begin{matrix} -n,\ n+1,\ 1;\ b^2 \\ \frac{3-ia}{2},\ \frac{3+ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

33. $$\int_0^\infty \frac{e^{-ax}}{\sin x} T_{2n+1}(b \sin x)\, dx = (-1)^n (2n+1) \frac{b}{a}\, {}_4F_3\left(\begin{matrix} -n,\ n+1,\ \frac{1}{2},\ 1;\ b^2 \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

34. $$\int_0^\infty \frac{e^{-ax}}{\cos x} T_{2n+1}(\cos x)\, dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -n,\ n+1,\ 1;\ 1 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

35. $$\int_0^\infty e^{-ax} T_n(1 + b \sin^2 x)\, dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -n,\ n,\ 1;\ -\frac{b}{2} \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

36. $$\int_0^\infty e^{-ax} (1 + b \sin^2 x)^{n/2}\, T_n\left(\frac{1}{\sqrt{1 + b \sin^2 x}}\right) dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ 1;\ -b \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

37. $$\int_0^\infty e^{-ax} T_{2n}\left(\sqrt{1 + b \sin^2 x}\right) dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -n,\ n,\ 1;\ -b \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

38. $$\int_0^\infty \frac{e^{-ax}}{\sqrt{1 + b \sin^2 x}} T_{2n+1}\left(\sqrt{1 + b \sin^2 x}\right) dx = \frac{1}{a}\, {}_3F_2\left(\begin{matrix} -n,\ n+1,\ 1;\ -b \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

39. $$\int_0^\infty e^{-ax} \sin^n x\, T_n\left(\frac{b}{\sin x}\right) dx = (1 + \delta_{0,n}) \frac{2^{n-1} b^n}{a}\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ b^{-2} \\ 1-n,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [$n \geq 2$; Re $a > 0$].

40. $$\int_0^\infty e^{-ax} \cos^n x\, T_n\left(\frac{1}{\cos x}\right) dx = \frac{1}{a}\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ 1;\ 1 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

41. $$\int_0^\infty e^{-ax}\sin^{2n}x\,T_n(\cot^2 x)\,dx = \frac{2^{n-1}}{a}\,{}_4F_3\left(\begin{matrix}-n,\ \frac{1}{2}-n,\ \frac{1}{2},\ 1;\ 2\\ 1-2n,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)$$
$[n \geq 2;\ \operatorname{Re} a > 0].$

4.15.3. Integrals containing $T_n(z)$ and special functions

1. $$\int_0^\infty K_1(ax)T_{2n+1}(bx)\,dx = \frac{(2n+1)\pi^2}{8a}\left[I^2_{-n-1/2}\left(\frac{a}{2b}\right) - I^2_{n+1/2}\left(\frac{a}{2b}\right)\right]$$
$[\operatorname{Re} a > 0].$

2. $$\int_a^1 x^{-1/2}(x-a)^{-1/2}P_{2n+1}(\sqrt{x})\,T_{2n}\left(\sqrt{\frac{x}{a}}\right)dx = 2a^n(1-a)^{1/2}$$
$[0 < a < 1].$

3. $$\int_a^1 x^{-1/2}(x-a)^{-1/2}P_{2n+2}(\sqrt{x})\,T_{2n+1}\left(\sqrt{\frac{x}{a}}\right)dx = 2a^{n+1/2}(1-a)^{1/2}$$
$[0 < a < 1].$

4.16. The Chebyshev Polynomials $U_n(z)$

4.16.1. Integrals containing $U_n(z)$ and algebraic functions

1. $$\int_0^a (a^2-x^2)^{-1/2}U_{2n}(bx)\,dx = (-1)^n\frac{\pi}{2}P_n(1-2a^2b^2)$$ $[a > 0].$

2. $$\int_0^a (a^2-x^2)^{-1/2}(b^2-x^2)^{-n-1}U_{2n}\left(\frac{x}{b}\right)dx$$
$$= (-1)^n\frac{\pi}{2}\,(b^2-a^2)^{-n-1}\,P_{2n+1}\left(\sqrt{1-\frac{a^2}{b^2}}\right)$$
$[a > 0;\ |a/b| < 1].$

3. $$\int_0^1 \frac{(1-x^2)^{1/2}}{(x^2+a^2)^{n+2}}U_{2n}(x)\,dx = (-1)^n\pi\frac{\left(\frac{3}{2}\right)_n}{4(n+1)!}a^{-2n-3}(a^2+1)^{-n-1/2}$$
$[\operatorname{Re} a > 0].$

4. $$\int_0^1 \frac{x(1-x^2)^{1/2}}{(x^2+a^2)^{n+3}}U_{2n+1}(x)\,dx = (-1)^n\pi\frac{\left(\frac{3}{2}\right)_n}{4n!\,(n+2)}a^{-2n-3}(a^2+1)^{-n-3/2}$$
$[\operatorname{Re} a > 0].$

5. $\displaystyle\int_0^a x^{-1/2}(a-x)^{-1/2}U_{2n}\left(b\sqrt{x(a-x)}\right)dx = (-1)^n \pi P_n\left(1-\frac{a^2b^2}{2}\right)$ $[a>0]$.

4.16.2. Integrals containing $U_n(z)$ and trigonometric functions

1. $$\int_0^\pi \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} U_{2n}(b\sin x)\,dx = (-1)^n \frac{2^{-\mu}\pi\Gamma(\mu+1)}{\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)} \times \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} {}_4F_3\left(\begin{matrix}-n,\ n+1,\ \frac{\mu+1}{2},\ \frac{\mu}{2}+1;\ b^2\\ \frac{1}{2},\ \frac{\mu-a}{2}+1,\ \frac{\mu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}\mu>-1].$$

2. $$\int_0^\pi \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} U_{2n+1}(b\sin x)\,dx = \frac{2^{1-\mu}(n+1)\pi b\,\Gamma(\mu+2)}{n!\,\Gamma\left(\frac{\mu-a+3}{2}\right)\Gamma\left(\frac{\mu+a+3}{2}\right)} \cos\frac{a\pi}{2} \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} \times {}_4F_3\left(\begin{matrix}-n,\ n+2,\ \frac{\mu}{2}+1,\ \frac{\mu+3}{2};\ b^2\\ \frac{3}{2},\ \frac{\mu-a+3}{2},\ \frac{\mu+a+3}{2}\end{matrix}\right) \quad [\operatorname{Re}\mu>-2].$$

3. $$\int_0^\pi \sin x\sin(ax)U_{2n}\left(\sqrt{1+b\sin^2 x}\right)dx = (2n+1)\frac{\sin(\pi a)}{1-a^2}\,{}_3F_2\left(\begin{matrix}-n,\ n+1,\ 1;\ -b\\ \frac{3-a}{2},\ \frac{3+a}{2}\end{matrix}\right).$$

4. $$\int_0^\pi \frac{\sin x\sin(ax)}{\sqrt{1+b\sin^2 x}}U_{2n+1}\left(\sqrt{1+b\sin^2 x}\right)dx = 2(n+1)\frac{\sin(\pi a)}{1-a^2}\,{}_3F_2\left(\begin{matrix}-n,\ n+2,\ 1;\ -b\\ \frac{3-a}{2},\ \frac{3+a}{2}\end{matrix}\right).$$

5. $$\int_0^\pi \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} U_n\left(\frac{b}{\sin x}\right)dx = \frac{2^{2n-\mu}\pi\Gamma(\mu-n+1)}{\Gamma\left(\frac{\mu-a-n}{2}+1\right)\Gamma\left(\frac{\mu+a-n}{2}+1\right)} \times \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} {}_4F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{\mu-n+1}{2},\ \frac{\mu-n}{2}+1;\ b^{-2}\\ -n,\ \frac{\mu-a-n}{2}+1,\ \frac{\mu+a-n}{2}+1\end{matrix}\right) \quad [\operatorname{Re}\mu>n-1].$$

6. $$\int_0^{m\pi} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} U_{2n}(b\sin x)\,dx = (-1)^n\frac{2}{a}\sin\frac{m\pi a}{2} \begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix} {}_3F_2\left(\begin{matrix}-n,\ n+1,\ 1;\ b^2\\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right).$$

7. $\displaystyle\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} U_{2n}(\cos x)\, dx$

$$= 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{2n+1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2},\, 1;\ 1 \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

8. $\displaystyle\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} U_{2n+1}(b\sin x)\, dx$

$$= 4(-1)^n \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{n+1}{a} b\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{1}{2},\, 1;\ b^2 \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

9. $\displaystyle\int_0^{m\pi} \frac{1}{\cos x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} U_{2n+1}(\cos x)\, dx$

$$= 4\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{n+1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{1}{2},\, 1;\ 1 \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

10. $\displaystyle\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} U_n(1+b\sin^2 x)\, dx$

$$= 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{n+1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{1}{2},\, 1;\ -\frac{b}{2} \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

11. $\displaystyle\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^n x\, U_n\left(\frac{b}{\sin x}\right) dx$

$$= 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(2b)^n}{a}\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2},\, 1;\ b^{-2} \\ -n,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

12. $\displaystyle\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} U_{2n}\left(\sqrt{1+b\sin^2 x}\right) dx$

$$= 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{2n+1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2},\, 1;\ -b \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

13. $\displaystyle\int_0^{m\pi} \frac{1}{\sqrt{1+b\sin^2 x}} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} U_{2n+1}\left(\sqrt{1+b\sin^2 x}\right) dx$

$$= 4\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{n+1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{1}{2},\, 1;\ -b \\ \frac{3}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

14. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} (1+b\sin^2 x)^{n/2} U_n\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right) dx$$
$$= 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{n+1}{a}\, {}_4F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}, \frac{1}{2}, 1;\ -b \\ \frac{3}{2}, 1-\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right).$$

15. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^{2n} x\, U_n(\cot^2 x)\, dx$$
$$= \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{2^{n+1}}{a}\, {}_4F_3\left(\begin{matrix} -n, -\frac{1}{2}-n, \frac{1}{2}, 1;\ 2 \\ -2n-1, 1-\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right).$$

16. $$\int_0^{m\pi} e^{-ax} U_{2n}(b\sin x)\, dx = \frac{(-1)^n}{a}(1-e^{-m\pi a})\, {}_3F_2\left(\begin{matrix} -n, n+1, 1;\ b^2 \\ 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

17. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} U_{2n+1}(b\sin x)\, dx$$
$$= 2(-1)^n(n+1)\frac{b}{a}(1-e^{-m\pi a})\, {}_4F_3\left(\begin{matrix} -n, n+2, \frac{1}{2}, 1;\ b^2 \\ \frac{3}{2}, 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

18. $$\int_0^{m\pi} e^{-ax} U_{2n}(\cos x)\, dx = \frac{2n+1}{a}(1-e^{-m\pi a})\, {}_4F_3\left(\begin{matrix} -n, n+1, \frac{1}{2}, 1;\ 1 \\ \frac{3}{2}, 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

19. $$\int_0^{m\pi} \frac{e^{-ax}}{\cos x} U_{2n+1}(\cos x)\, dx = 2(n+1)\frac{1-e^{-m\pi a}}{a}\, {}_4F_3\left(\begin{matrix} -n, n+2, \frac{1}{2}, 1;\ 1 \\ \frac{3}{2}, 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

20. $$\int_0^{m\pi} e^{-ax} U_n(1+b\sin^2 x)\, dx$$
$$= (n+1)\frac{1-e^{-m\pi a}}{a}\, {}_4F_3\left(\begin{matrix} -n, n+2, \frac{1}{2}, 1;\ -\frac{b}{2} \\ \frac{3}{2}, 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

21. $$\int_0^{m\pi} e^{-ax} U_{2n}\left(\sqrt{1+b\sin^2 x}\right) dx$$
$$= (2n+1)\frac{1-e^{-m\pi a}}{a}\, {}_4F_3\left(\begin{matrix} -n, n+1, \frac{1}{2}, 1;\ -b \\ \frac{3}{2}, 1-\frac{ia}{2}, 1+\frac{ia}{2} \end{matrix}\right).$$

22. $\displaystyle\int_0^{m\pi} \frac{e^{-ax}}{\sqrt{1+b\sin^2 x}} U_{2n+1}\left(\sqrt{1+b\sin^2 x}\right) dx$

$$= 2(n+1)\frac{1-e^{-m\pi a}}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{1}{2},\, 1;\; -b \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

23. $\displaystyle\int_0^{m\pi} e^{-ax}(1+b\sin^2 x)^{n/2}\, U_n\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right) dx$

$$= \frac{n+1}{a}\,(1-e^{-m\pi a})\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2},\, 1;\; -b \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

24. $\displaystyle\int_0^{m\pi} e^{-ax}\sin^n x\, U_n\left(\frac{b}{\sin x}\right) dx$

$$= \frac{(2b)^n}{a}\,(1-e^{-m\pi a})\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, \frac{1}{2},\, 1;\; b^{-2} \\ -n,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

25. $\displaystyle\int_0^{m\pi} e^{-ax}\sin^{2n} x\, U_n(b\cot^2 x)\, dx$

$$= \frac{2^n}{a}\,(1-e^{-m\pi a})\, {}_4F_3\left(\begin{matrix} -n,\, -n-\frac{1}{2},\, \frac{1}{2},\, 1;\; 2 \\ -2n-1,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

26. $\displaystyle\int_0^{\infty} e^{-ax} U_{2n}(b\sin x)\, dx = \frac{(-1)^n}{a}\, {}_3F_2\left(\begin{matrix} -n,\, n+1,\, 1;\; b^2 \\ 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$ [Re $a > 0$].

27. $\displaystyle\int_0^{\infty} e^{-ax} U_{2n+1}(b\sin x)\, dx = (-1)^n \frac{2(n+1)b}{a^2+1}\, {}_3F_2\left(\begin{matrix} -n,\, n+2,\, 1;\; b^2 \\ \frac{3-ia}{2},\, \frac{3+ia}{2} \end{matrix}\right)$

[Re $a > 0$].

28. $\displaystyle\int_0^{\infty} e^{-ax} U_{2n}(\cos x)\, dx = \frac{2n+1}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+1,\, \frac{1}{2},\, 1;\; 1 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$ [Re $a > 0$].

29. $\displaystyle\int_0^{\infty} \frac{e^{-ax}}{\sin x} U_{2n+1}(b\sin x)\, dx = 2(-1)^n \frac{n+1}{a} b\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{1}{2},\, 1;\; b^2 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$

[Re $a > 0$].

30. $\displaystyle\int_0^{\infty} \frac{e^{-ax}}{\cos x} U_{2n+1}(\cos x)\, dx = \frac{2(n+1)}{a}\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{1}{2},\, 1;\; 1 \\ \frac{3}{2},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$ [Re $a > 0$].

31. $$\int_0^\infty e^{-ax} U_n(1 + b\sin^2 x)\,dx = \frac{n+1}{a}\,{}_4F_3\left(\begin{matrix} -n,\ n+2,\ \frac{1}{2},\ 1;\ -\frac{b}{2} \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

32. $$\int_0^\infty e^{-ax} U_{2n}\left(\sqrt{1 + b\sin^2 x}\right)dx = \frac{2n+1}{a}\,{}_4F_3\left(\begin{matrix} -n,\ n+1,\ \frac{1}{2},\ 1;\ -b \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

33. $$\int_0^\infty \frac{e^{-ax}}{\sqrt{1 + b\sin^2 x}} U_{2n+1}\left(\sqrt{1 + b\sin^2 x}\right)dx = \frac{2(n+1)}{a}\,{}_4F_3\left(\begin{matrix} -n,\ n+2,\ \frac{1}{2},\ 1;\ -b \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

34. $$\int_0^\infty e^{-ax} (1 + b\sin^2 x)^{n/2} U_n\left(\frac{1}{\sqrt{1 + b\sin^2 x}}\right)dx = \frac{n+1}{a}\,{}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ -b \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

35. $$\int_0^\infty e^{-ax} \sin^n x\, U_n\left(\frac{b}{\sin x}\right)dx = \frac{(2b)^n}{a}\,{}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ b^{-2} \\ -n,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

36. $$\int_0^\infty e^{-ax} \sin^{2n} x\, U_n(b\cot^2 x)\,dx = \frac{2^n}{a}\,{}_4F_3\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ \frac{1}{2},\ 1;\ 2 \\ -2n-1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$ [Re $a > 0$].

4.16.3. Integrals containing $U_n(z)$ and $K_\nu(z)$

1. $$\int_0^\infty K_0(ax) U_{2n}(bx)\,dx = \frac{\pi^2}{8b}\left[I^2_{-n-1/2}\left(\frac{a}{2b}\right) - I^2_{n+1/2}\left(\frac{a}{2b}\right)\right]$$ [Re $a > 0$].

4.16.4. Integrals containing products of $U_n(z)$

1. $$\int_0^{m\pi} \left\{\begin{matrix} \sin(ax) \\ \cos(ax) \end{matrix}\right\} [U_n(\cos x)]^2\,dx = 2\sin\frac{m\pi a}{2} \left\{\begin{matrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{matrix}\right\} \frac{(n+1)^2}{a} \times \left[2\,{}_4F_3\left(\begin{matrix} -n,\ n+2,\ \frac{1}{2},\ 1;\ 1 \\ \frac{3}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right) - {}_4F_3\left(\begin{matrix} -n,\ n+2,\ 1,\ 1;\ 1 \\ 2,\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right)\right].$$

2. $$\int_0^{m\pi} e^{-ax}[U_n(\cos x)]^2\,dx = (1-e^{-m\pi a})\frac{(n+1)^2}{a} \times \left[2\,{}_4F_3\left(\begin{matrix}-n,\,n+2,\,\frac{1}{2},\,1;\,1\\ \frac{3}{2},\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right) - {}_4F_3\left(\begin{matrix}-n,\,n+2,\,1,\,1;\,1\\ 2,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)\right].$$

3. $$\int_0^{\infty} e^{-ax}[U_n(\cos x)]^2\,dx = \frac{(n+1)^2}{a}\left[2\,{}_4F_3\left(\begin{matrix}-n,\,n+2,\,\frac{1}{2},\,1;\,1\\ \frac{3}{2},\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right) - {}_4F_3\left(\begin{matrix}-n,\,n+2,\,1,\,1;\,1\\ 2,\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)\right]$$

[Re $a > 0$].

4.17. The Hermite Polynomials $H_n(z)$

4.17.1. Integrals containing $H_n(z)$ and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1}H_{2n}\left(b\sqrt{x(a-x)}\right)dx = (-1)^n\frac{(2n)!}{n!}a^{s+t-1}\,\mathrm{B}\,(s,\,t)\,{}_3F_3\left(\begin{matrix}-n,\,s,\,t;\,\frac{a^2b^2}{4}\\ \frac{1}{2},\,\frac{s+t}{2},\,\frac{s+t+1}{2}\end{matrix}\right)$$

[a, Re s, Re $t > 0$].

2. $$\int_0^a x^{s-1}(a-x)^{t-1}H_{2n+1}\left(b\sqrt{x(a-x)}\right)dx = 2(-1)^n\frac{(2n+1)!}{n!}a^{s+t}b\,\mathrm{B}\left(s+\frac{1}{2},\,t+\frac{1}{2}\right)\,{}_3F_3\left(\begin{matrix}-n,\,s+\frac{1}{2},\,t+\frac{1}{2};\,\frac{a^2b^2}{4}\\ \frac{3}{2},\,\frac{s+t+1}{2},\,\frac{s+t}{2}+1\end{matrix}\right)$$

[$a > 0$; Re s, Re $t > -1/2$].

3. $$\int_0^a x^{1/2}(a-x)^{1/2}H_{2n}\left(b\sqrt{x(a-x)}\right)dx = (-1)^n\frac{(2n-1)!}{(n+1)!}\frac{\pi a^2}{8}\left[2nL_n^1\left(\frac{a^2b^2}{4}\right) - na^2b^2L_{n-1}^2\left(\frac{a^2b^2}{4}\right)\right]$$

[$n \geq 1$; $a > 0$].

4. $$\int_0^a x^{1/2}(a-x)^{-1/2}H_{2n}\left(b\sqrt{x(a-x)}\right)dx = (-1)^n\frac{(2n)!}{n!}\frac{\pi a}{2}L_n\left(\frac{a^2b^2}{4}\right)$$

[$a > 0$].

5. $$\int_0^a x^{-1/2}(a-x)^{-1/2} H_{2n}\left(b\sqrt{x(a-x)}\right) dx = (-1)^n \frac{(2n)!}{n!} \pi L_n\left(\frac{a^2b^2}{4}\right)$$
$[a > 0]$.

6. $$\int_0^a H_{2n+1}\left(b\sqrt{x(a-x)}\right) dx = (-1)^n \frac{(2n+1)!}{(n+1)!} \frac{\pi a^2 b}{4} L_n^1\left(\frac{a^2b^2}{4}\right) \qquad [a > 0].$$

7. $$\int_0^a x H_{2n+1}\left(b\sqrt{x(a-x)}\right) dx = (-1)^n \frac{(2n+1)!}{(n+1)!} \frac{\pi a^3 b}{8} L_n^1\left(\frac{a^2b^2}{4}\right) \qquad [a > 0].$$

8. $$\int_0^a x^s (a-x)^{s+1/2} H_{2n}\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= (-1)^n 2^{-2s-1} \sqrt{\pi}\, a^{2s+3/2} \frac{(2n)!}{n!} \frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)} {}_2F_2\left(\begin{matrix} -n,\ 2s+2;\ \frac{ab^2}{2} \\ \frac{1}{2},\ 2s+\frac{5}{2} \end{matrix}\right)$$
$[a > 0;\ \operatorname{Re} s > -1]$.

9. $$\int_0^a x^s (a-x)^{s+1/2} H_{2n+1}\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= (-1)^n 2^{-2s-1/2} \sqrt{\pi}\, a^{2s+2} b \frac{(2n+1)!}{n!} \frac{\Gamma\left(2s+\frac{5}{2}\right)}{\Gamma(2s+3)} {}_2F_2\left(\begin{matrix} -n,\ 2s+\frac{5}{2};\ \frac{ab^2}{2} \\ \frac{3}{2},\ 2s+3 \end{matrix}\right)$$
$[a > 0;\ \operatorname{Re} s > -5/4]$.

10. $$\int_0^a x^{-1/2} H_{2n+1}\left(b\sqrt[4]{x(a-x)}\right) dx = (-1)^n \frac{(2n+1)!}{(n+1)!} \frac{\pi a b}{\sqrt{2}} L_n^1\left(\frac{ab^2}{2}\right)$$
$[a > 0]$.

11. $$\int_0^a x^{-1/4}(a-x)^{-3/4} H_{2n}\left(b\sqrt[4]{x(a-x)}\right) dx$$
$$= (-1)^n \frac{(2n)!}{n!} \sqrt{2}\, \pi L_n\left(\frac{ab^2}{2}\right) \qquad [a > 0].$$

4.17.2. Integrals containing $H_n(z)$ and the exponential function

1. $$\int_a^\infty x(x^2-a^2)^{n-3/2} e^{-b^2x^2} H_{2n}(bx)\, dx = 2^{2n-1} a^{2n} b\, \Gamma\left(n-\frac{1}{2}\right) e^{-a^2b^2}$$
$[a > 0;\ |\arg b| < \pi/4;\ n \geq 1]$.

2. $$\int_a^\infty (x^2-a^2)^{n-1/2} e^{-b^2x^2} H_{2n+1}(bx)\,dx = (2a)^{2n}\Gamma\left(n+\frac{1}{2}\right) e^{-a^2b^2}$$

$[a>0;\ |\arg b|<\pi/4].$

3. $$\int_0^\infty x^{n-1} e^{-ax^2} H_n(bx)\,dx = (n-1)!a^{-n/2} T_n\left(\frac{b}{\sqrt{a}}\right) \qquad [n\geq 1;\ \operatorname{Re} a>0].$$

4.17.3. Integrals containing $H_n(z)$ and trigonometric functions

1. $$\int_0^\pi \sin(ax) H_{2n+1}(b\sin x)\,dx$$
$$= (-1)^n \frac{2^{2n+1} b\sin(\pi a)}{1-a^2}\left(\frac{3}{2}\right)_n {}_2F_2\left(\begin{matrix}-n,\ 1;\ b^2\\ \frac{3-a}{2},\ \frac{3+a}{2}\end{matrix}\right).$$

2. $$\int_0^\pi \sin x \sin(ax) H_{2n}(b\sin x)\,dx$$
$$= (-4)^n \frac{\sin(\pi a)}{1-a^2}\left(\frac{1}{2}\right)_n {}_3F_3\left(\begin{matrix}-n,\ 1,\ \frac{3}{2};\ b^2\\ \frac{1}{2},\ \frac{3-a}{2},\ \frac{3+a}{2}\end{matrix}\right).$$

3. $$\int_0^{m\pi} \left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\} H_{2n}(b\sin x)\,dx$$
$$= 2\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\}\frac{(-4)^n}{a}\left(\frac{1}{2}\right)_n {}_2F_2\left(\begin{matrix}-n,\ 1;\ b^2\\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right).$$

4. $$\int_0^{m\pi} \frac{1}{\sin x}\left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\} H_{2n+1}(b\sin x)\,dx$$
$$= (-1)^n 2^{2n+1}\frac{b}{a}\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\}\left(\frac{3}{2}\right)_n {}_3F_3\left(\begin{matrix}-n,\ \frac{1}{2},\ 1;\ b^2\\ \frac{3}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right).$$

5. $$\int_0^{m\pi} \left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\} \sin^{2n} x\, H_{2n}\left(\frac{b}{\sin x}\right)dx$$
$$= 2\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\}\frac{(2b)^{2n}}{a}\ {}_4F_2\left(\begin{matrix}-n,\ \frac{1}{2}-n,\ \frac{1}{2},\ 1\\ 1-\frac{a}{2},\ 1+\frac{a}{2};\ -b^{-2}\end{matrix}\right).$$

6. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^{2n+1} x\, H_{2n+1}\left(\frac{b}{\sin x}\right) dx$$
$$= 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(2b)^{2n+1}}{a}\, {}_4F_2\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ \frac{1}{2},\ 1 \\ 1-\frac{a}{2},\ 1+\frac{a}{2};\ -b^{-2} \end{matrix}\right).$$

7. $$\int_0^{m\pi} e^{-ax} H_{2n}(b\sin x)\, dx = \frac{(-4)^n}{a}\left(\frac{1}{2}\right)_n (1-e^{-m\pi a})\, {}_2F_2\left(\begin{matrix} -n,\ 1;\ b^2 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

8. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} H_{2n+1}(b\sin x)\, dx$$
$$= (-1)^n 2^{2n+1}(1-e^{-m\pi a})\left(\frac{3}{2}\right)_n \frac{b}{a}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{1}{2},\ 1;\ b^2 \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

9. $$\int_0^{m\pi} e^{-ax} \sin^{2n} x\, H_{2n}\left(\frac{b}{\sin x}\right) dx$$
$$= \frac{(2b)^{2n}}{a}(1-e^{-m\pi a})\, {}_4F_2\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \frac{1}{2},\ 1;\ -b^{-2} \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

10. $$\int_0^{m\pi} e^{-ax} \sin^{2n+1} x\, H_{2n+1}\left(\frac{b}{\sin x}\right) dx$$
$$= \frac{(2b)^{2n+1}}{a}(1-e^{-m\pi a})\, {}_4F_2\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ \frac{1}{2},\ 1;\ -b^{-2} \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

11. $$\int_0^{\infty} e^{-ax} H_{2n}(b\sin x)\, dx = \frac{(-4)^n}{a}\left(\frac{1}{2}\right)_n {}_2F_2\left(\begin{matrix} -n,\ 1;\ b^2 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \qquad [\operatorname{Re} a > 0].$$

12. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} H_{2n+1}(b\sin x)\, dx$$
$$= (-1)^{n+1} 2^{2n+1}\left(\frac{3}{2}\right)_n \frac{b}{a}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{1}{2},\ 1;\ b^2 \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \qquad [\operatorname{Re} a > 0].$$

13. $$\int_0^{\infty} e^{-ax} \sin^{2n} x\, H_{2n}\left(\frac{b}{\sin x}\right) dx = \frac{(2b)^{2n}}{a}\, {}_4F_2\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \frac{1}{2},\ 1;\ -b^{-2} \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$
$$[\operatorname{Re} a > 0].$$

14. $\displaystyle\int_0^\infty e^{-ax}\sin^{2n+1}x\,H_{2n+1}\left(\frac{b}{\sin x}\right)dx$

$$= \frac{(2b)^{2n+1}}{a}\,{}_4F_2\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,\frac{1}{2},\,1\\ 1-\frac{ia}{2},\,1+\frac{ia}{2};\,-b^{-2}\end{matrix}\right)\quad [\operatorname{Re}a>0].$$

4.17.4. Integrals containing $H_n(z)$, erf (z) and erfc (z)

1. $\displaystyle\int_0^\infty \operatorname{erfc}(ax)H_{2n}(bx)\,dx = \frac{(-1)^n}{\sqrt{\pi}\,a}\frac{(2n)!}{\left(\frac{3}{2}\right)_n}P_n^{(1/2,\,-n-1/2)}\left(1-\frac{2b^2}{a^2}\right)$

$[|\arg a|<\pi/4]$.

2. $\displaystyle\int_0^\infty \operatorname{erfc}(ax)H_{2n+1}(bx)\,dx = \frac{(-1)^n b}{2a^2}\frac{(2n+1)!}{(n+1)!}P_n^{(1,\,-n-1)}\left(1-\frac{2b^2}{a^2}\right)$

$[|\arg a|<\pi/4]$.

3. $\displaystyle\int_0^\infty x\operatorname{erfc}(ax)H_{2n+1}(bx)\,dx$

$$= (-1)^n\frac{2b}{3\sqrt{\pi}\,a^3}\frac{(2n+1)!}{\left(\frac{5}{2}\right)_n}P_n^{(3/2,\,-n-1/2)}\left(1-\frac{2b^2}{a^2}\right)\quad [|\arg a|<\pi/4].$$

4. $\displaystyle\int_0^\infty e^{-2x^2}\operatorname{erf}(ax)H_{2n+1}(x)\,dx = (-1)^n\frac{2^{n-1}n!\,a}{\sqrt{a^2+2}}P_n^{(1/2,\,-n-1)}\left(\frac{3a^2+2}{a^2+2}\right).$

4.17.5. Integrals containing $H_n(z)$ and $K_\nu(z)$

1. $\displaystyle\int_0^\infty K_0(ax)H_{2n}(bx)\,dx = (-1)^n(2n)!\pi\frac{2^{2n-1}b^{2n}}{a^{2n+1}}L_n^{-n-1/2}\left(-\frac{a^2}{4b^2}\right)$

$[\operatorname{Re}a>0]$.

2. $\displaystyle\int_0^\infty x\,K_1(ax)H_{2n}(bx)\,dx = (-1)^n(2n)!\pi\frac{2^{2n-1}b^{2n}}{a^{2n+2}}L_n^{-n-3/2}\left(-\frac{a^2}{4b^2}\right)$

$[\operatorname{Re}a>0]$.

3. $\displaystyle\int_0^\infty K_1(ax)H_{2n+1}(bx)\,dx = (-1)^n(2n+1)!\pi\frac{2^{2n}b^{2n+1}}{a^{2n+2}}L_n^{-n-1/2}\left(-\frac{a^2}{4b^2}\right)$

$[\operatorname{Re}a>0]$.

4.17.6. Integrals containing products of $H_n(z)$

1. $$\int_0^a H_{2m+1}(b\sqrt{x})\, H_{2n+1}\big(b\sqrt{a-x}\big)\, dx = \frac{(-2)^{m+n}\pi(2m+1)!!\,(2n+1)!!}{(m+n+1)(m+n+2)} a^2 b^2 L_{m+n}^2(ab^2) \quad [a>0].$$

2. $$\int_0^a H_{2n+1}(b\sqrt{x})\, H_{2n+1}\big(ib\sqrt{a-x}\big)\, dx = 2^{4n+1} i\,\Gamma^2(n+\tfrac{3}{2}) a^2 b^2 \, {}_1F_2\left(\begin{matrix} -n;\ \frac{a^2b^4}{4} \\ \frac{3}{2},\ 2 \end{matrix}\right) \quad [a>0].$$

3. $$\int_0^a (a-x)^{-1/2} H_{2m+1}(b\sqrt{x})\, H_{2n}\big(b\sqrt{a-x}\big)\, dx = (-2)^{m+n}\pi \frac{(2m+1)!!\,(2n-1)!!}{m+n+1} ab L_{m+n}^1(ab^2) \quad [a>0].$$

4. $$\int_0^a x^{-1/2}(a-x)^{-1/2} H_{2m}(b\sqrt{x})\, H_{2n}\big(b\sqrt{a-x}\big)\, dx = (-2)^{m+n}(2m-1)!!\,(2n-1)!!\, L_{m+n}(ab^2) \quad [a>0].$$

5. $$\int_0^a x^{-1/2}(a-x)^{-1/2} H_{2n}(b\sqrt{x})\, H_{2n}\big(ib\sqrt{a-x}\big)\, dx = 2^{4n}\Gamma^2\left(n+\frac{1}{2}\right) {}_1F_2\left(\begin{matrix} -n;\ \frac{a^2b^4}{4} \\ \frac{1}{2},\ 1 \end{matrix}\right) \quad [a>0].$$

6. $$\int_0^a (a-x)^{-1/2} H_{2n+1}(b\sqrt{x})\, H_{2n}\big(ib\sqrt{a-x}\big)\, dx = 2^{4n+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right) ab\, {}_1F_2\left(\begin{matrix} -n;\ \frac{a^2b^4}{4} \\ 1,\ \frac{3}{2} \end{matrix}\right) \quad [a>0].$$

7. $$\int_0^{m\pi} \left\{\begin{matrix} \sin(ax) \\ \cos(ax) \end{matrix}\right\} H_{2n}\big(b\sqrt{\sin x}\big)\, H_{2n}\big(ib\sqrt{\sin x}\big)\, dx$$
$$= \frac{2^{4n+1}}{a}\left(\frac{1}{2}\right)_n^2 \sin\frac{m\pi a}{2} \left\{\begin{matrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{matrix}\right\} {}_4F_5\left(\begin{matrix} -n,\ n+\frac{1}{2},\ \frac{1}{2},\ 1;\ \frac{b^4}{4} \\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right).$$

8. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} H_{2n+1}\big(b\sqrt{\sin x}\big)\, H_{2n+1}\big(ib\sqrt{\sin x}\big)\, dx$$
$$= 2^{4n+3} i \frac{b^2}{a} \left(\frac{3}{2}\right)_n^2 \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_5\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{1}{2},\, 1;\; \frac{b^4}{4} \\ \frac{3}{2},\, \frac{3}{4},\, \frac{5}{4},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix} \right).$$

9. $$\int_0^{m\pi} e^{-ax} H_{2n}\big(b\sqrt{\sin x}\big)\, H_{2n}\big(ib\sqrt{\sin x}\big)\, dx$$
$$= (1-e^{-m\pi a}) \frac{2^{4n}}{a} \left(\frac{1}{2}\right)_n^2 {}_4F_5\left(\begin{matrix} -n,\, n+\frac{1}{2},\, \frac{1}{2},\, 1;\; \frac{b^4}{4} \\ \frac{1}{4},\, \frac{1}{2},\, \frac{3}{4},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right).$$

10. $$\int_0^{m\pi} \frac{e^{-ax}}{\sin x} H_{2n+1}\big(b\sqrt{\sin x}\big)\, H_{2n+1}\big(ib\sqrt{\sin x}\big)\, dx$$
$$= (1-e^{-m\pi a}) 2^{4n+2} i \frac{b^2}{a} \left(\frac{3}{2}\right)_n^2 {}_4F_5\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{1}{2},\, 1;\; \frac{b^4}{4} \\ \frac{3}{2},\, \frac{3}{4},\, \frac{5}{4},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right).$$

11. $$\int_0^{\infty} e^{-ax} H_{2n}\big(b\sqrt{\sin x}\big)\, H_{2n}\big(ib\sqrt{\sin x}\big)\, dx$$
$$= \frac{2^{4n}}{a} \left(\frac{1}{2}\right)_n^2 {}_4F_5\left(\begin{matrix} -n,\, n+\frac{1}{2},\, \frac{1}{2},\, 1;\; \frac{b^4}{4} \\ \frac{1}{4},\, \frac{1}{2},\, \frac{3}{4},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right) \quad [\operatorname{Re} a > 0].$$

12. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} H_{2n+1}\big(b\sqrt{\sin x}\big)\, H_{2n+1}\big(ib\sqrt{\sin x}\big)\, dx$$
$$= 2^{4n+2} i \frac{b^2}{a} \left(\frac{3}{2}\right)_n^2 {}_4F_5\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{1}{2},\, 1;\; \frac{b^4}{4} \\ \frac{3}{2},\, \frac{3}{4},\, \frac{5}{4},\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix} \right) \quad [\operatorname{Re} a > 0].$$

13. $$\int_0^{\infty} \operatorname{erfc}(x) H_n(x) H_{n+1}(x)\, dx$$
$$= 2^{n-1} n! + [(-1)^n - 1] \frac{n!\,(n+1)!}{8} \left(\left[\frac{n+1}{2}\right]!\right)^{-2}.$$

14. $$\int_0^{\infty} \operatorname{erfc}(\sqrt{2}\,x) H_n(x) H_{n+1}(x)\, dx$$
$$= 2^{n-2} \frac{\left(\frac{3}{2}\right)_n}{n+1} + [(-1)^n - 1] \frac{n!\,(n+1)!}{8} \left(\left[\frac{n+1}{2}\right]!\right)^{-2}.$$

15. $$\int_0^\infty e^{-x^2} H_{2m}(x) H_n^2\left(\frac{x}{\sqrt{2}}\right) dx = \frac{2^{n-1}\sqrt{\pi}\,(2m)!\,(-n)_m \left(\frac{1}{2}\right)_n}{m!\left(\frac{1}{2}-n\right)_m}.$$

16. $$\int_0^\infty e^{-x^2} H_m^2(x) H_n^2(x)\, dx = 2^{m+n-1} m!\, n! \sqrt{\pi}\; {}_3F_2\left(\begin{matrix} -m,\ -n,\ \frac{1}{2} \\ 1,\ 1;\ 4 \end{matrix}\right).$$

17. $$\int_0^\infty e^{-2x^2} H_m^2(x) H_n^2(x)\, dx = \sqrt{\frac{\pi}{8}}\,(2m-1)!!\,(2n-1)!!\; {}_3F_2\left(\begin{matrix} -m,\ -n,\ \frac{1}{2};\ 1 \\ \frac{1}{2}-m,\ \frac{1}{2}-n \end{matrix}\right).$$

18. $$\int_0^\infty \frac{1}{x} e^{-2x^2} H_n^2(x) H_{n+1}^2(x)\, dx = 2^{2n}(n!)^2 \sum_{k=0}^{[n/2]} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}.$$

4.18. The Laguerre Polynomials $L_n^\lambda(z)$

4.18.1. Integrals containing $L_n^\lambda(z)$ and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1} L_n^\lambda(bx(a-x))\, dx = \frac{(\lambda+1)_n}{n!} a^{s+t-1}\, \mathrm{B}\,(s,\, t)$$
$$\times\; {}_3F_3\left(\begin{matrix} -n,\ s,\ t;\ \frac{a^2 b}{4} \\ \lambda+1,\ \frac{s+t}{2},\ \frac{s+t+1}{2} \end{matrix}\right) \quad [\mathrm{Re}\, s,\ \mathrm{Re}\, t > 0].$$

2. $$\int_0^a x^\lambda (a-x)^\lambda L_n^\lambda(bx(a-x))\, dx$$
$$= \pi^{1/2}\left(\frac{a}{2}\right)^{2\lambda+1} \frac{\Gamma\,(n+\lambda+1)}{\Gamma\left(n+\lambda+\frac{3}{2}\right)} L_n^{\lambda+1/2}\left(\frac{a^2 b}{4}\right) \quad [\mathrm{Re}\,\lambda > -1].$$

3. $$\int_0^a L_n(bx(a-x))\, dx = (-1)^n \frac{n!}{(2n+1)!\, b^{1/2}} H_{2n+1}\left(\frac{a\sqrt{b}}{2}\right).$$

4. $$\int_0^a x^s (a-x)^{s+1/2} L_n^\lambda\left(b\sqrt{x(a-x)}\right) dx = \frac{\sqrt{\pi}\, a^{2s+3/2}}{2^{2s+1}} \frac{(\lambda+1)_n \Gamma(2s+2)}{n!\,\Gamma\left(2s+\frac{5}{2}\right)}$$
$$\times\; {}_2F_2\left(\begin{matrix} -n,\ 2s+2;\ \frac{ab}{2} \\ \lambda+1,\ 2s+\frac{5}{2} \end{matrix}\right) \quad [a > 0;\ \mathrm{Re}\, s > -1].$$

5. $$\int_0^a x^{-1/2} L_n\left(b\sqrt{x(a-x)}\right) dx = (-1)^n \frac{n!}{(2n+1)!} \sqrt{\frac{2}{b}}\, H_{2n+1}\left(\sqrt{\frac{ab}{2}}\right)$$
$$[a > 0].$$

4.18.2. Integrals containing $L_n^\lambda(z)$ and trigonometric functions

1. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} L_n^\lambda(b\sin^2 x)\,dx = \frac{2}{a}\sin\frac{m\pi a}{2} \times \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(\lambda+1)_n}{n!}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{1}{2},\ 1;\ b \\ \lambda+1,\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right).$$

2. $$\int_0^{\pi} \sin^\mu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} L_n^\lambda(b\sin^2 x)\,dx = \frac{2^{-\mu}\pi\Gamma(\mu+1)(\lambda+1)_n}{n!\,\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)} \times \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix} {}_3F_3\left(\begin{matrix} -n,\ \frac{\mu+1}{2},\ \frac{\mu}{2}+1;\ b \\ \frac{\mu-a}{2}+1,\ \frac{\mu+a}{2}+1,\ \lambda+1 \end{matrix}\right).$$

3. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^{2n} x\, L_n^\lambda\left(\frac{b}{\sin^2 x}\right) dx = \frac{2(-b)^n}{n!\,a}\sin\frac{m\pi a}{2} \times \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_4F_2\left(\begin{matrix} -n,\ -\lambda-n,\ \frac{1}{2},\ 1 \\ 1-\frac{a}{2},\ 1+\frac{a}{2};\ -\frac{1}{b} \end{matrix}\right).$$

4. $$\int_0^{m\pi} e^{-ax} L_n^\lambda(b\sin^2 x)\,dx = (1-e^{-m\pi a})\frac{(\lambda+1)_n}{n!\,a}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{1}{2},\ 1;\ b \\ \lambda+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

5. $$\int_0^{m\pi} e^{-ax} \sin^{2n} x\, L_n^\lambda\left(\frac{b}{\sin^2 x}\right) dx = \frac{(-b)^n}{n!\,a}(1-e^{-m\pi a})\, {}_4F_2\left(\begin{matrix} -n,\ -\lambda-n,\ \frac{1}{2},\ 1 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2};\ -\frac{1}{b} \end{matrix}\right).$$

6. $$\int_0^{\infty} e^{-ax} L_n^\lambda(b\sin^2 x)\,dx = \frac{(\lambda+1)_n}{n!\,a}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{1}{2},\ 1;\ b \\ \lambda+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

7. $$\int_0^{\infty} e^{-ax} \sin^{2n} x\, L_n^\lambda\left(\frac{b}{\sin^2 x}\right) dx = \frac{(-b)^n}{n!\,a}\, {}_4F_2\left(\begin{matrix} -n,\ -\lambda-n,\ \frac{1}{2},\ 1 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2};\ -\frac{1}{b} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

8. $$\int_0^{a} x^{s-1} \ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\, L_n^\lambda(bx)\,dx = \frac{\pi^{1/2}a^s(\lambda+1)_n\Gamma(s)}{n!2s\,\Gamma\left(s+\frac{1}{2}\right)}\, {}_3F_3\left(\begin{matrix} -n,\ s,\ s;\ ab \\ \lambda+1,\ s+\frac{1}{2},\ s+1 \end{matrix}\right) \quad [a > 0;\ \operatorname{Re} s > -1/2].$$

4.18.3. Integrals containing $L_n^\lambda(z)$ and $\operatorname{erfc}(z)$

1. $$\int_0^\infty \operatorname{erfc}(a\sqrt{x})\,L_n(bx)\,dx = \frac{1}{2(n+1)a^2}\,P_n^{(1,\,-n-1/2)}\left(1-\frac{2b}{a^2}\right) \qquad [\operatorname{Re} a > 0].$$

4.18.4. Integrals containing products of $L_n^\lambda(z)$

1. $$\int_0^a x^\lambda (a-x)^\mu L_m^\lambda(bx)\,L_n^\mu(b(a-x))\,dx = \frac{(m+n)!}{m!\,n!}$$
$$\times\, \mathrm{B}\,(\lambda+m+1,\,\mu+n+1)\,a^{\lambda+\mu+1} L_{m+n}^{\lambda+\mu+1}(ab) \quad [\operatorname{Re}\lambda,\ \operatorname{Re}\mu > -1].$$

2. $$\int_0^a x^\lambda (a-x)^\mu L_n^\lambda(bx)\,L_n^\mu(-b(a-x))\,dx$$
$$= \frac{a^{\lambda+\mu+1}}{(n!)^2}\,\frac{\Gamma(\lambda+n+1)\,\Gamma(\mu+n+1)}{\Gamma(\lambda+\mu+2)}\,{}_1F_2\left(\begin{matrix} -n;\ \frac{a^2b^2}{4} \\ \frac{\lambda+\mu}{2}+1,\ \frac{\lambda+\mu+3}{2} \end{matrix}\right)$$
$$[\operatorname{Re}\lambda,\ \operatorname{Re}\mu > -1].$$

3. $$\int_0^{m\pi} \left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\} L_n^\lambda(-b\sin x)\,L_n^\lambda(b\sin x)\,dx = \frac{2}{a}\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\}$$
$$\times\left[\frac{(\nu+1)_n}{n!}\right]^2 {}_4F_5\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ \frac{b^2}{4} \\ \frac{\lambda+1}{2},\ \frac{\lambda}{2}+1,\ \lambda+1,\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right).$$

4. $$\int_0^{m\pi} e^{-ax} L_n^\lambda(-b\sin x)\ L_n^\lambda(b\sin x)\,dx$$
$$= \frac{(\lambda+1)_n^2}{(n!)^2\,a}\,(1-e^{-m\pi a})\ {}_4F_5\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ \frac{b^2}{4} \\ \frac{\lambda+1}{2},\ \frac{\lambda}{2}+1,\ \lambda+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

5. $$\int_0^\infty e^{-ax} L_n^\lambda(-b\sin x)\ L_n^\lambda(b\sin x)\,dx$$
$$= \frac{(\lambda+1)_n^2}{(n!)^2\,a}\,{}_4F_5\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ \frac{b^2}{4} \\ \frac{\lambda+1}{2},\ \frac{\lambda}{2}+1,\ \lambda+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

6. $$\int_0^\infty x^{2\lambda} e^{-2x} \left[L_m^\lambda(x)\right]^2 \left[L_n^\lambda(x)\right]^2 dx = \frac{2^{-2\lambda-1}}{(m!\,n!)^2}\Gamma(2\lambda+1)\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n$$
$$\times (\lambda+1)_m(\lambda+1)_n\, {}_4F_3\left(\begin{matrix}-m,\,-n,\,\lambda+\frac{1}{2},\,\frac{1}{2};\,1\\ \frac{1}{2}-m,\,\frac{1}{2}-n,\,\lambda+1\end{matrix}\right) \quad [\operatorname{Re}\lambda > -1/2].$$

4.19. The Gegenbauer Polynomials $C_n^\lambda(z)$

4.19.1. Integrals containing $C_n^\lambda(z)$ and algebraic functions

1. $$\int_0^a x^{-1/2}(a-x)^{-1/2} C_{2n}^\lambda\left(b\sqrt{x(a-x)}\right) dx$$
$$= (-1)^n \pi \frac{(\lambda)_n}{n!} P_n^{(0,\,\lambda-1)}\left(1-\frac{a^2b^2}{2}\right).$$

2. $$\int_0^a C_{2n+1}^\lambda\left(b\sqrt{x(a-x)}\right) dx = (-1)^n \pi \frac{(\lambda)_{n+1}}{4(n+1)!} a^2 b P_n^{(1,\,\lambda-1)}\left(1-\frac{a^2b^2}{2}\right).$$

4.19.2. Integrals containing $C_n^\lambda(z)$ and trigonometric functions

1. $$\int_0^\pi \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} C_{2n}^\lambda(b\sin x)\, dx$$
$$= (-1)^n \frac{2^{-\mu}\pi\Gamma(\mu+1)(\lambda)_n}{n!\,\Gamma\left(\frac{\mu-a}{2}+1\right)\Gamma\left(\frac{\mu+a}{2}+1\right)} \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}$$
$$\times\, {}_4F_3\left(\begin{matrix}-n,\,\lambda+n,\,\frac{\mu+1}{2},\,\frac{\mu}{2}+1;\,b^2\\ \frac{1}{2},\,\frac{\mu-a}{2}+1,\,\frac{\mu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}\mu > -1].$$

2. $$\int_0^\pi \sin^\mu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} C_{2n+1}^\lambda(b\sin x)\, dx$$
$$= (-1)^n \frac{2^{-\mu}\pi b\Gamma(\mu+2)(\lambda)_{n+1}}{n!\,\Gamma\left(\frac{\mu-a+3}{2}\right)\Gamma\left(\frac{\mu+a+3}{2}\right)} \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}$$
$$\times\, {}_4F_3\left(\begin{matrix}-n,\,\lambda+n+1,\,\frac{\mu}{2}+1,\,\frac{\mu+3}{2};\,b^2\\ \frac{3}{2},\,\frac{\mu-a+3}{2},\,\frac{\mu+a+3}{2}\end{matrix}\right) \quad [\operatorname{Re}\mu > -2].$$

3. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} C_{2n}^{\lambda}(b \sin x)\, dx$$

$$= (-1)^n \frac{2(\lambda)_n}{n!\, a} \sin \frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ \lambda+n,\ 1;\ b^2 \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

4. $$\int_0^{m\pi} \frac{1}{\sin x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} C_{2n+1}^{\lambda}(b \sin x)\, dx$$

$$= (-1)^n \frac{4(\lambda)_{n+1}}{n!\, a} \sin \frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_3F_2\left(\begin{matrix} -n,\ \lambda+n+1,\ 1;\ b^2 \\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

5. $$\int_0^{m\pi} \frac{1}{\cos x} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} C_{2n+1}^{\lambda}(\cos x)\, dx$$

$$= 2 \sin \frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(2\lambda)_{2n+1}}{(2n+1)!a}\, {}_4F_3\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ 1 \\ \lambda+\frac{1}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

6. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} C_{2n}^{\lambda}\left(\sqrt{1+b\sin^2 x}\right) dx$$

$$= 2 \sin \frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(2\lambda)_{2n}}{(2n)!\, a}\, {}_4F_3\left(\begin{matrix} -n,\ \lambda+n,\ \frac{1}{2},\ 1;\ -b \\ \lambda+\frac{1}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

7. $$\int_0^{m\pi} \frac{1}{\sqrt{1+b\sin^2 x}} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} C_{2n+1}^{\lambda}\left(\sqrt{1+b\sin^2 x}\right) dx$$

$$= 2 \sin \frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(2\lambda)_{2n+1}}{(2n+1)!a}\, {}_4F_3\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ -b \\ \lambda+\frac{1}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

8. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} (1+b\sin^2 x)^{n/2} C_n^{\lambda}\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right) dx$$

$$= 2 \sin \frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(2\lambda)_n}{n!\, a}\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ -b \\ \lambda+\frac{1}{2},\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

9. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \sin^n x\, C_n^{\lambda}\left(\frac{b}{\sin x}\right) dx$$

$$= 2 \sin \frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(\lambda)_n}{n!\, a} (2b)^n\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ b^{-2} \\ 1-\lambda-n,\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix} \right).$$

10. $\displaystyle\int_0^{m\pi} e^{-ax} C_{2n}^\lambda(b\sin x)\,dx$

$$= (-1)^n \frac{(\lambda)_n}{n!\,a}\,(1-e^{-m\pi a})\;{}_3F_2\left(\begin{matrix}-n,\ \lambda+n,\ 1;\ b^2\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

11. $\displaystyle\int_0^{m\pi} \frac{e^{-ax}}{\sin x} C_{2n+1}^\lambda\,(b\sin x)\,dx$

$$= 2(-1)^n \frac{b(\lambda)_{n+1}}{n!\,a}\,(1-e^{-m\pi a})\;{}_4F_3\left(\begin{matrix}-n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ b^2\\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

12. $\displaystyle\int_0^{m\pi} e^{-ax} C_{2n}^\lambda(\cos x)\,dx$

$$= \frac{(2\lambda)_n}{(2n)!\,a}(1-e^{-m\pi a})\;{}_4F_3\left(\begin{matrix}-n,\ \lambda+n,\ \frac{1}{2},\ 1;\ 1\\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

13. $\displaystyle\int_0^{m\pi} \frac{e^{-ax}}{\cos x} C_{2n+1}^\lambda\,(\cos x)\,dx$

$$= (1-e^{-m\pi a})\,\frac{(2\lambda)_{2n+1}}{(2n+1)!a}\;{}_4F_3\left(\begin{matrix}-n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ 1\\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

14. $\displaystyle\int_0^{m\pi} e^{-ax}\sin^n x\, C_n^\lambda\left(\frac{b}{\sin x}\right)dx$

$$= \frac{(\lambda)_n}{n!\,a}(2b)^n\,(1-e^{-m\pi a})\;{}_4F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ b^{-2}\\ 1-\lambda-n,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

15. $\displaystyle\int_0^{m\pi} e^{-ax} C_n^\lambda(1+b\sin^2 x)\,dx$

$$= \frac{(2\lambda)_n}{n!\,a}\,(1-e^{-m\pi a})\;{}_4F_3\left(\begin{matrix}-n,\ 2\lambda+n,\ \frac{1}{2},\ 1;\ -\frac{b}{2}\\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

16. $\displaystyle\int_0^{m\pi} e^{-ax} C_{2n}^\lambda\left(\sqrt{1+b\sin^2 x}\right)dx$

$$= \frac{(2\lambda)_{2n}}{(2n)!\,a}\,(1-e^{-m\pi a})\;{}_4F_3\left(\begin{matrix}-n,\ \lambda+n,\ \frac{1}{2},\ 1;\ -b\\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

17. $$\int_0^{m\pi} \frac{e^{-ax}}{\sqrt{1+b\sin^2 x}} C_{2n+1}^{\lambda}\left(\sqrt{1+b\sin^2 x}\right) dx$$
$$= \frac{(2\lambda)_{2n+1}}{(2n+1)!a}\left(1-e^{-m\pi a}\right) {}_4F_3\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ -b \\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

18. $$\int_0^{m\pi} e^{-ax}\left(1+b\sin^2 x\right)^{n/2} C_n^{\lambda}\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right) dx$$
$$= \frac{(2\lambda)_n}{n!\,a}\left(1-e^{-m\pi a}\right) {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ -b \\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

19. $$\int_0^{\infty} e^{-ax} C_{2n}^{\lambda}(b\sin x)\, dx = (-1)^n \frac{(\lambda)_n}{n!\,a}\, {}_3F_2\left(\begin{matrix} -n,\ \lambda+n,\ 1;\ b^2 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

20. $$\int_0^{\infty} e^{-ax} C_{2n+1}^{\lambda}(b\sin x)\, dx$$
$$= (-1)^n \frac{2b(\lambda)_{n+1}}{n!\left(a^2+1\right)}\, {}_3F_2\left(\begin{matrix} -n,\ \lambda+n+1,\ 1;\ b^2 \\ \frac{3-ia}{2},\ \frac{3+ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

21. $$\int_0^{\infty} \frac{e^{-ax}}{\sin x} C_{2n+1}^{\lambda}(b\sin x)\, dx$$
$$= 2(-1)^n \frac{(\lambda)_{n+1} b}{n!\,a}\, {}_4F_3\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ b^2 \\ \frac{3}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

22. $$\int_0^{\infty} e^{-ax} C_{2n}^{\lambda}(\cos x)\, dx = \frac{(2\lambda)_{2n}}{(2n)!\,a}\, {}_4F_3\left(\begin{matrix} -n,\ \lambda+n,\ \frac{1}{2},\ 1;\ 1 \\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

23. $$\int_0^{\infty} \frac{e^{-ax}}{\cos x} C_{2n+1}^{\lambda}(\cos x)\, dx = \frac{(2\lambda)_{2n+1}}{(2n+1)!a}\, {}_4F_3\left(\begin{matrix} -n,\ \lambda+n+1,\ \frac{1}{2},\ 1;\ 1 \\ \lambda+\frac{1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right)$$
$$[\operatorname{Re} a > 0].$$

24. $$\int_0^{\infty} e^{-ax} \sin^n x\, C_n^{\lambda}\left(\frac{b}{\sin x}\right) dx$$
$$= \frac{(\lambda)_n}{n!\,a}(2b)^n\, {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2},\ 1;\ b^{-2} \\ 1-\lambda-n,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

25. $$\int_0^\infty e^{-ax} C_n^\lambda(1+b\sin^2 x)\,dx = \frac{(2\lambda)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\,2\lambda+n,\,\frac{1}{2},\,1;\,-\frac{b}{2}\\ \lambda+\frac{1}{2},\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)$$
$[\operatorname{Re} a > 0]$.

26. $$\int_0^\infty e^{-ax} C_{2n}^\lambda\left(\sqrt{1+b\sin^2 x}\right)dx = \frac{(2\lambda)_{2n}}{(2n)!\,a}\,{}_4F_3\left(\begin{matrix}-n,\,\lambda+n,\,\frac{1}{2},\,1;\,-b\\ \lambda+\frac{1}{2},\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right)$$
$[\operatorname{Re} a > 0]$.

27. $$\int_0^\infty e^{-ax}(1+b\sin^2 x)^{n/2} C_n^\lambda\left(\frac{1}{\sqrt{1+b\sin^2 x}}\right)dx$$
$$= \frac{(2\lambda)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,\frac{1}{2},\,1;\,-b\\ \lambda+\frac{1}{2},\,1-\frac{ia}{2},\,1+\frac{ia}{2}\end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

4.19.3. Integrals containing products of $C_n^\lambda(z)$

1. $$\int_0^{\pi/2} \cos^\nu x\cos(ax)\left[C_n^\lambda\left(\sqrt{1+b\cos^2 x}\right)\right]^2 dx$$
$$= \frac{2^{-\nu-1}\pi\Gamma(\nu+1)(2\lambda)_n^2}{(n!)^2\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}\,{}_5F_4\left(\begin{matrix}-n,\,\lambda,\,2\lambda+n,\,\frac{\nu+1}{2},\,1+\frac{\nu}{2};\,-b\\ \lambda+\frac{1}{2},\,2\lambda,\,1+\frac{\nu-a}{2},\,1+\frac{\nu+a}{2}\end{matrix}\right)$$
$[\operatorname{Re}\nu > -1]$.

2. $$\int_0^{\pi} \sin^\nu x\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\left[C_n^\lambda\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx$$
$$= \frac{2^{-\nu}\pi\Gamma(\nu+1)(2\lambda)_n^2}{(n!)^2\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix}$$
$$\times\,{}_5F_4\left(\begin{matrix}-n,\,\lambda,\,2\lambda+n,\,\frac{\nu+1}{2},\,1+\frac{\nu}{2};\,-b\\ \lambda+\frac{1}{2},\,2\lambda,\,1+\frac{\nu-a}{2},\,1+\frac{\nu+a}{2}\end{matrix}\right) \quad [\operatorname{Re}\nu > -1].$$

3. $$\int_0^{m\pi}\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\left[C_n^\lambda(\cos x)\right]^2 dx = \frac{2}{a}\sin\frac{m\pi a}{2}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix}$$
$$\times\left[\frac{(2\lambda)_n}{n!}\right]^2{}_5F_4\left(\begin{matrix}-n,\,n+2\lambda,\,\lambda,\,\frac{1}{2},\,1;\,1\\ 2\lambda,\,\lambda+\frac{1}{2},\,1-\frac{a}{2},\,1+\frac{a}{2}\end{matrix}\right).$$

4. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \left[C_n^\lambda\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx$$

$$= \frac{2}{a}\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \left[\frac{(2\lambda)_n}{n!}\right]^2 {}_5F_4\left(\begin{matrix} -n,\, n+2\lambda,\, \lambda,\, \frac{1}{2},\, 1;\; -b \\ 2\lambda,\, \lambda+\frac{1}{2},\, 1-\frac{a}{2},\, 1+\frac{a}{2} \end{matrix}\right).$$

5. $$\int_0^{m\pi} e^{-ax}\left[C_n^\lambda(\cos x)\right]^2 dx$$

$$= \frac{1-e^{-m\pi a}}{a} \left[\frac{(2\lambda)_n}{n!}\right]^2 {}_5F_4\left(\begin{matrix} -n,\, \lambda,\, 2\lambda+n,\, \frac{1}{2},\, 1;\; 1 \\ \lambda+\frac{1}{2},\, 2\lambda,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

6. $$\int_0^{m\pi} e^{-ax}\left[C_n^\lambda\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx$$

$$= \frac{1-e^{-m\pi a}}{a} \left[\frac{(2\lambda)_n}{n!}\right]^2 {}_5F_4\left(\begin{matrix} -n,\, n+2\lambda,\, \lambda,\, \frac{1}{2},\, 1;\; -b \\ \lambda+\frac{1}{2},\, 2\lambda,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right).$$

7. $$\int_0^{\infty} e^{-ax}\left[C_n^\lambda(\cos x)\right]^2 dx = \frac{1}{a}\left[\frac{(2\lambda)_n}{n!}\right]^2 {}_5F_4\left(\begin{matrix} -n,\, n+2\lambda,\, \lambda,\, \frac{1}{2},\, 1;\; 1 \\ \lambda+\frac{1}{2},\, 2\lambda,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right)$$

$[\operatorname{Re} a > 0].$

8. $$\int_0^{\infty} e^{-ax}\left[C_n^\lambda\left(\sqrt{1+b\sin^2 x}\right)\right]^2 dx$$

$$= \frac{1}{a}\left[\frac{(2\lambda)_n}{n!}\right]^2 {}_5F_4\left(\begin{matrix} -n,\, n+2\lambda,\, \lambda,\, \frac{1}{2},\, 1;\; -b \\ \lambda+\frac{1}{2},\, 2\lambda,\, 1-\frac{ia}{2},\, 1+\frac{ia}{2} \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

4.20. The Jacobi Polynomials $P_n^{(\rho,\sigma)}(z)$

4.20.1. Integrals containing $P_n^{(\rho,\sigma)}(z)$ and algebraic functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1} P_n^{(\rho,\sigma)}(1+bx(a-x))\, dx = \frac{(\rho+1)_n}{n!}\,\mathrm{B}(s,\, t)$$

$$\times\, a^{s+t-1}\, {}_4F_3\left(\begin{matrix} -n,\, \rho+\sigma+n+1,\, s,\, t \\ \rho+1,\, \frac{s+t}{2},\, \frac{s+t+1}{2};\; -\frac{a^2 b}{8} \end{matrix}\right) \quad [\operatorname{Re} s,\, \operatorname{Re} t > 0].$$

2. $$\int_0^a x^s(a-x)^{s+1/2}P_n^{(\rho,\sigma)}\left(1+b\sqrt{x(a-x)}\right)dx = 2^{-2s-1}\frac{(\rho+1)_n}{n!}$$
$$\times \mathrm{B}\left(\frac{1}{2},\,2s+2\right)a^{2s+3/2}\,{}_3F_2\left(\begin{matrix}-n,\,\rho+\sigma+n+1,\,2s+2\\ \rho+1,\,2s+\frac{5}{2};\,-\frac{ab}{4}\end{matrix}\right)\quad [\operatorname{Re} s>-1].$$

4.20.2. Integrals containing $P_n^{(\rho,\sigma)}(z)$ and trigonometric functions

1. $$\int_0^{2m\pi}\left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\}P_n^{(\rho,\sigma)}(\cos x)\,dx$$
$$= 2\sin(m\pi a)\left\{\begin{matrix}\sin(m\pi a)\\ \cos(m\pi a)\end{matrix}\right\}\frac{(\rho+1)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\,\rho+\sigma+n+1,\,\frac{1}{2},\,1\\ \rho+1,\,1-a,\,1+a;\,1\end{matrix}\right).$$

2. $$\int_0^{m\pi}\left\{\begin{matrix}\sin(ax)\\ \cos(ax)\end{matrix}\right\}P_n^{(\rho,\sigma)}(\cos^2 x)\,dx$$
$$= 2\sin\frac{m\pi a}{2}\left\{\begin{matrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{matrix}\right\}\frac{(\rho+1)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\,\rho+\sigma+n+1,\,\frac{1}{2},\,1\\ \rho+1,\,1-\frac{a}{2},\,1+\frac{a}{2};\,\frac{1}{2}\end{matrix}\right).$$

3. $$\int_0^{m\pi}e^{-ax}P_n^{(\rho,\sigma)}(\cos^2 x)\,dx$$
$$= \left(1-e^{-m\pi a}\right)\frac{(\rho+1)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\,\rho+\sigma+n+1,\,\frac{1}{2},\,1\\ \rho+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2};\,\frac{1}{2}\end{matrix}\right).$$

4. $$\int_0^{\infty}e^{-ax}P_n^{(\rho,\sigma)}(\cos^2 x)\,dx = \frac{(\rho+1)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\,\rho+\sigma+n+1,\,\frac{1}{2},\,1\\ \rho+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2};\,\frac{1}{2}\end{matrix}\right)$$
$$[\operatorname{Re} a>0].$$

5. $$\int_0^{\infty}e^{-ax}P_n^{(\rho,\sigma)}(1+b\sin^2 x)\,dx$$
$$= \frac{(\rho+1)_n}{n!\,a}\,{}_4F_3\left(\begin{matrix}-n,\,\rho+\sigma+n+1,\,\frac{1}{2},\,1\\ \rho+1,\,1-\frac{ia}{2},\,1+\frac{ia}{2};\,-\frac{b}{2}\end{matrix}\right)\quad [\operatorname{Re} a>0].$$

4.20.3. Integrals containing $P_n^{(\rho,\sigma)}(z)$ and $J_\nu(z)$

1. $$\int_{-1}^{1}(1-x)^{\rho/2}(1+x)^{\sigma}J_\rho\left(a\sqrt{1-x}\right)P_n^{(\rho,\sigma)}(x)\,dx$$
$$= 2^{(\rho+3\sigma+3)/2}a^{-\sigma-1}\frac{\Gamma(\sigma+n+1)}{n!}J_{\rho+\sigma+2n+1}\left(\sqrt{2}\,a\right)\quad [\operatorname{Re}\rho,\,\operatorname{Re}\sigma>-1].$$

4.20.4. Integrals containing products of $P_n^{(\rho,\sigma)}(z)$

1. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} P_n^{(\rho,\sigma)}(-\cos x) P_n^{(\rho,\sigma)}(\cos x)\,dx$$
$$= (-1)^n 2\sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} \frac{(\rho+1)_n(\sigma+1)_n}{(n!)^2\,a}$$
$$\times\, {}_6F_5\left(\begin{matrix} -n,\ \frac{\rho+\sigma+1}{2},\ \frac{\rho+\sigma}{2}+1,\ \rho+\sigma+n+1,\ \frac{1}{2},\ 1 \\ \rho+1,\ \sigma+1,\ \rho+\sigma+1,\ 1-\frac{a}{2},\ 1+\frac{a}{2};\ 1 \end{matrix}\right).$$

2. $$\int_0^{m\pi} e^{-ax} P_n^{(\rho,\sigma)}(-\cos x) P_n^{(\rho,\sigma)}(\cos x)\,dx$$
$$= (-1)^n (1-e^{-m\pi a}) \frac{(\rho+1)_n(\sigma+1)_n}{(n!)^2\,a}$$
$$\times\, {}_6F_5\left(\begin{matrix} -n,\ \frac{\rho+\sigma+1}{2},\ \frac{\rho+\sigma}{2}+1,\ \rho+\sigma+n+1,\ \frac{1}{2},\ 1;\ 1 \\ \rho+1,\ \sigma+1,\ \rho+\sigma+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2} \end{matrix}\right).$$

3. $$\int_0^{m\pi} e^{-ax} P_n^{(\rho,\sigma)}\left(-\sqrt{1+b\sin^2 x}\right) P_n^{(\rho,\sigma)}\left(\sqrt{1+b\sin^2 x}\right) dx$$
$$= (-1)^n (1-e^{-m\pi a}) \frac{(\rho+1)_n(\sigma+1)_n}{(n!)^2\,a}$$
$$\times\, {}_6F_5\left(\begin{matrix} -n,\ \frac{\rho+\sigma+1}{2},\ \frac{\rho+\sigma}{2}+1,\ \rho+\sigma+n+1,\ \frac{1}{2},\ 1 \\ \rho+1,\ \sigma+1,\ \rho+\sigma+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2};\ -b \end{matrix}\right).$$

4. $$\int_0^{\infty} e^{-ax} P_n^{(\rho,\sigma)}(-\cos x) P_n^{(\rho,\sigma)}(\cos x)\,dx = (-1)^n \frac{\Gamma(\rho+1)\Gamma(\sigma+1)}{(n!)^2\,a}$$
$$\times\, {}_6F_5\left(\begin{matrix} -n,\ \frac{\rho+\sigma+1}{2},\ \frac{\rho+\sigma}{2}+1,\ \rho+\sigma+n+1,\ \frac{1}{2},\ 1 \\ \rho+1,\ \sigma+1,\ \rho+\sigma+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2};\ 1 \end{matrix}\right) \quad [\operatorname{Re} a > 0].$$

5. $$\int_0^{\infty} e^{-ax} P_n^{(\rho,\sigma)}\left(-\sqrt{1+b\sin^2 x}\right) P_n^{(\rho,\sigma)}\left(\sqrt{1+b\sin^2 x}\right) dx$$
$$= (-1)^n \frac{\Gamma(\rho+1)\Gamma(\sigma+1)}{(n!)^2\,a}\, {}_6F_5\left(\begin{matrix} -n,\ \frac{\rho+\sigma+1}{2},\ \frac{\rho+\sigma}{2}+1,\ \rho+\sigma+n+1,\ \frac{1}{2},\ 1 \\ \rho+1,\ \sigma+1,\ \rho+\sigma+1,\ 1-\frac{ia}{2},\ 1+\frac{ia}{2};\ -b \end{matrix}\right)$$
$$[\operatorname{Re} a > 0].$$

4.21. The Complete Elliptic Integral $\mathbf{K}(z)$

4.21.1. Integrals containing $\mathbf{K}(z)$ and algebraic functions

1. $$\int_0^1 \frac{x(1-x^2)^{s-1}}{(1-ax^2)^{s+1/2}}\,\mathbf{K}(x)\,dx = \frac{\pi}{4}\,\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)}(1-a)^{-1/2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\,s\\ s+\frac{1}{2};\,a\end{matrix}\right)$$
$[\operatorname{Re} s>0;\ |\arg(1-a)|<\pi]$.

2. $$\int_0^1 x(1-x^2)^{-1/2}\,\mathbf{K}(ax)\,dx = \frac{\pi}{2a}\arcsin a$$
$[|\arg(1-a^2)|<\pi]$.

3. $$\int_0^1 x^{s-1}(1+ax)^{\nu}\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi\Gamma^2(s)}{2\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_3F_2\left(\begin{matrix}-\nu,\,s,\,s;\,a\\ s+\frac{1}{2},\,s+\frac{1}{2}\end{matrix}\right)$$
$[\operatorname{Re} s>0;\ |\arg(1-a)|<\pi]$.

4. $$\int_0^a x^{s-1}(a-x)^{t-1}\,\mathbf{K}\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{\pi}{2}\,\mathrm{B}(s,t)\,a^{s+t-1}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,s,\,t;\,\frac{a^2b^2}{4}\\ 1,\,\frac{s+t}{2},\,\frac{s+t+1}{2}\end{matrix}\right)$$
$[\operatorname{Re} s,\,\operatorname{Re} t>0;\ |\arg(4-a^2b^2)|<\pi]$.

5. $$\int_0^a \mathbf{K}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{b}\arcsin\frac{ab}{2}$$
$[|\arg(4-a^2b^2)|<\pi]$.

6. $$\int_0^a x\,\mathbf{K}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi a}{2b}\arcsin\frac{ab}{2}$$
$[|\arg(4-a^2b^2)|<\pi]$.

7. $$\int_0^a x^2\,\mathbf{K}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{8b^3}\left[ab\sqrt{1-\frac{a^2b^2}{4}}+(3a^2b^2-2)\arcsin\frac{ab}{2}\right]$$
$[|\arg(4-a^2b^2)|<\pi]$.

8. $$\int_0^a x^{-1/2}(a-x)^{1/2}\,\mathbf{K}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi^2 a}{4}\,\psi_1\left(\frac{a^2b^2}{4}\right)$$
$[|\arg(4-a^2b^2)|<\pi]$.

9. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\,\mathbf{K}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi^2}{2}\,\psi_1\left(\frac{a^2b^2}{4}\right)$$
$[|\arg(4-a^2b^2)|<\pi]$.

10. $$\int_0^a x^{s+1/2}(a-x)^s\,\mathbf{K}\left(b\sqrt[4]{x(a-x)}\right)dx = 2^{-2s-2}\pi^{3/2}a^{2s+3/2}\frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)}$$
$$\times\ {}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ 2s+2\\ 1,\ 2s+\frac{5}{2};\ \frac{ab^2}{2}\end{matrix}\right)\quad\left[\operatorname{Re}s>-1;\ |\arg(2-ab^2)|<\pi\right].$$

11. $$\int_0^a x^{1/2}\,\mathbf{K}\left(b\sqrt[4]{x(a-x)}\right)dx = -\frac{\pi a^{1/2}}{4b^2}\sqrt{1-\frac{ab^2}{2}}$$
$$+\frac{\pi}{2^{3/2}b^3}(ab^2+1)\arcsin\left(b\sqrt{\frac{a}{2}}\right)\quad\left[|\arg(2-ab^2)|<\pi\right].$$

12. $$\int_0^a x^{-1/2}\,\mathbf{K}\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{2^{1/2}\pi}{b}\arcsin\left(b\sqrt{\frac{a}{2}}\right)$$
$$\left[|\arg(2-ab^2)|<\pi\right].$$

13. $$\int_0^a x^{1/4}(a-x)^{-1/4}\,\mathbf{K}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\frac{a\pi^2}{4\sqrt{2}}\left[2\psi_1\left(\frac{ab^2}{2}\right)-\psi_2\left(\frac{ab^2}{2}\right)\right]\quad\left[|\arg(2-ab^2)|<\pi\right].$$

14. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\,\mathbf{K}\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi^2}{\sqrt{2}}\psi_1\left(\frac{ab^2}{2}\right)$$
$$\left[|\arg(2-ab^2)|<\pi\right].$$

15. $$\int_0^2 x^{1/4}(2-x)^{-1/4}\,\mathbf{K}\left(\sqrt[4]{x(2-x)}\right)dx = \frac{1}{8\sqrt{2}\pi}\left[\Gamma^4\left(\frac{1}{4}\right)+16\Gamma^4\left(\frac{3}{4}\right)\right].$$

16. $$\int_0^2 x^{-1/4}(2-x)^{-3/4}\,\mathbf{K}\left(\sqrt[4]{x(2-x)}\right)dx = \frac{\pi^3}{\sqrt{2}}\Gamma^{-4}\left(\frac{3}{4}\right).$$

17. $$\int_0^a \frac{x^{-1/2}}{\sqrt{1+b^2\sqrt{x(a-x)}}}\,\mathbf{K}\left(\frac{b\sqrt[4]{x(a-x)}}{\sqrt{1+b^2\sqrt{x(a-x)}}}\right)dx$$
$$=\frac{2^{1/2}\pi}{b}\ln\left(b\sqrt{\frac{a}{2}}+\sqrt{1+\frac{ab^2}{2}}\right)\quad\left[|\arg(2+ab^2)|<\pi\right].$$

4.21.2. Integrals containing $\mathbf{K}(z)$, the exponential, hyperbolic and trigonometric functions

1. $$\int_0^1 x^{s-1} e^{ax}\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi\Gamma^2(s)}{2\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_2F_2\left(\begin{matrix} s,\, s;\; a \\ s+\frac{1}{2},\, s+\frac{1}{2}\end{matrix}\right) \qquad [\operatorname{Re} s > 0].$$

2. $$\int_0^1 x^{s-1} \left\{\begin{matrix}\sinh(a\sqrt{x}) \\ \sin(a\sqrt{x})\end{matrix}\right\} \mathbf{K}(\sqrt{1-x})\,dx$$
$$= \frac{\pi a\,\Gamma^2\left(s+\frac{1}{2}\right)}{2\Gamma^2(s+1)}\,{}_2F_3\left(\begin{matrix} s+\frac{1}{2},\, s+\frac{1}{2};\; \pm\frac{a^2}{4} \\ \frac{3}{2},\, s+1,\, s+1\end{matrix}\right) \qquad [\operatorname{Re} s > -1/2].$$

3. $$\int_0^1 x^{s-1} \left\{\begin{matrix}\cosh(a\sqrt{x}) \\ \cos(a\sqrt{x})\end{matrix}\right\} \mathbf{K}(\sqrt{1-x})\,dx$$
$$= \frac{\pi\Gamma^2(s)}{2\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_2F_3\left(\begin{matrix} s,\, s;\; \pm\frac{a^2}{4} \\ \frac{1}{2},\, s+\frac{1}{2},\, s+\frac{1}{2}\end{matrix}\right) \qquad [\operatorname{Re} s > 0].$$

4. $$\int_0^1 \sinh(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2} I_0\left(\frac{a}{2}\right) I_1\left(\frac{a}{2}\right).$$

5. $$\int_0^1 x^{-1/2}\cosh(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2} I_0^2\left(\frac{a}{2}\right).$$

6. $$\int_0^1 \sin(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2} J_0\left(\frac{a}{2}\right) J_1\left(\frac{a}{2}\right).$$

7. $$\int_0^1 x^{1/2}\cos(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx$$
$$= \frac{\pi^2}{4}\left[J_0^2\left(\frac{a}{2}\right) - \frac{2}{a} J_0\left(\frac{a}{2}\right) J_1\left(\frac{a}{2}\right) - J_1^2\left(\frac{a}{2}\right)\right].$$

8. $$\int_0^1 x^{-1/2}\cos(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2} J_0^2\left(\frac{a}{2}\right).$$

9. $$\int_0^{\pi/2} \cos(2nx)\,\mathbf{K}(a\sin x)\,dx = 2^{-2n-2}\pi(-a^2)^n$$
$$\times \frac{\Gamma^2\left(n+\frac{1}{2}\right)}{(n!)^2}\,{}_3F_2\left(\begin{matrix} \frac{n+1}{2},\, \frac{n+1}{2},\, \frac{n+1}{2} \\ n+1,\, 2n+1;\; a^2\end{matrix}\right) \qquad \left[|\arg(1-a^2)| < \pi\right].$$

10. $$\int_0^{\pi/2} \cos^\nu x \cos(ax)\, \mathbf{K}(b\cos x)\,dx = \frac{2^{-\nu-2}\pi^2\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)} \times {}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{\nu+1}{2}, 1+\frac{\nu}{2};\ b^2\\ 1, 1+\frac{\nu-a}{2}, 1+\frac{\nu+a}{2}\end{matrix}\right) \quad [\operatorname{Re}\nu > -1;\ |\arg b| < \pi].$$

11. $$\int_0^{\pi} \sin^\nu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} \mathbf{K}(b\sin x)\,dx = \frac{2^{-\nu-1}\pi^2\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)} \begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} \times {}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{\nu+1}{2}, 1+\frac{\nu}{2};\ b^2\\ 1, 1+\frac{\nu-a}{2}, 1+\frac{\nu+a}{2}\end{matrix}\right) \quad [\operatorname{Re}\nu > -1;\ |\arg b| < \pi].$$

12. $$\int_0^{m\pi} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix} \mathbf{K}(b\sin x)\,dx = \frac{\pi}{a}\sin\frac{m\pi a}{2} \times \begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix} {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2};\ b^2\\ 1-\frac{a}{2}, 1+\frac{a}{2}\end{matrix}\right) \quad \left[\left|\arg\left(1-b^2\right)\right| < \pi\right].$$

13. $$\int_0^{m\pi} e^{-ax}\,\mathbf{K}(b\sin x)\,dx = \frac{\pi}{2a}\left(1-e^{-m\pi a}\right) {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2};\ b^2\\ 1-\frac{ia}{2}, 1+\frac{ia}{2}\end{matrix}\right) \quad \left[\left|\arg\left(1-b^2\right)\right| < \pi\right].$$

14. $$\int_0^{\infty} e^{-ax}\,\mathbf{K}(b\sin x)\,dx = \frac{\pi}{2a}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2};\ b^2\\ 1-\frac{ia}{2}, 1+\frac{ia}{2}\end{matrix}\right) \quad \left[\operatorname{Re}a > 0;\ \left|\arg\left(1-b^2\right)\right| < \pi\right].$$

4.21.3. Integrals containing $\mathbf{K}(z)$ and the logarithmic function

1. $$\int_0^1 x(1-x^2)^{-3/2}\ln x\,\mathbf{K}(x)\,dx = -\frac{\pi}{2}.$$

2. $$\int_0^1 \frac{x^3}{\left(2-x^2\right)^{5/2}} \ln\frac{4x^4\left(1-x^2\right)}{\left(2-x^2\right)^4}\,\mathbf{K}(x)\,dx = -\frac{32}{9} + \frac{\sqrt{2}\,\pi}{3} + 2\ln 2.$$

3. $$\int_0^1 x^{s-1}\ln(1+ax)\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi a\,\Gamma^2(s+1)}{2\Gamma^2\left(s+\frac{3}{2}\right)} \times {}_4F_3\left(\begin{matrix}1,\,1,\,s+1,\,s+1\\ 2,\,s+\frac{3}{2},\,s+\frac{3}{2};\,-a\end{matrix}\right) \quad [\operatorname{Re} s > -1;\ |\arg(1+a)| < \pi].$$

4. $$\int_0^1 x^{-3/2}\ln(1-ax)\,\mathbf{K}(\sqrt{1-x})\,dx = 2\pi[(1-a)\,\mathbf{K}(\sqrt{a}) - \mathbf{E}(\sqrt{a})] \quad [|\arg(1-a)| < \pi].$$

5. $$\int_0^1 x^{-3/2}\ln(1+ax)\,\mathbf{K}\left(\sqrt{1-x}\right)dx = 2\pi(a+1)^{1/2}\left[\mathbf{K}\left(\sqrt{\frac{a}{a+1}}\right) - \mathbf{E}\left(\sqrt{\frac{a}{a+1}}\right)\right] \quad [|\arg(1+a)| < \pi].$$

6. $$\int_0^1 x^{s-1}\ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi a\,\Gamma^2\left(s+\frac{1}{2}\right)}{2\Gamma^2(s+1)} \times {}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,s+\frac{1}{2},\,s+\frac{1}{2}\\ \frac{3}{2},\,s+1,\,s+1;\,-a^2\end{matrix}\right) \quad \left[\operatorname{Re} s > -1/2;\ \left|\arg\left(1+a^2\right)\right| < \pi\right].$$

7. $$\int_0^1 \ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi a}{8}\,[2\psi_2(-a^2) - \psi_3(-a^2)] \quad \left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

8. $$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi a\,\Gamma^2\left(s+\frac{1}{2}\right)}{2\Gamma^2(s+1)} \times {}_4F_3\left(\begin{matrix}1,\,1,\,s+\frac{1}{2},\,s+\frac{1}{2}\\ \frac{3}{2},\,s+1,\,s+1;\,-a^2\end{matrix}\right) \quad \left[\operatorname{Re} s > -1/2;\ \left|\arg\left(1+a^2\right)\right| < \pi\right].$$

9. $$\int_0^1 \frac{1}{\sqrt{1+a^2x}}\ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi^2}{2a}\ln\frac{a+\sqrt{1+a^2}}{2} \quad \left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

10. $$\int_0^1 \frac{x^{-1}}{\sqrt{1+a^2x}}\ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi^2}{2}\ln\left(a+\sqrt{1+a^2}\right) \quad \left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

11. $$\int_0^1 x^{s-1} \ln \frac{1+a\sqrt{x}}{1-a\sqrt{x}}\, \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi a \Gamma^2\left(s+\frac{1}{2}\right)}{\Gamma^2(s+1)} \times {}_4F_3\left(\begin{matrix}\frac{1}{2},\ 1,\ s+\frac{1}{2},\ s+\frac{1}{2} \\ \frac{3}{2},\ s+1,\ s+1;\ a^2\end{matrix}\right) \qquad \left[\operatorname{Re} s > -1/2;\ \left|\arg\left(1-a^2\right)\right| < \pi\right].$$

12. $$\int_0^1 \ln \frac{1+a\sqrt{x}}{1-a\sqrt{x}}\, \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi}{a}\left[\pi - 2\,\mathbf{E}(a)\right] \qquad \left[\left|\arg\left(1-a^2\right)\right| < \pi\right].$$

13. $$\int_0^1 \frac{1}{x} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}\, \mathbf{K}\left(\sqrt{1-x}\right) dx = 4\pi\mathbf{G}.$$

14. $$\int_0^1 x^{s-1} \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\, \mathbf{K}\,(ax)\, dx = \frac{\pi^{3/2}\Gamma\left(\frac{s}{2}\right)}{2s\,\Gamma\left(\frac{s+1}{2}\right)}\, {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{s}{2},\ \frac{s}{2};\ a^2 \\ 1,\ \frac{s+1}{2},\ \frac{s}{2}+1\end{matrix}\right) \qquad \left[\operatorname{Re} s > 0;\ \left|\arg\left(1-a^2\right)\right| < \pi\right].$$

15. $$\int_0^1 x \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\, \mathbf{K}\,(ax)\, dx = \frac{\pi}{a^2}\left(a \arcsin a + \sqrt{1-a^2} - 1\right) \qquad \left[\left|\arg\left(1-a^2\right)\right| < \pi\right].$$

16. $$\int_0^a x^{s-1} \ln \frac{a+\sqrt{a^2-x^2}}{x}\, \mathbf{K}\,(bx)\, dx = \frac{\pi^{3/2} a^s \Gamma\left(\frac{s}{2}\right)}{4s\,\Gamma\left(\frac{s+1}{2}\right)}\, {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{s}{2},\ \frac{s}{2};\ a^2b^2 \\ 1,\ \frac{s+1}{2},\ \frac{s}{2}+1\end{matrix}\right) \qquad \left[\operatorname{Re} s > 0;\ \left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

17. $$\int_0^a x \ln \frac{a+\sqrt{a^2-x^2}}{x}\, \mathbf{K}\,(bx)\, dx = \frac{\pi}{2b^2}\left[ab \arcsin(ab) + \sqrt{1-a^2b^2} - 1\right] \qquad \left[\left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

18. $$\int_0^a x^3 \ln \frac{a+\sqrt{a^2-x^2}}{x}\, \mathbf{K}\,(bx)\, dx = \frac{\pi}{72b^4\sqrt{1-a^2b^2}}\left(a^4b^4 - 17a^2b^2 + 16\right) + \frac{\pi}{72b^4}\left[3ab(2a^2b^2+3)\arcsin(ab) - 16\right] \qquad \left[\left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

19. $$\int_0^a \frac{x^3}{\sqrt{b^2x^2+1}} \ln \frac{a+\sqrt{a^2-x^2}}{x}\, \mathbf{K}\left(\frac{bx}{\sqrt{b^2x^2+1}}\right) dx = \frac{\pi\sqrt{a^2b^2+1}}{72b^4} \times \left(a^2b^2+16\right) + \frac{\pi}{72b^4}\left[3ab(2a^2b^2-3)\ln\left(ab+\sqrt{a^2b^2+1}\right) - 16\right] \qquad \left[\left|\arg\left(1+a^2b^2\right)\right| < \pi\right].$$

20. $$\int_0^a \frac{x}{\sqrt{b^2x^2+1}} \ln\frac{a+\sqrt{a^2-x^2}}{x}\, \mathbf{K}\left(\frac{bx}{\sqrt{b^2x^2+1}}\right) dx$$
$$= \frac{\pi}{2b^2}\left[ab\ln\left(ab+\sqrt{a^2b^2+1}\right) - \sqrt{a^2b^2+1} + 1\right] \quad [|\arg(1+a^2b^2)| < \pi].$$

21. $$\int_0^1 x\ln\frac{1+a\sqrt{1-x^2}}{1-a\sqrt{1-x^2}}\, \mathbf{K}(x)\, dx = \frac{\pi}{a}\left[\frac{\pi}{2} - \mathbf{E}(a)\right] \quad [|\arg(1-a)| < \pi].$$

22. $$\int_0^1 x^{s-1}\ln^2\left(a\sqrt{x}+\sqrt{1+a^2x}\right) \mathbf{K}(\sqrt{1-x})\, dx = \frac{\pi a^2 \Gamma^2(s+1)}{2\Gamma^2\left(s+\frac{3}{2}\right)}$$
$$\times\, {}_5F_4\left(\begin{matrix}1,\, 1,\, 1,\, s+1,\, s+1\\ \frac{3}{2},\, 2,\, s+\frac{3}{2},\, s+\frac{3}{2};\, -a^2\end{matrix}\right) \quad [\operatorname{Re} s > -1;\ |\arg(1+a^2)| < \pi].$$

23. $$\int_0^1 x^{-1/2}\ln^2\left(a\sqrt{x}+\sqrt{1+a^2x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi^2}{4}\left[\ln^2 2 + \ln\left(1+\sqrt{1+a^2}\right)\ln\frac{1+\sqrt{1+a^2}}{4} - 2\operatorname{Li}_2\left(\frac{1-\sqrt{1+a^2}}{2}\right)\right]$$
$$[|\arg(1+a^2)| < \pi].$$

24. $$\int_0^1 x^{-3/2}\ln^2\left(a\sqrt{x}+\sqrt{1+a^2x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx$$
$$= \pi^2\left[1-\sqrt{1-a^2} + a\ln\left(a+\sqrt{1+a^2}\right)\right] \quad [|\arg(1+a^2)| < \pi].$$

25. $$\int_0^1 (x-x^2)^s \ln^n(x-x^2)\, \mathbf{K}(\sqrt{x})\, dx = \frac{\pi^2}{\Gamma^2(3/4)}\, \mathrm{D}_s^n\left[\frac{2^{-2s-2}\Gamma^2(s+1)}{\Gamma^2\left(s+\frac{5}{4}\right)}\right]$$
$$[\operatorname{Re} s > -1].$$

4.21.4. Integrals containing $\mathbf{K}(z)$ and inverse trigonometric functions

1. $$\int_0^1 \arcsin(a\sqrt{x})\, \mathbf{K}(\sqrt{1-x})\, dx = \frac{\pi^2 a}{8}[2\psi_2(a^2) - \psi_3(a^2)]$$
$$[|\arg(1-a^2)| < \pi].$$

2. $$\int_0^1 \arcsin(\sqrt{x})\, \mathbf{K}(\sqrt{1-x})\, dx = \frac{2}{\pi}\Gamma^4\left(\frac{3}{4}\right).$$

3. $$\int_0^1 x \arcsin\left(\sqrt{1-x}\right) \mathbf{K}\left(\sqrt{x}\right) dx = \frac{1}{\pi}\Gamma^4\left(\frac{3}{4}\right) - \frac{1}{432\pi}\Gamma^4\left(\frac{1}{4}\right).$$

4. $$\int_0^1 \frac{x^{s-1}}{\sqrt{1-a^2x}} \arcsin\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi a \Gamma^2\left(s+\frac{1}{2}\right)}{2\Gamma^2(s+1)}$$
$$\times\ {}_4F_3\left(\begin{matrix} 1,\, 1,\, s+\frac{1}{2},\, s+\frac{1}{2} \\ \frac{3}{2},\, s+1,\, s+1;\ a^2 \end{matrix}\right) \quad [\operatorname{Re} s > -1/2;\ |\arg\left(1-a^2\right)| < \pi].$$

5. $$\int_0^1 \frac{x^{-1}}{\sqrt{1-a^2x}} \arcsin\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi^2}{2} \arcsin a$$
$$\left[|\arg\left(1-a^2\right)| < \pi\right].$$

6. $$\int_0^1 \frac{1}{\sqrt{1-a^2x}} \arcsin\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx = -\frac{\pi^2}{2a} \ln \frac{1+\sqrt{1-a^2}}{2}$$
$$\left[|\arg\left(1-a^2\right)| < \pi\right].$$

7. $$\int_0^1 x^{s-1} \arccos x\ \mathbf{K}(ax)\, dx = \frac{\pi^{3/2}\Gamma\left(\frac{s+1}{2}\right)}{2s^2\Gamma\left(\frac{s}{2}\right)}\, {}_4F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{s}{2},\, \frac{s+1}{2};\ a^2 \\ 1,\, \frac{s}{2}+1,\, \frac{s}{2}+1 \end{matrix}\right)$$
$$\left[\operatorname{Re} s > 0;\ |\arg\left(1-a^2\right)| < \pi\right].$$

8. $$\int_0^1 \arccos x\ \mathbf{K}(x)\, dx = \frac{\pi^2}{4} \ln 2.$$

9. $$\int_0^1 x \arccos x\ \mathbf{K}(ax)\, dx = \frac{\pi^2}{16}\left[2\psi_2(a^2) - \psi_3(a^2)\right] \qquad \left[|\arg\left(1-a^2\right)| < \pi\right].$$

10. $$\int_0^1 x \arccos x\ \mathbf{K}(x)\, dx = 4\pi^3\Gamma^{-4}\left(\frac{1}{4}\right).$$

11. $$\int_0^1 x^2 \arccos x\ \mathbf{K}(x)\, dx = \frac{\pi^2}{16} \ln 2.$$

12. $$\int_0^1 \frac{\arccos x}{x^2} \left[\frac{\pi}{2} - \mathbf{K}(ax)\right] dx = \frac{\pi}{2}\left(\sqrt{1-a^2} - \ln\frac{1+\sqrt{1-a^2}}{2} - 1\right)$$
$$\left[|\arg\left(1-a^2\right)| < \pi\right].$$

13. $$\int_0^1 \frac{x^{s-1}}{\sqrt{a^2x^2+1}} \arccos x \, \mathbf{K}\left(\frac{ax}{\sqrt{a^2x^2+1}}\right) dx = \frac{\pi^{3/2}\Gamma\left(\frac{s+1}{2}\right)}{2s^2\Gamma\left(\frac{s}{2}\right)}$$
$$\times\, {}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{s}{2}, \frac{s+1}{2}; -a^2 \\ 1, \frac{s}{2}+1, \frac{s}{2}+1\end{matrix}\right) \quad \left[\operatorname{Re} s > -1/2;\ \left|\arg\left(1+a^2\right)\right| < \pi\right].$$

14. $$\int_0^1 \frac{x}{\sqrt{a^2x^2+1}} \arccos x \, \mathbf{K}\left(\frac{ax}{\sqrt{a^2x^2+1}}\right) dx = \frac{\pi^2}{16}\left[2\psi_2(-a^2) - \psi_3(-a^2)\right]$$
$$\left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

15. $$\int_0^1 x^{s-1} \arctan\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi a \Gamma^2\left(s+\frac{1}{2}\right)}{2\Gamma^2(s+1)}\, {}_4F_3\left(\begin{matrix}\frac{1}{2}, 1, s+\frac{1}{2}, s+\frac{1}{2}; -a^2 \\ \frac{3}{2}, s+1, s+1\end{matrix}\right)$$
$$\left[\operatorname{Re} s > -1/2;\ \left|\arg\left(1+a^2\right)\right| < \pi\right].$$

16. $$\int_0^1 \arctan\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi}{2a}\left[2\sqrt{1+a^2}\, \mathbf{E}\left(\frac{a}{\sqrt{1+a^2}}\right) - \pi\right]$$
$$\left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

17. $$\int_0^1 \arctan\left(\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\sqrt{2\pi}}{8}\left[\Gamma^2\left(\frac{1}{4}\right) + 8\pi^2\Gamma^{-2}\left(\frac{1}{4}\right) - (2\pi)^{3/2}\right].$$

18. $$\int_0^1 \arctan\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi}{a}\left[(1+a^2)^{1/2}\, \mathbf{E}\left(\frac{a}{\sqrt{1+a^2}}\right) - \frac{\pi}{2}\right]$$
$$\left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

19. $$\int_0^1 x^{s-1} \arcsin^2\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi a^2 \Gamma^2(s+1)}{2\Gamma^2\left(s+\frac{3}{2}\right)}$$
$$\times\, {}_5F_4\left(\begin{matrix}1, 1, 1, s+1, s+1; a^2 \\ \frac{3}{2}, 2, s+\frac{3}{2}, s+\frac{3}{2}\end{matrix}\right) \quad \left[\operatorname{Re} s > -1;\ \left|\arg\left(1-a^2\right)\right| < \pi\right].$$

20. $$\int_0^1 x^{-1/2} \arcsin^2\left(a\sqrt{x}\right) \mathbf{K}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi^2}{4}\left[2\operatorname{Li}_2\left(\frac{1-\sqrt{1-a^2}}{2}\right) - \ln^2\left(\frac{1+\sqrt{1-a^2}}{2}\right)\right] \quad \left[\left|\arg\left(1-a^2\right)\right| < \pi\right].$$

21. $\displaystyle\int_0^1 x^{-3/2}\arcsin^2(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \pi^2\left(\sqrt{1-a^2}+a\arcsin a-1\right)$

$\left[|\arg\left(1-a^2\right)|<\pi\right].$

22. $\displaystyle\int_0^1 x^{-1/2}\arcsin^2\sqrt{x}\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{24}\left(\pi^2-12\ln^2 2\right).$

23. $\displaystyle\int_0^1 x^{-3/2}\arcsin^2(\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2}(\pi-2).$

4.21.5. Integrals containing $\mathbf{K}(z)$ and $\mathrm{Li}_2(z)$

1. $\displaystyle\int_0^1 x^{s-1}\,\mathrm{Li}_2(ax)\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi a\,\Gamma^2(s+1)}{2\Gamma^2\left(s+\frac{3}{2}\right)}$

$\displaystyle\times\,{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,s+1,\,s+1;\;a\\ 2,\,2,\,s+\frac{3}{2},\,s+\frac{3}{2}\end{matrix}\right)\quad\left[\mathrm{Re}\,s>-1;\;|\arg\left(1-a^2\right)|<\pi\right].$

2. $\displaystyle\int_0^1 x^{-3/2}\,\mathrm{Li}_2(ax)\,\mathbf{K}(\sqrt{1-x})\,dx = 2\pi\left[2(a-1)\,\mathbf{K}(\sqrt{a})+4\,\mathbf{E}(\sqrt{a})-\pi\right]$

$\left[|\arg\left(1-a^2\right)|<\pi\right].$

3. $\displaystyle\int_0^1 x^{-3/2}\,\mathrm{Li}_2(-ax)\,\mathbf{K}(\sqrt{1-x})\,dx = 4\pi\sqrt{a+1}$

$\displaystyle\times\left[2\,\mathbf{E}\left(\sqrt{\frac{a}{a+1}}\right)-\mathbf{K}\left(\sqrt{\frac{a}{a+1}}\right)\right]-2\pi^2\quad\left[|\arg\left(1+a^2\right)|<\pi\right].$

4. $\displaystyle\int_0^1 x^{-3/2}\,\mathrm{Li}_2(x)\,\mathbf{K}(\sqrt{1-x})\,dx = 2\pi(4-\pi).$

5. $\displaystyle\int_0^1 x^{-3/2}\,\mathrm{Li}_2(-x)\,\mathbf{K}(\sqrt{1-x})\,dx = 2^{5/2}\sqrt{\pi}\,\Gamma^2\left(\frac{3}{4}\right)-2\pi^2.$

4.21.6. Integrals containing $\mathbf{K}(z)$, $\mathrm{shi}\,(z)$ and $\mathrm{Si}\,(z)$

1. $\displaystyle\int_0^1 x^{s-1}\left\{\begin{matrix}\mathrm{shi}\,(a\sqrt{x})\\ \mathrm{Si}\,(a\sqrt{x})\end{matrix}\right\}\mathbf{K}(\sqrt{1-x})\,dx$

$\displaystyle=\frac{\pi a\,\Gamma^2\left(s+\frac{1}{2}\right)}{2\Gamma^2(s+1)}\,{}_3F_4\left(\begin{matrix}\frac{1}{2},\,s+\frac{1}{2},\,s+\frac{1}{2};\;\pm\frac{a^2}{4}\\ \frac{3}{2},\,\frac{3}{2},\,s+1,\,s+1\end{matrix}\right)\quad[\mathrm{Re}\,s>-1/2].$

2. $$\int_0^1 \operatorname{Si}(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi^2}{4}\left[a J_0^2\left(\frac{a}{2}\right) - 2J_0\left(\frac{a}{2}\right) J_1\left(\frac{a}{2}\right) + a J_1^2\left(\frac{a}{2}\right)\right].$$

4.21.7. Integrals containing $\mathbf{K}(z)$ and $\operatorname{erf}(z)$

1. $$\int_0^1 x^{s-1} \operatorname{erf}(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx$$
$$= \frac{\sqrt{\pi}\,a\,\Gamma^2\left(s+\frac{1}{2}\right)}{\Gamma^2(s+1)}\,{}_3F_3\left(\begin{matrix}\frac{1}{2},\, s+\frac{1}{2},\, s+\frac{1}{2};\; -a^2\\ \frac{3}{2},\, s+1,\, s+1\end{matrix}\right) \quad [\operatorname{Re} s > -1/2].$$

2. $$\int_0^1 x^{s-1} e^{a^2 x} \operatorname{erf}(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)\,dx$$
$$= \frac{\sqrt{\pi}\,a\,\Gamma^2\left(s+\frac{1}{2}\right)}{\Gamma^2(s+1)}\,{}_3F_3\left(\begin{matrix}1,\, s+\frac{1}{2},\, s+\frac{1}{2};\; a^2\\ \frac{3}{2},\, s+1,\, s+1\end{matrix}\right) \quad [\operatorname{Re} s > -1/2].$$

4.21.8. Integrals containing $\mathbf{K}(z)$, $S(z)$ and $C(z)$

1. $$\int_0^1 x^{s-1} S(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)\,dx$$
$$= \frac{1}{3}\sqrt{\frac{\pi a^3}{2}}\,\frac{\Gamma^2\left(s+\frac{3}{4}\right)}{\Gamma^2\left(s+\frac{5}{4}\right)}\,{}_3F_4\left(\begin{matrix}\frac{3}{4},\, s+\frac{3}{4},\, s+\frac{3}{4};\; -\frac{a^2}{4}\\ \frac{3}{2},\, \frac{7}{4},\, s+\frac{5}{4},\, s+\frac{5}{4}\end{matrix}\right) \quad [\operatorname{Re} s > -3/4].$$

2. $$\int_0^1 x^{s-1} C(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)\,dx$$
$$= \sqrt{\frac{\pi a}{2}}\,\frac{\Gamma^2\left(s+\frac{1}{4}\right)}{\Gamma^2\left(s+\frac{3}{4}\right)}\,{}_3F_4\left(\begin{matrix}\frac{1}{4},\, s+\frac{1}{4},\, s+\frac{1}{4};\; -\frac{a^2}{4}\\ \frac{1}{2},\, \frac{5}{4},\, s+\frac{3}{4},\, s+\frac{3}{4}\end{matrix}\right) \quad [\operatorname{Re} s > -1/4].$$

4.21.9. Integrals containing $\mathbf{K}(z)$ and $\gamma(\nu, z)$

1. $$\int_0^1 x^{s-1} \gamma(\nu,\, ax)\,\mathbf{K}\left(\sqrt{1-x}\right)\,dx$$
$$= \frac{\pi a^{\nu}\Gamma^2(s+\nu)}{2\nu\Gamma^2\left(s+\nu+\frac{1}{2}\right)}\,{}_3F_3\left(\begin{matrix}\nu,\, s+\nu,\, s+\nu;\; -a\\ \nu+1,\, s+\nu+\frac{1}{2},\, s+\nu+\frac{1}{2}\end{matrix}\right) \quad [\operatorname{Re}(s+\nu) > 0].$$

2. $$\int_0^1 x^{s-1} e^{ax} \gamma(\nu, ax) \, \mathbf{K}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi a^\nu \Gamma^2(s+\nu)}{2\nu \Gamma^2\left(s+\nu+\frac{1}{2}\right)} \, {}_3F_3\left(\begin{matrix} 1, \, s+\nu, \, s+\nu; \; a \\ \nu+1, \, s+\nu+\frac{1}{2}, \, s+\nu+\frac{1}{2} \end{matrix}\right) \quad [\mathrm{Re}\,(s+\nu) > 0].$$

4.21.10. Integrals containing $\mathbf{K}(z)$, $J_\nu(z)$ and $I_\nu(z)$

1. $$\int_0^1 x^{s-1} \left\{\begin{matrix} J_\nu(a\sqrt{x}) \\ I_\nu(a\sqrt{x}) \end{matrix}\right\} \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi a^\nu \Gamma^2\left(s+\frac{\nu}{2}\right)}{2^{\nu+1} \Gamma^2\left(s+\frac{\nu+1}{2}\right) \Gamma(\nu+1)}$$
$$\times \, {}_2F_3\left(\begin{matrix} s+\frac{\nu}{2}, \, s+\frac{\nu}{2}; \; \mp\frac{a^2}{4} \\ s+\frac{\nu+1}{2}, \, s+\frac{\nu+1}{2}, \, \nu+1 \end{matrix}\right) \quad [\mathrm{Re}\,(2s+\nu) > 0].$$

2. $$\int_0^1 \left\{\begin{matrix} I_0(a\sqrt{x}) \\ J_0(a\sqrt{x}) \end{matrix}\right\} \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi}{a} \left\{\begin{matrix} \mathbf{L}_0(a) \\ \mathbf{H}_0(a) \end{matrix}\right\}.$$

3. $$\int_0^1 \sqrt{x} \, J_1(a\sqrt{x}) \, \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi}{a^2} \left[\mathbf{H}_0(a) - a\,\mathbf{H}_{-1}(a)\right].$$

4. $$\int_0^1 \sqrt{x} \, I_1(a\sqrt{x}) \, \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi}{a^2} \left[a\,\mathbf{L}_{-1}(a) - \mathbf{L}_0(a)\right].$$

5. $$\int_0^1 x^{s-1} e^{ax} I_\nu(ax) \, \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{\pi \Gamma^2(s+\nu)}{2\Gamma^2\left(s+\nu+\frac{1}{2}\right) \Gamma(\nu+1)} \left(\frac{a}{2}\right)^\nu$$
$$\times \, {}_3F_3\left(\begin{matrix} \nu+\frac{1}{2}, \, s+\nu, \, s+\nu; \; 2a \\ 2\nu+1, \, s+\nu+\frac{1}{2}, \, s+\nu+\frac{1}{2} \end{matrix}\right) \quad [\mathrm{Re}\,(s+\nu) > 0].$$

6. $$\int_0^1 \left\{\begin{matrix} I_0(a\sqrt{x}) \\ J_0(a\sqrt{x}) \end{matrix}\right\}^2 \mathbf{K}\left(\sqrt{1-x}\right) dx = \frac{1}{a} \left\{\begin{matrix} \mathrm{shi}\,(2a) \\ \mathrm{Si}\,(2a) \end{matrix}\right\}.$$

7. $$\int_0^1 J_1^2(a\sqrt{x}) \, \mathbf{K}(\sqrt{1-x}) \, dx = \frac{1}{a^2} \left[\cos(2a) + a\,\mathrm{Si}\,(2a) - 1\right].$$

8. $$\int_0^1 I_1^2(a\sqrt{x}) \, \mathbf{K}(\sqrt{1-x}) \, dx = \frac{1}{a^2} \left[2\sinh^2 a - a\,\mathrm{shi}\,(2a)\right].$$

9. $$\int_0^1 x\, J_0^2(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{1}{16a^3}[5\sin(2a) - 6a\cos(2a) + 2(2a^2-1)\operatorname{Si}(2a)].$$

10. $$\int_0^1 x\, I_0^2(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{1}{16a^3}[-5\sinh(2a) + 6a\cosh(2a) + 2(2a^2+1)\operatorname{shi}(2a)].$$

11. $$\int_0^1 \frac{1}{x} J_1^2(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \mathbf{C} - \frac{1}{2} - \frac{1}{4a^2} + \frac{1}{2a}\sin(2a) + \frac{1}{4a^2}\cos(2a) + \ln(2a) - \operatorname{ci}(2a).$$

12. $$\int_0^1 \frac{1}{x} I_1^2(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = -\mathbf{C} + \frac{1}{2} - \frac{1}{4a^2} - \frac{1}{2a}\sinh(2a) + \frac{1}{4a^2}\cosh(2a) - \ln(2a) + \operatorname{chi}(2a).$$

13. $$\int_0^1 x\, J_1^2(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{1}{16a^3}[10a\cos(2a) - 11\sin(2a) + (4a^2+6)\operatorname{Si}(2a)].$$

14. $$\int_0^1 x\, I_1^2(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{1}{16a^3}[10a\cosh(2a) - 11\sinh(2a) + (6-4a^2)\operatorname{shi}(2a)].$$

4.21.11. Integrals containing $\mathbf{K}(z)$, $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

1. $$\int_0^1 x^{s-1}\begin{Bmatrix}\mathbf{H}_\nu(a\sqrt{x})\\ \mathbf{L}_\nu(a\sqrt{x})\end{Bmatrix}\mathbf{K}(\sqrt{1-x})\,dx = \frac{\sqrt{\pi}\,\Gamma^2\left(s+\frac{\nu+1}{2}\right)\left(\frac{a}{2}\right)^{\nu+1}}{2\Gamma^2\left(s+\frac{\nu}{2}+1\right)\Gamma\left(\nu+\frac{3}{2}\right)} \times {}_3F_4\left(\begin{matrix}1,\ s+\frac{\nu+1}{2},\ s+\frac{\nu+1}{2};\ \mp\frac{a^2}{4}\\ \frac{3}{2},\ \nu+\frac{3}{2},\ s+\frac{\nu}{2}+1,\ s+\frac{\nu}{2}+1\end{matrix}\right) \quad [\operatorname{Re}(2s+\nu) > -1].$$

2. $$\int_0^1 \mathbf{H}_0(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a}[1 - J_0(a)].$$

3. $$\int_0^1 \mathbf{L}_0(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a}[I_0(a) - 1].$$

4. $$\int_0^1 x\,\mathbf{H}_0(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{4a^3}[a^2-4+4(1-a^2)J_0(a)+8a\,J_1(a)].$$

5. $$\int_0^1 x\,\mathbf{L}_0(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{4a^3}[4(1+a^2)\,I_0(a)-8a\,I_1(a)-a^2-4].$$

4.21.12. Integrals containing $\mathbf{K}(z)$ and $L_n^\lambda(z)$

1. $$\int_0^1 x^{s-1}L_n^\lambda(ax)\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi\Gamma^2(s)\,(\lambda+1)_n}{n!2\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_3F_3\left(\begin{matrix}-n,\ s,\ s;\ a\\ \lambda+1,\ s+\frac{1}{2},\ s+\frac{1}{2}\end{matrix}\right)\quad [\operatorname{Re} s>0].$$

4.21.13. Integrals containing products of $\mathbf{K}(z)$

1. $$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathbf{K}^2(x)\,dx = \frac{\pi^{-3}}{64}\left[-\frac{1}{2}\Gamma^8\left(\frac{1}{4}\right)+\Gamma^8\left(\frac{1}{4}\right)\,{}_5F_4\left(\begin{matrix}\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ 1\\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4};\ 1\end{matrix}\right)\right.$$
$$\left.+\,256\Gamma^8\left(\frac{3}{4}\right)\,{}_5F_4\left(\begin{matrix}\frac{3}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4},\ 1\\ \frac{5}{4},\ \frac{5}{4},\ \frac{5}{4},\ \frac{5}{4};\ 1\end{matrix}\right)\right].$$

2. $$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,\mathbf{K}^2(x)dx = \frac{\pi^4}{16}\,{}_7F_6\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{5}{4}\\ \frac{1}{4},\ 1,\ 1,\ 1,\ 1,\ 1;\ 1\end{matrix}\right).$$

3. $$\int_0^1 x^{s-1}\,\mathbf{K}(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{4}\,\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ s,\ s;\ a^2\\ 1,\ s+\frac{1}{2},\ s+\frac{1}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re} s>0;\ \left|\arg\left(1-a^2\right)\right|<\pi\right].$$

4. $$\int_0^1 \mathbf{K}(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{2a}\left[\operatorname{Li}_2(a)-\operatorname{Li}_2(-a)\right]\qquad \left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

5. $$\int_0^1 \mathbf{K}(\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi^3}{8}.$$

6. $$\int_0^1 x\,\mathbf{K}(a\sqrt{x})\,\mathbf{K}(\sqrt{1-x})\,dx = \frac{\pi}{8a^3}\left\{(1-a^2)\ln\frac{1-a}{1+a}\right.$$
$$\left.+\,(1+a^2)[\operatorname{Li}_2(a)-\operatorname{Li}_2(-a)]\right\}\quad \left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

7. $\displaystyle\int_0^1 x\,\mathbf{K}(\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^3}{16}.$

8. $\displaystyle\int_0^1 x^2\,\mathbf{K}(\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{11\pi^3}{256}.$

9. $\displaystyle\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\,\mathbf{K}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)\mathbf{K}\left(\sqrt{1-x}\right)dx$

$$= \frac{\pi^2}{4}\,\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ s,\ s;\ -a^2\\ 1,\ s+\frac{1}{2},\ s+\frac{1}{2}\end{matrix}\right) \qquad \left[\operatorname{Re} s>0;\ \left|\arg\left(1+a^2\right)\right|<\pi\right].$$

10. $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x}}\,\mathbf{K}\left(\sqrt{\frac{x}{1+x}}\right)\mathbf{K}\left(\sqrt{1-x}\right)dx = \pi\mathbf{G}.$

11. $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x}}\,\mathbf{K}\left(\sqrt{\frac{x}{1+x}}\right)\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{8}.$

12. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x}}\,\mathbf{K}\left(\sqrt{\frac{x}{1+x}}\right)\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{64}\,(3+14\,\mathbf{G}).$

13. $\displaystyle\int_0^{\pi/2} \mathbf{K}(\sin x)\,\mathbf{K}(\sin(2x))\,dx = -\frac{\pi^{-3}}{128}\Gamma^8\left(\frac{1}{4}\right)$

$$+\frac{\pi^{-3}}{64}\Gamma^8\left(\frac{1}{4}\right)\,{}_7F_6\left(\begin{matrix}\frac{1}{8},\ \frac{1}{8},\ \frac{1}{4},\ \frac{1}{4},\ \frac{5}{8},\ \frac{5}{8},\ 1\\ \frac{3}{8},\ \frac{3}{8},\ \frac{3}{4},\ \frac{3}{4},\ \frac{7}{8},\ \frac{7}{8};\ 1\end{matrix}\right) - \frac{\pi}{9}\,{}_7F_6\left(\begin{matrix}\frac{5}{8},\ \frac{5}{8},\ \frac{3}{4},\ \frac{3}{4},\ 1,\ \frac{9}{8},\ \frac{9}{8}\\ \frac{7}{8},\ \frac{7}{8},\ \frac{5}{4},\ \frac{5}{4},\ \frac{11}{8},\ \frac{11}{8};\ 1\end{matrix}\right).$$

4.22. The Complete Elliptic Integral $\mathbf{E}(z)$

4.22.1. Integrals containing $\mathbf{E}(z)$ and algebraic functions

1. $\displaystyle\int_0^1 \frac{x(1-x^2)^{s-1}}{(1-ax^2)^{s+1/2}}\,\mathbf{E}(x)\,dx$

$$= \frac{\pi}{2}\,\frac{s\,\Gamma^2(s)}{(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}\,(1-a)^{-1/2}\,{}_2F_1\left(\begin{matrix}1/2,\ s\\ s+3/2;\ a\end{matrix}\right)$$

$$[\operatorname{Re} s>0;\ |\arg(1-a)|<\pi].$$

2. $$\int_0^1 \frac{x}{\sqrt{1-x^2}}\, \mathbf{E}(ax)\, dx = \frac{\pi}{96a^3}\left[3a(2a^2+1)\sqrt{1-a^2} + 3(4a^2-1)\arcsin a\right] \quad [|\arg(1-a^2)| < \pi].$$

3. $$\int_0^a x^{s-1}(a-x)^{t-1}\, \mathbf{E}\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi}{2}\, \mathrm{B}(s,t)\, a^{s+t-1}\, {}_4F_3\left(\begin{matrix} -\frac{1}{2}, \frac{1}{2}, s, t; \frac{a^2b^2}{4} \\ 1, \frac{s+t}{2}, \frac{s+t+1}{2} \end{matrix}\right)$$
$$[a, \operatorname{Re} s, \operatorname{Re} t > 0;\ |\arg(4-a^2b^2)| < \pi].$$

4. $$\int_0^a \mathbf{E}\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a}{4}\left(\frac{2}{ab}\arcsin\frac{ab}{2} + \sqrt{1-\frac{a^2b^2}{4}}\right)$$
$$[a > 0;\ |\arg(4-a^2b^2)| < \pi].$$

5. $$\int_0^a x\, \mathbf{E}\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi a^2}{8}\left(\frac{2}{ab}\arcsin\frac{ab}{2} + \sqrt{1-\frac{a^2b^2}{4}}\right)$$
$$[a > 0;\ |\arg(4-a^2b^2)| < \pi].$$

6. $$\int_0^a x^2\, \mathbf{E}\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi}{64b^3}\left[4(3a^2b^2-1)\arcsin\frac{ab}{2} + ab(5a^2b^2+2)\sqrt{1-\frac{a^2b^2}{4}}\right]$$
$$[a > 0;\ |\arg(4-a^2b^2)| < \pi].$$

7. $$\int_0^a x^{1/2}(a-x)^{1/2}\, \mathbf{E}\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi^2 a^2}{48}\left[(4-a^2b^2)\psi_1\left(\frac{a^2b^2}{4}\right)\right.$$
$$\left. + (2+a^2b^2)\psi_2\left(\frac{a^2b^2}{4}\right) - \frac{a^2b^2}{4}\psi_3\left(\frac{a^2b^2}{4}\right)\right] \quad [a > 0;\ |\arg(4-a^2b^2)| < \pi].$$

8. $$\int_0^a x^{-1/2}(a-x)^{1/2}\, \mathbf{E}\left(b\sqrt{x(a-x)}\right) dx = \frac{\pi^2 a}{16}\left[(4-a^2b^2)\psi_1\left(\frac{a^2b^2}{4}\right)\right.$$
$$\left. + a^2b^2\psi_2\left(\frac{a^2b^2}{4}\right) - \frac{a^2b^2}{4}\psi_3\left(\frac{a^2b^2}{4}\right)\right] \quad [a > 0;\ |\arg(4-a^2b^2)| < \pi].$$

9. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\,\mathbf{E}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi^2}{8}\left[(4-a^2b^2)\psi_1\left(\frac{a^2b^2}{4}\right) + a^2b^2\psi_2\left(\frac{a^2b^2}{4}\right) - \frac{a^2b^2}{4}\psi_3\left(\frac{a^2b^2}{4}\right)\right] \quad \left[a>0;\ |\arg(4-a^2b^2)|<\pi\right].$$

10. $$\int_0^a \frac{x^{s-1}(a-x)^{t-1}}{1-b^2x(a-x)}\,\mathbf{E}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{2}\,\mathrm{B}\,(s,\,t)\,a^{s+t-1}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,s,\,t;\ \frac{a^2b^2}{4}\\ 1,\,\frac{s+t}{2},\,\frac{s+t+1}{2}\end{matrix}\right)$$
$$\left[a,\,\mathrm{Re}\,s,\,\mathrm{Re}\,t>0;\ |\arg(4-a^2b^2)|<\pi\right].$$

11. $$\int_0^a \frac{1}{1-b^2x(a-x)}\,\mathbf{E}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi a}{\sqrt{4-a^2b^2}} \quad \left[a>0;\ |\arg(4-a^2b^2)|<\pi\right].$$

12. $$\int_0^a \frac{x}{1-b^2x(a-x)}\,\mathbf{E}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi a^2}{2\sqrt{4-a^2b^2}} \quad \left[a>0;\ |\arg(4-a^2b^2)|<\pi\right].$$

13. $$\int_0^a \frac{x^{1/2}(a-x)^{1/2}}{1-b^2x(a-x)}\,\mathbf{E}\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{\pi^2a^2}{8(4-a^2b^2)}\left[(2+a^2b^2)\psi_2\left(\frac{a^2b^2}{4}\right) - a^2b^2\psi_1\left(\frac{a^2b^2}{4}\right) - \frac{a^2b^2}{4}\psi_3\left(\frac{a^2b^2}{4}\right)\right]$$
$$\left[a>0;\ |\arg(4-a^2b^2)|<\pi\right].$$

14. $$\int_0^a \frac{x^{-1/2}(a-x)^{1/2}}{1-b^2x(a-x)}\,\mathbf{E}\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{\pi^2a}{4(4-a^2b^2)}\left[2(2-a^2b^2)\psi_1\left(\frac{a^2b^2}{4}\right) + \frac{3a^2b^2}{2}\psi_2\left(\frac{a^2b^2}{4}\right) - \frac{a^2b^2}{4}\psi_3\left(\frac{a^2b^2}{4}\right)\right]$$
$$\left[a>0;\ |\arg(4-a^2b^2)|<\pi\right].$$

15. $$\int_0^a \frac{x^{-1/2}(a-x)^{-1/2}}{1-b^2x(a-x)}\,\mathbf{E}\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{\pi^2}{2(4-a^2b^2)}\left[2(2-a^2b^2)\psi_1\left(\frac{a^2b^2}{4}\right) + \frac{3a^2b^2}{2}\psi_2\left(\frac{a^2b^2}{4}\right) - \frac{a^2b^2}{4}\psi_3\left(\frac{a^2b^2}{4}\right)\right]$$
$$\left[a>0;\ |\arg(4-a^2b^2)|<\pi\right].$$

16. $$\int_0^a x^{s+1/2}(a-x)^s\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx = 2^{-2s-2}\pi^{3/2}a^{2s+3/2}\frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)}$$
$$\times\ {}_3F_2\left(\begin{matrix}-\frac{1}{2},\ \frac{1}{2},\ 2s+2\\ 1,\ 2s+\frac{5}{2};\ \frac{ab^2}{2}\end{matrix}\right)\qquad\left[a>0;\ \operatorname{Re}s>-1;\ |\arg(2-a^2b^2)|<\pi\right].$$

17. $$\int_0^a x^{1/2}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\frac{\pi a^{1/2}}{16b^3}\left[\sqrt{\frac{2}{a}}\,(2ab^2+1)\arcsin\left(b\sqrt{\frac{a}{2}}\right)+b(3ab^2-1)\sqrt{1-\frac{ab^2}{2}}\right]$$
$$\left[a>0;\ |\arg(2-a^2b^2)|<\pi\right].$$

18. $$\int_0^a x^{-1/2}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\frac{\pi\sqrt{a}}{2}\left[\frac{1}{b}\sqrt{\frac{2}{a}}\arcsin\left(b\sqrt{\frac{a}{2}}\right)+\sqrt{1-\frac{ab^2}{2}}\right]\qquad\left[|\arg(2-a^2b^2)|<\pi\right].$$

19. $$\int_0^a x^{1/4}(a-x)^{-1/4}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\frac{\pi^2 a}{12\sqrt{2}}\left[4\left(1-\frac{ab^2}{2}\right)\psi_1\left(\frac{ab^2}{2}\right)-(1-2ab^2)\,\psi_2\left(\frac{ab^2}{2}\right)-\frac{ab^2}{2}\psi_3\left(\frac{ab^2}{2}\right)\right]$$
$$\left[a>0;\ |\arg(2-a^2b^2)|<\pi\right].$$

20. $$\int_0^a x^{-1/4}(a-x)^{-3/4}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\frac{\pi^2}{\sqrt{2}}\left[\left(1-\frac{ab^2}{2}\right)\psi_1\left(\frac{ab^2}{2}\right)+\frac{ab^2}{2}\psi_2\left(\frac{ab^2}{2}\right)-\frac{ab^2}{8}\psi_3\left(\frac{ab^2}{2}\right)\right]$$
$$\left[a>0;\ |\arg(2-a^2b^2)|<\pi\right].$$

21. $$\int_0^a \frac{x^{s+1/2}(a-x)^s}{1-b^2\sqrt{x(a-x)}}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx=\frac{\pi^{3/2}a^{2s+3/2}\Gamma(2s+2)}{2^{2s+2}\Gamma\left(2s+\frac{5}{2}\right)}$$
$$\times\ {}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{3}{2},\ 2s+2\\ 1,\ 2s+\frac{5}{2};\ \frac{ab^2}{2}\end{matrix}\right)\qquad\left[a>0;\ \operatorname{Re}s>-1;\ |\arg(2-a^2b^2)|<\pi\right].$$

22. $$\int_0^a \frac{x^{1/2}}{1-b^2\sqrt{x(a-x)}}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx=\frac{\pi\sqrt{a}\,(ab^2+2)}{4b^2}\left(1-\frac{ab^2}{2}\right)^{-1/2}$$
$$-\frac{\pi}{\sqrt{2}\,b^3}\arcsin\left(b\sqrt{\frac{a}{2}}\right)\qquad\left[a>0;\ |\arg(2-a^2b^2)|<\pi\right].$$

23. $$\int_0^a \frac{x^{-1/2}}{1-b^2\sqrt{x(a-x)}}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx = \pi\sqrt{\frac{2a}{2-ab^2}}$$

$\left[a>0;\ |\arg(2-a^2b^2)|<\pi\right].$

24. $$\int_0^a \frac{x^{1/4}(a-x)^{-1/4}}{1-b^2\sqrt{x(a-x)}}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{a\pi^2}{2^{5/2}}\left(1-\frac{ab^2}{2}\right)^{-1}$$
$$\times\left[(1+ab^2)\,\psi_2\left(\frac{ab^2}{2}\right)-ab^2\psi_2\left(\frac{ab^2}{2}\right)-\frac{ab^2}{4}\psi_3\left(\frac{ab^2}{2}\right)\right]$$

$\left[a>0;\ |\arg(2-a^2b^2)|<\pi\right].$

25. $$\int_0^a \frac{x^{-1/4}(a-x)^{-3/4}}{1-b^2\sqrt{x(a-x)}}\,\mathbf{E}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$=\frac{\pi^2}{2^{5/2}}\left(1-\frac{ab^2}{2}\right)^{-1}\left[4(1-ab^2)\,\psi_1\left(\frac{ab^2}{2}\right)\right.$$
$$\left.+3ab^2\psi_2\left(\frac{ab^2}{2}\right)-\frac{ab^2}{2}\psi_3\left(\frac{ab^2}{2}\right)\right]$$

$\left[a>0;\ |\arg(2-a^2b^2)|<\pi\right].$

26. $$\int_0^a \frac{x^{-1/2}}{\left[1+b^2\sqrt{x(a-x)}\right]^{1/2}}\,\mathbf{E}\left(\frac{b\sqrt[4]{x(a-x)}}{\sqrt{1+b^2\sqrt{x(a-x)}}}\right)dx = \pi\sqrt{\frac{2a}{2+ab^2}}$$

$\left[a>0;\ |\arg(2+a^2b^2)|<\pi\right].$

27. $$\int_0^1 x^{s-1}(1+ax)^{\nu}\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi s^2\Gamma(s)}{(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}$$
$$\times\,{}_3F_2\left(\begin{matrix}-\nu,\ s,\ s+1;\ a\\ s+\frac{1}{2},\ s+\frac{3}{2}\end{matrix}\right)$$

$[\operatorname{Re} s>0;\ |\arg(1-a)|<\pi].$

4.22.2. Integrals containing $\mathbf{E}(z)$, the exponential, hyperbolic and trigonometric functions

1. $$\int_0^1 x^{s-1}e^{ax}\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi\Gamma(s)\,\Gamma(s+1)}{2\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}\,{}_2F_3\left(\begin{matrix}s,\ s+1;\ a\\ s+\frac{1}{2},\ s+\frac{3}{2}\end{matrix}\right)$$

$[\operatorname{Re} s>0].$

2. $$\int_0^1 x^{s-1}\left\{\begin{matrix}\sinh(a\sqrt{x})\\ \sin(a\sqrt{x})\end{matrix}\right\}\mathbf{E}\left(\sqrt{1-x}\right)dx$$
$$=\frac{\pi a(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}{4(s+1)\Gamma^2(s+1)}\,{}_2F_3\left(\begin{matrix}s+\frac{1}{2},\ s+\frac{3}{2};\ \mp\frac{a^2}{4}\\ \frac{3}{2},\ s+1,\ s+2\end{matrix}\right)$$

$[\operatorname{Re} s>-1/2].$

3. $$\int_0^1 x^{s-1} \left\{ \begin{matrix} \cosh(a\sqrt{x}) \\ \cos(a\sqrt{x}) \end{matrix} \right\} \mathbf{E}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi s \Gamma^2(s)}{(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)} \, {}_2F_3\left(\begin{matrix} s,\ s+1;\ \pm\frac{a^2}{4} \\ \frac{1}{2},\ s+\frac{1}{2},\ s+\frac{3}{2} \end{matrix} \right) \quad [\operatorname{Re} s > 0].$$

4. $$\int_0^1 \sinh(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^2}{2a} \left[I_0\left(\frac{a}{2}\right) I_1\left(\frac{a}{2}\right) - I_1^2\left(\frac{a}{2}\right) \right].$$

5. $$\int_0^1 \frac{1}{x} \sinh(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^2 a}{4} \left[I_0^2\left(\frac{a}{2}\right) - I_1^2\left(\frac{a}{2}\right) \right].$$

6. $$\int_0^1 \frac{1}{\sqrt{x}} \cosh(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^2}{4} \left[I_0^2\left(\frac{a}{2}\right) + I_1^2\left(\frac{a}{2}\right) \right].$$

7. $$\int_0^1 \sin(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^2}{2a} \left[J_0\left(\frac{a}{2}\right) J_1\left(\frac{a}{2}\right) - J_1^2\left(\frac{a}{2}\right) \right].$$

8. $$\int_0^1 \frac{1}{x} \sin(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^2 a}{4} \left[J_0^2\left(\frac{a}{2}\right) + J_1^2\left(\frac{a}{2}\right) \right].$$

9. $$\int_0^1 \frac{1}{\sqrt{x}} \cos(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^2}{4} \left[J_0^2\left(\frac{a}{2}\right) - J_1^2\left(\frac{a}{2}\right) \right].$$

10. $$\int_0^{\pi/2} \cos(2nx)\, \mathbf{E}(a \sin x)\, dx = -2^{-2n-3}\pi(-a^2)^n \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{(n!)^2}$$
$$\times\, {}_3F_2\left(\begin{matrix} n-\frac{1}{2},\ n+\frac{1}{2},\ n+\frac{1}{2} \\ n+1,\ 2n+1;\ a^2 \end{matrix} \right) \quad [|\arg(1-a^2)| < \pi].$$

11. $$\int_0^{\pi/2} \cos^\nu x \cos(ax)\, \mathbf{E}(b \cos x)\, dx = \frac{2^{-\nu-2}\pi^2\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}$$
$$\times\, {}_4F_3\left(\begin{matrix} -\frac{1}{2},\ \frac{1}{2},\ \frac{\nu+1}{2},\ 1+\frac{\nu}{2};\ b^2 \\ 1,\ 1+\frac{\nu-a}{2},\ 1+\frac{\nu+a}{2} \end{matrix} \right) \quad [\operatorname{Re}\nu > -1;\ |\arg(1-b^2)| < \pi].$$

12. $$\int_0^{\pi/2} \frac{\cos^\nu x}{1-b^2\cos^2 x}\cos(ax)\,\mathbf{E}(b\cos x)\,dx = \frac{2^{-\nu-2}\pi^2\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)} \times {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{3}{2},\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1;\ b^2\\ 1,\ \frac{\nu-a}{2}+1,\ \frac{\nu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}\nu > -1;\ |\arg(1-b^2)| < \pi].$$

13. $$\int_0^{\pi} \sin^\nu x \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\mathbf{E}(b\sin x)\,dx$$
$$= \frac{2^{-\nu-1}\pi^2\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} {}_4F_3\left(\begin{matrix}-\frac{1}{2},\ \frac{1}{2},\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1;\ b^2\\ 1,\ \frac{\nu-a}{2}+1,\ \frac{\nu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}\nu > -1;\ |\arg(1-b^2)| < \pi].$$

14. $$\int_0^{\pi} \frac{\sin^\nu x}{1-b^2\sin^2 x}\begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\mathbf{E}(b\sin x)\,dx$$
$$= \frac{2^{-\nu-1}\pi^2\Gamma(\nu+1)}{\Gamma\left(\frac{\nu-a}{2}+1\right)\Gamma\left(\frac{\nu+a}{2}+1\right)}\begin{Bmatrix}\sin(a\pi/2)\\ \cos(a\pi/2)\end{Bmatrix} {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{3}{2},\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1;\ b^2\\ 1,\ \frac{\nu-a}{2}+1,\ \frac{\nu+a}{2}+1\end{matrix}\right) \quad [\operatorname{Re}\nu > -1;\ |\arg(1-b^2)| < \pi].$$

15. $$\int_0^{\pi} \cos(nx)\,\mathbf{E}(a\cos x)\,dx = -2^{-n-2}\pi a^n \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma^2\left(\frac{n}{2}+1\right)}\cos^2\frac{n\pi}{2} \times {}_3F_2\left(\begin{matrix}\frac{n-1}{2},\ \frac{n+1}{2},\ \frac{n+1}{2}\\ \frac{n}{2}+1,\ n+1;\ a^2\end{matrix}\right) \quad [|\arg(1-a^2)| < \pi].$$

16. $$\int_0^{m\pi} \begin{Bmatrix}\sin(ax)\\ \cos(ax)\end{Bmatrix}\mathbf{E}(b\sin x)\,dx = \frac{\pi}{a}\sin\frac{m\pi a}{2}\begin{Bmatrix}\sin(m\pi a/2)\\ \cos(m\pi a/2)\end{Bmatrix} \times {}_3F_2\left(\begin{matrix}-\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ b^2\\ 1-\frac{a}{2},\ 1+\frac{a}{2}\end{matrix}\right) \quad [|\arg(1-b^2)| < \pi].$$

17. $$\int_0^{m\pi} e^{-ax}\,\mathbf{E}(b\sin x)\,dx = \frac{\pi}{2a}(1-e^{-m\pi a})\,{}_3F_2\left(\begin{matrix}-\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ b^2\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [|\arg(1-b^2)| < \pi].$$

18. $$\int_0^{m\pi} \frac{e^{-ax}}{1-b^2\sin^2 x}\,\mathbf{E}(b\sin x)\,dx = \frac{\pi}{2a}(1-e^{-m\pi a})\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2};\ b^2\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) \quad [|\arg(1-b^2)| < \pi].$$

19. $\displaystyle\int_0^\infty e^{-ax}\,\mathbf{E}(b\sin x)\,dx = \frac{\pi}{2a}\,{}_3F_2\left(\begin{matrix} -\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ b^2 \\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)$

$\left[\operatorname{Re} a > 0;\ \left|\arg\left(1-b^2\right)\right| < \pi\right].$

4.22.3. Integrals containing E(z) and the logarithmic function

1. $\displaystyle\int_0^1 x(1-x^2)^{-3/2}\ln x\,\mathbf{E}(x)\,dx = \frac{\pi}{2} - \frac{\pi^2}{4}.$

2. $\displaystyle\int_0^1 x^{s-1}\ln(1+ax)\,\mathbf{E}\left(\sqrt{1-x}\right)dx$

$$= \frac{\pi a(s+1)\Gamma^2(s+1)}{(2s+3)\Gamma^2\left(s+\frac{3}{2}\right)}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ s+1,\ s+2;\ -a \\ 2,\ s+\frac{3}{2},\ s+\frac{5}{2}\end{matrix}\right)$$

$[\operatorname{Re} s > -1;\ |\arg(1+a)| < \pi].$

3. $\displaystyle\int_0^1 x^{-3/2}\ln(1+ax)\,\mathbf{E}\left(\sqrt{1-x}\right)dx = 2\pi(a+1)^{1/2}\,\mathbf{E}\left(\sqrt{\frac{a}{a+1}}\right) - \pi^2$

$[|\arg(1+a)| < \pi].$

4. $\displaystyle\int_0^1 x^{-3/2}\ln(1-ax)\,\mathbf{E}\left(\sqrt{1-x}\right)dx = 2\pi\,\mathbf{E}(a) - \pi^2$ $\quad[|\arg(1-a)| < \pi].$

5. $\displaystyle\int_0^1 x^{s-1}\ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi a(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}{4(s+1)\Gamma^2(s+1)}$

$$\times\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ s+\frac{1}{2},\ s+\frac{3}{2} \\ \frac{3}{2},\ s+1,\ s+2;\ -a^2\end{matrix}\right)$$ $\left[\operatorname{Re} s > -1/2;\ \left|\arg\left(1+a^2\right)\right| < \pi\right].$

6. $\displaystyle\int_0^1 \ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx$

$$= \frac{\pi^2}{72a}\left[4(a^2+1)\psi_1(-a^2) - 4(1-2a^2)\psi_2(-a^2) - 5a^2\psi_3(-a^2)\right]$$

$\left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$

7. $\displaystyle\int_0^1 \frac{1}{\sqrt{1+a^2x}}\ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx$

$$= \frac{\pi^2}{8a^3}\left(2+a^2-2\sqrt{1+a^2}+2a^2\ln\frac{1+\sqrt{1+a^2}}{2}\right)$$ $\left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$

8. $$\int_0^1 \frac{x^{-1}}{\sqrt{1+a^2x}} \ln\left(a\sqrt{x}+\sqrt{1+a^2x}\right) \mathbf{E}(\sqrt{1-x})\,dx$$
$$= \frac{\pi^2}{2a}\left(\sqrt{1+a^2}-1\right) \quad \left[\left|\arg\left(1+a^2\right)\right|<\pi\right].$$

9. $$\int_0^1 x^{s-1} \ln\frac{1+a\sqrt{x}}{1-a\sqrt{x}}\, \mathbf{E}(\sqrt{1-x})\,dx = \frac{\pi a(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}{2(s+1)\Gamma^2(s+1)}$$
$$\times\, {}_4F_3\left(\begin{matrix}\frac{1}{2},\, 1,\, s+\frac{1}{2},\, s+\frac{3}{2}\\ \frac{3}{2},\, s+1,\, s+2;\; a^2\end{matrix}\right) \quad \left[\operatorname{Re} s>-1/2;\ \left|\arg\left(1-a^2\right)\right|<\pi\right].$$

10. $$\int_0^1 \ln\frac{1+a\sqrt{x}}{1-a\sqrt{x}}\, \mathbf{E}(\sqrt{1-x})\,dx$$
$$= \frac{\pi}{6a^3}\left[3\pi a^2+4(a^2-1)\,\mathbf{K}(a)+4(1-2a^2)\,\mathbf{E}(a)\right] \quad \left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

11. $$\int_0^1 \ln\frac{1+\sqrt{x}}{1-\sqrt{x}}\, \mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2}-\frac{2\pi}{3}.$$

12. $$\int_0^1 \frac{1}{x}\ln\frac{1+a\sqrt{x}}{1-a\sqrt{x}}\, \mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{2\pi}{a}\left[(a^2-1)\,\mathbf{K}(a)+\mathbf{E}(a)\right]$$
$$\left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

13. $$\int_0^1 \frac{1}{x}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}\, \mathbf{E}\left(\sqrt{1-x}\right)dx = 2\pi.$$

14. $$\int_0^1 x^{s-1}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\, \mathbf{E}\,(ax)\,dx = \frac{\pi^{3/2}\Gamma\left(\frac{s}{2}\right)}{2s\,\Gamma\left(\frac{s+1}{2}\right)}\, {}_4F_3\left(\begin{matrix}-\frac{1}{2},\, \frac{1}{2},\, \frac{s}{2},\, \frac{s}{2};\; a^2\\ 1,\, \frac{s+1}{2},\, \frac{s}{2}+1\end{matrix}\right)$$
$$\left[\operatorname{Re} s>0;\ \left|\arg\left(1-a^2\right)\right|<\pi\right].$$

15. $$\int_0^1 x\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\, \mathbf{E}(ax)\,dx$$
$$= \frac{\pi}{6a^2}\left[3a\arcsin a+(a^2+2)\sqrt{1-a^2}-2\right] \quad \left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

16. $$\int_0^1 \frac{x}{1-a^2x^2}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\, \mathbf{E}(ax)\,dx = \frac{\pi}{a^2}\left(1-\sqrt{1-a^2}\right)$$
$$\left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

17. $$\int_0^1 \frac{x}{\sqrt{1+a^2x^2}} \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \mathbf{E}\left(\frac{ax}{\sqrt{1+a^2x^2}}\right) dx = \frac{\pi}{a^2}\left(\sqrt{1+a^2}-1\right)$$

$$\left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

18. $$\int_0^a x^{s-1} \ln \frac{a+\sqrt{a^2-x^2}}{x} \mathbf{E}(bx)\, dx$$

$$= \frac{\pi^{3/2} a^s \Gamma\left(\frac{s}{2}\right)}{4s\,\Gamma\left(\frac{s+1}{2}\right)} {}_4F_3\left(\begin{matrix} -\frac{1}{2}, \frac{1}{2}, \frac{s}{2}, \frac{s}{2};\ a^2b^2 \\ 1, \frac{s+1}{2}, \frac{s}{2}+1 \end{matrix}\right)$$

$$\left[a, \operatorname{Re} s > 0;\ \left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

19. $$\int_0^a x \ln \frac{a+\sqrt{a^2-x^2}}{x} \mathbf{E}(bx)\, dx$$

$$= \frac{\pi}{12b^2}\left[3ab \arcsin(ab) + (a^2b^2+2)\sqrt{1-a^2b^2} - 2\right]$$

$$\left[a > 0;\ \left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

20. $$\int_0^a x^3 \ln \frac{a+\sqrt{a^2-x^2}}{x} \mathbf{E}(bx)\, dx = \frac{\pi}{1440b^4}$$

$$\times\left[15ab(4a^2b^2+3)\arcsin(ab) + (54a^4b^4 - 13a^2b^2 + 64)\sqrt{1-a^2b^2} - 64\right]$$

$$\left[a > 0;\ \left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

21. $$\int_0^a \frac{x^{s-1}}{1-b^2x^2} \ln \frac{a+\sqrt{a^2-x^2}}{x} \mathbf{E}(bx)\, dx$$

$$= \frac{\pi^{3/2} a^s \Gamma\left(\frac{s}{2}\right)}{4s\,\Gamma\left(\frac{s+1}{2}\right)} {}_4F_3\left(\begin{matrix} \frac{1}{2}, \frac{3}{2}, \frac{s}{2}, \frac{s}{2};\ a^2b^2 \\ 1, \frac{s+1}{2}, \frac{s}{2}+1 \end{matrix}\right)$$

$$\left[a, \operatorname{Re} s > 0;\ \left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

22. $$\int_0^a \frac{1}{1-b^2x^2} \ln \frac{a+\sqrt{a^2-x^2}}{x} \mathbf{E}(bx)\, dx = \frac{\pi^2 a}{4} \psi_1(a^2b^2)$$

$$\left[a > 0;\ \left|\arg\left(1-a^2b^2\right)\right| < \pi\right].$$

23. $$\int_0^a \frac{x}{1-b^2x^2} \ln \frac{a+\sqrt{a^2-x^2}}{x} \, \mathbf{E}(bx)\, dx = \frac{\pi}{2b^2}\left(1-\sqrt{1-a^2b^2}\right)$$
$$\left[a>0;\ \left|\arg\left(1-a^2b^2\right)\right|<\pi\right].$$

24. $$\int_0^1 \frac{1}{1-x^2} \ln \frac{1+\sqrt{1-x^2}}{x} \, \mathbf{E}(x)\, dx = \frac{1}{16\pi}\Gamma^4\left(\frac{1}{4}\right).$$

25. $$\int_0^a x\sqrt{b^2x^2+1} \ln \frac{a+\sqrt{a^2-x^2}}{x} \, \mathbf{E}\left(\frac{bx}{\sqrt{b^2x^2+1}}\right) dx$$
$$= \frac{\pi}{12b^2}\left[3ab\ln\left(ab+\sqrt{a^2b^2+1}\right)+(a^2b^2-2)\sqrt{a^2b^2+1}+2\right]$$
$$\left[a>0;\ \left|\arg\left(1+a^2b^2\right)\right|<\pi\right].$$

26. $$\int_0^a x^3\sqrt{b^2x^2+1} \ln \frac{a+\sqrt{a^2-x^2}}{x} \, \mathbf{E}\left(\frac{bx}{\sqrt{b^2x^2+1}}\right) dx$$
$$= \frac{\pi}{1440b^4}\left[15ab(4a^2b^2-3)\ln\left(ab+\sqrt{a^2b^2+1}\right)\right.$$
$$\left.+(54a^4b^4+13a^2b^2+64)\sqrt{a^2b^2+1}-64\right] \quad \left[a>0;\ \left|\arg\left(1+a^2b^2\right)\right|<\pi\right].$$

27. $$\int_0^1 x^{s-1}\ln^2\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx$$
$$= \frac{\pi a^2(s+1)\Gamma^2(s+1)}{(2s+3)\Gamma^2\left(s+\frac{3}{2}\right)}\, {}_5F_4\left(\begin{matrix}1,\ 1,\ 1,\ s+1,\ s+2\\ \frac{3}{2},\ 2,\ s+\frac{3}{2},\ s+\frac{5}{2};\ -a^2\end{matrix}\right)$$
$$\left[a>0;\ \operatorname{Re}s>-1;\ \left|\arg\left(1+a^2\right)\right|<\pi\right].$$

28. $$\int_0^1 x^{-3/2}\ln^2\left(a\sqrt{x}+\sqrt{1+a^2x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx$$
$$= \pi^2\left(\sqrt{1+a^2}-\ln\frac{1+\sqrt{1+a^2}}{2}-1\right) \quad \left[a>0;\ \left|\arg\left(1+a^2\right)\right|<\pi\right].$$

4.22.4. Integrals containing $\mathbf{E}(z)$ and inverse trigonometric functions

1. $$\int_0^1 x^{s-1}\arcsin\left(a\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi a\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{2(s+1)\Gamma^2(s+1)}$$
$$\times\, {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ s+\frac{1}{2},\ s+\frac{3}{2}\\ \frac{3}{2},\ s+1,\ s+2;\ a^2\end{matrix}\right) \quad \left[\operatorname{Re}s>-1/2;\ \left|\arg\left(1-a^2\right)\right|<\pi\right].$$

2. $\displaystyle\int_0^1 \arcsin\left(a\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx$

$$= \frac{\pi^2}{72a}\left[4(a^2-1)\psi_1(a^2)+4(2a^2+1)\psi_2(a^2)-5a^2\psi_3(a^2)\right]$$

$\left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$

3. $\displaystyle\int_0^1 \arcsin\left(a\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{1}{144\pi}\Gamma^4\left(\frac{1}{4}\right)+\frac{1}{\pi}\Gamma^4\left(\frac{3}{4}\right)$

$\left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$

4. $\displaystyle\int_0^1 \frac{1}{x}\arcsin\left(a\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi^2 a}{4}\psi_2(a^2).$

5. $\displaystyle\int_0^1 \frac{1}{x}\arcsin\left(\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{1}{16\pi}\Gamma^4\left(\frac{1}{4}\right)-\frac{1}{\pi}\Gamma^4\left(\frac{3}{4}\right).$

6. $\displaystyle\int_0^1 \frac{x^{s-1}}{\sqrt{1-a^2x}}\arcsin\left(a\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi a\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{2(s+1)\Gamma^2(s+1)}$

$$\times\,{}_4F_3\left(\begin{matrix}1,\,1,\,s+\frac{1}{2},\,s+\frac{3}{2}\\ \frac{3}{2},\,s+1,\,s+2;\;a^2\end{matrix}\right)\quad\left[\operatorname{Re}s>-1/2;\;\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

7. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-a^2x}}\arcsin\left(a\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx$

$$= \frac{\pi^2}{8a^3}\left(2-a^2-2\sqrt{1-a^2}-2a^2\ln\frac{1+\sqrt{1-a^2}}{2}\right)\quad\left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$$

8. $\displaystyle\int_0^1 \frac{x^{-1}}{\sqrt{1-a^2x}}\arcsin\left(a\sqrt{x}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2a}\left(1-\sqrt{1-a^2}\right)$

$\left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$

9. $\displaystyle\int_0^1 x^{s-1}\arccos x\;\mathbf{E}(ax)\,dx = \frac{\pi^{3/2}\Gamma\left(\frac{s+1}{2}\right)}{2s^2\Gamma\left(\frac{s}{2}\right)}\,{}_4F_3\left(\begin{matrix}-\frac{1}{2},\,\frac{1}{2},\,\frac{s}{2},\,\frac{s+1}{2};\;a^2\\ 1,\,\frac{s}{2}+1,\,\frac{s}{2}+1\end{matrix}\right)$

$\left[\operatorname{Re}s>0;\;\left|\arg\left(1-a^2\right)\right|<\pi\right].$

10. $\displaystyle\int_0^1 \arccos x\;\mathbf{E}(x)\,dx = \frac{\pi^2}{16}\,(1+2\ln 2).$

11. $$\int_0^1 x \arccos x \, \mathbf{E}(x)\, dx = \frac{1}{288\pi}\Gamma^4\left(\frac{1}{4}\right) + \frac{1}{2\pi}\Gamma^4\left(\frac{3}{4}\right).$$

12. $$\int_0^1 x^2 \arccos x \, \mathbf{E}(x)\, dx = \frac{\pi^2}{256}(5 + 4\ln 2).$$

13. $$\int_0^1 x^{s-1}\sqrt{1+a^2x^2} \arccos x \, \mathbf{E}\left(\frac{ax}{\sqrt{1+a^2x^2}}\right) dx = \frac{\pi^{3/2}\Gamma\left(\frac{s+1}{2}\right)}{2s^2\Gamma\left(\frac{s}{2}\right)}$$
$$\times\, {}_4F_3\left(\begin{matrix} -\frac{1}{2}, \frac{1}{2}, \frac{s}{2}, \frac{s+1}{2} \\ 1, \frac{s}{2}+1, \frac{s}{2}+1;\ -a^2 \end{matrix}\right) \quad \left[\operatorname{Re} s > 0;\ \left|\arg\left(1+a^2\right)\right| < \pi\right].$$

14. $$\int_0^1 \frac{x^{s-1}}{1-a^2x^2} \arccos x \, \mathbf{E}(ax)\, dx$$
$$= \frac{\pi^{3/2}\Gamma\left(\frac{s+1}{2}\right)}{2s^2\Gamma\left(\frac{s}{2}\right)}\, {}_4F_3\left(\begin{matrix} \frac{1}{2}, \frac{3}{2}, \frac{s}{2}, \frac{s+1}{2} \\ 1, \frac{s}{2}+1, \frac{s}{2}+1;\ a^2 \end{matrix}\right) \quad \left[\operatorname{Re} s > 0;\ \left|\arg\left(1-a^2\right)\right| < \pi\right].$$

15. $$\int_0^1 \frac{1}{1-a^2x^2} \arccos x \, \mathbf{E}(ax)\, dx = \frac{\pi}{2a}\arcsin a \qquad \left[\left|\arg\left(1-a^2\right)\right| < \pi\right].$$

16. $$\int_0^1 \frac{x}{1-x^2} \arccos x \, \mathbf{E}(x)\, dx = \frac{1}{32\pi}\left[\Gamma^4\left(\frac{1}{4}\right) - 16\Gamma^4\left(\frac{3}{4}\right)\right].$$

17. $$\int_0^1 \frac{x^2}{1-x^2} \arccos x \, \mathbf{E}(x)\, dx = \frac{\pi^2}{16}(3 - 2\ln 2).$$

18. $$\int_0^1 \frac{1}{\sqrt{1+a^2x^2}} \arccos x \, \mathbf{E}\left(\frac{ax}{\sqrt{1+a^2x^2}}\right) dx = \frac{\pi}{2a}\ln\left(a + \sqrt{1+a^2}\right)$$
$$\left[\left|\arg\left(1+a^2\right)\right| < \pi\right].$$

19. $$\int_0^1 x^{s-1}\arctan\left(a\sqrt{x}\right) \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi a\, \Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{2(s+1)\Gamma^2(s+1)}$$
$$\times\, {}_4F_3\left(\begin{matrix} \frac{1}{2}, 1, s+\frac{1}{2}, s+\frac{3}{2} \\ \frac{3}{2}, s+1, s+2;\ -a^2 \end{matrix}\right) \quad \left[\operatorname{Re} s > 0;\ \left|\arg\left(1+a^2\right)\right| < \pi\right].$$

20. $\displaystyle\int_0^1 \arctan(a\sqrt{x})\,\mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{\pi(a^2+1)^{1/2}}{12a^3}\left[4(2a^2+1)\,\mathbf{E}\left(\frac{a}{\sqrt{a^2+1}}\right) - 4\,\mathbf{K}\left(\frac{a}{\sqrt{a^2+1}}\right) - \frac{3\pi a^2}{\sqrt{a^2+1}}\right]$$

$$\left[|\arg(1+a^2)| < \pi\right].$$

21. $\displaystyle\int_0^1 \arctan\sqrt{x}\;\mathbf{E}\left(\sqrt{1-x}\right)dx = -\frac{\pi^2}{4} + \frac{1}{12}\sqrt{\frac{\pi}{2}}\left[\Gamma^4\left(\frac{1}{4}\right) + 12\Gamma^4\left(\frac{3}{4}\right)\right].$

22. $\displaystyle\int_0^1 \frac{1}{x}\arctan(a\sqrt{x})\,\mathbf{E}(\sqrt{1-x})\,dx = \frac{\pi}{a}(a^2+1)^{1/2}$

$$\times\left[\mathbf{K}\left(\frac{a}{\sqrt{a^2+1}}\right) - \mathbf{E}\left(\frac{a}{\sqrt{a^2+1}}\right)\right] \quad \left[|\arg(1+a^2)| < \pi\right].$$

23. $\displaystyle\int_0^1 x^{s-1}\arcsin^2(a\sqrt{x})\,\mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{\pi a^2(s+1)\Gamma^2(s+1)}{(2s+3)\Gamma^2\left(s+\frac{3}{2}\right)}\,{}_5F_4\left(\begin{matrix}1,\,1,\,1,\,s+1,\,s+2\\ \frac{3}{2},\,2,\,s+\frac{3}{2},\,s+\frac{5}{2};\;a^2\end{matrix}\right)$$

$$\left[\operatorname{Re} s > -1;\ |\arg(1-a^2)| < \pi\right].$$

24. $\displaystyle\int_0^1 x^{-3/2}\arcsin^2(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx$

$$= \pi^2\left(1 - \sqrt{1-a^2} + \ln\frac{1+\sqrt{1-a^2}}{2}\right) \quad \left[|\arg(1-a^2)| < \pi\right].$$

25. $\displaystyle\int_0^1 x^{-3/2}\arcsin^2\sqrt{x}\;\mathbf{E}\left(\sqrt{1-x}\right)dx = \pi^2(1-\ln 2).$

4.22.5. Integrals containing $\mathbf{E}(z)$ and $\mathrm{Li}_2(z)$

1. $\displaystyle\int_0^1 x^{s-1}\,\mathrm{Li}_2(ax)\,\mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{\pi a(s+1)\Gamma^2(s+1)}{(2s+3)\Gamma^2\left(s+\frac{3}{2}\right)}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,s+1,\,s+2\\ 2,\,2,\,s+\frac{3}{2},\,s+\frac{5}{2};\;a\end{matrix}\right)$$

$$[\operatorname{Re} s > -1;\ |\arg(1-a)| < \pi].$$

4.22.6. Integrals containing $\mathbf{E}(z)$, $\mathrm{shi}(z)$ and $\mathrm{Si}(z)$

1. $\displaystyle\int_0^1 x^{s-1} \left\{ \begin{matrix} \mathrm{shi}(a\sqrt{x}) \\ \mathrm{Si}(a\sqrt{x}) \end{matrix} \right\} \mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{\pi a(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}{4(s+1)\Gamma^2(s+1)}\, {}_3F_4\left(\begin{matrix} \frac{1}{2},\, s+\frac{1}{2},\, s+\frac{3}{2};\, \pm\frac{a^2}{4} \\ \frac{3}{2},\, \frac{3}{2},\, s+1,\, s+2 \end{matrix} \right) \quad [\mathrm{Re}\, s > -1/2].$$

2. $\displaystyle\int_0^1 \mathrm{Si}(a\sqrt{x})\, \mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{\pi^2}{6a}\left[a^2 J_0^2\left(\frac{a}{2}\right) - 2a\, J_0\left(\frac{a}{2}\right) J_1\left(\frac{a}{2}\right) + (1+a^2)\, J_1^2\left(\frac{a}{2}\right)\right].$$

4.22.7. Integrals containing $\mathbf{E}(z)$ and $\mathrm{erf}(z)$

1. $\displaystyle\int_0^1 x^{s-1}\, \mathrm{erf}(a\sqrt{x})\, \mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{\sqrt{\pi}\, a(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}{2(s+1)\Gamma^2(s+1)}\, {}_3F_3\left(\begin{matrix} \frac{1}{2},\, s+\frac{1}{2},\, s+\frac{3}{2} \\ \frac{3}{2},\, s+1,\, s+2;\, -a^2 \end{matrix} \right) \quad [\mathrm{Re}\, s > -1/2].$$

2. $\displaystyle\int_0^1 x^{s-1} e^{a^2 x}\, \mathrm{erf}(a\sqrt{x})\, \mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{\sqrt{\pi}\, a(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}{2(s+1)\Gamma^2(s+1)}\, {}_3F_3\left(\begin{matrix} 1,\, s+\frac{1}{2},\, s+\frac{3}{2} \\ \frac{3}{2},\, s+1,\, s+2;\, a^2 \end{matrix} \right) \quad [\mathrm{Re}\, s > -1/2].$$

4.22.8. Integrals containing $\mathbf{E}(z)$, $S(z)$ and $C(z)$

1. $\displaystyle\int_0^1 x^{s-1} S(a\sqrt{x})\, \mathbf{E}(\sqrt{1-x})\,dx$

$$= \frac{1}{3}\sqrt{\frac{\pi a^3}{2}}\, \frac{(4s+3)\Gamma^2\left(s+\frac{3}{4}\right)}{(4s+5)\Gamma^2\left(s+\frac{5}{4}\right)}\, {}_3F_4\left(\begin{matrix} \frac{3}{4},\, s+\frac{3}{4},\, s+\frac{7}{4};\, -\frac{a^2}{4} \\ \frac{3}{2},\, \frac{7}{4},\, s+\frac{5}{4},\, s+\frac{9}{4} \end{matrix} \right) \quad [\mathrm{Re}\, s > -3/4].$$

2. $\displaystyle\int_0^1 x^{s-1} C(a\sqrt{x})\, \mathbf{E}(\sqrt{1-x})\,dx$

$$= \sqrt{\frac{\pi a}{2}}\, \frac{(4s+1)\Gamma^2\left(s+\frac{1}{4}\right)}{(4s+3)\Gamma^2\left(s+\frac{3}{4}\right)}\, {}_3F_4\left(\begin{matrix} \frac{1}{4},\, s+\frac{1}{4},\, s+\frac{5}{4};\, -\frac{a^2}{4} \\ \frac{1}{2},\, \frac{5}{4},\, s+\frac{3}{4},\, s+\frac{7}{4} \end{matrix} \right) \quad [\mathrm{Re}\, s > -1/4].$$

4.22.9. Integrals containing $\mathbf{E}(z)$ and $\gamma(\nu, z)$

1. $$\int_0^1 x^{s-1}\gamma(\nu, ax)\,\mathbf{E}(\sqrt{1-x})\,dx$$
$$= \frac{\pi a^{\nu}(s+\nu)\Gamma^2(s+\nu)}{\nu(2s+\nu+1)\Gamma^2\left(s+\nu+\frac{1}{2}\right)}\,{}_3F_3\left(\begin{matrix}\nu,\ s+\nu,\ s+\nu+1;\ -a\\ \nu+1,\ s+\nu+\frac{1}{2},\ s+\nu+\frac{3}{2}\end{matrix}\right)$$
$[\operatorname{Re}(s+\nu)>0]$.

2. $$\int_0^1 x^{s-1}e^{ax}\gamma(\nu, ax)\,\mathbf{E}(\sqrt{1-x})\,dx$$
$$= \frac{\pi a^{\nu}(s+\nu)\Gamma^2(s+\nu)}{\nu(2s+\nu+1)\Gamma^2\left(s+\nu+\frac{1}{2}\right)}\,{}_3F_3\left(\begin{matrix}1,\ s+\nu,\ s+\nu+1;\ a\\ \nu+1,\ s+\nu+\frac{1}{2},\ s+\nu+\frac{3}{2}\end{matrix}\right)$$
$[\operatorname{Re}(s+\nu)>0]$.

4.22.10. Integrals containing $\mathbf{E}(z)$, $J_\nu(z)$ and $I_\nu(z)$

1. $$\int_0^1 x^{s-1}\left\{\begin{matrix}J_\nu(a\sqrt{x})\\ I_\nu(a\sqrt{x})\end{matrix}\right\}\mathbf{E}\left(\sqrt{1-x}\right)dx$$
$$= \frac{\pi a^{\nu}(2s+\nu)\Gamma^2\left(s+\frac{\nu}{2}\right)}{2^{\nu+1}(2s+\nu+1)\Gamma^2\left(s+\frac{\nu}{2}+1\right)\Gamma(\nu+1)}\,{}_2F_3\left(\begin{matrix}s+\frac{\nu}{2},\ s+\frac{\nu}{2}+1;\ \mp\frac{a^2}{4}\\ s+\frac{\nu+1}{2},\ s+\frac{\nu+3}{2},\ \nu+1\end{matrix}\right)$$
$[\operatorname{Re}(s+\nu/2)>0]$.

2. $$\int_0^1 J_0(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a^2}[a\,\mathbf{H}_0(a)-\mathbf{H}_1(a)].$$

3. $$\int_0^1 \frac{1}{\sqrt{x}}J_1(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a}\,\mathbf{H}_1(a).$$

4. $$\int_0^1 I_0(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a^2}[a\,\mathbf{L}_0(a)-\mathbf{L}_1(a)].$$

5. $$\int_0^1 \frac{1}{\sqrt{x}}I_1(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a}\,\mathbf{L}_1(a).$$

6. $$\int_0^1 x^{s-1} e^{ax} I_\nu(ax)\, \mathbf{E}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi(s+\nu)\Gamma^2(s+\nu)}{(2s+2\nu+1)\Gamma^2\left(s+\nu+\frac{1}{2}\right)\Gamma(\nu+1)} \left(\frac{a}{2}\right)^\nu$$
$$\times\ {}_3F_3\left(\begin{matrix}\nu+\frac{1}{2},\ s+\nu,\ s+\nu+1;\ 2a\\ 2\nu+1,\ s+\nu+\frac{1}{2},\ s+\nu+\frac{3}{2}\end{matrix}\right) \quad [\operatorname{Re}(s+\nu)>0].$$

7. $$\int_0^1 x^{s-1} \left\{\begin{matrix} J_\mu(a\sqrt{x})\, J_\nu(a\sqrt{x}) \\ I_\mu(a\sqrt{x})\, I_\nu(a\sqrt{x}) \end{matrix}\right\} \mathbf{E}\left(\sqrt{1-x}\right) dx$$
$$= \frac{(2s+\mu+\nu)\Gamma^2\left(s+\frac{\mu+\nu}{2}\right)}{(2s+\mu+\nu+1)\Gamma^2\left(s+\frac{\mu+\nu+1}{2}\right)} \frac{2^{-\mu-\nu-1}\pi a^{\mu+\nu}}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$
$$\times\ {}_4F_5\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1,\ s+\frac{\mu+\nu}{2},\ s+\frac{\mu+\nu}{2}+1;\ \mp a^2\\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ s+\frac{\mu+\nu+1}{2},\ s+\frac{\mu+\nu+3}{2}\end{matrix}\right)$$
$$[\operatorname{Re}(2s+\mu+\nu)>0].$$

8. $$\int_0^1 J_0^2(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{1}{8a^3}[\sin(2a) - 2a\cos(2a) + 4a^2 \operatorname{Si}(2a)].$$

9. $$\int_0^1 J_1^2(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{1}{8a^3}[6a\cos(2a) - 3\sin(2a) + 4a^2 \operatorname{Si}(2a)].$$

10. $$\int_0^1 \frac{1}{x} J_1^2(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{1}{2a^2}[\cos(2a) + 2a^2 - 1].$$

11. $$\int_0^1 I_0^2(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{1}{8a^3}[2a\cosh(2a) - \sinh(2a) + 4a^2 \operatorname{shi}(2a)].$$

12. $$\int_0^1 I_1^2(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx$$
$$= \frac{1}{8a^3}[6a\cosh(2a) - 3\sinh(2a) - 4a^2 \operatorname{shi}(2a)].$$

13. $$\int_0^1 \frac{1}{x} I_1^2(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{1}{2a^2}[\cosh(2a) - 2a^2 - 1].$$

4.22.11. Integrals containing $\mathbf{E}(z)$, $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

1. $$\int_0^1 x^{s-1} \begin{Bmatrix} \mathbf{H}_\nu(a\sqrt{x}) \\ \mathbf{L}_\nu(a\sqrt{x}) \end{Bmatrix} \mathbf{E}(\sqrt{1-x})\,dx$$
$$= \frac{\sqrt{\pi}\,(2s+\nu+1)\Gamma^2\left(s+\frac{\nu+1}{2}\right)}{(2s+\nu+2)\Gamma^2\left(s+\frac{\nu}{2}+1\right)\Gamma\left(\nu+\frac{3}{2}\right)} \left(\frac{a}{2}\right)^{\nu+1}$$
$$\times {}_3F_4\left(\begin{matrix} 1,\, s+\frac{\nu+1}{2},\, s+\frac{\nu+3}{2};\, \mp\frac{a^2}{4} \\ \frac{3}{2},\, \nu+\frac{3}{2},\, s+\frac{\nu}{2}+1,\, s+\frac{\nu}{2}+2 \end{matrix}\right) \quad [\operatorname{Re}(2s+\nu) > -1].$$

2. $$\int_0^1 \mathbf{H}_0(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a}[1-J_0(a)].$$

3. $$\int_0^1 \mathbf{L}_0(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a}[I_0(a)-1].$$

4. $$\int_0^1 \frac{1}{x}\,\mathbf{H}_0(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \pi[a\,J_0(a)-J_1(a)]$$
$$+\frac{\pi^2 a}{2}[J_1(a)\,\mathbf{H}_0(a)-J_0(a)\,\mathbf{H}_1(a)].$$

5. $$\int_0^1 \frac{1}{x}\,\mathbf{L}_0(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \pi[a\,I_0(a)-I_1(a)]$$
$$-\frac{\pi^2 a}{2}[I_1(a)\,\mathbf{L}_0(a)-I_0(a)\,\mathbf{L}_1(a)].$$

4.22.12. Integrals containing $\mathbf{E}(z)$ and $L_n^\lambda(z)$

1. $$\int_0^1 x^{s-1} L_n^\lambda(ax)\,\mathbf{E}(\sqrt{1-x})\,dx$$
$$= \frac{\pi s\,\Gamma^2(s)\,(\lambda+1)_n}{n!\,(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_3F_3\left(\begin{matrix} -n,\, s,\, s+1;\, a \\ \nu+1,\, s+\frac{1}{2},\, s+\frac{3}{2} \end{matrix}\right) \quad [\operatorname{Re} s > 0].$$

4.22.13. Integrals containing products of $\mathbf{E}(z)$ and $\mathbf{K}(z)$

1. $$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,\mathbf{K}(x)\,\mathbf{E}(x)dx = \frac{\pi^4}{32}\,{}_7F_6\left(\begin{matrix} -\frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{5}{4} \\ \frac{1}{4},\, 1,\, 1,\, 1,\, 1,\, 2;\, 1 \end{matrix}\right).$$

2. $$\int_0^1 x^{s-1}\,\mathbf{K}(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{2}\,\frac{s\,\Gamma^2(s)}{(2s+1)\Gamma^2\left(s+\frac{1}{2}\right)}\;{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,s,\,s+1;\;a^2\\ 1,\,s+\frac{1}{2},\,s+\frac{3}{2}\end{matrix}\right)$$

$\left[\operatorname{Re} s > 0;\ \left|\arg\left(1-a^2\right)\right| < \pi\right]$.

3. $$\int_0^1 x^{s-1}\,\mathbf{K}\left(\sqrt{1-x}\right)\mathbf{E}(a\sqrt{x})\,dx = \frac{\pi^2}{4}\,\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)}\;{}_4F_3\left(\begin{matrix}-\frac{1}{2},\,\frac{1}{2},\,s,\,s;\;a^2\\ 1,\,s+\frac{1}{2},\,s+\frac{1}{2}\end{matrix}\right)$$

$\left[\operatorname{Re} s > 0;\ \left|\arg\left(1-a^2\right)\right| < \pi\right]$.

4. $$\int_0^1 \mathbf{K}\left(\sqrt{1-x}\right)\mathbf{E}(a\sqrt{x})\,dx = \frac{\pi}{4a}\left[a + \frac{1-a^2}{2}\ln\frac{1+a}{1-a} + \operatorname{Li}_2(a) - \operatorname{Li}_2(-a)\right]$$

$\left[\left|\arg\left(1-a^2\right)\right| < \pi\right]$.

5. $$\int_0^1 \mathbf{K}(a\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{4a}\left[\frac{1}{a} + \frac{1-a^2}{2a^2}\ln\frac{1-a}{1+a} + \operatorname{Li}_2(a) - \operatorname{Li}_2(-a)\right]$$

$\left[\left|\arg\left(1-a^2\right)\right| < \pi\right]$.

6. $$\int_0^1 \mathbf{K}(\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi^3}{16} + \frac{\pi}{4}.$$

7. $$\int_0^1 x\,\mathbf{K}(\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{5\pi^3}{128} + \frac{\pi}{8}.$$

8. $$\int_0^1 x^2\,\mathbf{K}(\sqrt{x})\,\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{15\pi^3}{512} + \frac{\pi}{12}.$$

9. $$\int_0^1 \frac{x^{s-1}}{1-a^2x}\,\mathbf{E}(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{4}\,\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)} \times {}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,s,\,s;\;a^2\\ 1,\,s+\frac{1}{2},\,s+\frac{1}{2}\end{matrix}\right)$$

$\left[\operatorname{Re} s > 0;\ \left|\arg\left(1-a^2\right)\right| < \pi\right]$.

10. $$\int_0^1 \frac{1}{1-a^2x}\,\mathbf{E}(a\sqrt{x})\,\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{2a}\ln\frac{1+a}{1-a}$$

$\left[\left|\arg\left(1-a^2\right)\right| < \pi\right]$.

11. $$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\,\mathbf{E}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{4}\,\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)} \times {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{3}{2},\ s,\ s;\ -a^2\\ 1,\ s+\frac{1}{2},\ s+\frac{1}{2}\end{matrix}\right) \quad \left[\operatorname{Re} s>0;\ \left|\arg\left(1+a^2\right)\right|<\pi\right].$$

12. $$\int_0^1 \frac{1}{\sqrt{1+a^2x}}\,\mathbf{E}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)\mathbf{K}\left(\sqrt{1-x}\right)dx = \frac{\pi}{a}\arctan a.$$

13. $$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\,\mathbf{K}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi^2}{4}\,\frac{\Gamma(s)\Gamma(s+1)}{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)} \times {}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ s,\ s+1;\ -a^2\\ 1,\ s+\frac{1}{2},\ s+\frac{3}{2}\end{matrix}\right) \quad \left[\operatorname{Re} s>0;\ \left|\arg\left(1+a^2\right)\right|<\pi\right].$$

14. $$\int_0^1 \frac{1}{\sqrt{1+a^2x}}\,\mathbf{K}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = -\frac{\pi}{4a^2}+\frac{\pi}{4a^3}\left(a^2-1\right)\arctan a-\frac{\pi i}{4a}\left[\operatorname{Li}_2(ia)-\operatorname{Li}_2(-ia)\right] \quad \left[\left|\arg\left(1+a^2\right)\right|<\pi\right].$$

15. $$\int_0^1 \frac{1}{\sqrt{1+x}}\,\mathbf{K}\left(\sqrt{\frac{x}{1+x}}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{8}\left(\pi+4\mathbf{G}-2\right).$$

16. $$\int_0^1 \sqrt{1+x}\,\mathbf{K}\left(\sqrt{1-x}\right)\mathbf{E}\left(\sqrt{\frac{x}{1+x}}\right)dx = \frac{\pi}{8}\left(\pi+4\mathbf{G}+2\right).$$

17. $$\int_0^1 \frac{x}{\sqrt{1+x}}\,\mathbf{K}\left(\sqrt{\frac{x}{1+x}}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{8}\left(\pi+2\mathbf{G}+5\right).$$

18. $$\int_0^1 x\sqrt{1+x}\,\mathbf{K}\left(\sqrt{1-x}\right)\mathbf{E}\left(\sqrt{\frac{x}{1+x}}\right)dx = \frac{\pi}{32}\left(3\pi+2\mathbf{G}+5\right).$$

19. $$\int_0^1 x^{s-1}\left(1+a^2x\right)^{1/2}\mathbf{K}\left(\sqrt{1-x}\right)\mathbf{E}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)dx = \frac{\pi^2}{4}\,\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)} \times {}_4F_3\left(\begin{matrix}-\frac{1}{2},\ \frac{1}{2},\ s,\ s;\ -a^2\\ 1,\ s+\frac{1}{2},\ s+\frac{1}{2}\end{matrix}\right) \quad \left[\operatorname{Re} s>0;\ \left|\arg\left(1+a^2\right)\right|<\pi\right].$$

4.22.14. Integrals containing products of $\mathbf{E}(z)$

1. $$\int_0^1 \frac{x}{\sqrt{1-x^2}} \mathbf{E}^2(x)dx = \frac{\pi^4}{64} {}_7F_6\left(\begin{matrix} -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{4} \\ \frac{1}{4}, 1, 1, 1, 2, 2;\ 1 \end{matrix}\right).$$

2. $$\int_0^1 x^{s-1} \mathbf{E}(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi^2}{4} \frac{\Gamma(s)\Gamma(s+1)}{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)} {}_4F_3\left(\begin{matrix} -\frac{1}{2}, \frac{1}{2}, s, s+1;\ a^2 \\ 1, s+\frac{1}{2}, s+\frac{3}{2} \end{matrix}\right)$$
$[\operatorname{Re} s > 0;\ |\arg(1-a^2)| < \pi]$.

3. $$\int_0^1 \mathbf{E}(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx$$
$$= \frac{\pi}{8a}\left[\frac{3a^2+1}{2a} - \frac{4a^2-3a^4-1}{2a^2} \ln\frac{1-a}{1+a} + \operatorname{Li}_2(a) - \operatorname{Li}_2(-a)\right]$$
$[|\arg(1-a^2)| < \pi]$.

4. $$\int_0^1 \mathbf{E}(\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^3}{32} + \frac{\pi}{4}.$$

5. $$\int_0^1 x\, \mathbf{E}(\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^3}{64} + \frac{\pi}{8}.$$

6. $$\int_0^1 x^2\, \mathbf{E}(\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{21\pi^3}{2048} + \frac{\pi}{12}.$$

7. $$\int_0^1 \frac{x^{s-1}}{1-a^2x} \mathbf{E}(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi^2}{4} \frac{\Gamma(s)\Gamma(s+1)}{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}$$
$$\times {}_4F_3\left(\begin{matrix} \frac{1}{2}, \frac{3}{2}, s, s+1;\ a^2 \\ 1, s+\frac{1}{2}, s+\frac{3}{2} \end{matrix}\right)$$
$[\operatorname{Re} s > 0;\ |\arg(1-a^2)| < \pi]$.

8. $$\int_0^1 \frac{1}{1-a^2x} \mathbf{E}(a\sqrt{x})\, \mathbf{E}\left(\sqrt{1-x}\right) dx = \frac{\pi}{4a^3}\left[(a^2+1)\ln\frac{1+a}{1-a} - 2a\right]$$
$[|\arg(1-a^2)| < \pi]$.

9. $$\int_0^1 (1+a^2x)^{1/2}\,\mathbf{E}\left(\sqrt{1-x}\right)\mathbf{E}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)dx$$
$$= \frac{\pi}{16}\left\{3 - \frac{1}{a^2} + \frac{3a^4+4a^2+1}{a^3}\arctan a - \frac{2i}{a}\left[\mathrm{Li}_2(ia) - \mathrm{Li}_2(-ia)\right]\right\}$$
$$\left[\left|\arg\left(1-a^2\right)\right| < \pi\right].$$

10. $$\int_0^1 \frac{1}{1+a^2x}\,\mathbf{E}\left(\frac{a\sqrt{x}}{\sqrt{1+a^2x}}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{2a^3}\left[a + (a^2-1)\arctan a\right]$$
$$\left[\left|\arg\left(1-a^2\right)\right| < \pi\right].$$

11. $$\int_0^1 (1+x)^{1/2}\,\mathbf{E}\left(\sqrt{\frac{x}{1+x}}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{8}\,(\pi + 2\mathbf{G} + 1).$$

12. $$\int_0^1 x(1+x)^{1/2}\,\mathbf{E}\left(\sqrt{\frac{x}{1+x}}\right)\mathbf{E}\left(\sqrt{1-x}\right)dx = \frac{\pi}{32}\,(2\pi + 2\,\mathbf{G} + 5).$$

4.23. The Complete Elliptic Integral $\mathbf{D}(z)$

4.23.1. Integrals containing $\mathbf{D}(z)$ and elementary functions

1. $$\int_0^a x^{s-1}(a-x)^{t-1}\,\mathbf{D}\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{\pi}{4}\,\mathrm{B}\,(s,\,t)\,a^{s+t-1}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,s,\,t;\,\frac{a^2b^2}{4}\\ 2,\,\frac{s+t}{2},\,\frac{s+t+1}{2}\end{matrix}\right)$$
$$\left[\mathrm{Re}\,s,\,\mathrm{Re}\,t > 0;\ \left|\arg\left(4-a^2b^2\right)\right| < \pi\right].$$

2. $$\int_0^a \mathbf{D}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{ab^2}\left(2-\sqrt{4-a^2b^2}\right) \qquad \left[\left|\arg\left(4-a^2b^2\right)\right| < \pi\right].$$

3. $$\int_0^a x\,\mathbf{D}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi}{2b^2}\left(2-\sqrt{4-a^2b^2}\right) \qquad \left[\left|\arg\left(4-a^2b^2\right)\right| < \pi\right].$$

4. $$\int_0^a x^{1/2}(a-x)^{1/2}\,\mathbf{D}\left(b\sqrt{x(a-x)}\right)dx$$
$$= \frac{\pi^2a^2}{32}\left[4\psi_1\left(\frac{a^2b^2}{4}\right) - 4\psi_2\left(\frac{a^2b^2}{4}\right) + \psi_3\left(\frac{a^2b^2}{4}\right)\right]$$
$$\left[\left|\arg\left(4-a^2b^2\right)\right| < \pi\right].$$

5. $$\int_0^a x^{-1/2}(a-x)^{1/2}\,\mathbf{D}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi^2 a}{8}\left[2\psi_1\left(\frac{a^2b^2}{4}\right) - \psi_2\left(\frac{a^2b^2}{4}\right)\right]$$
$$\left[|\arg(4-a^2b^2)| < \pi\right].$$

6. $$\int_0^a x^{-1/2}(a-x)^{-1/2}\,\mathbf{D}\left(b\sqrt{x(a-x)}\right)dx = \frac{\pi^2}{4}\left[2\psi_1\left(\frac{a^2b^2}{4}\right) - \psi_2\left(\frac{a^2b^2}{4}\right)\right]$$
$$\left[|\arg(4-a^2b^2)| < \pi\right].$$

7. $$\int_0^a x^{s+1/2}(a-x)^{s}\,\mathbf{D}\left(b\sqrt[4]{x(a-x)}\right)dx = 2^{-2s-3}\pi^{3/2}a^{2s+3/2}\,\frac{\Gamma(2s+2)}{\Gamma\left(2s+\frac{5}{2}\right)}$$
$$\times\ {}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{3}{2},\ 2s+2;\ \frac{ab^2}{2}\\ 2,\ 2s+\frac{5}{2}\end{matrix}\right)\quad\left[\operatorname{Re} s > -1;\ |\arg(2-ab^2)| < \pi\right].$$

8. $$\int_0^a x^{1/2}\,\mathbf{D}\left(b\sqrt[4]{x(a-x)}\right)dx = \frac{\pi}{\sqrt{2}\,b^3}\arcsin\left(b\sqrt{\frac{a}{2}}\right) - \frac{\pi\sqrt{a}}{2b^2}\sqrt{1-\frac{ab^2}{2}}$$
$$\left[|\arg(2-ab^2)| < \pi\right].$$

9. $$\int_0^a x^{-1/2}\,\mathbf{D}\left(b\sqrt[4]{x(a-x)}\right)dx = \pi a^{1/2}\left(1+\sqrt{1-\frac{ab^2}{2}}\right)^{-1}$$
$$\left[|\arg(2-ab^2)| < \pi\right].$$

10. $$\int_0^a x^{-1/4}(a-x)^{1/4}\,\mathbf{D}\left(b\sqrt[4]{x(a-x)}\right)dx = 2^{-7/2}\pi^2 a$$
$$\times\left[4\psi_1\left(\frac{ab^2}{2}\right) - 4\psi_2\left(\frac{ab^2}{2}\right) + \psi_3\left(\frac{ab^2}{2}\right)\right]\quad\left[|\arg(2-ab^2)| < \pi\right].$$

11. $$\int_0^a x^{-3/4}(a-x)^{-1/4}\,\mathbf{D}\left(b\sqrt[4]{x(a-x)}\right)dx$$
$$= 2^{-3/2}\pi^2\left[2\psi_1\left(\frac{ab^2}{2}\right) - \psi_2\left(\frac{ab^2}{2}\right)\right]\quad\left[|\arg(2-ab^2)| < \pi\right].$$

12. $$\int_0^{m\pi} e^{-ax}\,\mathbf{D}(b\sin x)\,dx = \frac{\pi a}{2b^2}\left(1-e^{-m\pi a}\right)\left[{}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2};\ b^2\\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right) - 1\right]$$
$$\left[|\arg(1-b^2)| < \pi\right].$$

13. $$\int_0^\infty e^{-ax}\,\mathbf{D}(b\sin x)\,dx = \frac{\pi a}{2b^2}\left[{}_3F_2\left(\begin{matrix}-\frac{1}{2},\,-\frac{1}{2},\,\frac{1}{2};\,b^2\\ -\frac{ia}{2},\,\frac{ia}{2}\end{matrix}\right)-1\right]$$
$$\left[\operatorname{Re}a>0;\ |\arg(1-b^2)|<\pi\right].$$

14. $$\int_0^a x^{s-1}\ln\frac{a+\sqrt{a^2-x^2}}{x}\,\mathbf{D}(bx)\,dx = \frac{\pi^{3/2}a^s\Gamma\left(\frac{s}{2}\right)}{8s\,\Gamma\left(\frac{s+1}{2}\right)}$$
$$\times\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,\frac{s}{2},\,\frac{s}{2};\,a^2b^2\\ 2,\,\frac{s+1}{2},\,\frac{s}{2}+1\end{matrix}\right)\quad\left[a,\,\operatorname{Re}s>0;\ |\arg(1-a^2b^2)|<\pi\right].$$

15. $$\int_0^a \ln\frac{a+\sqrt{a^2-x^2}}{x}\,\mathbf{D}(bx)\,dx = \frac{\pi^2 a}{8}\psi_2(a^2b^2)$$
$$\left[a>0;\ |\arg(1-a^2b^2)|<\pi\right].$$

16. $$\int_0^a x\ln\frac{a+\sqrt{a^2-x^2}}{x}\,\mathbf{D}(bx)\,dx$$
$$=\frac{\pi}{2b^2}\left(1-\sqrt{1-a^2b^2}+\ln\frac{1+\sqrt{1-a^2b^2}}{2}\right)$$
$$\left[a>0;\ |\arg(1-a^2b^2)|<\pi\right].$$

17. $$\int_0^1 x^{s-1}\arccos x\,\mathbf{D}(ax)\,dx = \frac{\pi^{3/2}\Gamma\left(\frac{s+1}{2}\right)}{4s^2\Gamma\left(\frac{s}{2}\right)}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,\frac{s}{2},\,\frac{s+1}{2};\,a^2\\ 2,\,\frac{s}{2}+1,\,\frac{s}{2}+1\end{matrix}\right)$$
$$\left[\operatorname{Re}s>0;\ |\arg(1-a^2)|<\pi\right].$$

18. $$\int_0^1 \arccos x\,\mathbf{D}(ax)\,dx = \frac{\pi}{2a^2}\left(a\arcsin a+\sqrt{1-a^2}-1\right)$$
$$\left[|\arg(1-a^2)|<\pi\right].$$

4.23.2. Integrals containing products of $\mathbf{D}(z)$, $\mathbf{K}(z)$ and $\mathbf{E}(z)$

1. $$\int_0^1 x^{s-1}\,\mathbf{K}\left(\sqrt{1-x}\right)\mathbf{D}(a\sqrt{x})\,dx = \frac{\pi^2\Gamma^2(s)}{8\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,s,\,s;\,a^2\\ 2,\,s+\frac{1}{2},\,s+\frac{1}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re}s>0;\ |\arg(1-a^2)|<\pi\right].$$

2. $$\int_0^1 \mathbf{K}\left(\sqrt{1-x}\right)\mathbf{D}(a\sqrt{x})\,dx = \frac{\pi}{2a^2}\left[\ln(1-a^2)+a\ln\frac{1+a}{1-a}\right]$$
$$\left[|\arg(1-a^2)|<\pi\right].$$

3. $\displaystyle\int_0^1 \mathbf{K}\left(\sqrt{1-x}\right)\mathbf{D}\left(\sqrt{x}\right)dx = \pi\ln 2.$

4. $\displaystyle\int_0^1 x^{s-1}\,\mathbf{E}\left(\sqrt{1-x}\right)\,\mathbf{D}\left(a\sqrt{x}\right)dx$

$$= \frac{\pi^2\Gamma(s)\Gamma(s+1)}{8\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,s,\,s+1;\;a^2\\ 2,\,s+\frac{1}{2},\,s+\frac{3}{2}\end{matrix}\right)$$

$\left[\operatorname{Re}s>0;\ \left|\arg\left(1-a^2\right)\right|<\pi\right].$

5. $\displaystyle\int_0^1 \mathbf{E}\left(\sqrt{1-x}\right)\mathbf{D}\left(a\sqrt{x}\right)dx = \frac{\pi}{2a^3}\left(a+\frac{a^2-1}{2}\ln\frac{1+a}{1-a}\right)$

$\left[\left|\arg\left(1-a^2\right)\right|<\pi\right].$

6. $\displaystyle\int_0^1 \mathbf{E}\left(\sqrt{1-x}\right)\mathbf{D}\left(\sqrt{x}\right)dx = \frac{\pi}{2}.$

7. $\displaystyle\int_0^1 x\,\mathbf{E}\left(\sqrt{1-x}\right)\mathbf{D}\left(\sqrt{x}\right)dx = \frac{\pi^3}{32}.$

4.24. The Generalized Hypergeometric Function ${}_pF_q((a_p);\ (b_q);\ z)$

4.24.1. Integrals containing ${}_pF_q((a_p);\ (b_q);\ z)$ and algebraic functions

1. $\displaystyle\int_0^a x^{s-1}(a-x)^{t-1}\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ bx(a-x)\\ (b_q)\end{matrix}\right)dx = \mathrm{B}\,(s,\,t)a^{s+t-1}$

$$\times\ {}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\,s,\,t;\ \frac{a^2b}{4}\\ (b_q),\,\frac{s+t}{2},\,\frac{s+t+1}{2}\end{matrix}\right)\quad\left[a,\,\operatorname{Re}s,\,\operatorname{Re}t>0;\ \left|\arg\left(4-a^2b\right)\right|<\pi\right].$$

2. $\displaystyle\int_0^a x^{s-1}(a-x)^{s-1/2}\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ b\sqrt{x(a-x)}\\ (b_q)\end{matrix}\right)dx = \frac{\sqrt{\pi}\,\Gamma(2s)a^{2s-1/2}}{2^{2s-1}\Gamma\left(2s+\frac{1}{2}\right)}$

$$\times\ {}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\,2s;\ \frac{ab}{2}\\ (b_q),\,2s+\frac{1}{2}\end{matrix}\right)\quad\left[a,\,\operatorname{Re}s>0;\ \left|\arg(2-ab)\right|<\pi\right].$$

4.24.2. Integrals containing ${}_pF_q((a_p);\ (b_q);\ z)$ and trigonometric functions

1. $$\int_0^{m\pi} \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} {}_pF_q\left(\begin{matrix} (a_p);\ b\sin^2 x \\ (b_q) \end{matrix}\right) dx = \frac{2}{a} \sin\frac{m\pi a}{2} \begin{Bmatrix} \sin(m\pi a/2) \\ \cos(m\pi a/2) \end{Bmatrix} {}_{p+2}F_{q+2}\left(\begin{matrix} (a_p),\ \frac{1}{2},\ 1;\ b \\ (b_q),\ 1-\frac{a}{2},\ 1+\frac{a}{2} \end{matrix}\right).$$

2. $$\int_0^{\pi/2} \cos(2nx)\, {}_pF_q\left(\begin{matrix} (a_p);\ a\sin^2 x \\ (b_q) \end{matrix}\right) dx = \frac{\pi(-a)^n}{2^{2n+1} n!} \frac{\prod (a_p)_n}{\prod (b_q)_n} {}_{p+1}F_{q+1}\left(\begin{matrix} (a_p)+n,\ n+\frac{1}{2};\ a \\ (b_q)+n,\ 2n+1 \end{matrix}\right).$$

3. $$\int_0^{\pi/2} \sin x \sin(2n+1)x\, {}_pF_q\left(\begin{matrix} (a_p);\ b\sin^2 x \\ (b_q) \end{matrix}\right) dx = \frac{2^{-2n-2}\pi}{n!} (-b)^n \frac{\prod (a_p)_n}{\prod (b_q)_n} {}_{p+1}F_{q+1}\left(\begin{matrix} (a_p)+n,\ n+\frac{3}{2};\ b \\ (b_q)+n,\ 2n+2 \end{matrix}\right).$$

4. $$\int_0^{\pi} \sin^\nu x \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} {}_pF_q\left(\begin{matrix} (a_p);\ b\sin^2 x \\ (b_q) \end{matrix}\right) dx = \frac{2^{-\nu}\pi\Gamma(\nu+1)}{\Gamma\left(\frac{\nu+a}{2}+1\right)\Gamma\left(\frac{\nu-a}{2}+1\right)} \begin{Bmatrix} \sin(a\pi/2) \\ \cos(a\pi/2) \end{Bmatrix}$$
$$\times\ {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1;\ b \\ (b_q),\ \frac{\nu-a}{2}+1,\ \frac{\nu+a}{2}+1 \end{matrix}\right) \qquad [\operatorname{Re}\nu > -1].$$

5. $$\int_0^{\pi} \cos(nx)\, {}_pF_q\left(\begin{matrix} (a_p);\ a\cos^2 x \\ (b_q) \end{matrix}\right) dx = \frac{2^{-n}\pi(-a)^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \cos\frac{n\pi}{2} \frac{\prod (a_p)_{n/2}}{\prod (b_q)_{n/2}} {}_pF_q\left(\begin{matrix} (a_p)+\frac{n}{2},\ \frac{n+1}{2};\ a \\ (b_q)+\frac{n}{2},\ n+1 \end{matrix}\right).$$

6. $$\int_0^{\pi/2} \cos^\nu x \cos(ax)\, {}_pF_q\left(\begin{matrix} (a_p);\ b\cos^2 x \\ (b_q) \end{matrix}\right) dx = \frac{2^{-\nu-1}\pi\Gamma(\nu+1)}{\Gamma\left(\frac{\nu+a}{2}+1\right)\Gamma\left(\frac{\nu-a}{2}+1\right)} {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1;\ b \\ (b_q),\ \frac{\nu-a}{2}+1,\ \frac{\nu+a}{2}+1 \end{matrix}\right)$$
$$[\operatorname{Re}\nu > -1].$$

7. $$\int_0^{m\pi} e^{-ax}\,{}_pF_q\left(\begin{matrix}(a_p);\ b\sin x\\ (b_q)\end{matrix}\right)dx$$

$$= \frac{1-e^{-m\pi a}}{a}\,{}_{2p+1}F_{2q+2}\left(\begin{matrix}\frac{(a_p)}{2},\ \frac{(a_p)+1}{2},\ 1;\ \frac{b^2}{4^{q-p+1}}\\ \frac{(b_q)}{2},\ \frac{(b_q)+1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)$$

$$+\frac{b\left(1-e^{-m\pi}\right)}{a^2+1}\frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j}\,{}_{2p+1}F_{2q+2}\left(\begin{matrix}\frac{(a_p)+1}{2},\ \frac{(a_p)}{2}+1,\ 1;\ \frac{b^2}{4^{q-p+1}}\\ \frac{(b_q)+1}{2},\ \frac{(b_q)}{2}+1,\ \frac{3}{2}-\frac{ia}{2},\ \frac{3}{2}+\frac{ia}{2}\end{matrix}\right)$$

[$\operatorname{Re} a > 0$].

8. $$\int_0^{m\pi} e^{-ax}\,{}_pF_q\left(\begin{matrix}(a_p);\ b\sin^2 x\\ (b_q)\end{matrix}\right)dx$$

$$= \frac{1-e^{-m\pi a}}{a}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\ \frac{1}{2},\ 1;\ b\\ (b_q),\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right).$$

9. $$\int_0^{m\pi} e^{-ax}\sin x\,{}_pF_q\left(\begin{matrix}(a_p);\ b\sin^2 x\\ (b_q)\end{matrix}\right)dx$$

$$= \frac{1-(-1)^m e^{-m\pi a}}{a^2+1}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\ 1,\ \frac{3}{2};\ b\\ (b_q),\ \frac{3-ia}{2},\ \frac{3+ia}{2}\end{matrix}\right).$$

10. $$\int_0^{\infty} e^{-ax}\,{}_pF_q\left(\begin{matrix}(a_p);\ b\sin x\\ (b_q)\end{matrix}\right)dx$$

$$= \frac{1}{a}\,{}_{2p+1}F_{2q+2}\left(\begin{matrix}\frac{(a_p)}{2},\ \frac{(a_p)+1}{2},\ 1;\ \frac{b^2}{4^{q-p+1}}\\ \frac{(b_q)}{2},\ \frac{(b_q)+1}{2},\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)$$

$$+\frac{b}{a^2+1}\frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j}\,{}_{2p+1}F_{2q+2}\left(\begin{matrix}\frac{(a_p)+1}{2},\ \frac{(a_p)}{2}+1,\ 1;\ \frac{b^2}{4^{q-p+1}}\\ \frac{(b_q)+1}{2},\ \frac{(b_q)}{2}+1,\ \frac{3-ia}{2},\ \frac{3+ia}{2}\end{matrix}\right)\quad [\operatorname{Re} a > 0].$$

11. $$\int_0^{\infty} e^{-ax}\,{}_pF_q\left(\begin{matrix}(a_p);\ b\sin^2 x\\ (b_q)\end{matrix}\right)dx = \frac{1}{a}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\ \frac{1}{2},\ 1;\ b\\ (b_q);\ 1-\frac{ia}{2},\ 1+\frac{ia}{2}\end{matrix}\right)$$

[$\operatorname{Re} a > 0$].

12. $$\int_0^\infty e^{-ax}\sin x\,{}_pF_q\left(\begin{matrix}(a_p);\ b\sin^2 x\\ (b_q)\end{matrix}\right)dx = \frac{1}{a^2+1}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\,1,\,\frac{3}{2};\ b\\ (b_q),\,\frac{3-ia}{2},\,\frac{3+ia}{2}\end{matrix}\right)\quad [\operatorname{Re} a>0].$$

4.24.3. Integrals containing ${}_pF_q((a_p);\,(b_q);\,z)$ and the logarithmic function

1. $$\int_0^a x^{s-1}\ln\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\,{}_pF_q\left(\begin{matrix}(a_p);\ bx\\ (b_q)\end{matrix}\right)dx = \frac{\sqrt{\pi}}{2s}a^s\frac{\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\,s,\,s;\ ab\\ (b_q),\,s+\frac{1}{2},\,s+1\end{matrix}\right)\quad [a,\,\operatorname{Re} s>0].$$

4.24.4. Integrals containing ${}_pF_q((a_p);\,(b_q);\,z)$, $\mathbf{K}(z)$ and $\mathbf{E}(z)$

1. $$\int_0^1 x^{s-1}\,\mathbf{K}\left(\sqrt{1-x}\right)\,{}_pF_q\left(\begin{matrix}(a_p);\ ax\\ (b_q)\end{matrix}\right)dx = \frac{\pi}{2}\frac{\Gamma^2(s)}{\Gamma^2\left(s+\frac{1}{2}\right)}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\,s,\,s;\ a\\ (b_q),\,s+\frac{1}{2},\,s+\frac{1}{2}\end{matrix}\right)\quad [\operatorname{Re} s>0].$$

2. $$\int_0^1 x^{s-1}\,\mathbf{E}\left(\sqrt{1-x}\right)\,{}_pF_q\left(\begin{matrix}(a_p);\ ax\\ (b_q)\end{matrix}\right)dx = \frac{\pi}{2}\frac{\Gamma(s)\Gamma(s+1)}{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p),\,s,\,s+1;\ a\\ (b_q),\,s+\frac{1}{2},\,s+\frac{3}{2}\end{matrix}\right)\quad [\operatorname{Re} s>0].$$

4.24.5. Integrals containing products of ${}_pF_q((a_p);\,(b_q);\,z)$

1. $$\int_0^a x^{s-1}(a-x)^{t-1}\,{}_1F_1\left(\begin{matrix}b;\ wx\\ s\end{matrix}\right){}_1F_1\left(\begin{matrix}c;\ w(a-x)\\ t\end{matrix}\right)dx = \mathrm{B}\,(s,\,t)a^{s+t-1}\,{}_1F_1\left(\begin{matrix}b+c;\ aw\\ s+t\end{matrix}\right)\quad [a,\,\operatorname{Re} s,\,\operatorname{Re} t>0].$$

2. $$\int_0^a x^{s-1}(a-x)^{t-1}\,{}_1F_1\left(\begin{matrix}b;\ wx\\ s\end{matrix}\right){}_1F_1\left(\begin{matrix}b;\ -w(a-x)\\ t\end{matrix}\right)dx = \mathrm{B}\,(s,\,t)a^{s+t-1}\,{}_1F_2\left(\begin{matrix}b;\ \frac{a^2w^2}{4}\\ \frac{s+t}{2},\,\frac{s+t+1}{2}\end{matrix}\right)\quad [a,\,\operatorname{Re} s,\,\operatorname{Re} t>0].$$

3. $$\int_0^a x^{s-1}(a-x)^{t-1}\,{}_1F_2\left(\begin{matrix} b;\ wx \\ c,\ s \end{matrix}\right){}_1F_2\left(\begin{matrix} b;\ -w(a-x) \\ c,\ t \end{matrix}\right)dx$$
$$= \mathrm{B}\,(s,\,t)a^{s+t-1}\,{}_2F_5\left(\begin{matrix} b,\ c-b;\ \frac{a^2w^2}{16} \\ c,\ \frac{c}{2},\ \frac{c+1}{2},\ \frac{s+t}{2},\ \frac{s+t+1}{2} \end{matrix}\right) \quad [a,\,\mathrm{Re}\,s,\,\mathrm{Re}\,t>0].$$

4. $$\int_0^a x^{\mu+m-1}(a-x)^{\nu+n-1}\,{}_0F_1(\mu;\ -bx)\,{}_0F_1(\nu;\ -c(a-x))\,dx$$
$$= (-1)^{m+n}\Gamma(\mu)\Gamma(\nu)\left(\frac{a}{b+c}\right)^{(\mu+\nu+m+n-1)/2}$$
$$\times \sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}(1-\mu-m)_{m-j}(1-\nu-n)_{n-k}b^jc^k\left(\frac{a}{b+c}\right)^{(j+k)/2}$$
$$\times J_{\mu+\nu+j+k+m+n-1}\left[2\sqrt{a(b+c)}\right] \quad [a>0].$$

5. $$\int_0^a x^{n+1/2}(a-x)^{(m+\nu)/2-1}\,{}_0F_1(\nu;\ -b(a-x))\,{}_1F_2\left(\begin{matrix} \frac{1}{2};\ -cx \\ n+\frac{3}{2},\ \frac{1}{2}-n \end{matrix}\right)$$
$$= (-1)^n\left(n+\frac{1}{2}\right)\pi\Gamma(\nu)(1-\nu)_m\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)_{n-k}c^k$$
$$\times \mathrm{D}_u^m\,\mathrm{D}_w^{k+n+1}\left[b^{(1-\nu)/2}J_{(\nu-1)/2}\left(\sqrt{a}\sqrt{u+w}+\sqrt{aw}\right)\right.$$
$$\left.\times J_{(\nu-1)/2}\left(\sqrt{a}\sqrt{u+w}-\sqrt{aw}\right)\right]\Big|_{\substack{u=b\\w=c}} \quad [a>0].$$

6. $$\int_1^\infty x^{-m-n-2}\,{}_2F_2\left(\begin{matrix} -m,\ -m,\ -m-\frac{1}{2};\ x \\ \frac{1}{2}-m,\ -2m-1 \end{matrix}\right){}_2F_2\left(\begin{matrix} -n,\ -n,\ -n-\frac{1}{2};\ x \\ \frac{1}{2}-n,\ -2n-1 \end{matrix}\right)dx$$
$$= (-1)^{m+n}\frac{(m!)^2(n!)^2}{2(2m)!\,(2n)!}\left[\mathbf{C}+2\ln 2+\psi\left(m+\frac{1}{2}\right)\right] \quad [m\le n].$$

7. $$\int_a^\infty (a-x)^{\nu-1}\,{}_0F_1(\nu;\ ab-bx)\,{}_pF_{p+1}\left(\begin{matrix} (a_p);\ -cx \\ (b_{p+1}) \end{matrix}\right)=0$$
$$\left[a>0;\ 0<c<4b;\ \mathrm{Re}\left(\nu+\sum a_p-\sum b_{p+1}\right)<-1\right].$$

Chapter 5

Finite Sums

5.1. The Psi Function $\psi(z)$

5.1.1. Sums containing $\psi(k+a)$

1. $\displaystyle\sum_{k=1}^{n} \psi(k) = n\psi(n+1) - n.$

2. $\displaystyle\sum_{k=0}^{n} \psi(k+a) = (a-1)\left[1-\psi(a-1)\right] - (a+n)\left[1-\psi(a+n)\right].$

3. $$\sum_{k=0}^{n} k\psi(k+a) = \frac{a-1}{4}\left[2-3a+2a\psi(a)\right] + \frac{a+n}{4}\left[3a-3-n+2(1-a+n)\psi(a+n+1)\right].$$

4. $$\sum_{k=0}^{n} k^2\psi(k+a) = \frac{n}{36}\left[1+12a\,(1-a)-n\,(3-6a+4n)\right] - \frac{(a-1)a(2a-1)}{6}\psi(a) + \frac{1}{6}\left[(a-1)a(2a-1)+n(n+1)(2n+1)\right]\psi(a+n).$$

5. $$\sum_{k=0}^{n} k^3\psi(k+a) = \frac{n+1}{48}\left[12a^3-6(n+4)a^2 + 2(n+2)(2n+3)a - n(n+2)(3n+5)\right] + \frac{1}{4}(a-1)^2a^2\psi(a) + \frac{1}{4}\left[n^2(n+1)^2-(a-1)^2a^2\right]\psi(a+n+1).$$

6. $$\sum_{k=1}^{n} \frac{k}{2^k}\psi(k+1) = 2(1-\mathbf{C}+\ln 2) - 2^{-n}(n+2)\psi(n+3) - \frac{2^{-n}}{n+2}\,{}_2F_1\left(\begin{matrix}1,\,1;\ -1\\ n+3\end{matrix}\right) \quad [n\ge 1].$$

7. $\displaystyle\sum_{k=1}^{n} 2^{-k}(k+a-2)\psi(k+a) = 2 + a\psi(a) - 2^{-n}\left[2+(a+n)\psi(a+n)\right].$

8. $\displaystyle\sum_{k=1}^{n}(-1)^k\binom{n}{k}\psi(k)=\left(2-\frac{1}{n}\right)\mathbf{C}+\left(1-\frac{1}{n}\right)\psi(n+1)\qquad [n\geq 1].$

9. $\displaystyle\sum_{k=0}^{n}\frac{(a)_k}{k!}\psi(k+1)=\frac{1}{a}+\frac{(a+1)_n}{n!\,a}[a\psi(n+1)-1].$

10. $\displaystyle\sum_{k=0}^{n}\frac{(b)_k}{k!}\psi(k+b)=\frac{(b+1)_n}{n!}\,[\psi(b)-\psi(b+1)+\psi(b+n+1)].$

11. $\displaystyle\sum_{k=0}^{n}\frac{(a)_k}{(b)_k}\psi(k+a)$

$$=\frac{1}{(a-b+1)^2}\left\{b-1-\frac{a(a+1)_n}{(b)_n}+(a-b+1)\left[1-b+\frac{a(a+1)_n}{(b)_n}\right]\psi(a)\right.$$
$$\left.+\frac{(b-a-1)(a+1)_n}{(b)_n}\,[a\psi(a+1)-a\psi(a+n+1)-1]\right\}.$$

12. $\displaystyle\sum_{k=0}^{n}\frac{(b)_k}{(a)_k}\psi(k+a)=\frac{a-1}{(b-a+1)^2}[1+(a-b-1)\psi(a-1)]$

$$-\frac{(b)_{n+1}}{(b-a+1)^2(a)_n}\,[1+(a-b-1)\psi(a+n)].$$

13. $\displaystyle\sum_{k=0}^{n}\frac{(a)_k}{k!\,(n-k+1)}\psi(k+a)$

$$=\frac{(a)_{n+1}}{(n+1)!}\,\{\psi(a+n+1)\,[\psi(-a-n)-\psi(1-a)]$$
$$+\psi'(1-a)-\psi'(-a-n)\}.$$

14. $\displaystyle\sum_{k=0}^{n}2^k\binom{n}{k}(2n-k)!\,(a)_k\,\psi(k+a)$

$$=2^{2n-1}n!\left(\frac{a+1}{2}\right)_n\left[2\ln 2+\psi\left(\frac{a}{2}\right)+\psi\left(n+\frac{a+1}{2}\right)\right].$$

15. $\displaystyle\sum_{k=1}^{n}\frac{(2n-k-1)!}{(n-k)!}\psi(k)$

$$=\frac{2^{2n-1}\left(\frac{1}{2}\right)_n}{n}\left[2\ln 2-\mathbf{C}-\frac{1}{n}+\psi\left(n+\frac{1}{2}\right)-\psi(2n)\right]\qquad [n\geq 1].$$

16. $\displaystyle\sum_{k=0}^{n}2^k\frac{(2n-k)!}{(n-k)!}\psi(k+1)=2^{2n-1}n!\,[\psi(n+1)-\mathbf{C}].$

17. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{k}^{-1}\frac{(a-1)_k}{k!}\,\psi(k+a)$

$$= \frac{2^{-2n}(a)_{2n}}{\left(\frac{1}{2}\right)_n(a)_n}\left[2\psi(a)-\psi(a+n)-\psi\left(\frac{a+1}{2}\right)+\psi\left(\frac{a+n+1}{2}\right)\right].$$

5.1.2. Sums containing products of $\psi(k+a)$

1. $\displaystyle\sum_{k=0}^{n}\psi^2(k+a) = 2n+(2a-1)\psi(a)+(1-a)\psi^2(a)$

$$+(1-2n-2a)\psi(n+a)+(n+a)\psi^2(n+a).$$

2. $\displaystyle\sum_{k=0}^{n}\frac{(b)_k}{(a)_k}\,\psi(k+a)\,\psi(k+b)$

$$= \frac{a-1}{(a-b-1)^2}\left\{\frac{2}{a-b-1}+\psi(b)+\psi(a-1)\left[1+(a-b-1)\psi(b)\right]\right.$$

$$-\frac{(b)_{n+1}}{(a)_n\,(a-b-1)^2}\left[\frac{2}{a-b-1}+\psi(b+n+1)\right.$$

$$\left.\left.+\,\psi(a+n)\left(1+(a-b-1)\psi(b+n+1)\right)\right]\right\}.$$

3. $\displaystyle\sum_{k=0}^{n}\psi^3(k+a) = -6n+3(1-2a)\psi(a)+\frac{3}{2}(2a-1)\psi^2(a)$

$$+(1-a)\psi^3(a)+3(2n+2a-1)\psi(n+a)+\frac{3}{2}(1-2n-2a)\psi^2(n+a)$$

$$+(n+a)\psi^3(n+a)+\frac{1}{2}\psi'(a)-\frac{1}{2}\psi'(n+a).$$

5.1.3. Sums containing $\psi'(k+a,\,z)$

1. $\displaystyle\sum_{k=1}^{n}(-1)^k\binom{n}{k}\psi'\left(k+\frac{1}{2}\right) = -\frac{\pi^2}{2}+\frac{(n-1)!}{\left(\frac{1}{2}\right)_n}\left[\mathrm{C}+2\ln 2+\psi\left(n+\frac{1}{2}\right)\right]$

$$[n\ge 1].$$

2. $\displaystyle\sum_{k=1}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}\psi'\left(k+\frac{1}{2}\right) = \frac{n!\,\pi^2}{2\left(\frac{3}{2}\right)_n}-\frac{\pi^2}{2}$

$$+\frac{2n}{\left(\frac{3}{2}\right)_n}\,n!\,{}_4F_3\left(\begin{matrix}1-n,\,1,\,1,\,1\\ \frac{3}{2},\,2,\,2;\;1\end{matrix}\right)\quad[n\ge 1].$$

3. $$\sum_{k=0}^{n} \psi(k+a)\psi'(k+a) = \psi(a) - \frac{1}{2}\psi^2(a) - \psi(n+a) + \frac{1}{2}\psi^2(n+a) + \frac{1}{2}\left[2a-1+2(1-a)\psi(a)\right]\psi'(a) + \frac{1}{2}\left[1-2n-2a+2(n+a)\psi(n+a)\right]\psi'(n+a).$$

5.2. The Incomplete Gamma Functions $\gamma(\nu, z)$ and $\Gamma(\nu, z)$

5.2.1. Sums containing $\gamma(nk+\nu, z)$

1. $$\sum_{k=0}^{n} \frac{z^{-k}}{\nu+k-1}\gamma(\nu+k, z) = \frac{z^{\nu}e^{-z}}{\nu-1}\, {}_2F_2\left(\begin{matrix}1, 1;\ z\\ \nu, 2\end{matrix}\right) - \frac{z^{\nu}e^{-z}}{\nu+n}\, {}_2F_2\left(\begin{matrix}1, 1;\ z\\ \nu+n+1, 2\end{matrix}\right).$$

2. $$\sum_{k=0}^{n} \frac{(-1)^k}{k!}\gamma(k+1, z) = \frac{1+(-1)^n}{2} - \frac{(-z)^n}{n!}e^{-z}\, {}_3F_0\left(\begin{matrix}-\frac{n}{2}, \frac{1-n}{2}, 1\\ 4z^{-2}\end{matrix}\right).$$

3. $$\sum_{k=0}^{n} \binom{n}{k}(-z)^{-k}\gamma(\nu+k, z) = \frac{n!\,z^{\nu}e^{-z}}{(\nu)_{n+1}}\, {}_1F_1\left(\begin{matrix}n+1;\ z\\ \nu+n+1\end{matrix}\right).$$

4. $$\sum_{k=0}^{n} (-1)^k\binom{n}{k}\frac{1}{(\nu)_k\,(k+1)}\gamma(\nu+k, z) = \frac{1}{n+1}\left[(\nu-1)\,\gamma(\nu-1, z) - \frac{n!}{(\nu)_n}z^{\nu-1}e^{-z}L_n^{\nu-1}(z)\right].$$

5. $$\sum_{k=0}^{n} (-1)^k\binom{n}{k}\frac{(a-1)_k}{(a-n)_k(\nu)_k}\gamma(\nu+k, z) = \frac{z^{\nu}e^{-z}}{\nu}\, {}_2F_2\left(\begin{matrix}1-n, a;\ z\\ \nu+1, a-n\end{matrix}\right).$$

6. $$\sum_{k=0}^{n} \frac{(-1)^k}{(k+m)!}\binom{n}{k}\gamma(k+m+1, z) = \delta_{n,0} - (-1)^m e^{-z}L_{m+n}^{-m-1}(z).$$

7. $$\sum_{k=0}^{n} \binom{n}{k}t^k\gamma(k+1, z) = t^n e^{1/t}\left[\gamma\left(n+1, z+\frac{1}{t}\right) - \gamma\left(n+1, \frac{1}{t}\right)\right].$$

8. $$\sum_{k=0}^{n} \binom{n}{k}\frac{(k+n+1)!}{(k+1)!}(-z)^{-k}\gamma(k+1, z) = \frac{n!\,z}{n+1} - \frac{(n!)^2}{(2n+2)!}z^{n+2}e^{-z}\, {}_1F_1\left(\begin{matrix}n+2;\ z\\ 2n+3\end{matrix}\right).$$

9. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k}{(b)_k} (-z)^{-k} \gamma(k+1, z) = {}_3F_1\left(\begin{matrix} -n, a, 1 \\ b;\ z^{-1} \end{matrix}\right) - \frac{(b-a)_n}{(b)_n} e^{-z} {}_3F_1\left(\begin{matrix} -n, a, 1;\ -z^{-1} \\ a-b-n+1 \end{matrix}\right).$$

10. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a+m)_k}{(k+m)!} (-z)^{-k} \gamma(k+m+1, z) = (-z)^{a+m} \Psi\left(\begin{matrix} a+m;\ -z \\ a+m+n+1 \end{matrix}\right) - \frac{(1-a)_n}{(m+n)!} z^m e^{-z} {}_3F_1\left(\begin{matrix} -m-n, a, 1 \\ a-n;\ -z^{-1} \end{matrix}\right).$$

11. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\nu+n)_k}{(\nu+1)_k (\nu)_{2k}} \gamma(\nu+2k, z) = \frac{(n-1)!n!}{(\nu+1)_n (\nu+1)_{n-1}} \frac{z^\nu e^{-z}}{\nu} L_n^\nu(-z) L_{n-1}^\nu(z) \quad [n \geq 1].$$

5.2.2. Sums containing products of $\gamma(\nu \pm k, z)$

1. $$\sum_{k=0}^{2n} \binom{2n}{k} \gamma(\nu+k, z) \gamma(\nu+2n-k, -z) = (2n)! \frac{e^{\nu\pi i} z^{2\nu+2n}}{(\nu)_{2n+1}(\nu+n)} {}_3F_4\left(\begin{matrix} n+\frac{1}{2}, n+1, \nu+n;\ \frac{z^2}{4} \\ \frac{\nu+1}{2}+n, \frac{\nu}{2}+n+1, \nu+n+1, \frac{1}{2} \end{matrix}\right).$$

2. $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} \gamma(\nu+k, z) \gamma(\nu+2n+1-k, -z) = (2n+2)! \frac{e^{(\nu+1)\pi i} z^{2\nu+2n+2}}{(\nu)_{2n+3}(\nu+n+1)} {}_3F_4\left(\begin{matrix} n+\frac{3}{2}, n+2, \nu+n+1;\ \frac{z^2}{4} \\ \frac{\nu+3}{2}+n, \frac{\nu}{2}+n+2, \nu+n+2, \frac{3}{2} \end{matrix}\right).$$

5.2.3. Sums containing $\Gamma(\nu \pm k, z)$

1. $$\sum_{k=0}^{n} \binom{n}{k} (-z)^k \Gamma(\nu-k, z) = n!\, e^{-z} \Psi\left(\begin{matrix} 1-\nu+n \\ 1-\nu;\ z \end{matrix}\right).$$

2. $$\sum_{k=0}^{n} \binom{n}{k} (1-\nu)_k \Gamma(\nu-k, z) = (-1)^{n-1} (n-1)! z^{\nu-n} e^{-z} L_{n-1}^{\nu-n}(z) \quad [n \geq 1].$$

3. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(1-\nu)_k}{(1-\nu-n)_k} (-z)^k \Gamma(\nu-k, z) = \frac{n!}{(\nu)_n} z^{\nu+n} e^{-z} \Psi\left(\begin{matrix} n+1;\ z \\ \nu+n+1 \end{matrix}\right).$$

4. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(\nu)_k} \Gamma(\nu+k, z) = -\frac{(n-1)!}{(\nu)_n} z^\nu e^{-z} L_{n-1}^\nu(z) \quad [n \geq 1].$$

5.3. The Bessel Function $J_\nu(z)$

5.3.1. Sums containing $J_{\nu\pm nk}(z)$

1. $$\sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!}(2z)^{-k} J_{k-n+1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z + \frac{n\pi}{2}\right).$$

2. $$\sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!}(-2z)^{-k} J_{n-k-1/2}(z) = (-1)^n \sqrt{\frac{2}{\pi z}} \cos\left(z + \frac{n\pi}{2}\right).$$

3. $$\sum_{k=0}^{n} \binom{n}{k}(2k+\nu)\frac{(\nu)_k}{(\nu+n+1)_k} J_{2k+\nu}(z) = (\nu)_{n+1}\left(\frac{2}{z}\right)^n J_{\nu+n}(z).$$

4. $$\sum_{k=0}^{n}(-1)^k \binom{n}{k}(2k+\nu)\frac{(\nu)_k}{(\nu+n+1)_k} J_{4k+2\nu}(z)$$
$$= \frac{2^{-2\nu-1} z^{2\nu}}{\Gamma(2\nu)}\, {}_1F_2\left(\begin{matrix}\nu+n+\frac{1}{2};\ -\frac{z^2}{4}\\ \nu+\frac{1}{2},\ 2\nu+2n+1\end{matrix}\right).$$

5. $$\sum_{k=0}^{n} \binom{n}{k}\frac{\left(-\frac{z}{2}\right)^k}{(\nu-n+1)_k} J_{k+\nu}(z) = \frac{\left(-\frac{z}{2}\right)^n}{(-\nu)_n} J_{\nu-n}(z).$$

5.3.2. Sums containing products of $J_{\nu\pm nk}(z)$

1. $$\sum_{k=1}^{n}(-1)^k (k+\nu) J_{k+\nu}^2(z) = \frac{1}{2} J_{\nu+1}(z)\left[z J_{\nu+2}(z) - 2(\nu+1) J_{\nu+1}(z)\right]$$
$$+ \frac{(-1)^n}{2} J_{\nu+n+1}(z)\left[2(\nu+n+1) J_{\nu+n+1}(z) - z J_{\nu+n+2}(z)\right].$$

2. $$\sum_{k=0}^{n} \binom{n}{k}(k+\nu)\frac{(2\nu)_k}{(2\nu+n+1)_k} J_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{\nu\Gamma^2(\nu)}\, {}_1F_2\left(\begin{matrix}\nu+\frac{1}{2};\ -z^2\\ \nu+1,\ 2\nu+n+1\end{matrix}\right).$$

3. $$\sum_{k=0}^{2n+1}(2n-2k+1)\binom{2n+1}{k} J_{k-n-1/2}(w)\, J_{k-n-1/2}(z)$$
$$= -\frac{2^{n+3/2}}{\Gamma\left(-n-\frac{1}{2}\right)}\left(\frac{w+z}{wz}\right)^{n+1/2} J_{-n-1/2}(w+z).$$

5.4. The Modified Bessel Function $I_\nu(z)$

5.4.1. Sums containing $I_{\nu\pm nk}(z)$

1. $$\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\frac{z}{2}\right)^k}{(\nu-n+1)_k}I_{k+\nu}(z)=\frac{\left(-\frac{z}{2}\right)^n}{(-\nu)_n}I_{\nu-n}(z).$$

2. $$\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}}{k!}(2z)^{-k}I_{\pm n\mp k\mp 1/2}(z)=\frac{1}{\sqrt{2\pi z}}[e^z\pm(-1)^n e^{-z}].$$

3. $$\sum_{k=1}^{n}(2k+\nu)\,I_{2k+\nu}(z)=\frac{z}{2}\,[I_{\nu+1}(z)-I_{\nu+2n+1}(z)].$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2k+\nu)\frac{(\nu)_k}{(\nu+n+1)_k}I_{2k+\nu}(z)=(\nu)_{n+1}\left(\frac{2}{z}\right)^n I_{\nu+n}(z).$$

5.4.2. Sums containing products of $J_{\nu\pm nk}(z)$ and $I_{\nu\pm nk}(z)$

1. $$\sum_{k=0}^{n}\binom{n}{k}J_{\nu-n+k}(z)\,I_{\nu+k}(z)$$
$$=\frac{(-1)^n(-\nu)_n}{\Gamma^2(\nu+1)}\left(\frac{z}{2}\right)^{2\nu-n}{}_0F_3\left(\begin{matrix}-\frac{z^4}{64}\\ \nu+1,\ \frac{\nu-n+1}{2},\ \frac{\nu-n}{2}+1\end{matrix}\right).$$

2. $$\sum_{k=0}^{n}\binom{n}{k}J_{\nu-n+k}(z)\,I_{\nu-k}(z)$$
$$=\frac{(-1)^n(-2\nu)_n}{\Gamma^2(\nu+1)}\left(\frac{z}{2}\right)^{2\nu-n}{}_1F_4\left(\begin{matrix}\nu+\frac{1}{2};\ -\frac{z^4}{64}\\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ \nu-\frac{n+1}{2},\ \nu-\frac{n}{2}+1\end{matrix}\right).$$

3. $$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{z}{w}\right)^k J_{k-n+1/2}(w)\,I_{k-1/2}(z)$$
$$=\frac{w^{-n-1/2}}{\sqrt{2\pi z}}[(w+iz)^{n+1/2}J_{1/2-n}(w+iz)+(w-iz)^{n+1/2}J_{1/2-n}(w-iz)].$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{z}{w}\right)^k J_{n-k-1/2}(w)\,I_{k-1/2}(z)$$
$$=\frac{w^{-n-1/2}}{\sqrt{2\pi z}}[(w+iz)^{n+1/2}J_{n-1/2}(w+iz)+(w-iz)^{n+1/2}J_{n-1/2}(w-iz).$$

5. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{k-1/2}(z)\,I_{n-k+1/2}(z)=(-1)^n\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\pi}{8}\operatorname{ber}_{n+1/2}\left(\sqrt{2}\,z\right)+\cos\frac{(6n+3)\pi}{8}\operatorname{bei}_{n+1/2}\left(\sqrt{2}\,z\right)\right].$$

6. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{k-1/2}(z)\,I_{k-n+1/2}(z)=\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{1/2-n}\left(\sqrt{2}\,z\right)+\cos\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{1/2-n}\left(\sqrt{2}\,z\right)\right].$$

7. $$\sum_{k=0}^{n}\binom{n}{k}J_{-k-1/2}(z)\,I_{k-n+1/2}(z)=(-1)^n\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{n+1/2}\left(\sqrt{2}\,z\right)+\cos\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{n+1/2}\left(\sqrt{2}\,z\right)\right].$$

8. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{k+1/2}(z)\,I_{k-n+1/2}(z)=\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{-n-1/2}\left(\sqrt{2}\,z\right)+\cos\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{-n-1/2}\left(\sqrt{2}\,z\right)\right].$$

9. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{k-1/2}(z)\,I_{k-n-1/2}(z)=\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\cos\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{-n-1/2}\left(\sqrt{2}\,z\right)-\sin\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{-n-1/2}\left(\sqrt{2}\,z\right)\right].$$

10. $$\sum_{k=0}^{n}\binom{n}{k}J_{1/2-k}(z)\,I_{n-k+1/2}(z)=\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{-n-1/2}\left(\sqrt{2}\,z\right)+\cos\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{-n-1/2}\left(\sqrt{2}\,z\right)\right].$$

11. $$\sum_{k=0}^{n}\binom{n}{k}J_{-k-1/2}(z)\,I_{n-k-1/2}(z)=\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\cos\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{-n-1/2}\left(\sqrt{2}\,z\right)-\sin\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{-n-1/2}\left(\sqrt{2}\,z\right)\right].$$

12. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}J_{k+1/2}(z)\,I_{n-k-1/2}(z)=(-1)^n\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{n+1/2}\left(\sqrt{2}\,z\right)-\cos\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{n+1/2}\left(\sqrt{2}\,z\right)\right].$$

13. $$\sum_{k=0}^{n}\binom{n}{k}J_{1/2-k}(z)\,I_{n-k-1/2}(z)=\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{1/2-n}\left(\sqrt{2}\,z\right)-\cos\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{1/2-n}\left(\sqrt{2}\,z\right)\right].$$

14. $$\sum_{k=0}^{n}\binom{n}{k}J_{1/2-k}(z)\,I_{k-n-1/2}(z)=(-1)^n\frac{2^{(2n+3)/4}}{\sqrt{\pi z}}$$
$$\times\left[\sin\frac{(6n+3)\,\pi}{8}\operatorname{bei}_{n+1/2}\left(\sqrt{2}\,z\right)-\cos\frac{(6n+3)\,\pi}{8}\operatorname{ber}_{n+1/2}\left(\sqrt{2}\,z\right)\right].$$

5.4.3. Sums containing products of $I_{\nu\pm nk}(z)$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\nu+k)\frac{(2\nu)_k}{(2\nu+n+1)_k}I_{\nu+k}^2(z)$$
$$=\frac{\left(\frac{z}{2}\right)^{2\nu}}{\nu\Gamma^2(\nu)}\,{}_1F_2\left(\begin{matrix}\nu+\frac{1}{2};\ z^2\\ \nu+1,\,2\nu+n+1\end{matrix}\right).$$

2. $$\sum_{k=0}^{n}\binom{n}{k}I_{\mu-k}(z)\,I_{\nu+k}(z)=(-1)^n\frac{(\mu+\nu-n+1)_{2n}}{\Gamma(\mu+1)\Gamma(\nu+n+1)(-\mu-\nu)_n}\left(\frac{z}{2}\right)^{\mu+\nu}$$
$$\times\,{}_2F_3\left(\begin{matrix}\frac{\mu+\nu+n+1}{2},\ \frac{\mu+\nu+n}{2}+1;\ z^2\\ \mu+1,\,\nu+n+1,\,\mu+\nu+1\end{matrix}\right).$$

3. $$\sum_{k=0}^{n}\binom{n}{k}I_{\mu-k}(z)\,I_{\nu-k}(z)$$
$$=\frac{(-\mu)_n(-\nu)_n}{\Gamma(\mu+1)\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\mu+\nu-2n}{}_2F_3\left(\begin{matrix}\frac{\mu+\nu-n+1}{2},\ \frac{\mu+\nu-n}{2}+1;\ z^2\\ \mu-n+1,\,\nu-n+1,\,\mu+\nu-n+1\end{matrix}\right).$$

4. $$\sum_{k=0}^{2n+1}(-1)^k(2n-2k+1)\binom{2n+1}{k}I_{k-n-1/2}(w)\,I_{k-n-1/2}(z)$$
$$=-\frac{2^{n+3/2}}{\Gamma\left(-n-\frac{1}{2}\right)}\left(\frac{w+z}{wz}\right)^{n+1/2}I_{-n-1/2}(w+z).$$

5. $$\sum_{k=1}^{n}\frac{(-1)^k}{k^{1/2}}\binom{n}{k}\left[I_{-m-1/2}^2\left(\frac{z}{\sqrt{k}}\right)-I_{m+1/2}^2\left(\frac{z}{\sqrt{k}}\right)\right]$$
$$=(-1)^n2^{1-2n}z^{-2n-1}\frac{[(2n)!]^2}{n!\,\pi}\delta_{m,n}-(-1)^m\frac{2}{\pi z}\quad[m\le n].$$

6. $$\sum_{k=0}^{n}\frac{(-1)^k}{I_k(z)\,I_{k+1}(z)}=(-1)^n z\frac{K_{n+1}(z)}{I_{n+1}(z)}+z\frac{K_0(z)}{I_0(z)}.$$

5.5. The Macdonald Function $K_\nu(z)$

5.5.1. Sums containing $K_{\nu\pm nk}(z)$

1. $$\sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!} t^k K_{k+1/2}(z)$$
$$= \frac{1}{\sqrt{\pi t}} K_{n+1/2}\left(\frac{z}{2} - \frac{1}{2}\sqrt{z^2 - \frac{2z}{t}}\right) K_{n+1/2}\left(\frac{z}{2} + \frac{1}{2}\sqrt{z^2 - \frac{2z}{t}}\right) \quad [68].$$

2. $$\sum_{k=0}^{[n/2]} \frac{(-2z)^{-k}}{k!\,(n-2k)!} K_{n-k-1/2}(z) = \frac{1}{n!}\sqrt{\frac{\pi}{2z}}\, e^{-z}.$$

3. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-a-n\right)_k}{(n-k)!\,(1-a-2n)_k} \left(\frac{2}{z}\right)^k K_{k+1/2}(z)$$
$$= \Gamma\left(n+\frac{1}{2}\right)\left[\frac{2^{-a}\sqrt{\pi}\,\Gamma(2a+2n)}{\Gamma(a+2n)} z^{-a-2n} I_{a-1/2}(z) \right.$$
$$\left. - \frac{a+2n}{(2n+1)!\,(a+n)} 2^{2n-1/2} z^{1/2} e^{-z}\, {}_2F_2\left(\begin{matrix} a+2n+1,\ 1;\ 2z \\ 2a+2n+1,\ 2n+2 \end{matrix}\right)\right].$$

4. $$\sum_{k=0}^{n} \binom{n}{k} \frac{\left(-\frac{z}{2}\right)^k}{\left(\frac{1}{2}-m-n\right)_k} K_{m-k+1/2}(z) = (-1)^{m+n} \frac{2^{-n-1}\pi z^n}{\left(m+\frac{1}{2}\right)_n} I_{-m-n-1/2}(z).$$

5. $$\sum_{k=0}^{n} \binom{n}{k} (\nu+n)_k \left(\frac{2}{z}\right)^k K_{\nu+k}(z) = K_{\nu+2n}(z).$$

5.5.2. Sums containing $K_{\nu\pm nk}(z)$ and special functions

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} J_{\nu+k-n}(z)\, K_{\nu+k}(z)$$
$$= \frac{\Gamma\left(\frac{\nu-n}{2}\right)}{4\Gamma\left(\frac{\nu+n}{2}+1\right)} \left(-\frac{z}{2}\right)^n {}_0F_3\left(\begin{matrix} -\frac{z^4}{64} \\ \frac{n-\nu}{2}+1,\ \frac{n+\nu}{2}+1,\ \frac{1}{2} \end{matrix}\right)$$
$$- \frac{\Gamma\left(\frac{\nu-n-1}{2}\right)}{4\Gamma\left(\frac{\nu+n+3}{2}\right)} \left(-\frac{z}{2}\right)^{n+2} {}_0F_3\left(\begin{matrix} -\frac{z^4}{64} \\ \frac{n-\nu+3}{2},\ \frac{n+\nu+3}{2},\ \frac{3}{2} \end{matrix}\right)$$
$$+ (-1)^n \frac{\Gamma(n-\nu)}{2\Gamma(\nu+1)} \left(\frac{z}{2}\right)^{2\nu-n} {}_0F_3\left(\begin{matrix} -\frac{z^4}{64} \\ \frac{\nu-n+1}{2},\ \frac{\nu-n}{2}+1,\ \nu+1 \end{matrix}\right).$$

2. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{z}{w}\right)^k I_{\pm n\mp k\mp 1/2}(w)\,K_{k-1/2}(z)$$
$$= (-1)^n \frac{w^{-n-1/2}}{\sqrt{2\pi z}}\left[(z-w)^{n+1/2}K_{n-1/2}(z-w) \pm (z+w)^{n+1/2}K_{n-1/2}(z+w)\right].$$

3. $$\sum_{k=0}^{n}(-1)^k k\, I_{k+\nu}(z)\,K_{k-\nu}(z) = -\frac{z}{4}[I_\nu(z)\,K_{\nu-1}(z) - I_{\nu+1}(z)\,K_\nu(z)] + (-1)^n\frac{z}{4}[I_{n+\nu}(z)\,K_{n-\nu+1}(z) - I_{n+\nu+1}(z)\,K_{n-\nu}(z)].$$

5.5.3. Sums containing products of $K_{\nu\pm nk}(z)$

1. $$\sum_{k=0}^{n}(2k+1)\,K_{k+1/2}(w)\,K_{k+1/2}(z) = (n+1)\sqrt{\pi}\sum_{k=0}^{n}\frac{(k+n+1)!}{(k+1)!\,(n-k)!}\left(\frac{w+z}{2wz}\right)^{k+1/2}K_{k+1/2}(w+z) \quad [68].$$

2. $$\sum_{k=0}^{n}(-1)^k\frac{2k+1}{(n-k)!\,(k+n+1)!}K_{k+1/2}(w)\,K_{k+1/2}(z) = (-1)^n\frac{\sqrt{\pi}}{n!}\left(\frac{w+z}{2wz}\right)^{n+1/2}K_{n+1/2}(w+z).$$

5.6. The Struve Functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

5.6.1. Sums containing $\mathbf{H}_{k+\nu}(z)$ and $\mathbf{L}_{k+\nu}(z)$

1. $$\sum_{k=0}^{n}\frac{(-n)_k(n)_k}{k!}\left(\frac{2}{z}\right)^k\mathbf{H}_{k-1/2}(z) = (-1)^n\sqrt{\frac{\pi}{2z}}\,J_{n+1/2}\left(\frac{z}{2}\right)\left[2(2n+1)J_{n+1/2}\left(\frac{z}{2}\right) - z\,J_{n+3/2}\left(\frac{z}{2}\right)\right].$$

2. $$\sum_{k=0}^{n}\frac{(-n)_k(n)_k}{k!}\left(\frac{2}{z}\right)^k\mathbf{L}_{k-1/2}(z) = \sqrt{\frac{\pi}{2z}}\,I_{n+1/2}\left(\frac{z}{2}\right)\left[2(2n+1)I_{n+1/2}\left(\frac{z}{2}\right) + z\,I_{n+3/2}\left(\frac{z}{2}\right)\right].$$

3. $$\sum_{k=0}^{n}\binom{n}{k}(a)_k\left(-\frac{2}{z}\right)^k\mathbf{H}_{k+\nu}(z) = \frac{2^{-\nu}z^{\nu+1}\left(\nu-a+\frac{3}{2}\right)_n}{\sqrt{\pi}\,\Gamma\left(\nu+n+\frac{3}{2}\right)}\,{}_2F_3\left(\begin{matrix}\nu-a+n+\frac{3}{2},\ 1;\ -\frac{z^2}{4}\\ \nu+n+\frac{3}{2},\ \nu-a+\frac{3}{2},\ \frac{3}{2}\end{matrix}\right).$$

4. $$\sum_{k=0}^{n}\binom{n}{k}(a)_k\left(-\frac{2}{z}\right)^k \mathbf{L}_{k+\nu}(z)$$
$$= \frac{2^{-\nu}z^{\nu+1}\left(\nu-a+\frac{3}{2}\right)_n}{\sqrt{\pi}\,\Gamma\left(\nu+n+\frac{3}{2}\right)}\,{}_2F_3\left(\begin{matrix}\nu-a+n+\frac{3}{2},\,1;\ \frac{z^2}{4}\\ \nu+n+\frac{3}{2},\,\nu-a+\frac{3}{2},\,\frac{3}{2}\end{matrix}\right).$$

5.7. The Legendre Polynomials $P_n(z)$

5.7.1. Sums containing $P_{m\pm nk}(z)$

1. $$\sum_{k=0}^{n} P_k(z) = \frac{(2n)!}{(n!)^2}\left(\frac{z-1}{2}\right)^n {}_3F_2\left(\begin{matrix}-n,\,-n,\,-n-\frac{1}{2};\ \frac{2}{1-z}\\ \frac{1}{2}-n,\,-2n-1\end{matrix}\right).$$

2. $$\sum_{k=0}^{n}(2k+1)\,P_k(z) = \frac{n+1}{z-1}\,[P_{n+1}(z)-P_n(z)].$$

3. $$\sum_{k=0}^{n}\binom{n}{k}t^k P_k(z) = (1+2tz+t^2)^{n/2}P_n\left(\frac{1+tz}{\sqrt{1+2tz+t^2}}\right).$$

4. $$\sum_{k=1}^{n}\frac{k\,\Gamma(2n-k)}{(n-k)!}(2z)^k P_k(z) = \left(\frac{1}{2}\right)_n (2z)^n \qquad [n\ge 1].$$

5. $$\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!\,(a)_k}\left(z-\sqrt{z^2-1}\right)^k P_k(z)$$
$$= \frac{1}{(a)_n}P_n^{(a-3/2,\,1-a-n)}\left(3-4z^2+4z\sqrt{z^2-1}\right).$$

6. $$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^n\left(z+\sqrt{z^2-1}\right)^k P_k(z)$$
$$= \frac{2^n\left(\frac{1}{2}\right)_n}{n!}\left(1-\frac{z}{\sqrt{z^2-1}}\right)^{-n}\sum_{k=0}^{n}2^{-k}\sigma_n^k\,\frac{(-n)_k^2}{(1/2-n)_k}\left(\frac{z}{\sqrt{z^2-1}}-1\right)^k.$$

7. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{k!}\left(\frac{2}{1-z}\right)^k P_k(z)$$
$$= \frac{(a)_n}{n!}\left(\frac{1+z}{1-z}\right)^n {}_4F_3\left(\begin{matrix}-n,\,-n,\,\frac{1-a-n}{2},\,1-\frac{a+n}{2}\\ 1-a-n,\,1-a-n,\,1;\ \frac{4z-4}{z+1}\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}(-1)^k\frac{2k+1}{(n-k)!\,(k+n+1)!}P_k(z) = \frac{2^{-n}}{(n!)^2}\,(1-z)^n.$$

9. $$\sum_{k=0}^{n}\binom{n}{k}\frac{2^{-k}}{(n-k)!\left(\frac{1}{2}-n\right)_k}\left(\frac{\sqrt{z^2-1}-z}{1-z^2+z\sqrt{z^2-1}}\right)^k P_k(z)$$
$$=\frac{2^{-n}}{\left(\frac{1}{2}\right)_n}\left(1-z^2+z\sqrt{z^2-1}\right)^{-n}{}_3F_2\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{1}{2}\\ 1,\ 1;\ 4\left(1-z^2+z\sqrt{z^2-1}\right)^2\end{matrix}\right).$$

10. $$\sum_{k=0}^{n}\frac{(a)_k}{(n-k)!\,(k+n+1)!\,(2-a)_k}P_k(z)$$
$$=\frac{(a)_n}{(n!)^2\,(2n+1)(2-a)_n}\left(\frac{z+1}{2}\right)^n{}_3F_2\left(\begin{matrix}-n,\ -n,\ \frac{3}{2}-a;\ \frac{2}{z+1}\\ \frac{1}{2}-n,\ 1-a-n\end{matrix}\right).$$

11. $$\sum_{k=0}^{n}(2k+1)\,\frac{(a)_k(b)_k}{(n-k)!\,(k+n+1)!(2-a)_k(2-b)_k}P_k(z)$$
$$=\frac{(a)_n(b)_n}{(n!)^2\,(2-a)_n(2-b)_n}\left(\frac{1+z}{2}\right)^n{}_3F_2\left(\begin{matrix}-n,\ -n,\ 2-a-b;\ \frac{2}{1+z}\\ 1-a-n,\ 1-b-n\end{matrix}\right).$$

12. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\sqrt{z^2-1}-z\right)^{2k}P_{2k}(z)$$
$$=(1-z^2+z\sqrt{z^2-1})^nP_n^{(n,\,-n-1/2)}\left(2z^2-2z\sqrt{z^2-1}-1\right).$$

13. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{\left(\frac{1}{2}\right)_k}z^{-2k}P_{2k}(z)$$
$$=\frac{(a)_n}{\left(\frac{1}{2}\right)_n}z^{-2n}(1-z^2)^n{}_4F_3\left(\begin{matrix}-n,\ -n,\ \frac{1-2a-2n}{4},\ \frac{3-2a-2n}{4}\\ \frac{1}{2}-a-n,\ 1-a-n,\ \frac{1}{2};\ \frac{4z^2}{z^2-1}\end{matrix}\right).$$

14. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{\left(\frac{1}{2}\right)_k}(1-z^2)^{-k}P_{2k}(z)$$
$$=\frac{(a)_n}{\left(\frac{1}{2}\right)_n}z^{2n}(1-z^2)^{-n}{}_4F_3\left(\begin{matrix}-n,\ \frac{1}{2}-n,\ \frac{1-a-n}{2},\ 1-\frac{a+n}{2}\\ 1-a-n,\ 1-a-n,\ 1;\ 4-4z^{-2}\end{matrix}\right).$$

15. $$\sum_{k=0}^{n}\frac{4k+1}{(n-k)!\left(n+\frac{3}{2}\right)_k}P_{2k}(z)=\frac{2n+1}{n!}z^{2n}.$$

16. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(4k+1)\,\frac{\left(\frac{1}{2}\right)_k}{\left(n+\frac{3}{2}\right)_k}P_{2k}(z)=\frac{2\left(\frac{1}{2}\right)_{n+1}}{n!}\left(1-z^2\right)^n.$$

17. $$\sum_{k=0}^{n}(-1)^k \frac{(4k+1)\,(n)_k}{(n-k)!\left(\frac{3}{2}-n\right)_k\left(\frac{3}{2}+n\right)_k} P_{2k}(z) = \frac{1-4n^2}{n!} T_{2n}(z).$$

18. $$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(a)_k}{\left(\frac{1}{2}\right)_k} z^{-2k} P_{2k}(z)$$
$$= \frac{(a)_n}{n!} z^{-2n}\left(1-z^2\right)^n {}_4F_3\left(\begin{matrix}-n,\, -n,\, \frac{1-2a-2n}{4},\, \frac{3-2a-2n}{4} \\ \frac{1}{2}-a-n,\, 1-a-n,\, \frac{1}{2};\ \frac{4z^2}{z^2-1}\end{matrix}\right).$$

19. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k}{\left(\frac{1}{2}\right)_k} \left(1-z^2\right)^{-k} P_{2k}(z)$$
$$= \frac{(a)_n}{\left(\frac{1}{2}\right)_n} z^{2n}\left(1-z^2\right)^{-n} {}_4F_3\left(\begin{matrix}-n,\, \frac{1}{2}-n,\, \frac{1-a-n}{2},\, 1-\frac{a+n}{2} \\ 1-a-n,\, 1-a-n,\, 1;\ 4-4z^{-2}\end{matrix}\right).$$

20. $$\sum_{k=0}^{n} \frac{4k+1}{(n-k)!\,(4k-1)(4k+3)\left(n+\frac{3}{2}\right)_k} P_{2k}(z)$$
$$= \frac{2n+1}{n!\,(4n-1)(4n+3)} z^{2n}\, {}_3F_2\left(\begin{matrix}-n,\, \frac{1}{2}-n,\, 1;\ z^{-2} \\ \frac{1}{4}-n,\, \frac{5}{4}-n\end{matrix}\right).$$

21. $$\sum_{k=0}^{n}(-1)^k \binom{n}{k}(4k+1) \frac{(k+n)!\left(\frac{1}{2}\right)_k^2}{k!\left(\frac{1}{2}-n\right)_k\left(\frac{3}{2}+n\right)_k} P_{2k}(z)$$
$$= n!\,(2n+1)\left[P_n(z)\right]^2.$$

22. $$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{4k+1}{(4k-1)(4k+3)} \frac{\left(\frac{1}{2}\right)_k}{\left(n+\frac{3}{2}\right)_k} P_{2k}(z)$$
$$= \frac{\left(\frac{3}{2}\right)_n}{n!\,(4n-1)(4n+3)} \left(1-z^2\right)^n {}_3F_2\left(\begin{matrix}-n,\, -n,\, 1;\ \frac{1}{1-z^2} \\ \frac{1}{4}-n,\, \frac{5}{4}-n\end{matrix}\right).$$

23. $$\sum_{k=0}^{n}(-1)^k \binom{n}{k}(4k+1) \frac{(a)_k(b)_k\left(\frac{1}{2}\right)_k}{\left(n+\frac{3}{2}\right)_k\left(\frac{3}{2}-a\right)_k\left(\frac{3}{2}-b\right)_k} P_{2k}(z)$$
$$= \frac{2(a)_n(b)_n\left(\frac{1}{2}\right)_{n+1}}{n!\left(\frac{3}{2}-a\right)_n\left(\frac{3}{2}-b\right)_n} \left(1-z^2\right)^n {}_3F_2\left(\begin{matrix}-n,\, -n,\, \frac{3}{2}-a-b;\ \frac{1}{1-z^2} \\ 1-a-n,\, 1-b-n\end{matrix}\right).$$

24. $$\sum_{k=0}^{n} \frac{4k+3}{(n-k)!\left(n+\frac{5}{2}\right)_k} P_{2k+1}(z) = \frac{2n+3}{n!} z^{2n+1}.$$

25. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{\left(\frac{3}{2}\right)_k}(1-z^2)^{-k}P_{2k+1}(z)$$
$$=\frac{(a)_n}{\left(\frac{3}{2}\right)_n}z^{2n+1}(1-z^2)^{-n}\,{}_4F_3\left(\begin{matrix}-n,\,-\frac{1}{2}-n,\,\frac{1-a-n}{2},\,1-\frac{a+n}{2}\\1-a-n,\,1-a-n,\,1;\;4-4z^{-2}\end{matrix}\right).$$

26. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{\left(\frac{3}{2}\right)_k}z^{-2k}P_{2k+1}(z)$$
$$=\frac{(a)_n}{n!}z^{1-2n}(1-z^2)^n\,{}_4F_3\left(\begin{matrix}-n,\,-n,\,\frac{3-2a-2n}{4},\,\frac{5-2a-2n}{4}\\1-a-n,\,\frac{3}{2}-a-n,\,\frac{3}{2};\;\frac{4z^2}{z^2-1}\end{matrix}\right).$$

27. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{\left(\frac{3}{2}\right)_k}(1-z^2)^{-k}P_{2k+1}(z)$$
$$=\frac{(a)_n}{\left(\frac{3}{2}\right)_n}z^{2n+1}(1-z^2)^{-n}\,{}_4F_3\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,\frac{1-a-n}{2},\,1-\frac{a+n}{2}\\1-a-n,\,1-a-n,\,1;\;4-4z^{-2}\end{matrix}\right).$$

28. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(2k+\frac{3}{2}\right)\frac{\left(\frac{3}{2}\right)_k}{\left(n+\frac{5}{2}\right)_k}P_{2k+1}(z)=\frac{\left(\frac{3}{2}\right)_{n+1}}{n!}z(1-z^2)^n.$$

29. $$\sum_{k=0}^{n}\frac{4k+3}{(n-k)!\,(4k+1)(4k+5)}\frac{1}{\left(n+\frac{5}{2}\right)_k}P_{2k+1}(z)$$
$$=\frac{2n+3}{n!\,(4n+1)(4n+5)}z^{2n+1}\,{}_3F_2\left(\begin{matrix}-n,\,-\frac{1}{2}-n,\,1;\;z^{-2}\\-\frac{1}{4}-n,\,\frac{3}{4}-n\end{matrix}\right).$$

30. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{4k+3}{(4k+1)(4k+5)}\frac{\left(\frac{3}{2}\right)_k}{\left(n+\frac{5}{2}\right)_k}P_{2k+1}(z)$$
$$=\frac{2\left(\frac{3}{2}\right)_{n+1}z}{n!\,(4n+1)(4n+5)}(1-z^2)^n\,{}_3F_2\left(\begin{matrix}-n,\,-n,\,1;\;\frac{1}{1-z^2}\\-\frac{1}{4}-n,\,\frac{3}{4}-n\end{matrix}\right).$$

31. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(4k+3)\frac{(a)_k(b)_k\left(\frac{3}{2}\right)_k}{\left(n+\frac{5}{2}\right)_k\left(\frac{5}{2}-a\right)_k\left(\frac{5}{2}-b\right)_k}P_{2k+1}(z)$$
$$=\frac{2(a)_n(b)_n\left(\frac{3}{2}\right)_{n+1}}{n!\left(\frac{5}{2}-a\right)_n\left(\frac{5}{2}-b\right)_n}z(1-z^2)^n\,{}_3F_2\left(\begin{matrix}-n,\,-n,\,\frac{5}{2}-a-b;\;\frac{1}{1-z^2}\\1-a-n,\,1-b-n\end{matrix}\right).$$

32. $$\sum_{k=0}^{[n/2]}(-1)^k\binom{m}{k}(2n-4k+1)\frac{\left(-n-\frac{1}{2}\right)_k}{\left(m-n+\frac{1}{2}\right)_k}P_{n-2k}(z)$$
$$=-2^{2m+1}\frac{(n-2m)!}{n!}\left(\frac{1}{2}\right)_m\left(-n-\frac{1}{2}\right)_{m+1}\left(1-z^2\right)^m C_{n-2m}^{m+1/2}(z)$$
$$[2m\le n].$$

33. $$\sum_{k=0}^{[n/2]}(-1)^k(2n-4k+1)\frac{\left(-n-\frac{1}{2}\right)_k}{k!}P_{n-2k}(z)=\left(\frac{3}{2}\right)_n\frac{(2z)^n}{n!}.$$

34. $$\sum_{k=0}^{[n/2]}(2n-4k+1)\frac{(a)_k\left(-\frac{1}{2}-n\right)_k}{k!\left(\frac{1}{2}-a-n\right)_k}P_{n-2k}(z)=\frac{\left(\frac{3}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n}C_n^{a+1/2}(z).$$

35. $$\sum_{k=0}^{[n/3]}\frac{2n-6k+1}{k!}\left(-\frac{2n+1}{3}\right)_k P_{n-3k}(z)=3^nP_n^{\left(\frac{1-n}{3},\,\frac{1-4n}{6}\right)}\left(\frac{4z-1}{3}\right).$$

36. $$\sum_{k=0}^{[n/3]}(4n-12k+1)\frac{(-n)_{3k}\left(-\frac{4n+1}{6}\right)_k}{k!\left(-n+\frac{1}{2}\right)_{3k}}P_{2n-6k}(z)$$
$$=2^{2n}\frac{\left(\frac{3}{2}\right)_{2n}}{(2n)!}\left(z^2-1\right)^n{}_3F_2\left(\begin{matrix}-n,\,-n,\,-\frac{4n+1}{6};\ \frac{3}{4-4z^2}\\ -n-\frac{1}{4},\,-n+\frac{1}{4}\end{matrix}\right).$$

37. $$\sum_{k=0}^{[n/3]}(4n-12k+3)\frac{(-n)_{3k}\left(-\frac{4n+3}{6}\right)_k}{k!\left(-n-\frac{1}{2}\right)_{3k}}P_{2n-6k+1}(z)$$
$$=2^{2n+1}\frac{\left(\frac{3}{2}\right)_{2n+1}}{(2n+1)!}z\left(z^2-1\right)^n{}_3F_2\left(\begin{matrix}-n,\,-n,\,-\frac{4n+3}{6};\ \frac{3}{4-4z^2}\\ -n-\frac{3}{4},\,-n-\frac{1}{4}\end{matrix}\right).$$

38. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(-m)_k}{(a)_k}\left(z+\sqrt{z^2-1}\right)^kP_{m-k}(z)$$
$$=\frac{(a+m)_n}{(a)_n}\left(z+\sqrt{z^2-1}\right)^m$$
$$\times\,{}_3F_2\left(\begin{matrix}-m,\,1-a-m,\,\frac{1}{2};\ 2\left(1-z^2+z\sqrt{z^2-1}\right)\\ 1-a-m-n,\,1\end{matrix}\right)\quad[m\ge n].$$

39. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(m+1)_k}{\left(m-n+\frac{3}{2}\right)_k}\left(z+\sqrt{z^2-1}\right)^{-k}P_{k+m}(z)$$
$$=\frac{\left(\frac{1}{2}\right)_n}{\left(-m-\frac{1}{2}\right)_n}\left(z+\sqrt{z^2-1}\right)^{-m-2n}$$
$$\times P_m^{(0,\,-m-n-1/2)}\left(4z^2+4z\sqrt{z^2-1}-3\right).$$

5.7.2. Sums containing $P_n(z)$ and special functions

1. $\displaystyle\sum_{k=0}^{n} \binom{2n}{2k} \frac{2^{2n-2k}-1}{(2z)^{2k}} B_{2n-2k} P_{2k}(z) = \frac{n}{(2z)^{2n-1}} P_{2n-1}(z) \qquad [n \geq 1].$

2. $\displaystyle\sum_{k=0}^{n} \binom{2n+1}{2k+1} (2z)^{-2k} B_{2n-2k} P_{2k+1}(z) = \frac{2n+1}{2^{2n} z^{2n-1}} z^{1-2n} P_{2n}(z).$

3. $$\sum_{k=0}^{n} \left(z-\sqrt{z^2-1}\right)^k \psi(k+1)\, P_k(z)$$
$$= \psi(n+2)\, P_n^{(1,\,-n-3/2)}\left(4z^2-3-4z\sqrt{z^2-1}\right)$$
$$+ \left(1-z^2+z\sqrt{z^2-1}\right)^{-1}\left[1-P_{n+1}^{(0,\,-n-5/2)}\left(4z^2-3-4z\sqrt{z^2-1}\right)\right].$$

5.7.3. Sums containing products of $P_{m\pm nk}(z)$

1. $$\sum_{k=0}^{n} (2k+1)\left[P_k(z)\right]^2$$
$$= \frac{(n+1)^2}{1-z^2}\left\{\left[P_n(z)\right]^2 - 2zP_n(z)P_{n+1}(z) + \left[P_{n+1}(z)\right]^2\right\}.$$

2. $\displaystyle\sum_{k=0}^{n} (-1)^k \frac{2k+1}{(n-k)!\,(k+n+1)!}\left[P_k(z)\right]^2 = \frac{\left(\frac{1}{2}\right)_n}{(n!)^3}\left(1-z^2\right)^n.$

3. $\displaystyle\sum_{k=0}^{n} (-1)^k \frac{(2k+1)\left(n+\frac{1}{2}\right)_k}{(n-k)!\,(k+n+1)!\left(\frac{3}{2}-n\right)_k}\left[P_k(z)\right]^2 = \frac{1-2n}{(n!)^2} P_{2n}(z).$

4. $$\sum_{k=0}^{n} (-1)^k (4k+1) \frac{\left(\frac{1}{2}\right)_k}{k!} P_{2k}\left(\sqrt{\frac{1-z}{2}}\right) P_{2k}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$= 2(-1)^{n+1}(n+1)\frac{\left(\frac{3}{2}\right)_n}{n!} P_{2n+2}\left(\sqrt{\frac{1-z}{2}}\right) P_{2n+2}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$+ \frac{9\left(\frac{5}{2}\right)_n^2}{8(n!)^2}\left(1-z^2\right) {}_4F_3\left(\begin{matrix}-n,\ n+\frac{5}{2},\ \frac{5}{4},\ \frac{7}{4}\\ \frac{3}{2},\ 2,\ \frac{5}{2};\ 1-z^2\end{matrix}\right).$$

5. $$\sum_{k=0}^{n}(-1)^k (4k+3)\frac{\left(\frac{3}{2}\right)_k}{k!} P_{2k+1}\left(\sqrt{\frac{1-z}{2}}\right) P_{2k+1}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$= 2(-1)^{n+1}(n+1)\frac{\left(\frac{5}{2}\right)_n}{n!} P_{2n+3}\left(\sqrt{\frac{1-z}{2}}\right) P_{2n+3}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$+ \frac{25\left(\frac{7}{2}\right)_n^2}{16(n!)^2}\left(1-z^2\right)^{3/2} {}_4F_3\left(\begin{matrix} -n,\ n+\frac{7}{2},\ \frac{7}{4},\ \frac{9}{4} \\ 2,\ \frac{5}{2},\ \frac{7}{2};\ 1-z^2 \end{matrix}\right).$$

5.7.4. Sums containing $P_m(\varphi(k,\,z))$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_m(w+kz) = 0 \qquad [m < n].$$

2. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}P_m(1+kz) = (-2z)^m\left(\frac{1}{2}\right)_m \delta_{m,n} - 1 \qquad [n \geq m].$$

3. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_{m+n}(w+kz)$$
$$= (2n)!\left(-\frac{z}{2}\right)^n \sum_{k=0}^{m}\sigma_{k+n}^{n}\frac{\left(n+\frac{1}{2}\right)_k}{(k+n)!}(2z)^k C_{m-k}^{k+n+1/2}(w).$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_{2m}\left(\sqrt{k}\,z\right) = (-1)^m z^{2m}\frac{\left(\frac{1}{2}\right)_{2m}}{\left(\frac{1}{2}\right)_m}\delta_{m,n} \qquad [n \geq m].$$

5. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1/2}P_{2m+1}\left(\sqrt{k}\,z\right) = (-1)^m z^{2m+1}\frac{\left(\frac{3}{2}\right)_{2m}}{\left(\frac{3}{2}\right)_m}\delta_{m,n}$$
$$-(-1)^m\frac{\left(\frac{3}{2}\right)_m}{m!}z \qquad [n \geq m].$$

6. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_{2m}\left(\sqrt{1+kz}\right) = (-z)^m\frac{\left(\frac{1}{2}\right)_{2m}}{\left(\frac{1}{2}\right)_m}\delta_{m,n} \qquad [n \geq m].$$

7. $$\sum_{k=1}^{n}\frac{(-1)^k}{\sqrt{1+kz}}\binom{n}{k}P_{2m+1}\left(\sqrt{1+kz}\right) = (-4z)^m\frac{m!\left(\frac{3}{2}\right)_{2m}}{(2m+1)!}\delta_{m,n} - 1 \qquad [n \geq m].$$

8. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m P_m\left(1+\frac{z}{k}\right) = (-1)^m n!\,\delta_{m,n} - \frac{\left(\frac{1}{2}\right)_m}{m!}(2z)^m \qquad [n \geq m].$$

9. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^m P_{2m}\left(\sqrt{\frac{z}{k+z}}\right) = \left(\frac{1}{2}\right)_m \delta_{m,n} - z^m \qquad [n \geq m].$$

10. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^{m+1/2} P_{2m+1}\left(\sqrt{\frac{z}{k+z}}\right) = \sqrt{z}\left(\frac{3}{2}\right)_m \delta_{m,n} - z^{m+1/2} \qquad [n \geq m].$$

11. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k-z)^m P_m\left(\frac{k+z}{k-z}\right) = (-1)^m m!\,\delta_{m,n} - z^m \qquad [n \geq m].$$

12. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m P_{2m}\left(\frac{z}{\sqrt{k}}\right) = \left(\frac{1}{2}\right)_m \delta_{m,n} - \frac{\left(\frac{1}{2}\right)_{2m}}{(2m)!}(2z)^{2m} \qquad [n \geq m].$$

13. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m+1/2} P_{2m+1}\left(\frac{z}{\sqrt{k}}\right) = z\left(\frac{3}{2}\right)_m \delta_{m,n} - \frac{\left(\frac{3}{2}\right)_{2m}}{(2m+1)!}2^{2m}z^{2m+1} \qquad [n \geq m].$$

14. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m P_{2m}\left(\sqrt{1+\frac{z}{k}}\right) = (-1)^m m!\,\delta_{m,n} - \frac{\left(\frac{1}{2}\right)_{2m}}{(2m)!}(4z)^m \qquad [n \geq m].$$

15. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{k^{m+1/2}}{\sqrt{k+z}} P_{2m+1}\left(\sqrt{1+\frac{z}{k}}\right) = (-1)^m m!\,\delta_{m,n} - \frac{\left(\frac{3}{2}\right)_{2m}}{(2m+1)!}(4z)^m \qquad [n \geq m].$$

16. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^m P_{2m}\left(\sqrt{\frac{k}{k+z}}\right) = (-1)^m m!\,\delta_{m,n} - \frac{\left(\frac{1}{2}\right)_m}{m!}(-z)^m \qquad [n \geq m].$$

17. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1/2}(k+z)^{m+1/2} P_{2m+1}\left(\sqrt{\frac{k}{k+z}}\right) = (-1)^m m!\,\delta_{m,n} - \frac{\left(\frac{3}{2}\right)_m}{m!}(-z)^m \qquad [n \geq m].$$

18. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m/2}P_m\left(\frac{k+z}{2\sqrt{kz}}\right)=(-1)^m\left(\frac{1}{2}\right)_m z^{-m/2}\delta_{m,n}-\frac{\left(\frac{1}{2}\right)_m}{m!}z^{m/2}\quad[n\geq m].$$

19. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m/2}(k+z)^{m/2}P_m\left(\frac{2k+z}{2\sqrt{k(k+z)}}\right)=-z^m\frac{\left(\frac{1}{2}\right)_m}{m!}+(-1)^n m!\,n!\,z^{m-n}\frac{\left(\frac{1}{2}\right)_{m-n}}{(m-n)!^2}\sum_{k=0}^{m-n}\sigma_{k+n}^{n}\frac{(n-m)_k^2}{(k+n)!}\frac{(-z)^{-k}}{\left(n-m+\frac{1}{2}\right)_k}.$$

5.7.5. Sums containing $P_k(\varphi(k,z))$

1. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}a^k(ka+b)^{n-k-1}(k^2+kz)^{k/2}P_k\left(\frac{2k+z}{2\sqrt{k^2+kz}}\right)=-b^{n-1}+\frac{\left(b^2-abz\right)^{n/2}}{na+b}P_n\left(\frac{2b-az}{2\sqrt{b^2-abz}}\right).$$

2. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(ka+1)^{n-k/2-1}(ka+z+1)^{k/2}\times P_k\left(\frac{2ka+z+2}{2\sqrt{(ka+1)(ka+z+1)}}\right)=\frac{\left(\frac{1}{2}\right)_n(-z)^n}{n!\,(na+1)}.$$

5.7.6. Sums containing products of $P_m(\varphi(k,z))$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}[P_m(kw+z)]^2=0\quad[2m<n].$$

2. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m P_{2m}\left(\sqrt{1+\frac{z}{k}}+\sqrt{\frac{z}{k}}\right)P_{2m}\left(\sqrt{1+\frac{z}{k}}-\sqrt{\frac{z}{k}}\right)=m!\,(-1)^m\,\delta_{m,n}-\frac{\left(\frac{1}{2}\right)_{2m}}{(m!)^2}(-4z)^m\quad[n\geq m].$$

3. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m P_{2m+1}\left(\sqrt{1+\frac{z}{k}}+\sqrt{\frac{z}{k}}\right)P_{2m+1}\left(\sqrt{1+\frac{z}{k}}-\sqrt{\frac{z}{k}}\right)=m!\,(-1)^m\,\delta_{m,n}-\frac{\left(\frac{3}{2}\right)_{2m}}{(m!)^2}(-4z)^m\quad[n\geq m].$$

5.8. The Chebyshev Polynomials $T_n(z)$ and $U_n(z)$

5.8.1. Sums containing $T_{m+nk}(z)$

1. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} t^k T_k(z) = \left(1+2tz+t^2\right)^{n/2} T_n\left(\frac{1+tz}{\sqrt{1+2tz+t^2}}\right).$

2. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k}{\left(\frac{1}{2}\right)_k} \left(\frac{2}{1-z}\right)^k T_k(z) = \frac{(a)_n}{\left(\frac{1}{2}\right)_n} \left(\frac{1+z}{1-z}\right)^n {}_4F_3\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \frac{1-2a-2n}{4},\ \frac{3-2a-2n}{4} \\ \frac{1}{2}-a-n,\ 1-a-n,\ \frac{1}{2};\ \frac{4z-4}{z+1} \end{matrix}\right).$$

3. $\displaystyle\sum_{k=0}^{n} T_{2k}(z) = \frac{1}{2} U_{2n}(z) + \frac{1}{2}.$

4. $\displaystyle\sum_{k=0}^{n} (n^2-k^2) T_{2k}(z) = \frac{1}{4(z^2-1)} \left[2n^2(z^2-1) + 2nT_{2n}(z) - zU_{2n-1}(z)\right]$ $[n \geq 1].$

5. $\displaystyle\sum_{k=0}^{n} \binom{2n}{n-k} T_{2k}(z) = 2^{2n-1} z^{2n} + \frac{1}{2}\binom{2n}{n}.$

6. $\displaystyle\sum_{k=0}^{n} \frac{1}{(n-k)!\,(n+k)!} T_{2k}(z) = \frac{(2z)^{2n}}{2(2n)!} + \frac{1}{2(n!)^2}.$

7. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k}{\left(\frac{1}{2}\right)_k} z^{-2k} T_{2k}(z) = \frac{(a)_n}{\left(\frac{1}{2}\right)_n} z^{-2n} (1-z^2)^n {}_4F_3\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \frac{1-2a-2n}{4},\ \frac{3-2a-2n}{4} \\ \frac{1}{2}-a-n,\ 1-a-n,\ \frac{1}{2};\ \frac{4z^2}{z^2-1} \end{matrix}\right).$$

8. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k}{\left(\frac{1}{2}\right)_k} (1-z^2)^{-k} T_{2k}(z) = \frac{(a)_n}{\left(\frac{1}{2}\right)_n} z^{2n} (1-z^2)^{-n} {}_4F_3\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \frac{1-2a-2n}{4},\ \frac{3-2a-2n}{4} \\ \frac{1}{2}-a-n,\ 1-a-n,\ \frac{1}{2};\ 4-4z^{-2} \end{matrix}\right).$$

9. $\displaystyle\sum_{k=0}^{n} T_{2k+1}(z) = \frac{1}{2} U_{2n-1}(z)$ $[n \geq 1].$

10. $\displaystyle\sum_{k=0}^{n} \binom{2n+1}{n-k} T_{2k+1}(z) = 2^{2n} z^{2n+1}.$

11. $$\sum_{k=0}^{n} \frac{1}{k+n+1}\binom{2n}{n-k}T_{2k+1}(z) = \frac{2^{2n}z^{2n+1}}{2n+1}.$$

12. $$\sum_{k=0}^{n} (n-k)(n+k+1)T_{2k+1}(z) = \frac{1}{4(z^2-1)}\left[(2n+1)\,T_{2n+1}(z) - zU_{2n}(z)\right].$$

13. $$\sum_{k=0}^{n} \frac{1}{(n-k)!\,(k+n+1)!}T_{2k+1}(z) = \frac{2^{2n}z^{2n+1}}{(2n+1)!}.$$

14. $$\sum_{k=0}^{n}(-1)^k \frac{2k+1}{(n-k)!\,(k+n+1)!}T_{2k+1}(z) = \frac{2^{2n}z}{(2n)!}\left(1-z^2\right)^n.$$

15. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{\left(\frac{3}{2}\right)_k}\left(1-z^2\right)^{-k}T_{2k+1}(z) = \frac{(a)_n}{\left(\frac{3}{2}\right)_n} z^{2n+1}\left(1-z^2\right)^{-n} {}_4F_3\left(\begin{matrix}-n,\, -n-\frac{1}{2},\, \frac{1-2a-2n}{4},\, \frac{3-2a-2n}{4}\\ \frac{1}{2}-a-n,\, 1-a-n,\, \frac{1}{2};\; 4-4z^{-2}\end{matrix}\right).$$

16. $$\sum_{k=0}^{n}(-1)^k \frac{(a)_k}{(n-k)!\,(2k+1)!}\left(\frac{2}{z}\right)^{2k}T_{2k+1}(z) = \frac{(a)_n}{(2n)!}z^{1-2n}\left(\frac{1-z^2}{4}\right)^n {}_4F_3\left(\begin{matrix}-n,\, \frac{1}{2}-n,\, \frac{3-2a-2n}{4},\, \frac{5-2a-2n}{4}\\ 1-a-n,\, \frac{3}{2}-a-n,\, \frac{3}{2};\; \frac{4z^2}{z^2-1}\end{matrix}\right).$$

17. $$\sum_{k=0}^{n}(-1)^k \frac{(1+a+n)_k}{(n-k)!\,(k+n+1)!\,(1-a-n)_k}T_{2k+1}(z) = \frac{1}{2(a+n)(a)_n^2}C_{2n+1}^{a}(z).$$

18. $$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\frac{(a)_k}{(1-a-n)_k}T_{n-2k}(z) = \frac{n!}{2(a)_n}C_n^a(z) + \frac{(-1)^{[n/2]}}{2}\delta_{2[n/2],n}\binom{n}{\left[\frac{n}{2}\right]}\frac{(a)_{[n/2]}}{(1-a-n)_{[n/2]}}.$$

19. $$\sum_{k=0}^{[n/3]}\frac{\left(-\frac{2n}{3}\right)_k}{k!}T_{n-3k}(z) = \frac{3^n}{2}P_n^{(-n/3,\,-2n/3)}\left(\frac{4z-1}{3}\right) + \frac{1}{2}\delta_{3[n/3],n}\frac{\left(-\frac{2n}{3}\right)_{[n/3]}}{\left[\frac{n}{3}\right]!}.$$

20. $$\sum_{k=0}^{[n/2]} (-1)^k \frac{\left(\frac{1}{2}-n\right)_k}{k!\,(2n-2k+1)} T_{2n-4k+1}(z) = \frac{(-2)^n z}{2n+1} P_n^{(1/2,\,-n-1/2)}(1-4z^2).$$

21. $$\sum_{k=0}^{n} \binom{m}{k} T_{m+n-2k}(z) = (2z)^m T_n(z).$$

22. $$\sum_{k=0}^{n} \binom{2n+1}{2k+1} (2z)^{-2k}\, B_{2n-2k} T_{2k+1}(z) = \frac{2n+1}{2^{2n} z^{2n-1}} T_{2n}(z).$$

23. $$\sum_{k=0}^{n} \binom{2n}{2k} \frac{2^{2n-2k}-1}{(2z)^{2k}} B_{2n-2k} T_{2k}(z) = \frac{n}{(2z)^{2n-1}} z^{1-2n} T_{2n-1}(z) \qquad [n \geq 1].$$

5.8.2. Sums containing products of $T_{m+nk}(z)$

1. $$\sum_{k=0}^{n} (-1)^k \frac{\left(n-\frac{1}{2}\right)_k}{(n-k)!(k+n)!\left(\frac{3}{2}-n\right)_k} [T_k(z)]^2 = \frac{1}{2(n!)^2} + \frac{1}{4\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n} [P_{2n}(z) - zP_{2n-1}(z)] \quad [n \geq 1].$$

2. $$\sum_{k=0}^{n} (-1)^k T_{2k+1}\left(\sqrt{\frac{1-z}{2}}\right) T_{2k+1}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$= (-1)^{n+1} \frac{(n+1)^2}{2n+3} T_{2n+3}\left(\sqrt{\frac{1-z}{2}}\right) T_{2n+3}\left(\sqrt{\frac{1+z}{2}}\right) + (-1)^n \frac{(n+1)(n+2)}{2(2n+3)} T_{2n+3}\left(\sqrt{1-z^2}\right) + \frac{1}{4\sqrt{1-z^2}} \left[1 - {}_2F_1\left(\begin{matrix}-n-2,\, n+1\\ \frac{1}{2};\, 1-z^2\end{matrix}\right)\right].$$

5.8.3. Sums containing $T_n(\varphi(k, z))$

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} T_m(w+kz) = 0 \qquad [m < n].$$

2. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} T_{m+n}(w+kz) = 2^{n-1} n!\,(m+n)\,(-z)^n \sum_{k=0}^{m} \sigma_{k+n}^n \frac{(2z)^k}{k+n} C_{m-k}^{k+n}(w).$$

3. $$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^m T_m\left(1+\frac{z}{k}\right) = (-1)^m m!\,\delta_{m,n} - 2^{m-1} z^m \qquad [n \geq m \geq 1].$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}T_{2m}\big(\sqrt{k}\,z\big) = (-1)^m m!2^{2m-1}z^{2m}\delta_{m,n} \qquad [n \ge m \ge 1].$$

5. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1/2}T_{2m+1}\big(\sqrt{k}\,z\big) = (-1)^m 2^{2m}z^{2m+1}m!\,\delta_{m,n} - (-1)^m(2m+1)z \qquad [n \ge m].$$

6. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m T_{2m}\left(\frac{z}{\sqrt{k}}\right) = m!\,\delta_{m,n} - 2^{2m-1}z^{2m} \qquad [n \ge m \ge 1].$$

7. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m+1/2}T_{2m+1}\left(\frac{z}{\sqrt{k}}\right) = m!\,(2m+1)z\,\delta_{m,n} - 2^{2m}z^{2m+1} \qquad [n \ge m].$$

8. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m T_{2m}\left(\sqrt{1+\frac{z}{k}}\right) = (-1)^m m!\,\delta_{m,n} - 2^{2m-1}z^m \qquad [n \ge m \ge 1].$$

9. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{k^{m+1/2}}{\sqrt{k+z}}T_{2m+1}\left(\sqrt{1+\frac{z}{k}}\right) = (-1)^m m!\,\delta_{m,n} - (4z)^m \qquad [n \ge m].$$

10. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^m T_{2m}\left(\sqrt{\frac{z}{k+z}}\right) = m!\,\delta_{m,n} - z^m \qquad [n \ge m].$$

11. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^{m+1/2}T_{2m+1}\left(\sqrt{\frac{z}{k+z}}\right) = (-1)^{m+1}m!\,(2m+1)\sqrt{z}\,\delta_{m,n} - z^{m+1/2} \qquad [n \ge m].$$

12. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^m T_{2m}\left(\sqrt{\frac{k}{k+z}}\right) = (-1)^m m!\,\delta_{m,n} - (-z)^m \qquad [n \ge m].$$

13. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1/2}(k+z)^{m+1/2}T_{2m+1}\left(\sqrt{\frac{k}{k+z}}\right) = (-1)^m m!\,\delta_{m,n} - (2m+1)(-z)^m \qquad [n \ge m].$$

5.8.4. Sums containing $U_{m+nk}(z)$

1. $$\sum_{k=0}^{n}\frac{t^k}{k+1}\binom{n}{k}U_k(z) = \frac{\left(1+2tz+t^2\right)^{n/2}}{n+1}U_n\left(\frac{1+tz}{\sqrt{1+2tz+t^2}}\right).$$

2. $$\sum_{k=0}^{n}(-1)^k(k+1)\binom{2n+2}{n-k}U_k(z) = 2^n(n+1)(1-z)^n.$$

3. $$\sum_{k=0}^{n}(-1)^k \frac{k+1}{(n-k)!\,(k+n+2)!}U_k(z) = \frac{2^{n-1}}{(2n+1)!}\,(1-z)^{\,n}.$$

4. $$\sum_{k=0}^{n} \frac{2^{3k}(a)_k}{(n-k)!\,(2k+2)!}\,(1-z)^{-k}\,U_k(z)$$
$$= \frac{2^{2n-1}(a)_n}{(2n+1)!}\left(\frac{1+z}{1-z}\right)^n {}_4F_3\left(\begin{matrix}-n,\ -\frac{1}{2}-n,\ \frac{3-2a-2n}{4},\ \frac{5-2a-2n}{4}\\ 1-a-n,\ \frac{3}{2}-a-n,\ \frac{3}{2};\ \frac{4z-4}{z+1}\end{matrix}\right).$$

5. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(2k+2)!}\left(\frac{8}{1-z}\right)^k U_k(z)$$
$$= \frac{2^{2n-1}(a)_n}{(2n+1)!}\left(\frac{1+z}{1-z}\right)^n {}_4F_3\left(\begin{matrix}-n,\ -n-\frac{1}{2},\ \frac{3-2a-2n}{4},\ \frac{5-2a-2n}{4}\\ 1-a-n,\ \frac{3}{2}-a-n,\ \frac{3}{2};\ \frac{4z-4}{z+1}\end{matrix}\right).$$

6. $$\sum_{k=0}^{n} \frac{k+1}{(n-k)!\,(k+n+2)!\,(2k+1)(2k+3)}U_k(z)$$
$$= -\frac{2^{-1/2}}{\left(\frac{3}{2}\right)_n^2 (2n+3)\sqrt{1-z}}\,C_{2n+1}^{-n-1/2}\left(\sqrt{\frac{1-z}{2}}\right).$$

7. $$\sum_{k=0}^{n}(k+1)\frac{(a)_k(b)_k}{(n-k)!\,(k+n+2)!\,(3-a)_k(3-b)_k}U_k(z)$$
$$= \frac{2^{n-1}(a)_n(b)_n}{(2n+1)!\,(3-a)_n(3-b)_n}(1+z)^n\ {}_3F_2\left(\begin{matrix}-n,\ -n-\frac{1}{2},\ 3-a-b\\ 1-a-n,\ 1-b-n;\ \frac{2}{1+z}\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}(-1)^k U_{2k}(z) = U_n(1-2z^2).$$

9. $$\sum_{k=0}^{n}(-1)^k(2k+1)U_{2k}(z)$$
$$= -\frac{1}{2z^2} + (-1)^n(n+1)U_{2n+2}(z) - \frac{2(n+2)!}{\left(\frac{3}{2}\right)_n}\,z^{-2}P_{n+2}^{(-3/2,\,-1/2)}(1-2z^2).$$

10. $$\sum_{k=0}^{n} \frac{2k+1}{(n-k)!\,(k+n+1)!}U_{2k}(z) = \frac{(2z)^{2n}}{(2n)!}.$$

11. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(2k+1)!} \left(\frac{4}{1-z^2}\right)^k U_{2k}(z)$$
$$= \frac{(a)_n}{(2n)!}(2z)^{2n}(1-z^2)^{-n}\ {}_4F_3\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \frac{3-2a-2n}{4},\ \frac{5-2a-2n}{4} \\ 1-a-n,\ \frac{3}{2}-a-n,\ \frac{3}{2};\ 4-4z^{-2} \end{matrix}\right).$$

12. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k}{\left(\frac{3}{2}\right)_k} z^{-2k} U_{2k}(z)$$
$$= \frac{(a)_n}{\left(\frac{3}{2}\right)_n} z^{-2n}\,(1-z^2)^n\ {}_4F_3\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ \frac{1-2a-2n}{4},\ \frac{3-2a-2n}{4} \\ \frac{1}{2}-a-n,\ 1-a-n,\ \frac{1}{2};\ \frac{4z^2}{z^2-1} \end{matrix}\right).$$

13. $$\sum_{k=0}^{n} \frac{k+1}{(n-k)!\,(k+n+2)!} U_{2k+1}(z) = \frac{2^{2n}z^{2n+1}}{(2n+1)!}.$$

14. $$\sum_{k=0}^{n} (-1)^k \frac{k+1}{(n-k)!\,(k+n+2)!} U_{2k+1}(z) = \frac{2^{2n}z}{(2n+1)!}(1-z^2)^n.$$

15. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(2k+2)!} \left(\frac{4}{1-z^2}\right)^k U_{2k+1}(z)$$
$$= \frac{(a)_n}{(2n+1)!} z^{2n+1}\left(\frac{4}{1-z^2}\right)^n\ {}_4F_3\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ \frac{3-2a-2n}{2},\ \frac{5-2a-2n}{2} \\ 1-a-n,\ \frac{3}{2}-a-n,\ \frac{3}{2};\ 4-4z^{-2} \end{matrix}\right).$$

16. $$\sum_{k=0}^{n} (-1)^k \frac{(a)_k}{(n-k)!\,(2k+2)!} \left(\frac{2}{z}\right)^{2k} U_{2k+1}(z)$$
$$= \frac{2^{2n}(a)_n}{(2n+1)!} z^{1-2n}(1-z^2)^n\ {}_4F_3\left(\begin{matrix} -n,\ -\frac{1}{2}-n,\ \frac{3-2a-2n}{4},\ \frac{5-2a-2n}{4} \\ 1-a-n,\ \frac{3}{2}-a-n,\ \frac{3}{2};\ \frac{4z^2}{z^2-1} \end{matrix}\right).$$

17. $$\sum_{k=0}^{n} (2k+1) \frac{(a)_k(b)_k}{(n-k)!\,(k+n+1)!\,(2-a)_k(2-b)_k} U_{2k}(z)$$
$$= \frac{(a)_n(b)_n}{(2n)!\,(2-a)_n(2-b)_n}(2z)^{2n}\ {}_3F_2\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ 2-a-b;\ z^{-2} \\ 1-a-n,\ 1-b-n \end{matrix}\right).$$

18. $$\sum_{k=0}^{[n/2]} \binom{n}{k} \frac{n-2k+1}{n-k+1} U_{n-2k}(z) = (2z)^n.$$

19. $$\sum_{k=0}^{[n/2]} (-1)^k (n-2k+1) \frac{(a-1)_k}{k!\,(n-k+1)!\,(1-a-n)_k} U_{n-2k}(z) = \frac{1}{(a)_n} C_n^a(z).$$

20. $$\sum_{k=0}^{[n/2]} \binom{n+1}{k} \frac{1}{(m-k)!\,(m-n)_k} U_{n-2k}(z)$$
$$= -2^{2m} \frac{(n-2m)!}{(n+1)!} (-n-1)_{m+1} \left(1-z^2\right)^m C_{n-2m}^{m+1}(z) \quad [2m \le n].$$

21. $$\sum_{k=0}^{[n/2]} (-1)^k (2n-4k+1) \frac{\left(-n-\frac{1}{2}\right)_k}{k!} U_{2n-4k}(z)$$
$$= (-2)^n (2n+1) P_n^{(-1/2,\,1/2-n)}\left(1-4z^2\right).$$

22. $$\sum_{k=0}^{[n/3]} \frac{n-3k+1}{k!} \left(-\frac{2n+2}{3}\right)_k U_{n-3k}(z) = 3^n P_n^{((2-n)/3,\,(1-n)/3)}\left(\frac{4z-1}{3}\right).$$

23. $$\sum_{k=0}^{n} U_{m+n-2k}(z) = U_m(z) U_n(z).$$

24. $$\sum_{k=0}^{n} \binom{2n+2}{2k+2} (2z)^{-2k} B_{2n-2k} U_{2k+1}(z) = (n+1)(2z)^{1-2n} U_{2n}(z).$$

25. $$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \frac{2^{2n-2k}-1}{(2z)^{2k}} B_{2n-2k} U_{2k}(z) = \frac{2n+1}{2^{2n} z^{2n-1}} U_{2n-1}(z) \qquad [n \ge 1].$$

5.8.5. Sums containing products of $U_n(z)$

1. $$\sum_{k=0}^{n} (k+1) [U_k(z)]^2$$
$$= \frac{1}{4\left(1-z^2\right)} \left\{(n+2)^2 [U_n(z)]^2 - 2(n+1)(n+2) z U_n(z) U_{n+1}(z) + (n+1)^2 [U_{n+1}(z)]^2\right\}.$$

2. $$\sum_{k=0}^{n} (-1)^k \frac{\left(\frac{3}{2}+n\right)_k}{(n-k)!\,(k+n+2)!\left(\frac{3}{2}-n\right)_k} [U_k(z)]^2$$
$$= \frac{2^{2n} n!\,\left(1-z^2\right)^{-1}}{(2n+2)!\left(-\frac{1}{2}\right)_n} [P_{2n}(z) - z P_{2n+1}(z)].$$

5.8.6. Sums containing $U_n(\varphi(k, z))$

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} U_m(w+kz) = 0 \qquad [m < n].$$

2. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} U_{m+n}(w+kz) = n!\,(-2z)^n \sum_{k=0}^{m} \sigma_{k+n}^{n} (2z)^k C_{m-k}^{k+n+1}(w).$$

3. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}U_m(1+kz) = n!\,(-2z)^m\,\delta_{m,n} \qquad [n \geq m].$$

4. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m U_m\left(1+\frac{z}{k}\right) = (-1)^m(m+1)!\,\delta_{m,n} - (2z)^m \qquad [n \geq m].$$

5. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k-z)^m U_m\left(\frac{k+z}{k-z}\right) = (-1)^m(m+1)!\,n!\,\delta_{m,n} - (m+1)\,z^m \qquad [n \geq m].$$

6. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}U_{2m}\left(\sqrt{k}\,z\right) = (-1)^m m!\,(2z)^{2m}\,\delta_{m,n} \qquad [n \geq m].$$

7. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1/2}U_{2m+1}\left(\sqrt{k}\,z\right) = (-1)^m m!\,(2z)^{2m+1}\,\delta_{m,n} - 2(-1)^m(m+1)z \qquad [n \geq m].$$

8. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m U_{2m}\left(\frac{z}{\sqrt{k}}\right) = m!\,\delta_{m,n} - (2z)^{2m} \qquad [n \geq m].$$

9. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m+1/2}U_{2m+1}\left(\frac{z}{\sqrt{k}}\right) = 2(m+1)!z\,\delta_{m,n} - (2z)^{2m+1} \qquad [n \geq m].$$

10. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m U_{2m}\left(\sqrt{1+\frac{z}{k}}\right) = (-1)^m m!\,(2m+1)\delta_{m,n} - (4z)^m \qquad [n \geq m].$$

11. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{k^{m+1/2}}{\sqrt{k+z}}U_{2m+1}\left(\sqrt{1+\frac{z}{k}}\right) = 2(-1)^m(m+1)!\,\delta_{m,n} - 2^{2m+1}z^m \qquad [n \geq m].$$

12. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^m U_{2m}\left(\sqrt{\frac{z}{k+z}}\right) = (-1)^{m+1}m!\,\delta_{m,n} - (2m+1)z^m \qquad [n \geq m].$$

13. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^{m+1/2}U_{2m+1}\left(\sqrt{\frac{z}{k+z}}\right) = 2(-1)^m(m+1)!\sqrt{z}\,\delta_{m,n} - 2(-1)^m(m+1)z^{m+1/2} \qquad [n \geq m].$$

14. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^m U_{2m}\left(\sqrt{\frac{k}{k+z}}\right) = (-1)^m m!\,(2m+1)\delta_{m,n} - (-z)^m \quad [n \geq m].$$

15. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1/2}(k+z)^{m+1/2}U_{2m+1}\left(\sqrt{\frac{k}{k+z}}\right) = 2(-1)^m(m+1)!\,\delta_{m,n} - 2(m+1)(-z)^m \quad [n \geq m].$$

16. $$\sum_{k=0}^{2n}\frac{(-1)^k}{k+1}\binom{2n}{k}(ka+1)^{2n-k-1}\left[(ka+1)^2+z\right]^{k/2}U_k\left(\frac{ka+1}{\sqrt{(ka+1)^2+z}}\right) = \frac{(-z)^n}{(2n+1)(2na+1)}.$$

5.9. The Hermite Polynomials $H_n(z)$

5.9.1. Sums containing $H_{m\pm nk}(z)$

1. $$\sum_{k=0}^{n}\frac{t^k}{(n-k)!\,(2k)!}H_{2k}(z) = \frac{(t-1)^n}{(2n)!}(t-1)^n\,H_{2n}\left(z\sqrt{\frac{t}{t-1}}\right).$$

2. $$\sum_{k=0}^{n-1}\frac{(n-k)^n}{(n-k)!\,(2k)!}H_{2k}(z) = (2z)^{2n}\sum_{k=0}^{n}\frac{\sigma_n^k}{(2n-2k)!}(2z)^{-2k}.$$

3. $$\sum_{k=0}^{n}\binom{n}{k}\frac{2^{-2k}}{(a)_k}H_{2k}(z) = (-1)^n\,\frac{n!}{(a)_n}L_n^{1/2-a-n}\left(z^2\right).$$

4. $$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(2k)!}z^{-2k}H_{2k}(z) = \frac{\left(\frac{1}{2}\right)_n}{n!}z^{-2n}\,{}_2F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2};\ -z^4\\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}\end{matrix}\right).$$

5. $$\sum_{k=0}^{n}\frac{(a)_k}{(n-k)!\,(2k)!}\left(-z^2\right)^{-k}H_{2k}(z) = \frac{(a)_n}{n!}z^{-2n}\,{}_3F_3\left(\begin{matrix}-n,\ \frac{1-2a-2n}{4},\ \frac{3-2a-2n}{4}\\ 1-a-n,\ \frac{1}{2}-a-n,\ \frac{1}{2};\ 4z^2\end{matrix}\right).$$

6. $$\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{k}^{-1}\frac{1}{(n-k)!\,(2k)!}H_{2k}(z) = \frac{n!}{(2n)!}\,{}_2F_2\left(\begin{matrix}-n,\ -n\\ \frac{1}{2},\ 1;\ z^2\end{matrix}\right).$$

7. $$\sum_{k=0}^{n}\frac{(a)_k}{(n-k)!\,(2k)!\,(b)_k}H_{2k}(z) = \frac{(b-a)_n}{n!\,(b)_n}\,{}_2F_2\left(\begin{matrix}-n,\ a;\ z^2\\ a-b-n+1,\ \frac{1}{2}\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}(-1)^k \frac{\sigma_m^{n-k+1}}{(2k)!} H_{2k}(z) = (-1)^n \frac{(2z)^{2n}}{(2n)!}\, {}_{m+1}F_{m-1}\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ 2,\ \ldots,\ 2 \\ 1,\ \ldots,\ 1;\ -z^{-2} \end{matrix}\right).$$

9. $$\sum_{k=0}^{n} \frac{t^k}{(n-k)!\,(2k+1)!} H_{2k+1}(z) = \frac{t^{-1/2}}{(2n+1)!}(t-1)^{n+1/2}\, H_{2n+1}\left(z\sqrt{\frac{t}{t-1}}\right).$$

10. $$\sum_{k=0}^{n} \frac{(n-k)^{n+1}}{(n-k)!\,(2k+1)!} H_{2k+1}(z) = (2z)^{2n+1} \sum_{k=0}^{n} \frac{\sigma_{n+1}^{k}}{(2n-2k+1)!}(2z)^{-2k}.$$

11. $$\sum_{k=0}^{n}(-1)^k \frac{(a)_k}{(n-k)!\,(2k+1)!} z^{-2k} H_{2k+1}(z) = 2\frac{(a)_n}{n!} z^{1-2n}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{3-2a-2n}{4},\ \frac{5-2a-2n}{4} \\ 1-a-n,\ \frac{3}{2}-a-n,\ \frac{3}{2};\ 4z^2 \end{matrix}\right).$$

12. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(2k+1)!}(-z^2)^{-k} H_{2k+1}(z) = 2\frac{(a)_n}{n!} z^{1-2n}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{3-2a-2n}{4},\ \frac{5-2a-2n}{4} \\ 1-a-n,\ \frac{3}{2}-a-n,\ \frac{3}{2};\ 4z^2 \end{matrix}\right).$$

13. $$\sum_{k=0}^{n} \binom{n}{k}\binom{2n}{k}^{-1} \frac{1}{(n-k)!\,(2k+1)!} H_{2k+1}(z) = 2z\frac{n!}{(2n)!}\, {}_2F_2\left(\begin{matrix} -n,\ -n \\ 1,\ \frac{3}{2};\ z^2 \end{matrix}\right).$$

14. $$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{-2k-1}}{(a)_k} H_{2k+1}(z) = (-1)^n \frac{n!\,z}{(a)_n} L_n^{3/2-a-n}(z^2).$$

15. $$\sum_{k=0}^{n} \frac{\left(-n-\frac{1}{2}\right)_k}{(n-k)!\,(2k+1)!} z^{-2k} H_{2k+1}(z) = 2\frac{\left(\frac{3}{2}\right)_n}{n!} z^{1-2n}\, {}_2F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2};\ -z^4 \\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2} \end{matrix}\right).$$

16. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(2k+1)!\,(b)_k} H_{2k+1}(z) = 2z\frac{(b-a)_n}{n!\,(b)_n}\, {}_2F_2\left(\begin{matrix} -n,\ a;\ z^2 \\ a-b-n+1,\ \frac{3}{2} \end{matrix}\right).$$

17. $$\sum_{k=0}^{n}(-1)^k \frac{\sigma_m^{n-k+1}}{(2k+1)!} H_{2k+1}(z) = (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!}\, {}_{m+1}F_{m-1}\left(\begin{matrix} -n,\ -\frac{1}{2}-n,\ 2,\ \ldots,\ 2 \\ 1,\ \ldots,\ 1;\ -z^{-2} \end{matrix}\right).$$

18. $$\sum_{k=0}^{n}\binom{n}{k}\frac{2^{-2k}}{(k+m)!} H_{2k+2m}(z) = (-1)^{m+n}\, 2^{2m} L_{m+n}^{-n-1/2}(z^2).$$

19. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a+m)_k}{(2k+2m)!} H_{2k+2m}(z) = (-1)^m \frac{(1-a)_n}{(m+n)!}\, {}_2F_2\left(\begin{matrix} -m-n,\ a \\ a-n,\ \frac{1}{2};\ z^2 \end{matrix}\right).$$

20. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a+m)_k}{(2k+2m+1)!} H_{2k+2m+1}(z) = 2(-1)^m \frac{(1-a)_n}{(m+n)!}\, z\, {}_2F_2\left(\begin{matrix} -m-n,\ a \\ a-n,\ \frac{3}{2};\ z^2 \end{matrix}\right).$$

21. $$\sum_{k=0}^{n}\binom{n}{k}\frac{2^{-2k}}{(k+m)!} H_{2k+2m+1}(z) = (-1)^{m+n}\, 2^{2m+1} z L_{m+n}^{1/2-n}(z^2).$$

22. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2k)!}{(4k)!} H_{4k}(z) = n!\,\frac{(2z^2)^n}{\left(\frac{1}{2}\right)_{2n}} L_n^{n-1/2}\left(\frac{z^2}{2}\right).$$

23. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2k)!}{(4k+1)!} H_{4k+1}(z) = \frac{2^{n+1} n!}{\left(\frac{3}{2}\right)_{2n}} z^{2n+1} L_n^{n+1/2}\left(\frac{z^2}{2}\right).$$

24. $$\sum_{k=0}^{[n/2]}\frac{2^{-2k}}{k!\,(2n-4k)!\,(a)_k} H_{2n-4k}(z) = \frac{(2z)^{2n}}{(2n)!}\, {}_3F_1\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ a-\frac{1}{2} \\ 2a-1;\ -2z^{-2} \end{matrix}\right).$$

25. $$\sum_{k=0}^{[n/2]}\frac{2^{-2k}}{k!\,(2n-4k+1)!\,(a)_k} H_{2n-4k+1}(z) = \frac{(2z)^{2n+1}}{(2n+1)!}\, {}_3F_1\left(\begin{matrix} -n,\ -\frac{1}{2}-n,\ a-\frac{1}{2} \\ 2a-1;\ -2z^{-2} \end{matrix}\right).$$

26. $$\sum_{k=0}^{n}\binom{n}{k}\frac{2^{2k}}{k+1}\left(\frac{1}{2}-n\right)_k H_{2n-2k}(z) P_k^{(\rho-k,1)}(3) = (2z)^{2n}\, {}_3F_1\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \rho+2 \\ 2;\ -2z^{-2} \end{matrix}\right).$$

27. $$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{2k}}{k+1} \left(-n-\frac{1}{2}\right)_k H_{2n-2k+1}(z) P_k^{(\rho-k,1)}(3) = (2z)^{2n+1} {}_3F_1\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ \rho+2 \\ 2;\ -2z^{-2} \end{matrix}\right).$$

28. $$\sum_{k=0}^{n} \binom{m}{k} \frac{2^k}{(n-k)!} H_{m+n-2k}(z) = \frac{1}{n!} H_m(z)\, H_n(z) \qquad [m \geq n;\ [34],\ (44)].$$

5.9.2. Sums containing $H_{m\pm nk}(z)$ and special functions

1. $$\sum_{k=0}^{n} \binom{2n+1}{2k+1} (4z)^{-2k} B_{2n-2k} H_{2k+1}(z) = \frac{2n+1}{2^{4n-1} z^{2n-1}} H_{2n}(z).$$

2. $$\sum_{k=0}^{n} \binom{2n}{2k} \frac{2^{2n-2k}-1}{(4z)^{2k}} B_{2n-2k} H_{2k}(z) = \frac{n}{(4z)^{2n-1}} H_{2n-1}(z) \qquad [n \geq 1].$$

3. $$\sum_{k=0}^{n} \frac{1}{(2k)!\,(n-k+1)!} \psi(a-k) H_{2k}(z) = \frac{1}{(2n+2)!} \psi(a-n-1) \times \left[(2z)^{2n+2} - H_{2n+2}(z)\right] + \frac{(2z)^{2n}}{(2n)!\,(a-n-1)}\, {}_4F_2\left(\begin{matrix} -n,\ -n+\frac{1}{2},\ 1,\ 1 \\ a-n,\ 2;\ -z^{-2} \end{matrix}\right).$$

4. $$\sum_{k=0}^{n} \frac{1}{(2k+1)!\,(n-k+1)!} \psi(a-k) H_{2k+1}(z) = \frac{1}{(2n+3)!} \psi(a-n-1) \times \left[(2z)^{2n+3} - H_{2n+3}(z)\right] + \frac{(2z)^{2n+1}}{(2n+1)!\,(a-n-1)}\, {}_4F_2\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ 1,\ 1 \\ a-n,\ 2;\ -z^{-2} \end{matrix}\right).$$

5. $$\sum_{k=0}^{n} (-1)^k \frac{(1-a)_k}{(2k)!} \psi(a-k) H_{2k}(z) = (-1)^{n+1} \frac{(1-a)_k}{(2n)!\,(n-a+1)} (2z)^{2n} \times \left[(a-n-1)\psi(a-n-1)\, {}_3F_1\left(\begin{matrix} -n,\ -n+\frac{1}{2},\ a-n-1;\ -z^{-2} \\ a-n \end{matrix}\right) + {}_4F_2\left(\begin{matrix} -n,\ -n+\frac{1}{2},\ a-n-1,\ a-n-1 \\ a-n,\ a-n;\ -z^{-2} \end{matrix}\right)\right].$$

6. $$\sum_{k=0}^{n} (-1)^k \frac{(1-a)_k}{(2k+1)!} \psi(a-k) H_{2k+1}(z) = (-1)^{n+1} \frac{(1-a)_n}{(2n+1)!\,(n-a+1)} (2z)^{2n+1}$$

$$\times\left[(a-n-1)\psi(a-n-1)\,{}_3F_1\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,a-n-1\\ a-n;\;-z^{-2}\end{matrix}\right)\right.$$
$$\left.+\,{}_4F_2\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,a-n-1,\,a-n-1\\ a-n,\,a-n;\;-z^{-2}\end{matrix}\right)\right].$$

7. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^{-k/2}}{(\nu+1)_k}D_{\nu+k}(\sqrt{2}\,z)H_k(z)=\frac{(-1)^n}{(\nu+1)_n}D_{\nu+2n}(\sqrt{2}\,z).$

8. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\left(\frac{i}{\sqrt{2}}\right)^k D_{\nu-k}(\sqrt{2}\,z)H_k(iz)=(n-\nu)_n D_{\nu-2n}(\sqrt{2}\,z).$

9. $\displaystyle\sum_{k=0}^{n}\frac{\left(w-\sqrt{w^2-1}\right)^k}{(n-k)!\,(2k)!}P_{n-k}(w)\,H_{2k}(z)$

$$=2^n\frac{\left(\frac{1}{2}\right)_n}{(n!)^2}(w^2-1)^{n/2}\,{}_2F_2\left(\begin{matrix}-n,\,-n;\;\frac{z^2}{2}-\frac{wz^2}{2\sqrt{w^2-1}}\\ \frac{1}{2}-n,\,\frac{1}{2}\end{matrix}\right).$$

10. $\displaystyle\sum_{k=0}^{n}\frac{\left(w-\sqrt{w^2-1}\right)^k}{(n-k)!\,(2k+1)!}P_{n-k}(w)\,H_{2k+1}(z)$

$$=2^{n+1}\frac{\left(\frac{1}{2}\right)_n}{(n!)^2}(w^2-1)^{n/2}\,z\,{}_2F_2\left(\begin{matrix}-n,\,-n;\;\frac{z^2}{2}-\frac{wz^2}{2\sqrt{w^2-1}}\\ \frac{1}{2}-n,\,\frac{3}{2}\end{matrix}\right).$$

11. $\displaystyle\sum_{k=0}^{[n/2]}\frac{1}{(n-2k)!\,(\lambda+1)_k}L_k^{\lambda}(-1)H_{n-2k}(z)=\frac{(-1)^n}{(\lambda+1)_n}C_n^{-\lambda-n}(z).$

5.9.3. Sums containing products of $H_{m\pm nk}(z)$

1. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{\left(-\frac{1}{4}\right)^k}{(2k)!}\left[H_{2k}(z)\right]^2=(-4)^n L_{2n}^{-1/2-n}(z^2).$

2. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{\left(-\frac{1}{4}\right)^k}{(2k+1)!}\left[H_{2k+1}(z)\right]^2=\frac{(-1)^n 2^{2n+2}}{2n+1}z^2 L_{2n}^{1/2-n}(z^2).$

3. $\displaystyle\sum_{k=1}^{n}(-i)^k\binom{n}{k}H_{n-k}(iz)\,H_{k-1}(z)$

$$=2^{(n-1)/2}n!\,(-i)^n e^{z^2/2}D_{-n-1}\left(\sqrt{2}\,z\right)-\frac{\sqrt{\pi}}{2}e^{z^2}\operatorname{erfc}(z)H_n(iz).$$

4. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}H_{2k}(w)\,H_{2n-2k}(z)=(-4)^n n!\,L_n(w^2+z^2).$

5. $\sum_{k=0}^{n} \binom{n}{k} H_{2n-2k+1}(w)\, H_{2k+1}(z) = (-1)^n 2^{2n+2} n!\, wz L_n^2(w^2+z^2).$

6. $\sum_{k=0}^{n} \binom{n}{k} H_{2n-2k}(w)\, H_{2k+1}(z) = (-1)^n 2^{2n+1} n!\, z L_n^1(w^2+z^2).$

7. $\sum_{k=0}^{n} 2^{2k} \frac{\left(\frac{1}{2}-n\right)_k}{(2k)!\,(n-k)!} H_{2k}(w) H_{2n-2k}(z) = \frac{2^{2n}}{n!}(w^2+z^2)^n T_n\left(\frac{z^2-w^2}{z^2+w^2}\right).$

8. $$\sum_{k=0}^{n} 2^{2k} \frac{\left(\frac{1}{2}-n\right)_k}{(2k+1)!\,(n-k)!} H_{2k+1}(w) H_{2n-2k}(z) = \frac{(-1)^n 2^{2n+1}}{n!\,(2n+1)}(w^2+z^2)^{n+1/2} T_{2n+1}\left(\frac{w}{\sqrt{w^2+z^2}}\right).$$

9. $$\sum_{k=0}^{n} 2^{2k} \frac{\left(-\frac{1}{2}-n\right)_k}{(2k+1)!\,(n-k)!} H_{2k+1}(w) H_{2n-2k+1}(z) = \frac{(-1)^n 2^{2n+1} z}{(n+1)!}(w^2+z^2)^{n+1/2} U_{2n+1}\left(\frac{w}{\sqrt{w^2+z^2}}\right).$$

5.9.4. Sums containing $H_n(\varphi(k, z))$

1. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} H_m(w+kz) = 0 \qquad [m<n].$

2. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{2m}\left(\sqrt{k}\, z\right) = (-1)^m (2m)!\, n!\, (2z)^{2n} \sum_{k=0}^{m-n} \sigma_{k+n}^n \frac{(-4z^2)^k}{(m-n-k)!\,(2k+2n)!}.$$

3. $$\sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}} \binom{n}{k} H_{2m+1}\left(\sqrt{k}\, z\right) = (-1)^{m+1} 2^{2m+1} \left(\frac{3}{2}\right)_m z + (-1)^m (2m+1)!\, n!\, (2z)^{2n+1} \sum_{k=0}^{m-n} \sigma_{k+n}^n \frac{(-4z^2)^k}{(m-n-k)!\,(2k+2n+1)!}.$$

4. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{2m+2n}\left(\sqrt{w+kz}\right) = (-4)^{m+n} n!\, (m+n)!\, z^n \sum_{k=0}^{m} \sigma_{k+n}^n \frac{(-z)^k}{(k+n)!} L_{m-k}^{k+n-1/2}(w).$$

5. $$\sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{kw+z}} \binom{n}{k} H_{2m+2n+1}\left(\sqrt{w+kz}\right) = (-1)^{m+n} 2^{2m+2n+1} n!\,(m+n)!\,z^n \sum_{k=0}^{m} \sigma_{k+n}^{n} \frac{(-z)^k}{(k+n)!} L_{m-k}^{k+n+1/2}(w).$$

6. $$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^m H_{2m}\left(\frac{z}{\sqrt{k}}\right) = -(2z)^{2m} + (2m)!\,n!\,(2z)^{2m-2n} \sum_{k=0}^{m-n} (-1)^k \frac{\sigma_{k+n}^{n}}{(k+n)!\,(2m-2n-2k)!} (2z)^{-2k}.$$

7. $$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^{m+1/2} H_{2m+1}\left(\frac{z}{\sqrt{k}}\right) = -(2z)^{2m+1} + (2m+1)!\,n!\,(2z)^{2m-2n+1} \sum_{k=0}^{m-n} (-1)^k \frac{\sigma_{k+n}^{n}}{(k+n)!\,(2m-2n-2k+1)!} (2z)^{-2k}.$$

8. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (ka+b)^{m+n} H_{2m+2n}\left(\frac{z}{\sqrt{ka+b}}\right) = \frac{2^{-2m} n!\,(2m+2n)!}{\left(\frac{1}{2}\right)_m} a^n b^m \sum_{k=0}^{m} \sigma_{k+n}^{n} \frac{\left(\frac{1}{2}-m\right)_k}{(n+k)!\,(m-k)!} \left(\frac{4a}{b}\right)^k H_{2m-2k}\left(\frac{z}{\sqrt{b}}\right).$$

9. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (ka+b)^{m+n+1/2} H_{2m+2n+1}\left(\frac{z}{\sqrt{ka+b}}\right) = \frac{2^{-2m} n!\,(2m+2n+1)!}{\left(\frac{3}{2}\right)_m} a^n b^{m+1/2} \times \sum_{k=0}^{m} \sigma_{k+n}^{n} \frac{\left(-\frac{1}{2}-m\right)_k}{(n+k)!\,(m-k)!} \left(\frac{4a}{b}\right)^k H_{2m-2k+1}\left(\frac{z}{\sqrt{b}}\right).$$

5.9.5. Sums containing $H_{m\pm nk}(\varphi(k, z))$

1. $$\sum_{k=1}^{n} \frac{(ka)^k (ka-b)^{n-k-1}}{(n-k)!\,(2k)!} H_{2k}\left(\frac{z}{\sqrt{k}}\right) = \frac{(-1)^n b^{n-1}}{n!} + \frac{b^n}{(2n)!\,(na-b)} H_{2n}\left(z\sqrt{\frac{a}{b}}\right).$$

2. $$\sum_{k=1}^{n} \frac{a^k k^{k+1/2} (ka-b)^{n-k-1}}{(n-k)!\,(2k+1)!} H_{2k+1}\left(\frac{z}{\sqrt{k}}\right) = \frac{2(-1)^n b^{n-1} z}{n!} + \frac{a^{-1/2} b^{n+1/2}}{(2n+1)!\,(na-b)} H_{2n+1}\left(z\sqrt{\frac{a}{b}}\right).$$

3. $$\sum_{k=1}^{n} \frac{k^{n-1}}{k!\,(2n-2k)!} H_{2n-2k}\left(\frac{z}{\sqrt{k}}\right) = \frac{(2z)^{2n-2}}{(2n-2)!} \qquad [n \geq 1].$$

4. $$\sum_{k=1}^{n} \frac{k^{n-k}(k+1)^{k-1}}{k!\,(2n-2k)!} H_{2n-2k}\left(\frac{z}{\sqrt{k}}\right) = -\frac{(2z)^{2n}}{(2n)!} + \frac{(-1)^n}{(2n)!} H_{2n}(iz).$$

5. $$\sum_{k=1}^{n} \frac{k^{n-k+1/2}(k+1)^{k-1}}{k!\,(2n-2k+1)!} H_{2n-2k+1}\left(\frac{z}{\sqrt{k}}\right) = -\frac{(2z)^{2n+1}}{(2n+1)!} - \frac{(-1)^n i}{(2n+1)!} H_{2n+1}(iz).$$

6. $$\sum_{k=1}^{n} \frac{k^{n-1/2}}{k!\,(2n-2k+1)!} H_{2n-2k+1}\left(\frac{z}{\sqrt{k}}\right) = \frac{(2z)^{2n-1}}{(2n-1)!} \qquad [n \geq 1].$$

7. $$\sum_{k=0}^{n} \frac{(ka+1)^{n-1}}{(n-k)!\,(2k)!} H_{2k}\left(\frac{z}{\sqrt{ka+1}}\right) = \frac{(2z)^{2n}}{(2n)!\,(na+1)}.$$

8. $$\sum_{k=0}^{n} \frac{(ka+1)^{n-1/2}}{(n-k)!\,(2k+1)!} H_{2k+1}\left(\frac{z}{\sqrt{ka+1}}\right) = \frac{(2z)^{2n+1}}{(2n+1)!\,(na+1)}.$$

5.9.6. Sums containing products of $H_{m\pm nk}(\varphi(k,z))$

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} [H_m(w+kz)]^2 = 0 \qquad [2m < n].$$

2. $$\sum_{k=0}^{n} \frac{(-1)^k (k+1)^{n-1/2}}{(2k+1)!\,(2n-2k)!} H_{2n-2k}\left(\frac{w}{\sqrt{k+1}}\right) H_{2k+1}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{2^{2n+2}}{(2n+2)!\,z} \left[w^{2n+2} - (w^2+z^2)^{n+1} T_{n+1}\left(\frac{w^2-z^2}{w^2+z^2}\right)\right].$$

3. $$\sum_{k=0}^{n} \frac{(-1)^k (k+1)^{n}}{(2k+1)!\,(2n-2k+1)!} H_{2n-2k+1}\left(\frac{w}{\sqrt{k+1}}\right) H_{2k+1}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{2^{2n+3} w}{(2n+3)!\,z} \left[w^{2n+2} + (-1)^n (w^2+z^2)^{n+1} U_{2n+2}\left(\frac{z}{\sqrt{w^2+z^2}}\right)\right].$$

5.10. The Laguerre Polynomials $L_n^\lambda(z)$

5.10.1. Sums containing $L_m^{\lambda\pm nk}(z)$

1. $$\sum_{k=0}^{n} \frac{(-n)_k (n)_k}{k!\,(k+m)!} L_m^k(z) = \frac{(-z)^n}{(m+n)!} L_{m-n}^{2n}(z) \qquad [m \geq n].$$

2. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_m^{\lambda+k}(z) = (-1)^n L_{m-n}^{\lambda+n}(z) \qquad [m \geq n].$

3. $$\sum_{k=0}^{n}\binom{n}{k}\frac{k^r}{(\lambda+m+1)_k}(-z)^k L_m^{\lambda+k}(z)$$
$$= \frac{1}{m!\,(\lambda+m+1)_n}\sum_{k=1}^{r}\sigma_r^k(-n)_k(m+n-k)!\,z^k L_{m+n-k}^{\lambda+k}(z).$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(\lambda+m+1)_k}L_m^{\lambda+k}(z)$$
$$= \frac{(\lambda+1)_m(\lambda-a+1)_n}{m!\,(\lambda+1)_n}\,{}_2F_2\left(\begin{matrix}-m,\,\lambda-a+n+1;\ z\\ \lambda+n+1,\,\lambda-a+1\end{matrix}\right).$$

5. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{(\lambda-n+1)_k}L_m^{\lambda+k}(z) = (-1)^n\frac{(m+n)!}{m!\,(-\lambda)_n}L_{m+n}^{\lambda-n}(z).$$

6. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{(\lambda+m+1)_k}L_m^{\lambda+k}(z) = \frac{(m+n)!}{m!\,(\lambda+m+1)_n}L_{m+n}^{\lambda}(z).$$

7. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a+m+n)_k}{(a)_k(\lambda+m+1)_k}(-z)^k L_m^{\lambda+k}(z) = \frac{(\lambda+1)_m}{m!}\,{}_2F_2\left(\begin{matrix}-m-n,\,a+n\\ \lambda+1,\,a;\ z\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(\lambda+m+n+1)_k}{(\lambda+1)_k(\lambda+m+1)_{2k}}(-z^2)^k L_m^{\lambda+2k}(z)$$
$$= \frac{n!\,(m+n)!\,(\lambda+1)_m}{m!\,(\lambda+1)_n(\lambda+1)_{m+n}}L_n^{\lambda}(-z)\,L_{m+n}^{\lambda}(z).$$

9. $$\sum_{k=0}^{n}L_m^{\lambda-k}(z) = L_{m+1}^{\lambda}(z) - L_{m+1}^{\lambda-n-1}(z).$$

10. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_m^{\lambda-k}(z) = L_{m-n}^{\lambda}(z) \qquad [m \geq n].$

11. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(-\lambda-m)_k}{(1-\lambda-n)_k}L_m^{\lambda-k}(z) = \frac{z^n}{(\lambda)_n}L_{m-n}^{\lambda+n}(z) \qquad [m \geq n].$

12. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(-\lambda-m)_k}{(1-a-n)_k}L_m^{\lambda-k}(z)$$
$$= \frac{(\lambda+1)_m(a-\lambda)_n}{m!\,(a)_n}\,{}_2F_2\left(\begin{matrix}-m,\,\lambda-a+1;\ z\\ \lambda+1,\,\lambda-a-n+1\end{matrix}\right).$$

13. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-\lambda-m)_k(a+m+n)_k}{(a)_k}z^{-k}L_m^{\lambda-k}(z) = \frac{(-z)^m}{m!}\,{}_3F_1\left(\begin{matrix}-m-n,\,-\lambda-m,\,a+n\\ a;\,-z^{-1}\end{matrix}\right).$$

14. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(-\lambda-m)_k}{(a)_k}L_m^{\lambda-k}(z) = \frac{(\lambda+a+m)_n(-z)^m}{m!\,(a)_n}\,{}_3F_1\left(\begin{matrix}-m,\,-\lambda-m,\,1-\lambda-a-m\\ 1-\lambda-a-m-n;\,-z^{-1}\end{matrix}\right).$$

15. $$\sum_{k=0}^{n}\binom{n}{k}(n-\lambda)_k z^{-k}L_m^{\lambda-k}(z) = \frac{(m+n)!}{m!}(-z)^{-n}L_{m+n}^{\lambda-2n}(z).$$

16. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{(\lambda+m+1)_k(k+1)}L_m^{\lambda+k}(z) = \frac{z^{-1}}{n+1}\left[(\lambda+m)L_m^{\lambda-1}(z)-\frac{(m+n+1)!\,z}{m!\,(\lambda+m+1)_n}L_{m+n+1}^{\lambda-1}(z)\right].$$

5.10.2. Sums containing $L_{m\pm nk}^{\lambda}(z)$

1. $$\sum_{k=0}^{n}\frac{t^k}{(n-k)!\,(\lambda+1)_k}L_k^{\lambda}(z) = \frac{(t+1)^n}{(\lambda+1)_n}L_n^{\lambda}\left(\frac{tz}{t+1}\right).$$

2. $$\sum_{k=0}^{n-1}(-1)^k\frac{(n-k)^n}{(n-k)!\,(\lambda+1)_k}L_k^{\lambda}(z) = \frac{z^n}{n!\,(\lambda+1)_n}\sum_{k=0}^{n}\sigma_n^k(-n)_k(-\lambda-n)_k z^{-k}.$$

3. $$\sum_{k=0}^{n}(-1)^k\frac{(2n-k)!}{[(n-k)!]^2\,(\lambda+1)_k}L_k^{\lambda}(z) = {}_2F_2\left(\begin{matrix}-n,\,-n;\,z\\ \lambda+1,\,1\end{matrix}\right).$$

4. $$\sum_{k=0}^{n}\frac{(a)_k}{(n-k)!\,(\lambda+1)_k}z^{-k}L_k^{\lambda}(z) = \frac{(a)_n}{n!}z^{-n}\,{}_3F_3\left(\begin{matrix}-n,\,\frac{\lambda-a-n+1}{2},\,\frac{\lambda-a-n}{2}+1;\,4z\\ \lambda+1,\,-a-n+1,\,\lambda-a-n+1\end{matrix}\right).$$

5. $$\sum_{k=0}^{n}(-1)^k\frac{(a)_k}{(n-k)!\,(b)_k(\lambda+1)_k}L_k^{\lambda}(z) = \frac{(b-a)_n}{n!\,(b)_n}\,{}_2F_2\left(\begin{matrix}-n,\,a;\,z\\ a-b-n+1,\,\lambda+1\end{matrix}\right).$$

6. $$\sum_{k=0}^{n}\frac{(-\lambda-n)_k}{(n-k)!\,(\lambda+1)_k}(-z)^{-k}L_k^{\lambda}(z) = \frac{(\lambda+1)_n}{n!}z^{-n}\,{}_2F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2};\,-z^2\\ \frac{\lambda+1}{2},\,\frac{\lambda}{2}+1,\,\lambda+1\end{matrix}\right).$$

7. $$\sum_{k=0}^{n} \frac{\sigma_m^{n-k+1}}{(\lambda+1)_k} L_k^\lambda(z) = \frac{(-z)^n}{n!\,(\lambda+1)_n}\, {}_{m+1}F_{m-1}\left(\begin{matrix}-n,\ -n-\lambda,\ 2,\ \ldots,\ 2\\ 1,\ \ldots,\ 1;\ -z^{-1}\end{matrix}\right).$$

8. $$\sum_{k=0}^{n} \frac{(-\lambda-n)_k}{(n-k)!\,(\lambda+1)_k} (-z)^{-k} L_k^\lambda(z) = \frac{(\lambda+1)_n}{n!}\, z^{-n}\, {}_2F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2};\ -z^2\\ \frac{\lambda+1}{2},\ \frac{\lambda}{2}+1,\ \lambda+1\end{matrix}\right).$$

9. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2k)!}{(\lambda+1)_{2k}} L_{2k}^\lambda(z) = \frac{n!\,(2z)^n}{(\lambda+1)_{2n}} L_n^{\lambda+n}\left(\frac{z}{2}\right).$$

10. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{k+m}^\lambda(z) = (-1)^n L_{m+n}^{\lambda-n}(z).$$

11. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(k+m)!}{(\lambda+m+1)_k} L_{k+m}^\lambda(z) = \frac{m!\,z^n}{(\lambda+m+1)_n} L_m^{\lambda+n}(z).$$

12. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+m)_k}{(\lambda+m+1)_k} L_{k+m}^\lambda(z) = \frac{(1-a)_n(\lambda+1)_m}{(m+n)!}\, {}_2F_2\left(\begin{matrix}-m-n,\ a;\ z\\ \lambda+1,\ a-n\end{matrix}\right).$$

13. $$\sum_{k=0}^{n} \sigma_{k+m}^m \frac{(-\lambda-n)_k}{(k+m)!} t^k L_{n-k}^\lambda(z) = \frac{t^{-m}}{(\lambda+n+1)_m} \sum_{k=0}^{m} (-1)^k \frac{(1-kt)^{m+n}}{k!\,(m-k)!} L_{m+n}^\lambda\left(\frac{z}{1-kt}\right).$$

14. $$\sum_{k=0}^{n} \frac{(-4)^k}{(n-k)!\,(2k)!} (-\lambda-m)_k L_{m-k}^\lambda(z) = \frac{(-z)^m}{m!\,n!}\, {}_3F_1\left(\begin{matrix}-m,\ -\lambda-m,\ n+\frac{1}{2}\\ \frac{1}{2};\ -z^{-1}\end{matrix}\right) \quad [m \geq n].$$

15. $$\sum_{k=0}^{n} \frac{(-4)^k}{(n-k)!\,(2k+1)!} (-\lambda-m)_k L_{m-k}^\lambda(z) = \frac{(-z)^m}{m!\,n!}\, {}_3F_1\left(\begin{matrix}-m,\ -\lambda-m,\ n+\frac{3}{2}\\ \frac{3}{2};\ -z^{-1}\end{matrix}\right) \quad [m \geq n].$$

16. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{m-k}^{-m-1/2}(z) = L_m^{-m-n-1/2}(z) \quad [m \geq n].$$

17. $$\sum_{k=0}^{[n/2]} \frac{\left(\frac{1}{2}\right)_k}{k!} L_{n-2k}(z) = (-1)^n \frac{2^{-n-1/2}}{n!\sqrt{z}} H_n\left(\sqrt{\frac{z}{2}}\right) H_{n+1}\left(\sqrt{\frac{z}{2}}\right).$$

5.10.3. Sums containing $L_{m\pm pk}^{\lambda\pm nk}(z)$

1. $$\sum_{k=0}^{n} (-1)^k \frac{(a+k\mu)_{n-1}}{(n-k)!\,(a+k\mu)_k} L_k^{\lambda+k\mu}(z) = \frac{(-1)^n}{a+n\mu+n-1} L_n^{\lambda-a-n+1}(z).$$

2. $$\sum_{k=0}^{n} \frac{(-z)^k}{(n-k)!\,(\lambda+1)_k(\mu+1)_k} L_k^{\lambda+\mu+k}(z) = \frac{n!}{(\lambda+1)_n(\mu+1)_n} L_n^{\lambda}(z)\, L_n^{\mu}(z).$$

3. $$\sum_{k=0}^{n} (-1)^k \frac{(a)_k}{(n-k)!\,(\lambda+1)_{2k}} L_k^{\lambda+k}(z) = \frac{(a)_n}{n!\,(\lambda+1)_{2n}} z^n\, {}_3F_1\left(\begin{matrix}-n,\ -\lambda-2n,\ a-\lambda-n\\ 1-a-n;\ z^{-1}\end{matrix}\right).$$

4. $$\sum_{k=0}^{n} \binom{n}{k} \frac{\left(n+\frac{1}{2}\right)_k}{(\lambda+1)_{2k}} (-4z)^k L_k^{\lambda+k}(z) = \frac{\left(n+\frac{1}{2}\right)_n}{n!\,(\lambda+1)_{2n}} (2z)^{2n}\, {}_3F_1\left(\begin{matrix}-2n,\ -2n,\ -\lambda-2n\\ -4n;\ -z^{-1}\end{matrix}\right).$$

5. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\left(a-n+\frac{1}{2}\right)_k(\lambda+1)_{2k}} (-4z)^k L_k^{\lambda+k}(z) = \frac{(a)_n}{n!\left(\frac{1}{2}-a\right)_n(\lambda+1)_{2n}} (-4z^2)^n\, {}_3F_1\left(\begin{matrix}-2n,\ \frac{1}{2}-a-n,\ -\lambda-2n\\ 1-2a-2n;\ -z^{-1}\end{matrix}\right).$$

6. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-a\right)_k}{(n-k)!\,(1-a-n)_k\,(\lambda+1)_{2k}} (-4z)^k L_k^{\lambda+k}(z) = \frac{\left(\frac{1}{2}-a\right)_n}{n!\,(a)_n(\lambda+1)_{2n}} (-4z^2)^n\, {}_3F_1\left(\begin{matrix}-2n,\ a-n,\ -\lambda-2n\\ 2a-2n;\ -z^{-1}\end{matrix}\right).$$

7. $$\sum_{k=0}^{2n} \frac{(a)_k}{(n-k)!\left(a-n+\frac{1}{2}\right)_k(\lambda+1)_{2k}} (-4z)^k L_k^{\lambda+k}(z) = \frac{1}{n!}\, {}_2F_2\left(\begin{matrix}-2n,\ 2a;\ z\\ a-n+\frac{1}{2},\ \lambda+1\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}\binom{n}{k}\frac{\left(m+n+\frac{1}{2}\right)_k}{k!\,(\lambda+m+1)_{2k}}(4z)^k L_{k+m}^{\lambda+k}(z) = \frac{(\lambda+1)_m}{m!}\,{}_2F_2\left(\begin{matrix}-m-2n,\ m+2n+1\\ m+1,\ \lambda+1;\ z\end{matrix}\right).$$

9. $$\sum_{k=0}^{n}\binom{n}{k}L_{m-k}^{\lambda+k}(z) = L_m^{\lambda+n}(z) \qquad [m \geq n].$$

10. $$\sum_{k=0}^{n}\frac{(-4z)^k}{(n-k)!\,(2k+1)!}L_{m-k}^{\lambda+k}(z) = \frac{(\lambda+1)_m}{m!\,n!}\,{}_2F_2\left(\begin{matrix}-m,\ n+\frac{3}{2}\\ \frac{3}{2},\ \lambda+1;\ z\end{matrix}\right).$$

11. $$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}L_{m-k}^{\lambda+k}(z) = \frac{1}{(n+1)(n+2)}\left[L_{m+2}^{\lambda-2}(z)+(n+1)L_{m+1}^{\lambda+n}(z)-L_{m+2}^{\lambda+n-1}(z)\right] \qquad [m \geq n].$$

12. $$\sum_{k=0}^{n}\sigma_{k+m}^{m}\frac{t^k}{(k+m)!}L_{n-k}^{\lambda+k}(z) = (-t)^{-m}\sum_{k=0}^{m}\frac{(-1)^k}{k!\,(m-k)!}L_{m+n}^{\lambda-m}(z-kt).$$

13. $$\sum_{k=0}^{n}\binom{n}{k}k^r L_{m-k}^{\lambda+k}(z) = n!\sum_{k=1}^{r}\frac{\sigma_r^k}{(n-k)!}L_{m-k}^{\lambda+n}(z) \qquad [m \geq n;\ m \geq r].$$

14. $$\sum_{k=0}^{n}\binom{n}{k}(m-k)!z^k L_{m-k}^{\lambda+k}(z) = (-1)^n(m-n)!\,(-\lambda-m)_n L_{m-n}^{\lambda}(z) \qquad [m \geq n].$$

15. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{(\lambda-n+1)_k}L_{m-k}^{\lambda+k}(z) = \frac{(\lambda+1)_m}{(\lambda-n+1)_m}L_m^{\lambda-n}(z) \qquad [m \geq n].$$

16. $$\sum_{k=0}^{n}\frac{(-4z)^k}{(n-k)!\,(2k)!}L_{m-k}^{n+k-1/2}(z) = (-1)^m\frac{\left(n+\frac{1}{2}\right)_m}{(2m)!\,n!}H_{2m}(\sqrt{z}).$$

17. $$\sum_{k=1}^{n}\frac{2k+\lambda}{(\lambda+m+1)_k}(-z)^k L_{m-k}^{\lambda+2k}(z) = -zL_{m-1}^{\lambda+1}(z) - \frac{(-z)^{n+1}}{(\lambda+m+1)_n}L_{m-n-1}^{\lambda+2n+1}(z) \qquad [m > n \geq 1].$$

18. $$\sum_{k=0}^{n}\frac{(-4z)^k}{(n-k)!\,(2k+1)!}L_{m-k}^{\lambda+k}(z) = \frac{(\lambda+1)_m}{m!\,n!}\,{}_2F_2\left(\begin{matrix}-m,\ n+\frac{3}{2}\\ \lambda+1,\ \frac{3}{2};\ z\end{matrix}\right) \qquad [m \geq n].$$

19. $$\sum_{k=0}^{n} \binom{n}{k} \frac{4^{-k}(-\lambda-2m)_k}{\left(\frac{1}{2}-\lambda-2m-n\right)_k} L_{2m-2k}^{\lambda+k}(z) = \frac{z^{2m}}{(2m)!}\, {}_3F_1\left(\begin{matrix}-2m,\, -2\lambda-4m,\, -\lambda-2m-n \\ -2\lambda-4m-2n;\; -z^{-1}\end{matrix}\right) \quad [m \geq n].$$

20. $$\sum_{k=0}^{n} \frac{t^k}{(n-k)!} L_k^{\lambda-k}(z) = t^n L_n^{\lambda-n}\left(z-\frac{1}{t}\right).$$

21. $$\sum_{k=0}^{n} (a)_k (-z)^{-k} L_k^{\lambda-k}(z) = \frac{(a)_n(-\lambda)_n}{n!} z^{-n}\, {}_2F_2\left(\begin{matrix}-n,\, -a-n;\; z \\ 1-a-n,\, \lambda-n+1\end{matrix}\right).$$

22. $$\sum_{k=0}^{n} (-\lambda-1)_k (-z)^{-k} L_k^{\lambda-k}(z) = (-\lambda)_n (-z)^{-n} L_n^{\lambda-n+1}(z).$$

23. $$\sum_{k=0}^{n} \sigma_m^{n-k+1} (-z)^{-k} L_k^{\lambda-k}(z) = \frac{(-\lambda)_n}{n!} z^{-n}\, {}_mF_m\left(\begin{matrix}-n,\, 2,\, \ldots,\, 2;\; z \\ \lambda-n+1,\, 1,\, \ldots,\, 1\end{matrix}\right).$$

24. $$\sum_{k=0}^{n} \frac{1}{(n-k)!\,(1-\lambda-n)_k} L_k^{\lambda-k}(z) = \frac{z^n}{n!\,(\lambda)_n}.$$

25. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\lambda-n+1)_k} L_k^{\lambda-k}(z) = \frac{(-z)^n}{n!\,(-\lambda)_n}\, {}_3F_0\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, -\lambda \\ 4z^{-2}\end{matrix}\right).$$

26. $$\sum_{k=0}^{n-1} \frac{(n-k)^n}{(n-k)!} z^{-k} L_k^{\lambda-k}(z) = \frac{(-\lambda)_n}{n!} (-z)^{-n} \sum_{k=0}^{n} \sigma_n^k \frac{(-n)_k}{(\lambda-n+1)_k} (-z)^k.$$

27. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(b)_k} z^{-k} L_k^{\lambda-k}(z) = \frac{(b-a)_n}{n!\,(b)_n}\, {}_3F_1\left(\begin{matrix}-n,\, a,\, -\lambda;\; -z^{-1} \\ a-b-n+1\end{matrix}\right).$$

28. $$\sum_{k=0}^{n} \frac{z^{-k}}{(n-k)!\,(a-k)} L_k^{\lambda-k}(z) = \frac{(-\lambda)_n}{n!\,(a-n)} (-z)^{-n}\, {}_2F_2\left(\begin{matrix}-n,\, 1;\; z \\ a-n+1,\, \lambda-n+1\end{matrix}\right).$$

29. $$\sum_{k=0}^{n} \frac{z^{-k}}{(n-k)!\,(k-a)} L_k^{n-k}(z) = \frac{n!}{(-a)_{n+1}} (-z)^{-n} L_n^{a-n}(z).$$

30. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-a-n\right)_k}{(n-k)!\left(\frac{3}{2}-n\right)_k} z^{-k} L_k^{\lambda-k}(z) = \frac{\left(a+\frac{1}{2}\right)_n (-\lambda)_n}{n!\left(-\frac{1}{2}\right)_n} \frac{(-z)^{-n}}{2a} \times \left[(2a-1)\, {}_2F_2\left(\begin{matrix}-n,\, a;\; z \\ a-\frac{1}{2},\, \lambda-n+1\end{matrix}\right) + {}_2F_2\left(\begin{matrix}-n,\, a;\; z \\ a+\frac{1}{2},\, \lambda-n+1\end{matrix}\right)\right].$$

31. $$\sum_{k=0}^{n} \frac{\left(-\lambda-\frac{1}{2}\right)_k}{(n-k)!\left(\frac{3}{2}-n\right)_k} z^{-k} L_k^{\lambda-k}(z)$$
$$= \frac{(-\lambda)_n(-z)^{-n}}{2(\lambda-n+1)\left(-\frac{1}{2}\right)_n}\left[(2\lambda+1)L_n^{\lambda-n-1/2}(z)+L_n^{\lambda-n+1/2}(z)\right].$$

32. $$\sum_{k=0}^{n}\binom{n}{k} L_{k+m}^{\lambda-k}(z) = L_{m+n}^{\lambda}(z).$$

33. $$\sum_{k=0}^{n}\binom{n}{k}(a)_k z^{-k} L_{k+m}^{\lambda-k}(z)$$
$$= \frac{(m-a+1)_n(-z)^m}{(m+n)!}\, {}_3F_1\left(\begin{matrix}-m-n,\ a-m,\ -\lambda-m\\ a-m-n;\ -z^{-1}\end{matrix}\right).$$

34. $$\sum_{k=0}^{n}\binom{n}{k}(n-\lambda)_k z^{-k} L_{k+m}^{\lambda-k}(z) = \frac{(\lambda+1)_m(-\lambda)_n z^{-n}}{(\lambda-n+1)_m} L_{m+n}^{\lambda-2n}(z).$$

35. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-\lambda-m)_{2k}}{\left(\frac{1}{2}-m-n\right)_k}(4z)^{-k} L_{m-k}^{\lambda-k}(z)$$
$$= \frac{(-z)^m}{m!}\, {}_3F_1\left(\begin{matrix}-2m,\ -m-n,\ -\lambda-m\\ -2m-2n;\ -z^{-1}\end{matrix}\right) \quad [m \geq n \geq 1].$$

36. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(2k+\lambda)(\lambda)_k}{(\lambda+m+1)_k(\lambda+n+1)_k} z^k L_{m-k}^{\lambda+2k}(z) = \frac{(\lambda)_{m+1}}{(\lambda+n+1)_m} L_m^{\lambda+n}(z)$$
$$[m \geq n].$$

37. $$\sum_{k=0}^{n}\frac{z^k}{(\lambda+m+1)_k} L_{m-k}^{\lambda+2k}(z) = \frac{(\lambda+1)_m}{m!}$$
$$\times\left[{}_2F_2\left(\begin{matrix}-m,\ \frac{\lambda}{2};\ z\\ \lambda,\ \frac{\lambda}{2}+1\end{matrix}\right) - \frac{(-m)_{n+1}(-z)^{n+1}}{(\lambda+1)_{2n+2}}\, {}_2F_2\left(\begin{matrix}n-m+1,\ n+\frac{\lambda}{2}+1;\ z\\ 2n+\lambda+2,\ n+\frac{\lambda}{2}+2\end{matrix}\right)\right]$$
$$[m \geq n].$$

38. $$\sum_{k=0}^{n}\binom{n}{k}\frac{\left(-\frac{z^2}{4}\right)^k}{\left(\frac{1}{2}-m-n\right)_k(\lambda+m+1)_k} L_{m-k}^{\lambda+2k}(z)$$
$$= \frac{(\lambda+1)_m}{m!}\, {}_2F_2\left(\begin{matrix}-2m,\ -m-n;\ z\\ -2m-2n,\ \lambda+1\end{matrix}\right) \quad [m \geq n].$$

39. $$\sum_{k=0}^{[n/2]}\frac{\left(\frac{z}{2}\right)^{2k}}{k!\,(a)_k} L_{n-2k}^{\lambda+2k}(z) = \frac{(\lambda+1)_n}{n!}\, {}_2F_2\left(\begin{matrix}-n,\ a-\frac{1}{2};\ 2z\\ 2a-1,\ \lambda+1\end{matrix}\right).$$

40. $$\sum_{k=0}^{n}(-\lambda)_k(-z)^{-k}L_k^{\lambda-2k}(z)=\frac{(-\lambda)_{2n}}{n!}z^{-n}\,{}_2F_2\left(\begin{matrix}-n,\,\frac{\lambda}{2}-n;\; z\\ \frac{\lambda}{2}-n+1,\,\lambda-2n\end{matrix}\right).$$

41. $$\sum_{k=0}^{n}\frac{\left(n-\lambda-\frac{1}{2}\right)_k}{(n-k)!}\left(-\frac{4}{z^2}\right)^k L_k^{\lambda-2k}(z)=\left(\frac{1}{2}\right)_n\left(\frac{2}{z}\right)^{2n}L_{2n}^{2\lambda-4n+1}(z).$$

42. $$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}-a-n\right)_k(-\lambda)_k}{(n-k)!\,(1-a-2n)_k}\left(-\frac{4}{z^2}\right)^k L_k^{\lambda-2k}(z)$$
$$=\frac{(2a)_{2n}(-\lambda)_{2n}}{n!\,(a)_{2n}}z^{-2n}\,{}_2F_2\left(\begin{matrix}-2n,\,a;\; z\\ 2a,\,\lambda-2n+1\end{matrix}\right).$$

43. $$\sum_{k=0}^{n}\frac{(-\lambda)_k}{(n-k)!\,(a)_k}z^{-k}L_k^{\lambda-2k}(z)$$
$$=\frac{(-\lambda)_{2n}}{n!\,(a)_n}(-z)^{-n}\,{}_3F_3\left(\begin{matrix}-n,\,\frac{a+\lambda-n}{2},\,\frac{a+\lambda-n+1}{2};\; z\\ \frac{\lambda+1}{2}-n,\,\frac{\lambda}{2}-n+1,\,a+\lambda-n\end{matrix}\right).$$

44. $$\sum_{k=0}^{n}\frac{(n-\lambda)_k}{(n-k)!}t^k L_k^{\lambda-2k}(z)$$
$$=n!\,(-t)^n L_n^{\lambda-2n}\left(\frac{\sqrt{t}\,z-\sqrt{4+tz^2}}{2\sqrt{t}}\right)L_n^{\lambda-2n}\left(\frac{\sqrt{t}\,z+\sqrt{4+tz^2}}{2\sqrt{t}}\right).$$

45. $$\sum_{k=0}^{n}(2k-\lambda)\frac{(-\lambda)_k z^{-k}}{(n-k)!\,(n-\lambda+1)_k}L_k^{\lambda-2k}(z)=\frac{(-\lambda)_{n+1}}{n!}(-z)^{-n}.$$

46. $$\sum_{k=0}^{n}\frac{\left(a+n-\frac{1}{2}\right)_k(-\lambda)_k}{(n-k)!\,(a)_k}\left(-\frac{4}{z^2}\right)^k L_k^{\lambda-2k}(z)$$
$$=\frac{1}{n!}\,{}_3F_1\left(\begin{matrix}-2n,\,2n+2a-1,\,-\lambda\\ a;\; -z^{-1}\end{matrix}\right).$$

47. $$\sum_{k=0}^{n}\frac{(2n-2k+a)(-\lambda)_k}{(n-k)!\,(1-a-n)_k}z^{-k}L_k^{\lambda-2k}(z)$$
$$=\frac{(-\lambda)_{2n}az^{-n}}{n!\,(a)_n}\,{}_3F_3\left(\begin{matrix}-n,\,\frac{\lambda-a}{2}-n,\,\frac{\lambda-a+1}{2}-n;\; z\\ \frac{\lambda+1}{2}-n,\,\frac{\lambda}{2}-n+1,\,\lambda-a-2n+1\end{matrix}\right).$$

48. $$\sum_{k=0}^{n}(2k-\lambda)\frac{(1-a-n)_k(-\lambda)_k}{(a-\lambda+n)_k}(-z)^{-k}L_k^{\lambda-2k}(z)$$
$$=\frac{(a)_n(-\lambda)_{2n+1}(a-\lambda)_n}{n!\,(a-\lambda)_{2n}}(-z)^{-n}\,{}_2F_2\left(\begin{matrix}-n,\,a-1;\; z\\ a,\,\lambda-2n\end{matrix}\right).$$

49. $$\sum_{k=0}^{n} \frac{2k-\lambda}{(\lambda-2k)^2-1} \frac{(-\lambda)_k}{(n-k)!\,(n-\lambda+1)_k} z^{-k} L_k^{\lambda-2k}(z)$$
$$= \frac{(-\lambda)_{n+1}}{n!\left[(\lambda-2n)^2-1\right]}(-z)^{-n}\,{}_2F_2\left(\begin{matrix}-n,\,1;\ z\\ \frac{\lambda+1}{2}-n,\ \frac{\lambda+3}{2}-n\end{matrix}\right).$$

50. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-a\right)_k(-\lambda)_k}{(n-k)!\,(1-a-n)_k}\left(-\frac{4}{z^2}\right)^k L_k^{\lambda-2k}(z)$$
$$= \frac{\left(\frac{1}{2}-a\right)_n(-\lambda)_{2n}}{n!\,(a)_n}\left(-\frac{4}{z^2}\right)^n {}_2F_2\left(\begin{matrix}-2n,\,a-n;\ z\\ 2a-2n,\,\lambda-2n+1\end{matrix}\right).$$

51. $$\sum_{k=0}^{n} \frac{(-\lambda)_k}{(n-k)!\,(a)_k\left(\frac{3}{2}-a-\lambda-n\right)_k} 2^{-2k} L_k^{\lambda-2k}(z)$$
$$= \frac{(-\lambda)_n}{n!\,(a)_n\left(a+\lambda-\frac{1}{2}\right)_n}\left(\frac{z}{4}\right)^n {}_3F_1\left(\begin{matrix}-n,\,2a+\lambda+n-1,\,2-2a-\lambda-n\\ \lambda-n+1;\ -\frac{1}{z}\end{matrix}\right).$$

52. $$\sum_{k=0}^{n} \frac{(-\lambda)_k}{(n-k)!\,(1-a-n)_k\left(a-\lambda+\frac{1}{2}\right)_k} 2^{-2k} L_k^{\lambda-2k}(z)$$
$$= \frac{(-\lambda)_n}{n!\,(a)_n\left(a-\lambda+\frac{1}{2}\right)_n}\left(\frac{z}{4}\right)^n {}_3F_1\left(\begin{matrix}-n,\,2a-\lambda+n,\,\lambda-2a-n+1\\ \lambda-n+1;\ -z^{-1}\end{matrix}\right).$$

53. $$\sum_{k=0}^{n}(2k-\lambda)\frac{(a)_k(b)_k(-\lambda)_k}{(n-k)!\,(1-a-\lambda)_k(1-b-\lambda)_k(n-\lambda+1)_k} z^{-k} L_k^{\lambda-2k}(z)$$
$$= \frac{(a)_n(b)_n(-\lambda)_{n+1}}{n!\,(1-a-\lambda)_n(1-b-\lambda)_n}(-z)^{-n}\,{}_2F_2\left(\begin{matrix}-n,\,1-a-b-\lambda;\ z\\ 1-a-n,\,1-b-n\end{matrix}\right).$$

54. $$\sum_{k=0}^{n}\binom{n}{k}\frac{\left(-\frac{z^2}{4}\right)^k}{\left(\frac{1}{2}-m-n\right)_k(\lambda+m+1)_k} L_{m-k}^{\lambda+2k}(z)$$
$$= \frac{(\lambda+1)_m}{m!}\,{}_2F_2\left(\begin{matrix}-2m,\,-m-n;\ z\\ -2m-2n,\,\lambda+1\end{matrix}\right) \quad [m \ge n \ge 1].$$

55. $$\sum_{k=0}^{n}\binom{n}{k}\left(m+n+\frac{1}{2}\right)_k(-\lambda-m)_k\left(-\frac{4}{z^2}\right)^k L_{k+m}^{\lambda-2k}(z)$$
$$= \frac{(-z)^m}{m!}\,{}_3F_1\left(\begin{matrix}-m-2n,\,m+2n+1,\,-\lambda-m\\ m+1;\ -z^{-1}\end{matrix}\right).$$

56. $$\sum_{k=0}^{n}\binom{2k}{k}\frac{1}{(n-k)!\,(\lambda+1)_k} L_{2k}^{\lambda-k}(z) = \frac{1}{(\lambda+1)_n}\left[L_n^{\lambda}(z)\right]^2$$ [[34], (42′)].

57. $$\sum_{k=0}^{n} \frac{(1-a-n)_k}{(n-k)!\,(\lambda+1)_k} z^{-k} L_{2k}^{\lambda-k}(z) = \frac{(a)_n(-r)_n}{(2n)!} z^{-n}\, {}_2F_2\left(\begin{matrix} -2n,\, 1-a-2n \\ a,\, \lambda-n+1;\; -z \end{matrix}\right).$$

58. $$\sum_{k=0}^{n} \binom{n}{k} \frac{\left(\lambda+n+\frac{1}{2}\right)_k}{(\lambda+1)_k} 2^{2k} L_{2k}^{\lambda-k}(z) = \frac{\left(\lambda+\frac{1}{2}\right)_{2n}}{(2\lambda+1)_{2n}} \frac{(4z)^{2n}}{(2n)!}\, {}_3F_1\left(\begin{matrix} -2n,\, -\lambda-2n,\, -2\lambda-2n \\ -2\lambda-4n;\; -z^{-1} \end{matrix}\right).$$

59. $$\sum_{k=0}^{n} \frac{(1-2a-n)_k \left(\frac{1}{2}\right)_k}{(n-k)!\,(1-a-n)_k(\lambda+1)_k} z^{-k} L_{2k}^{\lambda-k}(z) = \frac{(2a)_n(-\lambda)_n}{n!\,(a)_n} (-4z)^{-n}\, {}_2F_3\left(\begin{matrix} -n,\, \frac{1}{2}-a-n;\; \frac{z^2}{4} \\ a+\frac{1}{2},\, \frac{\lambda-n+1}{2},\, \frac{\lambda-n}{2}+1 \end{matrix}\right).$$

60. $$\sum_{k=0}^{n} \frac{\lambda-4k}{(n-k)!} \frac{\left(\frac{1}{2}\right)_k \left(-\frac{\lambda}{2}\right)_k \left(n+\frac{1-\lambda}{2}\right)_k}{\left(\frac{1}{2}-n\right)_k \left(n-\frac{\lambda}{2}+1\right)_k} \left(\frac{2}{z}\right)^{2k} L_{2k}^{\lambda-4k}(z) = 2(-1)^{n+1} \frac{n!\left(-\frac{\lambda}{2}\right)_{n+1}}{\left(\frac{1}{2}\right)_n} \left(\frac{2}{z}\right)^{2n} \left[L_n^{(\lambda-1)/2-2n}\left(\frac{z}{2}\right)\right]^2.$$

61. $$\sum_{k=0}^{n} \binom{2n}{n-k} (2k)!\, z^{-2k} L_{2k}^{-4k}(z) = 2^{4n-1}(2n)!\, z^{-2n} L_{2n}^{-2n-1/2}\left(\frac{z}{2}\right) - \frac{1}{2}\binom{2n}{n}.$$

62. $$\sum_{k=0}^{n} \binom{2n+1}{n-k} (2k+1)!\, z^{-2k} L_{2k+1}^{-4k-2}(z) = 2^{4n+1}(2n+1)! z^{-2n} L_{2n+1}^{-2n-3/2}\left(\frac{z}{2}\right).$$

63. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda-2m)_{2k}}{\left(\frac{1}{2}-\lambda-2m-n\right)_k} 4^{-k} L_{2m-2k}^{\lambda+k}(z) = \frac{z^{2m}}{(2m)!}\, {}_3F_1\left(\begin{matrix} -2m,\, -2\lambda-4m,\, -\lambda-2m-n \\ -2\lambda-4m-2n;\; -z^{-1} \end{matrix}\right) \quad [m \geq n].$$

64. $$\sum_{k=0}^{n} \frac{(m+n-2k)!}{(m-k)!\,(n-k)!\,(\lambda+1)_k} z^{2k} L_{m+n-2k}^{\lambda+2k}(z) = \frac{(\lambda+1)_{m+n}}{(\lambda+1)_m(\lambda+1)_n} L_m^{\lambda}(z)\, L_n^{\lambda}(z) \quad [m \geq n].$$

65. $$\sum_{k=0}^{[n/2]}(4k+\lambda)\frac{(a)_k\left(\frac{\lambda}{2}\right)_k}{k!\left(\frac{\lambda}{2}-a+1\right)_k(\lambda+n+1)_{2k}(\lambda)_{4k+1}}z^{2k}L_{n-2k}^{\lambda+4k}(z)$$
$$=\frac{(\lambda+1)_n}{n!}\,{}_2F_2\left(\begin{matrix}-n,\ \frac{\lambda+1}{2}-a;\ z\\ \frac{\lambda+1}{2},\ \lambda-2a+1\end{matrix}\right).$$

66. $$\sum_{k=0}^{[n/2]}(-1)^k(2n-4k+1)\frac{(n-2k)!\left(-n-\frac{1}{2}\right)_k}{k!}z^{2k}L_{n-2k}^{4k-2n-1}(z)$$
$$=2^{2n+1}\left(\frac{1}{2}\right)_{n+1}L_n^{-n-1}\left(\frac{z}{2}\right).$$

67. $$\sum_{k=0}^{[n/3]}(\lambda+6k)\frac{\left(\frac{\lambda}{3}\right)_k}{k!\,(\lambda+n+1)_{3k}}z^{3k}L_{n-3k}^{\lambda+6k}(z)=\frac{(\lambda)_{n+1}}{n!}\,{}_2F_2\left(\begin{matrix}-n,\ \frac{\lambda}{3};\ \frac{3z}{4}\\ \frac{\lambda}{2},\ \frac{\lambda+1}{2}\end{matrix}\right).$$

68. $$\sum_{k=0}^{n}\frac{(1-a-n)_k\left(a+2n-\frac{1}{2}\right)_k}{(n-k)!\,(\lambda+1)_k}\left(\frac{4}{z}\right)^kL_{3k}^{\lambda-k}(z)$$
$$=\frac{(a)_n\left(a+2n-\frac{1}{2}\right)_n}{(3n)!\,(\lambda+1)_{2n}}(2z)^{2n}\,{}_3F_2\left(\begin{matrix}-3n,\ a-\frac{1}{2},\ 1-a-3n,\ -\lambda-2n\\ 2a-1,\ 2-2a-6n;\ -\frac{4}{z}\end{matrix}\right).$$

69. $$\sum_{k=0}^{n}\frac{(a)_k\left(n-a+\frac{1}{2}\right)_k}{(n-k)!\,(\lambda+1)_k}\left(-\frac{4}{z^2}\right)^kL_{3k}^{\lambda-2k}(z)$$
$$=\frac{(a)_n\left(n-a+\frac{1}{2}\right)_n(-\lambda)_{2n}}{(3n)!}\left(-\frac{4}{z^2}\right)^n$$
$$\times\,{}_3F_3\left(\begin{matrix}-3n,\ a-2n,\ \frac{1}{2}-a-n;\ 4z\\ 2a-4n,\ 1-2a-2n,\ \lambda-2n+1\end{matrix}\right).$$

5.10.4. Sums containing $L_{m\pm pk}^{\lambda\pm nk}(z)$ and special functions

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\psi(k+1)\,L_k^\lambda(z)$$
$$=(-1)^n\left[\psi(n+1)\,L_n^{\lambda-n}(z)+\sum_{k=0}^{n-1}\frac{1}{n-k}L_k^{\lambda-n}(z)\right].$$

2. $$\sum_{k=0}^{n}\frac{(1-a)_k}{(\lambda+1)_k}\psi(a-k)L_k^\lambda(z)=\frac{(1-a)_n}{n!\,(\lambda+1)_n(n-a+1)}(-z)^n$$
$$\times\left[(n-a+1)\psi(a-n-1)\,{}_3F_1\left(\begin{matrix}-n,\ a-n-1,\ -\lambda-n\\ a-n;\ -z^{-1}\end{matrix}\right)\right.$$
$$\left.-\,{}_4F_2\left(\begin{matrix}-n,\ a-n-1,\ a-n-1,\ -\lambda-n\\ a-n,\ a-n;\ -z^{-1}\end{matrix}\right)\right].$$

3. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k+1)!\,(\lambda+1)_k}\,\psi(a-k)L_k^{\lambda}(z)$$
$$= \frac{1}{(\lambda+1)_{n+1}}\,\psi(a-n-1)\left[\frac{z^{n+1}}{(n+1)!} + (-1)^n L_{n+1}^{\lambda}(z)\right]$$
$$+ \frac{z^n}{n!\,(\lambda+1)_n\,(a-n-1)}\;{}_4F_2\left(\begin{matrix}-n,\,-\lambda-n,\,1,\,1;\; -\frac{1}{z}\\ a-n,\,2\end{matrix}\right).$$

4. $$\sum_{k=0}^{n} \frac{z^{-k}}{(n-k)!}\,\psi(a-k)L_k^{\lambda-k}(z) = \frac{(-\lambda)_n}{n!}(-z)^{-n}\psi(a-n)$$
$$-(-1)^n\frac{(-\lambda)_{n-1}}{(n-1)!\,(a-n)}z^{1-n}\,{}_3F_3\left(\begin{matrix}1-n,\,1,\,1;\; z\\ a-n+1,\,\lambda-n+2,\,2\end{matrix}\right)\quad [m\ge n].$$

5. $$\sum_{k=0}^{n} \frac{z^{-k}}{(n-k)!}\,\psi(a-2k)L_k^{\lambda-k}(z) = \frac{(-\lambda)_n}{n!}(-z)^{-n}$$
$$\times\left[\psi(a-2n) + \frac{nz}{(a-2n)(\lambda-n+1)}\,{}_3F_3\left(\begin{matrix}1-n,\,1,\,1;\; z\\ \frac{a}{2}-n+1,\,\lambda-n+2,\,2\end{matrix}\right)\right.$$
$$\left. + \frac{nz}{(a-2n+1)(\lambda-n+1)}\,{}_3F_3\left(\begin{matrix}1-n,\,1,\,1;\; z\\ \frac{a+3}{2}-n,\,\lambda-n+2,\,2\end{matrix}\right)\right].$$

6. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!}\,\gamma(a-k,\,z)L_k^{n-k}(-z) = (-1)^n\frac{z^{a-n}}{a-n}\,{}_1F_1\left(\begin{matrix}a-2n;\; -z\\ a-n+1\end{matrix}\right).$$

7. $$\sum_{k=0}^{n} \frac{(-1)^k}{k!}\,\gamma(\lambda+k,\,z)L_{n-k}^{\lambda-n+k}(-z) = \frac{(-z)^n}{n!}(1-\lambda)_n\gamma(\lambda-n,\,z).$$

8. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!}\Gamma(\lambda+n-k,\,z)L_k^{\lambda-k}(-z) = \frac{(1-\lambda)_n}{2n!}z^n\Gamma(\lambda-n,\,z).$$

9. $$\sum_{k=0}^{n} \frac{2^k}{(n-k)!\left(\lambda-2n+\frac{3}{2}\right)_{2k}}\,D_{\lambda-2n+2k+1/2}\left(\sqrt{2z}\right)\,L_k^{\lambda}(z)$$
$$= \frac{(-1)^n}{n!\left(-\lambda-\frac{1}{2}\right)_{2n}}\,D_{\lambda+2n+1/2}\left(\sqrt{2z}\right).$$

10. $$\sum_{k=0}^{n} \frac{\left(w+\sqrt{w^2-1}\right)^k z^k}{k!}\,P_k(w)L_{n-k}^{\lambda+k}(z)$$
$$= \frac{2^{-n}(2n)!}{(n!)^3}\left(\frac{z\sqrt{w^2-1}}{\sqrt{w^2-1}-w}\right)^n\,{}_3F_1\left(\begin{matrix}-n,\,-n,\,-\lambda-n\\ \frac{1}{2}-n;\; \frac{w-\sqrt{w^2-1}}{2z\sqrt{w^2-1}}\end{matrix}\right).$$

11. $$\sum_{k=0}^{n} \frac{2k+1}{(k+n+1)!} z^k P_k(w) L_{n-k}^{2k+1}(z) = \frac{1}{n!} L_n\left(\frac{z-wz}{2}\right).$$

12. $$\sum_{k=0}^{n} \frac{2k+1}{(k+n+1)!} z^k \left[P_k(w)\right]^2 L_{n-k}^{2k+1}(z) = \frac{1}{n!}\, {}_2F_2\left(\begin{matrix} -n,\ \frac{1}{2} \\ 1,\ 1;\ z-w^2 z \end{matrix}\right).$$

13. $$\sum_{k=0}^{n} \frac{2k+1}{(k+n+1)!} z^k T_{2k+1}(w) L_{n-k}^{2k+1}(z) = \frac{(-1)^n w}{(2n)!} H_{2n}\left(\sqrt{z-w^2 z}\right).$$

14. $$\sum_{k=0}^{n} \frac{2k+1}{(k+n+1)!} (-z)^k U_{2k}(w) L_{n-k}^{2k+1}(z) = \frac{(-1)^n}{(2n)!} H_{2n}(w\sqrt{z}).$$

15. $$\sum_{k=0}^{n} (-1)^k \frac{(-\lambda-n)_k}{(2k)!} H_{2k}(w) L_{n-k}^{\lambda}(z) = \frac{(w^2+z)^n}{\left(\lambda+\frac{1}{2}\right)_n} C_{2n}^{\lambda+1/2}\left(\frac{w}{\sqrt{w^2+z}}\right).$$

16. $$\sum_{k=0}^{n} (-1)^k \frac{(-\lambda-n)_k}{(2k+1)!} H_{2k+1}(w) L_{n-k}^{\lambda}(z)$$
$$= \frac{(w^2+z)^{n+1/2}}{\left(\lambda+\frac{1}{2}\right)_{n+1}} C_{2n+1}^{\lambda+1/2}\left(\frac{w}{\sqrt{w^2+z}}\right).$$

17. $$\sum_{k=0}^{n} \frac{(-z)^k}{(k+1)!} P_k^{(\rho-k,\,1)}(3) L_{n-k}^{\lambda+k}(z) = \frac{(\lambda+1)_n}{n!}\, {}_2F_2\left(\begin{matrix} -n,\ \rho+2;\ 2z \\ 2,\ \lambda+1 \end{matrix}\right).$$

5.10.5. Sums containing products of $L_{m\pm pk}^{\lambda\pm nk}(z)$

1. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\lambda+1)_k} [L_k^\lambda(z)]^2 = \frac{(-1)^n}{(\lambda+1)_n} \binom{2n}{n} L_{2n}^{\lambda-n}(z)$$ [[34], (43′)].

2. $$\sum_{k=0}^{2n} (-1)^k \frac{k!\, z^{-2k}}{(2n-k)!} [L_k^{n-k-1/2}(z)]^2 = \frac{(-1)^n \pi z^{-4n}}{2^{2n}\Gamma^2\left(\frac{1}{2}-n\right)} L_n^{-n-1/2}(4z).$$

3. $$\sum_{k=0}^{n} (-1)^k \frac{k!}{(n-k)!} z^{-2k} L_k^{\lambda-k}(z) L_k^{\mu-k}(z)$$
$$= (-1)^n z^{-2n} \frac{(-\lambda)_n(-\mu)_n}{n!}\, {}_3F_3\left(\begin{matrix} -n,\ \frac{\lambda+\mu}{2}-n,\ \frac{\lambda+\mu}{2}-n+1;\ 4z \\ \lambda-n+1,\ \mu-n+1,\ \lambda+\mu-2n+1 \end{matrix}\right).$$

4. $$\sum_{k=0}^{n} \frac{k!}{(n-k)!} z^{-2k} L_k^{\lambda-k}(z) L_k^{\lambda-k}(-z)$$
$$= \frac{(-\lambda)_n^2}{n!} z^{-2n}\, {}_2F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2};\ -z^2 \\ \frac{\lambda-n+1}{2},\ \frac{\lambda-n}{2}+1,\ \lambda-n+1 \end{matrix}\right).$$

5. $$\sum_{k=0}^{n}(\lambda-2k)\frac{k!\,(-\lambda)_k}{(n-k)!\,(n-\lambda+1)_k}(-wz)^{-k}\,L_k^{\lambda-2k}(w)\,L_k^{\lambda-2k}(z) = -(-\lambda)_{n+1}\,(wz)^{-n}L_n^{\lambda-2n}(w+z).$$

6. $$\sum_{k=0}^{n}(\lambda-2k)\frac{(-\lambda)_k^2}{(n-k)!\,(n-\lambda+1)_k}z^{-2k}\left[L_k^{\lambda-2k}(z)\right]^2 = -\frac{(-\lambda)_{n+1}\left(\frac{1}{2}\right)_n}{n!}\left(\frac{2}{z}\right)^{2n}L_{2n}^{\lambda-2n}(z).$$

7. $$\sum_{k=0}^{n}\frac{(-\mu-n)_k}{(\lambda+1)_k}L_k^{\lambda}(w)L_{n-k}^{\mu}(z) = \frac{(-z)^n}{(\lambda+1)_n}P_n^{(\lambda,\,-\lambda-\mu-2n-1)}\left(1+\frac{2w}{z}\right).$$

8. $$\sum_{k=0}^{n}(-\mu-n)_k(-w)^{-k}L_k^{\lambda-k}(w)L_{n-k}^{\mu}(z) = \frac{(-z)^n}{n!}\,{}_3F_0\left(\begin{matrix}-n,\,-\lambda,\,-\mu-n\\ w^{-1}z^{-1}\end{matrix}\right).$$

9. $$\sum_{k=0}^{n}\frac{z^k}{(\lambda+1)_k}L_k^{\lambda}(w)L_{n-k}^{\mu+k}(z) = \frac{(\mu+1)_n}{n!}\,{}_1F_2\left(\begin{matrix}-n;\ wz\\ \lambda+1,\,\mu+1\end{matrix}\right).$$

10. $$\sum_{k=0}^{n}\frac{z^{-k}}{(n-k)!}L_m^{\lambda-k}(-z)L_k^{\mu-k}(z) = \frac{(\lambda-n+1)_m(-\mu)_n}{m!\,n!}(-z)^{-n}\,{}_2F_2\left(\begin{matrix}-m,\,\mu+1;\,-z\\ \lambda-n+1,\,\mu-n+1\end{matrix}\right).$$

11. $$\sum_{k=0}^{n}\frac{(-z)^k}{k!}L_m^{\lambda+k}(z)L_{n-k}^{k-m-1}(-z) = \frac{(-\lambda-m)_n}{n!}L_{m-n}^{\lambda}(z) \qquad [m\ge n].$$

12. $$\sum_{k=0}^{n}\frac{(-z)^k}{k!}L_m^{\lambda+k}(z)L_{n-k}^{\lambda-n+k}(-z) = \frac{(-1)^n}{n!}(-\lambda-m)_nL_m^{\lambda-n}(z) \qquad [m\ge n].$$

13. $$\sum_{k=0}^{n}\frac{(-1)^k}{k!}(-\lambda-m)_kL_m^{\lambda-k}(z)L_{n-k}^{m+k}(z) = \binom{m+n}{m}L_{m+n}^{\lambda}(z).$$

14. $$\sum_{k=0}^{n}\frac{(-1)^k}{k!}(-\lambda-m)_kL_m^{\lambda-k}(z)L_{n-k}^{k-n-\lambda}(z) = \frac{(-z)^n}{n!}L_m^{\lambda+n}(z).$$

15. $$\sum_{k=0}^{n}\frac{(-z)^k}{k!}L_{m-k}^{\lambda+k}(z)L_{n-k}^{\mu+k}(z) = \frac{(\lambda+1)_m(\mu+1)_n}{m!\,n!}e^z\,{}_2F_2\left(\begin{matrix}\lambda+m+1,\,\mu+n+1;\ -z\\ \lambda+1,\,\mu+1\end{matrix}\right) \qquad [m\ge n].$$

16. $$\sum_{k=0}^{n}\frac{(-z)^k}{k!}L_{m-k}^{\lambda+k}(z)L_{n-k}^{\lambda+m+k}(z) = \binom{m+n}{m}L_{m+n}^{\lambda}(z) \qquad [m\ge n].$$

17. $$\sum_{k=0}^{n} \binom{m+k}{k} L_{m+k}^{\lambda-k}(z) L_{n-k}^{k-n-\lambda}(-z) = \frac{(-z)^n}{n!} L_{m-n}^{\lambda+n}(z) \qquad [m \ge n].$$

18. $$\sum_{k=0}^{n} \frac{(-z)^k}{k!} L_{m-k}^{\lambda+k}(z) L_{n-k}^{\lambda-n+k}(z) = \binom{m+n}{m} L_{m+n}^{\lambda-n}(z) \qquad [m \ge n].$$

19. $$\sum_{k=0}^{n} \binom{m+k}{k} L_{m+k}^{\lambda-k}(z) L_{n-k}^{k-m-\lambda-1}(-z) = \frac{(-\lambda-m)_n}{n!} L_{m-n}^{\lambda}(z) \qquad [m \ge n].$$

20. $$\sum_{k=0}^{n} \frac{k!\,(\lambda-2k)}{(n-k)!} \frac{(-\lambda)_k}{(n-\lambda+1)_k} (-wz)^{-k} L_k^{\lambda-2k}(w) L_k^{\lambda-2k}(z)$$
$$= -(-\lambda)_{n+1} (wz)^{-n} L_n^{\lambda-2n}(w+z).$$

21. $$\sum_{k=0}^{n} (2k+\lambda) \frac{(\lambda)_k}{(\lambda+n+1)_k} \left(-\frac{w}{z}\right)^k L_{n-k}^{\lambda+2k}(w) L_k^{-\lambda-2k}(z)$$
$$= \frac{(\lambda)_{n+1}}{n!} \left(1+\frac{w}{z}\right)^n.$$

22. $$\sum_{k=0}^{n} (2k-\lambda) \frac{(k!)^2\,(a+n)_k (-2\lambda)_{2k}}{(n-k)!\,(2k)!\,(1-\lambda+n)_k (1-a-\lambda-n)_k}$$
$$\times\, z^{-2k} L_k^{\lambda-2k}(z) L_k^{\lambda-2k}(-z)$$
$$= \frac{(2a)_{2n}(-\lambda)_{n+1}}{(2n)!\,(a)_n (a+\lambda)_n}\, {}_5F_2\left(\begin{matrix} -n,\, n+a,\, -\frac{\lambda}{2},\, \frac{1-\lambda}{2},\, \frac{1}{2}-\lambda \\ a+\frac{1}{2},\, -\lambda;\ 4z^{-2} \end{matrix}\right).$$

23. $$\sum_{k=0}^{n} (2k+\lambda) \frac{k!\,(\lambda)_k}{(\lambda+n+1)_k} \left(-\frac{z}{w^2}\right)^k L_k^{-\lambda-2k}(-w) L_k^{-\lambda-2k}(w) L_{n-k}^{\lambda+2k}(z)$$
$$= \frac{(\lambda)_{n+1}}{n!}\, {}_3F_1\left(\begin{matrix} -n,\, \frac{\lambda}{2},\, \frac{\lambda+1}{2} \\ \lambda;\ 4w^{-2}z \end{matrix}\right).$$

5.10.6. Sums containing $L_{m\pm pk}^{\lambda\pm nk}(\varphi(k,z))$

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_m^\lambda(w+kz) = 0 \qquad [m < n].$$

2. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_n^\lambda(kz) = z^n.$$

3. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_m^\lambda(kz) = \frac{n!\,(\lambda+1)_m}{m!\,(\lambda+1)_n} z^n \sum_{k=0}^{m-n} \sigma_{k+n}^n \binom{m}{k+n} \frac{(-z)^k}{(\lambda+n+1)_k}.$$

4. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m L_m^{\lambda}\left(\frac{z}{k}\right) = (-1)^{m+1}\frac{z^m}{m!}$$
$$+(-1)^m\frac{n!\,(\lambda+1)_m}{m!\,(\lambda+1)_{m-n}}z^{m-n}\sum_{k=0}^{m-n}\sigma_{k+n}^{n}\binom{m}{k+n}(n-m-\lambda)_k z^{-k}\quad [m\geq n].$$

5. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{m+n}^{\lambda}(w+kz) = n!\,z^n\sum_{k=0}^{m}\sigma_{k+n}^{n}\frac{(-z)^k}{(k+n)!}L_{m-k}^{\lambda+k+n}(w).$$

6. $$\sum_{k=0}^{n}\frac{(ka+b)^{n-k-1}}{(n-k)!}\left(\frac{a}{z}\right)^k L_k^{\lambda-k}(kz) = \frac{1}{na+b}\left(\frac{a}{z}\right)^n L_n^{\lambda-n}\left(-\frac{bz}{a}\right).$$

7. $$\sum_{k=1}^{n}\frac{k^{k-1}}{k!}z^k L_{n-k}^{\lambda+k}(kz) = \frac{(\lambda+2)_{n-1}z}{(n-1)!}\quad [n\geq 1].$$

8. $$\sum_{k=0}^{n}\frac{(ka+1)^{n-k-1}}{(n-k)!}z^{-k}L_k^{\lambda-k}((ka+1)z) = \frac{(-\lambda)_n(-z)^{-n}}{n!\,(na+1)}.$$

9. $$\sum_{k=1}^{n}\frac{k^{2k}}{(k+n)!}z^k L_{n-k}^{2k}(k^2z) = -\frac{1}{2(n!)}+\frac{(-z)^n}{2}L_n^{-n-1}\left(\frac{1}{z}\right).$$

10. $$\sum_{k=1}^{n}\frac{k^{2k+2}}{(k+n)!}z^k L_{n-k}^{2k}(k^2z) = \frac{(-z)^n}{2}\left[L_{n-2}^{-n-2}\left(\frac{1}{z}\right)-L_{n-1}^{-n-3}\left(\frac{1}{z}\right)\right]\quad [n\geq 2].$$

11. $$\sum_{k=1}^{n}\frac{k^{2k-2}}{(k+n)!}z^k L_{n-k}^{2k}(k^2z) = \frac{z}{2(n-1)!}.$$

12. $$\sum_{k=1}^{n}\frac{k^{2k-4}}{(k+n)!}z^k L_{n-k}^{2k}(k^2z) = -\frac{z^2}{8(n-2)!}+\frac{z}{2(n-1)!}.$$

13. $$\sum_{k=1}^{n}\frac{k^{2k-6}}{(k+n)!}z^k L_{n-k}^{2k}(k^2z) = \frac{z^3}{72(n-3)!}-\frac{5z^2}{32(n-2)!}+\frac{z}{2(n-1)!}.$$

14. $$\sum_{k=1}^{n}\frac{k^{2k}}{k^2+a^2}\frac{z^k}{(k+n)!}L_{n-k}^{2k}(k^2z) = \frac{a^{-2}}{2(n!)}\left[{}_2F_2\left(\begin{matrix}-n,\,1;\ -a^2z\\ 1-ia,\,1+ia\end{matrix}\right)-1\right].$$

15. $$\sum_{k=0}^{n}\frac{(2k+1)^{2k-1}}{(k+n+1)!}z^k L_{n-k}^{2k+1}((2k+1)^2z) = \frac{1}{n!}.$$

16. $$\sum_{k=0}^{n}\frac{(2k+1)^{2k-3}}{(k+n+1)!}z^k L_{n-k}^{2k+1}((2k+1)^2z) = \frac{1}{n!}-\frac{4z}{9(n-1)!}.$$

17. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k-5}}{(k+n+1)!} z^k L_{n-k}^{2k+1}((2k+1)^2 z) = \frac{1}{n!} - \frac{40z}{81(n-1)!} + \frac{16z^2}{225(n-2)!}.$$

18. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k+1}}{(2k+1)^2+a^2} \frac{z^k}{(k+n+1)!} L_{n-k}^{2k+1}((2k+1)^2 z)$$
$$= \frac{1}{n!\,(1+a^2)}\, {}_2F_2\left(\begin{matrix}-n,\ 1;\ -a^2 z \\ \frac{3-ia}{2},\ \frac{3+ia}{2}\end{matrix}\right).$$

19. $$\sum_{k=0}^{n} \frac{(2k+\lambda)^{2k-1}(\lambda)_k}{k!\,(\lambda+n+1)_k} z^k L_{n-k}^{\lambda+2k}((2k+\lambda)^2 z) = \frac{(\lambda)_{n+1}}{n!\,a^2}.$$

20. $$\sum_{k=1}^{n} \frac{(-ka)^k (ka+b)^{n-k-1}}{(n-k)!\,(\lambda+1)_k} L_k^\lambda\left(\frac{z}{k}\right) = -\frac{b^{n-1}}{n!}$$
$$+ \frac{b^n}{(\lambda+1)_n(na+b)} L_n^\lambda\left(-\frac{az}{b}\right).$$

21. $$\sum_{k=1}^{n} \frac{k^{n-1}}{k!} (-\lambda-n)_k L_{n-k}^\lambda\left(\frac{z}{k}\right) = (-1)^n \frac{\lambda+n}{(n-1)!} z^{n-1}.$$

22. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\lambda+1)_k} (ka+1)^{n-1} L_k^\lambda\left(\frac{z}{ka+1}\right) = \frac{z^n}{n!\,(na+1)(\lambda+1)_n}.$$

23. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (ka+b)^{m+n} L_{m+n}^\lambda\left(\frac{z}{ka+b}\right)$$
$$= n!\,(\lambda+m+1)_n(-a)^n b^m \sum_{k=0}^{m} \sigma_{k+n}^n \frac{(-\lambda-m)_k}{(k+n)!} \left(-\frac{a}{b}\right)^k L_{m-k}^\lambda\left(\frac{z}{b}\right).$$

5.10.7. Sums containing $L_{m\pm pk}^{\lambda\pm nk}(\varphi(k,z))$ and special functions

1. $$\sum_{k=0}^{n} \frac{(k+1)^{n-1}}{(2n-2k)!\,(\lambda+1)_k} H_{2n-2k}\left(\frac{w}{\sqrt{k+1}}\right) L_k^\lambda\left(\frac{z}{k+1}\right)$$
$$= \frac{\lambda w^{2n+2} z^{-1}}{3(n+1)!\left(\frac{1}{2}\right)_{n+1}} - \frac{2^{2n+2}\lambda z^{-1}(w^2+z)^{n+1}}{(2\lambda-1)_{2n+2}} C_{2n+2}^{\lambda-1/2}\left(\frac{w}{\sqrt{w^2+z}}\right).$$

2. $$\sum_{k=0}^{n} \frac{(k+1)^{n-1/2}}{(2n-2k+1)!\,(\lambda+1)_k} H_{2n-2k+1}\left(\frac{w}{\sqrt{k+1}}\right) L_k^\lambda\left(\frac{z}{k+1}\right)$$
$$= \frac{4\lambda w^{2n+3} z^{-1}}{3(n+1)!\left(\frac{5}{2}\right)_n} - \frac{2^{2n+3}\lambda z^{-1}(w^2+z)^{n+3/2}}{(2\lambda-1)_{2n+3}} C_{2n+3}^{\lambda-1/2}\left(\frac{w}{\sqrt{w^2+z}}\right).$$

3. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(2k)!} z^k H_{2k}\left(\frac{w}{\sqrt{k+1}}\right) L_{n-k}^{\lambda+k}((n-k)z) = \sum_{k=0}^{n} \frac{\left(4w^2 z\right)^k}{(2k)!\,(k+1)} L_{n-k}^{\lambda+k}((n+1)z).$$

4. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1/2}}{(2k+1)!} z^k H_{2k+1}\left(\frac{w}{\sqrt{k+1}}\right) L_{n-k}^{\lambda+k}((n-k)z) = 4w \sum_{k=0}^{n} \frac{\left(4w^2 z\right)^k}{(2k+2)!} L_{n-k}^{\lambda+k}((n+1)z).$$

5. $$\sum_{k=0}^{n-1} \frac{(k+1)^{k-1}}{(2k)!} (n-k)^{n-k} (-\lambda-n)_k H_{2k}\left(\frac{w}{\sqrt{k+1}}\right) L_{n-k}^{\lambda}\left(\frac{z}{n-k}\right)$$
$$= (-1)^{n-1} (n+1)^{n-1} \frac{(\lambda+1)_n}{(2n)!} H_{2n}\left(\frac{w}{\sqrt{n+1}}\right) + (n+1)^n \sum_{k=0}^{n} \left(\frac{4w^2}{n+1}\right)^k \frac{(-\lambda-n)_k}{(2k)!\,(k+1)} L_{n-k}^{\lambda}\left(\frac{z}{n+1}\right).$$

6. $$\sum_{k=0}^{n-1} \frac{(k+1)^{k-1/2}}{(2k+1)!} (n-k)^{n-k} (-\lambda-n)_k H_{2k+1}\left(\frac{w}{\sqrt{k+1}}\right) L_{n-k}^{\lambda}\left(\frac{z}{n-k}\right)$$
$$= (-1)^{n-1} (n+1)^{n-1/2} \frac{(\lambda+1)_n}{(2n+1)!} H_{2n+1}\left(\frac{w}{\sqrt{n+1}}\right) + 4w(n+1)^n \sum_{k=0}^{n} \left(\frac{4w^2}{n+1}\right)^k \frac{(-\lambda-n)_k}{(2k+2)!} L_{n-k}^{\lambda}\left(\frac{z}{n+1}\right).$$

5.10.8. Sums containing products of $L_{m\pm pk}^{\lambda\pm nk}(\varphi(k,\,z))$

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[L_m^{\lambda}(w+kz)\right]^2 = 0 \qquad [2m < n].$$

2. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_m^{\lambda}\left(\sqrt{k}\,z\right) L_m^{\lambda}\left(-\sqrt{k}\,z\right) = \frac{z^{2m}}{m!} \delta_{m,n} \qquad [n \geq m].$$

3. $$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^m L_m^{\lambda}\left(\frac{z}{\sqrt{k}}\right) L_m^{\lambda}\left(-\frac{z}{\sqrt{k}}\right) = (-1)^m \frac{(\lambda+1)_m^2}{m!} \delta_{m,n} - (-1)^m \frac{z^{2m}}{(m!)^2} \qquad [n \geq m].$$

4. $$\sum_{k=0}^{n} (k+1)^{n-k-1}(-\mu-n)_k(-w)^{-k} L_k^{\lambda-k}((k+1)w)\, L_{n-k}^{\mu}\left(\frac{z}{k+1}\right) = \frac{w(-z)^{n+1}}{(n+1)!\,(\lambda+1)(\mu+n+1)}\left[{}_3F_0\left(\begin{matrix}-n-1,\,-\lambda-1,\,-\mu-n-1\\ w^{-1}z^{-1}\end{matrix}\right)-1\right].$$

5. $$\sum_{k=0}^{n} (k+1)^{n-1}\frac{(-\lambda-n)_k}{(\mu+1)_k} L_{n-k}^{\lambda}\left(\frac{w}{k+1}\right) L_k^{\mu}\left(\frac{z}{k+1}\right) = \frac{(-1)^n\mu w^{n+1}z^{-1}}{(n+1)!\,(\lambda+n+1)} + \frac{z^n}{(\mu+1)_n(\lambda+n+1)} P_{n+1}^{(\lambda,\,-\lambda-\mu-2n-2)}\left(1+\frac{2w}{z}\right).$$

6. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(\lambda+1)_k}(-z)^k L_k^{\lambda}\left(\frac{w}{k+1}\right) L_{n-k}^{\mu+k}((n-k)z) = \sum_{k=0}^{n}\frac{(wz)^k}{(k+1)!\,(\lambda+1)_k} L_{n-k}^{\mu+k}((n+1)z).$$

7. $$\sum_{k=0}^{n} \frac{\left(\frac{z}{w}\right)^k}{k+1} L_k^{\lambda-k}((k+1)w)\, L_{n-k}^{\mu+k}((n-k)z) = \sum_{k=0}^{n}\frac{(-\lambda)_k}{(k+1)!}\left(-\frac{z}{w}\right)^k L_{n-k}^{\mu+k}((n+1)z).$$

5.11. The Gegenbauer Polynomials $C_n^\lambda(z)$

5.11.1. Sums containing $C_m^{\lambda\pm nk}(z)$

1. $$\sum_{k=0}^{n} \frac{(-n)_k(n)_k}{k!\,(k+m)!} C_m^{-k-m}(z) = (-1)^m\frac{(-m)_{2n}}{m!\,(2n)!}(2z)^{m-2n}\,{}_2F_1\left(\begin{matrix}n-\frac{m}{2},\,n+\frac{1-m}{2}\\ 2n+1;\; z^{-2}\end{matrix}\right).$$

2. $$\sum_{k=0}^{n} (-1)^k\binom{n}{k}\frac{(a)_k}{(1-\lambda)_k} C_m^{\lambda-k}(z) = \frac{(2z)^m(\lambda)_m(1-\lambda-a-m)_n}{m!\,(1-\lambda-m)_n}\,{}_3F_2\left(\begin{matrix}-\frac{m}{2},\,\frac{1-m}{2},\,1-\lambda-a-m+n;\; z^{-2}\\ 1-\lambda-m+n,\,1-\lambda-a-m\end{matrix}\right).$$

3. $$\sum_{k=0}^{n} \frac{(\lambda)_k}{(\lambda+m)_k} C_{2m}^{\lambda+k}(z) = (-1)^{m+1}z^{-2}\frac{(\lambda)_m}{\left(\frac{1}{2}\right)_m} \times\left[P_{m+1}^{(-3/2,\,\lambda+n-1/2)}(1-2z^2) - P_{m+1}^{(-3/2,\,\lambda-3/2)}(1-2z^2)\right].$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda)_k}{(a)_k}C_{2m}^{\lambda+k}(z)$$
$$=(-1)^m\frac{(\lambda)_m(a-\lambda-m)_n}{m!\,(a)_n}\,{}_3F_2\left(\begin{matrix}-m,\ \lambda+m,\ \lambda-a+m+1\\ \lambda-a+m-n+1,\ \frac{1}{2};\ z^2\end{matrix}\right).$$

5. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda)_k}{(\lambda+m+1)_k}C_{2m+1}^{\lambda+k}(z)$$
$$=(-1)^n\frac{2(\lambda)_{m+1}}{\left(\frac{3}{2}\right)_m}z^{2n+1}P_{m-n}^{(\lambda+n-1/2,\,n+1/2)}(2z^2-1)\quad[m\geq n].$$

6. $$\sum_{k=0}^{n}\frac{(\lambda)_k}{(\lambda+m+1)_k}C_{2m+1}^{\lambda+k}(z)$$
$$=z^{-1}\left[\frac{(\lambda)_{m+1}}{(\lambda+n)_{m+1}}C_{2m+2}^{\lambda+n}(z)-\frac{\lambda+m}{\lambda-1}C_{2m+2}^{\lambda-1}(z)\right].$$

7. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda)_k}{(a)_k}C_{2m+1}^{\lambda+k}(z)$$
$$=(-1)^m2z\frac{(\lambda)_{m+1}(a-\lambda-m-1)_n}{m!\,(a)_n}\,{}_3F_2\left(\begin{matrix}-m,\ \lambda+m+1,\ \lambda-a+m+2\\ \lambda-a+m-n+2,\ \frac{3}{2};\ z^2\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda)_k}{\left(\lambda+m+\frac{1}{2}\right)_k}C_{2m+1}^{\lambda+k}(z)$$
$$=2^{1-2m}\lambda z\frac{\left(-m-\frac{1}{2}\right)_n(2\lambda+1)_{2m}}{\left(\frac{3}{2}\right)_m\left(\lambda+\frac{1}{2}\right)_{m+n}}P_m^{(\lambda+n-1/2,\,1/2-n)}(2z^2-1).$$

9. $$\sum_{k=0}^{n}\binom{n}{k}k^r\frac{\left(\frac{1}{2}-\lambda-m\right)_k}{}(1-\lambda)_k(z^2-1)^{-k}C_{2m+1}^{\lambda-k}(z)$$
$$=\frac{2^{2m+1}(\lambda)_{m+1}}{(2m+1)!\,(-\lambda-m)_n}$$
$$\times z(1-z^2)^m\sum_{k=1}^{r}\sigma_r^k(m+n-k)!\,(-n)_k\left(\frac{1}{2}-\lambda-m\right)_k(1-z^2)^{-k}$$
$$\times P_{m+n-k}^{(-\lambda+k-2m-1,\,k-n+1/2)}\left(\frac{z^2+1}{z^2-1}\right)\quad[m+n\geq r].$$

10. $$\sum_{k=0}^{n}\binom{n}{k}k^r\frac{(\lambda)_k}{\left(\lambda+m+\frac{1}{2}\right)_k}\left(z^2-1\right)^k C_{2m+1}^{\lambda+k}(z)=\frac{(2\lambda)_{2m+1}z}{(2m+1)!\left(\lambda+\frac{1}{2}\right)_{m+n}}$$
$$\times\sum_{k=1}^{r}\sigma_r^k(m+n-k)!\,(-n)_k(\lambda+m+1)_k(1-z^2)^k$$
$$\times P_{m+n-k}^{(\lambda+k-1/2,\,k-n+1/2)}(2z^2-1)\quad [m+n\geq r].$$

5.11.2. Sums containing $C_{m\pm pk}^{\lambda}(z)$

1. $$\sum_{k=0}^{n}C_k^\lambda(z)=\frac{(2\lambda)_{2n}}{n!\left(\lambda+\frac{1}{2}\right)_n}\left(\frac{z-1}{2}\right)^n{}_3F_2\left(\begin{matrix}-n,\ -n-\lambda,\ \frac{1}{2}-n-\lambda;\ \frac{2}{1-z}\\ 1-n-\lambda,\ -2n-2\lambda\end{matrix}\right).$$

2. $$\sum_{k=0}^{n}(-1)^k\frac{(2k+\lambda)\left(\frac{1}{2}\right)_k}{(n-k)!\left(\lambda+\frac{1}{2}\right)_k(\lambda+n+1)_k}C_{2k}^\lambda(z)=\frac{(\lambda)_{n+1}}{n!\left(\lambda+\frac{1}{2}\right)_n}\left(1-z^2\right)^n$$
$$[n\geq 1].$$

3. $$\sum_{k=0}^{n}(-1)^k\frac{(2k+\lambda)\,(n)_k}{(n-k)!\,(\lambda-n+1)_k(\lambda+n+1)_k}C_{2k}^\lambda(z)=\frac{(\lambda)_{n+1}}{n!\,(-\lambda)_n}T_{2n}(z).$$

4. $$\sum_{k=0}^{n}(-1)^k\frac{(2k+\lambda+1)\left(\frac{3}{2}\right)_k}{(n-k)!\left(\lambda+\frac{1}{2}\right)_k(\lambda+n+2)_k}C_{2k+1}^\lambda(z)$$
$$=2\lambda z\frac{(\lambda+1)_{n+1}}{n!\left(\lambda+\frac{1}{2}\right)_n}\left(1-z^2\right)^n.$$

5. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(m-2k)!\,(m+\lambda-2k)\frac{(-m-\lambda)_k(1-m-2\lambda)_{2k}}{(n-m-\lambda+1)_k}C_{m-2k}^\lambda(z)$$
$$=-2^{2n}(m-2n)!\,(\lambda)_n(-m-\lambda)_{n+1}(1-z^2)^nC_{m-2n}^{\lambda+n}(z)\quad [m\geq 2n].$$

6. $$\sum_{k=0}^{[n/2]}(-1)^k\binom{m}{k}(\lambda+n-2k)\frac{(n-2k)!\,(-\lambda-n)_k(1-2\lambda-n)_{2k}}{(m-n-\lambda+1)_k}C_{n-2k}^\lambda(z)$$
$$=-2^{2m}(n-2m)!\,(\lambda)_m(-\lambda-n)_{m+1}(1-z^2)^mC_{n-2m}^{\lambda+m}(z)\quad [2m\leq n].$$

5.11.3. Sums containing $C_{m\pm pk}^{\lambda\pm nk}(z)$

1. $$\sum_{k=0}^{n-1}\frac{2^{-k}(n-k)^n}{(n-k)!\,(1-\lambda)_k}\left(z-1\right)^{-k}C_k^{\lambda-k}(z)$$
$$=\frac{\left(\frac{1}{2}-\lambda\right)_n}{n!\,(1-2\lambda)_n}\left(\frac{2}{1-z}\right)^n\sum_{k=0}^{n}\sigma_n^k\frac{(-n)_k(2\lambda-n)_k}{\left(\lambda-n+\frac{1}{2}\right)_k}\left(\frac{z-1}{2}\right)^k.$$

2. $$\sum_{k=1}^{n} \frac{k\,\Gamma(2n-k)}{(n-k)!} \frac{(1-2\lambda)_k}{(1-\lambda)_k} \left(\frac{z}{1-z^2}\right)^k C_k^{\lambda-k}(z) = \left(\frac{1}{2}-\lambda\right)_n \left(\frac{4z^2}{z^2-1}\right)^n \qquad [n \ge 1].$$

3. $$\sum_{k=0}^{n} \frac{(a)_k}{(1-\lambda)_k} 2^{-k}(1-z)^{-k} C_k^{\lambda-k}(z) = \frac{(a+1)_n}{n!}\, {}_3F_2\left(\begin{matrix} -n,\, a,\, \frac{1}{2}-\lambda;\ \frac{2}{1-z} \\ a+1,\, 1-2\lambda \end{matrix}\right).$$

4. $$\sum_{k=0}^{n} \frac{k!\,2^{-k}}{(1-\lambda)_k} (1-z)^{-k} C_k^{\lambda-k}(z) = \frac{2\lambda(1-z)}{2\lambda+1} \left[1 - \frac{(n+1)!}{(-2\lambda)_{n+1}} P_{n+1}^{(-2\lambda-1,\,\lambda-n-3/2)}\left(\frac{z+3}{z-1}\right)\right].$$

5. $$\sum_{k=0}^{n} \sigma_{k+m}^{m} \frac{(\lambda)_k}{(k+m)!} t^k C_{n-k}^{\lambda+k}(z) = \frac{t^{-m}}{m!\,(1-\lambda)_m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} C_{m+n}^{\lambda-m}\left(\frac{kt}{2}+z\right).$$

6. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(\lambda)_k}{(2\lambda+m)_k} 2^k (z+1)^k C_{m-k}^{\lambda+k}(z) = \left(\frac{z+1}{2}\right)^m \frac{(2\lambda)_m}{\left(\lambda+\frac{1}{2}\right)_m} P_m^{(\lambda+n-1/2,\,-2\lambda-2m-n)}\left(\frac{3-z}{1+z}\right).$$

7. $$\sum_{k=0}^{n} \frac{\sigma_m^{n-k+1}}{(1-\lambda)_k} C_{2k}^{\lambda-k}(z) = (-1)^n \frac{(\lambda)_n}{(2n)!} (2z)^{2n}\, {}_{m+1}F_m\left(\begin{matrix} -n,\, \frac{1}{2}-n,\, 2,\, \dots,\, 2;\ z^{-2} \\ 1-\lambda-n,\, 1,\, \dots,\, 1 \end{matrix}\right).$$

8. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k}{(1-\lambda)_k} C_{2k}^{\lambda-k}(z) = -\frac{\left(\frac{3}{2}\right)_n}{2z(1-\lambda)_{n+1}} C_{2n+1}^{\lambda-n-1}(z).$$

9. $$\sum_{k=0}^{n-1} (-1)^k \frac{(n-k)^n}{(n-k)!\,(1-\lambda)_k} C_{2k}^{\lambda-k}(z) = \frac{(\lambda)_n}{n!\left(\frac{1}{2}\right)_n} z^{2n} \sum_{k=0}^{n} (-1)^k \sigma_n^k \frac{(-n)_k \left(\frac{1}{2}-n\right)_k}{(1-\lambda-n)_k} z^{-2k} \qquad [n \ge 1].$$

10. $$\sum_{k=0}^{n} \frac{t^k}{(n-k)!\,(1-\lambda)_k} C_{2k}^{\lambda-k}(z) = \frac{(t+1)^n}{(1-\lambda)_n} C_{2n}^{\lambda-n}\left(z\sqrt{\frac{t}{t+1}}\right).$$

11. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{k}^{-1}\frac{1}{(n-k)!\,(1-\lambda)_k}C_{2k}^{\lambda-k}(z)$$
$$=\frac{n!}{(2n)!}\,{}_3F_2\left(\begin{matrix}-n,\,-n,\,\lambda\\ \frac{1}{2},\,1;\ z^2\end{matrix}\right).$$

12. $$\sum_{k=0}^{n}(-1)^k\frac{(a)_k}{(n-k)!\,(b)_k(1-\lambda)_k}C_{2k}^{\lambda-k}(z)$$
$$=\frac{(b-a)_n}{n!\,(b)_n}\,{}_3F_2\left(\begin{matrix}-n,\,a,\,\lambda;\ z^2\\ a-b-n+1,\ \frac{1}{2}\end{matrix}\right).$$

13. $$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)_k}{(1-\lambda)_k}\left(1-z^2\right)^{-k}C_{2k}^{\lambda-k}(z)=-\frac{\left(\frac{3}{2}\right)_n}{2(-\lambda)_{n+1}z}\left(1-z^2\right)^{-n}C_{2n+1}^{\lambda-n}(z).$$

14. $$\sum_{k=0}^{n}\frac{1}{(n-k)!\,(1-\lambda)_k}\left(z^2-1\right)^{-k}C_{2k}^{\lambda-k}(z)=\frac{\left(\frac{1}{2}-\lambda\right)_n}{n!\left(\frac{1}{2}\right)_n}\left(\frac{z^2}{z^2-1}\right)^n.$$

15. $$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k(1-\lambda-n)_k}z^{-2k}C_{2k}^{\lambda-k}(z)$$
$$=\frac{\left(\frac{1}{2}\right)_n}{n!\,(\lambda)_n}z^{-2n}\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,\lambda,\,\frac{1}{2}-\lambda\\ \frac{1}{4},\,\frac{1}{2},\,\frac{3}{4};\ z^4\end{matrix}\right).$$

16. $$\sum_{k=0}^{n}(-1)^k\frac{\left(\frac{1}{2}\right)_k}{(n-k)!\,(a)_k(1-\lambda)_k}C_{2k}^{\lambda-k}(z)$$
$$=\frac{1}{(a)_n}P_n^{(\lambda+a-3/2,\,-a-n+1/2)}(2z^2-1).$$

17. $$\sum_{k=0}^{n}\binom{n}{k}\frac{2^{2k}\left(\frac{1}{2}-\lambda\right)_k\left(\frac{1}{2}\right)_k}{(1-2\lambda)_{2k}\,(n-k+1)}\left(z^2-1\right)^{-k}C_{2k}^{\lambda-k}(z)=(-1)^n\frac{\left(z^2-1\right)^{-n-1}}{n+1}$$
$$\times\left[\frac{\left(\frac{3}{2}\right)_n}{2(1-\lambda)_{n+1}}C_{2n+2}^{\lambda-n-1}(z)+(-1)^nP_{n+1}^{(\lambda-n-3/2,\,-n-3/2)}(2z^2-1)\right].$$

18. $$\sum_{k=0}^{n}(-1)^k\frac{(1-a-n)_k}{(n-k)!\,(1-\lambda)_{2k}}z^{-2k}C_{2k}^{\lambda-2k}(z)$$
$$=\frac{(a)_n}{n!\,(1-\lambda)_n}z^{-2n}\,{}_4F_3\left(\begin{matrix}-n,\,\lambda-n,\,\frac{2a-1}{4},\,\frac{2a+1}{4}\\ a,\,a-\frac{1}{2},\,\frac{1}{2};\ 4z^2\end{matrix}\right).$$

19. $$\sum_{k=0}^{n} \frac{4k-\lambda+1}{(4k-2\lambda-1)(4k-2\lambda+3)} \frac{\left(\frac{1}{2}-\lambda\right)_k \left(\frac{1}{2}\right)_k}{(n-k)!\left(n-\lambda+\frac{3}{2}\right)_k (1-\lambda)_{2k}}$$
$$\times \left(1-z^2\right)^{-k} C_{2k}^{\lambda-2k}(z) = -\frac{2(1/2-\lambda)_{n+1}}{n!\,(4n-2\lambda-1)(4n-2\lambda+3)(1-\lambda)_n} \left(1-z^2\right)^{-n}$$
$$\times\, {}_3F_2\left(\begin{matrix} -n,\ \lambda-n,\ 1;\ 1-z^2 \\ \frac{2\lambda+1}{4}-n,\ \frac{2\lambda+5}{4}-n \end{matrix}\right).$$

20. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\lambda)_k}{\left(\frac{1}{2}-m\right)_k} C_{2m-2k}^{\lambda+k}(z)$$
$$= \frac{(\lambda)_m}{\left(\frac{1}{2}\right)_m} P_m^{(n+\lambda-1/2,\,-n-1/2)}(2z^2-1) \quad [m \geq n].$$

21. $$\sum_{k=0}^{n} \binom{n}{k} k^r \frac{(\lambda)_k}{\left(\frac{1}{2}-m\right)_k} \left(z^2-1\right)^k C_{2m-2k}^{\lambda+k}(z)$$
$$= \frac{(\lambda)_m}{\left(\frac{1}{2}\right)_m} \sum_{k=1}^{r} \sigma_r^k (-n)_k \left(z^2-1\right)^k P_{m-k}^{(\lambda+k-1/2,\,k-n-1/2)}(2z^2-1)$$
$$[m \geq n;\ m \geq r].$$

22. $$\sum_{k=1}^{n} \binom{2n}{n-k} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+m)!\left(\frac{1}{2}-m\right)_k} \left(z^2-1\right)^k C_{2m-2k}^{2k+1/2}(z)$$
$$= \frac{1}{2}\binom{2n}{n}\left[\frac{n!}{(m+n)!} P_m^{(n,\,-n-1/2)}(2z^2-1) - \frac{1}{m!} P_{2m}(z)\right] \quad [m \geq n].$$

23. $$\sum_{k=0}^{n} \sigma_{k+m}^m \frac{(\lambda)_k}{(k+m)!} t^k C_{2n-2k}^{\lambda+k}(z)$$
$$= \frac{t^{-m}}{m!\,(1-\lambda)_m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (1-kt)^{m+n} C_{2m+2n}^{\lambda-m}\left(\frac{z}{\sqrt{1-kt}}\right).$$

24. $$\sum_{k=0}^{m} \binom{m+n-2k}{m-k} \frac{2^{2k}(\lambda)_k^2}{k!\,(2\lambda)_k} \left(1-z^2\right)^k C_{m+n-2k}^{\lambda+k}(z)$$
$$= \frac{(m+2\lambda)_n}{(2\lambda)_n} C_m^{\lambda}(z)\, C_n^{\lambda}(z) \quad [n \geq m].$$

25. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k}{(1-\lambda)_k} C_{2k+2m}^{\lambda-k}(z)$$
$$= (-1)^m \frac{(m-a+1)_n\,(\lambda)_m}{(m+n)!}\, {}_3F_2\left(\begin{matrix} -m-n,\ a-m,\ \lambda+m \\ a-m-n,\ \frac{1}{2};\ z^2 \end{matrix}\right).$$

26. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\left(m+\frac{1}{2}\right)_k}{(1-\lambda)_k}C_{2k+2m}^{\lambda-k}(z) = (-1)^{m+n}\frac{(\lambda)_m}{\left(\frac{1}{2}\right)_m}P_{m+n}^{(-n-1/2,\,\lambda-1/2)}(1-2z^2).$$

27. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda+2m+n)_k}{(1-\lambda)_k}C_{2k+2m}^{\lambda-k}(z) = C_{2m+2n}^{\lambda}(z).$$

28. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{(1-\lambda)_k}(z^2-1)^{-k}C_{2k+2m}^{\lambda-k}(z) = \frac{(1-a+m)_n(\lambda)_m}{(m+n)!}(z^2-1)^m\,{}_3F_2\left(\begin{matrix}-m-n,\ a-m,\ \frac{1}{2}-\lambda-m\\ a-m-n,\ \frac{1}{2};\ \frac{z^2}{z^2-1}\end{matrix}\right).$$

29. $$\sum_{k=0}^{n}\frac{\left(\frac{3}{2}\right)_k}{(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z) = 2\lambda z P_n^{(3/2,\,\lambda-n-3/2)}(1-2z^2).$$

30. $$\sum_{k=0}^{n}\frac{\sigma_m^{n-k+1}}{(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z) = (-1)^n\frac{(\lambda)_{n+1}}{(2n+1)!}(2z)^{2n+1}\,{}_{m+1}F_m\left(\begin{matrix}-n,\ -\frac{1}{2}-n,\ 2,\ \dots,\ 2\\ -\lambda-n,\ 1,\ \dots,\ 1;\ z^{-2}\end{matrix}\right).$$

31. $$\sum_{k=0}^{n}\frac{t^k}{(n-k)!\,(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z) = \frac{t^{-1/2}(t+1)^{n+1/2}}{(1-\lambda)_n}C_{2n+1}^{\lambda-n}\left(z\sqrt{\frac{t}{t+1}}\right).$$

32. $$\sum_{k=0}^{n}\frac{1}{(n-k)!\,(1-\lambda)_k}\left(z^2-1\right)^{-k}C_{2k+1}^{\lambda-k}(z) = \frac{2\lambda z\left(\frac{1}{2}-\lambda\right)_n}{n!\left(\frac{3}{2}\right)_n}\left(\frac{z^2}{z^2-1}\right)^n.$$

33. $$\sum_{k=0}^{n}\frac{\left(\frac{3}{2}\right)_k}{(1-\lambda)_k}\left(1-z^2\right)^{-k}C_{2k+1}^{\lambda-k}(z) = (-1)^n 2\lambda z\left(1-z^2\right)^{-n}P_n^{(\lambda-n-1/2,\,3/2)}(2z^2-1).$$

34. $$\sum_{k=0}^{n-1}(-1)^k\frac{(n-k)^n}{(n-k)!\,(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z) = \frac{2(\lambda)_{n+1}}{n!\left(\frac{3}{2}\right)_n}z^{2n+1}\sum_{k=0}^{n}(-1)^k\sigma_n^k\frac{(-n)_k\left(-\frac{1}{2}-n\right)_k}{(-\lambda-n)_k}z^{-2k} \quad [n\ge 1].$$

35. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{k}^{-1}\frac{1}{(n-k)!\,(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z)$$
$$=2\lambda z\frac{n!}{(2n)!}\,{}_3F_2\left(\begin{matrix}-n,\,-n,\,\lambda+1\\ 1,\,\frac{3}{2};\;z^2\end{matrix}\right).$$

36. $$\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\frac{3}{2}\right)_k}{(1-\lambda)_k(n-k+1)}\left(z^2-1\right)^{-k}C_{2k+1}^{\lambda-k}(z)=\frac{2\lambda}{n+1}\left(z^2-1\right)^{-n-1}$$
$$\times\left[\frac{3\left(\frac{5}{2}\right)_n}{4(-\lambda)_{n+2}}C_{2n+3}^{\lambda-n-1}(z)-(-1)^n z P_{n+1}^{(\lambda-n-3/2,\,-n-1/2)}(2z^2-1)\right].$$

37. $$\sum_{k=0}^{n}(-1)^k\frac{(a)_k}{(n-k)!\,(b)_k(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z)$$
$$=2\lambda z\frac{(b-a)_n}{n!\,(b)_n}\,{}_3F_2\left(\begin{matrix}-n,\,a,\,\lambda+1;\;z^2\\ a-b-n+1,\,\frac{3}{2}\end{matrix}\right).$$

38. $$\sum_{k=0}^{n}(-1)^k\frac{\left(\frac{3}{2}\right)_k}{(n-k)!\,(a)_k(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z)$$
$$=\frac{2\lambda z}{(a)_n}P_n^{(\lambda+a-3/2,\,-a-n+3/2)}(2z^2-1).$$

39. $$\sum_{k=0}^{n}\frac{k!}{(1-\lambda)_k}C_{2k+1}^{\lambda-k}(z)=z^{-1}\left[1-\frac{(n+1)!}{(1-\lambda)_{n+1}}C_{2n+2}^{\lambda-n-1}(z)\right].$$

40. $$\sum_{k=0}^{n}\frac{\left(-\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k(-\lambda-n)_k}z^{-2k}C_{2k+1}^{\lambda-k}(z)$$
$$=2\lambda\frac{\left(\frac{3}{2}\right)_n}{n!\,(\lambda+1)_n}z^{1-2n}\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,\lambda+1,\,\frac{1}{2}-\lambda\\ \frac{3}{4},\,\frac{5}{4},\,\frac{3}{2};\;z^4\end{matrix}\right).$$

41. $$\sum_{k=0}^{n}\frac{4k-\lambda+1}{(4k-2\lambda-1)(4k-2\lambda+3)}\frac{\left(\frac{1}{2}-\lambda\right)_k\left(\frac{3}{2}\right)_k}{(n-k)!\left(n-\lambda+\frac{3}{2}\right)_k(-\lambda)_{2k+1}}$$
$$\times\left(1-z^2\right)^{-k}C_{2k+1}^{\lambda-2k}(z)=-\frac{4z\left(\frac{1}{2}-\lambda\right)_{n+1}}{n!\,(4n-2\lambda-1)(4n-2\lambda+3)(-\lambda)_n}\left(1-z^2\right)^{-n}$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,\,1,\,\lambda-n+1;\;1-z^2\\ \frac{2\lambda+1}{4}-n,\,\frac{2\lambda+5}{4}-n\end{matrix}\right).$$

42. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(1-\lambda)_k}C_{2k+2m+1}^{\lambda-k}(z) = 2(-1)^m\frac{(m-a+1)_n\,(\lambda)_{m+1}}{(m+n)!}\,z\,{}_3F_2\left(\begin{matrix}-m-n,\; a-m,\; \lambda+m+1\\ a-m-n,\; \frac{3}{2};\; z^2\end{matrix}\right).$$

43. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda+2m+n+1)_k}{(1-\lambda)_k}C_{2k+2m+1}^{\lambda-k}(z) = C_{2m+2n+1}^{\lambda}(z).$$

44. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\left(m+\frac{3}{2}\right)_k}{(1-\lambda)_k}C_{2k+2m+1}^{\lambda-k}(z) = 2(-1)^{m+n}\frac{(\lambda)_{m+1}}{\left(\frac{3}{2}\right)_m}\,z\,P_{m+n}^{(1/2-n,\,\lambda-1/2)}(1-2z^2).$$

45. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(2m+2k)!}{(1-\lambda)_k(\lambda+2m-n+1)_k}2^{-2k}(z^2-1)^{-k}C_{2k+2m+1}^{\lambda-k}(z) = \frac{(2m)!\,(\lambda)_m(1/2-\lambda-m)_n}{(\lambda-n)_m(-\lambda-2m)_n}(1-z^2)^{-n}C_{2m}^{\lambda-n}(z).$$

46. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(2m+2k+1)!}{(1-\lambda)_k(\lambda+2m-n+2)_k}2^{-2k}(z^2-1)^{-k}C_{2k+2m+1}^{\lambda-k}(z) = \frac{(2m+1)!\,(\lambda)_{m+1}\left(\frac{1}{2}-\lambda-m\right)_n}{(\lambda-n)_{m+1}(-\lambda-2m-1)_n}(1-z^2)^{-n}C_{2m+1}^{\lambda-n}(z).$$

47. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{(1-\lambda)_k}(z^2-1)^{-k}C_{2k+2m+1}^{\lambda-k}(z) = \frac{(1-a+m)_n(\lambda)_{m+1}}{(m+n)!}2z(z^2-1)^m\,{}_3F_2\left(\begin{matrix}-m-n,\; a-m,\; \frac{1}{2}-\lambda-m\\ a-m-n,\; \frac{3}{2};\; \frac{z^2}{z^2-1}\end{matrix}\right).$$

48. $$\sum_{k=0}^{n}\sigma_{k+m}^{m}\frac{(\lambda)_k}{(k+m)!}t^kC_{2n-2k+1}^{\lambda+k}(z) = \frac{t^{-m}}{m!\,(1-\lambda)_m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(1-kt)^{m+n+1/2}C_{2m+2n+1}^{\lambda-m}\left(\frac{z}{\sqrt{1-kt}}\right).$$

49. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda)_k}{\left(-m-\frac{1}{2}\right)_k}C_{2m-2k+1}^{\lambda+k}(z) = 2z\frac{(\lambda)_{m+1}}{\left(\frac{3}{2}\right)_m}P_m^{(n+\lambda-1/2,\,1/2-n)}(2z^2-1)\quad [m\ge n].$$

50. $$\sum_{k=0}^{n}\binom{n}{k}k^r\frac{(\lambda)_k}{\left(-m-\frac{1}{2}\right)_k}\left(z^2-1\right)^k C_{2m-2k+1}^{\lambda+k}(z)$$
$$=2z\frac{(\lambda)_{m+1}}{\left(\frac{3}{2}\right)_m}\sum_{k=1}^{r}\sigma_r^k(-n)_k\left(z^2-1\right)^k P_{m-k}^{(\lambda+k-1/2,\,k-n+1/2)}\left(2z^2-1\right)$$
$$[m\ge n;\ m\ge r].$$

51. $$\sum_{k=1}^{n}\binom{2n}{n-k}\frac{\left(\frac{1}{2}\right)_{2k}}{(k+m)!\left(-m-\frac{1}{2}\right)_k}\left(z^2-1\right)^k C_{2m-2k+1}^{2k+1/2}(z)$$
$$=\frac{1}{2}\binom{2n}{n}\left[\frac{n!\,z}{(m+n)!}P_m^{(n,\,1/2-n)}\left(2z^2-1\right)-\frac{1}{m!}P_{2m+1}(z)\right]\quad [m\ge n].$$

52. $$\sum_{k=0}^{[n/2]}(n-2k+\lambda)\frac{(a-\lambda)_k(-n-\lambda)_k}{k!\,(1-a-n)_k}C_{n-2k}^{\lambda}(z)=\frac{(\lambda)_{n+1}}{(a)_n}C_n^a(z).$$

53. $$\sum_{k=0}^{[n/3]}\frac{\lambda+n-3k}{k!}\left(-\frac{2\lambda+2n}{3}\right)_k C_{n-3k}^{\lambda}(z)=3^n\lambda P_n^{\left(\frac{2\lambda-n}{3},\,\frac{\lambda-2n}{3}\right)}\left(\frac{4z-1}{3}\right).$$

54. $$\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k)!}{k!}(\lambda)_{2k}\left(\frac{1-z^2}{2}\right)^{2k}C_{2n-4k}^{\lambda+2k}(z)$$
$$=\frac{(2\lambda)_{2n}}{\left(\frac{1}{2}\right)_n}\left(\frac{z}{2}\right)^{2n}{}_3F_2\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,-n+\frac{1}{2}\\ -2n-1,\,\lambda+\frac{1}{2};\ 2-2z^{-2}\end{matrix}\right).$$

55. $$\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k)!}{k!}(\lambda)_{2k}\left(\frac{1-z^2}{2}\right)^{2k}C_{2n-4k+1}^{\lambda+2k}(z)$$
$$=\frac{2^{-2n}(2\lambda)_{2n+1}}{\left(\frac{3}{2}\right)_n}z^{2n+1}{}_3F_2\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,-n-\frac{1}{2}\\ -2n-1,\,\lambda+\frac{1}{2};\ 2-2z^{-2}\end{matrix}\right).$$

56. $$\sum_{k=0}^{[n/2]}\frac{(8k+2\lambda-1)(a)_k\left(\frac{2\lambda-1}{4}\right)_k(\lambda)_{4k}}{k!\left(\frac{2\lambda+3}{4}-a\right)_k\left(\lambda+n+\frac{1}{2}\right)_{2k}\left(\frac{1}{2}-n\right)_{2k}}\left(1-z^2\right)^{2k}C_{2n-4k}^{\lambda+4k}(z)$$
$$=\frac{(2\lambda-1)_{2n+1}}{(2n)!}{}_3F_2\left(\begin{matrix}-n,\,n+\lambda,\,\frac{2\lambda+1}{4}-a\\ \frac{2\lambda+1}{4},\,\lambda-2a+\frac{1}{2};\ 1-z^2\end{matrix}\right).$$

57. $$\sum_{k=0}^{[n/2]}\frac{(8k+2\lambda-1)(a)_k\left(\frac{2\lambda-1}{4}\right)_k(\lambda)_{4k}}{k!\left(\frac{2\lambda+3}{4}-a\right)_k\left(\lambda+n+\frac{1}{2}\right)_{2k}\left(-\frac{1}{2}-n\right)_{2k}}\left(1-z^2\right)^{2k}C_{2n-4k+1}^{\lambda+4k}(z)$$
$$=\frac{(2\lambda-1)_{2n+2}z}{(2n+1)!}{}_3F_2\left(\begin{matrix}-n,\,n+\lambda+1,\,\frac{2\lambda+1}{4}-a\\ \frac{2\lambda+1}{4},\,\lambda-2a+\frac{1}{2};\ 1-z^2\end{matrix}\right).$$

58. $$\sum_{k=0}^{[n/3]}(-1)^k\frac{(2\lambda+12k-1)\left(\frac{2\lambda-1}{6}\right)_k(\lambda)_{6k}}{k!\left(\lambda+n+\frac{1}{2}\right)_{3k}\left(\frac{1}{2}-n\right)_{3k}}\left(1-z^2\right)^{3k}C_{2n-6k}^{\lambda+6k}(z)$$
$$=\frac{(2\lambda-1)_{2n+1}}{(2n)!}\,{}_3F_2\left(\begin{matrix}-n,\,\frac{2\lambda-1}{6},\,\lambda+n\\ \frac{2\lambda-1}{4},\,\frac{2\lambda+1}{4};\,\frac{3-3z^2}{4}\end{matrix}\right).$$

59. $$\sum_{k=0}^{[n/3]}(-1)^k\frac{(2\lambda+12k-1)\left(\frac{2\lambda-1}{6}\right)_k(\lambda)_{6k}}{k!\left(\lambda+n+\frac{1}{2}\right)_{3k}\left(-\frac{1}{2}-n\right)_{3k}}\left(1-z^2\right)^{3k}C_{2n-6k+1}^{\lambda+6k}(z)$$
$$=\frac{(2\lambda-1)_{2n+2}}{(2n+1)!}\,z\,{}_3F_2\left(\begin{matrix}-n,\,\frac{2\lambda-1}{6},\,\lambda+n+1\\ \frac{2\lambda-1}{4},\,\frac{2\lambda+1}{4};\,\frac{3-3z^2}{4}\end{matrix}\right).$$

5.11.4. Sums containing $C_{m\pm pk}^{\lambda\pm nk}(z)$ and special functions

1. $$\sum_{k=0}^{n}\frac{\left(2^{2n-2k}-1\right)}{(2n-2k)!\,(1-\lambda)_{2k}}(4z)^{-2k}B_{2n-2k}C_{2k}^{\lambda-2k}(z)$$
$$=-\frac{(4z)^{1-2n}}{2(1-\lambda)_{2n-1}}C_{2n-1}^{\lambda-2n+1}(z)\quad[n\geq 1].$$

2. $$\sum_{k=0}^{n}\frac{(4z)^{-2k}}{(2n-2k)!\,(-\lambda)_{2k+1}}B_{2n-2k}C_{2k+1}^{\lambda-2k}(z)=-\frac{(4z)^{1-2n}}{2(-\lambda)_{2n}}C_{2n}^{\lambda-2n+1}(z).$$

3. $$\sum_{k=0}^{n}\frac{(a)_k}{(1-\lambda)_k}\psi(k+a)\,C_{2k}^{\lambda-k}(z)=\frac{(a+1)_n}{n!}$$
$$\times\left\{\left[\psi(a+n+1)-\frac{1}{a}\right]{}_3F_2\left(\begin{matrix}-n,\,a,\,\lambda;\,z^2\\ a+1,\,\frac{1}{2}\end{matrix}\right)\right.$$
$$\left.-\frac{2n\lambda z^2}{(a+1)^2}\,{}_4F_3\left(\begin{matrix}1-n,\,a+1,\,a+1,\,\lambda+1\\ a+2,\,a+2,\,\frac{3}{2};\,z^2\end{matrix}\right)\right\}.$$

4. $$\sum_{k=0}^{n}\frac{(a)_k}{(1-\lambda)_k}\psi(k+a)\,C_{2k+1}^{\lambda-k}(z)=\frac{2\lambda(a+1)_n z}{n!}$$
$$\times\left\{\left[\psi(a+n+1)-\frac{1}{a}\right]{}_3F_2\left(\begin{matrix}-n,\,a,\,\lambda+1;\,z^2\\ a+1,\,3/2\end{matrix}\right)\right.$$
$$\left.-\frac{2n(\lambda+1)z^2}{3(a+1)^2}\,{}_4F_3\left(\begin{matrix}1-n,\,a+1,\,a+1,\,\lambda+2\\ a+2,\,a+2,\,\frac{5}{2};\,z^2\end{matrix}\right)\right\}.$$

5. $$\sum_{k=0}^{n} \frac{2^{-k}(z-1)^{-k}}{(n-k+1)!\,(1-\lambda)_k}\,\psi(a-k)C_k^{\lambda-k}(z) = \left(\frac{2}{1-z}\right)^{n+1}$$
$$\times\,\psi(a-n-1)\left[\frac{\left(\frac{1}{2}-\lambda\right)_{n+1}}{(n+1)!\,(1-2\lambda)_{n+1}} - \frac{\left(-\frac{1}{4}\right)^{n+1}}{(1-\lambda)_{n+1}}C_{n+1}^{\lambda-n-1}(z)\right]$$
$$+\;\frac{\left(\frac{1}{2}-\lambda\right)_n}{n!\,(1-2\lambda)_n(a-n-1)}\left(\frac{2}{1-z}\right)^n {}_4F_3\left(\begin{matrix}-n,\,2\lambda-n,\,1,\,1;\;\frac{1-z}{2}\\ \lambda-n+\frac{1}{2},\,a-n,\,2\end{matrix}\right).$$

6. $$\sum_{k=0}^{n} \frac{(1-a)_k}{(1-\lambda)_k}\,\frac{2^{-k}}{(z-1)^k}\,\psi(a-k)C_k^{\lambda-k}(z)$$
$$= \frac{\left(\frac{1}{2}-\lambda\right)_n(1-a)_n}{n!\,(1-2\lambda)_n(a-n-1)}\left(\frac{2}{z-1}\right)^n$$
$$\times\left[(a-n-1)\psi(a-n-1)\,{}_3F_2\left(\begin{matrix}-n,\,2\lambda-n,\,a-n-1,\,1\\ \lambda-n+\frac{1}{2},\,a-n;\;\frac{1-z}{2}\end{matrix}\right)\right.$$
$$\left.+\,{}_4F_3\left(\begin{matrix}-n,\,2\lambda-n,\,a-n-1,\,a-n-1\\ \lambda-n+\frac{1}{2},\,a-n,\,a-n;\;\frac{1-z}{2}\end{matrix}\right)\right].$$

7. $$\sum_{k=0}^{n}\frac{(-n)_k}{(1-\lambda)_k}\left(w-\sqrt{w^2-1}\right)^k P_{n-k}(w)C_{2k}^{\lambda-k}(z)$$
$$= 2^n\frac{\left(\frac{1}{2}\right)_n}{n!}(w^2-1)^{n/2}\,{}_3F_2\left(\begin{matrix}-n,\,-n,\,\lambda\\ \frac{1}{2}-n,\,\frac{1}{2};\;\frac{z^2}{2}-\frac{z^2}{2}w(w^2-1)^{-1/2}\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}\frac{(-n)_k}{(1-\lambda)_k}\left(w-\sqrt{w^2-1}\right)^k P_{n-k}(w)C_{2k+1}^{\lambda-k}(z)$$
$$= 2^{n+1}\frac{\left(\frac{1}{2}\right)_n}{n!}\lambda z(w^2-1)^{n/2}\,{}_3F_2\left(\begin{matrix}-n,\,-n,\,\lambda+1\\ \frac{1}{2}-n,\,\frac{3}{2};\;\frac{z^2}{2}-\frac{z^2}{2}w(w^2-1)^{-1/2}\end{matrix}\right).$$

9. $$\sum_{k=0}^{n}\frac{(2k+1)\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(\frac{1}{2}-n\right)_k}(z^2-1)^k\left[P_k(w)\right]^2 C_{2n-2k}^{2k+3/2}(z)$$
$$= \frac{2n+1}{n!}\,{}_3F_2\left(\begin{matrix}-n,\,n+\frac{3}{2},\,\frac{1}{2}\\ 1,\,1;\;(1-w^2)(1-z^2)\end{matrix}\right).$$

10. $$\sum_{k=0}^{n}\frac{(2k+1)\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(-\frac{1}{2}-n\right)_k}(z^2-1)^k\left[P_k(w)\right]^2 C_{2n-2k+1}^{2k+3/2}(z)$$
$$= \frac{(2n+3)\,z}{n!}\,{}_3F_2\left(\begin{matrix}-n,\,n+\frac{5}{2},\,\frac{1}{2}\\ 1,\,1;\;(1-w^2)(1-z^2)\end{matrix}\right).$$

11. $$\sum_{k=0}^{n} \frac{(4k+1)(2k)!}{\left(n+\frac{3}{2}\right)_k \left(\frac{1}{2}-n\right)_k} (1-z^2)^k P_{2k}\left(\sqrt{\frac{1-w}{2}}\right) P_{2k}\left(\sqrt{\frac{1+w}{2}}\right) \times C_{2n-2k}^{2k+1}(z) = (2n+1)\, {}_4F_3\left(\begin{matrix} -n,\, n+1,\, \frac{1}{4},\, \frac{3}{4} \\ \frac{1}{2},\, \frac{1}{2},\, 1;\; (1-w^2)(1-z^2) \end{matrix}\right).$$

12. $$\sum_{k=0}^{n} \frac{(4k+3)(2k+1)!}{\left(n+\frac{5}{2}\right)_k \left(\frac{1}{2}-n\right)_k} (1-z^2)^k P_{2k+1}\left(\sqrt{\frac{1-w}{2}}\right) P_{2k+1}\left(\sqrt{\frac{1+w}{2}}\right) \times C_{2n-2k}^{2k+2}(z) = \frac{(2n+3)!}{4(2n)!}\sqrt{1-w^2}\, {}_4F_3\left(\begin{matrix} -n,\, n+2,\, \frac{3}{4},\, \frac{5}{4} \\ 1,\, \frac{3}{2},\, \frac{3}{2};\; (1-w^2)(1-z^2) \end{matrix}\right).$$

13. $$\sum_{k=0}^{n} \frac{(4k+3)(2k+1)!}{\left(n+\frac{5}{2}\right)_k \left(-\frac{1}{2}-n\right)_k} (1-z^2)^k P_{2k+1}\left(\sqrt{\frac{1-w}{2}}\right) P_{2k+1}\left(\sqrt{\frac{1+w}{2}}\right) \times C_{2n-2k+1}^{2k+2}(z) = (n+1)(n+2)(2n+3)\sqrt{1-w^2}\, z \times {}_4F_3\left(\begin{matrix} -n,\, n+3,\, \frac{3}{4},\, \frac{5}{4} \\ 1,\, \frac{3}{2},\, \frac{3}{2};\; (1-w^2)(1-z^2) \end{matrix}\right).$$

14. $$\sum_{k=0}^{n} \frac{\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(\frac{1}{2}-n\right)_k} (1-z^2)^k T_{2k+1}\left(\sqrt{\frac{1-w}{2}}\right) T_{2k+1}\left(\sqrt{\frac{1+w}{2}}\right) \times C_{2n-2k}^{2k+3/2}(z) = \frac{(-1)^n}{2\left(\frac{1}{2}\right)_n \sqrt{1-z^2}} P_{2n+1}\left(\sqrt{(1-w^2)(1-z^2)}\right).$$

15. $$\sum_{k=0}^{n} \frac{\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(-\frac{1}{2}-n\right)_k} (1-z^2)^k \times T_{2k+1}\left(\sqrt{\frac{1-w}{2}}\right) T_{2k+1}\left(\sqrt{\frac{1+w}{2}}\right) C_{2n-2k+1}^{2k+3/2}(z) = \frac{(2n+3)\sqrt{1-w^2}\, z}{2n!} \times {}_2F_1\left(\begin{matrix} -n,\, n+\frac{5}{2} \\ \frac{3}{2};\; (1-w^2)(1-z^2) \end{matrix}\right).$$

16. $$\sum_{k=0}^{n} \frac{\left(-n+\frac{1}{2}\right)_k}{(n-k)!\,(1-\lambda)_k} \left(\frac{2}{z-1}\right)^k H_{2n-2k}(w) \times C_k^{\lambda-k}(z) = 2^{3n} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-\lambda\right)_n}{n!\,(1-2\lambda)_n} (z-1)^{-n}\, {}_2F_2\left(\begin{matrix} -n,\, 2\lambda-n;\; \frac{w^2-w^2z}{2} \\ \lambda-n+\frac{1}{2},\, \frac{1}{2} \end{matrix}\right).$$

17. $$\sum_{k=0}^{n} \frac{\left(-n-\frac{1}{2}\right)_k}{(n-k)!\,(1-\lambda)_k} \left(\frac{2}{z-1}\right)^k H_{2n-2k+1}(w)\, C_k^{\lambda-k}(z)$$
$$= 2^{3n+1} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}-\lambda\right)_n}{n!\,(1-2\lambda)_n} w\,(z-1)^{-n}\; {}_2F_2\left(\begin{matrix} -n,\, 2\lambda-n;\; \frac{w^2-w^2z}{2} \\ \lambda-n+\frac{1}{2},\, \frac{3}{2} \end{matrix}\right).$$

18. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k} \left(\frac{2}{z}\right)^k H_{2n-2k}(w) C_k^{\lambda-k}(z)$$
$$= \frac{(2w)^{2n}}{n!}\; {}_4F_1\left(\begin{matrix} -\frac{n}{2},\, \frac{1-2n}{4},\, \frac{1-n}{2},\, \frac{3-2n}{4} \\ 1-\lambda;\; 4w^{-4}z^{-2} \end{matrix}\right).$$

19. $$\sum_{k=0}^{n} \frac{\left(-\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k} \left(\frac{2}{z}\right)^k H_{2n-2k+1}(w) C_k^{\lambda-k}(z)$$
$$= \frac{(2w)^{2n+1}}{n!}\; {}_4F_1\left(\begin{matrix} -\frac{n}{2},\, -\frac{1+2n}{4},\, \frac{1-n}{2},\, \frac{1-2n}{4} \\ 1-\lambda;\; 4w^{-4}z^{-2} \end{matrix}\right).$$

20. $$\sum_{k=0}^{n} \frac{\left(1-z^2\right)^{-k}}{(2n-2k)!\,(1-\lambda)_k} H_{2n-2k}(w) C_{2k}^{\lambda-k}(z)$$
$$= \frac{\left(\frac{1}{2}-\lambda\right)_n}{n!\left(\frac{1}{2}\right)_n} z^{2n}\left(1-z^2\right)^{-n}\; {}_2F_2\left(\begin{matrix} -n,\, \frac{1}{2}-n;\; w^2(1-z^{-2}) \\ \lambda-n+\frac{1}{2},\, \frac{1}{2} \end{matrix}\right).$$

21. $$\sum_{k=0}^{n} \frac{\left(1-z^2\right)^{-k}}{(2n-2k+1)!\,(1-\lambda)_k} H_{2n-2k+1}(w) C_{2k}^{\lambda-k}(z)$$
$$= 2w \frac{\left(\frac{1}{2}-\lambda\right)_n}{n!\left(\frac{1}{2}\right)_n} z^{2n}\left(1-z^2\right)^{-n}\; {}_2F_2\left(\begin{matrix} -n,\, \frac{1}{2}-n;\; w^2(1-z^{-2}) \\ \lambda-n+\frac{1}{2},\, \frac{3}{2} \end{matrix}\right).$$

22. $$\sum_{k=0}^{n} \frac{\left(1-z^2\right)^{-k}}{(2n-2k)!\,(1-\lambda)_k} H_{2n-2k}(w) C_{2k+1}^{\lambda-k}(z)$$
$$= 2\lambda \frac{\left(\frac{1}{2}-\lambda\right)_n}{n!\left(\frac{3}{2}\right)_n} z^{2n+1}\left(1-z^2\right)^{-n}\; {}_2F_2\left(\begin{matrix} -n,\, -n-\frac{1}{2};\; w^2(1-z^{-2}) \\ \lambda-n+\frac{1}{2},\, \frac{1}{2} \end{matrix}\right).$$

23. $$\sum_{k=0}^{n} \frac{\left(1-z^2\right)^{-k}}{(2n-2k+1)!\,(1-\lambda)_k} H_{2n-2k+1}(w) C_{2k+1}^{\lambda-k}(z)$$
$$= 4\lambda w \frac{\left(\frac{1}{2}-\lambda\right)_n}{n!\left(\frac{3}{2}\right)_n} z^{2n+1}\left(1-z^2\right)^{-n}\; {}_2F_2\left(\begin{matrix} -n,\, -n-\frac{1}{2};\; w^2(1-z^{-2}) \\ \lambda-n+\frac{1}{2},\, \frac{3}{2} \end{matrix}\right).$$

24. $$\sum_{k=0}^{n} \frac{(-\lambda-n)_k}{(1-\mu)_k} (-2z)^{-k} L_{n-k}^{\lambda}(w) C_k^{\mu-k}(z) = \frac{(-w)^n}{n!} {}_4F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}, -\frac{\lambda+n}{2}, \frac{1-\lambda-n}{2} \\ 1-\mu;\ 4w^{-2}z^{-2} \end{matrix}\right).$$

25. $$\sum_{k=0}^{n} \frac{(-\lambda-n)_k}{(1-\mu)_k} 2^{-k} (1-z)^{-k} L_{n-k}^{\lambda}(w)\, C_k^{\mu-k}(z) = \frac{(\lambda+1)_n \left(\frac{1}{2}-\mu\right)_n}{n!\,(1-2\mu)_n} \left(\frac{2}{1-z}\right)^n {}_2F_2\left(\begin{matrix} -n,\, 2\mu-n;\ \frac{w-wz}{2} \\ \lambda+1,\, \mu-n+\frac{1}{2} \end{matrix}\right).$$

26. $$\sum_{k=0}^{n} \frac{(-2)^{-k}}{(1-\mu)_k} \left(\frac{w}{z-1}\right)^k L_{n-k}^{\lambda+k}(w)\, C_k^{\mu-k}(z) = \frac{\left(\frac{1}{2}-\mu\right)_n}{n!\,(1-2\mu)_n} \left(\frac{2w}{z-1}\right)^n {}_3F_1\left(\begin{matrix} -n,\, -\lambda-n,\, 2\mu-n \\ \mu-n+\frac{1}{2};\ \frac{z-1}{2w} \end{matrix}\right).$$

27. $$\sum_{k=0}^{n} \frac{(-\lambda-n)_k}{(1-\mu)_k} (1-z^2)^{-k} L_{n-k}^{\lambda}(w) C_{2k}^{\mu-k}(z) = \frac{(\lambda+1)_n \left(\frac{1}{2}-\mu\right)_n}{n!\left(\frac{1}{2}\right)_n} z^{2n}(z^2-1)^{-n} {}_2F_2\left(\begin{matrix} -n,\, \frac{1}{2}-n;\ w(1-z^{-2}) \\ \lambda+1,\, \mu-n+\frac{1}{2} \end{matrix}\right).$$

28. $$\sum_{k=0}^{n} \frac{(-\lambda-n)_k}{(1-\mu)_k} (1-z^2)^{-k} L_{n-k}^{\lambda}(w) C_{2k+1}^{\mu-k}(z) = 2\mu \frac{(\lambda+1)_n \left(\frac{1}{2}-\mu\right)_n}{n!\left(\frac{3}{2}\right)_n} z^{2n+1} (z^2-1)^{-n} {}_2F_2\left(\begin{matrix} -n,\, -n-\frac{1}{2};\ w(1-z^{-2}) \\ \lambda+1,\, \mu-n+\frac{1}{2} \end{matrix}\right).$$

29. $$\sum_{k=0}^{n} \frac{(4k-2\lambda+1)\left(\frac{1}{2}-\lambda\right)_k \left(\frac{1}{2}\right)_k}{\left(n-\lambda+\frac{3}{2}\right)_k (1-\lambda)_{2k}} \left(\frac{w}{z^2-1}\right)^k L_{n-k}^{2k-\lambda+1/2}(w) C_{2k}^{\lambda-2k}(z) = \frac{2\left(\frac{1}{2}-\lambda\right)_{n+1}}{(1-\lambda)_n} L_n^{-\lambda}\left(\frac{w}{1-z^2}\right).$$

30. $$\sum_{k=0}^{n} \frac{(4k-2\lambda+1)\left(\frac{1}{2}-\lambda\right)_k \left(\frac{3}{2}\right)_k}{\left(n-\lambda+\frac{3}{2}\right)_k (1-\lambda)_{2k}} \left(\frac{w}{z^2-1}\right)^k L_{n-k}^{2k-\lambda+1/2}(w) C_{2k+1}^{\lambda-2k}(z) = \frac{4\lambda z\left(\frac{1}{2}-\lambda\right)_{n+1}}{(-\lambda)_n} L_n^{-\lambda-1}\left(\frac{w}{1-z^2}\right).$$

31. $$\sum_{k=0}^{n} \frac{1}{(1-\mu)_k} \left(\frac{w}{1-z^2}\right)^k L_{n-k}^{\lambda+k}(w) C_{2k}^{\mu-k}(z) = \frac{\left(\frac{1}{2}-\mu\right)_n}{(2n)!} \left(\frac{4wz^2}{1-z^2}\right)^n {}_3F_1\left(\begin{matrix} -n,\ -n-\lambda,\ \frac{1}{2}-n \\ \mu-n+\frac{1}{2};\ \frac{z^{-2}-1}{w} \end{matrix}\right).$$

32. $$\sum_{k=0}^{n} \frac{1}{(1-\mu)_k} \left(\frac{w}{1-z^2}\right)^k L_{n-k}^{\lambda+k}(w) C_{2k+1}^{\mu-k}(z) = \frac{\mu\left(\frac{1}{2}-\mu\right)_n}{(2n)!} (2z)^{2n+1} \left(\frac{w}{1-z^2}\right)^n {}_3F_1\left(\begin{matrix} -n,\ -n-\lambda,\ -n-\frac{1}{2} \\ \mu-n+\frac{1}{2};\ \frac{z^{-2}-1}{w} \end{matrix}\right).$$

33. $$\sum_{k=0}^{[n/2]} (\mu)_k (-w)^{-k} L_k^{\lambda-k}(w) C_{n-2k}^{\mu+k}(z) = \frac{(\mu)_n}{n!} (2z)^n {}_3F_1\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ -\lambda \\ 1-\mu-n;\ -\frac{1}{wz^2} \end{matrix}\right).$$

34. $$\sum_{k=0}^{[n/2]} \frac{(\mu)_k}{(\lambda+1)_k} L_k^{\lambda}(w) C_{n-2k}^{\mu+k}(z) = \frac{(\mu)_n}{n!} (2z)^n {}_2F_2\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2};\ \frac{w}{z^2} \\ \lambda+1,\ 1-\mu-n \end{matrix}\right).$$

35. $$\sum_{k=0}^{[n/2]} (2k-n-\lambda)(-n-\lambda)_k L_k^{\lambda-2k+n}(1) C_{n-2k}^{\lambda}(z) = -\frac{(\lambda+1)_n}{n!} H_n(z).$$

5.11.5. Sums containing products of $C_{m\pm pk}^{\lambda\pm nk}(z)$

1. $$\sum_{k=0}^{n} (-1)^k \frac{(\lambda+k)(\lambda+n)_k}{(n-k)!\,(\lambda-n+1)_k(2\lambda+n+1)_k} \left[C_k^{\lambda}(z)\right]^2 = \frac{(2\lambda)_{n+1}\left(\frac{1}{2}\right)n}{2n!\,(-\lambda)_n\left(\lambda+\frac{1}{2}\right)_n} C_{2n}^{\lambda}(z) \quad [[69], (23)].$$

2. $$\sum_{k=0}^{n} (-1)^k \frac{(k+\lambda)k!\left(2\lambda+n-\frac{1}{2}\right)_k}{(n-k)!\left(\frac{3}{2}-n\right)_k (2\lambda)_k(2\lambda+n+1)_k} \left[C_k^{\lambda}(z)\right]^2 = 2^{2n}\lambda(1-2n) \frac{(\lambda)_n(2\lambda+1)_n}{\left(\lambda+\frac{1}{2}\right)_n (4\lambda-1)_{2n}} C_{2n}^{2\lambda-1/2}(z) \quad [[69], (24)].$$

3. $$\sum_{k=0}^{n} \frac{(2k+2\lambda-1)(2\lambda-1)_k}{k!\left(\lambda+\frac{1}{2}\right)_k^2 (2\lambda+2k)_{m-k}(2\lambda+2k)_{n-k}} \left(\frac{z^2-1}{4}\right)^k C_{m-k}^{\lambda+k}(z) C_{n-k}^{\lambda+k}(z) = \frac{2\lambda-1}{(2\lambda)_{m+n}} \binom{m+n}{n} C_{m+n}^{\lambda}(z) \quad [m \geq n;\ [25], (3)].$$

4. $$\sum_{k=0}^{n} \frac{(\mu)_k}{(1-\lambda)_k} \left(1-z^2\right)^k C_{2k}^{\lambda-k}(w) C_{2n-2k+1}^{\mu+k}(z) = z^{2n+1} \frac{(2\mu)_{2n+1}}{(2n+1)!} \, {}_3F_2\left(\begin{matrix} -n,\, -n-\frac{1}{2},\, \lambda \\ \mu+\frac{1}{2},\, \frac{1}{2};\; w^2(1-z^{-2}) \end{matrix} \right).$$

5. $$\sum_{k=0}^{n} \frac{(\mu)_k}{(1-\lambda)_k} \left(1-z^2\right)^k C_{2k}^{\lambda-k}(w) C_{2n-2k}^{\mu+k}(z) = z^{2n} \frac{(2\mu)_{2n}}{(2n)!} \, {}_3F_2\left(\begin{matrix} -n,\, \frac{1}{2}-n,\, \lambda \\ \mu+\frac{1}{2},\, \frac{1}{2};\; w^2(1-z^{-2}) \end{matrix} \right).$$

6. $$\sum_{k=0}^{n} \frac{(\mu)_k}{(1-\lambda)_k} \left(1-z^2\right)^k C_{2k+1}^{\lambda-k}(w) C_{2n-2k}^{\mu+k}(z) = 2\lambda w z^{2n} \frac{(2\mu)_{2n}}{(2n)!} \, {}_3F_2\left(\begin{matrix} -n,\, \frac{1}{2}-n,\, \lambda+1 \\ \mu+\frac{1}{2},\, \frac{3}{2};\; w^2(1-z^{-2}) \end{matrix} \right).$$

7. $$\sum_{k=0}^{n} \frac{(\mu)_k}{(1-\lambda)_k} \left(1-z^2\right)^k C_{2k+1}^{\lambda-k}(w) C_{2n-2k+1}^{\mu+k}(z) = 2\lambda w z^{2n+1} \frac{(2\mu)_{2n+1}}{(2n+1)!} \, {}_3F_2\left(\begin{matrix} -n,\, -n-\frac{1}{2},\, \lambda+1 \\ \mu+\frac{1}{2},\, \frac{3}{2};\; w^2(1-z^{-2}) \end{matrix} \right).$$

8. $$\sum_{k=0}^{n} \frac{k!\,(\lambda+k)}{(2\lambda)_k(2\lambda+n+1)_k} w^k L_{n-k}^{2\lambda+2k}(w) \left[C_k^\lambda(z)\right]^2 = \frac{(2\lambda)_{n+1}}{2n!} \, {}_2F_2\left(\begin{matrix} -n,\, \lambda;\; w-wz^2 \\ \lambda+\frac{1}{2},\, 2\lambda \end{matrix} \right).$$

9. $$\sum_{k=0}^{n} \frac{(k+\lambda)k!\left(2\lambda+\frac{1}{2}\right)_{2k}}{(2\lambda)_k(2\lambda+n+1)_k\left(\frac{1}{2}-n\right)_k} \left(w^2-1\right)^k C_{2n-2k}^{2\lambda+2k+1/2}(w) \left[C_k^\lambda(z)\right]^2 = \frac{\lambda(4\lambda+1)_{2n}}{(2n)!} \, {}_3F_2\left(\begin{matrix} -n,\, \lambda,\, 2\lambda+n+\frac{1}{2} \\ \lambda+\frac{1}{2},\, 2\lambda;\; (1-w^2)(1-z^2) \end{matrix} \right).$$

10. $$\sum_{k=0}^{n} \frac{(k+\lambda)k!\left(2\lambda+\frac{1}{2}\right)_{2k}}{(2\lambda)_k(2\lambda+n+1)_k\left(-\frac{1}{2}-n\right)_k} \left(w^2-1\right)^k C_{2n-2k+1}^{2\lambda+2k+1/2}(w) \left[C_k^\lambda(z)\right]^2 = \frac{(4\lambda)_{2n+2}w}{4(2n+1)!} \, {}_3F_2\left(\begin{matrix} -n,\, \lambda,\, 2\lambda+n+\frac{3}{2} \\ \lambda+\frac{1}{2},\, 2\lambda;\; (1-w^2)(1-z^2) \end{matrix} \right).$$

11. $$\sum_{k=0}^{n} \frac{(2k-2\lambda+1)k!\,(1-2\lambda)_k\left(\frac{3}{2}-2\lambda\right)_{2k}}{(1-\lambda)_k^2(n-2\lambda+2)_k\left(\frac{1}{2}-n\right)_k} 2^{-2k}\left(\frac{1-w^2}{1-z^2}\right)^k C_{2n-2k}^{2k-2\lambda+3/2}(w)$$
$$\times\left[C_k^{\lambda-k}(z)\right]^2 = \frac{(1-2\lambda)(3-4\lambda)_{2n}}{(2n)!}\,{}_3F_2\left(\begin{matrix}-n,\, n-2\lambda+\frac{3}{2},\, \frac{1}{2}-\lambda\\ 1-\lambda,\, 1-2\lambda;\ \frac{1-w^2}{1-z^2}\end{matrix}\right).$$

12. $$\sum_{k=0}^{n} \frac{(2k-2\lambda+1)k!\,(1-2\lambda)_k\left(\frac{3}{2}-2\lambda\right)_{2k}}{(1-\lambda)_k^2(n-2\lambda+2)_k\left(-\frac{1}{2}-n\right)_k} 2^{-2k}\left(\frac{1-w^2}{1-z^2}\right)^k C_{2n-2k+1}^{2k-2\lambda+3/2}(w)$$
$$\times\left[C_k^{\lambda-k}(z)\right]^2 = \frac{(2-4\lambda)_{2n+2}w}{2(2n+1)!}\,{}_3F_2\left(\begin{matrix}-n,\, n-2\lambda+\frac{5}{2},\, \frac{1}{2}-\lambda\\ 1-\lambda,\, 1-2\lambda;\ \frac{1-w^2}{1-z^2}\end{matrix}\right).$$

13. $$\sum_{k=0}^{n} \frac{(\lambda+2k)(2k)!\left(\lambda+\frac{1}{2}\right)_{2k}}{(\lambda+n+1)_k\left(\frac{1}{2}-n\right)_k(2\lambda)_{2k}}\left(1-w^2\right)^k$$
$$\times C_{2n-2k}^{\lambda+2k+1/2}(w)C_{2k}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right)C_{2k}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$= \frac{(2\lambda)_{2n+1}}{2(2n)!}\,{}_4F_3\left(\begin{matrix}-n,\, \frac{\lambda}{2},\, \frac{\lambda+1}{2},\, \lambda+n+\frac{1}{2}\\ \lambda,\, \lambda+\frac{1}{2},\, \frac{1}{2};\ (1-w^2)(1-z^2)\end{matrix}\right).$$

14. $$\sum_{k=0}^{n} \frac{(\lambda+2k)(2k)!\left(\lambda+\frac{1}{2}\right)_{2k}}{(\lambda+n+1)_k\left(-\frac{1}{2}-n\right)_k(2\lambda)_{2k}}\left(1-w^2\right)^k$$
$$\times C_{2n-2k+1}^{\lambda+2k+1/2}(w)C_{2k}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right)C_{2k}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$= \frac{(2\lambda)_{2n+2}w}{2(2n+1)!}\,{}_4F_3\left(\begin{matrix}-n,\, \frac{\lambda}{2},\, \frac{\lambda+1}{2},\, \lambda+n+\frac{3}{2}\\ \lambda,\, \lambda+\frac{1}{2},\, \frac{1}{2};\ (1-w^2)(1-z^2)\end{matrix}\right).$$

15. $$\sum_{k=0}^{n} \frac{(\lambda+2k+1)(2k+1)!\left(\lambda+\frac{3}{2}\right)_{2k}}{(\lambda+n+2)_k\left(\frac{1}{2}-n\right)_k(2\lambda+1)_{2k}}\left(1-w^2\right)^k$$
$$\times C_{2n-2k}^{\lambda+2k+3/2}(w)C_{2k+1}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right)C_{2k+1}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$= \frac{2\lambda^2(\lambda+1)(2\lambda+3)_{2n}}{(2n)!}\sqrt{1-z^2}\,{}_4F_3\left(\begin{matrix}-n,\, \frac{\lambda+1}{2},\, \frac{\lambda}{2}+1,\, \lambda+n+\frac{3}{2}\\ \lambda+\frac{1}{2},\, \lambda+1,\, \frac{3}{2};\ (1-w^2)(1-z^2)\end{matrix}\right).$$

16. $$\sum_{k=0}^{n} \frac{(\lambda+2k+1)(2k+1)!\left(\lambda+\frac{3}{2}\right)_{2k}}{(\lambda+n+2)_k\left(-\frac{1}{2}-n\right)_k(2\lambda+1)_{2k}}\left(1-w^2\right)^k \times C_{2n-2k+1}^{\lambda+2k+3/2}(w) C_{2k+1}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right) C_{2k+1}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)$$
$$= \frac{\lambda^2(2\lambda+2)_{2n+2}}{(2n+1)!} w\sqrt{1-z^2}\, {}_4F_3\left(\begin{matrix} -n,\ \frac{\lambda+1}{2},\ \frac{\lambda}{2}+1,\ \lambda+n+\frac{5}{2} \\ \lambda+\frac{1}{2},\ \lambda+1,\ \frac{3}{2};\ (1-w^2)(1-z^2) \end{matrix}\right).$$

5.11.6. Sums containing $C_{mk+n}^{\lambda k+\mu}(\varphi(k, z))$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k} C_m^{\lambda}(w+kz)=0 \qquad [m<n].$$

2. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k} C_m^{\lambda}(1+kz) = \frac{n!\,(2\lambda)_{m+n}}{(m-n)!\left(\lambda+\frac{1}{2}\right)_n}\left(-\frac{z}{2}\right)^n \sum_{k=0}^{m-n} \sigma_{k+n}^{n} \frac{(n-m)_k(2\lambda+m+n)_k}{(k+n)!\left(\lambda+n+\frac{1}{2}\right)_k}\left(-\frac{z}{2}\right)^k.$$

3. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k} C_{m+n}^{\lambda}(w+kz) = n!\,(\lambda)_n(-2z)^n \sum_{k=0}^{m} \sigma_{k+n}^{n} \frac{(\lambda+n)_k}{(k+n)!}(2z)^k C_{m-k}^{\lambda+k+n}(w).$$

4. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k} k^m C_m^{\lambda}\left(1+\frac{z}{k}\right)=(-1)^m(2\lambda)_m \delta_{m,n}-\frac{(\lambda)_m}{m!}(2z)^m \qquad [n \geq m].$$

5. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k-z)^m C_m^{\lambda}\left(\frac{k+z}{k-z}\right)=(-1)^m(2\lambda)_m \delta_{m,n}-\frac{(2\lambda)_m}{m!} z^m \qquad [n \geq m].$$

6. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k} C_{2m}^{\lambda}(\sqrt{k}\, z)=(-1)^m \frac{m!\,(\lambda)_{2m}}{(2m)!}(2z)^{2m} \delta_{m,n}-(-1)^m \frac{(\lambda)_m}{m!} \qquad [n \geq m].$$

7. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k} k^{-1/2} C_{2m+1}^{\lambda}(\sqrt{k}\, z)=(-1)^m \frac{m!\,(\lambda)_{2m+1}}{(2m+1)!}(2z)^{2m+1} \delta_{m,n} -(-1)^m \frac{2(\lambda)_{m+1} z}{m!} \qquad [n \geq m].$$

8. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k} k^m C_{2m}^{\lambda}\left(\frac{z}{\sqrt{k}}\right)=(\lambda)_m \delta_{m,n}-\frac{(\lambda)_{2m}}{(2m)!}(2z)^{2m} \qquad [n \geq m].$$

9. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m+1/2}C_{2m+1}^{\lambda}\left(\frac{z}{\sqrt{k}}\right) = 2(\lambda)_{m+1}z\delta_{m,n} - \frac{(\lambda)_{2m+1}}{(2m+1)!}(2z)^{2m+1} \quad [n \ge m].$$

10. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}C_{2m}^{\lambda}\left(\sqrt{1+kz}\right) = \frac{m!}{(2m)!}(\lambda)_{2m}(-4z)^m\delta_{m,n} - \frac{(2\lambda)_{2m}}{(2m)!} \quad [n \ge m].$$

11. $$\sum_{k=1}^{n}\frac{(-1)^k}{\sqrt{1+kz}}\binom{n}{k}C_{2m+1}^{\lambda}\left(\sqrt{1+kz}\right) = (-1)^m\frac{2^{2m+1}m!}{(2m+1)!}(\lambda)_{2m+1}z^m\delta_{m,n} - \frac{(2\lambda)_{2m+1}}{(2m+1)!} \quad [n \ge m].$$

12. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^mC_{2m}^{\lambda}\left(\frac{z}{\sqrt{k}}\right) = -\frac{(\lambda)_{2m}}{(2m)!}(2z)^{2m} + \frac{n!\,(\lambda)_m(\lambda+m)_{m-n}}{(2m-2n)!}(2z)^{2m-2n}\sum_{k=0}^{m-n}\sigma_{k+n}^{n}\frac{(2n-2m)_{2k}}{(k+n)!\,(n-2m-\lambda+1)_k}(2z)^{-2k} \quad [n \ge m].$$

13. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m+1/2}C_{2m+1}^{\lambda}\left(\frac{z}{\sqrt{k}}\right) = -\frac{(\lambda)_{2m+1}}{(2m+1)!}(2z)^{2m+1} + \frac{n!\,(\lambda)_{m+1}(\lambda+m+1)_{m-n}}{(2m-2n)!}(2z)^{2m-2n+1} \times\sum_{k=0}^{m-n}\sigma_{k+n}^{n}\frac{(2n-2m)_{2k}}{(k+n)!\,(n-2m-\lambda)_k(2m-2n-2k+1)}(2z)^{-2k} \quad [n \ge m].$$

14. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^mC_{2m}^{\lambda}\left(\sqrt{1+\frac{z}{k}}\right) = \frac{(-1)^m m!}{(2m)!}(2\lambda)_{2m}\delta_{m,n} - \frac{(\lambda)_{2m}}{(2m)!}(4z)^m \quad [n \ge m].$$

15. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{k^{m+1/2}}{\sqrt{k+z}}C_{2m+1}^{\lambda}\left(\sqrt{1+\frac{z}{k}}\right) = \frac{(-1)^m m!}{(2m+1)!}(2\lambda)_{2m+1}\delta_{m,n} - \frac{(\lambda)_{2m+1}}{(2m+1)!}2^{2m+1}z^m \quad [n \ge m].$$

16. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^mC_{2m}^{\lambda}\left(\sqrt{\frac{k}{k+z}}\right) = (-1)^m\frac{m!}{(2m)!}(2\lambda)_{2m}\delta_{m,n} - \frac{(\lambda)_m}{m!}(-z)^m \quad [n \ge m].$$

17. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-1/2}(k+z)^{m+1/2}C_{2m+1}^{\lambda}\left(\sqrt{\frac{k}{k+z}}\right)$$
$$= (-1)^m\frac{m!}{(2m+1)!}(2\lambda)_{2m+1}\delta_{m,n} - \frac{2(\lambda)_{m+1}}{m!}(-z)^m \quad [n \geq m].$$

18. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^m C_{2m}^{\lambda}\left(\sqrt{\frac{z}{k+z}}\right) = (\lambda)_m\delta_{m,n} - \frac{(2\lambda)_{2m}}{(2m)!}z^m \quad [n \geq m].$$

19. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+z)^{m+1/2}C_{2m+1}^{\lambda}\left(\sqrt{\frac{z}{k+z}}\right)$$
$$= (-1)^{m+1}2\sqrt{z}(\lambda)_{m+1}\delta_{m,n} + (-1)^m\frac{(2\lambda)_{2m+1}}{(2m+1)!}z^{m+1/2} \quad [n \geq m].$$

20. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{m/2}C_m^{\lambda}\left(\frac{1}{2}\sqrt{\frac{k}{z}} + \frac{1}{2}\sqrt{\frac{z}{k}}\right) = (-1)^m(\lambda)_m z^{-m/2}\delta_{m,n} - \frac{(\lambda)_m}{m!}z^{m/2} \quad [n \geq m].$$

21. $$\sum_{k=1}^{n}\frac{k^{k-1}}{k!}(\lambda)_k(-2z)^k C_{n-k}^{\lambda+k}(1+kz) = -\frac{(2\lambda)_{n+1}z}{(n-1)!(2\lambda+1)}.$$

22. $$\sum_{k=0}^{n}\frac{(ka+b)^{n-k-1}}{(n-k)!(1-\lambda)_k}\left(\frac{a}{2z}\right)^k C_k^{\lambda-k}(1+kz) = \frac{\left(\frac{a}{2z}\right)^n}{(na+b)(1-\lambda)_n}C_n^{\lambda-n}\left(1-\frac{bz}{a}\right).$$

23. $$\sum_{k=0}^{n}\frac{2^{-k}}{(n-k)!(1-\lambda)_k}(ka+z+1)^{n-k-1}C_k^{\lambda-k}(ka+z) = \frac{2^n\left(\frac{1}{2}-\lambda\right)_n}{n!(na+z+1)(1-2\lambda)_n}.$$

24. $$\sum_{k=0}^{2n}\frac{(2z)^{-k}}{(2n-k)!(1-\lambda)_k}(ka+1)^{2n-k-1}C_k^{\lambda-k}((ka+1)z) = \frac{\left(\frac{1}{2}\right)_n z^{-2n}}{(2n)!(2na+1)(1-\lambda)_n}.$$

25. $$\sum_{k=1}^{n}\frac{k^{n-1}}{k!}(\lambda)_k C_{2n-2k}^{\lambda+k}\left(\frac{z}{\sqrt{k}}\right) = \frac{(\lambda)_{2n-1}}{(2n-2)!}(2z)^{2n-2} \quad [n \geq 1].$$

26. $$\sum_{k=1}^{n}\frac{k^{n-1/2}}{k!}(\lambda)_k C_{2n-2k+1}^{\lambda+k}\left(\frac{z}{\sqrt{k}}\right) = \frac{(\lambda)_{2n}}{(2n-1)!}(2z)^{2n-1} \quad [n \geq 1].$$

27. $$\sum_{k=1}^{n} k^{2k-2} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(\frac{1}{2}-n\right)_k} z^k C_{2n-2k}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{(2n+1)z}{4(n-1)!}.$$

28. $$\sum_{k=1}^{n} k^{2k-2} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(-n-\frac{1}{2}\right)_k} \frac{z^k}{\sqrt{1+k^2 z}} C_{2n-2k+1}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{(2n+3)z}{4(n-1)!}.$$

29. $$\sum_{k=1}^{n} k^{2k-4} \frac{\left(\frac{1}{2}\right)_{2k} z^k}{(k+n)!\left(\frac{1}{2}-n\right)_k} C_{2n-2k}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{(2n+1)z}{4}\left[\frac{1}{(n-1)!}+\frac{(2n+3)z}{8(n-2)!}\right].$$

30. $$\sum_{k=1}^{n} k^{2k-4} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(-n-\frac{1}{2}\right)_k} \frac{z^k}{\sqrt{1+k^2 z}} C_{2n-2k+1}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{(2n+3)z}{4}\left[\frac{1}{(n-1)!}+\frac{(2n+5)z}{8(n-2)!}\right].$$

31. $$\sum_{k=1}^{n} k^{2k-6} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(\frac{1}{2}-n\right)_k} z^k C_{2n-2k}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{(2n+1)(2n+3)(2n+5)z^3}{576(n-3)!} - \frac{5(2n+1)(2n+3)z^2}{128(n-2)!} - \frac{(2n+1)z}{4(n-1)!}.$$

32. $$\sum_{k=1}^{n} k^{2k-6} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(-\frac{1}{2}-n\right)_k} \frac{z^k}{\sqrt{1+k^2 z}} C_{2n-2k+1}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{(2n+3)(2n+5)(2n+7)z^3}{576(n-3)!} - \frac{5(2n+3)(2n+5)z^2}{128(n-2)!} - \frac{(2n+3)z}{4(n-1)!}.$$

33. $$\sum_{k=1}^{n} \frac{k^{2k}}{k^2+a^2} \frac{\left(-\frac{1}{2}\right)_{2k}}{(k+n)!\left(\frac{1}{2}-n\right)_k} z^k C_{2n-2k}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{a^{-2}}{2\,(n!)} - \frac{a^{-2}}{(n+1)!\,(2n-1)z}\left[{}_3F_2\left(\begin{matrix}-n-1,\ n-\frac{1}{2},\ 1\\ ia,\ -ia;\ a^2 z\end{matrix}\right)-1\right].$$

34. $$\sum_{k=1}^{n} \frac{k^{2k}}{k^2+a^2} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(-\frac{1}{2}-n\right)_k} \frac{z^k}{\sqrt{1+k^2 z}} C_{2n-2k+1}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = -\frac{a^{-2}}{2\,(n!)} - \frac{a^{-2}}{(n+1)!\,(2n+1)z}\left[{}_3F_2\left(\begin{matrix}-n-1,\ n+\frac{1}{2},\ 1\\ ia,\ -ia;\ a^2 z\end{matrix}\right)-1\right].$$

35. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k-1}\left(\frac{1}{2}\right)_{2k+1}}{(k+n+1)!\left(\frac{1}{2}-n\right)_k} z^k C_{2n-2k}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = \frac{2n+!}{2(n!)}.$$

36. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k-1}\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(-\frac{1}{2}-n\right)_k} \frac{z^k}{\sqrt{1+(2k+1)^2 z}} \times C_{2n-2k+1}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = \frac{2n+3}{n!}.$$

37. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k-3}\left(\frac{1}{2}\right)_{2k+1}}{(k+n+1)!\left(\frac{1}{2}-n\right)_k} z^k C_{2n-2k}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = \frac{2n+1}{2(n!)}\left[1+\frac{2}{9}n(2n+3)z\right].$$

38. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k-3}\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(-\frac{1}{2}-n\right)_k} \frac{z^k}{\sqrt{1+(2k+1)^2 z}} \times C_{2n-2k+1}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = (2n+3)\left[\frac{1}{n!}+\frac{2(2n+5)z}{9(n-1)!}\right].$$

39. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k-5}\left(\frac{1}{2}\right)_{2k+1}}{(k+n+1)!\left(\frac{1}{2}-n\right)_k} z^k C_{2n-2k}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = \frac{2n+1}{2(n!)}\left[1+\frac{20}{81}n(2n+3)z+\frac{4}{225}n(n+1)(2n+3)(2n+5)z^2\right].$$

40. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k-5}\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(-n-\frac{1}{2}\right)_k} \frac{z^k}{\sqrt{1+(2k+1)^2 z}} \times C_{2n-2k+1}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = (2n+3)\left[\frac{1}{n!}+\frac{20(2n+5)z}{81(n-1)!}+\frac{4(2n+5)(2n+7)z^2}{225(n-2)!}\right].$$

41. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k+1}}{(2k+1)^2+a^2} \frac{\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\,(1/2-n)_k} z^k \times C_{2n-2k}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = \frac{2n+1}{n!\,(1+a^2)}\, {}_3F_2\left(\begin{matrix} -n,\, 1,\, n+\frac{3}{2};\; a^2 z \\ \frac{3-ia}{2},\, \frac{3+ia}{2} \end{matrix}\right).$$

42. $$\sum_{k=0}^{n} \frac{(2k+1)^{2k+1}}{(2k+1)^2+a^2} \frac{\left(\frac{3}{2}\right)_{2k}}{(k+n+1)!\left(-\frac{1}{2}-n\right)_k} \frac{z^k}{\sqrt{1+(2k+1)^2 z}} \times C_{2n-2k+1}^{2k+3/2}\left(\sqrt{1+(2k+1)^2 z}\right) = \frac{2n+3}{n!\,(1+a^2)}\, {}_3F_2\left(\begin{matrix} -n,\, n+\frac{5}{2},\, 1 \\ \frac{3-ia}{2},\, \frac{3+ia}{2};\; a^2 z \end{matrix}\right).$$

43. $$\sum_{k=0}^{n} \frac{(2k+\lambda)^{2k-1}(\lambda)_k\left(\lambda+\frac{1}{2}\right)_{2k}}{k!\,(\lambda+n+1)_k\left(-\frac{1}{2}-n\right)_k} \frac{z^k}{\sqrt{1+(2k+\lambda)^2 z}} \times C_{2n-2k+1}^{\lambda+2k+1/2}\left(\sqrt{1+(2k+\lambda)^2 z}\right) = \frac{2^{-2n}(2\lambda+1)_{2n+1}}{n!\lambda\left(\frac{3}{2}\right)_n}.$$

44. $$\sum_{k=1}^{n} \frac{k^{2k+2}\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(\frac{1}{2}-n\right)_k} z^k C_{2n-2k}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = \frac{(2n+1)\,z}{8(n-2)!}\left[(2n+3)\,z\,{}_3F_0\left(\begin{matrix}2-n,\ n+\frac{5}{2},\ 4\\ z\end{matrix}\right) - \frac{2}{n-1}\,{}_3F_0\left(\begin{matrix}1-n,\ n+\frac{3}{2},\ 4\\ z\end{matrix}\right)\right] \quad [n \geq 2].$$

45. $$\sum_{k=1}^{n} \frac{k^{2k+2}}{\sqrt{1+k^2 z}} \frac{\left(\frac{1}{2}\right)_{2k}}{(k+n)!\left(-\frac{1}{2}-n\right)_k} z^k C_{2n-2k+1}^{2k+1/2}\left(\sqrt{1+k^2 z}\right) = \frac{(2n+3)\,z}{8(n-2)!}\left[(2n+5)\,z\,{}_3F_0\left(\begin{matrix}2-n,\ n+\frac{7}{2},\ 4\\ z\end{matrix}\right) - \frac{2}{n-1}\,{}_3F_0\left(\begin{matrix}1-n,\ n+\frac{5}{2},\ 4\\ z\end{matrix}\right)\right] \quad [n \geq 2].$$

46. $$\sum_{k=0}^{n} (2k+\lambda)^{2k-1} \frac{(\lambda)_k\left(\lambda+\frac{1}{2}\right)_{2k} z^k}{k!\,(\lambda+n+1)_k\left(\frac{1}{2}-n\right)_k} C_{2n-2k}^{\lambda+2k+1/2}\left(\sqrt{1+(2k+\lambda)^2 z}\right) = \frac{2^{-2n}(2\lambda+1)_{2n}}{n!\lambda\left(\frac{1}{2}\right)_n}.$$

47. $$\sum_{k=1}^{n} \frac{(-ka)^k(ka-b)^{n-k-1}}{(n-k)!\,(1-\lambda)_k} C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k}}\right) = \frac{(-1)^n b^{n-1}}{n!} + \frac{(-b)^n}{(na-b)(1-\lambda)_n} C_{2n}^{\lambda-n}\left(z\sqrt{\frac{a}{b}}\right).$$

48. $$\sum_{k=1}^{n} \frac{k^{k+1/2}(-a)^k(ka-b)^{n-k-1}}{(n-k)!\,(1-\lambda)_k} C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k}}\right) = \frac{2(-1)^n b^{n-1}\lambda z}{n!} + \frac{(-1)^n a^{-1/2} b^{n+1/2}}{(na-b)(1-\lambda)_n} C_{2n+1}^{\lambda-n}\left(z\sqrt{\frac{a}{b}}\right).$$

49. $$\sum_{k=0}^{n} \frac{(-a)^k(ka+b)^{n-k-1}}{(n-k)!\,(1-\lambda)_k} (k+z)^k C_{2k}^{\lambda-k}\left(\sqrt{\frac{z}{k+z}}\right) = \frac{(b-az)^n}{(na+b)(1-\lambda)_n} C_{2n}^{\lambda-n}\left(\sqrt{\frac{az}{az-b}}\right).$$

50. $$\sum_{k=0}^{n} \frac{(-a)^k (ka+b)^{n-k-1}}{(n-k)!\,(1-\lambda)_k} (k+z)^{k+1/2} C_{2k+1}^{\lambda-k}\left(\sqrt{\frac{z}{k+z}}\right) = \frac{(-1)^{n+1} a^{-1/2} (az-b)^{n+1/2}}{(na+b)(1-\lambda)_n} C_{2n+1}^{\lambda-n}\left(\sqrt{\frac{az}{az-b}}\right).$$

51. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(1-\lambda)_k} (ka+1)^{n-1} C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{ka+1}}\right) = \frac{(\lambda)_n (2z)^{2n}}{(2n)!\,(na+1)}.$$

52. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(1-\lambda)_k} (ka+1)^{n-1/2} C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{ka+1}}\right) = \frac{(\lambda)_{n+1} (2z)^{2n+1}}{(2n+1)!\,(na+1)}.$$

53. $$\sum_{k=0}^{n} \frac{(ka+1)^{n-k-1}}{(n-k)!\,(1-\lambda)_k} (-z)^{-k} (1+(ka+1)z)^k C_{2k}^{\lambda-k}\left(\frac{1}{\sqrt{1+(ka+1)z}}\right) = \frac{\left(\frac{1}{2}-\lambda\right)_n}{(2n)!\,(na+1)} \left(-\frac{4}{z}\right)^n.$$

54. $$\sum_{k=0}^{n} \frac{(ka+1)^{n-k-1}}{(n-k)!\,(1-\lambda)_k} (-z)^{-k} (1+(ka+1)z)^{k+1/2} \times C_{2k+1}^{\lambda-k}\left(\frac{1}{\sqrt{1+(ka+1)z}}\right) = \frac{2\lambda\left(\frac{1}{2}-\lambda\right)_n}{(2n+1)!\,(na+1)} \left(-\frac{4}{z}\right)^n.$$

55. $$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^{n/2} C_n^\lambda\left(\frac{k+z}{2\sqrt{kz}}\right) = (\lambda)_n \left[(-1)^n z^{-n/2} - \frac{z^{n/2}}{n!}\right].$$

56. $$\sum_{k=0}^{2n} (-1)^k \frac{(ka+1)^{2n-k-1}}{(2n-k)!\,(2\lambda)_k} \left[(ka+1)^2+z\right]^{k/2} C_k^\lambda\left(\frac{ka+1}{\sqrt{(ka+1)^2+z}}\right) = \frac{\left(\frac{1}{2}\right)_n (-z)^n}{(2n)!\,(2na+1)\left(\lambda+\frac{1}{2}\right)_n}.$$

57. $$\sum_{k=0}^{n} (-1)^k \frac{(ka+1)^{n-1}}{(n-k)!\,(1-\lambda)_k} C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{ka+1}}\right) = \frac{(\lambda)_n z^{2n}}{n!\,(na+1)\left(\frac{1}{2}\right)_n}.$$

58. $$\sum_{k=0}^{n} (-1)^k \frac{(ka+1)^{n-1/2}}{(n-k)!\,(1-\lambda)_k} C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{ka+1}}\right) = \frac{2(\lambda)_{n+1} z^{2n+1}}{n!\,(na+1)\left(\frac{3}{2}\right)_n}.$$

5.11.7. Sums containing $C_{mk+n}^{\lambda k+\mu}(\varphi(k,\,z))$ and special functions

1. $$\sum_{k=1}^{n}(4k)^k(n-k+1)^{n-k-1/2}\frac{\left(-n-\frac{1}{2}\right)_k}{(n-k)!\,(1-\lambda)_k}H_{2n-2k+1}\left(\frac{w}{\sqrt{n-k+1}}\right)$$
$$\times C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k}}\right)=-\frac{(n+1)^{n-1/2}}{n!}H_{2n+1}\left(w\sqrt{\frac{1}{n+1}}\right)$$
$$+(2w)^{2n+1}\sum_{k=0}^{n}\left(\frac{n+1}{w^2}\right)^k\frac{\left(-\frac{1}{2}-n\right)_k}{(n-k+1)!\,(1-\lambda)_k}C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{n+1}}\right).$$

2. $$\sum_{k=1}^{n}2^{2k}k^{k+1/2}(n-k+1)^{n-k-1/2}\frac{\left(-n-\frac{1}{2}\right)_k}{(n-k)!\,(1-\lambda)_k}H_{2n-2k+1}\left(\frac{w}{\sqrt{n-k+1}}\right)$$
$$\times C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k}}\right)=-\frac{2\lambda z}{n!}(n+1)^{n-1/2}H_{2n+1}\left(w\sqrt{\frac{1}{n+1}}\right)$$
$$+\sqrt{n+1}\,(2w)^{2n+1}\sum_{k=0}^{n}\left(\frac{n+1}{w^2}\right)^k\frac{\left(-\frac{1}{2}-n\right)_k}{(n-k+1)!\,(1-\lambda)_k}C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{n+1}}\right).$$

3. $$\sum_{k=1}^{n}2^{2k}k^{k+1/2}(n-k+1)^{n-k-1}\frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k}H_{2n-2k}\left(\frac{w}{\sqrt{n-k+1}}\right)$$
$$\times C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k}}\right)=-\frac{2\lambda z}{n!}(n+1)^{n-1}H_{2n}\left(\frac{w}{\sqrt{n+1}}\right)$$
$$+\sqrt{n+1}\,(4w^2)^n\sum_{k=0}^{n}\left(\frac{n+1}{w^2}\right)^k\frac{\left(\frac{1}{2}-n\right)_k}{(n-k+1)!\,(1-\lambda)_k}C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{n+1}}\right).$$

4. $$\sum_{k=1}^{n}(4k)^k(n-k+1)^{n-k-1}\frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k}H_{2n-2k}\left(\frac{w}{\sqrt{n-k+1}}\right)$$
$$\times C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k}}\right)=-\frac{(n+1)^{n-1}}{n!}H_{2n}\left(w\sqrt{\frac{1}{n+1}}\right)$$
$$-(2w)^{2n}\sum_{k=0}^{n}\left(\frac{n+1}{w^2}\right)^k\frac{\left(\frac{1}{2}-n\right)_k}{(n-k+1)!\,(1-\lambda)_k}C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{n+1}}\right).$$

5. $$\sum_{k=0}^{n}(-4)^k(k+1)^{n-1/2}\frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k}H_{2n-2k}\left(\frac{w}{\sqrt{k+1}}\right)C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{2^{2n+1}w^{2n+2}z^{-1}}{(n+1)!\,(2n+1)}\left[1-{}_3F_1\left(\begin{matrix}-n-1,\,-n-\frac{1}{2},\,\lambda\\ \frac{1}{2};\,-w^{-2}z^2\end{matrix}\right)\right].$$

$$\textbf{6.}\ \sum_{k=0}^{n}(-4)^k(k+1)^{n-1}\frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k}H_{2n-2k}\left(\frac{w}{\sqrt{k+1}}\right)C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{2^{2n}w^{2n+2}z^{-2}}{(n+1)!\,(2n+1)(\lambda-1)}\left[{}_3F_1\left(\begin{matrix}-n-1,\ -n-\frac{1}{2},\ \lambda\\ -\frac{1}{2};\ -w^{-2}z^2\end{matrix}\right)-1\right].$$

$$\textbf{7.}\ \sum_{k=0}^{n}(-4)^k(k+1)^{n-1/2}\frac{\left(-\frac{1}{2}-n\right)_k}{(n-k)!\,(1-\lambda)_k}$$
$$\times H_{2n-2k+1}\left(\frac{w}{\sqrt{k+1}}\right)C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{2^{2n+1}w^{2n+3}z^{-2}}{(n+1)!\,(2n+3)(\lambda-1)}\left[{}_3F_1\left(\begin{matrix}-n-1,\ -n-\frac{3}{2},\ \lambda-1\\ -\frac{1}{2};\ -w^{-2}z^2\end{matrix}\right)-1\right].$$

$$\textbf{8.}\ \sum_{k=0}^{n}(-4)^k(k+1)^{n}\frac{\left(-n-\frac{1}{2}\right)_k}{(n-k)!\,(1-\lambda)_k}H_{2n-2k+1}\left(\frac{w}{\sqrt{k+1}}\right)C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{2^{2n+2}w^{2n+3}z^{-1}}{(n+1)!\,(2n+3)}\left[1-{}_3F_1\left(\begin{matrix}-n-1,\ -n-\frac{3}{2},\ \lambda\\ \frac{1}{2};\ -w^{-2}z^2\end{matrix}\right)\right].$$

$$\textbf{9.}\ \sum_{k=0}^{n}\frac{(k+1)^{-1}}{(1-\mu)_k}\left(\frac{w}{2z}\right)^k L_{n-k}^{\lambda+k}((n-k)w)\,C_k^{\mu-k}(1+(k+1)z)=\frac{\left(\frac{1}{2}-\mu\right)_n}{(1-2\mu)_n}$$
$$\times\left(-\frac{2w}{z}\right)^n\sum_{k=0}^{n}\left(-\frac{z}{2w}\right)^k\frac{(2\mu-n)_k}{(n-k+1)!\left(\mu-n+\frac{1}{2}\right)_k}L_k^{\lambda-k+n}((n+1)w).$$

$$\textbf{10.}\ \sum_{k=1}^{n}(-1)^k k^k(n-k+1)^{n-k-1}\frac{(-n-\lambda)_k}{(1-\mu)_k}L_{n-k}^{\lambda}\left(\frac{w}{n-k+1}\right)$$
$$\times C_{2k}^{\mu-k}\left(\frac{z}{\sqrt{k}}\right)=-(n+1)^{n-1}L_n^{\lambda}\left(\frac{w}{n+1}\right)$$
$$+(-w)^n\sum_{k=0}^{n}\left(\frac{n+1}{w}\right)^k\frac{(-n-\lambda)_k}{(1-\mu)_k(n-k+1)!}C_{2k}^{\mu-k}\left(\frac{z}{\sqrt{n+1}}\right).$$

$$\textbf{11.}\ \sum_{k=1}^{n}(-1)^k k^{k+1/2}(n-k+1)^{n-k-1}\frac{(-n-\lambda)_k}{(1-\mu)_k}L_{n-k}^{\lambda}\left(\frac{w}{n-k+1}\right)$$
$$\times C_{2k+1}^{\mu-k}\left(\frac{z}{\sqrt{k}}\right)=-2\mu z(n+1)^{n-1}L_n^{\lambda}\left(\frac{w}{n+1}\right)$$
$$+\sqrt{n+1}\,w^n\sum_{k=0}^{n}\left(\frac{n+1}{w}\right)^k\frac{(-n-\lambda)_k}{(1-\mu)_k(n-k+1)!}C_{2k+1}^{\mu-k}\left(\frac{z}{\sqrt{n+1}}\right).$$

12. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(1-\mu)_k} w^k L_{n-k}^{\lambda+k}((k+1)w) \times C_{2k}^{\mu-k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{(\lambda)_{n+1}(wz^2)^{-1}}{2(n+1)!\,(\mu-1)} \left[{}_2F_2\left(\begin{matrix} -n-1,\ \mu-1 \\ \lambda,\ -\frac{1}{2};\ wz^2 \end{matrix}\right) - 1 \right].$$

13. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1/2}}{(1-\mu)_k} w^k L_{n-k}^{\lambda+k}((k+1)w)\, C_{2k+1}^{\mu-k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{(\lambda)_{n+1}(wz)^{-1}}{(n+1)!} \left[1 - {}_2F_2\left(\begin{matrix} -n-1,\ \mu \\ \lambda,\ \frac{1}{2};\ wz^2 \end{matrix}\right) \right].$$

14. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(1-\lambda)_k} (-w)^k L_{n-k}^{\lambda+k}((n-k)w)\, C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right) = \sum_{k=0}^{n} (4wz^2)^k \frac{(\lambda)_k}{(2k)!\,(k+1)} L_{n-k}^{\lambda+k}((n+1)w).$$

15. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1/2}}{(1-\lambda)_k} (-w)^k L_{n-k}^{\lambda+k}((n-k)w)\, C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right) = 4\lambda z \sum_{k=0}^{n} (4wz^2)^k \frac{(\lambda+1)_k}{(2k+2)!} L_{n-k}^{\lambda+k}((n+1)w).$$

16. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(1-\lambda)_k} w^k L_{n-k}^{\mu+k}((k+1)w)\, C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{(\mu)_{n+1}}{2(n+1)!\,(\lambda-1)wz^2} \left[{}_2F_2\left(\begin{matrix} -n-1,\ \lambda-1;\ wz^2 \\ \mu,\ -\frac{1}{2} \end{matrix}\right) - 1 \right].$$

17. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1/2}}{(1-\lambda)_k} w^k L_{n-k}^{\lambda+k}((k+1)w)\, C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{(\lambda)_{n+1}}{wz} \left[\frac{1}{(n+1)!} + \frac{(-1)^n}{(2n+2)!} H_{2n+2}(\sqrt{w}\, z) \right].$$

5.11.8. Sums containing products of $C_{mk+n}^{\lambda k+\mu}(\varphi(k, z))$

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[C_m^{\lambda}(w+kz) \right]^2 = 0 \qquad [2m < n].$$

2. $$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^m \left[C_m^{\lambda}\left(\sqrt{1+\frac{z}{k}}\right) \right]^2 = (-1)^n \frac{(2\lambda)_n^2}{n!} \delta_{m,n} - \frac{(\lambda)_m^2}{(m!)^2} (4z)^m \qquad [m \le n].$$

3. $$\sum_{k=0}^{n}(k+1)^{k-1}\frac{(\lambda)_k}{(1-\mu)_k}(-2w)^k C_{n-k}^{\lambda+k}((k+1)w+1)\,C_{2k}^{\mu-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{(2\lambda-1)_n(2wz^2)^{-1}}{(n+1)!\,(\mu-1)}\left[1-{}_3F_2\left(\begin{matrix}-n-1,\,2\lambda+n-1,\,\mu-1\\ \lambda-\frac{1}{2},\,-\frac{1}{2};\,-\frac{wz^2}{2}\end{matrix}\right)\right].$$

4. $$\sum_{k=0}^{n}(k+1)^{k-1/2}\frac{(\lambda)_k}{(1-\mu)_k}(-2w)^k C_{n-k}^{\lambda+k}((k+1)w+1)\,C_{2k+1}^{\mu-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{(2\lambda-1)_n}{(n+1)!}(wz)^{-1}\left[{}_3F_2\left(\begin{matrix}-n-1,\,2\lambda+n-1,\,\mu\\ \lambda-\frac{1}{2},\,\frac{1}{2};\,-\frac{wz^2}{2}\end{matrix}\right)-1\right].$$

5. $$\sum_{k=0}^{n}(k+1)^{n-1/2}\frac{(\lambda)_k}{(1-\mu)_k}C_{2n-2k}^{\lambda+k}\left(\frac{w}{\sqrt{k+1}}\right)C_{2k+1}^{\mu-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{(\lambda)_{2n+1}}{(2n+2)!}(2w)^{2n+2}z^{-1}\left[1-{}_3F_2\left(\begin{matrix}-n-1,\,-n-\frac{1}{2},\,\mu\\ -\lambda-2n,\,\frac{1}{2};\,\frac{z^2}{w^2}\end{matrix}\right)\right].$$

6. $$\sum_{k=0}^{n}(k+1)^{n}\frac{(\lambda)_k}{(1-\mu)_k}C_{2n-2k+1}^{\lambda+k}\left(\frac{w}{\sqrt{k+1}}\right)C_{2k+1}^{\mu-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$=\frac{(\lambda)_{2n+2}}{(2n+3)!}(2w)^{2n+3}z^{-1}\left[1-{}_3F_2\left(\begin{matrix}-n-1,\,-n-\frac{3}{2},\,\mu\\ -\lambda-2n-1,\,\frac{1}{2};\,\frac{z^2}{w^2}\end{matrix}\right)\right].$$

5.12. The Jacobi Polynomials $P_n^{(\rho,\sigma)}(z)$

5.12.1. Sums containing $P_m^{(\rho\pm pk,\,\sigma\pm qk)}(z)$

1. $$\sum_{k=0}^{n}P_m^{(\rho+k,\sigma)}(z)=\frac{2}{z+1}\left[P_{m+1}^{(\rho+n,\sigma-1)}(z)-P_{m+1}^{(\rho-1,\sigma-1)}(z)\right].$$

2. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_m^{(\rho+k,\sigma)}(z)=\left(-\frac{1+z}{2}\right)^n P_{m-n}^{(\rho+n,\sigma+n)}(z) \qquad [m\ge n].$$

3. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(\rho+\sigma+m+1)_k}{(\rho+m+1)_k(k+1)}\left(\frac{z-1}{2}\right)^k P_m^{(\rho+k,\sigma)}(z)$$
$$=\frac{2}{(n+1)(\rho+\sigma+m)(1-z)}$$
$$\times\left[(\rho+m)P_m^{(\rho-1,\sigma)}(z)-\frac{(m+n+1)!}{m!\,(\rho+m+1)_n}P_{m+n+1}^{(\rho-1,\sigma-n-1)}(z)\right].$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\rho+\sigma+m+1)_k}{(\rho+m+1)_k}P_m^{(\rho+k,\,\sigma)}(z) = \frac{(\rho+1)_m\,(-\sigma-m)_n}{(\rho+1)_n(\rho+n+1)_m}P_m^{(\rho+n,\,\sigma-n)}(z).$$

5. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+n-1)!}{(k+m)!}P_m^{(k,\,\sigma-k)}(z) = \frac{(n-1)!\,(\sigma+m+1)_n}{(m+n)!}\left(\frac{z-1}{2}\right)^n P_{m-n}^{(2n,\,\sigma)}(z) \quad [m\geq n].$$

6. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_m^{(\rho-k,\,\sigma+k)}(z) = P_{m-n}^{(\rho,\,\sigma+n)}(z) \quad [m\geq n].$$

7. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(-\sigma-m)_k}{(\rho+m+1)_k}P_m^{(\rho+k,\,\sigma-k)}(z) = \frac{(\rho+\sigma+m+1)_n}{(\rho+m+1)_n}P_m^{(\rho+n,\,\sigma)}(z).$$

8. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(-\sigma-m)_k}{(1-\sigma-n)_k}P_m^{(\rho+k,\,\sigma-k)}(z) = \frac{(\rho+\sigma+m+1)_n}{(\sigma)_n}\left(-\frac{z+1}{2}\right)^n P_{m-n}^{(\rho+n,\,\sigma+n)}(z) \quad [m\geq n].$$

9. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-\sigma-m)_k}{(\rho-n+1)_k}\left(\frac{1-z}{1+z}\right)^k P_m^{(\rho+k,\,\sigma-k)}(z) = \frac{(m+n)!}{m!\,(-\rho)_n}\left(-\frac{2}{1+z}\right)^n P_{m+n}^{(\rho-n,\,\sigma-n)}(z).$$

5.12.2. Sums containing $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(z)$

1. $$\sum_{k=0}^{n}(-1)^k(2k+\rho+\sigma+1)\frac{(\rho+\sigma+1)_k}{(n-k)!\,(\rho+1)_k(\rho+\sigma+n+2)_k}P_k^{(\rho,\,\sigma)}(z) = \frac{(\rho+\sigma+1)_{n+1}}{n!\,(\rho+1)_n}\left(\frac{1-z}{2}\right)^n.$$

2. $$\sum_{k=0}^{n}(2k+\rho+\sigma+1)\frac{(\rho+\sigma+1)_k^2}{k!\,(\rho+1)_k}P_k^{(\rho,\,\sigma)}(z) = \frac{(n+1)\,(\rho+\sigma+1)_{n+1}^2}{n!\,(\rho+1)_{n+1}(\rho+\sigma+1)}P_{n+1}^{(\rho,\,\sigma)}(z) + \frac{(\rho+\sigma+1)(\rho+\sigma+2)}{2(\rho+1)} \times\left[\frac{(\rho+\sigma+3)_n}{n!}\right]^2(1-z)\,{}_3F_2\left(\begin{matrix}-n,\ \rho+\sigma+2,\ \rho+\sigma+n+3\\ \rho+2,\ \rho+\sigma+3;\ \frac{1-z}{2}\end{matrix}\right).$$

3. $$\sum_{k=0}^{[n/2]}(-1)^k(2n-4k+1)\frac{(n-2k)!\left(-n-\frac{1}{2}\right)_k(-\rho-n)_{2k}}{k!}P_{n-2k}^{(\rho,-\rho)}(z) = 2^n\left(\frac{3}{2}\right)_n P_n^{(\rho,-\rho-n)}(2z-1).$$

4. $$\sum_{k=0}^{[n/2]}\binom{n}{k}\frac{(n-2k+1)^2(n-2k)!}{(n-k+1)(\rho+1)_{n-2k}}P_{n-2k}^{(\rho,1-\rho)}(z) = 2^n\frac{n!}{(\rho+1)_n}P_n^{(\rho,1/2-\rho-n)}(2z-1).$$

5. $$\sum_{k=0}^{n}\binom{2n+1}{n-k}\frac{(2k+1)!}{(\rho+1)_{2k+1}}P_{2k+1}^{(\rho,-\rho-1)}(z) = \frac{2^{2n}(2n+1)!}{(\rho+1)_{2n+1}}P_{2n+1}^{(\rho,-\rho-2n-3/2)}(2z-1).$$

6. $$\sum_{k=0}^{n}\sigma_{k+m}^{m}t^k\frac{(-\rho-n)_k}{(k+m)!}P_{n-k}^{(\rho,\sigma+k)}(z) = \frac{(-1)^{m+n}t^{-m}}{m!(\rho+n+1)_m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kt-1)^{m+n}P_{m+n}^{(\rho,\sigma-m)}\left(\frac{kt-z}{kt-1}\right).$$

7. $$\sum_{k=0}^{n}\left(\frac{2}{z-1}\right)^k P_k^{(\rho-k,\sigma)}(z) = \left(\frac{2}{z-1}\right)^n P_n^{(\rho-n,\sigma+1)}(z).$$

8. $$\sum_{k=0}^{n}(-1)^k\frac{(a)_k}{(\sigma+1)_k}P_k^{(\rho-k,\sigma)}(z) = \frac{(a+1)_n}{n!}\,{}_3F_2\left(\begin{matrix}-n,\,a,\,\rho+\sigma+1\\ a+1,\,\sigma+1;\ \frac{1+z}{2}\end{matrix}\right).$$

9. $$\sum_{k=0}^{n}\frac{1}{(n-k)!\,(a)_k}P_k^{(\rho-k,\sigma)}(z) = \frac{1}{(a)_n}P_n^{(\rho+a-1,\sigma-a-n+1)}(z).$$

10. $$\sum_{k=0}^{n}\frac{1}{(n-k)!\,(\sigma+1)_k}P_k^{(\rho-k,\sigma)}(z) = \frac{(\rho+\sigma+1)_n}{n!\,(\sigma+1)_n}\left(\frac{z+1}{2}\right)^n.$$

11. $$\sum_{k=0}^{n}\frac{1}{(n-k)!\,(\sigma+1)_k}\left(\frac{2}{1-z}\right)^k P_k^{(\rho-k,\sigma)}(z) = \frac{(-\rho)_n}{n!\,(\sigma+1)_n}\left(\frac{z+1}{z-1}\right)^n.$$

12. $$\sum_{k=0}^{n}\frac{(-1)^k k!}{(\sigma+1)_k}P_k^{(\rho-k,\sigma)}(z) = \frac{2\sigma}{\rho+\sigma}(1+z)^{-1}\left[1+(-1)^n\frac{(n+1)!}{(\sigma)_{n+1}}P_{n+1}^{(\rho-n-1,\sigma-1)}(z)\right].$$

13. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(b)_k(\sigma+1)_k} P_k^{(\rho-k,\,\sigma)}(z) = \frac{(b-a)_n}{n!\,(b)_n}\, {}_3F_2\left(\begin{matrix} -n,\ a,\ \rho+\sigma+1;\ \frac{1+z}{2} \\ a-b-n+1,\ \sigma+1 \end{matrix}\right).$$

14. $$\sum_{k=0}^{n} \binom{n}{k}\binom{2n}{k}^{-1} \frac{1}{(n-k)!\,(\sigma+1)_k} P_k^{(\rho-k,\,\sigma)}(z) = \frac{n!}{(2n)!}\, {}_3F_2\left(\begin{matrix} -n,\ -n,\ \rho+\sigma+1 \\ \sigma+1,\ 1;\ \frac{1+z}{2} \end{matrix}\right).$$

15. $$\sum_{k=0}^{n} \frac{(-\sigma-n)_k}{(n-k)!\,(\sigma+1)_k(-\rho-\sigma-n)_k} \left(-\frac{2}{1+z}\right)^k P_k^{(\rho-k,\,\sigma)}(z) = \frac{(\sigma+1)_n}{n!\,(\rho+\sigma+1)_n} \left(\frac{2}{1+z}\right)^n {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ -\rho,\ \rho+\sigma+1 \\ \frac{\sigma+1}{2},\ \frac{\sigma}{2}+1,\ \sigma+1;\ \frac{(1+z)^2}{4} \end{matrix}\right).$$

16. $$\sum_{k=0}^{n} (2k-\rho)\frac{(-\rho)_k}{(\sigma+1)_k} \left(\frac{2}{1-z}\right)^k P_k^{(\rho-2k,\,\sigma)}(z) = \frac{(-\rho)_{n+1}}{(\sigma+1)_n} \left(\frac{2}{1-z}\right)^n P_n^{(\rho-2n-1,\,\sigma)}(z).$$

17. $$\sum_{k=0}^{n} (2k-\rho)\frac{(a)_k(-\rho)_k}{(1-a-\rho)_k(-\rho-\sigma)_k} \left(\frac{2}{1-z}\right)^k P_k^{(\rho-2k,\,\sigma)}(z) = \frac{(a)_n(-\rho)_{2n+1}}{n!\,(1-a-\rho)_n(-\rho-\sigma)_n} \left(\frac{2}{z-1}\right)^n {}_3F_2\left(\begin{matrix} -n,\ -n-a,\ \rho+\sigma-n+1 \\ 1-n-a,\ \rho-2n;\ \frac{1-z}{2} \end{matrix}\right).$$

18. $$\sum_{k=0}^{n} (2k-\rho)\frac{(-n-1)_k(-\rho)_k}{(n-\rho+2)_k(-\rho-\sigma)_k} \left(\frac{2}{1-z}\right)^k P_k^{(\rho-2k,\,\sigma)}(z) = \frac{(-\rho)_{n+2}}{(-\rho-\sigma)_{n+1}} \left(\frac{2}{1-z}\right)^{n+1} \left[1+(-1)^n(n+1)!\,\frac{(-\rho)_{n+1}}{(-\rho)_{2n+2}} P_{n+1}^{(\rho-2n-2,\,\sigma)}(z)\right].$$

19. $$\sum_{k=0}^{n} \frac{(-\rho)_k}{(n-k)!\,(a)_k(-\rho-\sigma)_k} \left(\frac{2}{z-1}\right)^k P_k^{(\rho-2k,\,\sigma)}(z) = \frac{(-\rho)_{2n}}{n!\,(a)_n(-\rho-\sigma)_n} \left(\frac{2}{1-z}\right)^n \times {}_4F_3\left(\begin{matrix} -n,\ \frac{a+\rho-n}{2},\ \frac{a+\rho-n+1}{2},\ \rho+\sigma-n+1 \\ \frac{\rho+1}{2}-n,\ \frac{\rho}{2}-n+1,\ a+\rho-n;\ \frac{1-z}{2} \end{matrix}\right).$$

20. $$\sum_{k=0}^{n} \frac{(-\rho)_k}{(-\rho-\sigma)_k} \left(\frac{2}{1-z}\right)^k P_k^{(\rho-2k,\,\sigma)}(z) = \frac{(-\rho)_{2n}}{n!\,(-\rho-\sigma)_n} \left(\frac{2}{z-1}\right)^n {}_3F_2\left(\begin{matrix} -n,\ \frac{\rho}{2}-n,\ \rho+\sigma-n+1 \\ \frac{\rho}{2}-n+1,\ \rho-2n;\ \frac{1-z}{2} \end{matrix}\right).$$

21. $$\sum_{k=0}^{n} \sigma_{k+m}^{m} t^k \frac{(\rho+\sigma+n+1)_k}{(k+m)!} P_{n-k}^{(\rho+k,\,\sigma+k)}(z) = \frac{t^{-m}}{m!\,(-\rho-\sigma-n)_m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} P_{m+n}^{(\rho-m,\,\sigma-m)}(2kt+z).$$

22. $$\sum_{k=0}^{n} \frac{(\rho+\sigma+1)_k}{(n-k)!\,(1-a-n)_k\left(a+\rho+\sigma+\frac{3}{2}\right)_k} (-2)^{-k}(1+z)^{-k} P_k^{(\rho+k,\,\sigma-k)}(z)$$
$$= \frac{(-\sigma)_n(\rho+\sigma+1)_n}{n!\,(a)_n\left(a+\rho+\sigma+\frac{3}{2}\right)_n} 2^{-n}(1+z)^{-n} \times {}_3F_2\left(\begin{matrix} -n,\ 2a+\rho+\sigma+n+1,\ -2a-\rho-\sigma-n \\ \sigma-n+1,\ -\rho-\sigma-n;\ \frac{1+z}{2} \end{matrix}\right).$$

23. $$\sum_{k=0}^{n} \frac{(\rho+\sigma+1)_k}{(n-k)!\,(a)_k\left(\rho+\sigma-a-n+\frac{5}{2}\right)_k} (-2)^{-k}\,(1+z)^{-k}\, P_k^{(\rho+k,\,\sigma-k)}(z)$$
$$= \frac{(-\sigma)_n(\rho+\sigma+1)_n}{n!\,(a)_n\left(a-\rho-\sigma-\frac{3}{2}\right)_n} 2^{-n}\,(1+z)^{-n} \times {}_3F_2\left(\begin{matrix} -n,\ 2a-\rho-\sigma+n-2,\ \rho+\sigma-2a-n+3 \\ \sigma-n+1,\ -\rho-\sigma-n;\ \frac{1+z}{2} \end{matrix}\right).$$

24. $$\sum_{k=0}^{n} \frac{\left(\rho+\sigma+n+\frac{1}{2}\right)_k}{(n-k)!\,(\rho+1)_{2k}} 2^k(z-1)^k P_k^{(\rho+k,\,\sigma-k)}(z) = \frac{(2n)!}{n!\,(\rho+1)_{2n}} P_{2n}^{(\rho,\,\rho+2\sigma)}(z).$$

25. $$\sum_{k=0}^{n} \frac{\left(a+n-\frac{1}{2}\right)_k(\rho+\sigma+1)_k}{(n-k)!\,(a)_k(\rho+1)_{2k}} 2^k(z-1)^k P_k^{(\rho+k,\,\sigma-k)}(z) = \frac{1}{n!}\,{}_3F_2\left(\begin{matrix} -2n,\ 2n+2a-1,\ \rho+\sigma+1 \\ a,\ \rho+1;\ \frac{1-z}{2} \end{matrix}\right).$$

26. $$\sum_{k=0}^{n} \frac{(a)_k(\rho+\sigma+1)_k}{(n-k)!\,(a-n+\frac{1}{2})_k(\rho+1)_{2k}} 2^k(z-1)^k P_k^{(\rho+k,\,\sigma-k)}(z)$$
$$= (-1)^n \frac{(a)_n(\rho+\sigma+1)_{2n}}{\left(\frac{1}{2}-a\right)_n (\rho+1)_{2n}} (z-1)^{2n}\ {}_3F_2\left(\begin{matrix} -2n,\ \frac{1}{2}-a-n,\ -\rho-2n \\ 1-2a-2n,\ -\rho-\sigma-2n;\ \frac{2}{1-z}\end{matrix}\right).$$

27. $$\sum_{k=0}^{n} \frac{(1-a-n)_k}{(-\rho-\sigma)_k} \left(\frac{2}{1-z}\right)^k P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(a)_n(-\rho)_n}{n!\,(-\rho-\sigma)_n} \left(\frac{2}{1-z}\right)^n {}_3F_2\left(\begin{matrix} -n,\ a-1,\ \rho+\sigma-n+1 \\ a,\ \rho-n+1;\ \frac{1-z}{2}\end{matrix}\right).$$

28. $$\sum_{k=0}^{n} \frac{t^k}{(n-k)!\,(-\rho-\sigma)_k} P_k^{(\rho-k,\,\sigma-k)}(z) = \frac{t^n}{(-\rho-\sigma)_n} P_n^{(\rho-n,\,\sigma-n)}\left(z-\frac{2}{t}\right).$$

29. $$\sum_{k=0}^{n} \frac{1}{(n-k)!\,(-\rho-\sigma)_k(n-k+a)} \left(\frac{2}{z-1}\right)^k P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(-\rho)_n}{n!\,a(-\rho-\sigma)_n} \left(\frac{2}{z-1}\right)^n {}_3F_2\left(\begin{matrix} -n,\ 1,\ \rho+\sigma-n+1 \\ a+1,\ \rho-n+1;\ \frac{1-z}{2}\end{matrix}\right).$$

30. $$\sum_{k=0}^{n} \frac{(\rho+\sigma-n+1)_k}{(n-k)!\,(\rho-n+1)_k(-\rho-\sigma)_k} P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(-\rho-\sigma)_n}{n!\,(-\rho)_n} \left(\frac{1-z}{2}\right)^n {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ -\rho,\ -\sigma;\ \frac{4}{(1-z)^2} \\ -\frac{\rho+\sigma}{2},\ \frac{1-\rho-\sigma}{2},\ -\rho-\sigma \end{matrix}\right).$$

31. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-a-n\right)_k}{(n-k)!\left(\frac{3}{2}-n\right)_k (-\rho-\sigma)_k} \left(\frac{2}{z-1}\right)^k P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{\left(a+\frac{1}{2}\right)_n (-\rho)_n}{2an!\left(-\frac{1}{2}\right)_n (-\rho-\sigma)_n} \left(\frac{2}{1-z}\right)^n$$
$$\times \left[(2a-1)\,{}_3F_2\left(\begin{matrix} -n,\ a,\ \rho+\sigma-n+1 \\ a-\frac{1}{2},\ \rho-n+1;\ \frac{1-z}{2}\end{matrix}\right) + {}_3F_2\left(\begin{matrix} -n,\ a,\ \rho+\sigma-n+1 \\ a+\frac{1}{2},\ \rho-n+1;\ \frac{1-z}{2}\end{matrix}\right)\right].$$

32. $$\sum_{k=0}^{n} \binom{m}{n-k} \frac{(m-\sigma)_k}{(-\rho-\sigma)_k} \left(\frac{2}{z+1}\right)^k P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(-\sigma)_n}{(-\rho-\sigma)_n} \left(\frac{2}{z+1}\right)^n P_n^{(\rho+m-n,\,\sigma-m-n)}(z).$$

33. $$\sum_{k=0}^{n} \frac{\sigma_m^{n-k+1}}{(-\rho-\sigma)_k} \left(\frac{2}{1-z}\right)^k P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(-\rho)_n}{n!\,(-\rho-\sigma)_n} \left(\frac{2}{z-1}\right)^n {}_{m+1}F_m\left(\begin{matrix} -n,\ \rho+\sigma-n+1,\ 2,\ \ldots,\ 2 \\ \rho-n+1,\ 1,\ \ldots,\ 1;\ \frac{1-z}{2} \end{matrix}\right) \quad [m \geq 1].$$

34. $$\sum_{k=0}^{n} \frac{(a)_k}{(n-k)!\,(-\rho-\sigma)_{2k}} \left(\frac{2}{z-1}\right)^k P_k^{(\rho-k,\,\sigma-2k)}(z)$$
$$= \frac{(a)_n(-\rho)_n}{n!\,(-\rho-\sigma)_{2n}} \left(\frac{2}{1-z}\right)^n {}_3F_2\left(\begin{matrix} -n,\ \rho+\sigma-2n+1,\ \rho+\sigma+a-n+1 \\ \rho-n+1,\ 1-a-n;\ \frac{z-1}{2} \end{matrix}\right).$$

35. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-\rho-n\right)_k(-\sigma)_k}{(n-k)!\,(1-\rho-2n)_k(-\rho-\sigma)_{2k}} \left(\frac{4}{1+z}\right)^{2k} P_k^{(\rho-k,\,\sigma-2k)}(z)$$
$$= \frac{(2\rho)_{2n}(-\sigma)_{2n}}{n!\,(\rho)_{2n}(-\rho-\sigma)_{2n}} \left(\frac{2}{1+z}\right)^{2n} {}_3F_2\left(\begin{matrix} -2n,\ \rho,\ \rho+\sigma-2n+1;\ \frac{1+z}{2} \\ 2\rho,\ \sigma-2n+1 \end{matrix}\right).$$

36. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-\rho-n\right)_k(-\sigma)_k}{(n-k)!\,(1-\rho-2n)_k(-\rho-\sigma)_{2k}} \left(\frac{4}{1+z}\right)^{2k} P_k^{(\rho-k,\,\sigma-2k)}(z)$$
$$= \frac{(2\rho)_{2n}(-\sigma)_{2n}}{n!\,(\rho)_{2n}(-\rho-\sigma)_{2n}} \left(\frac{2}{1+z}\right)^{2n} {}_3F_2\left(\begin{matrix} -2n,\ \rho,\ \rho+\sigma-2n+1;\ \frac{1+z}{2} \\ 2\rho,\ \sigma-2n+1 \end{matrix}\right).$$

37. $$\sum_{k=0}^{n} (-1)^k \frac{(a+k\sigma)_{n-1}}{(n-k)!\,(a+k\sigma)_k} P_k^{(\rho+k\sigma,\,\tau-k(\sigma+1))}(z)$$
$$= \frac{(-1)^n}{a+n\sigma+n-1} P_n^{(\rho-a-n+1,\,a+\tau-1)}(z).$$

38. $$\sum_{k=0}^{n} \frac{(a+k\sigma)_{n-1}}{(n-k)!\,(a+k\sigma)_k} \left(\frac{2}{z-1}\right)^k P_k^{(\rho-k,\,\tau-k(\sigma+1))}(z)$$
$$= \frac{1}{a+n\sigma+n-1} \left(\frac{2}{z-1}\right)^n P_n^{(\rho-n,\,a+\tau-1)}(z).$$

39. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2k)!}{(\rho+1)_{2k}} P_{2k}^{(\rho,\,\sigma-2k)}(z)$$
$$= \frac{n!\,(\rho+\sigma+1)_n}{(\rho+1)_{2n}} (1-z)^n P_n^{(\rho+n,\,\sigma-n)}\left(\frac{1+z}{2}\right).$$

40. $$\sum_{k=0}^{n} \binom{n}{k} P_{k+m}^{(\rho-k,\,\sigma)}(z) = P_{m+n}^{(\rho,\,\sigma-n)}(z).$$

41. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{(\sigma+m+1)_k}P_{k+m}^{(\rho-k,\,\sigma)}(z)$$
$$=(-1)^m\frac{(m-a+1)_n(\sigma+1)_m}{(m+n)!}\,{}_3F_2\left(\begin{matrix}-m-n,\,a-m,\,\rho+\sigma+m+1\\ a-m-n,\,\sigma+1;\ \frac{1+z}{2}\end{matrix}\right).$$

42. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(m+1)_k}{(\rho+\sigma+2m-n+2)_k}\left(\frac{2}{1-z}\right)^k P_{k+m}^{(\rho-k,\,\sigma)}(z)$$
$$=\frac{(-\rho-m)_n}{(-\rho-\sigma-2m-1)_n}\left(\frac{2}{1-z}\right)^n P_m^{(\rho-n,\,\sigma)}(z).$$

43. $$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{z+1}\right)^k P_{k+m}^{(\rho-k,\,\sigma-k)}(z)=\left(\frac{2}{z+1}\right)^n P_{m+n}^{(\rho,\,\sigma-n)}(z).$$

44. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-\sigma-m)_k}{(\rho-n+1)_k}\left(\frac{1-z}{2}\right)^k P_{m-k}^{(\rho+k,\,\sigma)}(z)$$
$$=\frac{(\rho+1)_m}{(\rho-n+1)_m}P_m^{(\rho-n,\,\sigma)}(z)\quad[m\ge n].$$

45. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a-\rho-\sigma-m+n-1)_k}{(a)_k}\left(\frac{1-z}{2}\right)^k P_{m-k}^{(\rho+k,\,\sigma)}(z)$$
$$=\frac{(\rho+1)_m}{m!}\,{}_3F_2\left(\begin{matrix}-m,\,\rho+\sigma+m-n+1,\,a+n\\ \rho+1,\,a;\ \frac{1-z}{2}\end{matrix}\right)\quad[m\ge n].$$

46. $$\sum_{k=0}^{n}\binom{n}{k}P_{m-k}^{(\rho+k,\,\sigma)}(z)=P_m^{(\rho+n,\,\sigma-n)}(z)\quad[m\ge n].$$

47. $$\sum_{k=0}^{n}\binom{n}{k}k^r P_{m-k}^{(\rho+k,\,\sigma)}(z)$$
$$=\left(\frac{1-z}{2}\right)^m\sum_{k=1}^{r}\sigma_r^k(-n)_k\left(\frac{2}{z-1}\right)^k P_{m-k}^{(k-\rho-\sigma-2m-1,\,\sigma+k-n)}\left(\frac{z+3}{z-1}\right)$$
$$[m\ge n;\ m\ge r].$$

48. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(m-k)!\,(\rho+\sigma+m+1)_k\left(\frac{z+1}{2}\right)^k P_{m-k}^{(\rho+k,\,\sigma+k)}(z)$$
$$=(m-n)!\,(-\sigma-m)_n P_{m-n}^{(\rho+n,\,\sigma)}(z)\quad[m\ge n].$$

49. $$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{z+1}{2}\right)^k P_{m-k}^{(\rho+k,\,\sigma+k)}(z)=P_m^{(\rho+n,\,\sigma)}(z)\quad[m\ge n].$$

50. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(\rho+\sigma+m+1)_k}{(\sigma-n+1)_k}\left(\frac{z+1}{2}\right)^k P_{m-k}^{(\rho+k,\,\sigma+k)}(z) = \frac{(-\sigma-m)_n}{(-\sigma)_n}P_m^{(\rho+n,\,\sigma-n)}(z) \quad [m\geq n].$$

51. $$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}-a\right)_k(\rho+\sigma+1)_k}{(n-k)!\,(1-a-n)_k(\rho+1)_{2k}}2^k(z-1)^k P_k^{(\rho+k,\,\sigma-k)}(z) = (-1)^n\frac{\left(\frac{1}{2}-a\right)_n(\rho+\sigma+1)_{2n}}{n!\,(a)_n(\rho+1)_{2n}}(z-1)^{2n}\,{}_3F_2\left(\begin{matrix}-2n,\,a-n,\,-\rho-2n\\2a-2n,\,-\rho-\sigma-2n;\,\frac{2}{1-z}\end{matrix}\right).$$

52. $$\sum_{k=0}^{n}\frac{(\rho+\sigma+1)_k}{(n-k)!\,(a)_k\left(\rho+\sigma-a-n-\frac{5}{2}\right)_k}(-2)^{-k}(1+z)^{-k}P_k^{(\rho+k,\,\sigma-k)}(z) = \frac{(-\rho)_n(\rho+\sigma+1)_n}{n!\,(a)_n\left(a-\rho-\sigma-\frac{3}{2}\right)_n}2^{-n}(1+z)^{-n}$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,\,2a-\rho-\sigma+n,\,\rho+\sigma-2a-n+3\\\rho-n-1,\,-\rho-\sigma-n;\,\frac{1+z}{2}\end{matrix}\right).$$

53. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-\sigma-m)_{2k}}{\left(\frac{1}{2}-m-n\right)_k(-\rho-\sigma-m)_k}2^{-k}(z+1)^{-k}P_{m-k}^{(\rho+k,\,\sigma-k)}(z) = \frac{(\rho+\sigma+m+1)_m}{m!}\left(\frac{z+1}{2}\right)^m{}_2F_2\left(\begin{matrix}-2m,\,-m-n,\,-\sigma-m;\,\frac{2}{1+z}\\-2m-2n,\,-\rho-\sigma-2m\end{matrix}\right)$$
$$[m\geq n\geq 1].$$

54. $$\sum_{k=0}^{n}\binom{n}{k}\frac{\left(m+n+\frac{1}{2}\right)_k(\rho+\sigma+m+1)_k}{(\rho+m+1)_{2k}}2^k(z-1)^k P_{k+m}^{(\rho+k,\,\sigma-k)}(z) = \frac{(\rho+1)_m}{m!}\,{}_3F_2\left(\begin{matrix}-m-2n,\,m+2n+1,\,\rho+\sigma+m+1\\m+1,\,\rho+1;\,\frac{1-z}{2}\end{matrix}\right).$$

55. $$\sum_{k=0}^{n}\frac{(-\sigma)_k}{(n-k)!\,(1-a-n)_k\left(a-\sigma+\frac{1}{2}\right)_k}(-4)^{-k}P_k^{(\rho+k,\,\sigma-2k)}(z) = \frac{(-\sigma)_n(\rho+\sigma+1)_n}{n!\,(a)_n\left(a-\sigma+\frac{1}{2}\right)_n}\left(\frac{z+1}{8}\right)^n{}_3F_2\left(\begin{matrix}-n,\,2a-\sigma+n,\,\sigma-2a-n+1\\\sigma-n+1,\,-\rho-\sigma-n;\,\frac{2}{z+1}\end{matrix}\right).$$

56. $$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}-a\right)_k(-\sigma)_k}{(n-k)!\,(1-a-n)_k(-\rho-\sigma)_{2k}}\left(\frac{4}{z+1}\right)^{2k}P_k^{(\rho-k,\,\sigma-2k)}(z) = (-1)^n\frac{\left(\frac{1}{2}-a\right)_n(-\sigma)_{2n}}{n!\,(a)_n(-\rho-\sigma)_{2n}}\left(\frac{4}{z+1}\right)^{2n}{}_3F_2\left(\begin{matrix}-2n,\,a-n,\,\rho+\sigma-2n+1\\2a-2n,\,\sigma-2n+1;\,\frac{z+1}{2}\end{matrix}\right).$$

57. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(\rho+\sigma+m+1)_{2k}}{\left(\frac{1}{2}-m-n\right)_k(\sigma+m+1)_k}\left(\frac{z+1}{4}\right)^{2k}P_{m-k}^{(\rho+k,\,\sigma+2k)}(z)$$
$$=(-1)^m\frac{(\sigma+1)_m}{m!}\,{}_3F_2\left(\begin{matrix}-2m,\,-m-n,\,\rho+\sigma+m+1\\-2m-2n,\,\sigma+1;\;\frac{z+1}{2}\end{matrix}\right)\quad[m\geq n\geq 1].$$

58. $$\sum_{k=0}^{n}\frac{(-\sigma)_k}{(n-k)!\,(a)_k\left(\frac{3}{2}-a-\sigma-n\right)_k}(-4)^{-k}P_k^{(\rho+k,\,\sigma-2k)}(z)$$
$$=\frac{(-\sigma)_n(\rho+\sigma+1)_n}{n!\,(a)_n\left(a+\sigma-\frac{1}{2}\right)_n}\left(\frac{1+z}{8}\right)^n{}_3F_2\left(\begin{matrix}-n,\,2a+\sigma+n-1,\,2-2a-\sigma-n\\\sigma-n+1,\,-\rho-\sigma-n;\;\frac{2}{1+z}\end{matrix}\right).$$

59. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-\rho-2m)_k}{\left(\frac{1}{2}-\rho-2m-n\right)_k}2^{-2k}P_{2m-2k}^{(\rho+k,\,\sigma+k)}(z)$$
$$=\frac{(\rho+\sigma+2m+1)_{2m}}{(2m)!}\left(\frac{1-z}{2}\right)^{2m}{}_3F_2\left(\begin{matrix}-2m,\,-\rho-2m-n,\,-2\rho-4m\\-2\rho-4m-2n,\,-\rho-\sigma-4m;\;\frac{2}{1-z}\end{matrix}\right)$$
$[m\geq n]$.

60. $$\sum_{k=0}^{n}\frac{(1-2a-n)_k\left(\frac{1}{2}\right)_k}{(n-k)!\,(1-a-n)_k(\sigma+1)_k(-\rho-\sigma)_k}\left(-\frac{2}{1+z}\right)^kP_{2k}^{(\rho-2k,\,\sigma-k)}(z)$$
$$=\frac{(2a)_n(\rho+\sigma+1)_n}{n!\,(a)_n(\sigma+1)_n}\left(\frac{1+z}{8}\right)^n$$
$$\times\,{}_4F_3\left(\begin{matrix}-n,\,\frac{1}{2}-a-n,\,-\frac{\sigma+n}{2},\,\frac{1-\sigma-n}{2}\\a+\frac{1}{2},\,-\frac{\rho+\sigma+n}{2},\,\frac{1-\rho-\sigma-n}{2};\;\frac{4}{(1+z)^2}\end{matrix}\right).$$

61. $$\sum_{k=0}^{n}\frac{(a)_k\left(\frac{1}{2}\right)_k}{(n-k)!\left(\frac{a-n+1}{2}\right)_k(\sigma+1)_k(-\rho-\sigma)_k}\left(-\frac{2}{1+z}\right)^kP_{2k}^{(\rho-2k,\,\sigma-k)}(z)$$
$$=\frac{(a)_n(-\sigma)_n}{n!\left(\frac{1-a-n}{2}\right)_n(-\rho-\sigma)_n}(-2)^{-n}(1+z)^{-n}$$
$$\times\,{}_4F_3\left(\begin{matrix}-n,\,\frac{a-n}{2},\,\frac{\rho+\sigma-n+1}{2},\,\frac{\rho+\sigma-n}{2}+1\\1-\frac{a+n}{2},\,\frac{\sigma-n+1}{2},\,\frac{\sigma-n}{2}+1;\;\frac{(1+z)^2}{4}\end{matrix}\right).$$

62. $$\sum_{k=0}^{n}\frac{(1-a-n)_k}{(n-k)!\,(\sigma+1)_k(-\rho-\sigma)_k}\left(-\frac{2}{1+z}\right)^kP_{2k}^{(\rho-2k,\,\sigma-k)}(z)$$
$$=\frac{(a)_n(-\sigma)_n}{(2n)!\,(-\rho-\sigma)_n}\left(-\frac{2}{1+z}\right)^n{}_3F_2\left(\begin{matrix}-2n,\,1-a-2n,\,\rho+\sigma-n+1\\a,\,\sigma-n+1;\;-\frac{1+z}{2}\end{matrix}\right).$$

63. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(\rho+\sigma+2m+1)_k}{\left(\rho+\sigma+2m-n+\frac{3}{2}\right)_k}\left(\frac{1+z}{4}\right)^{2k}P_{2m-2k}^{(\rho+k,\,\sigma+2k)}(z)$$
$$=\frac{(\sigma+1)_{2m}}{(2m)!}\,{}_3F_2\left(\begin{matrix}-2m,\,\rho+\sigma+2m-n+1,\,2\rho+2\sigma+4m+2\\ \sigma+1,\,2\rho+2\sigma+4m-2n+2;\ \frac{1+z}{2}\end{matrix}\right)\quad[m\ge n].$$

64. $$\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k)!}{k!}(-\rho-n)_{2k}\left(\frac{1+z}{4}\right)^{2k}P_{n-2k}^{(\rho,\,\sigma+2k)}(z)$$
$$=(\sigma+1)_n\left(\frac{z-1}{2}\right)^n{}_3F_2\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,-n-\rho\\ -2n-1,\,\sigma+1;\ \frac{2z+2}{z-1}\end{matrix}\right).$$

65. $$\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k)!}{k!}(\rho+\sigma+n+1)_{2k}\left(\frac{1+z}{4}\right)^{2k}P_{n-2k}^{(\rho+2k,\,\sigma+2k)}(z)$$
$$=(-1)^n(\sigma+1)_n\,{}_3F_2\left(\begin{matrix}-n,\,-n-\frac{1}{2},\,\rho+\sigma+n+1\\ -2n-1,\,\sigma+1;\ z+1\end{matrix}\right).$$

66. $$\sum_{k=0}^{[n/3]}(\rho+\sigma+2n-6k+1)\frac{(-\rho-n)_{3k}}{k!\,(-\rho-\sigma-n)_{3k}}\left(-\frac{\rho+\sigma+2n+1}{3}\right)_k$$
$$\times P_{n-3k}^{(\rho,\sigma)}(z)=-\frac{(-\rho-\sigma-2n-1)_{n+1}}{n!}\left(\frac{1-z}{2}\right)^n$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,\,-\rho-n,\,-\frac{\rho+\sigma+2n+1}{3}\\ -\frac{\rho+\sigma+2n+1}{2},\,-\frac{\rho+\sigma+2n}{2};\ \frac{3}{2-2z}\end{matrix}\right).$$

67. $$\sum_{k=0}^{[n/3]}(\rho+6k)\frac{\left(\frac{\rho}{3}\right)_k(\rho+\sigma+n+1)_{3k}}{k!\,(\rho+n+1)_{3k}}\left(\frac{1-z}{2}\right)^{3k}P_{n-3k}^{(\rho+6k,\,\sigma)}(z)$$
$$=\frac{(\rho)_{n+1}}{n!}\,{}_3F_2\left(\begin{matrix}-n,\,\frac{\rho}{3},\,\rho+\sigma+n+1\\ \frac{\rho}{2},\,\frac{\rho+1}{2};\ \frac{3-3z}{8}\end{matrix}\right).$$

68. $$\sum_{k=0}^{n}\frac{(1-a-n)_k\left(a+2n-\frac{1}{2}\right)_k}{(n-k)!\,(\sigma+1)_{2k}(-\rho-\sigma)_k}\left(\frac{8}{1+z}\right)^kP_{3k}^{(\rho-3k,\,\sigma-k)}(z)$$
$$=\frac{(a)_n\left(a+2n-\frac{1}{2}\right)_n(\rho+\sigma+1)_{2n}}{(3n)!\,(\sigma+1)_{2n}}(1+z)^{2n}$$
$$\times\,{}_4F_3\left(\begin{matrix}-3n,\,a-\frac{1}{2},\,1-a-3n,\,-\sigma-2n\\ 2a-1,\,2-2a-6n,\,-\rho-\sigma-2n;\ \frac{8}{1+z}\end{matrix}\right).$$

69. $$\sum_{k=0}^{n} \frac{(1-a-n)_k \left(a+2n-\frac{1}{2}\right)_k}{(n-k)!\,(\sigma+1)_k(-\rho-\sigma)_{2k}} \left(\frac{4}{1+z}\right)^{2k} P_{3k}^{(\rho-3k,\,\sigma-2k)}(z)$$
$$= \frac{(a)_n\left(a+2n-\frac{1}{2}\right)_n(-\sigma)_{2n}}{(3n)!\,(-\rho-\sigma)_{2n}}\left(\frac{4}{1+z}\right)^{2n}$$
$$\times\, {}_4F_3\left(\begin{matrix}-3n,\, a-\frac{1}{2},\, 1-a-3n,\, \rho+\sigma-2n+1\\ 2a-1,\, 2-2a-6n,\, \sigma-2n+1;\; 2z+2\end{matrix}\right).$$

5.12.3. Sums containing $P_{n\pm mk}^{(\rho\pm pk,\,\sigma\pm qk)}(z)$ and special functions

1. $$\sum_{k=0}^{n} \frac{1}{(n-k+1)!\,(-\rho-\sigma)_k}\left(\frac{2}{z-1}\right)^{k} \psi(a-k) P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{1}{(-\rho-\sigma)_{n+1}}\left(\frac{2}{1-z}\right)^{n+1}\psi(a-n-1)\left[\frac{(-\rho)_{n+1}}{(n+1)!} - P_{n+1}^{(\rho-n-1,\,\sigma-n-1)}(z)\right]$$
$$+ \frac{(-\rho)_n}{n!\,(-\rho-\sigma)_n(a-n-1)}\left(\frac{2}{1-z}\right)^{n} {}_4F_3\left(\begin{matrix}-n,\, \rho+\sigma-n+1,\, 1,\, 1;\; \frac{1-z}{2}\\ \rho-n+1,\, a-n,\, 2\end{matrix}\right).$$

2. $$\sum_{k=0}^{n} \frac{(1-a)_k}{(-\rho-\sigma)_k}\left(\frac{2}{1-z}\right)^{k} \psi(a-k) P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(-\rho)_n(1-a)_n}{n!\,(-\rho-\sigma)_n(a-n-1)}\left(\frac{2}{z-1}\right)^{n}$$
$$\times\left[(a-n-1)\psi(a-n-1)\, {}_3F_2\left(\begin{matrix}-n,\, \rho+\sigma-n+1,\, a-n-1\\ \rho-n+1,\, a-n;\; \frac{1-z}{2}\end{matrix}\right)\right.$$
$$\left. + \, {}_4F_3\left(\begin{matrix}-n,\, \rho+\sigma-n+1,\, a-n-1,\, a-n-1\\ \rho-n+1,\, a-n,\, a-n;\; \frac{1-z}{2}\end{matrix}\right)\right].$$

3. $$\sum_{k=0}^{n}\left(\sqrt{w^2-1}-w\right)^{k} \frac{(-n)_k}{(\sigma+1)_k} P_{n-k}(w) P_k^{(\rho-k,\,\sigma)}(z)$$
$$= 2^n \frac{\left(\frac{1}{2}\right)_n}{n!}(w^2-1)^{n/2}\, {}_3F_2\left(\begin{matrix}-n,\, -n,\, \rho+\sigma+1\\ \sigma+1,\, \frac{1}{2}-n;\; \frac{z+1}{4}-\frac{z+1}{4}w(w^2-1)^{-1/2}\end{matrix}\right).$$

4. $$\sum_{k=0}^{n} \frac{(-2i)^k}{(1-z^2)^{k/2}} P_{n-k}\left(\frac{iz}{\sqrt{1-z^2}}\right) P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(-2i)^n}{(1-z^2)^{n/2}} P_n^{(\rho-n-1/2,\,\sigma-n-1/2)}(z).$$

5. $$\sum_{k=0}^{n} \frac{1}{(2n-2k)!\,(\sigma+1)_k}\left(\frac{2}{z-1}\right)^k H_{2n-2k}(w)\,P_k^{(\rho-k,\sigma)}(z)$$
$$= \frac{(-\rho)_n}{n!\,(\sigma+1)_n}\left(\frac{1+z}{1-z}\right)^n {}_2F_2\left(\begin{matrix}-n,\ -n-\sigma;\ \dfrac{w^2(z-1)}{z+1}\\ \frac{1}{2},\ \rho-n+1\end{matrix}\right).$$

6. $$\sum_{k=0}^{n} \frac{1}{(2n-2k+1)!\,(\sigma+1)_k}\left(\frac{2}{z-1}\right)^k H_{2n-2k+1}(w)\,P_k^{(\rho-k,\sigma)}(z)$$
$$= \frac{2(-\rho)_n w}{n!\,(\sigma+1)_n}\left(\frac{1+z}{1-z}\right)^n {}_2F_2\left(\begin{matrix}-n,\ -n-\sigma;\ \dfrac{w^2(z-1)}{z+1}\\ \frac{3}{2},\ \rho-n+1\end{matrix}\right).$$

7. $$\sum_{k=0}^{n} \frac{(-w)^k}{(\sigma+1)_k} L_{n-k}^{\lambda+k}(w)\,P_k^{(\rho-k,\sigma)}(z)$$
$$= \frac{(\rho+\sigma+1)_n}{n!\,(\sigma+1)_n}(-w)^n\left(\frac{z+1}{2}\right)^n {}_3F_1\left(\begin{matrix}-n,\ -\lambda-n,\ -\sigma-n\\ -\rho-\sigma-n;\ -\dfrac{2}{w(z+1)}\end{matrix}\right).$$

8. $$\sum_{k=0}^{n} \frac{(-\lambda-n)_k}{(-\rho-\sigma)_k}\left(\frac{2}{1-z}\right)^k L_{n-k}^{\lambda}(w)\ P_k^{(\rho-k,\sigma-k)}(z)$$
$$= \frac{(\lambda+1)_n(-\rho)_n}{n!\,(-\rho-\sigma)_n}\left(\frac{2}{1-z}\right)^n {}_2F_2\left(\begin{matrix}-n,\ \rho+\sigma-n+1;\ \dfrac{w-wz}{2}\\ \lambda+1,\ \rho-n+1\end{matrix}\right).$$

9. $$\sum_{k=0}^{n} \frac{\left(\dfrac{2w}{1-z}\right)^k}{(-\rho-\sigma)_k} L_{n-k}^{\lambda+k}(w)\,P_k^{(\rho-k,\sigma-k)}(z)$$
$$= \frac{(-\rho)_n}{n!\,(-\rho-\sigma)_n}\left(\frac{2w}{z-1}\right)^n {}_3F_1\left(\begin{matrix}-n,\ -\lambda-n,\ \rho+\sigma-n+1\\ \rho-n+1;\ \dfrac{z-1}{2w}\end{matrix}\right).$$

10. $$\sum_{k=0}^{n}(2k+\rho+\sigma+1)\frac{(\rho+\sigma+1)_k}{(\rho+1)_k(\rho+\sigma+n+2)_k} w^k L_{n-k}^{2k+\rho+\sigma+1}(w)\,P_k^{(\rho,\sigma)}(z)$$
$$= \frac{(\rho+\sigma+1)_{n+1}}{(\rho+1)_n} L_n^{\rho}\left(\frac{w-wz}{2}\right).$$

11. $$\sum_{k=0}^{n} \frac{(-\lambda-n)_k}{(\sigma+1)_k}\left(\frac{2}{z-1}\right)^k L_{n-k}^{\lambda}(w)\,P_k^{(\rho-k,\sigma)}(z)$$
$$= \frac{(-\rho)_n(\lambda+1)_n}{n!\,(\sigma+1)_n}\left(\frac{z+1}{z-1}\right)^n {}_2F_2\left(\begin{matrix}-n,\ -n-\sigma;\ \dfrac{w(z-1)}{z+1}\\ \lambda+1,\ \rho-n+1\end{matrix}\right).$$

12. $$\sum_{k=0}^{n}\left(\frac{2}{z-1}\right)^k L_{n-k}^{-\sigma-n-1}(w)\,P_k^{(\rho-k,\sigma)}(z) = \left(\frac{z+1}{z-1}\right)^n L_n^{\rho-n}\left(w\,\frac{z-1}{z+1}\right).$$

13. $$\sum_{k=0}^{n} \frac{1}{(\sigma+1)_k} \left(\frac{2w}{z-1}\right)^k L_{n-k}^{\lambda+k}(w) P_k^{(\rho-k,\,\sigma)}(z)$$
$$= \frac{(-\rho)_n}{n!\,(\sigma+1)_n} \left(w\,\frac{1+z}{1-z}\right)^n {}_3F_1\left(\begin{matrix} -n,\ -n-\lambda,\ -n-\sigma \\ \rho-n+1;\ \frac{1-z}{w(1+z)} \end{matrix}\right).$$

14. $$\sum_{k=0}^{n} \left(\frac{2}{z-1}\right)^k L_{n-k}^{k-n-\sigma}(w) P_k^{(\rho-k,\,\sigma-k)}(z) = \left(\frac{z+1}{z-1}\right)^n L_n^{\rho-n}\left(w\,\frac{z-1}{z+1}\right).$$

15. $$\sum_{k=0}^{n} 2^{2k} \frac{(\lambda)_k}{(-\rho-\sigma)_k} \left(\frac{1-w}{1-z}\right)^k C_{n-k}^{\lambda+k}(w) P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{2^{2n}(\lambda)_n(-\rho)_n}{n!\,(-\rho-\sigma)_n} \left(\frac{w-1}{1-z}\right)^n {}_3F_2\left(\begin{matrix} -n,\ \frac{1}{2}-\lambda-n,\ \rho+\sigma-n+1 \\ 1-2\lambda-2n,\ \rho-n+1;\ \frac{1-z}{1-w} \end{matrix}\right).$$

16. $$\sum_{k=0}^{n} (w^2-1)^k \frac{(\lambda)_k}{(\sigma+1)_k} C_{2n-2k}^{\lambda+k}(w) P_k^{(\rho-k,\,\sigma)}(z)$$
$$= w^{2n} \frac{(2\lambda)_{2n}}{(2n)!} {}_3F_2\left(\begin{matrix} -n,\ \frac{1}{2}-n,\ \rho+\sigma+1 \\ \lambda+\frac{1}{2},\ \sigma+1;\ \frac{1}{2}(1-w^{-2})(z+1) \end{matrix}\right).$$

17. $$\sum_{k=0}^{n} (w^2-1)^k \frac{(\lambda)_k}{(\sigma+1)_k} C_{2n-2k+1}^{\lambda+k}(w) P_k^{(\rho-k,\,\sigma)}(z)$$
$$= w^{2n+1} \frac{(2\lambda)_{2n+1}}{(2n+1)!} {}_3F_2\left(\begin{matrix} -n,\ -n-\frac{1}{2},\ \rho+\sigma+1 \\ \lambda+\frac{1}{2},\ \sigma+1;\ \frac{1}{2}(1-w^{-2})(z+1) \end{matrix}\right).$$

18. $$\sum_{k=0}^{n} \frac{(2\lambda)_{2k}}{\left(\lambda+\frac{1}{2}\right)_k (2\lambda+n)_k} C_{n-k}^{\lambda+k}(z) P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{(\lambda+\rho-1/2,\,\lambda+\sigma-1/2)}(z).$$

19. $$\sum_{k=0}^{n} \frac{(\lambda)_k}{(-\rho-\sigma)_k} \left(\frac{4w}{z-1}\right)^k C_{n-k}^{\lambda+k}(w)\, P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \frac{(\lambda)_n(-\rho)_n}{n!\,(-\rho-\sigma)_n} \left(\frac{4w}{z-1}\right)^n {}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2}\ \frac{\rho+\sigma-n+1}{2},\ \frac{\rho+\sigma-n+2}{2};\ \frac{(1-z)^2}{4w^2} \\ \frac{\rho-n+1}{2},\ \frac{\rho-n+2}{2},\ 1-\lambda-n \end{matrix}\right).$$

5.12.4. Sums containing products of $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(z)$

1. $$\sum_{k=0}^{n} \frac{(\rho+\sigma+n+1)_k}{(\mu+1)_k} \left(\frac{1-z}{2}\right)^k P_k^{(\mu,\,\nu-k)}(w) P_{n-k}^{(\rho+k,\,\sigma+k)}(z) = \frac{(\rho+1)_n}{n!}\, {}_3F_2\left(\begin{matrix} -n,\, \mu+\nu+1,\, \rho+\sigma+n+1 \\ \mu+1,\, \rho+1;\ \frac{(1-w)(1-z)}{4}\end{matrix}\right).$$

2. $$\sum_{k=0}^{n} \frac{(\rho+\sigma+n+1)_k}{(-\mu-\nu)_k} \left(\frac{1-z}{1-w}\right)^k P_k^{(\mu-k,\,\nu-k)}(w) P_{n-k}^{(\rho+k,\,\sigma+k)}(z) = \frac{(\rho+1)_n}{n!}\, {}_3F_2\left(\begin{matrix} -n,\, -\mu,\, \rho+\sigma+n+1 \\ -\mu-\nu,\, \rho+1;\ \frac{1-z}{1-w}\end{matrix}\right).$$

3. $$\sum_{k=0}^{n} \frac{(-n-\sigma)_k}{(\nu+1)_k} \left(\frac{1-z}{2}\right)^k P_k^{(\mu-k,\,\nu)}(w) P_{n-k}^{(\rho-k,\,\sigma)}(z) = \frac{(\rho+1)_n}{n!} \left(\frac{1+z}{2}\right)^n {}_3F_2\left(\begin{matrix} -n,\, -n-\sigma,\, \mu+\nu+1 \\ \nu+1,\, \rho+1;\ \frac{(w+1)(z-1)}{2(z+1)}\end{matrix}\right).$$

4. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\left(\rho+m+\frac{3}{2}\right)_k}{\left(\frac{1}{2}-\rho-m-n\right)_k} P_{m-k}^{(\rho+k,\,k+1/2)}(z) P_{m-n+k}^{(\rho+n-k,\,n-k-1/2)}(z)$$
$$= (-1)^m \frac{(-2m)_n \left(\rho+\frac{1}{2}\right)_n (\rho+1)_m (2\rho+2m+2)_n (-\rho-m)_{m-n}}{(m!)^2 \left(\rho+m+\frac{1}{2}\right)_n (2\rho+1)_n}$$
$$\times\, {}_3F_2\left(\begin{matrix} n-2m,\, \rho+n+\frac{1}{2},\, 2\rho+2m+n+2 \\ \rho+n+1,\, 2\rho+n+1;\ \frac{1-z}{2}\end{matrix}\right) \quad [m \ge n].$$

5. $$\sum_{k=0}^{n} \frac{(\rho+\sigma+m+1)_k}{k!} \left(\frac{z^2-1}{4}\right)^k P_{m-k}^{(\rho+k,\,\sigma+k)}(z) P_{n-k}^{(\rho+k-n,\,\sigma+k-n)}(z) = \binom{m+n}{m} P_{m+n}^{(\rho-n,\,\sigma-n)}(z).$$

6. $$\sum_{k=0}^{n} \frac{(\rho+\sigma+m+1)_k}{k!} \left(\frac{z^2-1}{4}\right)^k P_{m-k}^{(\rho+k,\,\sigma+k)}(z) P_{n-k}^{(\rho+k+m,\,\sigma+k-n)}(z) = \binom{m+n}{m} P_{m+n}^{(\rho,\,\sigma-n)}(z).$$

5.12.5. Sums containing $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(\varphi(k,\,z))$

1. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P_m^{(\rho,\,\sigma)}(w+kz) = 0 \quad [m < n].$$

2. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_{m+n}^{(\rho,\sigma)}(w+kz)=n!\,(\rho+\sigma+m+n+1)_n\left(-\frac{z}{2}\right)^n \times\sum_{k=0}^{m}\sigma_{k+n}^{n}\frac{(\rho+\sigma+m+2n+1)_k}{(k+n)!}\left(\frac{z}{2}\right)^k P_{m-k}^{(\rho+k+n,\,\sigma+k+n)}(w).$$

3. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k-z)^m\,P_m^{(\rho,\sigma)}\left(\frac{k+z}{k-z}\right)=(-1)^m(\rho+1)_m\delta_{m,n} -\frac{(\sigma+1)_m}{m!}z^m\quad [n\geq m].$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(ka+z)^{m+n}P_{m+n}^{(\rho,\sigma)}\left(\frac{ka+w}{ka+z}\right) =n!\,(\rho+m+1)_n(-a)^n z^m\sum_{k=0}^{m}\sigma_{k+n}^{n}\left(-\frac{a}{z}\right)^k\frac{(-\rho-m)_k}{(k+n)!}P_{m-k}^{(\rho,\,\sigma+k+n)}\left(\frac{w}{z}\right).$$

5. $$\sum_{k=1}^{n}\frac{k^{k-1}}{k!}(\rho+\sigma+n+1)_k\left(-\frac{z}{2}\right)^k P_{n-k}^{(\rho+k,\,\sigma+k)}(1+kz) =-\frac{(\rho+2)_{n-1}(\rho+\sigma+n+1)z}{2(n-1)!}.$$

6. $$\sum_{k=0}^{n}\frac{(ka+b)^{n-k-1}}{(n-k)!\,(-\rho-\sigma)_k}\left(\frac{2a}{z}\right)^k P_k^{(\rho-k,\,\sigma-k)}(1+kz) =\frac{\left(\frac{2a}{z}\right)^n}{(na+b)(-\rho-\sigma)_n}P_n^{(\rho-n,\,\sigma-n)}\left(1-\frac{bz}{a}\right).$$

7. $$\sum_{k=0}^{n}\frac{(ka+1)^{n-k-1}}{(n-k)!\,(-\rho-\sigma)_k}\left(\frac{2}{z}\right)^k P_k^{(\rho-k,\,\sigma-k)}(1+(ka+1)z) =\frac{(-\rho)_n\left(-\frac{2}{z}\right)^n}{n!\,(na+1)(-\rho-\sigma)_n}.$$

8. $$\sum_{k=1}^{n}\frac{k^{2k}}{k^2+a^2}(\sigma+n+1)_k\frac{\left(-\frac{z}{2}\right)^k}{(k+n)!}P_{n-k}^{(2k,\,\sigma)}(1+k^2z) =-\frac{a^{-2}}{2(n!)}-\frac{a^{-2}}{(n+1)!\,(\sigma+n)z}\left[{}_3F_2\left(\begin{matrix}-n-1,\,\sigma+n,\,1\\ ia,\,-ia;\ \frac{a^2z}{2}\end{matrix}\right)-1\right].$$

9. $$\sum_{k=1}^{n}k^{2k}\frac{(\sigma+n+1)_k}{(k+n)!}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k,\,\sigma)}(1+k^2z) =\frac{1}{2(n!)}\left[{}_3F_0\left(\begin{matrix}-n,\,\sigma+n+1,\,1\\ \frac{z}{2}\end{matrix}\right)-1\right].$$

10. $$\sum_{k=1}^{n} k^{2k-2}\frac{(\sigma+n+1)_k}{(k+n)!}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k,\,\sigma)}(1+k^2z) = -\frac{(\sigma+n+1)z}{4(n-1)!}.$$

11. $$\sum_{k=1}^{n} k^{2k-4}\frac{(\sigma+n+1)_k}{(k+n)!}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k,\,\sigma)}(1+k^2z)$$
$$= -\frac{(\sigma+n+1)(\sigma+n+2)z^2}{32(n-2)!} - \frac{(\sigma+n+1)z}{4(n-1)!}.$$

12. $$\sum_{k=1}^{n} k^{2k-6}\frac{(\sigma+n+1)_k}{(k+n)!}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k,\,\sigma)}(1+k^2z)$$
$$= -\frac{(\sigma+n+1)(\sigma+n+2)(\sigma+n+3)z^3}{576(n-3)!} - \frac{5(\sigma+n+1)(\sigma+n+2)z^2}{128(n-2)!} - \frac{(\sigma+n+1)z}{4(n-1)!}.$$

13. $$\sum_{k=0}^{n}(2k+\rho)^{2k-1}\frac{(\rho)_k(\sigma+a+n+1)_k}{k!\,(a+n+1)_k}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k+\rho,\,\sigma)}(1+(2k+\rho)^2z)$$
$$= \frac{(\rho+1)_n}{n!\rho}.$$

14. $$\sum_{k=0}^{n}(2k+1)^{2k-3}\frac{(\sigma+n+2)_k}{(k+n+1)!}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k+1,\,\sigma)}(1+(2k+1)^2z)$$
$$= \frac{1}{n!} + \frac{2(\sigma+n+2)z}{9(n-1)!}.$$

15. $$\sum_{k=0}^{n}(2k+1)^{2k-5}\frac{(\sigma+n+2)_k}{(k+n+1)!}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k+1,\,\sigma)}(1+(2k+1)^2z)$$
$$= \frac{1}{n!} + \frac{20(\sigma+n+2)z}{81(n-1)!} + \frac{4(\sigma+n+2)(\sigma+n+3)z^2}{225(n-2)!}.$$

16. $$\sum_{k=0}^{n}\frac{(2k+1)^{2k+1}}{(2k+1)^2+a^2}\,\frac{(\sigma+n+2)_k}{(k+n+1)!}\left(-\frac{z}{2}\right)^k P_{n-k}^{(2k+1,\,\sigma)}(1+(2k+1)^2z)$$
$$= \frac{z^{-1}}{2(n+1)!\,(\sigma+n+1)\,a^2}\left[1 - {}_3F_2\left(\begin{matrix}-n-1,\ 1,\ \sigma+n+1\\ \frac{1-ia}{2},\ \frac{1+ia}{2};\ \frac{a^2z}{2}\end{matrix}\right)\right].$$

17. $$\sum_{k=1}^{n}\frac{k^{n-1}}{k!}(-\rho-n)_k P_{n-k}^{(\rho,\,\sigma+k)}\left(1+\frac{z}{k}\right)$$
$$= -\frac{(\rho+\sigma+n+1)_{n-1}(\rho+n)}{(n-1)!}\left(\frac{z}{2}\right)^{n-1}.$$

18. $$\sum_{k=1}^{n} \frac{(-ka)^k (ka+b)^{n-k-1}}{(n-k)!\,(\rho+1)_k} P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{k}\right) = -\frac{b^{n-1}}{n!} + \frac{b^n}{(na+b)(\rho+1)_n} P_n^{(\rho,\,\sigma-n)}\left(1-\frac{az}{b}\right).$$

19. $$\sum_{k=0}^{n} \frac{a^k (ka+b)^{n-k-1}}{(n-k)!\,(\sigma+1)_k} (z+k)^k P_k^{(\rho-k,\,\sigma)}\left(\frac{z-k}{z+k}\right) = \frac{(az-b)^n}{(na+b)(\sigma+1)_n} P_n^{(\rho-n,\,\sigma)}\left(\frac{az+b}{az-b}\right).$$

20. $$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\rho+1)_k} (ka+1)^{n-1} P_k^{(\rho,\,\sigma-k)}\left(\frac{ka+z}{ka+1}\right) = \frac{(\rho+\sigma+1)_n}{n!\,(na+1)(\rho+1)_n} \left(\frac{1-z}{2}\right)^n.$$

5.12.6. Sums containing $P_{n\pm mk}^{(\rho\pm pk,\,\sigma\pm qk)}(\varphi(k,\,z))$ and special functions

1. $$\sum_{k=1}^{n} (4k)^k (n-k+1)^{n-k-1} \frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(\rho+1)_k} H_{2n-2k}\left(\frac{w}{\sqrt{n-k+1}}\right) \times P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{k}\right) = \frac{(n+1)^{n-1}}{n!} H_{2n}\left(\frac{w}{\sqrt{n+1}}\right) + (-1)^n (2w)^{2n} \sum_{k=0}^{n} \left(\frac{n+1}{w^2}\right)^k \frac{\left(\frac{1}{2}-n\right)_k}{(n-k+1)!\,(\rho+1)_k} P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{n+1}\right)$$
$[n \geq 1]$.

2. $$\sum_{k=0}^{n} (-4)^k (k+1)^{n-1} \frac{\left(\frac{1}{2}-n\right)_k}{(n-k)!\,(\rho+1)_k} \times H_{2n-2k}\left(\frac{w}{\sqrt{k+1}}\right) P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{k+1}\right) = \frac{\rho(2w)^{2n+2} z^{-1}}{(n+1)!\,(2n+1)(\rho+\sigma)} \left[{}_3F_1\left(\begin{matrix}-n-1,\ -n-\frac{1}{2},\ \rho+\sigma\\ \rho;\ \frac{z}{2w^2}\end{matrix}\right) - 1\right].$$

3. $$\sum_{k=0}^{n} (-4)^k (k+1)^{n-1/2} \frac{\left(-\frac{1}{2}-n\right)_k}{(n-k)!\,(\rho+1)_k} \times H_{2n-2k+1}\left(\frac{w}{\sqrt{k+1}}\right) P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{k+1}\right) = \frac{\rho(2w)^{2n+3} z^{-1}}{(n+1)!\,(2n+3)(\rho+\sigma)} \left[{}_3F_1\left(\begin{matrix}-n-1,\ -n-\frac{3}{2},\ \rho+\sigma\\ \rho;\ \frac{z}{2w^2}\end{matrix}\right) - 1\right].$$

4. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(\rho+1)_k} w^k L_{n-k}^{k+\rho+\sigma}((k+1)w)\, P_k^{(\rho,\,\sigma-k)}\Big(1+\frac{z}{k+1}\Big) = \frac{2\rho}{wz}(\rho+\sigma+1)_n \left[\frac{1}{(\rho)_{n+1}} L_{n+1}^{\rho-1}\Big(-\frac{wz}{2}\Big) - \frac{1}{(n+1)!}\right].$$

5. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(\rho+1)_k} w^k L_{n-k}^{k+\lambda}((k+1)w)\, P_k^{(\rho,\,\sigma-k)}\Big(1+\frac{z}{k+1}\Big) = \frac{2\lambda\rho(wz)^{-1}}{(n+1)!\,(\rho+\sigma)} \left[1 - {}_2F_2\left(\begin{matrix}-n-1,\ \rho+\sigma\\ \lambda,\ \rho;\ -\frac{wz}{2}\end{matrix}\right)\right].$$

6. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{(\rho+1)_k} (-w)^k L_{n-k}^{\lambda+k}((n-k)w)\, P_k^{(\rho,\,\sigma-k)}\Big(1+\frac{z}{k+1}\Big) = \sum_{k=0}^{n}\Big(-\frac{wz}{2}\Big)^k \frac{(\rho+\sigma+1)_k}{(k+1)!\,(\rho+1)_k} L_{n-k}^{\lambda+k}((n+1)w).$$

7. $$\sum_{k=0}^{n} (k+1)^{n-1} \frac{(-\lambda-n)_k}{(\rho+1)_k} L_{n-k}^{\lambda}\Big(\frac{w}{k+1}\Big)\, P_k^{(\rho,\,\sigma-k)}\Big(1+\frac{z}{k+1}\Big) = (-1)^n \frac{2\rho w^{n+1} z^{-1}}{(n+1)!\,(\lambda+n+1)(\rho+\sigma)} \left[{}_3F_1\left(\begin{matrix}-n-1,\ -\lambda-n-1,\ \rho+\sigma\\ \rho;\ \frac{z}{2w}\end{matrix}\right) - 1\right].$$

8. $$\sum_{k=0}^{n} (k+1)^{n-1} L_{n-k}^{-\rho-n-1}\Big(\frac{w}{k+1}\Big)\, P_k^{(\rho,\,\sigma-k)}\Big(1+\frac{z}{k+1}\Big) = \frac{2(-1)^n z^{-1}}{\rho+\sigma} \left[\frac{w^{n+1}}{(n+1)!} - \Big(\frac{z}{2}\Big)^{n+1} L_{n+1}^{-\rho-\sigma-n-1}\Big(-\frac{2w}{z}\Big)\right].$$

9. $$\sum_{k=0}^{n} (k+1)^{n-1} \frac{(-\rho-n)_k}{(1-\lambda)_k} C_{2k}^{\lambda-k}\Big(\frac{w}{\sqrt{k+1}}\Big)\, P_{n-k}^{(\rho,\,\sigma+k)}\Big(1+\frac{z}{k+1}\Big) = \frac{2^{-n-2}(\rho+\sigma+1)_{2n+1} w^{-2} z^{n+1}}{(n+1)!\,(\rho+\sigma+1)_n(\rho+n+1)(\lambda-1)} \times \left[1 - {}_3F_2\left(\begin{matrix}-n-1,\ \lambda-1,\ -\rho-n-1\\ -\frac{1}{2},\ -\rho-\sigma-2n-1;\ -\frac{2w^2}{z}\end{matrix}\right)\right].$$

10. $$\sum_{k=0}^{n} (k+1)^{k-1} \frac{(\rho+\sigma+n+1)_k}{(1-\lambda)_k} \Big(-\frac{z}{2}\Big)^k \times C_{2k}^{\lambda-k}\Big(\frac{w}{\sqrt{k+1}}\Big)\, P_{n-k}^{(\rho+k,\,\sigma+k)}((k+1)z+1) = \frac{\rho(\rho+1)_n (w^2 z)^{-1}}{(n+1)!\,(\lambda-1)(\rho+\sigma+n)} \left[1 - {}_3F_2\left(\begin{matrix}-n-1,\ \lambda-1,\ \rho+\sigma+n\\ \rho,\ -\frac{1}{2};\ -\frac{w^2 z}{2}\end{matrix}\right)\right].$$

11. $$\sum_{k=0}^{n}(k+1)^{n-1/2}\frac{(-\rho-n)_k}{(1-\lambda)_k}C_{2k+1}^{\lambda-k}\left(\frac{w}{\sqrt{k+1}}\right)P_{n-k}^{(\rho,\,\sigma+k)}\left(1+\frac{z}{k+1}\right)$$
$$=\frac{(\rho+\sigma+1)_{2n+1}w^{-1}\left(\frac{z}{2}\right)^{n+1}}{(n+1)!\,(\rho+\sigma+1)_n(\rho+n+1)}\left[{}_3F_2\left(\begin{matrix}-n-1,\,\lambda,\,-\rho-n-1;\;-\frac{2w^2}{z}\\ -\rho-\sigma-2n-1,\,\frac{1}{2}\end{matrix}\right)-1\right].$$

12. $$\sum_{k=0}^{n}(k+1)^{k-1/2}\frac{(\rho+\sigma+n+1)_k}{(1-\lambda)_k}\left(-\frac{z}{2}\right)^k$$
$$\times C_{2k+1}^{\lambda-k}\left(\frac{w}{\sqrt{k+1}}\right)P_{n-k}^{(\rho+k,\,\sigma+k)}((k+1)z+1)$$
$$=\frac{2\rho(\rho+1)_n(wz)^{-1}}{(n+1)!\,(\rho+\sigma+n)}\left[{}_3F_2\left(\begin{matrix}-n-1,\,\lambda,\,\rho+\sigma+n\\ \rho,\,\frac{1}{2};\;-\frac{w^2z}{2}\end{matrix}\right)-1\right].$$

13. $$\sum_{k=0}^{n-1}(-1)^k(k+1)^{k-1}(n-k)^{n-k}\frac{(\lambda)_k}{(\rho+1)_k}$$
$$\times C_{2n-2k}^{\lambda+k}\left(\frac{w}{\sqrt{n-k}}\right)P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{k+1}\right)$$
$$=(-1)^{n+1}\frac{(n+1)^{n-1}(\lambda)_n}{(\rho+1)_n}P_n^{(\rho,\,\sigma-n)}\left(1+\frac{z}{n+1}\right)+\frac{(\lambda)_n(\rho+\sigma+1)_n}{(\rho+1)_n}\left(-\frac{z}{2}\right)^n$$
$$\times\sum_{k=0}^{n}\frac{2^k(-\rho-n)_k}{(n-k+1)!\,(1-\lambda-n)_k(-\rho-\sigma-n)_k}\left(\frac{n+1}{z}\right)^kC_{2k}^{\lambda+n-k}\left(\frac{w}{\sqrt{n+1}}\right).$$

14. $$\sum_{k=0}^{n-1}(-1)^k(k+1)^{k-1}(n-k)^{n-k+1/2}\frac{(\lambda)_k}{(\rho+1)_k}$$
$$\times C_{2n-2k+1}^{\lambda+k}\left(\frac{w}{\sqrt{n-k}}\right)P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{k+1}\right)$$
$$=2(-1)^{n+1}w\frac{(n+1)^{n-1}(\lambda)_{n+1}}{(\rho+1)_n}P_n^{(\rho,\,\sigma-n)}\left(1+\frac{z}{n+1}\right)$$
$$+\sqrt{n+1}\,\frac{(\lambda)_n(\rho+\sigma+1)_n}{(\rho+1)_n}\left(-\frac{z}{2}\right)^n$$
$$\times\sum_{k=0}^{n}\frac{2^k(-\rho-n)_k}{(n-k+1)!\,(1-\lambda-n)_k(-\rho-\sigma-n)_k}\left(\frac{n+1}{z}\right)^kC_{2k+1}^{\lambda+n-k}\left(\frac{w}{\sqrt{n+1}}\right).$$

5.12.7. Sums containing products of $P_{n\pm mk}^{(\rho\pm pk,\,\sigma\pm qk)}(\varphi(k,\,z))$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[P_m^{(\rho,\,\sigma)}(w+kz)\right]^2=0\qquad[2m<n].$$

2. $$\sum_{k=0}^{n} \frac{(\mu+\nu+n+1)_k}{(\sigma+1)_k}(k+1)^{k-1}\left(\frac{w}{2}\right)^k \times P_{n-k}^{(k+\mu,\,k+\nu)}((k+1)w+1)P_k^{(\rho-k,\,\sigma)}\left(\frac{z}{k+1}-1\right) = \frac{4\mu\sigma(\mu+1)_n(wz)^{-1}}{(n+1)!\,(\mu+\nu+n)(\rho+\sigma)}\left[{}_3F_2\left(\begin{matrix}-n-1,\,\mu+\nu+n,\,\rho+\sigma\\ \mu,\,\sigma;\,-\frac{wz}{4}\end{matrix}\right)-1\right].$$

3. $$\sum_{k=0}^{n-1}(-1)^k\frac{(-\rho-n)_k}{(\mu+1)_k}(k+1)^{k-1}(n-k)^{n-k} \times P_k^{(\mu,\,\nu-k)}\left(1+\frac{w}{k+1}\right)P_{n-k}^{(\rho,\,\sigma+k)}\left(1+\frac{z}{n-k}\right) = -\frac{(n+1)^{n-1}(\rho+1)_n}{(\mu+1)_n}P_n^{(\mu,\,\nu-n)}\left(1+\frac{w}{n+1}\right) + (\rho+1)_n\frac{(\mu+\nu+1)_n}{(\mu+1)_n}\left(\frac{w}{2}\right)^n \times\sum_{k=0}^{n}\frac{2^k(-\mu-n)_k}{(n-k+1)!\,(-\mu-\nu-n)_k(\rho+1)_k}\left(\frac{n+1}{w}\right)^k P_k^{(\rho,\,\sigma+n-k)}\left(1+\frac{z}{n+1}\right).$$

5.13. The Legendre Function $P_\nu^\mu(z)$

5.13.1. Sums containing $P_{\nu\pm k}^{\mu\pm k}(z)$

1. $$\sum_{k=0}^{n}(-2)^{-k}\binom{n}{k}\frac{(\nu-\mu+1)_{2k}}{\left(\nu-n+\frac{3}{2}\right)_k}(1-z^2)^{-k/2}P_{\nu+k}^{\mu-k}(z) = 2^{-n}\frac{(-\mu-\nu)_{2n}}{\left(-\nu-\frac{1}{2}\right)_n}(1-z^2)^{-n/2}P_{\nu-n}^{\mu-n}(z) \quad [\arg(1\pm z)<\pi].$$

2. $$\sum_{k=0}^{n}(-2)^{-k}\binom{n}{k}\frac{(-\mu-\nu)_{2k}}{\left(\frac{1}{2}-\nu-n\right)_k}(1-z^2)^{-k/2}P_{\nu-k}^{\mu-k}(z) = 2^{-n}\frac{(\nu-\mu+1)_{2n}}{\left(\nu+\frac{1}{2}\right)_n}(1-z^2)^{-n/2}P_{\nu+n}^{\mu-n}(z) \quad [\arg(1\pm z)<\pi].$$

5.14. The Kummer Confluent Hypergeometric Function ${}_1F_1(a;\ b;\ z)$

5.14.1. Sums containing ${}_1F_1(a;\ b;\ z)$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(b-a)_k}{(b)_k}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b+k\end{matrix}\right) = \frac{(a)_n}{(b)_n}\,{}_1F_1\left(\begin{matrix}a+n;\ z\\ b+n\end{matrix}\right).$$

2. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{(b-a)_k}{(b)_k(1-c-n)_k}(-z)^k\,{}_1F_1\left(\begin{matrix}a;\ z\\ b+k\end{matrix}\right)=e^z\,{}_2F_2\left(\begin{matrix}b-a,\,1-c;\ z\\ b,\,1-c-n\end{matrix}\right).$

3. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{(b)_k}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b+k\end{matrix}\right)={}_1F_1\left(\begin{matrix}a-n\\ b;\ z\end{matrix}\right).$

4. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-b)_k}{(1-c-n)_k}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b-k\end{matrix}\right)=\frac{(c-b+1)_n}{(c)_n}\,{}_2F_2\left(\begin{matrix}a,\,b-c;\ z\\ b,\,b-c-n\end{matrix}\right).$

5. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-b)_k}{(2-b-n)_k}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b-k\end{matrix}\right)$

$$=\frac{(a)_n}{(b-1)_n(b)_n}(-z)^n\,{}_1F_1\left(\begin{matrix}a+n;\ z\\ b+n\end{matrix}\right).$$

6. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-b)_k}{(a-b+1)_k}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b-k\end{matrix}\right)=\frac{(a)_n}{(a-b+1)_n}\,{}_1F_1\left(\begin{matrix}a+n\\ b;\ z\end{matrix}\right).$

7. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}(1-b)_kz^{-k}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b-k\end{matrix}\right)=(1-b)_nz^{-n}\,{}_1F_1\left(\begin{matrix}a-n;\ z\\ b-n\end{matrix}\right).$

5.14.2. Sums containing ${}_1F_1(a;\ b;\ z)$ and special functions

1. $\displaystyle\sum_{k=0}^{n}\frac{(b-a)_k}{k!\,(b)_k}(-z)^kL_{n-k}^{b+k-n-1}(-z)\,{}_1F_1\left(\begin{matrix}a;\ z\\ b+k\end{matrix}\right)$

$$=(-1)^n\frac{(1-b)_n}{n!}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b-n\end{matrix}\right).$$

2. $\displaystyle\sum_{k=0}^{n}\frac{(b-a)_k}{k!\,(b)_k}(-z)^kL_{n-k}^{a+k-1}(-z)\,{}_1F_1\left(\begin{matrix}a;\ z\\ b+k\end{matrix}\right)=\frac{(a)_n}{n!}\,{}_1F_1\left(\begin{matrix}a+n\\ b;\ z\end{matrix}\right).$

3. $\displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1-b)_kL_{n-k}^{k-a}(z)\,{}_1F_1\left(\begin{matrix}a;\ z\\ b-k\end{matrix}\right)=\frac{(b-a)_n}{n!}\,{}_1F_1\left(\begin{matrix}a-n\\ b;\ z\end{matrix}\right).$

4. $\displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1-b)_kL_{n-k}^{k-b-n+1}(z)\,{}_1F_1\left(\begin{matrix}a;\ z\\ b-k\end{matrix}\right)=\frac{(b-a)_n}{n!\,(b)_n}(-z)^n\,{}_1F_1\left(\begin{matrix}a;\ z\\ b+n\end{matrix}\right).$

5. $\displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1-b)_kL_{n-k}^{k-b-n+1}(-z)\,{}_1F_1\left(\begin{matrix}a-k;\ z\\ b-k\end{matrix}\right)$

$$=\frac{(a)_n}{n!\,(b)_n}z^n\,{}_1F_1\left(\begin{matrix}a+n;\ z\\ b+n\end{matrix}\right).$$

6. $$\sum_{k=0}^{n} \frac{(1-b)_k}{(n-k)!\,(a-b+1)_k}(-z)^{-k} L_k^{\lambda-k}(-z)\,{}_1F_1\binom{a;\ z}{b-k} = \frac{(1-b)_n(-\lambda)_n}{n!\,(a-b+1)_n} z^{-n}\,{}_2F_2\binom{a,\ \lambda+1;\ z}{b-n,\ \lambda-n+1}.$$

5.14.3. Sums containing products of ${}_1F_1(a;\ b;\ z)$

1. $$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{(b-a)_k(1-b-2n)_k}{(b)_k(a-b-2n+1)_k}\,{}_1F_1\binom{a;\ -z}{b+2n-k}\,{}_1F_1\binom{a;\ z}{b+k} = 2^{2n}\frac{(a)_n(b-a)_n\left(\frac{1}{2}\right)_n}{(b)_n(b-a)_{2n}}\,{}_3F_4\left(\begin{matrix} a+n,\ b-a+n,\ n+\frac{1}{2};\ \frac{z^2}{4} \\ \frac{b}{2}+n,\ \frac{b+1}{2}+n,\ b+n,\ \frac{1}{2}\end{matrix}\right).$$

5.15. The Tricomi Confluent Hypergeometric Function $\Psi(a;\ b;\ z)$

5.15.1. Sums containing $\Psi(a;\ b;\ z)$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\Psi\binom{a;\ z}{b+k} = (-1)^n(a)_n\Psi\binom{a+n;\ z}{b+n}.$$

2. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{(b-a)_k}\Psi\binom{a;\ z}{b+k} = \frac{(-1)^n}{(b-a)_n}\Psi\binom{a-n}{b;\ z}.$$

3. $$\sum_{k=0}^{n}\binom{n}{k}(a)_k\Psi\binom{a+k;\ z}{b+k} = \Psi\binom{a;\ z}{b+n}.$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\Psi\binom{a;\ z}{b-k} = (a)_n\Psi\binom{a+n}{b;\ z}.$$

5. $$\sum_{k=0}^{n}\binom{n}{k}(a-b+1)_k z^{-k}\Psi\binom{a;\ z}{b-k} = z^{-n}\Psi\binom{a-n;\ z}{b-n}.$$

6. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a-b+1)_k}{(2-b-n)_k}\Psi\binom{a;\ z}{b-k} = \frac{(a)_n}{(b-1)_n}z^n\Psi\binom{a+n;\ z}{b+n}.$$

7. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{(b-n)_k}(-z)^k\Psi\binom{a+k;\ z}{b+k} = \frac{(a-b+1)_n}{(1-b)_n}\Psi\binom{a;\ z}{b-n}.$$

5.15.2. Sums containing $\Psi(a;\ b;\ z)$ and special functions

1. $\displaystyle\sum_{k=0}^{n} \frac{(-z)^k}{k!} L_{n-k}^{k+a-1}(-z)\Psi\binom{a;\ z}{b+k} = \frac{(a)_n}{n!}(a-b+1)_n \Psi\binom{a+n}{b;\ z}.$

2. $\displaystyle\sum_{k=0}^{n} \frac{(-z)^k}{k!} L_{n-k}^{b-n+k-1}(-z)\Psi\binom{a;\ z}{b+k} = (-1)^n \frac{(a-b+1)_n}{n!} \Psi\binom{a;\ z}{b-n}.$

3. $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!}(a-b+1)_k L_{n-k}^{k-n-b+1}(z)\Psi\binom{a;\ z}{b-k} = \frac{(-z)^n}{n!} \Psi\binom{a;\ z}{b+n}.$

4. $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!}(a-b+1)_k L_{n-k}^{k-a}(z)\Psi\binom{a;\ z}{b-k} = \frac{(-1)^n}{n!} \Psi\binom{a-n}{b;\ z}.$

5.16. The Gauss Hypergeometric Function ${}_2F_1(a, b;\ c;\ z)$

5.16.1. Sums containing ${}_2F_1(a, b;\ c;\ z)$

1. $\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(d)_k}{(c)_k} {}_2F_1\binom{a,\ b;\ z}{c+k} = \frac{(c-d)_n}{(c)_n} {}_3F_2\binom{a,\ b,\ c-d+n}{c+n,\ c-d;\ z}.$

2. $\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(b)_k}{(b-a-n+1)_k} {}_2F_1\binom{a,\ b+k;\ z}{c} = \frac{(a)_n}{(a-b)_n} {}_2F_1\binom{a+n,\ b}{c;\ z}.$

3. $\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(c-a)_k (c-b)_k}{(c)_k (1-n-a-b+c)_k} {}_2F_1\binom{a,\ b;\ z}{c+k}$

$$= \frac{(a)_n (b)_n}{(c)_n (a+b-c)_n} (1-z)^n {}_2F_1\binom{a+n,\ b+n}{c+n;\ z}.$$

4. $\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(c-a)_k}{(c)_k} {}_2F_1\binom{a,\ b;\ z}{c+k} = \frac{(a)_n}{(c)_n} {}_2F_1\binom{a+n,\ b}{c+n;\ z}.$

5. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{(c-b)_k}{(c)_k} \left(\frac{z}{1-z}\right)^k {}_2F_1\binom{a,\ b;\ z}{c+k} = (1-z)^{-n} {}_2F_1\binom{a-n,\ b}{c;\ z}.$

6. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{(b)_k}{(c)_k} z^k {}_2F_1\binom{a+k,\ b+k;\ z}{c+k} = {}_2F_1\binom{a+n,\ b}{c;\ z}.$

7. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{(b)_k}{(c)_k} (z-1)^k {}_2F_1\binom{a+k,\ b+k;\ z}{c+k} = \frac{(c-b)_n}{(c)_n} {}_2F_1\binom{a+n,\ b}{c+n;\ z}.$

8. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k(b)_k}{(c)_k(c-n)_k}z^k\,{}_2F_1\left(\begin{matrix}a+k,\ b+k;\ z\\ c+k\end{matrix}\right)={}_2F_1\left(\begin{matrix}a,\ b\\ c-n;\ z\end{matrix}\right).$$

9. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-c)_k}{(2-c-n)_k}\,{}_2F_1\left(\begin{matrix}a,\ b;\ z\\ c-k\end{matrix}\right)$$
$$=\frac{(a)_n(b)_n}{(c)_n(c-1)_n}(-z)^n\,{}_2F_1\left(\begin{matrix}a+n,\ b+n\\ c+n;\ z\end{matrix}\right).$$

10. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(1-c)_k}{(a+b-c-n+1)_k}\left(\frac{1-z}{z}\right)^k\,{}_2F_1\left(\begin{matrix}a,\ b;\ z\\ c-k\end{matrix}\right)$$
$$=\frac{(1-c)_n}{(c-a-b)_n}(-z)^{-n}\,{}_2F_1\left(\begin{matrix}a-n,\ b-n\\ c-n;\ z\end{matrix}\right).$$

11. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(1-c)_k}{(b-c+1)_k}\left(\frac{1-z}{z}\right)^k\,{}_2F_1\left(\begin{matrix}a,\ b;\ z\\ c-k\end{matrix}\right)$$
$$=\frac{(1-c)_n}{(b-c+1)_n}z^{-n}\,{}_2F_1\left(\begin{matrix}a-n,\ b\\ c-n;\ z\end{matrix}\right).$$

5.16.2. Sums containing ${}_2F_1(a, b;\ c;\ z)$ and special functions

1. $$\sum_{k=0}^{n}\frac{(c-a)_k(c-b)_k}{k!\,(c)_k}z^k P_{n-k}^{(k-n+c-1,\,k-n-a-b+c)}(1-2z)\,{}_2F_1\left(\begin{matrix}a,\ b;\ z\\ c+k\end{matrix}\right)$$
$$=\frac{(1-c)_n}{n!}(z-1)^n\,{}_2F_1\left(\begin{matrix}a,\ b\\ c-n;\ z\end{matrix}\right).$$

2. $$\sum_{k=0}^{n}\frac{(c-a)_k(c-b)_k}{k!\,(c)_k}z^k P_{n-k}^{(k-n+c-1,\,k-a)}(1-2z)\,{}_2F_1\left(\begin{matrix}a,\ b;\ z\\ c+k\end{matrix}\right)$$
$$=(-1)^n\frac{(1-c)_n}{n!}\,{}_2F_1\left(\begin{matrix}a-n,\ b\\ c-n;\ z\end{matrix}\right).$$

3. $$\sum_{k=0}^{n}\frac{(a)_k(b)_k}{k!\,(c)_k}(z-z^2)^k P_{n-k}^{(k-n+c-1,\,k+b-c)}(1-2z)\,{}_2F_1\left(\begin{matrix}a+k,\ b+k;\ z\\ c+k\end{matrix}\right)$$
$$=(-1)^n\frac{(1-c)_n}{n!}\,{}_2F_1\left(\begin{matrix}a-n,\ b\\ c-n;\ z\end{matrix}\right).$$

4. $$\sum_{k=0}^{n}\frac{(a)_k(b)_k}{k!\,(c)_k}(z-z^2)^k P_{n-k}^{(k-n+c-1,\,k-n+a+b-c)}(1-2z)$$
$$\times\,{}_2F_1\left(\begin{matrix}a+k,\ b+k;\ z\\ c+k\end{matrix}\right)=(-1)^n\frac{(1-c)_n}{n!}\,{}_2F_1\left(\begin{matrix}a-n,\ b-n\\ c-n;\ z\end{matrix}\right).$$

5. $$\sum_{k=0}^{n} \frac{(a)_k(b)_k}{k!\,(c)_k}(z-z^2)^k P_{n-k}^{(k-a+c-1,\,k-n+a+b-c)}(1-2z) \times {}_2F_1\left(\begin{matrix} a+k,\ b+k;\ z \\ c+k \end{matrix}\right) = \frac{(c-a)_n}{n!}\,{}_2F_1\left(\begin{matrix} a-n,\ b \\ c;\ z \end{matrix}\right).$$

6. $$\sum_{k=0}^{n} \frac{(1-c)_k}{k!}(z-1)^k P_{n-k}^{(k-n-c+1,\,k+b-1)}(1-2z)\,{}_2F_1\left(\begin{matrix} a,\ b;\ z \\ c-k \end{matrix}\right) = \frac{(b)_n(c-a)_n}{n!\,(c)_n}(-z)^n\,{}_2F_1\left(\begin{matrix} a,\ b+n \\ c+n;\ z \end{matrix}\right).$$

7. $$\sum_{k=0}^{n} \frac{(1-c)_k}{k!}(z-1)^k P_{n-k}^{(k-n-c+1,\,k-n+a+b-c)}(1-2z)\,{}_2F_1\left(\begin{matrix} a,\ b;\ z \\ c-k \end{matrix}\right) = \frac{(c-a)_n(c-b)_n}{n!\,(c)_n} z^n\,{}_2F_1\left(\begin{matrix} a,\ b \\ c+n;\ z \end{matrix}\right).$$

8. $$\sum_{k=0}^{n} \frac{(1-c)_k}{k!}(z-1)^k P_{n-k}^{(k-n-c+1,\,k+a-1)}(1-2z)\,{}_2F_1\left(\begin{matrix} a,\ b;\ z \\ c-k \end{matrix}\right) = \frac{(a)_n(c-b)_n}{n!\,(c)_n}(-z)^n\,{}_2F_1\left(\begin{matrix} a+n,\ b \\ c+n;\ z \end{matrix}\right).$$

9. $$\sum_{k=0}^{n} \frac{(1-c)_k}{k!}(z-1)^k P_{n-k}^{(k-a,\,k-n+a+b-c)}(1-2z)\,{}_2F_1\left(\begin{matrix} a,\ b;\ z \\ c-k \end{matrix}\right) = \frac{(c-a)_n}{n!}\,{}_2F_1\left(\begin{matrix} a-n,\ b \\ c;\ z \end{matrix}\right).$$

10. $$\sum_{k=0}^{n}(-1)^k \frac{(1-c)_k}{k!} P_{n-k}^{(k+a-c,\,k-n-a-b+c)}(1-2z)\,{}_2F_1\left(\begin{matrix} a-k,\ b-k;\ z \\ c-k \end{matrix}\right) = \frac{(a)_n}{n!}(1-z)^n\,{}_2F_1\left(\begin{matrix} a+n,\ b \\ c;\ z \end{matrix}\right).$$

11. $$\sum_{k=0}^{n}(-1)^k \frac{(1-c)_k}{k!} P_{n-k}^{(k-n-c+1,\,k-a-b+c-n)}(1-2z)\,{}_2F_1\left(\begin{matrix} a-k,\ b-k;\ z \\ c-k \end{matrix}\right) = \frac{(a)_n(b)_n}{n!\,(c)_n}(z-z^2)^n\,{}_2F_1\left(\begin{matrix} a+n,\ b+n \\ c+n;\ z \end{matrix}\right).$$

12. $$\sum_{k=0}^{n}(-1)^k \frac{(1-c)_k}{k!} P_{n-k}^{(k-n-c+1,\,k-b+c-1)}(1-2z)\,{}_2F_1\left(\begin{matrix} a-k,\ b-k;\ z \\ c-k \end{matrix}\right) = \frac{(a)_n(c-b)_n}{n!\,(c)_n}(-z)^n\,{}_2F_1\left(\begin{matrix} a+n,\ b \\ c+n;\ z \end{matrix}\right).$$

13. $$\sum_{k=0}^{n} \frac{(1-c)_k}{k!} (z-1)^k P_{n-k}^{(k-n-c+1,\, k-n+a+b-c)}(1-2z)\, {}_2F_1\left(\begin{matrix} a,\ b;\ z \\ c-k \end{matrix}\right) = \frac{(c-a)_n (c-b)_n}{n!\,(c)_n} (-z)^n\, {}_2F_1\left(\begin{matrix} a,\ b \\ c+n;\ z \end{matrix}\right).$$

5.16.3. Sums containing products of ${}_2F_1(a, b;\ c;\ z)$

1. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k (b)_k}{\left(a+b+\frac{1}{2}\right)_k \left(a+b-n+\frac{1}{2}\right)_k} z^k \times {}_2F_1\left(\begin{matrix} a+k,\ b+k;\ z \\ a+b+k+\frac{1}{2} \end{matrix}\right) {}_2F_1\left(\begin{matrix} a,\ b;\ z \\ a+b-n+k+\frac{1}{2} \end{matrix}\right) = {}_3F_2\left(\begin{matrix} 2a,\ 2b,\ a+b;\ z \\ 2a+2b,\ a+b-n+\frac{1}{2} \end{matrix}\right).$$

2. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k (b)_k}{\left(a+b-\frac{1}{2}\right)_k \left(a+b-n-\frac{1}{2}\right)_k} z^k \times {}_2F_1\left(\begin{matrix} a+k,\ b+k;\ z \\ a+b+k-\frac{1}{2} \end{matrix}\right) {}_2F_1\left(\begin{matrix} a,\ b-1;\ z \\ a+b-n+k-\frac{1}{2} \end{matrix}\right) = {}_3F_2\left(\begin{matrix} 2a,\ 2b-1,\ a+b-1;\ z \\ 2a+2b-2,\ a+b-n-\frac{1}{2} \end{matrix}\right).$$

3. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k (b)_k \left(\frac{1}{2}-a-b-n\right)_k}{\left(a+b+\frac{1}{2}\right)_k (1-a-n)_k (1-b-n)_k} \times {}_2F_1\left(\begin{matrix} a+k,\ b+k;\ z \\ a+b+k+\frac{1}{2} \end{matrix}\right) {}_2F_1\left(\begin{matrix} a+n-k,\ b+n-k;\ z \\ a+b+n-k+\frac{1}{2} \end{matrix}\right) = \frac{(2a)_n (2b)_n (a+b)_n}{(a)_n (b)_n (2a+2b)_n}\, {}_3F_2\left(\begin{matrix} 2a+n,\ 2b+n,\ a+b+n;\ z \\ 2a+2b+n,\ a+b+n+\frac{1}{2} \end{matrix}\right).$$

4. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k \left(b+\frac{1}{2}\right)_k}{(a+b)_k (a+b-n)_k} \left(\frac{z}{1-z}\right)^k {}_2F_1\left(\begin{matrix} a-\frac{1}{2},\ b;\ z \\ a+b+k \end{matrix}\right) \times {}_2F_1\left(\begin{matrix} a,\ b-\frac{1}{2};\ z \\ a+b-n+k \end{matrix}\right) = (1-z)^{1/2}\, {}_3F_2\left(\begin{matrix} 2a,\ 2b,\ a+b-\frac{1}{2};\ z \\ 2a+2b-1,\ a+b-n \end{matrix}\right).$$

5. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k(b)_k\left(\frac{1}{2}-a-b-n\right)_k}{\left(a+b+\frac{1}{2}\right)_k(1-a-n)_k(1-b-n)_k}(z-1)^k$$
$$\times\ {}_2F_1\left(\begin{matrix}a+k,\ b+k;\ z\\ a+b+k+\frac{1}{2}\end{matrix}\right){}_2F_1\left(\begin{matrix}a+\frac{1}{2},\ b+\frac{1}{2};\ z\\ a+b+n-k+\frac{1}{2}\end{matrix}\right)$$
$$=\frac{(2a)_n(2b)_n(a+b)_n}{(a)_n(b)_n(2a+2b)_n}(1-z)^{n-1/2}\,{}_3F_2\left(\begin{matrix}2a+n,\ 2b+n,\ a+b+n;\ z\\ 2a+2b+n,\ a+b+n+\frac{1}{2}\end{matrix}\right).$$

6. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\left(a-\frac{1}{2}\right)_k(b)_k(1-a-b-n)_k}{\left(\frac{3}{2}-a-n\right)_k(1-b-n)_k(a+b)_k}$$
$$\times\ {}_2F_1\left(\begin{matrix}a,\ b+\frac{1}{2};\ z\\ a+b+k\end{matrix}\right){}_2F_1\left(\begin{matrix}a,\ b+\frac{1}{2};\ z\\ a+b+n-k\end{matrix}\right)$$
$$=\frac{(2a-1)_n(2b)_n\left(a+b-\frac{1}{2}\right)_n}{\left(a-\frac{1}{2}\right)_n(b)_n(2a+2b-1)_n}(1-z)^{n-1}$$
$$\times\ {}_3F_2\left(\begin{matrix}2a+n-1,\ 2b+n,\ a+b+n-\frac{1}{2}\\ 2a+2b+n-1,\ a+b+n;\ z\end{matrix}\right).$$

5.17. The Generalized Hypergeometric Function ${}_pF_q((a_p);\ (b_q);\ z)$

5.17.1. Sums containing ${}_pF_q((a_p)\pm mk;\ (b_q)\pm nk;\ z)$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(c)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ c+k\end{matrix}\right)$$
$$=\frac{(c-a)_n}{(c)_n}\,{}_{p+1}F_{p+2}\left(\begin{matrix}(a_p),\ c-a+n;\ z\\ (b_q),\ c+n,\ c-a\end{matrix}\right).$$

2. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(b+m+n)_k}{(b)_k}\,{}_{p+1}F_{q+1}\left(\begin{matrix}-m-n,\ (a_p);\ z\\ (b_q),\ b+k\end{matrix}\right)$$
$$=(-1)^{m+n}\frac{(m+n)!}{m!\,(b)_n}\,{}_{p+1}F_{q+1}\left(\begin{matrix}-m,\ (a_p);\ z\\ (b_q),\ b+n\end{matrix}\right).$$

3. $$\sum_{k=0}^{n}\frac{(-n)_k(n)_k}{(k!)^2}\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ k+1\end{matrix}\right)$$
$$=\frac{z^n}{(2n)!}\frac{\prod(a_p)_n}{\prod(b_q)_n}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+n;\ z\\ (b_q)+n,\ 2n+1\end{matrix}\right).$$

4. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-c)_k}{(a)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ c-k\end{matrix}\right) = \frac{(a+c-1)_n}{(a)_n}\,{}_{p+1}F_{q+2}\left(\begin{matrix}(a_p),\ a+c+n-1;\ z\\ (b_q),\ c,\ a+c-1\end{matrix}\right).$$

5. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(b)_k}\,{}_{p+1}F_q\left(\begin{matrix}(a_p),\ a+k;\ z\\ (b_q)\end{matrix}\right) = \frac{(b-a)_n}{(b)_n}\,{}_{p+2}F_{q+1}\left(\begin{matrix}(a_p),\ a,\ a-b+1;\ z\\ (b_q),\ a-b-n+1\end{matrix}\right).$$

6. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(b)_k}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\ k+a;\ z\\ (b_q),\ k+b\end{matrix}\right) = \frac{(b-a)_n}{(b)_n}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\ a;\ z\\ (b_q),\ b+n\end{matrix}\right).$$

7. $$\sum_{k=0}^{n}\binom{n}{k}(-z)^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}(a_p)+k,\ a\\ (b_q)+k;\ z\end{matrix}\right) = {}_{p+1}F_q\left(\begin{matrix}(a_p),\ a-n;\ z\\ (b_q)\end{matrix}\right).$$

8. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^k}{k+1}\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}(a_p)+k,\ a\\ (b_q)+k;\ z\end{matrix}\right)$$
$$= \frac{1}{(n+1)z}\,\frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)}\left[{}_{p+1}F_q\left(\begin{matrix}a,\ (a_p)-1\\ (b_q)-1;\ z\end{matrix}\right) - {}_{p+1}F_q\left(\begin{matrix}a-n-1,\ (a_p)-1\\ (b_q)-1;\ z\end{matrix}\right)\right].$$

9. $$\sum_{k=0}^{n}\binom{n}{k}k^r(-z)^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}-m,\ (a_p)+k\\ (b_q)+k;\ z\end{matrix}\right) = \sum_{k=1}^{r}\sigma_r^k(-n)_k z^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}-m-n+k,\ (a_p)+k\\ (b_q)+k;\ z\end{matrix}\right).$$

10. $$\sum_{k=0}^{n}\sigma_{k+m}^{m}\frac{(-n)_k}{(k+m)!}z^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}-n+k,\ (a_p)+k\\ (b_q)+k;\ az\end{matrix}\right)$$
$$= (-1)^{m(p+q+1)}\frac{z^{-m}}{m!\,(1+n)_m}\frac{\prod(1-(b_q))_m}{\prod(1-(a_p))_m}\sum_{k=0}^{m}(-1)^k\binom{m}{k} \times {}_{p+1}F_q\left(\begin{matrix}-m-n,\ (a_p)-m\\ (b_q)-m;\ (k+a)z\end{matrix}\right).$$

11. $$\sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)_k}{k!} \left(-\frac{z^2}{4}\right)^k \frac{\prod (a_p)_{2k}}{\prod (b_q)_{2k}} \, {}_{p+1}F_q\left(\begin{matrix} -n+k,\, (a_p)+2k \\ (b_q)+2k;\; z \end{matrix}\right) = {}_{p+2}F_{q+1}\left(\begin{matrix} -2n,\, -n-\frac{1}{2},\, (a_p) \\ -2n-1,\, (b_q);\; z \end{matrix}\right).$$

12. $$\sum_{k=1}^{n} \frac{a+2k-1}{(a)_{2k}(b)_{2k}} z^k \frac{\prod (a_p)_k}{\prod (b_q)_k} \, {}_pF_{q+3}\left(\begin{matrix} (a_p)+k;\; z \\ (b_q)+k,\, a+2k,\, b+2k,\, b-a+1 \end{matrix}\right)$$
$$= \frac{z}{ab(b+1)} \frac{\prod\limits_{i=1}^{p} a_i}{\prod\limits_{j=1}^{q} b_j} \, {}_{p+1}F_{q+4}\left(\begin{matrix} (a_p)+1,\, \frac{b+1}{2};\; z \\ (b_q)+1,\, a+1,\, b+1,\, \frac{b+3}{2},\, b-a+1 \end{matrix}\right)$$
$$- \frac{z^{n+1}}{(a)_{2n+1}(b)_{2n+2}} \frac{\prod (a_p)_{n+1}}{\prod (b_q)_{n+1}}$$
$$\times \, {}_{p+1}F_{q+4}\left(\begin{matrix} (a_p)+n+1,\, \frac{b+1}{2}+n;\; z \\ (b_q)+n+1,\, a+2n+1,\, b+2n+1,\, \frac{b+3}{2}+n,\, b-a+1 \end{matrix}\right).$$

13. $$\sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \, {}_{p+2}F_q\left(\begin{matrix} -n+2k,\, n-2k,\, (a_p) \\ (b_q);\; z \end{matrix}\right) = 2^{n-1} \, {}_{p+2}F_q\left(\begin{matrix} -n,\, \frac{1}{2},\, (a_p) \\ (b_q);\; 2z \end{matrix}\right) - \frac{1+(-1)^n}{4} \binom{n}{[n/2]}.$$

14. $$\sum_{k=0}^{[n/2]} \binom{n}{k} \frac{(n-2k+1)^2}{n-k+1} \, {}_{p+2}F_q\left(\begin{matrix} -n+2k,\, n-2k+2,\, (a_p) \\ (b_q);\; z \end{matrix}\right) = 2^n \, {}_{p+2}F_q\left(\begin{matrix} -n,\, \frac{3}{2},\, (a_p) \\ (b_q);\; 2z \end{matrix}\right).$$

15. $$\sum_{k=0}^{[n/2]} \binom{n}{2k} \frac{\left(\frac{1}{2}\right)_k}{(a)_k} (-4z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} \, {}_{p+1}F_q\left(\begin{matrix} -n+2k,\, (a_p)+k \\ (b_q)+k;\; z \end{matrix}\right) = {}_{p+2}F_{q+1}\left(\begin{matrix} -n,\, (a_p),\, 1-a-n \\ (b_q),\, a;\; -z \end{matrix}\right).$$

16. $$\sum_{k=0}^{n} \frac{1}{(n-k)!\,(k+n+1)!} \, {}_pF_{p+2}\left(\begin{matrix} (a_p);\; z \\ (b_p),\, \frac{1}{2}-k,\, \frac{3}{2}+k \end{matrix}\right) = \frac{2^{2n}}{(2n+1)!} \, {}_pF_{p+2}\left(\begin{matrix} (a_p);\; z \\ (b_p),\, n+\frac{3}{2},\, \frac{1}{2} \end{matrix}\right).$$

17. $$\sum_{k=0}^{n} \frac{(-z)^k}{(2k)!} \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k,\ 2k+2\end{matrix}\right)$$
$$= 1 - \frac{(-z)^{n+1}}{(2n+2)!} \frac{\prod(a_p)_{n+1}}{\prod(b_q)_{n+1}} {}_{p+1}F_{q+2}\left(\begin{matrix}(a_p)+n+1,\ n+1;\ z\\ (b_q)+n+1,\ n+2,\ 2n+3\end{matrix}\right).$$

18. $$\sum_{k=0}^{n} \frac{z^k}{(2k+1)!} \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k,\ 2k+3\end{matrix}\right) = {}_pF_{q+1}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ 2\end{matrix}\right)$$
$$- \frac{z^{n+1}}{(2n+3)!} \frac{\prod(a_p)_{n+1}}{\prod(b_q)_{n+1}} {}_pF_{q+1}\left(\begin{matrix}(a_p)+n+1;\ z\\ (b_q)+n+1,\ 2n+4\end{matrix}\right).$$

19. $$\sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_k (-z^2)^k \frac{\prod(a_p)_{2k}}{\prod(b_q)_{2k}}$$
$$\times {}_{p+2}F_q\left(\begin{matrix}-2n+2k,\ k-\frac{1}{2},\ (a_p)+2k\\ (b_q)+2k;\ z\end{matrix}\right) = 1 + nz \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j}.$$

20. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+k\mu)_{n-1}(\lambda+k\mu)_k}{(a+k\mu)_k} {}_{p+1}F_{q+1}\left(\begin{matrix}-k,\ (a_p);\ z\\ (b_q),\ \lambda+k\mu\end{matrix}\right)$$
$$= \frac{(a-\lambda)_n}{a+n\mu+n-1} {}_{p+1}F_{q+1}\left(\begin{matrix}-n,\ (a_p);\ z\\ (b_q),\ \lambda-a-n+1\end{matrix}\right).$$

5.17.2. Sums containing ${}_pF_q((a_p)\pm mk;\ (b_q)\pm nk;\ z)$ and special functions

1. $$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} (a)_k \psi(k+1)\ {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ z\end{matrix}\right)$$
$$= \frac{(1-a)_n}{n!} \psi(n+1)\ {}_{p+2}F_{q+1}\left(\begin{matrix}-n,\ (a_p),\ a;\ z\\ (b_q),\ a-n\end{matrix}\right)$$
$$+ (-1)^n \sum_{k=0}^{n-1} \frac{(a-n)_k}{k!\,(n-k)} {}_{p+2}F_{q+1}\left(\begin{matrix}-k,\ (a_p),\ a;\ z\\ (b_q),\ a-n\end{matrix}\right).$$

2. $$\sum_{k=0}^{n} \binom{n}{k} \frac{(-4z)^k}{\left(\frac{1}{2}\right)_k} B_{2k} \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_{p+1}F_{q+1}\left(\begin{matrix}-n+k,\ (a_p)+k;\ z\\ (b_q)+k,\ \frac{3}{2}\end{matrix}\right)$$
$$= {}_{p+1}F_{q+1}\left(\begin{matrix}-n,\ (a_p);\ z\\ (b_q),\ \frac{1}{2}\end{matrix}\right).$$

3. $$\sum_{k=0}^{n}\binom{n}{k}(-z)^k B_k(w)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_{q+1}\left(\begin{matrix}-n+k,\,(a_p)+k,\,1;\; z\\ (b_q)+k,\,2\end{matrix}\right) = {}_pF_q\left(\begin{matrix}-n,\,(a_p)\\ (b_q);\; wz\end{matrix}\right).$$

5.17.3. Sums containing ${}_pF_q((a_p)\pm mk;\;(b_q)\pm nk;\;\varphi(k,z))$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,{}_{p+1}F_q\left(\begin{matrix}-m,\,(a_p)\\ (b_q);\; w+kz\end{matrix}\right)=0 \qquad [m<n].$$

2. $$\sum_{k=1}^{n}\binom{n}{k}k^{k-1}z^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}-n+k,\,(a_p)+k\\ (b_q)+k;\; kz\end{matrix}\right)=nz\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}.$$

3. $$\sum_{k=1}^{n}\frac{k^{2k-4}}{(n-k)!\,(2k)!}z^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_{q+1}\left(\begin{matrix}-n+k,\,(a_p)+k;\; k^2z\\ (b_q)+k,\,2k+1\end{matrix}\right) = -\frac{z^2}{(n-2)!8}\frac{\prod_{i=1}^{p}a_i(a_i+1)}{\prod_{j=1}^{q}b_j(b_j+1)}+\frac{z}{(n-1)!2}\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j} \qquad [n\ge 2].$$

4. $$\sum_{k=0}^{n}\frac{(2k+1)^{2k-3}}{(n-k)!\,(2k+1)!}z^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}-n+k,\,(a_p)+k;\;(2k+1)^2z\\ (b_q)+k,\,2k+2\end{matrix}\right) = \frac{1}{n!}-\frac{4z}{9(n-1)!}\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}.$$

5. $$\sum_{k=0}^{n}\binom{n}{k}\frac{(ka+b)^k}{ka+1}z^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}-n+k,\,(a_p)+k\\ (b_q)+k;\;(ka+1)z\end{matrix}\right) = {}_{p+2}F_{q+1}\left(\begin{matrix}-n,\,(a_p),\,\frac{1}{a};\;(1-b)z\\ (b_q),\,\frac{1}{a}+1\end{matrix}\right).$$

6. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m\,{}_{p+1}F_q\left(\begin{matrix}-m,\,(a_p);\;\frac{z}{k}\\ (b_q)\end{matrix}\right)=(-1)^m m!\,\delta_{m,n} - \frac{\prod(a_p)_m}{\prod(b_q)_m}(-z)^m \qquad [m\le n].$$

7. $$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m\,{}_{p+1}F_q\left(\begin{matrix}-m,\,(a_p)\,;\ \frac{z}{k}\\(b_q)\end{matrix}\right)=(-1)^{m+1}z^m\frac{\prod(a_p)_m}{\prod(b_q)_m}$$
$$+(-1)^m n!\,z^{m-n}\frac{\prod(a_p)_{m-n}}{\prod(b_q)_{m-n}}\sum_{k=0}^{m-n}\sigma_{k+n}^{n}\binom{m}{k+n}(-1)^{k(p+q+1)}z^k$$
$$\times\frac{\prod(n-m-b_q+1)_k}{\prod(n-m-a_p+1)_k}\qquad[m\ge n].$$

8. $$\sum_{k=1}^{n}\binom{n}{k}(-ka)^k(ka+b)^{n-k-1}\,{}_{p+1}F_p\left(\begin{matrix}-k,\,(a_p)\\(b_p);\ \frac{z}{k}\end{matrix}\right)$$
$$=-b^{n-1}+\frac{b^n}{na+b}\,{}_{p+1}F_q\left(\begin{matrix}-n,\,(a_p)\\(b_q)\,;\ -\frac{az}{b}\end{matrix}\right).$$

9. $$\sum_{k=0}^{n}\binom{n}{k}(k+1)^{k-1}(a-k)^{n-k-1}\,{}_{p+1}F_p\left(\begin{matrix}-k,\,(a_p)\\(b_q);\ \frac{z}{k+1}\end{matrix}\right)$$
$$=\frac{(a+1)^n}{(n+1)\,(a-n)}\,{}_{p+1}F_p\left(\begin{matrix}-n,\,(a_p)\\(b_q);\ \frac{z}{a+1}\end{matrix}\right)$$
$$+\frac{(a+1)^n}{(n+1)z}\,\frac{\prod\limits_{j=1}^{q}(b_j-1)}{\prod\limits_{i=1}^{p}(a_i-1)}\left[1-{}_{p+1}F_p\left(\begin{matrix}-n,\,(a_p)-1\\(b_q)-1;\ \frac{z}{a+1}\end{matrix}\right)\right].$$

10. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(ka+1)^{n-1}\,{}_{p+1}F_q\left(\begin{matrix}-k,\,(a_p)\\(b_q);\ \frac{z}{ka+1}\end{matrix}\right)=\frac{z^n}{na+1}\,\frac{\prod(a_p)_n}{\prod(b_q)_n}.$$

11. $$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(ka+1)^{2n-1}\,{}_{p+2}F_q\left(\begin{matrix}-\frac{k}{2},\ \frac{1-k}{2},\,(a_p)\\(b_q);\ \frac{z}{(ka+1)^2}\end{matrix}\right)$$
$$=\frac{\left(\frac{1}{2}\right)_n}{2na+1}\,z^n\,\frac{\prod(a_p)_n}{\prod(b_q)_n}.$$

12. $$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}(ka+1)^{2n}\,{}_{p+2}F_q\left(\begin{matrix}-\frac{k}{2},\ \frac{1-k}{2},\,(a_p)\\(b_q);\ \frac{z}{(ka+1)^2}\end{matrix}\right)=0.$$

5.17.4. Sums containing ${}_pF_q((a_p) \pm mk;\ (b_q) \pm nk;\ \varphi(k, z))$ and special functions

1. $$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-n\right)_k}{k!\,(n-k)!}(-4)^k(k+1)^{n-1} H_{2n-2k}\left(\frac{w}{\sqrt{k+1}}\right)\ {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{z}{k+1}\end{matrix}\right)$$
$$= -\frac{2^{2n+1}w^{2n+2}z^{-1}}{(n+1)!\,(2n+1)} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_{p+2}F_q\left(\begin{matrix}-n-1,\ -n-\frac{1}{2},\ (a_p)-1\\ (b_q)-1;\ -w^{-2}z\end{matrix}\right)-1\right].$$

2. $$\sum_{k=0}^{n} \frac{\left(-n-\frac{1}{2}\right)_k}{k!\,(n-k)!}(-4)^k(k+1)^{n-1/2}$$
$$\times H_{2n-2k+1}\left(\frac{w}{\sqrt{k+1}}\right)\ {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{z}{k+1}\end{matrix}\right)$$
$$= -\frac{2^{2n+2}w^{2n+3}z^{-1}}{(n+1)!\,(2n+3)} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_{p+2}F_q\left(\begin{matrix}-n-1,\ -n-\frac{3}{2},\ (a_p)-1\\ (b_q)-1;\ -w^{-2}z\end{matrix}\right)-1\right].$$

3. $$\sum_{k=0}^{n} \frac{(k+1)^{n-1}}{k!}(-\lambda-n)_k L_{n-k}^{\lambda}\left(\frac{w}{k+1}\right)\ {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{z}{k+1}\end{matrix}\right)$$
$$= \frac{(-w)^{n+1}z^{-1}}{(n+1)!\,(\lambda+n+1)} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_{p+2}F_q\left(\begin{matrix}-n-1,\ -\lambda-n-1,\ (a_p)-1\\ (b_q)-1;\ -w^{-1}z\end{matrix}\right)-1\right].$$

4. $$\sum_{k=0}^{n} \frac{(k+1)^{k-1}}{k!}(-w)^k L_{n-k}^{\lambda+k}((n-k)w)\ {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{z}{k+1}\end{matrix}\right)$$
$$= \sum_{k=0}^{n} \frac{(wz)^k}{(k+1)!} \frac{\prod(a_p)_k}{\prod(b_q)_k} L_{n-k}^{\lambda+k}((n+1)w).$$

5. $$\sum_{k=1}^{n}(-k)^k(n-k+1)^{n-k-1}\frac{(-\lambda-n)_k}{k!} L_{n-k}^{\lambda}\left(\frac{w}{n-k+1}\right)\ {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{z}{k}\end{matrix}\right)$$
$$= -(n+1)^{n-1}L_n^{\lambda}\left(\frac{w}{n+1}\right)$$
$$+ \frac{(-w)^n}{(n+1)!}\sum_{k=0}^{n}\binom{n+1}{k}(-n-\lambda)_k\left(\frac{n+1}{w}\right)^k\ {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{z}{n+1}\end{matrix}\right).$$

5.17.5. Sums containing products of ${}_pF_q((a_p)\pm mk;(b_q)\pm nk;\varphi(k,z))$

1. $$\sum_{k=1}^{n}(-1)^k\binom{n+1}{k}k^n\,{}_2F_1\left(\begin{matrix}-n+k,\,a\\ b;\ \frac{n+1}{k}\end{matrix}\right){}_{p+1}F_q\left(\begin{matrix}-k,\,(a_p)\\ (b_q);\ \frac{z}{k}\end{matrix}\right)$$
$$=\frac{(-1)^{n+1}(n+1)^n(a)_{n+1}}{(a-b+1)(b)_n}$$
$$\times\left[\frac{a-b+1}{n+a}-{}_{p+2}F_{q+1}\left(\begin{matrix}-n-1,\,(a_p),\,1-b-n\\ (b_q),\,-a-n;\ \frac{z}{n+1}\end{matrix}\right)\right.$$
$$\left.+\frac{(b-1)(b)_n}{(a)_{n+1}}\,{}_{p+1}F_q\left(\begin{matrix}-n-1,\,(a_p)\\ (b_q);\ \frac{z}{n+1}\end{matrix}\right)\right].$$

2. $$\sum_{k=0}^{n-1}\binom{n}{k}(k+1)^{k-1}(n-k)^{n-k}\,{}_{p+1}F_q\left(\begin{matrix}-k,\,(a_p);\ \frac{w}{k+1}\\ (b_q)\end{matrix}\right)$$
$$\times\,{}_{r+1}F_s\left(\begin{matrix}-n+k,\,(c_r)\\ (d_s);\ \frac{z}{n-k}\end{matrix}\right)=-(n+1)^{n-1}\,{}_{p+1}F_q\left(\begin{matrix}-n,\,(a_p)\\ (b_q);\ \frac{w}{n+1}\end{matrix}\right)$$
$$+\frac{(-w)^n}{n+1}\frac{\prod(a_p)_n}{\prod(b_q)_n}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^{(p+q+1)k}\left(\frac{n+1}{w}\right)^k$$
$$\times\frac{\prod(1-b_q-n)_k}{\prod(1-a_p-n)_k}\,{}_{r+1}F_s\left(\begin{matrix}-k,\,(c_r)\\ (d_s);\ \frac{z}{n+1}\end{matrix}\right).$$

5.17.6. Various sums containing ${}_pF_q((a_p)+mk;(b_q)+nk;z)$

1. $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(c)_k(e-b)_k}{(e)_k(c-d+1)_k}\,{}_3F_2\left(\begin{matrix}a,\,b,\,c+k\\ d,\,e+k;\ 1\end{matrix}\right)$$
$$=\frac{(1-d)_n}{(c-d+1)_n}\,{}_3F_2\left(\begin{matrix}a-n,\,b,\,c\\ d-n,\,e;\ 1\end{matrix}\right)\quad[\operatorname{Re}(d+e-a-b)>0].$$

2. $$\sum_{k=0}^{n}\binom{n}{k}2^{-2k}(2k)!\frac{(a)_k(2a+n)_k}{\left(a+\frac{1}{2}\right)_k(2a)_k\left(\frac{1}{2}\right)_{2k}}$$
$$\times\,{}_4F_3\left(\begin{matrix}-n+k,\,k+1,\,k+a,\,k+n+2a\\ 2k+\frac{3}{2},\,k+a+\frac{1}{2},\,k+2a;\ 1\end{matrix}\right)P_{2k}(z)=\left[\frac{n!}{(2a)_n}C_n^a(z)\right]^2$$
[[69], (18)].

3. $$\sum_{k=0}^{n}\frac{(k!)^2\,(a+n)_k}{(n-k)!\,(2k)!\left(\frac{1}{2}\right)_k\left(a+\frac{1}{2}\right)_k}\,{}_4F_3\left(\begin{matrix}-n+k,\,k+1,\,k+1,\,k+n+a\\ k+\frac{1}{2},\,a+k+\frac{1}{2},\,2k+2;\ 1\end{matrix}\right)$$
$$\times[P_k(z)]^2=\frac{(2n)!}{n!\,(2a)_{2n}}C_{2n}^a(z).$$

4. $$\sum_{k=0}^{n} \frac{(k!)^2 (a)_k (2a+n)_k}{(n-k)!\,(2k)!\left(\frac{1}{2}\right)_k \left(a+\frac{1}{2}\right)_k (2a)_k}$$
$$\times\ {}_5F_4\left(\begin{matrix} -n+k,\ k+1,\ k+1,\ k+a,\ k+n+2a \\ k+\frac{1}{2},\ a+k+\frac{1}{2},\ k+2a,\ 2k+2;\ 1\end{matrix}\right)[P_k(z)]^2 = \frac{n!}{(2a)_n^2}\left[C_n^a(z)\right]^2.$$

5. $$\sum_{k=0}^{n} \binom{n}{k} 2^{-2k} \frac{(a)_k (2a+n)_k}{\left(a+\frac{1}{2}\right)_k (2a)_k}\ {}_4F_3\left(\begin{matrix} -n+k,\ k+\frac{3}{2},\ k+a,\ k+n+2a \\ 2k+2,\ k+a+\frac{1}{2},\ k+2a;\ 1\end{matrix}\right)$$
$$\times U_{2k}(z) = \left[\frac{n!}{(2a)_n} C_n^a(z)\right]^2 \qquad [[69], (18)].$$

6. $$\sum_{k=0}^{n} \frac{\left(\frac{3}{2}\right)_k (a+n)_k}{(n-k)!\,(2k+1)!\left(a+\frac{1}{2}\right)_k}\ {}_4F_3\left(\begin{matrix} -n+k,\ k+\frac{3}{2},\ k+2,\ k+n+a \\ k+1,\ a+k+\frac{1}{2},\ 2k+3;\ 1\end{matrix}\right)$$
$$\times [U_k(z)]^2 = \frac{(2n)!}{n!\,(2a)_{2n}} C_{2n}^a(z).$$

7. $$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{-2k}(a)_k (2a+n)_k}{\left(a+\frac{1}{2}\right)_k (2a)_k}\ {}_4F_3\left(\begin{matrix} -n+k,\ k+\frac{3}{2},\ k+2,\ k+a,\ k+n+2a \\ k+1,\ a+k+\frac{1}{2},\ k+2a,\ 2k+3;\ 1\end{matrix}\right)$$
$$\times [U_k(z)]^2 = \left[\frac{n!}{(2a)_n} C_n^a(z)\right]^2.$$

5.18. Multiple Sums

5.18.1. Sums containing Bessel functions

Condition: $k_i = 0, 1, 2, \ldots$

1. $$\sum_{k_1+\ldots+k_{2m}=n} \prod_{i=1}^{2m} \frac{1}{k_i!} J_{\pm k_i \mp 1/2}(z)$$
$$= \frac{(\pm 1)^n\, 2^{n-m+1}}{n!\,(\pi z)^{m-1/2}} \sum_{k=1}^{m} (\pm 1)^k \binom{2m}{m-k} k^{n+1/2} J_{n-1/2}(2kz) \qquad [n \geq 1].$$

2. $$\sum_{k_1+\ldots+k_{2m+1}=n} \prod_{i=1}^{2m+1} \frac{1}{k_i!} J_{\pm k_i \mp 1/2}(z)$$
$$= \frac{(\pm 1)^n}{n!\,(2\pi z)^m} \sum_{k=1}^{m} \binom{2m+1}{m-k} k^{n+1/2} J_{\pm n \mp 1/2}((2k+1)\,z) \qquad [n \geq 1].$$

3. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} \frac{z_i^{k_i}}{k_i!} K_{k_i-1/2}(z_i) = \frac{\left(\frac{\pi}{2}\right)^{(m-1)/2}}{n!} \frac{(z_1+\ldots+z_m)^{n+1/2}}{(z_1\ldots z_m)^{1/2}}$$
$$\times K_{n-1/2}(z_1+\ldots+z_m).$$

5.18.2. Sums containing orthogonal polynomials

Condition: $k_i = 0, 1, 2, \dots$

1. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} P_{k_i}(z) = C_n^{m/2}(z).$$

2. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} U_{k_i}(z) = C_n^{m}(z).$$

3. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} \frac{1}{k_i!} H_{2k_i}(z_i) = (-4)^n L_n^{m/2-1}(z_1^2 + \ldots + z_m^2).$$

4. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} \frac{1}{k_i!} H_{2k_i+1}(z_i) = (-1)^n 2^{2n+m} (z_1 \ldots z_m)^{1/2} L_n^{3m/2-1}(z_1^2 + \ldots + z_m^2).$$

5. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} L_{k_i}^{\lambda_i}(z_i) = L_n^{\lambda_1+\ldots+\lambda_m+m-1}(z_1 + \ldots + z_m).$$

6. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} \frac{z_i^{k_i}}{k_i!} L_{r_i}^{k_i-r_i}(z_i) = \frac{(r_1 + \ldots + r_m)!}{n!} \left(\prod_{i=1}^{m} \frac{z_i^{k_i}}{k_i!} \right) u^{n-r} L_r^{n-r}(u)$$
$$[r_i = 0, 1, \ldots, n;\ r = r_1 + \ldots + r_m;\ u = z_1 + \ldots + z_m].$$

7. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} C_{k_i}^{\lambda_i}(z) = C_n^{\lambda_1+\ldots+\lambda_m}(z).$$

8. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} \frac{(-\lambda_i)_{k_i}}{(k_i - 2\lambda_i)_{k_i}} C_{k_i}^{\lambda_i}(z) = \frac{(-\lambda_1 - \ldots - \lambda_m)_n}{(n - 2\lambda_1 - \ldots - 2\lambda_m)_n} C_n^{\lambda_1+\ldots+\lambda_m}(z).$$

9. $$\sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} P_{k_i}^{(\rho_i-k_i,\, \sigma_i-k_i)}(z) = P_n^{(\rho-n,\, \sigma-n)}(z)$$
$$[\rho = \rho_1 + \ldots + \rho_m;\ \sigma = \sigma_1 + \ldots + \sigma_m].$$

Chapter 6

Infinite Series

6.1. Elementary Functions

6.1.1. Series containing algebraic functions

1. $$\sum_{k=0}^{\infty} \frac{\sigma_{k+m}^{m}}{(k+m)!} z^k \frac{\prod (a_p)_k}{\prod (b_q)_k}$$
$$= (-1)^{m(p+q+1)} \frac{z^{-m}}{m!} \frac{\prod (1-(b_q))_m}{\prod (1-(a_p))_m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} {}_pF_q\left(\begin{matrix}(a_p)-m;\ kz\\ (b_q)-m\end{matrix}\right)$$
$[p \le q]$.

6.1.2. Series containing the exponential function

Notations: $c = \mathbf{K}(k')/\mathbf{K}(k)$, $k' = \sqrt{1-k^2}$.

1. $$\sum_{n=1}^{\infty} \frac{1}{n\left(e^{2n\pi c}-1\right)} = -\frac{\pi c}{12} - \frac{1}{6} \ln \frac{2kk'\,\mathbf{K}^3(k)}{\pi^3}$$ [[84], (T1.1)].

2. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\left(e^{2n\pi c}-1\right)} = -\frac{\pi c}{12} - \frac{1}{12} \ln \frac{k^2}{16k'}$$ [[84], (T1.2)].

3. $$\sum_{n=1}^{\infty} \frac{1}{n\left(e^{2n\pi c}+1\right)} = \frac{\pi c}{4} + \frac{1}{2} \ln \frac{k\,\mathbf{K}(k)}{2\pi}$$ [[84], (T1.5)].

4. $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\left[e^{(2n+1)\pi c}+1\right]} = \frac{1}{4} \ln \frac{2\,\mathbf{K}(k)}{\pi}$$ [[84], (T1.6)].

5. $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\left[e^{(2n+1)\pi c}-1\right]} = -\frac{1}{4} \ln \frac{2k'\,\mathbf{K}(k)}{\pi}$$ [[84], (T1.7)].

6. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\left(e^{2n\pi c}+1\right)} = \frac{\pi c}{4} + \frac{1}{2} \ln \frac{1-k'}{2k}$$ [[84], (T1.8)].

6.1.3. Series containing hyperbolic functions

Notations: $c = \mathbf{K}(k')/\mathbf{K}(k)$, $k' = \sqrt{1-k^2}$.

1. $$\sum_{n=1}^{\infty} \frac{\coth(nc\pi)-1}{n} = -\frac{c\pi}{6} - \frac{1}{3}\ln\frac{2kk'\,\mathbf{K}^3(k)}{\pi^3}$$ [[84], (T1.1)].

2. $$\sum_{n=1}^{\infty} (-1)^n \frac{\coth(nc\pi)-1}{n} = -\frac{\pi c}{6} - \frac{1}{6}\ln\frac{2kk'\,\mathbf{K}^3(k)}{\pi^3}$$ [[84], (T1.2)].

3. $$\sum_{n=1}^{\infty} \frac{\operatorname{csch}(nc\pi)}{n} = \frac{c\pi}{12} - \frac{1}{6}\ln\frac{4k'^2}{k}$$ [[84], (T1.4)].

4. $$\sum_{n=1}^{\infty} (-1)^n \frac{\operatorname{csch}(nc\pi)}{n} = \frac{c\pi}{12} + \frac{1}{6}\ln\frac{kk'}{4}$$ [[84], (T1.3)].

5. $$\sum_{n=1}^{\infty} \frac{1-\tanh(nc\pi)}{n} = \frac{c\pi}{2} - \ln\frac{k\,\mathbf{K}(k)}{2\pi}$$ [[84], (T1.5)].

6. $$\sum_{n=1}^{\infty} (-1)^n \frac{1-\tanh(nc\pi)}{n} = \frac{c\pi}{2} + \ln\frac{1-k'}{2k}$$ [[84], (T1.8)].

7. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\operatorname{csch}[(2n+1)c\pi] = -\frac{1}{4}\ln k'$$ [[84], (T1.9)].

8. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\operatorname{sech}\frac{(2n+1)c\pi}{2} = \frac{1}{2}\arcsin k$$ [[84], (T1.10)].

9. $$\sum_{n=1}^{\infty} (-1)^n \operatorname{csch}(nc\pi)\coth(nc\pi) = \frac{2k^2-1}{3\pi^2}\,\mathbf{K}^2(k) - \frac{1}{12}$$ [[84], (T1.18)].

10. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{6m+1}}\operatorname{sech}\frac{(2n+1)\sqrt{3}\,\pi}{2} = \frac{(-1)^{m+1}}{2}\pi^{6m+1}\sum_{p=0}^{3m}\frac{E_{2p+1}}{(2p+1)!}\,\frac{B_{6m-2p}}{(6m-2p)!}\cos\frac{(2p+1)\pi}{3}$$ [19].

6.1.4. Series containing trigonometric functions

1. $$\sum_{k=1}^{\infty} \frac{\sin(kx)\cos(ky)}{k^{2m-1}} = (-1)^m \frac{x^{2m-1}}{2(2m-1)!} + \sum_{k=0}^{m-2} \frac{x^{2k+1}}{(2k+1)!}$$
$$\times \left[(-1)^{m-1} \frac{\pi y^{2m-2k-3}}{2(2m-2k-3)!} + (-1)^k \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j}}{(2j)!} \zeta(2m-2j-2k-2)\right]$$
$[-\pi < x < \pi;\ |x| < y < 2\pi - |x|;\ [72]]$.

2. $$\sum_{k=1}^{\infty} (-1)^k \frac{\sin(kx)\cos(ky)}{k^{2m-1}} = (-1)^m \frac{x^{2m-1}}{2(2m-1)!} - \sum_{k=0}^{m-2} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$
$$\times \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j}}{(2j)!} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$
$[-\pi < x < \pi;\ |x| - \pi < y < \pi - |x|;\ [72]]$.

3. $$\sum_{k=0}^{\infty} \frac{(-1)^k \left(k+\frac{1}{2}\right)}{\left(k+\frac{1}{2}\right)^2 - b^2} \frac{\sin\sqrt{\left(k+\frac{1}{2}\right)^2 \pi^2 + a^2}}{\sqrt{\left(k+\frac{1}{2}\right)^2 \pi^2 + a^2}} = \frac{\pi}{2} \sec(b\pi) \frac{\sin\sqrt{a^2 + b^2\pi^2}}{\sqrt{a^2 + b^2\pi^2}}$$
[[39], (1.10)].

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\sqrt{k^2+a^2}} \sin\left(\sqrt{k^2+a^2}\, x\right) = \frac{3 - a^2\pi^2}{12a^3} \sin(ax) - \frac{x}{4a^2} \cos(ax)$$
$[-\pi < x < \pi]$.

5. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos\left(\sqrt{k^2+a^2}\, x\right) = \frac{x}{4a} \sin(ax) - \frac{\pi^2}{12} \cos(ax) \qquad [-\pi < x < \pi].$$

6.2. The Psi Function $\psi(z)$

6.2.1. Series containing $\psi(ka+b)$

1. $$\sum_{k=1}^{\infty} t^k \psi(k) = \frac{t}{t-1} [\mathbf{C} + \ln(1-t)] \qquad [|t| < 1].$$

2. $$\sum_{k=1}^{\infty} \frac{t^k}{k} \psi(k) = \mathbf{C} \ln(1-t) + \frac{1}{2} \ln^2(1-t) \qquad [|t| < 1].$$

3. $$\sum_{k=1}^{\infty} \frac{t^k}{k^2} \psi(k) = \frac{1}{2} \ln t \ln^2(1-t) - \mathbf{C}\, \mathrm{Li}_2(t) + \ln(1-t)\, \mathrm{Li}_2(1-t)$$
$$- \mathrm{Li}_3(1-t) + \zeta(3) \qquad [|t| < 1].$$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\psi(k) = \zeta(3) - \frac{\pi^2 \mathbf{C}}{6}.$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}\psi(k) = 1 - \mathbf{C}$ [[29], (7)].

6. $$\sum_{k=1}^{\infty} \frac{1}{(k+a)(k+b)}\psi(k) = \frac{1}{2(a-b)}[\psi'(b+1) - \psi'(a+1) - \psi^2(b+1) + \psi^2(a+1)].$$

7. $$\sum_{k=2}^{\infty} \frac{t^k}{k(k-1)}\psi(k) = -\mathbf{C}[t + \ln(1-t) - t\ln(1-t)] + \frac{1}{6}\left[\pi^2 t + 3(t-1)\ln^2(1-t) - 6t\ln t\ln(1-t) - 6t\operatorname{Li}_2(1-t)\right] \quad [|t| < 1].$$

8. $$\sum_{k=1}^{\infty} \frac{t^{k+1}}{k(k+1)}\psi(k) = -\frac{1}{2}(1-t)\ln^2(1-t) + (1-\mathbf{C})\,[t + (1-t)\ln(1-t)] \quad [|t| < 1].$$

9. $$\sum_{k=1}^{\infty} \frac{1}{k^2}\psi(k+m) = \zeta(3) + \frac{\pi^2}{6}\psi(m) - \frac{1}{2}\psi''(m) + \mathbf{C}\left[\psi'(m) - \frac{\pi^2}{6}\right] - \sum_{k=1}^{m-1} \frac{1}{k^2}\psi(k).$$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}\psi(k) = \frac{\pi^4}{360} - \mathbf{C}\zeta(3).$

11. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}\psi(k) = -\frac{\pi^4}{48} - \frac{\pi^2}{12}\ln^2 2 + \frac{1}{12}\ln^4 2 + \frac{1}{4}(7\ln 2 + 3\mathbf{C})\,\zeta(3) + 2\operatorname{Li}_4\left(\frac{1}{2}\right).$$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)}\psi(k) = \left(1 - \frac{\pi^2}{6}\right)\mathbf{C} + \zeta(3) - 1$ [[58], (10)].

13. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)^2}\psi(k) = \left(\frac{\pi^2}{6} - 2\right)\mathbf{C} - \frac{\pi^2}{6} - \zeta(3) + 3$ [[58], (13)].

14. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}\psi(k) = -\frac{\pi^4 \mathbf{C}}{90} - \frac{\pi^2}{6}\zeta(3) + 2\zeta(5).$

15. $$\sum_{k=1}^{\infty} \frac{1}{k^5}\psi(k) = \frac{\pi^6}{1260} - \frac{1}{2}\zeta^2(3) - \mathbf{C}\zeta(5).$$

16. $$\sum_{k=1}^{\infty} \frac{1}{k^{n+2}}\psi(k) = \frac{n+2}{2}\zeta(n+3) - \mathbf{C}\zeta(n+2) - \frac{1}{2}\sum_{k=0}^{n}\zeta(k+1)\,\zeta(n-k+2).$$

17. $$\sum_{k=1}^{\infty} \frac{1}{(k+a)^{n+2}}\psi(k) = \frac{(-1)^{n+1}}{(n+1)!}\left[\frac{1}{2}\psi^{(n+2)}(a+1) - \sum_{k=0}^{n}\binom{n}{k}\psi^{(n-k)}(a+1)\,\psi^{(k+1)}(a+1)\right].$$

18. $$\sum_{k=1}^{\infty} \frac{t^k}{k!}\psi(k) = 2\mathbf{C} + e^t\Gamma(0,t) + \Gamma(0,-t) + e^t\ln t + \ln(-t).$$

19. $$\sum_{k=0}^{\infty} \frac{t^k}{(k!)^2}\psi(k+1) = \frac{1}{2}\ln t\, I_0\big(2\sqrt{t}\big) + K_0\big(2\sqrt{t}\big).$$

20. $$\sum_{k=0}^{\infty} \frac{t^k}{k!\,(k+1)!}\psi(k+1) = \frac{1}{2t}\left[2 - I_0\big(2\sqrt{t}\big) + \sqrt{t}\ln t\, I_1\big(2\sqrt{t}\big) - 2\sqrt{t}\,K_1\big(2\sqrt{t}\big)\right].$$

21. $$\sum_{k=1}^{\infty} \frac{t^k}{k}\psi\left(k+\frac{1}{2}\right) = (\mathbf{C} + 2\ln 2)\ln(1-t) + \frac{1}{2}\ln^2\frac{1+\sqrt{t}}{1-\sqrt{t}} \qquad [|t|<1].$$

22. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\psi\left(k+\frac{1}{2}\right) = -\frac{\pi^2}{8} + \ln 2(\mathbf{C} + 2\ln 2).$$

23. $$\sum_{k=1}^{\infty} \frac{1}{k^2}\psi\left(k+\frac{1}{2}\right) = \frac{7}{2}\zeta(3) - \frac{\pi^2}{3}\ln 2 - \frac{\pi^2\mathbf{C}}{6} \qquad [[29], (15)].$$

24. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\psi\left(k+\frac{1}{2}\right) = \frac{7}{2}\zeta(3) + \frac{\pi^2}{6}\ln 2 + \frac{\pi^2\mathbf{C}}{12} - 2\pi\mathbf{G} \qquad [[29], (19)].$$

25. $$\sum_{k=1}^{\infty} \frac{1}{k\,(2k+1)}\psi\left(k+\frac{1}{2}\right) = \frac{\pi^2}{6} + 2(\ln 2 - 1)(\mathbf{C} + 2\ln 2).$$

26. $$\sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)}\psi\left(k+\frac{1}{2}\right) = -\frac{\pi^2}{6} + 2(2-\mathbf{C})\ln 2 - 4\ln^2 2.$$

27. $$\sum_{k=0}^{\infty} \frac{1}{(k+2)(2k+1)} \psi\left(k+\frac{1}{2}\right) = \frac{1}{18}\left[4-\pi^2+20\ln 2-24\ln^2 2-6(2\ln 2+1)\mathbf{C}\right].$$

28. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \psi\left(k+\frac{1}{2}\right) = \frac{7}{8}\zeta(3)+\frac{\pi^2}{4}\ln 2-\frac{\pi^2}{8}(\mathbf{C}+2\ln 2).$$

29. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \psi\left(k+\frac{1}{2}\right) = -\frac{1}{8}\left[\pi^2\mathbf{C}+7\zeta(3)\right].$$

30. $$\sum_{k=0}^{\infty} \frac{1}{(k+2)(2k+1)(2k+3)} \psi\left(k+\frac{1}{2}\right) = \frac{1}{18}\left[\pi^2+14-56\ln 2+24\ln^2 2+12\mathbf{C}(\ln 2-1)\right].$$

31. $$\sum_{k=0}^{\infty} \frac{t^k}{k!} \psi\left(k+\frac{1}{2}\right) = -(\mathbf{C}+2\ln 2)\, e^t+2te^t\, {}_2F_2\left(\begin{matrix}1,\ 1;\ -t\\ \frac{3}{2},\ 2\end{matrix}\right).$$

32. $$\sum_{k=1}^{\infty} \frac{1}{k!\,k} \psi\left(k+\frac{1}{2}\right) = \frac{1}{2}(\mathbf{C}+\pi^2)-2\mathbf{C}\ln 2+\ln 2-4\ln^2 2-1.$$

33. $$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!\,k} \psi\left(k+\frac{1}{2}\right) = \frac{1}{2}\left[\pi^2-4\ln 2(\mathbf{C}+2\ln 2)\right].$$

34. $$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(a)_k k} \psi\left(k+\frac{1}{2}\right) = (\mathbf{C}+2\ln 2)\left[\psi\left(a-\frac{1}{2}\right)-\psi(a)\right]+\psi'\left(a-\frac{1}{2}\right) \qquad [\operatorname{Re} a > 1/2].$$

35. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+1)!\,(k+1)} \psi\left(k+\frac{1}{2}\right) = 4(\ln 2-1)\mathbf{C}+8(2-2\ln 2+\ln^2 2)-\pi^2.$$

36. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+2)!\,(k+2)} \psi\left(k+\frac{1}{2}\right) = \frac{2}{27}\left[3(7-12\ln 2)\mathbf{C}+9\pi^2+138\ln 2-72\ln^2 2-154\right].$$

37. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+2)!\,(2k+3)} \psi\left(k+\frac{1}{2}\right) = \left(\pi-\frac{10}{3}\right)\mathbf{C}-\pi+\frac{4}{9}(17-15\ln 2).$$

38. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+3)!\,(2k+5)} \psi\left(k+\frac{1}{2}\right) = \left(\frac{23}{15}-\frac{\pi}{2}\right)\mathbf{C}+\frac{3\pi}{4}+\frac{46}{15}\ln 2-\frac{1018}{225}.$$

39. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!\,(2k+1)^2}\,\psi\left(k+\frac{1}{2}\right) = -\frac{\pi}{24}\left(\pi^2 + 12\mathbf{C}\ln 2\right).$$

40. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+1)!\,(2k+1)^2}\,\psi\left(k+\frac{1}{2}\right) = [\pi(1-\ln 2) - 2]\,\mathbf{C} - \frac{\pi^3}{12} + 4(1-\ln 2).$$

41. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+1)!\,(2k+1)(2k+3)}\,\psi\left(k+\frac{1}{2}\right) = \frac{\pi}{4} + \left(2 - \frac{3\pi}{4}\right)\mathbf{C} + 4(\ln 2 - 1).$$

42. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!\,(k+2)(2k+1)(2k+3)}\,\psi\left(k+\frac{1}{2}\right)$$
$$= \frac{1}{288}\left[153\pi + 6(64 - 21\pi)\,\mathbf{C} + 768\ln 2 - 1024\right].$$

43. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+1)!\,(k+3)(2k+5)(2k+7)}\,\psi\left(k+\frac{1}{2}\right) = \frac{398}{225} - \frac{29\pi}{96}$$
$$+ \frac{3}{80}\left(5\pi - 16\right)\mathbf{C} - \frac{6}{5}\ln 2.$$

44. $$\sum_{k=0}^{\infty} \frac{t^k}{k!\left(\frac{1}{2}\right)_k}\,\psi\left(k+\frac{1}{2}\right)$$
$$= \frac{1}{2}\left[\cosh(2\sqrt{t})\ln t + 2\sinh(2\sqrt{t})\,\mathrm{shi}\,(4\sqrt{t}) - 2\cosh(2\sqrt{t})\,\mathrm{chi}\,(4\sqrt{t})\right].$$

45. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{k!\,(k+1)!}\,\psi\left(k+\frac{1}{2}\right) = \frac{4}{\pi}\left(2 - \mathbf{C} - 4\ln 2\right).$$

46. $$\sum_{k=1}^{\infty} \frac{1}{(k+a)(k+a+1)}\,\psi\left(k+\frac{1}{2}\right)$$
$$= \frac{1}{a(a+1)(2a+1)}\left[2 + a(2 + \mathbf{C} + 2\ln 2)\right] + \frac{2}{2a+1}\,\psi(a).$$

47. $$\sum_{k=1}^{\infty} \frac{t^k}{\left(\frac{1}{2}\right)_k^2}\,\psi\left(k+\frac{1}{2}\right)$$
$$= \frac{\pi\sqrt{t}}{2}\left[\ln t\;\mathbf{L}_0(2\sqrt{t}) - 2K_0(2\sqrt{t}) + \frac{1}{\pi^2}\,G_{24}^{42}\left(t\,\middle|\,\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\end{matrix}\right)\right].$$

48. $$\sum_{k=0}^{\infty} \frac{t^k}{(2k+1)!}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{1}{\sqrt{t}}\,\Big[\frac{1}{2}\ln\frac{t}{4}\sinh\sqrt{t} - 2\operatorname{shi}\big(\sqrt{t}\big) + \cosh\sqrt{t}\operatorname{shi}\big(2\sqrt{t}\big) - \sinh\sqrt{t}\operatorname{chi}\big(2\sqrt{t}\big)\Big].$$

49. $$\sum_{k=0}^{\infty} \frac{\big(-\frac{1}{2}\big)_k\big(\frac{1}{2}\big)_k}{k!\,(k+1)!}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{8}{3\pi}\,(1-\mathbf{C}-4\ln 2).$$

50. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k^2}{k!\,(k+1)!}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{4}{\pi}\,(2-\mathbf{C}-4\ln 2).$$

51. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k^2}{[(k+1)!]^2}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{4}{\pi}\,[(\pi-4)\,\mathbf{C}+2\pi(\ln 2-1)+4(3-4\ln 2)].$$

52. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k\big(\frac{3}{2}\big)_k}{[(k+1)!]^2}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{4}{\pi}\,[(2-\pi)\,\mathbf{C}+2(4-\pi)\,(\ln 2-1)].$$

53. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k^2}{(k+1)!\,(k+2)!}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{4}{27\pi}\,[3\,(9\pi-32)\,\mathbf{C}+54\pi(\ln 2-1)-384\ln 2+304].$$

54. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k\big(\frac{3}{2}\big)_k}{(k+1)!\,(k+2)!}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{4}{9\pi}\,[3\,(8-3\pi)\,\mathbf{C}-18\pi(\ln 2-1)+8(12\ln 2-11)].$$

55. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k^3}{(k!)^3}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{1}{12\pi^3}\,(\pi-3\mathbf{C}-6\ln 2)\Gamma^4\Big(\frac{1}{4}\Big).$$

56. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k^3}{k!\,[(k+1)!]^2}\,\psi\Big(k+\frac{1}{2}\Big) = \frac{1}{6\pi^3}\Big[(\pi-6\ln 2-3\mathbf{C})\,\Gamma^4\Big(\frac{1}{4}\Big)+48\,(\pi+6\ln 2+3\mathbf{C}-8)\,\Gamma^4\Big(\frac{3}{4}\Big)\Big].$$

57. $$\sum_{k=0}^{\infty} \frac{\big(\frac{1}{2}\big)_k^3}{[(k+1)!]^3}\,\psi\Big(k+\frac{1}{2}\Big) = 8(2-\mathbf{C}-2\ln 2) + \frac{32}{\pi^3}(3\mathbf{C}+6\ln 2+\pi-10)\Gamma^4\Big(\frac{3}{4}\Big).$$

58. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k+1)!\,[(k+2)!]^2}\,\psi\left(k+\frac{1}{2}\right) = 8(2-\mathbf{C}-2\ln 2) - \frac{4}{243\pi^3}\,(22+15\pi-45\mathbf{C}-90\ln 2)\,\Gamma^4\left(\frac{1}{4}\right) - \frac{64}{27\pi^3}\,[78-7\pi-21\mathbf{C}-42\ln 2]\,\Gamma^4\left(\frac{3}{4}\right).$$

59. $$\sum_{k=0}^{\infty} \frac{k!}{(a)_k\,(k+1)}\,\psi(k+a) = [(a-1)\psi(a-1)-1]\,\psi'(a-1) - (a-1)\psi''(a-1) \qquad [\operatorname{Re} a > 1].$$

60. $$\sum_{k=1}^{\infty} \frac{t^k}{k!}\,\psi(k+a) = \frac{t}{a}\,e^t \left[a\psi(a)\frac{1-e^{-t}}{t} + {}_2F_2\left(\begin{matrix}1,\,1;\,-t\\ a+1,\,2\end{matrix}\right)\right] \qquad [[58],\,(1.1a)].$$

61. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!}\,t^k\psi(k+a) = (1-t)^{-a}[\psi(a)-\ln(1-t)] \qquad [|t|<1].$$

62. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!\,(k+b)}\,t^k\psi(k+a) = \frac{1}{b}\,\psi(a)\,{}_2F_1\left(\begin{matrix}a,\,b\\ b+1;\,t\end{matrix}\right) + t^{-b}(1-t)^{1-a}\left\{(1-t)^{a-1}\,\mathrm{B}\,(1-a,\,b)\left[\psi(b-a+1)-\psi(1-a)\right] + \frac{1}{1-a}\ln(1-t)\,{}_2F_1\left(\begin{matrix}1-a,\,1-b\\ 2-a;\,1-t\end{matrix}\right) - \frac{1}{(1-a)^2}\,{}_3F_2\left(\begin{matrix}1-a,\,1-a,\,1-b\\ 2-a,\,2-a;\,1-t\end{matrix}\right)\right\}.$$

63. $$\sum_{k=1}^{\infty} \frac{t^k}{(a)_k}\,\psi(k+a) = \frac{t}{a^2}\,e^t\left[a^2t^{-a}\psi(a)\gamma(a,\,t) + {}_2F_2\left(\begin{matrix}a,\,a;\,-t\\ a+1,\,a+1\end{matrix}\right)\right] \qquad [[58],\,(1.1b)].$$

64. $$\sum_{k=0}^{\infty} \frac{2^{-k}k!}{(a)_k}\,\psi(k+a) = [(a-1)\psi(a)-1]\left[\psi\left(\frac{a}{2}\right)-\psi\left(\frac{a-1}{2}\right)\right] + \frac{a-1}{2}\left[\psi'\left(\frac{a-1}{2}\right)-\psi'\left(\frac{a}{2}\right)\right].$$

65. $$\sum_{k=1}^{\infty} \frac{2^{-k}k!}{k^2(a)_k}\,\psi(k+a) = \frac{1}{8}\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(\frac{a}{2}\right)\right]\left[\psi'\left(\frac{a+1}{2}\right)-\psi'\left(\frac{a}{2}\right)\right] + \frac{1}{8}\,\psi(a)\left\{4\psi'(a)-\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(\frac{a}{2}\right)\right]^2\right\} + \frac{1}{8}\left[\zeta\left(3,\,\frac{a}{2}\right)+\zeta\left(3,\,\frac{a+1}{2}\right)\right].$$

66. $$\sum_{k=0}^{\infty} \frac{(k+n)!}{k!\,(a)_k} t^k \psi(k+a)$$
$$= n! \left[\psi(a)\, {}_1F_1\binom{n+1}{a;\ t} - t^{1-a} e^t L_n^{1-a}(-t) \gamma(a-1,\, t) \right.$$
$$- \sum_{k=1}^{n} \frac{1}{k} L_{n-k}^{k-a+1}(-t) L_{k-1}^{a-k-1}(t) + (a-1) e^t$$
$$\left. \times \sum_{k=0}^{n} \frac{(-t)^k}{k!\,(a+k-1)^2} L_{n-k}^{k}(-t)\, {}_2F_2\binom{a+k-1,\, a+k-1}{a+k,\, a+k;\ -t} \right].$$

67. $$\sum_{k=0}^{\infty} \frac{(a)_k \left(\frac{1}{2}\right)_k}{k!\,(k+1)!} \psi(k+a)$$
$$= \frac{2\Gamma\left(\frac{3}{2}-a\right)}{\sqrt{\pi}\,\Gamma(2-a)} \left[\frac{1}{1-a} + \pi \cot(a\pi) + 2\psi(a) - \psi\left(\frac{3}{2}-a\right) \right] \quad [\operatorname{Re} a < 1].$$

68. $$\sum_{k=0}^{\infty} \frac{(a)_k \left(\frac{1}{2}\right)_k}{(k+1)!(k+2)!} \psi(k+a)$$
$$= \frac{2}{a-1} \left\{ \psi(a-1) - \frac{4\Gamma\left(\frac{7}{2}-a\right)}{3\sqrt{\pi}\,\Gamma(3-a)} \left[\psi(3-a) - \psi\left(\frac{7}{2}-a\right) + \psi(a-1) \right] \right\}$$
$[\operatorname{Re} a < 1]$.

69. $$\sum_{k=0}^{\infty} \frac{(a)_k^3}{(k!)^3} \psi(k+a) = -\frac{2\Gamma\left(-\frac{3a}{2}\right)}{a^2 \Gamma^3\left(-\frac{a}{2}\right)} \cos\frac{a\pi}{2}$$
$$\times \left\{ \pi \tan\frac{a\pi}{2} - 6\psi(a) - 3\left[2\pi \cos\frac{a\pi}{2} \csc\frac{3a\pi}{2} + \psi\left(\frac{a}{2}\right) - \psi\left(\frac{3a}{2}\right) \right] \right\}$$
$[\operatorname{Re} a < 1]$.

70. $$\sum_{k=0}^{\infty} (-1)^k (2k+a) \frac{(a)_k^3}{(k!)^3} \psi(k+a) = \frac{1}{3} \cos(a\pi)$$
$$+ \left[a\psi(a) - \frac{1}{3} \right] {}_3F_2\binom{a,\, a,\, a}{1,\, 1;\ -1} - 2a^3 \psi(a)\, {}_3F_2\binom{a+1,\, a+1,\, a+1}{2,\, 2;\ -1}$$
$[\operatorname{Re} a < 1/3]$.

71. $$\sum_{k=1}^{\infty} \frac{1}{k^2} \psi(2k) = \frac{9}{4} \zeta(3) - \frac{\mathbf{C}\pi^2}{6}.$$

72. $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \psi(2k) = \frac{1}{2} - \mathbf{C} + 2 \ln 2 \quad [[62], (B.4)].$$

73. $$\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} \psi(2k) = \frac{9}{4}\zeta(3) + \left(1 - \frac{\pi^2}{6}\right) \mathbf{C} - 2\ln 2 - \frac{1}{2}.$$

74. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1} \psi(2k+1) = -\left[\mathbf{C} + \frac{1}{2}\ln\left(1+t^2\right)\right] \arctan t \qquad [|t| < 1].$$

75. $$\sum_{k=0}^{\infty} t^{2k} \psi(2k+1) = \frac{1}{t^2-1}\left[\mathbf{C} + \frac{1}{2}\ln\left(1-t^2\right) - \frac{1}{2t}\ln\frac{1+t}{1-t}\right] \qquad [|t| < 1].$$

76. $$\sum_{k=0}^{\infty} (-1)^k t^{2k} \psi(2k+1) = -\frac{1}{1+t^2}\left[\mathbf{C} + t\arctan t + \frac{1}{2}\ln\left(1+t^2\right)\right] \qquad [|t| < 1].$$

77. $$\sum_{k=0}^{\infty} \frac{(-1)^k t^{4k}}{[(2k)!]^2} \psi(2k+1) = \ln t \operatorname{ber}(2t) - \frac{\pi}{4} \operatorname{bei}(2t) + \operatorname{ker}(2t) \qquad [t > 0].$$

78. $$\sum_{k=0}^{\infty} \frac{(-1)^k t^{4k}}{[(2k+1)!]^2} \psi(2k+2) = \frac{1}{4t^2}\left[\pi \operatorname{ber}(2t) + 4\ln t \operatorname{bei}(2t) + 4\operatorname{kei}(2t)\right] \qquad [t > 0].$$

79. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{2k}}{(2k)!} t^{2k} \psi\left(2k + \frac{1}{2}\right) = \frac{2^{-3/2}}{\sqrt{1-t^2}}\left\{\left(1 - \sqrt{1-t^2}\right)^{1/2} \ln\frac{1+t}{1-t} - \left(1 + \sqrt{1-t^2}\right)^{1/2} \times \left[2\mathbf{C} + 4\ln 2 + \ln\left(1-t^2\right)\right]\right\} \qquad [|t| < 1].$$

80. $$\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \left(\frac{3}{2}\right)_{2k+1} \psi\left(2k + \frac{3}{2}\right) = \frac{2^{-1/2}}{\sqrt{1-t^2}}\left\{\left(1 + \sqrt{1-t^2}\right)^{1/2} \ln\frac{1+t}{1-t} - \left(1 - \sqrt{1-t^2}\right)^{1/2} \times \left[2\mathbf{C} + 4\ln 2 + \ln\left(1-t^2\right)\right]\right\} \qquad [|t| < 1].$$

81. $$\sum_{k=0}^{\infty} \frac{t^k}{k!} \psi(2k+a) = e^t \psi(a) + \frac{t}{a} e^t\, {}_2F_2\left(\begin{matrix} 1,\ 1;\ -t \\ 2,\ \frac{a}{2}+1 \end{matrix}\right) + \frac{t}{a+1} e^t\, {}_2F_2\left(\begin{matrix} 1,\ 1;\ -t \\ 2,\ \frac{a+3}{2} \end{matrix}\right).$$

82. $$\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}(a)_{2k}\psi(2k+a) = -\frac{1}{2}\left(t_-^{-a}\ln t_- + t_+^{-a}\ln t_+\right) + \frac{1}{2}\left(t_-^{-a} + t_+^{-a}\right)\psi(a) \quad [t_\pm = 1 \pm t;\ |t| < 1].$$

83. $$\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(a)_{2k}\psi(2k+a) = \frac{1}{2(1-a)}\left(t_-^{1-a}\ln t_- - t_+^{1-a}\ln t_+\right) + \frac{1}{2(1-a)}\left[\frac{1}{1-a} + \psi(a)\right]\left(t_+^{1-a} - t_-^{1-a}\right) \quad [t_\pm = 1 \pm t;\ |t| < 1].$$

84. $$\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}(a)_{2k}\psi(2k+a) = \left(1+t^2\right)^{-a/2} \times \left\{\cos(au)\left[\psi(a) - \frac{1}{2}\ln\left(1+t^2\right)\right] - u\sin(au)\right\} \quad [u = \arctan t;\ |t| < 1].$$

85. $$\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(a)_{2k+1}\psi(2k+a+1) = \frac{1}{2}\left[t_+^{-a}\ln t_+ - t_-^{-a}\ln t_- + \left(t_-^{-a} - t_+^{-a}\right)\psi(a)\right] \quad [t_\pm = 1 \pm t;\ |t| < 1].$$

86. $$\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}(a)_{2k+1}\psi(2k+a+1) = \frac{1}{2}\left(1+t^2\right)^{-a/2}\left\{2u\cos(au) + \sin(au)\left[2\psi(a) - \ln\left(1+t^2\right)\right]\right\} \quad [u = \arctan t;\ |t| < 1].$$

87. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{3k+1}\psi(3k+1) = -\frac{\pi^2}{54} - \frac{1}{6}\ln^2 2 - \frac{1}{3\sqrt{3}}\left(\pi + \sqrt{3}\ln 2\right).$$

88. $$\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)}\psi(3k+1) = \frac{\pi}{6\sqrt{3}}(2 - 2\mathrm{C} - \ln 3).$$

89. $$\sum_{k=0}^{\infty} \frac{(a)_{3k}}{(3k)!}\psi(3k+a) = 3^{-a/2-2}\left\{3\cos\frac{a\pi}{6}\left[2\psi(a) - \ln 3\right] - \pi\sin\frac{a\pi}{6}\right\} \quad [\operatorname{Re} a < 1].$$

90. $$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_{3k}}{(3k)!}\psi(3k+a) = \frac{1}{3}\left[\left(2^{-a} + 2\cos\frac{a\pi}{3}\right)\psi(a) - 2^{-a}\ln 2 - \frac{2\pi}{3}\sin\frac{a\pi}{3}\right] \quad [\operatorname{Re} a < 1].$$

91. $$\sum_{k=0}^{\infty} \frac{(a)_{3k}}{(3k+1)!}\psi(3k+a)$$
$$= \frac{3^{-(a+1)/2}}{1-a}\left\{\cos\frac{(a+1)\pi}{6}\left[\frac{2}{1-a} - \ln 3 + 2\psi(a)\right] - \frac{\pi}{3}\sin\frac{(a+1)\pi}{6}\right\}$$
$[\operatorname{Re} a < 1]$.

92. $$\sum_{k=0}^{\infty}(-1)^k \frac{(a)_{3k}}{(3k+1)!}\psi(3k+a)$$
$$= \frac{1}{3(a-1)}\left\{2^{1-a}\ln 2 + \left[2^{-a} + 2\cos\frac{(a+1)\pi}{3}\right]\left[\frac{1}{a-1} - \psi(a)\right] + \frac{2\pi}{3}\sin\frac{(a+1)\pi}{3}\right\}$$
$[\operatorname{Re} a < 2]$.

93. $$\sum_{k=0}^{\infty} \frac{(a)_{3k}}{(3k+2)!}\psi(3k+a) = -\frac{3^{-a/2}}{(a-1)(a-2)}$$
$$\times\left\{\cos\frac{(a+2)\pi}{6}\left[\ln 3 + \frac{4a-6}{(a-1)(a-2)} - 2\psi(a)\right] + \frac{\pi}{3}\sin\frac{(a+2)\pi}{6}\right\}$$
$[\operatorname{Re} a < 2]$.

94. $$\sum_{k=1}^{\infty} \frac{1}{k^2}\psi(nk) = \left(\frac{n^2}{2} + \frac{1}{2n}\right)\zeta(3) - \frac{\pi^2 \mathbf{C}}{6} + \pi\sum_{k=1}^{n-1} k\ \mathrm{Cl}_2\left(\frac{2k\pi}{n}\right)$$
[[61], (5)].

95. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\psi(nk) = \left(\frac{n^2}{2} - \frac{3}{8n}\right)\zeta(3) + \frac{\pi^2 \mathbf{C}}{12} + \pi\sum_{k=1}^{n-1} k\ \mathrm{Cl}_2\left(\frac{2k+1}{n}\pi\right)$$
[[61], (6)].

96. $$\sum_{k=1}^{\infty} \frac{1}{k}\left[\psi\left(\frac{k+1}{2}\right) - \psi\left(\frac{k}{2}\right)\right] = \ln^2 2 + \frac{\pi^2}{6}.$$

97. $$\sum_{k=1}^{\infty} \frac{1}{k^2}\left[\psi\left(\frac{k+1}{2}\right) - \psi\left(\frac{k}{2}\right)\right] = \frac{13}{4}\zeta(3) - \frac{\pi^2}{3}\ln 2$$ [[29], (31)].

98. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\left[\psi\left(\frac{k+1}{2}\right) - \psi\left(\frac{k}{2}\right)\right] = \frac{\pi^2}{6}\ln 2 - 2\zeta(3)$$ [[29], (32)].

99. $$\sum_{k=1}^{\infty} \frac{1}{k}\left[\psi\left(k+\frac{3}{4}\right) - \psi\left(k+\frac{1}{4}\right)\right] = 8\mathbf{G} - 3\pi\ln 2.$$

100. $$\sum_{k=1}^{\infty} \frac{1}{k}\left[\psi\left(\frac{k}{2}+\frac{3}{4}\right) - \psi\left(\frac{k}{2}+\frac{1}{4}\right)\right] = 4\mathbf{G} - \pi\ln 2.$$

101. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{3k}{2} + \frac{3}{4}\right) - \psi\left(\frac{3k}{2} + \frac{1}{4}\right) \right] = 12\mathbf{G} - \pi \ln 2 - 2\pi \ln\left(2 + \sqrt{3}\right).$

102. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{k}{3} + \frac{3}{4}\right) - \psi\left(\frac{k}{3} + \frac{1}{4}\right) \right] = \frac{8}{3}\mathbf{G} - \pi \ln \frac{9}{8}.$

103. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{k+3}{4}\right) - \psi\left(\frac{k+1}{4}\right) \right] = 4\mathbf{G} - \frac{\pi}{2} \ln 2.$

104. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{k}{5} + \frac{3}{4}\right) - \psi\left(\frac{k}{5} + \frac{1}{4}\right) \right] = \frac{8}{5}\mathbf{G} - \pi \ln \frac{32}{5} + 2\pi \ln\left(1 + \sqrt{5}\right).$

105. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{k}{6} + \frac{3}{4}\right) - \psi\left(\frac{k}{6} + \frac{1}{4}\right) \right] = \frac{4}{3}\mathbf{G} + \pi \ln 2.$

106. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{k}{8} + \frac{3}{4}\right) - \psi\left(\frac{k}{8} + \frac{1}{4}\right) \right] = 2\mathbf{G} - \frac{\pi}{4} \ln 2 + \pi \ln\left(1 + \sqrt{2}\right).$

107. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{3k}{8} + \frac{3}{4}\right) - \psi\left(\frac{3k}{8} + \frac{1}{4}\right) \right] = \frac{8}{3}\mathbf{G} - \frac{\pi}{4} \ln 2 - \pi \ln\left(1 + \sqrt{2}\right)$
$+ \frac{2\pi}{3} \ln\left(2 + \sqrt{3}\right).$

108. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{3k}{10} + \frac{3}{4}\right) - \psi\left(\frac{3k}{10} + \frac{1}{4}\right) \right] = \frac{12}{5}\mathbf{G} + 3\pi \ln 2 + 2\pi \ln\left(2 + \sqrt{3}\right)$
$- 4\pi \ln\left(1 + \sqrt{5}\right).$

109. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{k}{12} + \frac{3}{4}\right) - \psi\left(\frac{k}{12} + \frac{1}{4}\right) \right] = \frac{4}{3}\mathbf{G} + \frac{\pi}{2} \ln 2 + \frac{2\pi}{3} \ln\left(2 + \sqrt{3}\right).$

110. $\sum_{k=1}^{\infty} \frac{1}{k} \left[\psi\left(\frac{3k}{16} + \frac{3}{4}\right) - \psi\left(\frac{3k}{16} + \frac{1}{4}\right) \right] = \frac{4}{3}\mathbf{G} - \frac{3\pi}{8} \ln 2 - \frac{2\pi}{3} \ln\left(2 + \sqrt{3}\right)$
$+ \pi \ln\left[2\sqrt{2} + \left(3 - \sqrt{2}\right)\sqrt{2 + \sqrt{2}}\right].$

111. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{mk}{4n+2} + \frac{3}{4}\right) - \psi\left(\frac{mk}{4n+2} + \frac{1}{4}\right) \right] = \frac{4m\mathbf{G}}{2n+1} - 2m\pi \ln 2$
$- 2\pi \sum_{k=0}^{2n} \sum_{p=0}^{m-1} (-1)^k \ln\left|\sin\left(\frac{2k+1}{8n+4}\pi + \frac{2p+1}{4m}\pi\right)\right|.$

112. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(k + \frac{3}{4}\right) - \psi\left(k + \frac{1}{4}\right) \right] = 8\mathbf{G} - \pi \ln\left(6 + 4\sqrt{2}\right).$

113. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{3k}{2}+\frac{3}{4}\right) - \psi\left(\frac{3k}{2}+\frac{1}{4}\right)\right] = 12\mathbf{G} - 2\pi \ln 6.$$

114. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{k+3}{4}\right) - \psi\left(\frac{k+1}{4}\right)\right] = -\frac{\pi}{2} \ln 2.$$

115. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{3k}{4}+\frac{3}{4}\right) - \psi\left(\frac{3k}{4}+\frac{1}{4}\right)\right] = \frac{20}{3}\mathbf{G} - \frac{\pi}{2} \ln 2 - \frac{4\pi}{3} \ln\left(2+\sqrt{3}\right).$$

116. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{k}{5}+\frac{3}{4}\right) - \psi\left(\frac{k}{5}+\frac{1}{4}\right)\right] = \frac{8}{5}\mathbf{G} + 3\pi \ln 2 + 2\pi \ln\left(1+\sqrt{2}\right) - 4\pi \ln\left(1+\sqrt{5}\right).$$

117. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{k}{6}+\frac{3}{4}\right) - \psi\left(\frac{k}{6}+\frac{1}{4}\right)\right] = \frac{4}{3}\mathbf{G} - 2\pi \ln \frac{3}{2}.$$

118. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{k}{8}+\frac{3}{4}\right) - \psi\left(\frac{k}{8}+\frac{1}{4}\right)\right] = 2\mathbf{G} - \frac{\pi}{4} \ln 2 - \pi \ln\left(1+\sqrt{2}\right).$$

119. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{3k}{8}+\frac{3}{4}\right) - \psi\left(\frac{3k}{8}+\frac{1}{4}\right)\right] = \frac{8}{3}\mathbf{G} - \frac{\pi}{4} \ln 2 + \pi \ln\left(1+\sqrt{2}\right) - \frac{4\pi}{3} \ln\left(2+\sqrt{3}\right).$$

120. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{k}{10}+\frac{3}{4}\right) - \psi\left(\frac{k}{10}+\frac{1}{4}\right)\right] = \frac{4}{5}\mathbf{G} + \pi \ln 5 - 2\pi \ln\left(1+\sqrt{5}\right).$$

121. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{3k}{10}+\frac{3}{4}\right) - \psi\left(\frac{3k}{10}+\frac{1}{4}\right)\right] = \frac{12}{5}\mathbf{G} - \pi \ln \frac{144}{5} + 2\pi \ln\left(1+\sqrt{5}\right).$$

122. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{k}{12}+\frac{3}{4}\right) - \psi\left(\frac{k}{12}+\frac{1}{4}\right)\right] = \frac{\pi}{2} \ln 2 - \frac{2\pi}{3} \ln\left(2+\sqrt{3}\right).$$

123. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{3k}{16}+\frac{3}{4}\right) - \psi\left(\frac{3k}{16}+\frac{1}{4}\right)\right] = \frac{4}{3}\mathbf{G} + \frac{\pi}{8} \ln 2 + \frac{4\pi}{3} \ln\left(2+\sqrt{3}\right) - \pi \ln\left(4+2\sqrt{2}+\sqrt{26+17\sqrt{2}}\right).$$

124. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{\sqrt{3}+2}{2} k + 1 \right) - \psi\left(\frac{\sqrt{3}+2}{2} k + \frac{1}{2} \right) \right] = \left(\sqrt{3} - 1\right) \frac{\pi^2}{6} - 2 \ln 2 \ln\left(\sqrt{3}+1\right).$$

125. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{k}{2\sqrt{3}+4} + 1 \right) - \psi\left(\frac{k}{2\sqrt{3}+4} + \frac{1}{2} \right) \right] = \left(1 - \sqrt{3}\right) \frac{\pi^2}{6} + 2 \ln 2 \ln\left(\sqrt{3}+1\right) - 2 \ln^2 2.$$

126. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{3-\sqrt{2}}{2} k + 1 \right) - \psi\left(\frac{3-\sqrt{2}}{2} k + \frac{1}{2} \right) \right] = \left(3 - 4\sqrt{2}\right) \frac{\pi^2}{12} + 3 \ln 2 \ln\left(\sqrt{2}+1\right) - \ln^2 2.$$

127. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{4+\sqrt{15}}{2} k + 1 \right) - \psi\left(\frac{4+\sqrt{15}}{2} k + \frac{1}{2} \right) \right] = \left(\sqrt{15} - 2\right) \frac{\pi^2}{6} - 2 \ln 2 \ln\left(\sqrt{3}+\sqrt{5}\right) - 2 \ln\left(\frac{1+\sqrt{5}}{2}\right) \ln\left(2+\sqrt{3}\right).$$

128. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{4-\sqrt{15}}{2} k + 1 \right) - \psi\left(\frac{4-\sqrt{15}}{2} k + \frac{1}{2} \right) \right] = \left(2 - \sqrt{15}\right) \frac{\pi^2}{6} + 2 \ln 2 \ln\left(\sqrt{3}+\sqrt{5}\right) + 2 \ln\left(\frac{1+\sqrt{5}}{2}\right) \ln\left(2+\sqrt{3}\right) - 2 \ln^2 2.$$

129. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{5+\sqrt{24}}{2} k + 1 \right) - \psi\left(\frac{5+\sqrt{24}}{2} k + \frac{1}{2} \right) \right] = \left(\sqrt{\frac{2}{3}} - \frac{5}{12}\right) \pi^2 - \frac{3}{2} \ln 2 \ln\left(5+2\sqrt{6}\right) - \ln\left(1+\sqrt{2}\right) \ln\left(2+\sqrt{3}\right) - \ln^2 2.$$

130. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{5-\sqrt{24}}{2} k + 1 \right) - \psi\left(\frac{5-\sqrt{24}}{2} k + \frac{1}{2} \right) \right] = \left(\sqrt{\frac{2}{3}} - \frac{5}{12}\right) \pi^2 + \frac{3}{2} \ln 2 \ln\left(5+2\sqrt{6}\right) + \ln\left(1+\sqrt{2}\right) \ln\left(2+\sqrt{3}\right) - \ln^2 2.$$

131. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{6+\sqrt{35}}{2} k + 1 \right) - \psi\left(\frac{6+\sqrt{35}}{2} k + \frac{1}{2} \right) \right] = \left(\sqrt{35} - 3\right) \frac{\pi^2}{6} - 2 \ln 2 \ln\left(\sqrt{5}+\sqrt{7}\right) - 2 \ln \frac{1+\sqrt{5}}{2} \ln\left(8+3\sqrt{7}\right).$$

132. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{6-\sqrt{35}}{2}k+1\right) - \psi\left(\frac{6-\sqrt{35}}{2}k+\frac{1}{2}\right)\right]$$
$$= \left(3-\sqrt{35}\right)\frac{\pi^2}{6} + 2\ln 2 \ln\left(\sqrt{5}+\sqrt{7}\right) + 2\ln\frac{1+\sqrt{5}}{2}\ln\left(8+3\sqrt{7}\right) - 2\ln^2 2.$$

133. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{8+\sqrt{63}}{2}k+1\right) - \psi\left(\frac{8+\sqrt{63}}{2}k+\frac{1}{2}\right)\right]$$
$$= \left(\sqrt{63}-4\right)\frac{\pi^2}{6} - 2\ln 2 \ln\left(3+\sqrt{7}\right) - 2\ln\frac{5+\sqrt{21}}{2}\ln\left(2+\sqrt{3}\right).$$

134. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{8-\sqrt{63}}{2}k+1\right) - \psi\left(\frac{8-\sqrt{63}}{2}k+\frac{1}{2}\right)\right]$$
$$= \left(4-\sqrt{63}\right)\frac{\pi^2}{6} + 2\ln 2 \ln\left(3+\sqrt{7}\right) + 2\ln\frac{5+\sqrt{21}}{2}\ln\left(2+\sqrt{3}\right) - 2\ln^2 2.$$

135. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{11+\sqrt{120}}{2}k+1\right) - \psi\left(\frac{11+\sqrt{120}}{2}k+\frac{1}{2}\right)\right]$$
$$= \left(\sqrt{480}-11\right)\frac{\pi^2}{12} - \ln\left(2+\sqrt{3}\right)\ln\left(3+\sqrt{10}\right) - \ln\left(1+\sqrt{2}\right)\ln\left(4+\sqrt{15}\right)$$
$$- \frac{3}{2}\ln 2\ln\left(11+\sqrt{120}\right) - \ln\frac{1+\sqrt{5}}{2}\ln\left(5+\sqrt{24}\right) - \ln^2 2.$$

136. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{11-\sqrt{120}}{2}k+1\right) - \psi\left(\frac{11-\sqrt{120}}{2}k+\frac{1}{2}\right)\right]$$
$$= \left(11-\sqrt{480}\right)\frac{\pi^2}{12} + \ln\left(2+\sqrt{3}\right)\ln\left(3+\sqrt{10}\right) + \ln\left(1+\sqrt{2}\right)\ln\left(4+\sqrt{15}\right)$$
$$+ \frac{3}{2}\ln 2\ln\left(11+\sqrt{120}\right) + \ln\frac{1+\sqrt{5}}{2}\ln\left(5+\sqrt{24}\right) - \ln^2 2.$$

137. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{12+\sqrt{143}}{2}k+1\right) - \psi\left(\frac{12+\sqrt{143}}{2}k+\frac{1}{2}\right)\right]$$
$$= \left(\sqrt{143}-6\right)\frac{\pi^2}{6} - 2\ln 2\ln\left(\sqrt{11}+\sqrt{13}\right) - 2\ln\frac{3+\sqrt{13}}{2}\ln\left(10+3\sqrt{11}\right).$$

138. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{12-\sqrt{143}}{2}k+1\right) - \psi\left(\frac{12-\sqrt{143}}{2}k+\frac{1}{2}\right)\right]$$
$$= \left(6-\sqrt{143}\right)\frac{\pi^2}{6} + 2\ln 2\ln\left(\sqrt{11}+\sqrt{13}\right) + 2\ln\frac{3+\sqrt{13}}{2}\ln\left(10+3\sqrt{11}\right)$$
$$- 2\ln^2 2.$$

139. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{13+\sqrt{168}}{2} k + 1 \right) - \psi\left(\frac{13+\sqrt{168}}{2} k + \frac{1}{2} \right) \right]$$
$$= \left(\sqrt{672} - 13\right) \frac{\pi^2}{12} - \frac{1}{2} \ln\left(2+\sqrt{3}\right) \ln\left(15+\sqrt{224}\right) - \frac{1}{2} \ln\left(5+\sqrt{24}\right) \ln\left(8+\sqrt{63}\right) - \frac{3}{2} \ln 2 \ln\left(13+\sqrt{168}\right) - \ln \frac{5+\sqrt{21}}{2} \ln\left(1+\sqrt{2}\right) - \ln^2 2.$$

140. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{13-\sqrt{168}}{2} k + 1 \right) - \psi\left(\frac{13-\sqrt{168}}{2} k + \frac{1}{2} \right) \right]$$
$$= \left(13 - \sqrt{672}\right) \frac{\pi^2}{12} + \frac{1}{2} \ln\left(2+\sqrt{3}\right) \ln\left(15+\sqrt{224}\right) + \frac{1}{2} \ln\left(5+\sqrt{24}\right) \ln\left(8+\sqrt{63}\right) + \frac{3}{2} \ln 2 \ln\left(13+\sqrt{168}\right) + \ln \frac{5+\sqrt{21}}{2} \ln\left(1+\sqrt{2}\right) - \ln^2 2.$$

141. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{14+\sqrt{195}}{2} k + 1 \right) - \psi\left(\frac{14+\sqrt{195}}{2} k + \frac{1}{2} \right) \right]$$
$$= \left(\sqrt{195} - 7\right) \frac{\pi^2}{6} - 2 \ln 2 \ln\left(\sqrt{13}+\sqrt{15}\right) - 2 \ln \frac{3+\sqrt{13}}{2} \ln\left(4+\sqrt{15}\right) - 2 \ln \frac{1+\sqrt{5}}{2} \ln\left(25+\sqrt{39}\right).$$

142. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\psi\left(\frac{14-\sqrt{195}}{2} k + 1 \right) - \psi\left(\frac{14-\sqrt{195}}{2} k + \frac{1}{2} \right) \right]$$
$$= \left(7 - \sqrt{195}\right) \frac{\pi^2}{6} + 2 \ln 2 \ln\left(\sqrt{13}+\sqrt{15}\right) + 2 \ln \frac{3+\sqrt{13}}{2} \ln\left(4+\sqrt{15}\right) + {} + 2 \ln \frac{1+\sqrt{5}}{2} \ln\left(25+\sqrt{39}\right) - 2\ln^2 2.$$

143. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left[\psi\left(\frac{3+\sqrt{8}}{4}\,(2k+1) + \frac{3}{4} \right) - \psi\left(\frac{3+\sqrt{8}}{4}\,(2k+1) + \frac{1}{4} \right) \right] = \frac{1}{4} \ln 2 \ln\left(3+\sqrt{8}\right).$$

144. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left[\psi\left(\frac{5+\sqrt{24}}{4} (2k+1) + \frac{3}{4} \right) - \psi\left(\frac{5+\sqrt{24}}{4} (2k+1) + \frac{1}{4} \right) \right] = \frac{1}{2} \ln\left(1+\sqrt{2}\right) \ln\left(2+\sqrt{3}\right) - \frac{1}{4} \ln 2 \ln\left(5+\sqrt{24}\right).$$

145. $$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left[\psi\left(\frac{11+\sqrt{120}}{4}(2k+1)+\frac{3}{4}\right)\right.$$
$$\left.-\psi\left(\frac{11+\sqrt{120}}{4}(2k+1)+\frac{1}{4}\right)\right]=\frac{1}{4}\ln 2\ln\left(11+\sqrt{120}\right)$$
$$-\frac{1}{2}\ln\left(1+\sqrt{2}\right)\ln\left(4+\sqrt{15}\right)-\frac{1}{2}\ln\left(2+\sqrt{3}\right)\ln\left(3+\sqrt{10}\right)$$
$$+\frac{3}{2}\ln\frac{1+\sqrt{5}}{2}\ln\left(5+\sqrt{24}\right).$$

146. $$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left[\psi\left(\frac{13+\sqrt{168}}{4}(2k+1)+\frac{3}{4}\right)\right.$$
$$\left.-\psi\left(\frac{13+\sqrt{168}}{4}(2k+1)+\frac{1}{4}\right)\right]$$
$$=\frac{1}{4}\ln\left(2+\sqrt{3}\right)\ln\left(15+\sqrt{224}\right)+\frac{1}{4}\ln\left(5+\sqrt{24}\right)\ln\left(8+\sqrt{63}\right)$$
$$-\frac{3}{2}\ln\frac{5+\sqrt{21}}{2}\ln\left(1+\sqrt{2}\right)-\frac{1}{4}\ln 2\ln\left(13+\sqrt{168}\right).$$

6.2.2. Series containing $\psi(ka+b)$ and trigonometric functions

1. $$\sum_{k=0}^{\infty}\frac{t^k}{k!}\begin{Bmatrix}\sin(kz)\\ \cos(kz)\end{Bmatrix}\psi\left(k+\frac{1}{2}\right)=-\frac{i^{(1\pm1)/2}}{2}(\mathbf{C}+2\ln 2)\left(e^{u_-}\mp e^{u_+}\right)$$
$$+i^{(1\pm1)/2}\left[u_-e^{u_-}{}_2F_2\left(\begin{matrix}1,\,1;\;-u_-\\ \frac{3}{2},\,2\end{matrix}\right)\mp u_+e^{u_+}{}_2F_2\left(\begin{matrix}1,\,1;\;-u_+\\ \frac{3}{2},\,2\end{matrix}\right)\right]\qquad\left[u_\pm=te^{\pm iz}\right].$$

2. $$\sum_{k=0}^{\infty}\frac{t^k}{(k!)^2}\begin{Bmatrix}\sin(kz)\\ \cos(kz)\end{Bmatrix}\psi(k+1)$$
$$=\frac{i^{(1\pm1)/2}}{2}\left[K_0(2u_-)\mp K_0(2u_+)+\ln u_-I_0(2u_-)\mp\ln u_+I_0(2u_+)\right]$$
$$\left[u_\pm=\sqrt{t}\,e^{\pm iz/2}\right].$$

6.2.3. Series containing products of $\psi(ka+b)$

1. $$\sum_{k=1}^{\infty}t^k\psi^2(k)=\frac{t}{1-t}\left[\mathbf{C}^2+2\mathbf{C}\ln(1-t)+\ln^2(1-t)+\mathrm{Li}_2(t)\right]\qquad[|t|<1].$$

2. $$\sum_{k=1}^{\infty}\frac{t^k}{k}\psi^2(k)=\ln(1-t)\left[\mathrm{Li}_2(t)-\frac{\pi^2}{3}-\mathbf{C}^2\right]+\ln^2(1-t)(\ln t-\mathbf{C})$$
$$-\frac{1}{3}\ln^3(1-t)+2\,\mathrm{Li}_3(1-t)-2\zeta(3)\qquad[|t|<1].$$

3. $$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\psi^2(k)=\left(\frac{\pi^2}{12}-\mathbf{C}^2\right)\ln 2-\mathbf{C}\ln^2 2-\frac{1}{3}\ln^3 2-\frac{1}{4}\zeta(3).$$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \psi^2(k) = \frac{11\pi^4}{360} + \frac{\pi^2 \mathbf{C}^2}{6} - 2\mathbf{C}\zeta(3)$ [[29], (9)].

5. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \psi^2(k) = \frac{\pi^2}{480} \left(11\pi^2 - 40\mathbf{C}^2 + 40 \ln^2 2\right) - \frac{1}{12} \left[\ln^4 2 + 24 \operatorname{Li}_4(\tfrac{1}{2})\right]$
$$- \frac{1}{4} (\mathbf{C} + 7 \ln 2)\, \zeta(3).$$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \psi^2(k) = \mathbf{C}^2 - 2\mathbf{C} + \frac{\pi^2}{6} + 1$ [[63], (4.16)].

7. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3} \psi^2(k) = -\frac{\pi^4 \mathbf{C}}{180} + \left(\frac{\pi^2}{6} + \mathbf{C}^2\right) \zeta(3) - \frac{3}{2}\zeta(5).$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4} \psi^2(k) = \frac{37\pi^6}{22680} + \frac{\pi^4 \mathbf{C}^2}{90} + \frac{\pi^2 \mathbf{C}}{3}\zeta(3) - \zeta^2(3) - 4\mathbf{C}\zeta(5).$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5} \psi^2(k) = -\frac{\pi^4}{180}\zeta(3) + \mathbf{C}\left[\zeta^2(3) - \frac{\pi^6}{630}\right] + \left(\frac{\pi^2}{6} + \mathbf{C}^2\right) \zeta(5) - \zeta(7).$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \psi^2\left(k + \frac{1}{2}\right) = \frac{1}{4}(\mathbf{C} + 2\ln 2)\left[\pi^2 - 4(\mathbf{C} + 2\ln 2)\ln 2\right] - \frac{7}{4}\zeta(3)$
[[29], (19)].

11. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \psi^2\left(k + \frac{1}{2}\right) = \frac{\pi^4}{8} + \frac{\pi^2}{6} (\mathbf{C} + 2\ln 2)^2 - 7(\mathbf{C} + 2\ln 2)\, \zeta(3)$
[[29], (22)].

12. $\displaystyle\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!\,(2k+1)} \psi^2\left(k + \frac{1}{2}\right) = \frac{\pi}{4}\left(\pi^2 + 2\mathbf{C}^2\right).$

13. $\displaystyle\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!\,(2k+1)(2k+3)} \psi^2\left(k + \frac{1}{2}\right) = \frac{\pi}{16}\left(\pi^2 + 4\mathbf{C} + 2\mathbf{C}^2 - 6\right).$

14. $\displaystyle\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{k!\,(k+1)!} \psi^2\left(k + \frac{1}{2}\right)$
$$= \frac{2}{3\pi}\left[\pi^2 + 24(2\ln 2 - 1)\mathbf{C} + 6\mathbf{C}^2 + 24(2\ln 2 - 1)^2\right].$$

15. $\displaystyle\sum_{k=0}^{\infty} \frac{(a)_k}{k!} \psi^2(k + a) = -\frac{\pi}{a}\csc(a\pi)$ [$\operatorname{Re} a < 0$].

16. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} \psi(k+a)\,\psi(k+b) = \mathrm{B}\,(-a,\, b)\,[\psi(-a) - \psi(a) - \psi(b-a)]$$
$$[\operatorname{Re} a < 0;\ a,\, b \neq 0,\, -1,\, -2,\, \ldots].$$

17. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+1)!} \psi\left(k+\frac{1}{2}\right) \psi(k+1) = \pi^2 - 4\mathrm{C} + 2\mathrm{C}^2 + 8\ln 2(1-\ln 2).$$

18. $$\sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k}{(k+1)!} \psi\left(k-\frac{1}{2}\right) \psi(k+1)$$
$$= \frac{1}{9}\,[3\pi^2 - 4\mathrm{C} + 6\mathrm{C}^2 + 8\ln 2(7-3\ln 2) - 56].$$

19. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+2)!} \psi\left(k+\frac{1}{2}\right) \psi(k+2)$$
$$= \frac{2}{9}\,\left[3\mathrm{C}^2 + 2\mathrm{C}(9\ln 2 - 7) - 3\pi^2 + 56(1-\ln 2) + 24\ln^2 2\right].$$

20. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+1)!\,(k+1)} \psi\left(k+\frac{1}{2}\right) \psi(k+2) = 4\left(\frac{\pi^2}{3} - \mathrm{C}^2 - 4\right)(\ln 2 - 1)$$
$$+ \mathrm{C}\left[\frac{5\pi^2}{3} - 8\left(3 - 2\ln 2 + \ln^2 2\right)\right] + 4[8 - 7\zeta(3)].$$

21. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{k!\,(k+1)!} \psi^2\left(k+\frac{1}{2}\right)$$
$$= \frac{2}{3\pi}\,[6\mathrm{C}^2 + 24\mathrm{C}(2\ln 2 - 1) + \pi^2 + 24(2\ln 2 - 1)^2].$$

22. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_k}{(k+1)!\,(k+2)!} \psi\left(k+\frac{1}{2}\right) \psi\left(k+\frac{3}{2}\right)$$
$$= \frac{4}{9\pi}\,\{3(3\pi - 8)\mathrm{C}^2 + 2\,[(56 - 96\ln 2) + 9\pi\,(2\ln 2 - 1)]\,\mathrm{C}$$
$$- 64(6\ln^2 2 - 7\ln 2 + 1) + 36\pi\ln 2\,(\ln 2 - 1) - 4\pi^2\big\}.$$

6.2.4. Series containing $\psi'(ka+b)$

1. $$\sum_{k=1}^{\infty} t^k \psi'(k) = -\frac{\pi^2}{6} + \mathrm{Li}_2(t) + \frac{1}{1-t}\,[\ln t \ln(1-t) + \mathrm{Li}_2\,(1-t)] \quad [|t| < 1].$$

2. $$\sum_{k=1}^{\infty} (-1)^k \psi'(k) = -\frac{\pi^2}{8}.$$

3. $$\sum_{k=1}^{\infty} \frac{t^k}{k}\,\psi'(k) = 2\zeta(3) - \frac{\pi^2}{6}\ln t + \ln^2 t \ln(1-t) + \ln t \operatorname{Li}_2(t) + \ln(t-t^2)\operatorname{Li}_2(1-t) - 2\operatorname{Li}_3(1-t) \quad [|t|<1].$$

4. $$\sum_{k=1}^{\infty} \frac{1}{k}\,\psi'(k) = 2\zeta(3) \qquad [[63], (4.12)].$$

5. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\,\psi'(k) = \frac{1}{4}\zeta(3) - \frac{\pi^2}{4}\ln 2.$$

6. $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}\,\psi'(k) = 1 \qquad [[63], (4.14)].$$

7. $$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k+2)!}\,t^k\psi'(k) = -\frac{7}{2} - \frac{\pi^2}{12} + 3\frac{\sqrt{4-t}}{\sqrt{t}}\arcsin\frac{\sqrt{t}}{2} + \frac{3t+\pi^2+6}{3t}\arcsin^2\frac{\sqrt{t}}{2} - \frac{2}{3t}\arcsin^4\frac{\sqrt{t}}{2} \quad [|t|<4].$$

8. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\,\psi'\left(k+\frac{1}{2}\right) = \frac{5\pi^4}{96}.$$

9. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}\,t^k\psi'\left(k+\frac{1}{2}\right) = \frac{\pi^2}{2\sqrt{1-t}} - \frac{2}{\sqrt{1-t}}\arcsin^2\sqrt{t} \quad [|t|<1].$$

10. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!\,(2k+1)}\,t^k\psi'\left(k+\frac{1}{2}\right) = \frac{\pi^2}{2\sqrt{t}}\arcsin\sqrt{t} - \frac{2}{3\sqrt{t}}\arcsin^3\sqrt{t} \quad [|t|<1].$$

11. $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(k+1)!\,(2k+1)}\,t^k\psi'\left(k+\frac{1}{2}\right) = \frac{8+\pi^2}{t}\left(\sqrt{1-t}-1\right) + \frac{8+\pi^2}{\sqrt{t}}\arcsin\sqrt{t} - \frac{4}{t}\sqrt{1-t}\arcsin^2\sqrt{t} - \frac{4}{3\sqrt{t}}\arcsin^3\sqrt{t} \quad [|t|<1].$$

12. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!}\,\psi'(k+a) = \frac{\pi}{a}\csc(a\pi) \qquad [\operatorname{Re} a<1;\ a\neq 0,-1,-2,\ldots].$$

13. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!}\,\psi(k+a)\,\psi'(k+a) = -\frac{\pi}{a}\csc(a\pi)\,[2\mathbf{C}+\psi(-a)] \qquad [\operatorname{Re} a<1;\ a\neq 0,-1,-2,\ldots].$$

6.3. The Hurwitz Zeta Function $\zeta(s, z)$

6.3.1. Series containing $\zeta(k, z)$

1. $$\sum_{k=2}^{\infty} \frac{t^k}{k+n} \zeta(k, z) = \sum_{k=0}^{n} \binom{n}{k} t^{-k} \left. \frac{\partial \zeta(s, z-t)}{\partial s} \right|_{s=-k} - t^{-n} \left. \frac{\partial \zeta(s, z)}{\partial s} \right|_{s=-n}$$
$$- \sum_{k=0}^{n-1} \frac{t^{-k}}{n-k} \zeta(-k, z) - \frac{t}{n+1} [\psi(n+1) - \psi(z) + \mathbf{C}] \quad [|t| < |z|;\ [50]].$$

2. $$\sum_{k=2}^{\infty} \frac{(k-1)!}{(k+n)!} t^k \zeta(k, z) = \frac{(-t)^{-n}}{n!} \left. \frac{\partial \zeta(s, z-t)}{\partial s} \right|_{s=-n}$$
$$- \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-t)^{-k} \left. \frac{\partial \zeta(s, z)}{\partial s} \right|_{s=-k}$$
$$- \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} \frac{(-t)^{-k}}{k+1} B_{k+1}(z) [\psi(n+1) - \psi(k+1)]$$
$$+ \frac{t}{(n+1)!} [\psi(n+1) - \psi(z) + \mathbf{C}] \quad [|t| < |z|;\ [50]].$$

6.4. The Sine Si (z) and Cosine ci (z) Integrals

6.4.1. Series containing Si $(\varphi(k)x)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \operatorname{Si}((2k+1)x) = \operatorname{Si}(x) - 2\sin x \qquad [-\pi/2 < x < \pi/2].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2a^2+b^2)} \operatorname{Si}(kx) = -\frac{x}{2b^2} + \frac{\pi}{2b^2} \operatorname{csch} \frac{b\pi}{a} \operatorname{shi}\left(\frac{bx}{a}\right) \qquad [-\pi \le x \le \pi].$$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2a^2-b^2)} \operatorname{Si}(kx) = \frac{x}{2b^2} - \frac{\pi}{2b^2} \csc \frac{b\pi}{a} \operatorname{Si}\left(\frac{bx}{a}\right) \qquad [-\pi \le x \le \pi].$$

4. $$\sum_{k=2}^{\infty} \frac{(-1)^k}{k\left(k^2-1\right)} \operatorname{Si}(kx) = \frac{x}{2} + \frac{1}{2}\sin x - \frac{3}{4}\operatorname{Si}(x) \qquad [-\pi \le x \le \pi].$$

5. $$\sum_{k=0}^{\infty} \frac{(2k+1)^{-1}}{((2k+1)^2a^2+b^2)} \operatorname{Si}((2k+1)x)$$
$$= \frac{\pi}{4b^2}\left[\tanh\frac{b\pi}{2a} \operatorname{shi}\left(\frac{bx}{a}\right) - \operatorname{chi}\left(\frac{bx}{a}\right) + \ln\frac{bx}{a} + \mathbf{C}\right] \quad [0 \le x \le \pi].$$

6. $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)} \operatorname{Si}\left((2k+1)x\right) = 3\operatorname{Si}(x) + \pi \operatorname{ci}(x) - 2\sin x - \pi \ln x - \pi \mathbf{C} \qquad [0 < x < \pi/2].$$

7. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k^2+a^2}} \operatorname{Si}\left(\sqrt{k^2+a^2}\,x\right) = -\frac{1}{2a} \operatorname{Si}(ax) \qquad [-\pi < x < \pi].$$

8. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\sqrt{k^2+a^2}} \operatorname{Si}\left(\sqrt{k^2+a^2}\,x\right) = \frac{1}{12a^3}\left[(3-\pi^2 a^2)\operatorname{Si}(ax) - 3\sin(ax)\right] \qquad [-\pi < x < \pi].$$

9. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\sqrt{(2k+1)^2+a^2}} \operatorname{Si}\left(\sqrt{(2k+1)^2+a^2}\,x\right) = \frac{\pi}{4a} \operatorname{Si}(ax) \qquad [-\pi/2 < x < \pi/2].$$

6.4.2. Series containing $\operatorname{ci}(\varphi(k)x)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+a^2} \left[\operatorname{ci}\left(\sqrt{k^2+a^2}\,x\right) - \frac{1}{2}\ln(k^2+a^2)\right] = \frac{1}{2a^2}\operatorname{csch}(a\pi)\left[\sinh(a\pi)\ln a + a\pi(\mathbf{C}+\ln x) - \sinh(a\pi)\operatorname{ci}(ax)\right] \qquad [0 < a < \pi].$$

2. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}\left[\operatorname{ci}\left((2k+1)x\right) - \ln(2k+1)\right] = \frac{\pi^3}{32}(\mathbf{C}+\ln x) - \frac{\pi x^2}{16} \qquad [0 < x < \pi/2].$$

3. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k^2-1)(2k+3)}\left[\operatorname{ci}\left((2k+1)x\right) - \ln(2k+1)\right] = -\frac{\pi}{16}\left[\mathbf{C} + \ln\frac{x}{2} + \operatorname{ci}(2x)\right].$$

4. $$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^{-1}}{(2k+1)^2+a^2}\left[\operatorname{ci}\left(\sqrt{(2k+1)^2+a^2}\,x\right) - \frac{1}{2}\ln\left((2k+1)^2+a^2\right)\right] = -\frac{\pi}{4a^2}\left[\mathbf{C} + \ln(ax) - \operatorname{ci}(ax)\right] + \frac{\pi(e^{a\pi/2}-1)^2}{4a^2(e^{a\pi}+1)}(\mathbf{C}+\ln x) \qquad [0 < a < \pi].$$

5. $$\sum_{k=0}^{\infty} \frac{(2k+1)^{-2}}{(2k+1)^2 a^2 + b^2}\left[\operatorname{ci}\left((2k+1)x\right) - \ln(2k+1)\right] = \frac{\pi}{8b^3}\left\{b(\pi\mathbf{C} - 2x + \pi\ln x) + 2a\operatorname{shi}\left(\frac{bx}{a}\right) + 2a\tanh\frac{b\pi}{2a}\left[\ln\frac{b}{a} - \operatorname{chi}\left(\frac{bx}{a}\right)\right]\right\} \qquad [0 < x < \pi].$$

6. $\sum_{k=0}^{\infty} \frac{(2k+1)^{-2}}{(2k+1)^2 a^2 - b^2} [\operatorname{ci}((2k+1)x) - \ln(2k+1)]$

$$= -\frac{\pi}{8b^3} \left\{ 2a \operatorname{Si}\left(\frac{bx}{a}\right) + b(\pi \mathbf{C} - 2x + \pi \ln x) + 2a \tan \frac{b\pi}{2a} \left[\ln \frac{b}{a} - \operatorname{ci}\left(\frac{bx}{a}\right) \right] \right\}$$

$[0 < x < \pi]$.

6.4.3. Series containing $\operatorname{Si}(kx)$ and trigonometric functions

1. $\sum_{k=1}^{\infty} \frac{\cos(ky)}{k^{2m-1}} \operatorname{Si}(kx) = (-1)^m \frac{x^{2m-1}}{2(2m-1)!\,(2m-1)} + \sum_{k=0}^{m-2} \frac{x^{2k+1}}{(2k+1)!\,(2k+1)}$

$$\times \left[(-1)^{m-1} \frac{\pi y^{2m-2k-3}}{2(2m-2k-3)!} + (-1)^k \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j}}{(2j)!} \zeta(2m-2j-2k-2) \right]$$

$[m \geq 1;\ 0 < x < \pi;\ x < y < 2\pi - x]$.

2. $\sum_{k=1}^{\infty} (-1)^k \frac{\cos(ky)}{k^{2m-1}} \operatorname{Si}(kx) = (-1)^m \frac{x^{2m-1}}{2(2m-1)!\,(2m-1)}$

$$- \sum_{k=0}^{m-2} \frac{(-1)^k x^{2k+1}}{(2k+1)!\,(2k+1)} \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j}}{(2j)!} \left(1 - 2^{2j+2k-2m+3}\right)$$

$$\times \zeta(2m-2j-2k-2) \qquad [m \geq 1;\ -\pi < x < \pi;\ |x| - \pi < y < \pi - |x|].$$

6.4.4. Series containing products of $\operatorname{Si}(kx)$

1. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m}} \operatorname{Si}(kx) \operatorname{Si}(ky) = (-1)^m \frac{x^{2m-1} y}{2(2m-1)!\,(2m-1)}$

$$- \sum_{k=0}^{m-2} \frac{(-1)^k x^{2k+1}}{(2k+1)!\,(2k+1)} \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j+1}}{(2j+1)!\,(2j+1)}$$

$$\times \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$

$[m \geq 1;\ -\pi < x < \pi;\ |x| - \pi < y < \pi - |x|]$.

2. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^n} \prod_{i=1}^{n} \operatorname{Si}(kx_i) = -\frac{1}{2} \prod_{i=1}^{n} x_i \qquad \left[x_i > 0;\ \sum_{i=1}^{n} x_i < \pi \right]$.

6.5. The Fresnel Integrals $S(x)$ and $C(x)$

6.5.1. Series containing $S(\varphi(k)x)$, $C(\varphi(k)x)$ and algebraic functions

1. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^{3/2}(k^2-1)} S(kx) = \frac{1}{6} \sqrt{\frac{x}{2\pi}} (2x + 3 \sin x) - S(x) \qquad [-\pi \leq x \leq \pi]$.

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}\left(k^2 a^2 + b^2\right)}\, S(kx) = -\frac{2}{3b^2\sqrt{\pi}}\left(\frac{x}{2}\right)^{3/2} + \frac{a^{1/2}\pi}{(2b)^{5/2}} \operatorname{csch}\frac{b\pi}{a}\left[\operatorname{erfi}\left(\sqrt{\frac{bx}{a}}\right) - \operatorname{erf}\left(\sqrt{\frac{bx}{a}}\right)\right] \quad [-\pi < x < \pi].$$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}\left(k^2 a^2 - b^2\right)}\, S(kx) = \frac{2}{3b^2\sqrt{\pi}}\left(\frac{x}{2}\right)^{3/2} - \frac{a^{1/2}\pi}{2b^{5/2}} \csc\frac{b\pi}{a}\, S\left(\frac{bx}{a}\right) \quad [-\pi < x < \pi].$$

4. $$\sum_{k=1}^{\infty} \frac{(2k+1)^{-3/2}}{k(k+1)}\, S((2k+1)x) = 4S(x) + \pi C(x) - \sqrt{\frac{2x}{\pi}}\,(\pi + \sin x) \quad [0 \le x \le \pi].$$

5. $$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^{-1/2}}{(2k-1)(2k+3)}\, S((2k+1)x) = -\frac{\pi}{2^{7/2}}\, S(2x) \quad [-\pi/2 < x < \pi/2].$$

6. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{5/2}}\, S((2k+1)x) = \frac{1}{6}\sqrt{\frac{\pi x^3}{2}} \quad [-\pi/2 < x < \pi/2].$$

7. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)\,(2k+1)^{1/2}}\, S((2k+1)x) = 2S(x) - \sqrt{\frac{2x}{\pi}}\, \sin x \quad [-\pi/2 < x < \pi/2].$$

8. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{(k^2 + a^2)^{3/4}}\, S\left(\sqrt{k^2 + a^2}\, x\right) = \frac{1}{2a^{3/2}}\, S(ax) \quad [-\pi/2 < x < \pi/2].$$

9. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k^2 + a^2)^{3/4}}\, S\left(\sqrt{k^2 + a^2}\, x\right) = \frac{1}{24a^{7/2}}\left[(9 - 2\pi^2 a^2) S(ax) - 3\sqrt{\frac{2ax}{\pi}}\, \sin(ax)\right] \quad [-\pi < x < \pi].$$

10. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)((2k+1)^2 + a^2)^{3/4}}\, S\left(\sqrt{(2k+1)^2 + a^2}\, x\right) = \frac{\pi}{4a^{3/2}}\, S(ax) \quad [-\pi/2 < x < \pi/2].$$

11. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}\left(k^2 a^2 + b^2\right)}\, C(kx) = -\frac{1}{b^2}\sqrt{\frac{x}{2\pi}} + \frac{\pi}{2^{5/2} a^{1/2} b^{3/2}} \operatorname{csch}\frac{b\pi}{a}\left[\operatorname{erf}\left(\sqrt{\frac{bx}{a}}\right) + \operatorname{erfi}\left(\sqrt{\frac{bx}{a}}\right)\right] \quad [-\pi < x < \pi].$$

12. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}\left(k^2a^2-b^2\right)} C(kx) = \frac{1}{b^2}\sqrt{\frac{x}{2\pi}} - \frac{\pi}{2a^{1/2}b^{3/2}} \csc\frac{b\pi}{a} C\left(\frac{bx}{a}\right) \qquad [-\pi < x < \pi].$$

13. $$\sum_{k=2}^{\infty} \frac{(-1)^k}{k^{1/2}\left(k^2-1\right)} C(kx) = \sqrt{\frac{x}{8\pi}}(\cos x + 2) - \frac{1}{2}C(x) \qquad [-\pi < x < \pi].$$

14. $$\sum_{k=1}^{\infty} \frac{(2k+1)^{-1/2}}{k(k+1)} C((2k+1)x) = 2C(x) - \pi S(x) - \sqrt{\frac{2x}{\pi}} \cos x \qquad [0 \le x \le \pi].$$

15. $$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^{1/2}}{(2k-1)(2k+3)} C((2k+1)x) = -\frac{\pi}{2^{7/2}} C(2x) \qquad [0 \le x \le \pi].$$

16. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{3/2}} C((2k+1)x) = \frac{1}{2}\sqrt{\frac{\pi x}{2}} \qquad [-\pi/2 < x < \pi/2].$$

17. $$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^{1/2}}{k(k+1)} C((2k+1)x) = -\sqrt{\frac{2x}{\pi}} \cos x \qquad [-\pi/2 < x < \pi/2].$$

18. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{(k^2+a^2)^{1/4}} C\left(\sqrt{k^2+a^2}\,x\right) = -\frac{1}{2a^{1/2}} C(ax) \qquad [-\pi/2 < x < \pi/2].$$

19. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k^2+a^2)^{1/4}} C\left(\sqrt{k^2+a^2}\,x\right) = \frac{1}{24a^{5/2}}\left[(2\pi^2a^2-3)C(ax) - 3\sqrt{\frac{2ax}{\pi}} \cos(ax)\right] \qquad [-\pi < x < \pi].$$

20. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)((2k+1)^2+a^2)^{1/4}} C\left(\sqrt{(2k+1)^2+a^2}\,x\right) = \frac{\pi}{4a^{1/2}} C(ax) \qquad [-\pi/2 < x < \pi/2].$$

6.5.2. Series containing $S(\varphi(k)x)$, $C(\varphi(k)x)$ and trigonometric functions

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{5/2}} \sin(kx)\, S(ky) = -\frac{x}{3}\sqrt{\frac{y^3}{2\pi}} \qquad [x,\, y > 0;\; x+y < \pi].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}} \cos(kx)\, S(ky) = -\frac{1}{3}\sqrt{\frac{y^3}{2\pi}} \qquad [x,\, y > 0;\; x+y < \pi].$$

3. $$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2m-1/2}} S(ky) = (-1)^m \frac{y^{2m-1/2}}{(2m-1)!\,(4m-1)\sqrt{2\pi}} + \sqrt{\frac{2y^3}{\pi}} \sum_{k=0}^{m-2} \frac{y^{2k}}{(2k+1)!\,(4k+3)}$$
$$\times \left[(-1)^{m-1} \frac{\pi x^{2m-2k-3}}{2(2m-2k-3)!} + (-1)^k \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j}}{(2j)!} \zeta(2m-2j-2k-2)\right]$$
$$[m \geq 1;\ 0 < y < \pi;\ y < x < 2\pi - y].$$

4. $$\sum_{k=1}^{\infty} (-1)^k \frac{\cos(kx)}{k^{2m-1/2}} S(ky) = (-1)^m \frac{y^{2m-1/2}}{(2m-1)!\,(4m-1)\sqrt{2\pi}}$$
$$- \sqrt{\frac{2y^3}{\pi}} \sum_{k=0}^{m-2} \frac{(-1)^k y^{2k}}{(2k+1)!\,(4k+3)} \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j}}{(2j)!} \left(1 - 2^{2j+2k-2m+3}\right)$$
$$\times \zeta(2m-2j-2k-2) \qquad [m \geq 1;\ -\pi < y < \pi;\ |y| - \pi < x < \pi - |y|].$$

5. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}} \sin(kx)\, C(ky) = -x\sqrt{\frac{y}{2\pi}} \qquad [x,\, y > 0;\ x + y < \pi].$$

6. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}} \cos(kx)\, C(ky) = -\sqrt{\frac{y}{2\pi}} \qquad [x,\, y > 0;\ x + y < \pi].$$

7. $$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2m-3/2}} C(ky) = (-1)^m \frac{y^{2m-3/2}}{(2m-2)!\,(4m-3)\sqrt{2\pi}} + \sqrt{\frac{2y}{\pi}} \sum_{k=0}^{m-2} \frac{y^{2k}}{(2k)!\,(4k+1)}$$
$$\times \left[(-1)^m \frac{\pi x^{2m-2k-3}}{2(2m-2k-3)!} + (-1)^k \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j}}{(2j)!} \zeta(2m-2j-2k-2)\right]$$
$$[m \geq 2;\ 0 < y < \pi;\ y < x < 2\pi - y].$$

8. $$\sum_{k=1}^{\infty} (-1)^k \frac{\cos(kx)}{k^{2m-3/2}} C(ky) = (-1)^m \frac{y^{2m-3/2}}{(2m-2)!\,(4m-3)\sqrt{2\pi}}$$
$$- \sqrt{\frac{2y}{\pi}} \sum_{k=0}^{m-2} \frac{(-1)^k y^{2k}}{(2k)!\,(4k+1)} \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j}}{(2j)!} \left(1 - 2^{2j+2k-2m+3}\right)$$
$$\times \zeta(2m-2j-2k-2) \qquad [m \geq 2;\ -\pi < y < \pi;\ |y| - \pi < x < \pi - |y|].$$

9. $$\sum_{k=2}^{\infty} \frac{(-1)^k}{k^{1/2}(k^2-1)} \left[\sin(kx)S(kx) + \cos(kx)C(kx)\right]$$
$$= \frac{1}{2}\,(x\cos x - \sin x)\,S(x) - \frac{1}{2}\,(\cos x + x\sin x)\,C(x) + 3\sqrt{\frac{x}{8\pi}}$$
$$[-\pi/2 < x < \pi/2].$$

10. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}(k^2+a^2)} \left[\sin(kx)S(kx) + \cos(kx)C(kx)\right] = -\frac{1}{b^2}\sqrt{\frac{x}{2\pi}}$$
$$+ \frac{\pi}{2^{5/2}a^{1/2}b^{3/2}} \operatorname{csch}\frac{b\pi}{a}\left[e^{bx/a}\operatorname{erf}\left(\sqrt{\frac{bx}{a}}\right) + e^{-bx/a}\operatorname{erfi}\left(\sqrt{\frac{bx}{a}}\right)\right]$$
$$[-\pi < x < \pi].$$

11. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}(k^2+a^2)} \left[\cos(kx)S(kx) - \sin(kx)C(kx)\right] = \frac{4}{3\sqrt{\pi}\,b^2}\left(\frac{x}{2}\right)^{3/2}$$
$$- \frac{a^{1/2}\pi}{(2b)^{5/2}} \operatorname{csch}\frac{b\pi}{a}\left[e^{bz/a}\operatorname{erf}\left(\sqrt{\frac{bx}{a}}\right) - e^{-bz/a}\operatorname{erfi}\left(\sqrt{\frac{bx}{a}}\right)\right]$$
$$[-\pi < x < \pi].$$

12. $$\sum_{k=2}^{\infty} \frac{(-1)^k}{k^{1/2}(k^2-1)} \left[\sin(kx)S(kx) + \cos(kx)C(kx)\right]$$
$$= \frac{1}{2}\,(x\cos x - \sin x)\,S(x) - \frac{1}{2}(\cos x + x\sin x)\,C(x) + 3\sqrt{\frac{x}{8\pi}}$$
$$[-\pi/2 < x < \pi/2].$$

13. $$\sum_{k=2}^{\infty} \frac{(-1)^k}{k^{3/2}(k^2-1)} \left[\cos(kx)S(kx) - \sin(kx)C(kx)\right]$$
$$= -\frac{1}{2}\,(x\sin x + 2\cos x)\,S(x) + \frac{1}{2}\,(2\sin x - x\cos x)\,C(x) - \frac{4}{3\sqrt{\pi}}\left(\frac{x}{2}\right)^{3/2}$$
$$[-\pi < x < \pi].$$

14. $$\sum_{k=1}^{\infty}(-1)^k \frac{(2k+1)^{1/2}}{k(k+1)}$$
$$\times\left[\sin((2k+1)x)S((2k+1)x) + \cos((2k+1)x)C((2k+1)x)\right]$$
$$= 2x\left[\sin x\,C(x) - \cos x\,S(x) - \frac{1}{\sqrt{2\pi x}}\right] \quad [-\pi/2 < x < \pi/2].$$

15. $$\sum_{k=1}^{\infty} \frac{(2k+1)^{-3/2}}{k(k+1)}$$
$$\times\left[\cos((2k+1)x)S((2k+1)x) - \sin((2k+1)x)C((2k+1)x)\right]$$
$$= \left[4\cos x + (2x-\pi)\cos x\right]S(x) - \left[4\sin x + (\pi-2x)\cos x\right]C(x) + \sqrt{2\pi x}$$
$$[0 < x < \pi/2].$$

16. $$\sum_{k=1}^{\infty} \frac{(2k+1)^{-1/2}}{k(k+1)} \times [\sin((2k+1)x)S((2k+1)x) + \cos((2k+1)x)C((2k+1)x)]$$
$$= [2\sin x + (\pi - 2x)\cos x]\, S(x) + [2\cos x + (2x - \pi)\sin x]\, C(x) - \sqrt{\frac{2x}{\pi}} \qquad [0 < x < \pi/2].$$

17. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{3/2}} \times [\sin((2k+1)x)S((2k+1)x) + \cos((2k+1)x)C((2k+1)x)] = \sqrt{\frac{\pi x}{8}} \qquad [-\pi/2 < x < \pi/2].$$

18. $$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^{1/2}}{(2k-1)(2k+3)} \times [\sin((2k+1)x)S((2k+1)x) + \cos((2k+1)x)C((2k+1)x)]$$
$$= -\frac{\pi}{2^{5/2}} [\sin(2x)S(2x) + \cos(2x)C(2x)] \qquad [-\pi/2 < x < \pi/2].$$

19. $$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^{-1/2}}{(2k-1)(2k+3)} \times [\cos((2k+1)x)S((2k+1)x) - \sin((2k+1)x)C((2k+1)x)]$$
$$= -\frac{\pi}{2^{7/2}} [\cos(2x)S(2x) - \sin(2x)C(2x)] \qquad [-\pi/2 < x < \pi/2].$$

20. $$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^{-1/2}}{k(k+1)} \times [\cos((2k+1)x)S((2k+1)x) - \sin((2k+1)x)C((2k+1)x)]$$
$$= 2(\cos x + x\sin x)\, S(x) - 2(\sin x - x\cos x)\, C(x) \qquad [-\pi/2 < x < \pi/2].$$

21. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{5/2}} \times [\cos((2k+1)x)S((2k+1)x) - \sin((2k+1)x)C((2k+1)x)]$$
$$= -\frac{1}{3}\sqrt{\frac{\pi x^3}{2}} \qquad [-\pi/2 < x < \pi/2].$$

6.5.3. Series containing $S(kx)$, $C(kx)$ and $\operatorname{Si}(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{5/2}} \operatorname{Si}(kx)\, S(ky) = -\frac{x}{3}\sqrt{\frac{y^3}{2\pi}} \qquad [x, y > 0;\ x + y < \pi].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}} \operatorname{Si}(kx)\, C(ky) = -x\sqrt{\frac{y}{2\pi}} \qquad [x, y > 0;\ x + y < \pi].$$

6.5.4. Series containing products of $S(kx)$ and $C(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+1}} S(kx)\, S(ky) = (-1)^m \frac{x^{2m-1/2} y^{3/2}}{3(2m-1)!\,(4m-1)\pi}$$
$$- \frac{2}{\pi} \sum_{k=0}^{m-2} \frac{(-1)^k x^{2k+3/2}}{(2k+1)!\,(4k+3)} \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j+3/2}}{(2j+1)!\,(4j+3)} \left(1 - 2^{2j+2k-2m+3}\right)$$
$$\times \zeta(2m-2j-2k-2) \quad [m \geq 1;\ -\pi < x < \pi;\ |x| - \pi < y < \pi - |x|].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m}} S(kx)\, C(ky) = (-1)^m \frac{x^{2m-1/2} y^{1/2}}{(2m-1)!\,(4m-1)\pi}$$
$$- \frac{2}{\pi} \sum_{k=0}^{m-2} \frac{(-1)^k x^{2k+3/2}}{(2k+1)!\,(4k+3)} \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j+1/2}}{(2j)!\,(4j+1)} \left(1 - 2^{2j+2k-2m+3}\right)$$
$$\times \zeta(2m-2j-2k-2) \quad [m \geq 1;\ -\pi < x < \pi;\ |x| - \pi < y < \pi - |x|].$$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m-1}} C(kx)\, C(ky) = (-1)^m \frac{x^{2m-3/2} y^{1/2}}{(2m-2)!\,(4m-3)\pi}$$
$$- \frac{2}{\pi} \sum_{k=0}^{m-2} \frac{(-1)^k x^{2k+1/2}}{(2k)!\,(4k+1)} \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j+1/2}}{(2j)!\,(4j+1)} \left(1 - 2^{2j+2k-2m+3}\right)$$
$$\times \zeta(2m-2j-2k-2) \quad [m \geq 2;\ -\pi < x < \pi;\ |x| - \pi < y < \pi - |x|].$$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3n/2}} \prod_{i=1}^{n} S(kx_i) = -\frac{3^{-n}}{2} \left(\frac{2}{\pi}\right)^{n/2} \prod_{i=1}^{n} x_i^{3/2} \qquad \left[x_i > 0;\ \sum_{i=1}^{n} x_i < \pi\right].$$

5. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{n/2}} \prod_{i=1}^{n} C(kx_i) = -\frac{1}{2} \left(\frac{2}{\pi}\right)^{n/2} \prod_{i=1}^{n} x_i^{1/2} \qquad \left[x_i > 0;\ \sum_{i=1}^{n} x_i < \pi\right].$$

6.6. The Incomplete Gamma Function $\gamma(\nu, z)$

6.6.1. Series containing $\gamma(\nu \pm k, z)$

1. $$\sum_{k=0}^{\infty} \frac{t^k}{k!\,(\nu)_k} \gamma(\nu+k,\, z) = \frac{z^\nu}{\nu} e^{-z} \Phi_3(1;\ \nu+1;\ z,\, tz).$$

2. $$\sum_{k=0}^{\infty} \frac{1}{(\nu)_k} \gamma(\nu+k,\, z) = (\nu-1) z^{\nu-1} e^{-z} - (\nu-1)(\nu-z-1)\gamma(\nu-1,\, z).$$

3. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{(k!)^2} \gamma(k+1,\, z)$$
$$= \frac{e^{-z}}{2} [1 + (2z-1)\, J_0(2z) - \pi z\, J_0(2z)\, \mathbf{H}_1(2z) + \pi z\, J_1(2z)\, \mathbf{H}_0(2z)].$$

4. $$\sum_{k=0}^{\infty} \frac{(1-\nu)_k}{k!} t^k \gamma(\nu - k,\, z) = e^{-t} \gamma(\nu,\, z-t) \qquad [|t| < |z|].$$

5. $$\sum_{k=0}^{\infty} \frac{t^k}{k!\,(\nu)_k} \gamma(\nu + k,\, z) = \Gamma(\nu)\, (2t)^{-\nu} \int_0^{2\sqrt{tz}} x^\nu e^{-x^2/(4t)} I_{\nu-1}(x)\, dx.$$

6. $$\sum_{k=0}^{\infty} \frac{2^{-2k}}{k!\,(a)_k} \gamma(\nu + 2k,\, z) = \frac{z^\nu}{\nu}\, {}_2F_2\left(\begin{matrix} a - \frac{1}{2},\, \nu;\; -2z \\ 2a-1,\, \nu+1 \end{matrix}\right).$$

7. $$\sum_{k=0}^{\infty} \frac{2^k}{(\nu+1)_k (k+2\nu)} P_k^{(-k-\nu,\, \nu)}(0) \gamma(3\nu + k,\, z) = \frac{\nu}{6} \gamma^3(\nu,\, z).$$

8. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} E_k(w) \gamma(k+1,\, z) = \frac{2}{w} (1 - e^{-wz}) + \psi\left(\frac{w}{2}\right) - \psi\left(\frac{w+1}{2}\right)$$
$$+ 2e^{-wz} \Phi(-e^{-z},\, 1,\, w).$$

6.6.2. Series containing products of $\gamma(\nu + k,\, z)$

1. $$\sum_{k=0}^{\infty} \frac{1}{k!\,(\nu)_k} \gamma^2(\nu + k,\, z) = \frac{z^{2\nu}}{\nu^2}\, {}_2F_2\left(\begin{matrix} \nu,\, \nu + \frac{1}{2};\; -4z \\ \nu+1,\, 2\nu+1 \end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty} \frac{1}{k!\,(\nu)_k} \gamma(\nu + k,\, z) \gamma(\nu + k,\, -z) = \frac{e^{i\pi\nu}}{\nu^2} z^{2\nu}\, {}_1F_2\left(\begin{matrix} \frac{\nu}{2};\; -z^2 \\ \frac{\nu}{2} + 1,\, \nu+1 \end{matrix}\right).$$

6.7. The Parabolic Cylinder Function $D_\nu(z)$

6.7.1. Series containing $D_{\nu \pm nk}(z)$ and elementary functions

1. $$\sum_{k=0}^{\infty} \frac{t^k}{k!} D_{\nu+2k}(z) = (1+2t)^{-(\nu+1)/2} \exp \frac{tz^2}{2(2t+1)} D_\nu\left(\frac{z}{\sqrt{2t+1}}\right)$$
$$[|t| < 1/2;\; [80],\, (3.1)].$$

2. $$\sum_{k=0}^{\infty} \frac{(-\nu)_{2k}}{k!} t^k D_{\nu-2k}(z) = (1-2t)^{\nu/2} \exp \frac{tz^2}{2(1-2t)} D_\nu\left(\frac{z}{\sqrt{1-2t}}\right)$$
$$[|t| < 1/2;\; [80],\, (4.2)].$$

6.8. The Bessel Functions $J_\nu(z)$ and $Y_\nu(z)$

6.8.1. Series containing $J_{nk+\nu}(z)$

1. $$\sum_{k=0}^{\infty} J_{2k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_1F_2\left(\begin{matrix}\frac{\nu}{2};\ -\frac{z^2}{4}\\ \frac{\nu}{2}+1,\ \nu\end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty} (-1)^k (2k+\nu)^2 J_{2k+\nu}(z)$$
$$= \frac{\nu z^{\nu-1}}{2^\nu \Gamma(\nu-2)} \left[\frac{\sin z}{\nu-2}\, {}_3F_4\left(\begin{matrix}\frac{\nu}{2}-1,\ \frac{2\nu-1}{4},\ \frac{2\nu+1}{4}\\ \frac{\nu}{2},\ \nu-\frac{1}{2},\ \nu,\ \frac{1}{2};\ -z^2\end{matrix}\right) - \frac{z\cos z}{\nu-1}\, {}_3F_4\left(\begin{matrix}\frac{\nu-1}{2},\ \frac{2\nu+1}{4},\ \frac{2\nu+3}{4}\\ \frac{\nu+1}{2},\ \nu,\ \nu+\frac{1}{2},\ \frac{3}{2};\ -z^2\end{matrix}\right)\right].$$

3. $$\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!\,(k+a)} J_{k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{a\,\Gamma(\nu+1)}\, {}_1F_2\left(\begin{matrix}1;\ -\frac{z^2}{4}\\ a+1,\ \nu+1\end{matrix}\right).$$

4. $$\sum_{k=0}^{\infty} (2k+a)\frac{(a)_k}{k!} J_{2k+\nu}(z) = \frac{a}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_2F_3\left(\begin{matrix}\frac{\nu-a}{2},\ \frac{\nu-a+1}{2};\ -\frac{z^2}{4}\\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ \nu-a+1\end{matrix}\right).$$

5. $$\sum_{k=0}^{\infty} (2k+\nu)\frac{(\nu-\mu+1)_k}{(\mu)_k} J_{2k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu)}\, {}_1F_2\left(\begin{matrix}\mu-1;\ -\frac{z^2}{4}\\ \mu,\ \nu\end{matrix}\right).$$

6. $$\sum_{k=0}^{\infty} (\pm t)^k \frac{\left(\nu+\frac{1}{2}\right)_k}{k!\,(2\nu+1)_k} J_{k+\nu}(z)$$
$$= \Gamma(\nu+1)\left(\frac{4}{t}\right)^\nu J_\nu\left(\frac{z-\sqrt{z^2-2tz}}{2}\right) J_{\pm\nu}\left(\frac{z+\sqrt{z^2-2tz}}{2}\right) \quad [|t|<|z|/2].$$

7. $$\sum_{k=0}^{\infty} (-1)^k (2k+\nu)\frac{(\nu-\mu-1)_{2k}}{(\nu+\mu+2)_{2k}} J_{2k+\nu}(z) = \frac{(\mu+\nu)(\mu+\nu+1)z^{\nu-1}}{2^\nu\Gamma(\nu)}$$
$$\times\left[\frac{\sin z}{\mu+\nu}\, {}_3F_4\left(\begin{matrix}\frac{2\nu-1}{4},\ \frac{2\nu+1}{4},\ \frac{\mu+\nu}{2};\ -z^2\\ \nu-\frac{1}{2},\ \nu,\ \frac{\mu+\nu}{2}+1,\ \frac{1}{2}\end{matrix}\right) - \frac{z\cos z}{\mu+\nu+1}\, {}_3F_4\left(\begin{matrix}\frac{2\nu+1}{4},\ \frac{2\nu+3}{4},\ \frac{\mu+\nu+1}{2};\ -z^2\\ \frac{\nu+1}{2},\ \nu,\ \nu+\frac{1}{2},\ \frac{\mu+\nu+3}{2},\ \frac{3}{2}\end{matrix}\right)\right].$$

8. $$\sum_{k=0}^{\infty} (2k+\nu)\frac{(\nu)_k\left(\frac{1}{2}\right)_k}{k!\left(\nu+\frac{1}{2}\right)_k} J_{2k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu)}\, {}_1F_2\left(\begin{matrix}\frac{\nu}{2};\ -\frac{z^2}{4}\\ \frac{\nu+1}{2},\ \nu+\frac{1}{2}\end{matrix}\right).$$

9. $$\sum_{k=0}^{\infty}(4k+1)\frac{\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k\left(\frac{1}{2}\right)_k}{k!\,(1-\nu)_k(1+\nu)_k}\,J_{2k+1/2}(z) = \nu\sqrt{2\pi z}\,\csc(\nu\pi)\,J_{-\nu}\left(\frac{z}{2}\right)J_{\nu}\left(\frac{z}{2}\right).$$

10. $$\sum_{k=0}^{\infty}(-1)^k(2k+\nu)\frac{(2\nu)_{2k}(a)_{2k}(b)_{2k}}{(2k)!\,(2\nu-a+1)_{2k}(2\nu-b+1)_{2k}}\,J_{2k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{2\Gamma(\nu)}\left[e^{-iz}\,{}_2F_2\left(\begin{matrix}\nu+\frac{1}{2},\,2\nu-a-b+1;\;2iz\\ 2\nu-a+1,\,2\nu-b+1\end{matrix}\right) + e^{iz}\,{}_2F_2\left(\begin{matrix}\nu+\frac{1}{2},\,2\nu-a-b+1;\;-2iz\\ 2\nu-a+1,\,2\nu-b+1\end{matrix}\right)\right].$$

11. $$\sum_{k=0}^{\infty}(2k+\nu)\frac{(a)_k(\nu)_k}{k!\,(\nu-a+1)_k}\,J_{4k+2\nu}(z) = \frac{2^{-2\nu-1}z^{2\nu}}{\Gamma(2\nu)}\,{}_1F_2\left(\begin{matrix}\nu-a+\frac{1}{2};\;-\frac{z^2}{4}\\ \nu+\frac{1}{2},\,2\nu-2a+1\end{matrix}\right).$$

12. $$\sum_{k=0}^{\infty}(6k+\nu)\frac{\left(\frac{\nu}{3}\right)_k}{k!}\,J_{6k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu)}\,{}_1F_2\left(\begin{matrix}\frac{\nu}{3};\;-\frac{3z^2}{16}\\ \frac{\nu}{2},\,\frac{\nu+1}{2}\end{matrix}\right).$$

6.8.2. Series containing two Bessel functions $J_{nk+\nu}(z)$

1. $$\sum_{k=1}^{\infty}J_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu+2}}{\Gamma^2(\nu+2)}\,{}_2F_3\left(\begin{matrix}\nu+1,\,\nu+\frac{3}{2};\;-z^2\\ \nu+2,\,\nu+2,\,2\nu+2\end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(w)\,J_{2n(2k+1)}(z) = \frac{1}{2\pi}\int_0^{\pi}\sin\left(w\cos\left(2nx\right)\right)\cos\left(z\sin x\right)dx.$$

3. $$\sum_{k=0}^{\infty}(2k+1)\,J_{k+1/2}^2(z) = \frac{2z}{\pi}.$$

4. $$\sum_{k=0}^{\infty}(2k+3)\,J_{k+3/2}^2(z) = \frac{1}{\pi z}\left[\cos\left(2z\right)+2z^2-1\right].$$

5. $$\sum_{k=0}^{\infty}(k+\nu)\,J_{k+\nu}(w)\,J_{k+\nu}(z) = \frac{wz}{2(z-w)}\left[J_{\nu-1}(w)\,J_{\nu}(z)-J_{\nu}(w)\,J_{\nu-1}(z)\right].$$

6. $\sum_{k=0}^{\infty}(k+\nu)\,J_{k+\nu}^2(z)$

$$= \frac{z^2}{4}\,[J_{\nu-1}^2(z) + J_\nu^2(z) - J_{\nu-1}(z)\,J_{\nu+1}(z) - J_{\nu-2}(z)\,J_\nu(z)].$$

7. $\sum_{k=2}^{\infty} k^2(k^2-1)^2 J_k^2(z) = \frac{9z^4}{16} + \frac{5z^6}{32}.$

8. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k-1)(2k+3)}\,J_k(z)\,J_{k+1}(z) = \frac{\pi}{8z}\,[J_0(2z)\,\mathbf{H}_1(2z) - J_1(2z)\,\mathbf{H}_0(2z)].$

9. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}\,J_{\nu-k-1/2}(z)\,J_{\nu+k+1/2}(z) = \frac{1}{8}\int_0^\pi x(\pi - x)\,J_{2\nu}\,(2z\sin x)\,dx.$

10. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!}\,J_{\nu-k}(z)\,J_{\nu+k}(z) = \frac{2}{\pi}\int_0^{\pi/2} e^{\pm\cos(2x)}\cos(\sin 2x)\,J_{2\nu}(2z\cos x)\,dx.$

11. $\sum_{k=0}^{\infty}(-1)^k \frac{(2\nu)_k}{k!}\,J_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{\Gamma^2(\nu+1)}\,{}_1F_2\left(\begin{matrix}\nu+\frac{1}{2};\ -z^2\\ \nu+1,\ \nu+1\end{matrix}\right).$

12. $\sum_{k=0}^{\infty}(-1)^k(k+\nu)\,\frac{(2\nu)_k}{k!}\,J_{k+\nu}(w)\,J_{k+\nu}(z) = \frac{2^{-\nu}}{\Gamma(\nu)}\left(\frac{wz}{w+z}\right)^{\nu} J_\nu(w+z).$

13. $\sum_{k=0}^{\infty} \frac{(a)_k}{(1-a)_k}\,J_k^2(z) = \frac{1}{2}\,J_0^2(z) + \frac{1}{2}\,{}_1F_2\left(\begin{matrix}\frac{1}{2}-a;\ -z^2\\ 1,\ 1-a\end{matrix}\right).$

14. $\sum_{k=0}^{\infty}(-1)^k \frac{(a)_k}{(1-a)_k}\,J_k^2(z) = \frac{1}{2}\,J_0^2(z) + \frac{1}{2}\,{}_1F_2\left(\begin{matrix}\frac{1}{2};\ -z^2\\ 1,\ 1-a\end{matrix}\right).$

15. $\sum_{k=0}^{\infty} \frac{(a)_k}{(2-a)_k}\,J_{k+1/2}^2(z) = \frac{2z}{\pi}\,{}_2F_3\left(\begin{matrix}1,\ \frac{3}{2}-a;\ -z^2\\ \frac{3}{2},\ \frac{3}{2},\ 2-a\end{matrix}\right).$

16. $\sum_{k=0}^{\infty}(-1)^k(2k+1)\,\frac{(a)_k}{(2-a)_k}\,J_{k+1/2}^2(z) = \frac{\Gamma(2-a)}{\sqrt{\pi}}\,z^{a-1/2}\,\mathbf{H}_{1/2-a}(2z).$

17. $\sum_{k=0}^{\infty}(2k+a)\,\frac{(a)_k}{k!}\,J_{k+\nu}^2(z)$

$$= \frac{a}{\Gamma^2(\nu+1)}\left(\frac{z}{2}\right)^{2\nu}\,{}_2F_3\left(\begin{matrix}\nu-\frac{a}{2},\ \nu+\frac{1-a}{2};\ -z^2\\ \nu+1,\ \nu+1,\ 2\nu-a+1\end{matrix}\right).$$

18. $$\sum_{k=0}^{\infty} \frac{k+\nu}{4(k+\nu)^2-1} \frac{(2\nu)_k}{k!} J_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{(4\nu^2-1)\Gamma(\nu)\Gamma(\nu+1)} {}_1F_2\left(\begin{matrix} 1;\ -z^2 \\ \nu+1,\ \nu+\frac{3}{2} \end{matrix}\right).$$

19. $$\sum_{k=0}^{\infty} (-1)^k \frac{k+\nu}{4(k+\nu)^2-1} \frac{(2\nu)_k}{k!} J_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{(4\nu^2-1)\Gamma(\nu)\Gamma(\nu+1)} {}_1F_2\left(\begin{matrix} \nu+\frac{1}{2};\ -z^2 \\ \nu+1,\ \nu+\frac{3}{2} \end{matrix}\right).$$

20. $$\sum_{k=0}^{\infty} (-1)^k \frac{(1-a)_k}{(a)_k} J_{\nu-k}(z) J_{\nu+k}(z) = \frac{1}{2} J_\nu^2(z) + 2^{2a-3} z^{2\nu} \frac{\Gamma^2(a)\Gamma\left(a+\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(2a-1)\Gamma(a+\nu)\Gamma(2\nu+1)} {}_1F_2\left(\begin{matrix} a+\nu-\frac{1}{2};\ -z^2 \\ a+\nu,\ 2\nu+1 \end{matrix}\right).$$

21. $$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!} J_{k+\mu}(z) J_{\nu-k}(z) = \frac{\Gamma(\mu+\nu-a+1)}{\Gamma(\nu+1)\Gamma(\mu+\nu+1)\Gamma(\mu-a+1)} \times \left(\frac{z}{2}\right)^{\mu+\nu} {}_2F_3\left(\begin{matrix} \frac{\mu+\nu-a+1}{2},\ \frac{\mu+\nu-a}{2}+1;\ -z^2 \\ \nu+1,\ \mu+\nu+1,\ \mu-a+1 \end{matrix}\right).$$

22. $$\sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}-\nu\right)_k}{\left(\frac{3}{2}+\nu\right)_k} J_{\nu+k}(z) J_{\nu-k-1}(z) = \frac{\Gamma\left(\nu+\frac{3}{2}\right)}{2\Gamma(\nu+1)} z^{-1/2} J_{2\nu-1/2}(2z).$$

23. $$\sum_{k=0}^{\infty} \frac{(\mu+\nu)_k(\mu+\nu-a+1)_k}{k!\,(a)_k} J_{k+\mu}(z) J_{k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} {}_2F_3\left(\begin{matrix} \frac{\mu+\nu+1}{2},\ a-\frac{\mu+\nu}{2} \\ a,\ \mu+1,\ \nu+1;\ -z^2 \end{matrix}\right).$$

24. $$\sum_{k=0}^{\infty} \frac{(2\nu)_k(2\nu-a+1)_k}{k!\,(a)_k} J_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{\Gamma^2(\nu+1)} {}_2F_3\left(\begin{matrix} \nu+\frac{1}{2},\ a-\nu;\ -z^2 \\ a,\ \nu+1,\ \nu+1 \end{matrix}\right).$$

25. $$\sum_{k=0}^{\infty} (-1)^k (k+\nu) \frac{(2\nu)_k(2\nu-a+1)_k}{k!\,(a)_k} J_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{\nu\Gamma^2(\nu)} {}_1F_2\left(\begin{matrix} \nu+\frac{1}{2};\ -z^2 \\ a,\ \nu+1 \end{matrix}\right).$$

26. $$\sum_{k=0}^{\infty}(-1)^k \frac{(\mu-\nu)_k(\mu-\nu-a+1)_k}{k!\,(a)_k} J_{\mu+k}(z)\, J_{\nu-k}(z)$$
$$= \frac{\Gamma(a)\Gamma\left(\frac{\mu-\nu}{2}+1\right)\Gamma\left(a+\frac{3\nu-\mu}{2}\right)}{\Gamma(\nu+1)\Gamma(\mu-\nu+1)\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma(a+\nu)\Gamma\left(a+\frac{\nu-\mu}{2}\right)}\left(\frac{z}{2}\right)^{\mu+\nu}$$
$$\times\ {}_2F_3\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ a+\frac{3\nu-\mu}{2};\ -z^2\\ \nu+1,\ \mu+\nu+1,\ \nu+a\end{matrix}\right).$$

27. $$\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(w)\, J_{2n(2k+1)}(z) = \frac{1}{2\pi}\int_0^{\pi} \sin\left(w\cos\left(2nx\right)\right)\cos\left(z\sin x\right)dx.$$

28. $$\sum_{k=1}^{\infty}\frac{\Gamma^2\left(\frac{2k+1}{4}\right)}{\Gamma^2\left(\frac{2k+3}{4}\right)} J_k^2(z) = -\frac{1}{4\pi^2}\Gamma^4\left(\frac{1}{4}\right) J_0^2(z)$$
$$+\ \frac{4}{\pi}\int_0^{\pi/2} J_0(2z\sin x)\,\mathbf{K}(\cos x)\,dx.$$

29. $$\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma^2\left(\frac{2k+1}{4}\right)}{\Gamma^2\left(\frac{2k+3}{4}\right)} J_k^2(z) = -\frac{1}{4\pi^2}\Gamma^4\left(\frac{1}{4}\right) J_0^2(z)$$
$$+\ \frac{4}{\pi}\int_0^{1}\frac{1}{\sqrt{1-x^2}} J_0(2zx)\,\mathbf{K}(x)\,dx.$$

6.8.3. Series containing three Bessel functions $J_{nk+\nu}(z)$

1. $$\sum_{k=0}^{\infty}(-1)^k J_{k+\nu}(z)\, J_{\nu-k}(z)\, J_{2k}(2z)$$
$$= \frac{1}{2} J_\nu^2(z)\, J_0(2z) + \frac{\left(\frac{z}{2}\right)^{2\nu}}{2\Gamma^2(\nu+1)}\,{}_3F_4\left(\begin{matrix}\nu+\frac{1}{2},\ \nu+\frac{1}{4},\ \nu+\frac{3}{4};\ -4z^2\\ \nu+1,\ 2\nu+\frac{1}{2},\ 2\nu+1,\ \frac{1}{2}\end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty}(-1)^k J_{k+\nu}(z)\, J_{\nu-k-1}(z)\, J_{2k+1}(2z)$$
$$= \frac{2\left(\frac{z}{2}\right)^{2\nu}}{\Gamma(\nu)\Gamma(\nu+1)}\,{}_3F_4\left(\begin{matrix}\nu+\frac{1}{2},\ \nu+\frac{1}{4},\ \nu+\frac{3}{4};\ -4z^2\\ \nu+1,\ 2\nu,\ 2\nu+\frac{1}{2},\ \frac{3}{2}\end{matrix}\right).$$

3. $$\sum_{k=0}^{\infty} J_{k+1/2}^2(w)\, J_{2k+1}(z) = \frac{1}{\pi}\int_0^1 \frac{\sin(zx)}{\sqrt{1-x^2}}\,\mathbf{H}_0(2wx)\,dx.$$

4. $$\sum_{k=0}^{\infty} J_k(w)\, J_{k+1}(w)\, J_{2k+1}(z) = \frac{1}{\pi} \int_0^1 \frac{\sin(zx)}{\sqrt{1-x^2}}\, J_1(2wx)\, dx.$$

6.8.4. Series containing four Bessel functions $J_{nk+\nu}(z)$

1. $$\sum_{k=0}^{\infty} J_k^4(z) = \frac{1}{2} J_0^4(z) + \frac{1}{2}\, {}_2F_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2};\ -4z^2 \\ 1,\ 1,\ 1 \end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty} J_k^2(z)\, J_{k+1}^2(z) = \frac{z^2}{4}\, {}_2F_3\left(\begin{matrix} \frac{3}{2}, \frac{3}{2};\ -4z^2 \\ 2,\ 2,\ 3 \end{matrix}\right).$$

3. $$\sum_{k=0}^{\infty} J_{k+1/2}^2(w)\, J_{k+1/2}^2(z) = \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}}\, \mathbf{H}_0(2wx)\, \mathbf{H}_0(2zx)\, dx.$$

4. $$\sum_{k=0}^{\infty} J_{k+1/2}^2(w)\, J_k(z)\, J_{k+1}(z) = \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}}\, J_1(2zx)\, \mathbf{H}_0(2wx)\, dx.$$

5. $$\sum_{k=0}^{\infty} (-1)^k J_{k+1/2}^2(w)\, J_{(2k+1)n}^2(z) = \frac{1}{2\pi} \int_0^{\pi} J_0\left(2z \sin\frac{x}{2}\right) \mathbf{H}_0(2w\cos(nx))\, dx.$$

6. $$\sum_{k=0}^{\infty} J_{3k+3/2}^2(w)\, J_{2k+2}^2(z) = -\frac{1}{2\pi} \int_0^{\pi} J_0\left(2z \sin\frac{3x}{2}\right) \mathbf{H}_0(2w\cos(2x))\, dx.$$

7. $$\sum_{k=0}^{\infty} J_k^2(w)\, J_{nk}^2(z) = \frac{1}{2} J_0^2(w)\, J_0^2(z) + \frac{1}{\pi} \int_0^{\pi/2} J_0(2z\sin x)\, J_0(2w\sin(nx))\, dx.$$

8. $$\sum_{k=1}^{\infty} J_{\mu+k}(w)\, J_{\mu-k}(w)\, J_{\nu+k}(z)\, J_{\nu-k}(z)$$
$$= -\frac{1}{2} J_\mu^2(w)\, J_\nu^2(z) + \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}}\, J_{2\mu}(2wx)\, J_{2\nu}(2zx)\, dx.$$

9. $$\sum_{k=0}^{\infty} J_{\mu+k+1/2}(w)\, J_{\mu-k-1/2}(w)\, J_{\nu+k+1/2}(z)\, J_{\nu-k-1/2}(z)$$
$$= \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}}\, J_{2\mu}(2wx)\, J_{2\nu}(2zx)\, dx.$$

10. $$\sum_{k=1}^{\infty}(-1)^k J_{\mu+k}(w)\,J_{\mu-k}(w)\,J_{\nu+k}(z)\,J_{\nu-k}(z) = -\frac{1}{2}\,J_\mu^2(w)\,J_\nu^2(z) + \frac{1}{\pi}\int_0^{\pi/2} J_{2\mu}(2w\sin x)\,J_{2\nu}(2z\cos x)\,dx.$$

11. $$\sum_{k=1}^{\infty} J_k(z_1)\,J_k(z_2)\,J_k(z_3)\,J_k(z_4) = -\frac{1}{2}\,J_0(z_1)\,J_0(z_2)\,J_0(z_3)\,J_0(z_4) + \frac{1}{2\pi}\int_0^{\pi} J_0\Bigl(\sqrt{z_1^2+z_2^2-2z_1z_2\cos x}\Bigr)\,J_0\Bigl(\sqrt{z_3^2+z_4^2-2z_3z_4\cos x}\Bigr)\,dx.$$

6.8.5. Series containing $J_{k+\nu}(z)$ and $\psi(z)$

1. $$\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^k}{(k+1)!}\psi(k+a)\,J_k(z) = \frac{4}{z^2} + \frac{2}{z}\psi(a-1)J_1(z) - \Gamma(a-1)\left(\frac{2}{z}\right)^a J_{a-2}(z).$$

2. $$\sum_{k=0}^{\infty}(-1)^k\,(4k+1)\psi(2k+1)\,J_{2k+1/2}(z) = -\sqrt{\frac{z}{2\pi}}\,\sin z\,\mathrm{Si}\,(2z) - \sqrt{\frac{z}{2\pi}}\,\cos z\,[\mathrm{ci}\,(2z) - \ln(2z) + \mathbf{C}].$$

3. $$\sum_{k=0}^{\infty}(-1)^k\,(4k+3)\psi(2k+2)\,J_{2k+3/2}(z) = \sqrt{\frac{z}{2\pi}}\,\cos z\,\mathrm{Si}\,(2z) - \sqrt{\frac{z}{2\pi}}\,\sin z\,[\mathrm{ci}\,(2z) - \ln(2z) + \mathbf{C}].$$

6.8.6. Series containing $J_\nu(\varphi(k,x))$

1. $$\sum_{k=1}^{\infty} k^{m-2n-1/2}J_{1/2-m}(kx) = \frac{(-1)^{m+n+1}}{(2n+1)!\sqrt{\pi}}2^{m-1/2}x^{2n-m+1/2}\sum_{k=0}^{2n+1} B_k\binom{2n+1}{k}\left(\frac{k-1}{2}-n\right)_m\left(\frac{2\pi}{x}\right)^k$$
$[m \le 2n;\ 0 < x \le 2\pi]$.

2. $$\sum_{k=1}^{\infty} k^{m-2n-1/2}J_{m+1/2}(kx) = \frac{(-1)^{n+1}}{(2n+1)!\sqrt{\pi}}2^{m-1/2}x^{2n-m+1/2}\sum_{k=0}^{2n+1} B_k\binom{2n+1}{k}\left(\frac{k}{2}-n\right)_m\left(\frac{2\pi}{x}\right)^k$$
$[m \le 2n;\ 0 < x \le 2\pi]$.

3. $$\sum_{k=1}^{\infty} k^{m-2n+1/2} J_{-m-1/2}(kx)$$
$$= \frac{(-1)^{n+1}}{(2n)!\sqrt{\pi}} 2^{m-1/2} x^{2n-m-1/2} \sum_{k=0}^{2n} B_k \binom{2n}{k} \left(\frac{k+1}{2} - n\right)_m \left(\frac{2\pi}{x}\right)^k$$
$$[m \le 2n-1;\ 0 < x \le 2\pi].$$

4. $$\sum_{k=1}^{\infty} (-1)^k k^{m-2n+1/2} J_{m-1/2}(kx) = \frac{(-1)^{n+1}}{(2n)!} 2^{m+2n-1/2} \pi^{2n-1/2} x^{-m-1/2}$$
$$\times \sum_{k=0}^{2n} \frac{B_{2n-k}}{2^k} \binom{2n}{k} \sum_{p=0}^{k} \binom{k}{p} \left(-\frac{p}{2}\right)_m \left(\frac{x}{\pi}\right)^p \quad [m \le 2n-1;\ 0 < x \le \pi].$$

5. $$\sum_{k=1}^{\infty} (-1)^k k^{m-2n+1/2} J_{-m-1/2}(kx)$$
$$= \frac{(-1)^{m+n+1}}{(2n)!} 2^{m+2n-1/2} \pi^{2n-1/2} x^{-m-1/2}$$
$$\times \sum_{k=0}^{2n} \frac{B_{2n-k}}{2^k} \binom{2n}{k} \sum_{p=0}^{k} \binom{k}{p} \left(\frac{1-p}{2}\right)_m \left(\frac{x}{\pi}\right)^p \quad [m \le 2n-1;\ 0 < x \le \pi].$$

6. $$\sum_{k=1}^{\infty} (-1)^k k^{m-2n-1/2} J_{1/2-m}(kx) = \frac{(-1)^{m+n+1}}{(2n+1)!} 2^{m+2n+1/2} \pi^{2n+1/2} x^{-m-1/2}$$
$$\times \sum_{k=0}^{2n+1} \frac{B_{2n-k+1}}{2^k} \binom{2n+1}{k} \sum_{p=0}^{k} \binom{k}{p} \left(-\frac{p}{2}\right)_m \left(\frac{x}{\pi}\right)^p \quad [m \le 2n;\ 0 < x \le \pi].$$

7. $$\sum_{k=1}^{\infty} (-1)^k k^{m-2n-1/2} J_{m+1/2}(kx) = \frac{(-1)^{n+1}}{(2n+1)!} 2^{m+2n+1/2} \pi^{2n+1/2} x^{-m-1/2}$$
$$\times \sum_{k=0}^{2n+1} \frac{B_{2n-k+1}}{2^k} \binom{2n+1}{k} \sum_{p=0}^{k} \binom{k}{p} \left(\frac{1-p}{2}\right)_m \left(\frac{x}{\pi}\right)^p \quad [m \le 2n;\ 0 < x \le \pi].$$

8. $$\sum_{k=1}^{\infty} k^{m-2n+1/2} J_{m-1/2}(kx)$$
$$= \frac{(-1)^{n+1}}{(2n)!\sqrt{\pi}} 2^{m-1/2} x^{2n-m-1/2} \sum_{k=0}^{2n} B_k \binom{2n}{k} \left(\frac{k}{2} - n\right)_m \left(\frac{2\pi}{x}\right)^k$$
$$[m \le 2n-1;\ 0 < x \le 2\pi].$$

9. $$\sum_{k=2}^{\infty} (-1)^k \frac{k^{-\nu}}{k^2-1} J_\nu(kx) = \frac{\left(\frac{x}{2}\right)^\nu}{2\Gamma(\nu+1)} - \frac{1}{4} J_\nu(x) - \frac{x}{2} J_{\nu+1}(x)$$
$$[-\pi \le x \le \pi;\ \operatorname{Re}\nu > -2].$$

10. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{\left(k^2+a^2\right)^{\nu/2}} J_\nu\left(\sqrt{k^2+a^2}\,x\right) = -\frac{a^{-\nu}}{2} J_\nu(ax)$$

$[-\pi < x < \pi;\ \operatorname{Re}\nu > -1/2]$.

11. $$\sum_{k=0}^{\infty} (2k+1)^{m-2n-1/2} J_{1/2-m}((2k+1)x)$$
$$= \frac{(-1)^{m+n}}{(2n+1)!} 2^{m-3/2} \pi^{2n+1/2} x^{-m-1/2}$$
$$\times \sum_{k=0}^{2n} B_{2n-k+1} \binom{2n+1}{k} \left(2-2^{2n-k+2}\right) \left(-\frac{k}{2}\right)_m \left(\frac{x}{\pi}\right)^k \qquad [m \le 2n;\ 0 \le x \le \pi].$$

12. $$\sum_{k=0}^{\infty} (2k+1)^{m-2n-1/2} J_{m+1/2}((2k+1)x)$$
$$= \frac{(-1)^{n}}{(2n+1)!} 2^{m-3/2} \pi^{2n+1/2} x^{-m-1/2}$$
$$\times \sum_{k=0}^{2n} B_{2n-k+1} \binom{2n+1}{k} \left(2-2^{2n-k+2}\right) \left(\frac{1-k}{2}\right)_m \left(\frac{x}{\pi}\right)^k$$

$[m \le 2n;\ 0 \le x \le \pi]$.

13. $$\sum_{k=1}^{\infty} \frac{(2k+1)^{-\nu}}{k(k+1)} J_\nu((2k+1)x) = J_\nu(x) + 2x\,J_{\nu+1}(x) - \pi\,\mathbf{H}_\nu(x)$$

$[-\pi/2 \le x \le \pi/2;\ \operatorname{Re}\nu > -3/2]$.

14. $$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^{1-\nu}}{k(k+1)} J_\nu((2k+1)x) = 2x\,J_{\nu+1}(x) - J_\nu(x)$$

$[-\pi/2 \le x \le \pi/2;\ \operatorname{Re}\nu > -3/2]$.

15. $$\sum_{k=0}^{\infty} \frac{(2k+1)^{-\nu}}{(2k+1)^2 a^2 \pm b^2} J_\nu((2k+1)x)$$
$$= \frac{\pi a^{\nu-1}}{4b^{\nu+1}} \left[\left\{ \begin{matrix} \tanh(b\pi/(2a)) \\ \tan(b\pi/(2a)) \end{matrix} \right\} I_\nu\left(\frac{bx}{a}\right) - \left\{ \begin{matrix} \mathbf{L}_\nu(bx/a) \\ \mathbf{H}_\nu(bx/a) \end{matrix} \right\} \right]$$

$[0 < x < \pi;\ \operatorname{Re}\nu > -3/2]$.

16. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left[(2k+1)^2+a^2\right]^{-\nu/2} J_\nu\left(\sqrt{(2k+1)^2+a^2}\,x\right) = \frac{\pi a^{-\nu}}{4} J_\nu(ax)$$

$[0 < x < \pi/2;\ \operatorname{Re}\nu > -3/2]$.

17. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k+\frac{1}{2}\right)^{2n-1}} \left[\left(k+\frac{1}{2}\right)^2 \pi^2 + a^2\right]^{-\nu/2} J_\nu\left(\sqrt{\left(k+\frac{1}{2}\right)^2 \pi^2 + a^2}\right)$$
$$= (-1)^{n-1} \frac{\pi^{2n-1}}{2a^\nu} \sum_{k=0}^{n-1} \frac{(2a)^{-k}}{k!\,(2n-2k-2)!} E_{2n-2k-2} J_{\nu+k}(a) \quad [\operatorname{Re}\nu > 1/2 - 2n].$$

6.8.7. Series containing $J_\nu(kx)$ and trigonometric functions

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu+1}} \sin(kx)\, J_\nu(ky) = -\frac{xy^\nu}{2^{\nu+1}\Gamma(\nu+1)}$$
$[x,\, y > 0;\; x+y < \pi;\; \operatorname{Re}\nu > -3/2].$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu}} \cos(kx)\, J_\nu(ky) = -\frac{\left(\frac{y}{2}\right)^\nu}{2\Gamma(\nu+1)}$$
$[x,\, y > 0;\; x+y < \pi;\; \operatorname{Re}\nu > -1/2].$

3. $$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2m+\nu-2}} J_\nu(ky)$$
$$= \frac{\left(\frac{y}{2}\right)^\nu}{\Gamma(\nu+1)} \left\{ \frac{(-1)^m}{2(m-1)!\,(\nu+1)_{m-1}} \left(\frac{y}{2}\right)^{2m-2} + \sum_{k=0}^{m-2} \frac{\left(\frac{y}{2}\right)^{2k}}{k!\,(\nu+1)_k} \right.$$
$$\left. \times \left[\frac{(-1)^{m-1}\pi}{2(2m-2k-3)!} x^{2m-2k-3} + (-1)^k \sum_{j=0}^{m-k-1} \frac{(-x^2)^j}{(2j)!} \zeta(2m-2j-2k-2) \right] \right\}$$
$[m \ge 1;\; 0 < y < \pi;\; y < x < 2\pi - y].$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\nu-2}} \cos(kx)\, J_\nu(ky) = \frac{(-1)^m}{2(m-1)!\,\Gamma(m+\nu)} \left(\frac{y}{2}\right)^{2m+\nu-2}$$
$$- \sum_{k=0}^{m-2} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{y}{2}\right)^{2k+\nu}$$
$$\times \sum_{j=0}^{m-k-1} \frac{(-1)^j}{(2j)!} x^{2j} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$
$[m \ge 1;\; \operatorname{Re}\nu > 3/2 - 2m;\; -\pi < y < \pi;\; |y| - \pi < x < \pi - |y|].$

6.8.8. Series containing products of $J_\nu(\varphi(k,\, x))$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\mu+\nu}} J_\mu(kx)\, J_\nu(ky) = \frac{(-1)^{m+1}}{2m!\,\Gamma(m+\mu+1)\Gamma(\nu+1)} \left(\frac{x}{2}\right)^{2m+\mu} \left(\frac{y}{2}\right)^\nu$$
$$- \sum_{k=0}^{m-1} \frac{(-1)^k}{k!\,\Gamma(k+\mu+1)} \left(\frac{x}{2}\right)^{2k+\mu}$$

$$\times \sum_{j=0}^{m-k} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{y}{2}\right)^{2j+\nu} \left(1-2^{2j+2k-2m+1}\right) \zeta(2m-2j-2k)$$

$[-\pi < x < \pi;\ |x|-\pi < y < \pi-|x|;\ \operatorname{Re}(\mu+\nu) > -2m-1].$

2. $$\sum_{k=2}^{\infty} (-1)^k \frac{k^{-2\nu}}{k^2-1} J_\nu^2(kx) = \frac{\left(\frac{x}{2}\right)^{2\nu}}{2\Gamma^2(\nu+1)} - \frac{1}{4} J_\nu^2(x) - J_{\nu+1}(x)\left[2(\nu+1)\,J_{\nu+1}(x) - x\,J_{\nu+2}(x)\right]$$

$[-\pi \le x \le \pi;\ \operatorname{Re}\nu > -1].$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{\left(k^2+a^2\right)^\nu} J_\nu^2\left(\sqrt{k^2+a^2}\,x\right) = -\frac{a^{-2\nu}}{2} J_\nu^2(ax)$$

$[-\pi < x < \pi;\ \operatorname{Re}\nu > -1/2].$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\left(k^2+a^2\right)^\nu} J_\nu^2\left(\sqrt{k^2+a^2}\,x\right) = \frac{a^{-2\nu-2}}{12}\left\{6J_{\nu+1}(ax)\left[2(\nu+1)\,J_{\nu+1}(ax) - ax\,J_{\nu+2}(ax)\right] - \pi^2 a^2 J_\nu^2(ax)\right\}$$

$[-\pi < x < \pi;\ \operatorname{Re}\nu > -3/2].$

5. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left[(2k+1)^2+a^2\right]^{-\nu} J_\nu^2\left(\sqrt{(2k+1)^2+a^2}\,x\right) = \frac{\pi a^{-2\nu}}{4} J_\nu^2(ax)$$

$[0 < x < \pi/2;\ \operatorname{Re}\nu > -1].$

6.8.9. Series containing products of $J_\nu(kx)$ and trigonometric functions

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\mu+\nu+1}} \sin(kx)\,J_\mu(ky)\,J_\nu(ky) = -\frac{x\left(\frac{y}{2}\right)^{\mu+\nu}}{2\Gamma(\mu+1)\Gamma(\nu+1)}$$

$[x,\,y > 0;\ x+y < \pi;\ \operatorname{Re}(\mu+\nu) > -1].$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\mu+\nu}} \cos(kx)\,J_\mu(ky)\,J_\nu(ky) = -\frac{\left(\frac{y}{2}\right)^{\mu+\nu}}{2\Gamma(\mu+1)\Gamma(\nu+1)}$$

$[x,\,y > 0;\ x+y < \pi;\ \operatorname{Re}(\mu+\nu) > -1].$

3. $$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2m+\mu+\nu-2}} J_\mu(ky)\,J_\nu(ky) = \frac{\left(\frac{y}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} \times \left\{ \frac{(-1)^m 2^{2m-3}(\mu+\nu+1)_{2m-2}}{(\mu+1)_{m-1}(\nu+1)_{m-1}(\mu+\nu+1)_{m-1}} y^{2m-2} \right.$$

$$+ \sum_{k=0}^{m-2} \frac{(\mu+\nu+1)_{2k}}{k!\,(\mu+1)_k(\nu+1)_k(\mu+\nu+1)_k} \left(\frac{y}{2}\right)^{2k} \left[(-1)^{m-1} \frac{\pi x^{2m-2k-3}}{2(2m-2k-3)!}\right.$$

$$+ (-1)^k \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j}}{(2j)!} \zeta(2m-2j-2k-2)\Bigg]\Bigg]\Bigg\}$$

$$[m \geq 1;\ 0 < y < \pi/2;\ y < x/2 < \pi - y].$$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\mu+\nu-2}} \cos(kx)\, J_\mu(ky)\, J_\nu(ky) = \frac{\left(\frac{y}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}$$

$$\times \Bigg\{ \frac{(-1)^m (\mu+\nu+1)_{2m-2}}{2(m-1)!\,(\mu+1)_{m-1}(\nu+1)_{m-1}(\mu+\nu+1)_{m-1}} \left(\frac{y}{2}\right)^{2m-2}$$

$$- \sum_{k=0}^{m-2} \frac{(-1)^k (\mu+\nu+1)_{2k}}{k!\,(\mu+1)_k(\nu+1)_k(\mu+\nu+1)_k} \left(\frac{y}{2}\right)^{2k}$$

$$\times \sum_{j=0}^{m-k-1} \frac{(-x^2)^j}{(2j)!} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2) \Bigg\}$$

$$[m \geq 1;\ -\pi/2 < y < \pi/2;\ 2|y| - \pi < x < \pi - 2|y|;\ \operatorname{Re}(\mu+\nu) > 1-2m].$$

6.8.10. Series containing $J_\nu(kx)$ and $\operatorname{Si}(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\mu+\nu-1}} \operatorname{Si}(kx)\, J_\mu(ky)\, J_\nu(ky)$$

$$= \frac{(-1)^m \Gamma(2m+\mu+\nu-1)x}{2(m-1)!\,\Gamma(m+\mu)\Gamma(m+\nu)\Gamma(m+\mu+\nu)} \left(\frac{y}{2}\right)^{2m+\mu+\nu-2}$$

$$- \sum_{k=0}^{m-2} \frac{(-1)^k \Gamma(2k+\mu+\nu+1)}{k!\,\Gamma(2k+\mu+\nu+1)\Gamma(k+\mu+1)\Gamma(k+\nu+1)} \left(\frac{y}{2}\right)^{2k+\mu+\nu}$$

$$\times \sum_{j=0}^{m-k-1} \frac{(-1)^j}{(2j+1)!\,(2j+1)} x^{2j+1} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$

$$[m \geq 1;\ -\pi < x < \pi;\ |x| - \pi < 2y < \pi - |x|].$$

6.8.11. Series containing $J_\nu(kx)$, $S(kx)$ and $C(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu+1/2}} C(kx)\, J_\nu(ky) = -\frac{2^{-\nu-1/2} x^{1/2} y^\nu}{\sqrt{\pi}\,\Gamma(\nu+1)}$$

$$[x,\ y > 0;\ x + y < \pi;\ \operatorname{Re}\nu > -1].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu+3/2}} S(kx)\, J_\nu(ky) = -\frac{2^{-\nu-1/2} x^{3/2} y^\nu}{3\sqrt{\pi}\,\Gamma(\nu+1)}$$

$$[x,\ y > 0;\ x + y < \pi;\ \operatorname{Re}\nu > -2].$$

6.8.12. Series containing $J_{k\mu+\nu}(\varphi(k,z))$

Notation: $\Delta(z) = \left|\dfrac{z}{1+\sqrt{1-z^2}}\,e^{\sqrt{1-z^2}}\right|$.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+a^2} J_{2k}(kz) = \frac{1}{2a^2}\left[{}_1F_2\left(\begin{matrix} 1;\ \frac{a^2z^2}{4} \\ 1-ia,\ 1+ia \end{matrix}\right) - 1\right]$　　$[\Delta(z/2) < 1]$.

2. $\displaystyle\sum_{k=1}^{\infty} \frac{k^{-(k+\nu)/2}(k+a)^{k-1}}{k!}\left(\frac{z}{2}\right)^k J_{k+\nu}(\sqrt{k}\,z) = -\frac{\left(\frac{z}{2}\right)^\nu}{a\,\Gamma(\nu+1)} + a^{-\nu/2-1} I_\nu(\sqrt{a}\,z).$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{k^{(k-\nu)/2-1}}{k!}\left(\frac{z}{2}\right)^k J_{k+\nu}(\sqrt{k}\,z) = \frac{\left(\frac{z}{2}\right)^{\nu+2}}{\Gamma(\nu+2)}.$

4. $\displaystyle\sum_{k=1}^{\infty} (-1)^k J_{k/2}^2(kz) = -\frac{1}{2} + \frac{1}{2\sqrt{1-4z^2}} - \frac{\arcsin(2z)}{\pi\sqrt{1-4z^2}}$　　$[\Delta(z) < 1]$.

5. $\displaystyle\sum_{k=1}^{\infty} k^2 J_k^2(kz) = \frac{z^2(z^2+4)}{16(1-z^2)^{7/2}}$　　$[\Delta(z) < 1]$.

6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} J_k^2(kz) = \frac{z^2}{4}$　　$[\Delta(z) < 1]$.

7. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4} J_k^2(kz) = \frac{z^2}{4} - \frac{3z^4}{64}$　　$[\Delta(z) < 1]$.

8. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^6} J_k^2(kz) = \frac{5z^6}{1152} - \frac{15z^4}{256} + \frac{z^2}{4}$　　$[\Delta(z) < 1]$.

9. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} J_{k+1/2}^2((2k+1)z) = \frac{2z}{\pi}$　　$[\Delta(2z) < 1]$.

10. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} J_{k+1/2}^2((2k+1)z) = \frac{2z}{\pi}\left(1 - \frac{8z^2}{27}\right)$　　$[\Delta(2z) < 1]$.

11. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} J_{k+1/2}^2((2k+1)z) = \frac{2z}{\pi}\left(\frac{128z^4}{3375} - \frac{80z^3}{243} + 1\right)$　　$[\Delta(2z) < 1]$.

12. $\displaystyle\sum_{k=1}^{\infty} J_{k-1/2}(kz)\,J_{k+1/2}(kz) = \frac{\arcsin z}{\pi z\sqrt{1-z^2}} - \frac{1}{\pi}$　　$[\Delta(z) < 1]$.

13. $$\sum_{k=1}^{\infty} \frac{1}{k^2 - a^2} J_{k-1/2}(kz)\, J_{k+1/2}(kz) = \frac{1}{\pi a^2} - \frac{1}{\pi a^2}\, {}_2F_3\left(\begin{matrix} 1,\, 1;\, -a^2 z^2 \\ 1+a,\, 1-a,\, \frac{3}{2} \end{matrix}\right) \qquad [\Delta(z) < 1].$$

14. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} J_{k/2}^2(kz) = \frac{z^2}{4} - \frac{2z}{\pi} \qquad [\Delta(z) < 1].$$

15. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - a^2} J_{k/2}^2(kz) = \frac{1}{2a^2} - \frac{\pi}{4a} \csc\frac{\pi a}{2}\, J_{-a/2}(az)\, J_{a/2}(az) + \frac{2z}{\pi(a^2-1)}\, {}_2F_3\left(\begin{matrix} 1,\, 1;\, -a^2 z^2 \\ \frac{3}{2},\, \frac{3-a}{2},\, \frac{3+a}{2} \end{matrix}\right) \qquad [\Delta(z) < 1].$$

16. $$\sum_{k=0}^{\infty} \frac{(2\nu)_k}{k!} (k+\nu)^{-2\nu-1} J_{k+\nu}^2((k+\nu)z)) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{\nu\Gamma^2(\nu+1)} \qquad [\Delta(2z) < 1].$$

6.8.13. Various series containing $J_\nu(z)$

1. $$\sum_{k=1}^{\infty} J_k(w_1)\, J_k(w_2)\, J_0(kz) = -\frac{1}{2} J_0(w_1)\, J_0(w_2) + \frac{1}{\pi} \int_0^z (z^2 - x^2)^{-1/2} J_0\left(\sqrt{w_1^2 + w_2^2 - 2w_1 w_2 \cos x}\right) dx.$$

2. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\nu-1}} J_{k+1/2}^2(w)\, J_\nu((2k+1)z) = \frac{2(2z)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu - \frac{1}{2}\right)} \int_0^z x(z^2 - x^2)^{\nu-3/2}\, \mathbf{H}_0(2w \sin x)\, dx.$$

3. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\nu}} J_{k+1/2}^2(w)\, J_\nu((2k+1)z) = \frac{(2z)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^z (z^2 - x^2)^{\nu-1/2}\, \mathbf{H}_0(2w \cos x)\, dx.$$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{n\nu}} \prod_{i=1}^{n} J_\nu(kx_i) = -\frac{2^{-n\nu-1}}{\Gamma^n(\nu+1)} \prod_{i=1}^{n} x_i^\nu \qquad \left[x_i > 0;\ \sum_{i=1}^{n} x_i < \pi/2;\ \operatorname{Re}\nu > -1/2\right].$$

6.8.14. Series containing $Y_{k+\nu}(z)$

1. $$\sum_{k=0}^{\infty} J_{k+1/2}(w)\, J_{2k+1}(z)\, Y_{k+1/2}(w) = -\frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{1-x^2}} \sin(zx)\, J_0(2wx)\, dx.$$

2. $$\sum_{k=0}^{\infty} J_{k+1/2}^2(w)\, J_{k+1/2}(z)\, Y_{k+1/2}(z) = -\frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{1-x^2}} J_0(2zx)\, \mathbf{H}_0(2wx)\, dx.$$

3. $$\sum_{k=0}^{\infty} J_k(z)\, J_{k+1/2}(z)\, J_{k+1}(z)\, Y_{k+1/2}(z) = \frac{1}{4\pi z}[J_0(4z) - 1].$$

4. $$\sum_{k=1}^{\infty} J_k(x_1)\, J_k(x_2)\, Y_k(x_3)\, Y_k(x_4) = -\frac{1}{2} J_0(x_1)\, J_0(x_2)\, Y_0(x_3)\, Y_0(x_4)$$
$$+ \frac{1}{2\pi}\int_0^{\pi} Y_0\left(\sqrt{x_1^2 + x_3^2 - 2x_1x_3\cos x}\right) Y_0\left(\sqrt{x_2^2 + x_4^2 - 2x_2x_4\cos x}\right) dx$$
$$[x_1 < x_3,\ x_2 < x_4].$$

6.9. The Modified Bessel Function $I_\nu(z)$

6.9.1. Series containing $I_{nk+\nu}(z)$

1. $$\sum_{k=1}^{\infty} k\, I_k(z) = \frac{z}{2}\,[I_0(z) + I_1(z)].$$

2. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} I_{k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} e^{z}\, {}_2F_2\left(\begin{matrix} \nu + \frac{1-a}{2},\ \nu - \frac{a}{2} + 1 \\ \nu+1,\ 2\nu - a + 1;\ -2z \end{matrix}\right).$$

3. $$\sum_{k=0}^{\infty} \frac{k+\nu}{(2k+\nu)^2 - 1} \frac{(2\nu)_k}{k!} I_{k+\nu}(z)$$
$$= \frac{2^{-2\nu-3/2} z^{-1/2}}{(2\nu-1)\Gamma(\nu)} e^{(2\nu-3)\,\mathrm{sgn}\,(\mathrm{Im}\, z)\pi i/2 - z} \gamma\left(\nu + \frac{1}{2},\ -2z\right).$$

4. $$\sum_{k=0}^{\infty} (-1)^k (k+\nu) \frac{(a)_k (b)_k (2\nu)_k}{k!\,(2\nu - a + 1)_k (2\nu - b + 1)_k} I_{k+\nu}(z)$$
$$= \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu)} e^{-z}\, {}_2F_2\left(\begin{matrix} \nu + \frac{1}{2},\ 2\nu - a - b + 1;\ 2z \\ 2\nu - a + 1,\ 2\nu - b + 1 \end{matrix}\right).$$

5. $$\sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} I_{k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu} e^{z}}{\Gamma(\nu+1)}\, {}_2F_2\left(\begin{matrix} \nu + \frac{1-\mu}{2},\ \nu - \frac{\mu}{2} + 1 \\ \nu+1,\ 2\nu - \mu + 1;\ -2z \end{matrix}\right).$$

6. $$\sum_{k=1}^{\infty}(-1)^k \frac{\Gamma^2\left(\frac{2k+1}{4}\right)}{\Gamma^2\left(\frac{2k+3}{4}\right)} I_{k/2}(z)$$
$$= -\frac{1}{4\pi^2}\Gamma^4\left(\frac{1}{4}\right) I_0(z) + \frac{4}{\pi}\int_0^{\pi/2} e^{z\cos(4x)}\,\mathrm{erfc}\left(\sqrt{2z}\cos(2x)\right)\mathbf{K}(\cos x)\,dx.$$

7. $$\sum_{k=0}^{\infty}(-1)^k I_{2k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)}\,{}_1F_2\left(\begin{matrix}\frac{\nu}{2};\ \frac{z^2}{4}\\ \frac{\nu}{2}+1,\ \nu\end{matrix}\right).$$

8. $$\sum_{k=0}^{\infty}(-1)^k(2k+\nu)\frac{(\nu-\mu+1)_k}{(\mu)_k} I_{2k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu)}\,{}_1F_2\left(\begin{matrix}\mu-1;\ \frac{z^2}{4}\\ \mu,\ \nu\end{matrix}\right).$$

9. $$\sum_{k=0}^{\infty}(-1)^k(2k+\nu)\frac{(\nu)_k\left(\frac{2\nu+1}{4}\right)_k\left(\frac{1}{2}\right)_k}{k!\left(\nu+\frac{1}{2}\right)_k\left(\frac{2\nu+3}{4}\right)_k} I_{2k+\nu}(z)$$
$$= 2^{\nu-1}z^{1/2}\frac{\Gamma^2\left(\frac{2\nu+3}{4}\right)}{\Gamma(\nu)} I^2_{(2\nu-1)/4}\left(\frac{z}{2}\right).$$

10. $$\sum_{k=0}^{\infty}(2k+\nu)\frac{(a)_k(\nu)_k}{k!\,(\nu-a+1)_k} I_{2k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu)} e^{-z}\,{}_1F_1\left(\begin{matrix}\nu-a+\frac{1}{2};\ 2z\\ 2\nu-2a+1\end{matrix}\right).$$

11. $$\sum_{k=0}^{\infty}\sigma_{k+m}^{m}\frac{t^k}{(k+m)!} I_{k+\nu}(z)$$
$$= (-t)^{-m}z^{(\nu-m)/2}\sum_{k=0}^{m}(-1)^k\frac{(2kt+z)^{(m-\nu)/2}}{k!\,(m-k)!} I_{\nu-m}\left(\sqrt{z(2kt+z)}\right).$$

6.9.2. Series containing $I_{k+\nu}(z)$ and $\psi(z)$

1. $$\sum_{k=0}^{\infty}\frac{\left(-\frac{z}{2}\right)^k}{k!}\psi(k+a) I_{k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)}\psi(a)$$
$$- \frac{\left(\frac{z}{2}\right)^{\nu+2}}{\Gamma(\nu+2)a}\,{}_2F_3\left(\begin{matrix}1,\ 1;\ \frac{z^2}{4}\\ a+1,\ \nu+2,\ 2\end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty}\frac{\left(-\frac{z}{2}\right)^k}{(k+1)!}\psi(k+a) I_k(z) = -\frac{4}{z^2} + \frac{2}{z}\psi(a-1)I_1(z)$$
$$+ \Gamma(a-1)\left(\frac{2}{z}\right)^{a} I_{a-2}(z).$$

3. $$\sum_{k=0}^{\infty} \frac{\left(-\frac{z}{2}\right)^k}{k!} \psi(2k+a)\, I_{k+\nu}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \psi(a) - \frac{\left(\frac{z}{2}\right)^{\nu+2}}{\Gamma(\nu+2)} \left[\frac{1}{a}\, {}_2F_3\left(\begin{matrix} 1,\, 1;\; \frac{z^2}{4} \\ \frac{a}{2}+1,\, \nu+2,\, 2 \end{matrix} \right) + \frac{1}{a+1}\, {}_2F_3\left(\begin{matrix} 1,\, 1;\; \frac{z^2}{4} \\ \frac{a+3}{2},\, \nu+2,\, 2 \end{matrix} \right) \right].$$

6.9.3. Series containing products of $I_{nk+\nu}(z)$

1. $$\sum_{k=2}^{\infty} (-1)^k k^2 (k^2-1)^2 I_k^2(z) = \frac{9z^4}{16} - \frac{5z^6}{32}.$$

2. $$\sum_{k=1}^{\infty} (-1)^k \frac{\Gamma^2\left(\frac{2k+1}{4}\right)}{\Gamma^2\left(\frac{2k+3}{4}\right)} I_k^2(z) = -\frac{1}{4\pi^2} \Gamma^4\left(\frac{1}{4}\right) I_0^2(z) + \frac{4}{\pi} \int_0^{\pi/2} I_0(2z \sin x)\, \mathbf{K}(\cos x)\, dx.$$

3. $$\sum_{k=0}^{\infty} \frac{(a)_k}{(1-a)_k} I_k^2(z) = \frac{1}{2} I_0^2(z) + \frac{1}{2}\, {}_1F_2\left(\begin{matrix} \frac{1}{2};\; z^2 \\ 1,\, 1-a \end{matrix} \right).$$

4. $$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{(1-a)_k} I_k^2(z) = \frac{1}{2} I_0^2(z) + \frac{1}{2}\, {}_1F_2\left(\begin{matrix} \frac{1}{2}-a;\; z^2 \\ 1,\, 1-a \end{matrix} \right).$$

5. $$\sum_{k=0}^{\infty} (-1)^k (4k+1) \frac{\left(\frac{1}{2}-a\right)_k \left(\frac{1}{2}+a\right)_k \left(\frac{1}{2}\right)_k}{k!\,(1-a)_k (1+a)_k} I_{k+1/4}^2(z) = \frac{2^{7/2} z^{1/2}}{\Gamma^2\left(\frac{1}{4}\right)}\, {}_2F_3\left(\begin{matrix} \frac{1}{2},\, \frac{3}{4};\; z^2 \\ 1-a,\, 1+a,\, \frac{5}{4} \end{matrix} \right).$$

6. $$\sum_{k=0}^{\infty} (-1)^k (2k+1)\, I_{k+1/2}^2(z) = \frac{2z}{\pi}.$$

7. $$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{(2-a)_k} I_{k+1/2}^2(z) = \frac{2z}{\pi}\, {}_2F_3\left(\begin{matrix} \frac{3}{2}-a,\, 1;\; z^2 \\ 2-a,\, \frac{3}{2},\, \frac{3}{2} \end{matrix} \right).$$

8. $$\sum_{k=0}^{\infty} (2k+1) \frac{(a)_k}{(2-a)_k} I_{k+1/2}^2(z) = \frac{\Gamma(2-a)}{\sqrt{\pi}} z^{a-1/2}\, \mathbf{L}_{1/2-a}(2z).$$

9. $$\sum_{k=0}^{\infty} \frac{1}{(2k-1)(2k+3)} I_k(z)\, I_{k+1}(z) = \frac{\pi}{8z} [I_0(2z)\, \mathbf{L}_1(2z) - I_1(2z)\, \mathbf{L}_0(2z)].$$

10. $$\sum_{k=0}^{\infty}(-1)^k \frac{(\nu)_k}{(2-\nu)_k} I_k(z)\, I_{k+1}(z) = \frac{z}{2}\, {}_1F_2\left(\begin{matrix} \frac{3}{2}-\nu;\ z^2 \\ 2-\nu,\ 2 \end{matrix}\right).$$

11. $$\sum_{k=0}^{\infty}(-1)^k (2k+3)\, I_{k+3/2}^2(z) = \frac{1}{\pi z}\left[\cosh(2z) - 2z^2 - 1\right].$$

12. $$\sum_{k=1}^{\infty}(-1)^k I_{k+\nu}^2(z) = -\frac{\left(\frac{z}{2}\right)^{2\nu+2}}{\Gamma^2(\nu+2)}\, {}_2F_3\left(\begin{matrix} \nu+1,\ \nu+\frac{3}{2};\ z^2 \\ \nu+2,\ \nu+2,\ 2\nu+2 \end{matrix}\right).$$

13. $$\sum_{k=0}^{\infty} \frac{k+\nu}{4(k+\nu)^2-1} \frac{(2\nu)_k}{k!} I_{k+\nu}^2(z)$$
$$= \frac{\left(\frac{z}{2}\right)^{2\nu}}{(4\nu^2-1)\Gamma(\nu)\Gamma(\nu+1)}\, {}_1F_2\left(\begin{matrix} \nu+\frac{1}{2};\ z^2 \\ \nu+1,\ \nu+\frac{3}{2} \end{matrix}\right).$$

14. $$\sum_{k=0}^{\infty}(-1)^k \frac{k+\nu}{4(k+\nu)^2-1} \frac{(2\nu)_k}{k!} I_{k+\nu}^2(z)$$
$$= \frac{\left(\frac{z}{2}\right)^{2\nu}}{(4\nu^2-1)\Gamma(\nu)\Gamma(\nu+1)}\, {}_1F_2\left(\begin{matrix} 1;\ z^2 \\ \nu+1,\ \nu+\frac{3}{2} \end{matrix}\right).$$

15. $$\sum_{k=0}^{\infty}(-1)^k \frac{(2\nu)_k(2\nu-a+1)_k}{k!\,(a)_k} I_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{\Gamma^2(\nu+1)}\, {}_2F_3\left(\begin{matrix} \nu+\frac{1}{2},\ a-\nu;\ z^2 \\ a,\ \nu+1,\ \nu+1 \end{matrix}\right).$$

16. $$\sum_{k=0}^{\infty}(k+\nu) \frac{(2\nu)_k(2\nu-a+1)_k}{k!\,(a)_k} I_{k+\nu}^2(z) = \frac{\left(\frac{z}{2}\right)^{2\nu}}{\nu\Gamma^2(\nu)}\, {}_1F_2\left(\begin{matrix} \nu+\frac{1}{2};\ z^2 \\ a,\ \nu+1 \end{matrix}\right).$$

17. $$\sum_{k=0}^{\infty}(-1)^k (k+\nu) \frac{(2\nu)_k(a)_k(b)_k}{k!\,(2\nu-a+1)_k(2\nu-b+1)_k} I_{k+\nu}^2(z)$$
$$= \frac{\left(\frac{z}{2}\right)^{2\nu}}{\nu\Gamma^2(\nu)}\, {}_2F_3\left(\begin{matrix} \nu+\frac{1}{2},\ 2\nu-a-b+1;\ z^2 \\ \nu+1,\ 2\nu-a+1,\ 2\nu-b+1 \end{matrix}\right).$$

18. $$\sum_{k=0}^{\infty}(-1)^k \frac{(a)_k}{k!} I_{k+\mu}(z)\, I_{\nu-k}(z)$$
$$= \frac{\Gamma(\mu+\nu-a+1)}{\Gamma(\nu+1)\Gamma(\mu+\nu+1)\Gamma(\mu-a+1)} \left(\frac{z}{2}\right)^{\mu+\nu}$$
$$\times\, {}_2F_3\left(\begin{matrix} \frac{\mu+\nu-a+1}{2},\ \frac{\mu+\nu-a}{2}+1;\ z^2 \\ \nu+1,\ \mu+\nu+1,\ \mu-a+1 \end{matrix}\right).$$

19. $$\sum_{k=0}^{\infty}(k+\nu)\frac{(2\nu)_k}{k!}\,I_{k+\nu}(w)\,I_{k+\nu}(z)=\frac{2^{-\nu}}{\Gamma(\nu)}\left(\frac{wz}{w+z}\right)^{\nu}I_\nu(w+z).$$

20. $$\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{1}{2}-\nu\right)_k}{\left(\frac{3}{2}+\nu\right)_k}\,I_{\nu+k}(z)\,I_{\nu-k-1}(z)=\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{2\Gamma(\nu+1)}\,z^{-1/2}I_{2\nu-1/2}(2z).$$

21. $$\sum_{k=0}^{\infty}I_{k+\nu}(z)\,I_{\nu-k}(z)\,I_{2k}(2z)$$
$$=\frac{1}{2}\,I_\nu^2(z)\,I_0(2z)+\frac{\left(\frac{z}{2}\right)^{2\nu}}{2\Gamma^2(\nu+1)}\,{}_3F_4\left(\begin{matrix}\nu+\frac{1}{4},\ \nu+\frac{1}{2},\ \nu+\frac{3}{4};\ 4z^2\\ \nu+1,\ 2\nu+\frac{1}{2},\ 2\nu+1,\ \frac{1}{2}\end{matrix}\right).$$

22. $$\sum_{k=0}^{\infty}I_{k+\nu}(z)\,I_{\nu-k-1}(z)\,I_{2k+1}(2z)$$
$$=\frac{2\left(\frac{z}{2}\right)^{2\nu}}{\Gamma(\nu)\Gamma(\nu+1)}\,{}_3F_4\left(\begin{matrix}\nu+\frac{1}{4},\ \nu+\frac{1}{2},\ \nu+\frac{3}{4};\ 4z^2\\ \nu+1,\ 2\nu,\ 2\nu+\frac{1}{2},\ \frac{3}{2}\end{matrix}\right).$$

23. $$\sum_{k=0}^{\infty}I_k^2(z)\,I_{k+1}^2(z)=\frac{z^2}{4}\,{}_2F_3\left(\begin{matrix}\frac{3}{2},\ \frac{3}{2};\ 4z^2\\ 2,\ 2,\ 3\end{matrix}\right).$$

24. $$\sum_{k=0}^{\infty}I_k^2(w)\,I_{nk}^2(z)=\frac{1}{2}\,I_0^2(w)\,I_0^2(z)+\frac{1}{\pi}\int_0^{\pi/2}I_0(2z\sin x)\,I_0(2w\sin(nx))\,dx.$$

25. $$\sum_{k=1}^{\infty}(-1)^kJ_0(kz)\,I_k(w_1)\,I_k(w_2)$$
$$=-\frac{1}{2}\,I_0(w_1)\,I_0(w_2)+\frac{1}{\pi}\int_0^z(z^2-x^2)^{-1/2}I_0\left(\sqrt{w_1^2+w_2^2-2w_1w_2\cos x}\right)dx.$$

26. $$\sum_{k=0}^{\infty}(-1)^kJ_{nk}^2(w)\,I_k^2(z)=\frac{1}{2}\,J_0^2(w)\,I_0^2(z)$$
$$+\frac{1}{\pi}\int_0^{\pi/2}J_0(2w\sin x)\,I_0(2z\sin(nx))\,dx.$$

27. $$\sum_{k=1}^{\infty} J_k(x_1)\,Y_k(x_2)\,I_k(x_3)\,K_k(x_4) = -\frac{1}{2}\,J_0(x_1)\,Y_0(x_2)\,I_0(x_3)\,K_0(x_4)$$
$$+\frac{1}{2\pi}\int_0^{\pi} Y_0\left(\sqrt{x_1^2+x_2^2-2x_1x_2\cos x}\right) K_0\left(\sqrt{x_3^2+x_4^2-2x_3x_4\cos x}\right) dx$$
$[x_1 < x_3;\ x_2 < x_4].$

6.9.4. Series containing $I_{nk+\mu}((nk+\nu)z)$

Notation: $\Delta(z) = \left|\dfrac{z}{1+\sqrt{1+z^2}}\,e^{\sqrt{1+z^2}}\right|.$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\,I_{2k}(kz) = -\frac{z^2}{8} \qquad [\Delta(z/2) < 1].$$

2. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^6}\,I_{2k+1}((2k+1)z) = \frac{z^5}{450} + \frac{5z^3}{81} + \frac{z}{2} \qquad [\Delta(z) < 1].$$

3. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+a^2}\,I_{2k}(kz) = \frac{2}{a^2z^2}\left[1+\frac{z^2}{4} - {}_1F_2\left(\begin{matrix}1;\ -\frac{a^2z^2}{4}\\ -ia,\ ia\end{matrix}\right)\right] \qquad [\Delta(z/2) < 1].$$

6.9.5. Series containing products of $I_{nk+\nu}((nk+\nu)z)$

Notation: $\Delta(z) = \left|\dfrac{z}{1+\sqrt{1+z^2}}\,e^{\sqrt{1+z^2}}\right|.$

1. $$\sum_{k=1}^{\infty} (-1)^k k^2\,I_k^2(kz) = \frac{z^2(z^2-4)}{16(z^2+1)^{7/2}} \qquad [\Delta(z) < 1].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\,I_k^2(kz) = -\frac{z^2}{4} \qquad [\Delta(z) < 1].$$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}\,I_k^2(kz) = -\frac{3z^4}{64} - \frac{z^2}{4} \qquad [\Delta(z) < 1].$$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^6}\,I_k^2(kz) = -\frac{5z^6}{1152} - \frac{15z^4}{256} - \frac{z^2}{4} \qquad [\Delta(z) < 1].$$

5. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\,I_{k+1/2}^2((2k+1)z) = \frac{2z}{\pi} \qquad [\Delta(z) < 1].$$

6. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^4}\,I_{k+1/2}^2((2k+1)z) = \frac{2z}{\pi}\left(\frac{8z^2}{27}+1\right) \qquad [\Delta(z) < 1].$$

7. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^6} I_{k+1/2}^2((2k+1)z) = \frac{2z}{\pi}\left(\frac{128z^4}{3375} + \frac{80z^3}{243} + 1\right) \qquad [\Delta(z) < 1].$$

8. $$\sum_{k=1}^{\infty} (-1)^k I_{k-1/2}(kz)\, I_{k+1/2}(kz) = \frac{1}{\pi z\sqrt{z^2+1}} \ln\left(z + \sqrt{z^2+1}\right) - \frac{1}{\pi} \qquad [\Delta(z) < 1].$$

9. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - a^2} I_{k-1/2}(kz)\, I_{k+1/2}(kz) = \frac{1}{\pi a^2} - \frac{1}{\pi a^2}\, {}_2F_3\left(\begin{matrix} 1,\, 1;\ a^2 z^2 \\ 1+a,\, 1-a,\, \frac{3}{2} \end{matrix}\right) \qquad [\Delta(z) < 1].$$

6.10. The Struve Functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

6.10.1. Series containing $\mathbf{H}_{k+\nu}(z)$ and $\mathbf{L}_{k+\nu}(z)$

1. $$\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \mathbf{H}_k(z) = \frac{2}{\pi} \operatorname{shi}(z).$$

2. $$\sum_{k=0}^{\infty} \frac{\left(-\frac{z}{2}\right)^k}{k!} \mathbf{L}_k(z) = \frac{2}{\pi} \operatorname{Si}(z).$$

3. $$\sum_{k=0}^{\infty} \frac{\left(\pm\frac{z}{2}\right)^k}{\left(\frac{3}{2}\right)_k} \left\{\begin{matrix} \mathbf{H}_{k+\nu}(z) \\ \mathbf{L}_{k+\nu}(z) \end{matrix}\right\} = \frac{z^{\nu+1}}{2^\nu \sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\, {}_2F_5\left(\begin{matrix} \frac{1}{2},\, 1;\ \frac{z^4}{256} \\ \frac{2\nu+3}{4},\, \frac{2\nu+5}{4},\, \frac{3}{4},\, \frac{5}{4},\, \frac{3}{2} \end{matrix}\right).$$

4. $$\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \mathbf{H}_{k+1/2}(z) = \frac{z^{3/2}}{\sqrt{2\pi}} \left\{2I_0(z) - \frac{2}{z} I_1(z) + \pi\left[I_0(z)\,\mathbf{L}_1(z) - I_1(z)\,\mathbf{L}_0(z)\right]\right\}.$$

5. $$\sum_{k=0}^{\infty} \frac{\left(-\frac{z}{2}\right)^k}{k!} \mathbf{L}_{k+1/2}(z) = \frac{z^{3/2}}{\sqrt{2\pi}} \left\{2J_0(z) - \frac{2}{z} J_1(z) + \pi\left[J_1(z)\,\mathbf{H}_0(z) - J_0(z)\,\mathbf{H}_1(z)\right]\right\}.$$

6. $$\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{(k+1)!} \mathbf{H}_{k+1/2}(z) = -\sqrt{\frac{8}{\pi z^3}} \sin z + \sqrt{\frac{8}{\pi z}} \left\{I_0(z) - \frac{\pi}{2}\left[I_1(z)\,\mathbf{L}_0(z) - I_0(z)\,\mathbf{L}_1(z)\right]\right\}.$$

7. $$\sum_{k=0}^{\infty} \frac{\left(-\frac{z}{2}\right)^k}{(k+1)!} \mathbf{L}_{k+1/2}(z) = \sqrt{\frac{8}{\pi z^3}} \sinh z - \sqrt{\frac{8}{\pi z}} \left\{ J_0(z) + \frac{\pi}{2} \left[J_1(z) \mathbf{H}_0(z) - J_0(z) \mathbf{H}_1(z) \right] \right\}.$$

6.10.2. Series containing $\mathbf{H}_\nu(\varphi(k)x)$

1. $$\sum_{k=1}^{\infty} (-1)^k \frac{k^{-\nu-1}}{k^2 a^2 + b^2} \mathbf{H}_\nu(kx) = \frac{\pi a^\nu}{2b^{\nu+2}} \operatorname{csch}\left(\frac{b\pi}{a}\right) \mathbf{L}_\nu\left(\frac{bx}{a}\right) - \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\sqrt{\pi}\, b^2 \Gamma\left(\nu+\frac{3}{2}\right)} \quad [-\pi \le x \le \pi;\ \operatorname{Re}\nu > -7/2].$$

2. $$\sum_{k=1}^{\infty} (-1)^k \frac{k^{-\nu-1}}{k^2 a^2 - b^2} \mathbf{H}_\nu(kx) = \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\sqrt{\pi}\, b^2 \Gamma\left(\nu+\frac{3}{2}\right)} - \frac{\pi a^\nu}{2b^{\nu+2}} \csc\left(\frac{b\pi}{a}\right) \mathbf{H}_\nu\left(\frac{bx}{a}\right) \quad [-\pi \le x \le \pi;\ \operatorname{Re}\nu > -7/2].$$

3. $$\sum_{k=2}^{\infty} (-1)^k \frac{k^{-\nu-1}}{k^2 - 1} \mathbf{H}_\nu(kx) = \frac{(x/2)^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} - \left(\nu+\frac{3}{4}\right) \mathbf{H}_\nu(x) + \frac{x}{2} \mathbf{H}_{\nu-1}(x) \quad [-\pi \le x \le \pi;\ \operatorname{Re}\nu > -7/2].$$

4. $$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^{-\nu}}{(2k-1)(2k+3)} \mathbf{H}_\nu((2k+1)x) = -\frac{\pi}{2^{\nu+3}} \mathbf{H}_\nu(2x) \quad [-\pi/2 \le x \le \pi/2;\ \operatorname{Re}\nu > -5/2].$$

5. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\nu+2}} \mathbf{H}_\nu((2k+1)x) = \frac{2^{-\nu-2}\sqrt{\pi}}{\Gamma\left(\nu+\frac{3}{2}\right)} x^{\nu+1} \quad [-\pi/2 < x < \pi/2;\ \operatorname{Re}\nu > -7/2].$$

6. $$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^{-\nu}}{k(k+1)} \mathbf{H}_\nu((2k+1)x) = (4\nu+1) \mathbf{H}_\nu(x) - 2z\, \mathbf{H}_{\nu-1}(x) \quad [-\pi/2 \le x \le \pi/2;\ \operatorname{Re}\nu > -5/2].$$

7. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{\left(k^2+a^2\right)^{(\nu+1)/2}} \mathbf{H}_\nu\left(\sqrt{k^2+a^2}\, x\right) = -\frac{a^{-\nu-1}}{2} \mathbf{H}_\nu(ax) \quad [-\pi \le x \le \pi;\ \operatorname{Re}\nu > -3/2].$$

8. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\left(k^2+a^2\right)^{(\nu+1)/2}} \mathbf{H}_\nu\left(\sqrt{k^2+a^2}\,x\right) = \frac{a^{-\nu-3}}{12}\left\{\left[3(2\nu+1)-\pi^2 a^2\right]\mathbf{H}_\nu(ax) - 3ax\,\mathbf{H}_{\nu-1}(ax)\right\}$$
$[-\pi \le x \le \pi;\ \operatorname{Re}\nu > -7/2]$.

6.10.3. Series containing $\mathbf{H}_\nu(kx)$ and trigonometric functions

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu+2}} \sin(kx)\,\mathbf{H}_\nu(ky) = -\frac{x}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{y}{2}\right)^{\nu+1}$$
$[x,\, y > 0;\ x+y < \pi;\ \operatorname{Re}\nu > -3/2]$.

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\nu}} \sin(kx)\,\mathbf{H}_\nu(ky) = \frac{(-1)^m x\left(\frac{y}{2}\right)^{2m+\nu-1}}{2\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\nu+\frac{1}{2}\right)} - \sum_{k=0}^{m-2} \frac{(-1)^k\left(\frac{y}{2}\right)^{2k+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} \sum_{j=0}^{m-k-1} \frac{(-1)^j}{(2j+1)!} x^{2j+1} \times \left(1-2^{2j+2k-2m+3}\right)\zeta(2m-2j-2k-2)$$
$[-\pi < x < \pi;\ |x|-\pi < y < \pi-|x|;\ \operatorname{Re}\nu > -2m-1/2]$.

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu+1}} \cos(kx)\,\mathbf{H}_\nu(ky) = -\frac{\left(\frac{y}{2}\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}$$
$[x,\, y > 0;\ x+y < \pi;\ \operatorname{Re}\nu > -3/2]$.

4. $$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2m+\nu-1}} \mathbf{H}_\nu(ky) = \frac{\left(\frac{y}{2}\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left\{(-1)^m \frac{\left(\frac{y}{2}\right)^{2m-2}}{\left(\frac{3}{2}\right)_{m-1}\left(\nu+\frac{3}{2}\right)_{m-1}} + \sum_{k=0}^{m-2} \frac{\left(\frac{y}{2}\right)^{2k}}{\left(\frac{3}{2}\right)_k\left(\nu+\frac{3}{2}\right)_k}\right.$$
$$\left.\times\left[(-1)^{m-1}\frac{\pi x^{2m-2k-3}}{(2m-2k-3)!} + 2(-1)^k \sum_{j=0}^{m-k-1} \frac{\left(-x^2\right)^j}{(2j)!}\zeta(2m-2j-2k-2)\right]\right\}$$
$[0 < y < \pi;\ y < x < 2\pi - x;\ \operatorname{Re}\nu > 3/2-2m]$.

5. $$\sum_{k=1}^{\infty} (-1)^k \frac{\cos(kx)}{k^{2m+\nu-1}} \mathbf{H}_\nu(ky) = \frac{\left(\frac{y}{2}\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} \times\left\{\frac{(-1)^m\left(\frac{y}{2}\right)^{2m+v-1}}{2\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\nu+\frac{1}{2}\right)} - \sqrt{\pi}\sum_{k=0}^{m-2}\frac{(-1)^k\left(\frac{y}{2}\right)^{2k+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}\right.$$

$$\times \sum_{j=0}^{m-k-1} \frac{(-x^2)^j}{(2j)!} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2) \Bigg\}$$

$[-\pi < y < \pi;\ |y| - \pi < x < \pi - |y|;\ \operatorname{Re}\nu > 3/2 - 2m].$

6.10.4. Series containing $\mathbf{H}_\nu(kx)$ and $\operatorname{Si}(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\nu}} \operatorname{Si}(kx)\, \mathbf{H}_\nu(ky)$$

$$= \frac{(-1)^m x \left(\frac{y}{2}\right)^{2m+\nu-1}}{2\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\nu+\frac{1}{2}\right)} - \sum_{k=0}^{m-2} \frac{(-1)^k \left(\frac{y}{2}\right)^{2k+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

$$\times \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j+1}}{(2j+1)!\,(2j+1)} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$

$[m \ge 1;\ -\pi < y < \pi;\ |y| - \pi < x < \pi - |y|].$

6.10.5. Series containing $\mathbf{H}_\nu(kx)$, $S(kx)$ and $C(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu+5/2}} S(kx)\, \mathbf{H}_\nu(ky) = -\frac{2^{-\nu-1/2} x^{3/2} y^{\nu+1}}{3\pi\Gamma\left(\nu+\frac{3}{2}\right)}$$

$[x,\ y > 0;\ x + y < \pi;\ \operatorname{Re}\nu > -3].$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\nu+1/2}} S(kx)\, \mathbf{H}_\nu(ky)$$

$$= \frac{(-1)^m x^{3/2} \left(\frac{y}{2}\right)^{2m+\nu-1}}{3\sqrt{2\pi}\,\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\nu+\frac{1}{2}\right)} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{m-2} \frac{(-1)^k \left(\frac{y}{2}\right)^{2k+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

$$\times \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j+3/2}}{(2j+1)!\,(4j+3)} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$

$[m \ge 1;\ -\pi < y < \pi;\ |y| - \pi < x < \pi - |y|].$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\nu+3/2}} C(kx)\, \mathbf{H}_\nu(ky) = -\frac{2^{-\nu-1/2} x^{1/2} y^{\nu+1}}{\pi\Gamma\left(\nu+\frac{3}{2}\right)}$$

$[x,\ y > 0;\ x + y < \pi;\ \operatorname{Re}\nu > -2].$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\nu-1/2}} C(kx)\, \mathbf{H}_\nu(ky)$$

$$= \frac{(-1)^m x^{1/2} \left(\frac{y}{2}\right)^{2m+\nu-1}}{\sqrt{2\pi}\,\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\nu+\frac{1}{2}\right)} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{m-2} \frac{(-1)^k \left(\frac{y}{2}\right)^{2k+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

$$\times \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j+1/2}}{(2j)!\,(4j+1)} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$

$$[m \geq 1;\ -\pi < y < \pi;\ |y| - \pi < x < \pi - |y|].$$

6.10.6. Series containing $\mathbf{H}_\nu(\varphi(k)x)$ and $J_\mu(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\mu+\nu+1}} J_\mu(kx)\, \mathbf{H}_\nu(ky) = -\frac{\left(\frac{x}{2}\right)^\mu \left(\frac{y}{2}\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\mu+1)\Gamma\left(\nu+\frac{3}{2}\right)}$$

$$[x,\, y > 0;\ x + y < \pi;\ \operatorname{Re}(\mu+\nu) > -2].$$

2. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^\nu} J_k(w)\, J_{k+1}(w)\, \mathbf{H}_\nu((2k+1)z)$$

$$= \frac{(2z)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^z (z^2 - x^2)^{\nu-1/2} J_1(2w\sin x)\, dx.$$

3. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^\nu} J_{k+1/2}^2(w)\, \mathbf{H}_\nu((2k+1)z)$$

$$= \frac{(2z)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^z (z^2 - x^2)^{\nu-1/2}\, \mathbf{H}_0(2w\sin x)\, dx.$$

4. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\mu+\nu-1}} J_\mu(kx)\, \mathbf{H}_\nu(ky)$$

$$= \frac{(-1)^m \left(\frac{x}{2}\right)^\mu \left(\frac{y}{2}\right)^{2m+\nu-1}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma(\mu+1)\Gamma\left(m+\nu+\frac{1}{2}\right)} - \sum_{k=0}^{m-2} \frac{(-1)^k \left(\frac{y}{2}\right)^{2k+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

$$\times \sum_{j=0}^{m-k-1} (-1)^j \frac{\left(\frac{x}{2}\right)^{2j+\mu}}{j!\,\Gamma(j+\mu+1)} \left(1 - 2^{2j+2k-2m+3}\right) \zeta(2m-2j-2k-2)$$

$$[m \geq 1;\ -\pi < y < \pi;\ |y| - \pi < x < \pi - |y|].$$

6.10.7. Series containing product of $\mathbf{H}_\nu(kx)$

1. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2m+\mu+\nu}} \mathbf{H}_\mu(kx)\, \mathbf{H}_\nu(ky)$$

$$= \frac{(-1)^m \pi^{-1/2} \left(\frac{x}{2}\right)^{2m+\mu-1} \left(\frac{y}{2}\right)^{\nu+1}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(m+\mu+\frac{1}{2}\right)} - \sum_{k=0}^{m-2} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\mu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\mu+\frac{3}{2}\right)}$$

$$\times \sum_{j=0}^{m-k-1} (-1)^j \frac{\left(\frac{y}{2}\right)^{2j+\nu+1}}{\Gamma\left(j+\frac{3}{2}\right)\Gamma\left(j+\nu+\frac{3}{2}\right)} \left(1-2^{2j+2k-2m+3}\right)$$
$$\times \zeta(2m-2j-2k-2) \quad [m \geq 1;\ -\pi < x < \pi;\ |x|-\pi < y < \pi-|x|].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{n\nu+\nu}} \prod_{i=1}^{n} \mathrm{H}_\nu(kx_i) = -\frac{2^{-n\nu-1}}{\pi^{n/2}\Gamma^n\left(\nu+\frac{3}{2}\right)} \prod_{i=1}^{n} x_i^{\nu+1}$$
$$\left[x_i > 0;\ \sum_{i=1}^{n} x_i < \pi;\ \nu > -1\right].$$

6.11. The Legendre Polynomials $P_n(z)$

6.11.1. Series containing $P_{nk+m}(z)$

1. $$\sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)} P_k(x) = \ln\frac{2}{1-x} - 1 \qquad [-1 \leq x < 1;\ [47]].$$

2. $$\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2} P_k(x) = 1 - \frac{\pi^2}{6} + \mathrm{Li}_2\left(\frac{1+x}{2}\right) \qquad [-1 \leq x \leq 1;\ [47]].$$

3. $$\sum_{k=1}^{\infty} \frac{2k+1}{k^3(k+1)^3} P_k(x) = \frac{\pi^2}{6} - 2 + 2\zeta(3) + \ln\frac{1-x}{2}\,\mathrm{Li}_2\left(\frac{1-x}{2}\right)$$
$$-\mathrm{Li}_2\left(\frac{1+x}{2}\right) - 2\,\mathrm{Li}_3\left(\frac{1-x}{2}\right) \qquad [-1 \leq x < 1;\ [47],\ (15)].$$

6.11.2. Series containing $P_{nk+m}(z)$ and Bessel functions

1. $$\sum_{k=0}^{\infty} (2k+1)\, J_{2k+1}(w) P_k(z) = \frac{w}{2} J_0\left(w\sqrt{\frac{1-z}{2}}\right).$$

2. $$\sum_{k=0}^{\infty} (-1)^k (4k+1)\, J_{2k+1/2}(w) P_{2k}(z) = \sqrt{\frac{2w}{\pi}}\cos(wz).$$

3. $$\sum_{k=0}^{\infty} (-1)^k (4k+3)\, J_{2k+3/2}(w) P_{2k+1}(z) = \sqrt{\frac{2w}{\pi}}\sin(wz).$$

4. $$\sum_{k=0}^{\infty} (4k+1) \frac{\left(\frac{1}{2}\right)_k}{k!} J_{2k+1/2}(w) P_{2k}(z) = \sqrt{\frac{2w}{\pi}}\, J_0\left(w\sqrt{1-z^2}\right).$$

5. $$\sum_{k=0}^{\infty} (4k+3) \frac{\left(\frac{3}{2}\right)_k}{k!} J_{2k+3/2}(w) P_{2k+1}(z) = z\sqrt{\frac{2w^3}{\pi}}\, J_0\left(w\sqrt{1-z^2}\right).$$

6. $\displaystyle\sum_{k=0}^{\infty}(2k+1)\,J_{k+1/2}^{2}(w)P_k(z)=\frac{\sqrt{2}}{\pi\sqrt{1-z}}\sin\left(w\sqrt{2-2z}\right).$

7. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(4k+1)\,J_{k+\nu}(w)\,J_{k-\nu+1/2}(w)P_{2k}(z)$

$$=\frac{1}{\Gamma(\nu+1)\Gamma\left(\frac{3}{2}-\nu\right)}\sqrt{\frac{w}{2}}\;{}_2F_3\left(\begin{matrix}\frac{3}{4},\,\frac{5}{4};\ -w^2z^2\\ \frac{1}{2},\,\frac{3}{2}-\nu,\,1+\nu\end{matrix}\right).$$

8. $\displaystyle\sum_{k=0}^{\infty}(4k+1)\frac{\left(\frac{1}{2}\right)_k}{k!}\,J_{k+\nu}(w)\,J_{k-\nu+1/2}(w)P_{2k}(z)$

$$=\frac{1}{\Gamma(\nu+1)\Gamma\left(\frac{3}{2}-\nu\right)}\sqrt{\frac{w}{2}}\;{}_2F_3\left(\begin{matrix}\frac{3}{4},\,\frac{5}{4};\ w^2z^2-w^2\\ 1,\,\frac{3}{2}-\nu,\,1+\nu\end{matrix}\right).$$

9. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(4k+1)\,J_{k+1/4}^{2}(w)P_{2k}(z)$

$$=\frac{1}{\sqrt{z}}J_{1/4}(wz)\left[J_{1/4}(wz)-2wzJ_{5/4}(wz)\right].$$

10. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(4k+3)\,J_{k+3/4}^{2}(w)P_{2k+1}(z)$

$$=\frac{1}{\sqrt{z}}J_{3/4}(wz)\left[3J_{3/4}(wz)-2wz\,J_{7/4}(wz)\right].$$

11. $\displaystyle\sum_{k=0}^{\infty}(2k+1)\,I_{k+1/2}(w)P_k(z)=\sqrt{\frac{2w}{\pi}}\,e^{wz}.$

12. $\displaystyle\sum_{k=0}^{\infty}(4k+1)\,I_{2k+1/2}(w)P_{2k}(z)=\sqrt{\frac{2w}{\pi}}\,\cosh(wz).$

13. $\displaystyle\sum_{k=0}^{\infty}(4k+3)\,I_{2k+3/2}(w)P_{2k+1}(z)=\sqrt{\frac{2w}{\pi}}\,\sinh(wz).$

14. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(4k+1)\,\frac{\left(\frac{1}{2}\right)_k}{k!}\,I_{k+1/4}(w)P_{2k}(z)$

$$=\frac{2^{5/4}}{\pi}\Gamma\left(\frac{3}{4}\right)w^{1/4}e^{-w}\;{}_1F_1\left(\begin{matrix}\frac{3}{4};\ 2w-2wz^2\\ 1\end{matrix}\right).$$

15. $$\sum_{k=0}^{\infty}(-1)^k(4k+3)\,\frac{\left(\frac{3}{2}\right)_k}{k!}\,I_{k+3/4}(w)P_{2k+1}(z) = \frac{2^{3/4}}{\pi}\Gamma\left(\frac{1}{4}\right)w^{3/4}ze^{-w}\,{}_1F_1\left(\begin{matrix}\frac{5}{4};\ 2w-2wz^2\\ 1\end{matrix}\right).$$

6.11.3. Series containing products of $P_{nk+m}(z)$

1. $$\sum_{k=1}^{\infty}\frac{2k+1}{k(k+1)}P_k(x)\,P_k(y) = \ln\frac{4}{(1-x)(1+y)} - 1 \qquad [-1 \le x \le y < 1].$$

2. $$\sum_{k=1}^{\infty}\frac{2k+1}{k^2(k+1)^2}P_k(x)\,P_k(y) = 1 - \ln\frac{1+y}{2}\ln\frac{(1-x)(1-y)}{4} + \mathrm{Li}_2\left(\frac{1+x}{2}\right) - \mathrm{Li}_2\left(\frac{1+y}{2}\right) \qquad [-1 \le x \le y < 1].$$

3. $$\sum_{k=0}^{\infty}(2k+1)\,J_{2k+1}(w)\,[P_k(z)]^2 = \frac{w}{2}J_0^2\left(\frac{w}{2}\sqrt{1-z^2}\right).$$

4. $$\sum_{k=0}^{\infty}(2k+1)\,J_{k+\nu}(w)\,J_{k-\nu+1}(w)\,[P_k(z)]^2 = \frac{w\sin(\nu\pi)}{2\nu(1-\nu)\pi}\,{}_2F_3\left(\begin{matrix}\frac{1}{2},\ \frac{3}{2};\ w^2z^2-w^2\\ 1,\ \nu+1,\ 2-\nu\end{matrix}\right).$$

5. $$\sum_{k=0}^{\infty}(2k+1)\,J_{k+1/2}^2(w)\,[P_k(z)]^2 = \frac{2w}{\pi}I_0\left(2w\sqrt{z^2-1}\right) - wI_1\left(2w\sqrt{z^2-1}\right)\mathbf{L}_0\left(2w\sqrt{z^2-1}\right) + wI_0\left(2w\sqrt{z^2-1}\right)\mathbf{L}_1\left(2w\sqrt{z^2-1}\right).$$

6.11.4. Series containing $P_{nk+m}(\varphi(k,z))$

1. $$\sum_{k=1}^{\infty}\frac{k^n}{\sqrt{k+z}}t^kP_{2n}\left(\sqrt{1+\frac{z}{k}}\right) = \sum_{k=0}^{n}\binom{n}{k}\frac{\left(\frac{1}{2}-n\right)_k}{k!}(-z)^k\Phi\left(t,\,k-n+\frac{1}{2},\,z\right) - \frac{\left(n+\frac{1}{2}\right)_n}{n!}z^{n-1/2} \qquad [|t|<1].$$

2. $$\sum_{k=0}^{\infty}\frac{(k+1)^{k/2-1}}{k!}t^k(k+z+1)^{k/2}P_k\left(\frac{2k+z+2}{2\sqrt{(k+1)(k+z+1)}}\right) = e^{-w-wz/2}\left[I_0\left(\frac{wz}{2}\right)+I_1\left(\frac{wz}{2}\right)\right] \qquad \left[t=-we^w;\ |we^{w+1}|<1\right].$$

6.12. The Chebyshev Polynomials $T_k(z)$ and $U_k(z)$

6.12.1. Series containing $T_{nk+m}(\varphi(k, z))$

1. $$\sum_{k=0}^{\infty} \frac{t^k}{k!} T_k(z) = e^{tz} \cos\left(t\sqrt{1-z^2}\right)$$

2. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)\left[(2k+1)^2-a^2\right]^{n+1/2}} T_{2n+1}\left(\frac{2k+1}{\sqrt{(2k+1)^2-a^2}}\right) = \frac{a^{-2n-1}}{4} \sum_{k=0}^{2n} \frac{\left(\frac{a}{2}\right)^k}{k!} \left[(-1)^k \psi^{(k)}\left(\frac{1+a}{2}\right) - \psi^{(k)}\left(\frac{1-a}{2}\right)\right].$$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{\left(k^2a^2-b^2\right)^{n+1/2}} \sin kx\, T_{2n+1}\left(\frac{ka}{\sqrt{k^2a^2-b^2}}\right) = -\frac{\pi}{(2n)!2a} \mathrm{D}_b^{2n}\left[\sin\frac{bx}{a} \csc\frac{b\pi}{a}\right] \quad [-\pi < x < \pi].$$

6.12.2. Series containing $T_{nk+m}(z)$ and Bessel functions

1. $$\sum_{k=0}^{\infty} (-1)^k J_{2k}(w) T_{2k}(z) = \frac{1}{2}\left[\cos(wz) + J_0(w)\right].$$

2. $$\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(w) T_{2k+1}(z) = \frac{1}{2} \sin(wz).$$

3. $$\sum_{k=0}^{\infty} (2k+1) J_{2k+1}(w) T_{2k+1}(z) = \frac{wz}{2} \cos\left(w\sqrt{1-z^2}\right).$$

4. $$\sum_{k=0}^{\infty} (-1)^k J_k^2(w) T_{2k}(z) = \frac{1}{2}\left[J_0(2wz) + J_0^2(wz)\right].$$

5. $$\sum_{k=0}^{\infty} (-1)^k J_{k+1/2}^2(w) T_{2k+1}(z) = \frac{1}{2} \mathbf{H}_0(2wz).$$

6. $$\sum_{k=0}^{\infty} (2k+1) J_{k+1/2}^2(w) T_{2k+1}(z) = wz\, \mathbf{H}_{-1}\left(2w\sqrt{1-z^2}\right).$$

7. $$\sum_{k=0}^{\infty} (-1)^k J_k(w) J_{k+1}(w) T_{2k+1}(z) = \frac{1}{2} J_1(2wz).$$

8. $$\sum_{k=0}^{\infty} I_k(w) T_k(z) = \frac{1}{2}\left[e^{wz} + I_0(w)\right].$$

9. $\sum_{k=0}^{\infty} I_{2k}(w)T_{2k}(z) = \frac{1}{2}\,[\cosh(wz) + I_0(w)].$

10. $\sum_{k=0}^{\infty} I_{2k+1}(w)T_{2k+1}(z) = \frac{1}{2}\sinh(wz).$

11. $\sum_{k=0}^{\infty} I_{k+1/2}^2(w)T_{2k+1}(z) = \frac{1}{2}\,\mathbf{L}_0(2wz).$

12. $\sum_{k=0}^{\infty} I_{k/2}^2(w)T_k(z) = \frac{1}{2}\,[I_0(2wz) + I_0^2(w) + \mathbf{L}_0(2wz)].$

13. $\sum_{k=0}^{\infty} I_k^2(w)T_{2k}(z) = \frac{1}{2}\,[I_0(2wz) + I_0^2(w)].$

6.12.3. Series containing $U_{nk+m}(\varphi(k, z))$

1. $\sum_{k=0}^{\infty} \frac{t^k}{k!} U_k(z) = e^{tz}\left[\cos\left(t\sqrt{1-z^2}\right) + \frac{z}{\sqrt{1-z^2}}\sin\left(t\sqrt{1-z^2}\right)\right].$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{\left(k^2a^2-b^2\right)^{n+3/2}} \sin kx\, U_{2n+1}\left(\frac{ka}{\sqrt{k^2a^2-b^2}}\right) = -\frac{\pi}{(2n+1)!2ab}\,\mathrm{D}_b^{2n+1}\left[\sin\frac{bx}{a}\csc\frac{b\pi}{a}\right] \quad [-\pi < x < \pi].$$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k\left(k^2a^2-b^2\right)^{n+3/2}} \cos kx\, U_{2n+1}\left(\frac{ka}{\sqrt{k^2a^2-b^2}}\right)$$
$$= -\frac{\pi}{(2n+1)!2b}\,\mathrm{D}_b^{2n+1}\left[\frac{1}{b}\cos\frac{bx}{a}\csc\frac{b\pi}{a}\right] - (n+1)ab^{-2n-4} \quad [-\pi \le x \le \pi].$$

4. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)\left[(2k+1)^2-a^2\right]^{n+3/2}} U_{2n+1}\left(\frac{2k+1}{\sqrt{(2k+1)^2-a^2}}\right)$$
$$= -\frac{a^{-2n-3}}{4}\sum_{k=0}^{2n+1}\frac{\left(\frac{a}{2}\right)^k}{k!}\left[(-1)^k\psi^{(k)}\left(\frac{1+a}{2}\right) - \psi^{(k)}\left(\frac{1-a}{2}\right)\right].$$

6.12.4. Series containing $U_{nk+m}(z)$ and Bessel functions

1. $\sum_{k=0}^{\infty} (k+1)J_{2k+2}(w)U_k(z) = \frac{w}{8}\sqrt{\frac{2}{1-z}}\sin\left(w\sqrt{\frac{1-z}{2}}\right).$

2. $\sum_{k=0}^{\infty} J_{2k+1}(w)U_{2k}(z) = \frac{1}{2\sqrt{1-z^2}}\sin\left(w\sqrt{1-z^2}\right).$

3. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(2k+1)\,J_{2k+1}(w)U_{2k}(z) = \frac{w}{2}\cos(wz).$

4. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(k+1)J_{2k+2}(w)U_{2k+1}(z) = \frac{w}{4}\sin(wz).$

5. $\displaystyle\sum_{k=0}^{\infty}(k+1)J_{2k+2}(w)U_{2k+1}(z) = \frac{wz}{4\sqrt{1-z^2}}\sin\left(w\sqrt{1-z^2}\right).$

6. $$\sum_{k=0}^{\infty}(k+1)J_{k+\nu}(w)\,J_{k-\nu+2}(w)U_k(z) = \frac{w^2\sin(\nu\pi)}{4\pi\nu(\nu-1)(\nu-2)}\,{}_1F_2\left(\begin{matrix}2;\ \frac{w^2(z-1)}{2}\\ 3-\nu,\,1+\nu\end{matrix}\right).$$

7. $$\sum_{k=0}^{\infty}(k+1)J_k(w)\,J_{k+2}(w)U_k(z) = \frac{1}{2^{3/2}(1-z)}\left[\sqrt{2}\,J_2\left(w\sqrt{2-2z}\right) - w\sqrt{1-z}\,J_3\left(w\sqrt{2-2z}\right)\right].$$

8. $\displaystyle\sum_{k=0}^{\infty}J_{k+1/2}^2(w)U_{2k}(z) = \frac{1}{2\sqrt{1-z^2}}\,\mathbf{H}_0\left(2w\sqrt{1-z^2}\right).$

9. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(2k+1)\,J_{k+1/2}^2(w)U_{2k}(z) = w\,\mathbf{H}_{-1}(2wz).$

10. $$\sum_{k=0}^{\infty}(-1)^k(2k+1)\,J_{k+\nu}(w)\,J_{k-\nu+1}(w)U_{2k}(z) = \frac{w\sin(\nu\pi)}{2\pi\nu(\nu^2-1)(\nu-2)}$$
$$\times\left[(2+\nu-\nu^2)\,{}_1F_2\left(\begin{matrix}1;\ -w^2z^2\\ 2-\nu,\,1+\nu\end{matrix}\right) - 2w^2z^2\,{}_1F_2\left(\begin{matrix}2;\ -w^2z^2\\ 3-\nu,\,2+\nu\end{matrix}\right)\right].$$

11. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(2k+1)\,J_k(w)\,J_{k+1}(w)U_{2k}(z) = \frac{1}{2z}J_1(2wz) - w\,J_2(2wz).$

12. $\displaystyle\sum_{k=0}^{\infty}J_{k+\nu}(w)\,J_{k-\nu+1}(w)U_{2k}(z) = \frac{w\sin(\nu\pi)}{2\pi\nu(1-\nu)}\,{}_1F_2\left(\begin{matrix}1;\ w^2z^2-w^2\\ 2-\nu,\,1+\nu\end{matrix}\right).$

13. $$\sum_{k=0}^{\infty}J_{k-1/2}(w)\,J_{k+3/2}(w)U_{2k}(z) = \frac{1}{2w(1-z^2)}\left[w\sqrt{1-z^2}\,\mathbf{H}_0\left(2w\sqrt{1-z^2}\right) - \mathbf{H}_1\left(2w\sqrt{1-z^2}\right)\right].$$

14. $\displaystyle\sum_{k=0}^{\infty}J_k(w)J_{k+1}(w)U_{2k}(z) = \frac{1}{2\sqrt{1-z^2}}\,J_1\left(2w\sqrt{1-z^2}\right).$

15. $\sum_{k=0}^{\infty}(k+1)J_{k+1}^{2}(w)U_{2k+1}(z)=\frac{wz}{2\sqrt{1-z^{2}}}J_{1}\left(2w\sqrt{1-z^{2}}\right).$

16. $\sum_{k=0}^{\infty}(-1)^{k}(k+1)J_{k+1}^{2}(w)U_{2k+1}(z)=\frac{w}{2}J_{1}(2wz).$

17. $\sum_{k=0}^{\infty}(-1)^{k}(2k+1)\,J_{k+1/2}^{2}(w)U_{2k}(z)=w\,\mathbf{H}_{-1}(2wz).$

18. $\sum_{k=0}^{\infty}(-1)^{k}(k+1)J_{k+1}^{2}(w)U_{2k+1}(z)=\frac{w}{2}J_{1}(2wz).$

19. $\sum_{k=0}^{\infty}(2k+1)\,I_{k+1/2}(w)U_{2k}(z)=\sqrt{\frac{2w}{\pi}\,e^{-w}+2wze^{2wz^{2}-w}\operatorname{erf}\left(\sqrt{2w}\,z\right)}.$

20. $\sum_{k=0}^{\infty}(k+1)\,I_{k+1}(w)U_{k}(z)=\frac{w}{2}e^{wz}.$

21. $\sum_{k=0}^{\infty}(2k+1)\,I_{2k+1}(w)U_{2k}(z)=\frac{w}{2}\cosh(wz).$

22. $\sum_{k=0}^{\infty}(k+1)\,I_{2k+2}(w)U_{2k+1}(z)=\frac{w}{4}\sinh(wz).$

23. $\sum_{k=0}^{\infty}(-1)^{k}(k+1)\,I_{k+1}(w)U_{2k+1}(z)=wze^{w-2wz^{2}}.$

24. $\sum_{k=0}^{\infty}(k+1)I_{(k+1)/2}^{2}(w)U_{k}(z)=w[I_{1}(2wz)+\mathbf{L}_{-1}(2wz)].$

25. $\sum_{k=0}^{\infty}(2k+1)\,I_{k+1/2}^{2}(w)U_{2k}(z)=w\,\mathbf{L}_{-1}(2wz).$

26. $\sum_{k=0}^{\infty}(k+1)I_{k+1}^{2}(w)U_{2k+1}(z)=\frac{w}{2}I_{1}(2wz).$

6.13. Hermite Polynomials $H_n(z)$

6.13.1. Series containing $H_{nk+m}(z)$ and Bessel functions

1. $\sum_{k=0}^{\infty}\frac{\left(\mp\frac{w}{2}\right)^{k}}{(2k)!}\left\{\begin{matrix}J_{\nu+k}(w)\\ I_{\nu+k}(w)\end{matrix}\right\}H_{2k}(z)=\frac{\left(\frac{w}{2}\right)^{\nu}}{\Gamma(\nu+1)}\,{}_{0}F_{2}\left(\begin{matrix}\mp\frac{w^{2}z^{2}}{4}\\ \nu+1,\,\frac{1}{2}\end{matrix}\right).$

2. $$\sum_{k=0}^{\infty} \frac{\left(\mp\frac{w}{2}\right)^k}{(2k+1)!} \left\{ \begin{matrix} J_{\nu+k}(w) \\ I_{\nu+k}(w) \end{matrix} \right\} H_{2k+1}(z) = \frac{2\left(\frac{w}{2}\right)^{\nu} z}{\Gamma(\nu+1)} \, {}_0F_2\left(\begin{matrix} \mp\frac{w^2z^2}{4} \\ \nu+1,\ \frac{3}{2} \end{matrix} \right).$$

6.13.2. Series containing products of $H_{nk+m}(z)$

1. $$\sum_{k=0}^{\infty} \frac{t^k}{(2k)!} H_{2k}(w)\, H_{2k}(z)$$
$$= (1-4t)^{-1/2} \exp\left(\frac{4tw^2+4tz^2}{4t-1} \right) \cosh\left(\frac{4\sqrt{t}\,wz}{1-4t} \right) \quad [|t| < 1/4].$$

2. $$\sum_{k=0}^{\infty} \frac{t^k}{(2k+1)!} H_{2k+1}(w)\, H_{2k+1}(z)$$
$$= t^{-1/2}(1-4t)^{-1/2} \exp\left(\frac{4tw^2+4tz^2}{4t-1} \right) \sinh\left(\frac{4\sqrt{t}\,wz}{1-4t} \right) \quad [|t| < 1/4].$$

3. $$\sum_{k=0}^{\infty} \frac{t^k}{(k!)^2} \left[H_{2k}(z)\right]^2$$
$$= \frac{4}{\pi} e^{2z^2} \int_0^{\infty}\int_0^{\infty} e^{-x-y} \cos(2xz)\cos(2yz)\, I_0\left(8z\sqrt{t}\,xy\right) dx\,dy$$
$$[|t| < 1/16;\ [34]].$$

4. $$\sum_{k=0}^{\infty} \frac{t^k}{(k!)^2} \left[H_{2k+1}(z)\right]^2$$
$$= \frac{16}{\pi} e^{2z^2} \int_0^{\infty}\int_0^{\infty} xye^{-x^2-y^2} \sin(2xz)\sin(2yz)\, I_0\left(8z\sqrt{t}\,xy\right) dx\,dy$$
$$[|t| < 1/16;\ [34]].$$

6.13.3. Series containing $H_{nk+m}(\varphi(k, z))$

1. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(2k)!} t^k H_{2k}\left(\frac{z}{\sqrt{k+1}} \right)$$
$$= \frac{e^{-w}}{2wz^2} \left[2z\sqrt{w}\sinh(2z\sqrt{w}) - \cosh(2z\sqrt{w}) + 1\right]$$
$$\left[t = we^{w};\ |we^{w+1}| < 1\right].$$

2. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1/2}}{(2k+1)!} t^k H_{2k+1}\left(\frac{z}{\sqrt{k+1}} \right) = \frac{e^{-w}}{wz} \left[\cosh(2z\sqrt{w}) - 1\right]$$
$$\left[t = we^{w};\ |we^{w+1}| < 1\right].$$

6.13.4. Series containing $H_{nk+m}(\varphi(k, z))$ and special functions

1. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(k+1)^{-3/2}\gamma\left(k+\frac{1}{2},\,(k+1)w\right)H_{2k}\left(\frac{z}{\sqrt{k+1}}\right) = w^{-1/2}z^{-2}[1-\cos(2\sqrt{w}\,z)].$$

2. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(k+1)^{-2}\gamma\left(k+\frac{3}{2},\,(k+1)w\right)H_{2k+1}\left(\frac{z}{\sqrt{k+1}}\right) = 2w^{1/2}z^{-1}\left[1-\frac{\sin(2\sqrt{w}\,z)}{2\sqrt{w}\,z}\right].$$

3. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{(2k)!}\left(-\frac{w}{2}\right)^k J_{k+\nu}(\sqrt{k+1}\,w)H_{2k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{\left(\frac{w}{2}\right)^{\nu-2}z^{-2}}{2\Gamma(\nu)}\left[{}_0F_2\left(\begin{matrix}-\frac{w^2z^2}{4}\\ \nu,\,-\frac{1}{2}\end{matrix}\right)-1\right].$$

4. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu-1)/2}}{(2k+1)!}\left(-\frac{w}{2}\right)^k J_{k+\nu}(\sqrt{k+1}\,w)H_{2k+1}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{\left(\frac{w}{2}\right)^{\nu-2}z^{-1}}{\Gamma(\nu)}\left[1-{}_0F_2\left(\begin{matrix}-\frac{w^2z^2}{4}\\ \nu,\,\frac{1}{2}\end{matrix}\right)\right].$$

5. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{(2k)!}\left(\frac{w}{2}\right)^k I_{k+\nu}(\sqrt{k+1}\,w)H_{2k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{\left(\frac{w}{2}\right)^{\nu-2}z^{-2}}{2\Gamma(\nu)}\left[1-{}_0F_2\left(\begin{matrix}\frac{w^2z^2}{4}\\ \nu,\,-\frac{1}{2}\end{matrix}\right)\right].$$

6. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu-1)/2}}{(2k+1)!}\left(\frac{w}{2}\right)^k I_{k+\nu}(\sqrt{k+1}\,w)H_{2k+1}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{\left(\frac{w}{2}\right)^{\nu-2}z^{-1}}{\Gamma(\nu)}\left[{}_0F_2\left(\begin{matrix}\frac{w^2z^2}{4}\\ \nu,\,\frac{1}{2}\end{matrix}\right)-1\right].$$

6.13.5. Series containing products of $H_{nk+m}(\varphi(k, z))$

1. $$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2m-2}}H_{2n}(\sqrt{k}\,y)H_{2n}(i\sqrt{k}\,y) = (-1)^{n+1}\frac{[(2n)!]^2}{n!}\frac{2^{6m-7}\left(n+\frac{1}{2}\right)_{m-1}y^{4m-4}}{(n-m+1)!\,(4m-4)!}$$

$$+ \frac{[(2n)!]^2}{n!} \sum_{k=0}^{m-2} \frac{2^{6k} y^{4k}}{(n-k)!\,(4k)!} \left(n+\frac{1}{2}\right)_k$$

$$\times \left[(-1)^{m-1} \frac{\pi x^{2m-2k-3}}{2(2m-2k-3)!} + (-1)^k \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j}}{(2j)!} \zeta(2m-2j-2k-2)\right]$$

$$\left[m \geq 1;\ 0 < y < \sqrt{\pi};\ y^2 < x < 2\pi - y^2\right].$$

2. $$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2m-1}} H_{2n+1}(\sqrt{k}\,y) H_{2n+1}(i\sqrt{k}\,y)$$

$$= (-1)^m 4i \frac{[(2n+1)!]^2}{n!} \frac{2^{8m-8}\left(n+\frac{3}{2}\right)_{m-1} y^{4m-2}}{(n-m+1)!\,(4m-2)!}$$

$$+ 4i \frac{[(2n+1)!]^2}{n!} y^2 \sum_{k=0}^{m-2} \frac{2^{6k} y^{4k}}{(n-k)!\,(4k+2)!} \left(n+\frac{3}{2}\right)_k$$

$$\times \left[(-1)^{m-1} \frac{\pi x^{2m-2k-3}}{(2m-2k-3)!} + 2(-1)^k \sum_{j=0}^{m-k-1} (-1)^j \frac{x^{2j}}{(2j)!} \zeta(2m-2j-2k-2)\right]$$

$$\left[m \geq 1;\ 0 < y < \sqrt{\pi};\ y^2 < x < 2\pi - y^2\right].$$

6.14. The Laguerre Polynomials $L_n^\lambda(z)$

6.14.1. Series containing $L_{nk+m}^{\lambda \pm lk}(z)$

1. $$\sum_{k=0}^{\infty} \frac{z^k}{(\lambda+n+1)_k} L_n^{\lambda+k}(z) = \frac{\lambda+n}{n} L_{n-1}^{\lambda}(z) \qquad [n \geq 1].$$

2. $$\sum_{k=0}^{\infty} \frac{(ka+b)^m}{(\lambda+m+1)_k} z^k L_m^{\lambda+k}(z)$$

$$= \frac{(\lambda+1)_m}{m!} b^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a}{b}\right)^k \sum_{j=0}^{k} \sigma_k^j \frac{j!}{(\lambda+1)_j} z^j\, {}_1F_1\left(\begin{matrix} -m+j+1 \\ \lambda+j+1;\ 1-z^2 \end{matrix}\right)$$

$$[|z| < 1].$$

3. $$\sum_{k=0}^{\infty} \frac{(-\lambda-n)_k}{(k+n)!} t^k L_n^{\lambda-k}(z) = (n+\lambda+1)_n (-t)^n$$

$$\times \left[(1-t)^{\lambda+n} L_n^{\lambda+n}(z-tz) - (-t)^n \sum_{k=0}^{n-1} \frac{(-\lambda-2n)_k}{k!} t^k L_n^{\lambda+n-k}(z)\right]$$

$$[|t| < 1].$$

4. $$\sum_{k=0}^{\infty} \frac{t^k}{(\lambda+1)_k} L_k^{\lambda}(z) = \Gamma(\lambda+1)(tz)^{-\lambda/2} e^t J_\lambda\left(2\sqrt{tz}\right).$$

5. $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{(\lambda+2)_k} L_k^{\lambda+k}(z) = e^z.$

6. $\displaystyle\sum_{k=0}^{\infty} \frac{(a)_{2k}}{(a+1)_k(\lambda+1)_{2k}} z^k L_k^{\lambda+k}(z) = {}_1F_1\left(\begin{matrix} a;\ z \\ \lambda+1 \end{matrix}\right).$

7. $\displaystyle\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{\left(a+b+\frac{1}{2}\right)_k (\lambda+1)_{2k}} (4z)^k L_k^{\lambda+k}(z) = {}_2F_2\left(\begin{matrix} 2a,\, 2b \\ a+b+\frac{1}{2};\ z \end{matrix}\right).$

8. $\displaystyle\sum_{k=0}^{\infty} \binom{k+n}{k} t^k L_{k+n}^{\lambda}(z) = (1-t)^{-\lambda-n-1} e^{tz/(t-1)} L_n^{\lambda}\left(\frac{z}{1-t}\right) \qquad [|t|<1].$

9. $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!} L_k^{k}(z) = e^{2t} I_0\left(2\sqrt{t(t-z)}\right) \qquad [34].$

10. $\displaystyle\sum_{k=0}^{\infty} \frac{(k+n)!}{k!\,(a)_k} z^k L_{k+n}^{\lambda+k}(z) = (\lambda+1)_n e^z {}_2F_2\left(\begin{matrix} \lambda+n+1,\, \lambda-a+n+2 \\ \lambda+1,\, a;\ z \end{matrix}\right).$

11. $\displaystyle\sum_{k=0}^{\infty} \frac{(a)_k\left(\frac{1}{2}-a+n\right)_k}{k!\,(\lambda+n+1)_{2k}} (4z)^k L_{k+n}^{\lambda+k}(z) = \frac{(\lambda+1)_n}{n!} {}_2F_2\left(\begin{matrix} 2a-n,\, n-2a+1 \\ n+1,\, \lambda+1;\ z \end{matrix}\right).$

6.14.2. Series containing $L_{nk+m}^{\lambda\pm lk}(z)$ and special functions

1. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(\lambda+1)_k} \gamma(k+\lambda+1,\, w)\, L_k^{\lambda}(z) = \Gamma(\lambda+1)\left(\frac{w}{z}\right)^{(\lambda+1)/2} J_{\lambda+1}\left(2\sqrt{wz}\right).$

2. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(\lambda+1)_k} \gamma(\nu+k,\, w) L_k^{\lambda}(z) = \frac{w^{\nu}}{\nu}\, {}_1F_2\left(\begin{matrix} \nu;\ -wz \\ \lambda+1,\, \nu+1 \end{matrix}\right).$

3. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} \gamma(k+\nu,\, z) L_n^{\lambda+k}(z) = \frac{(\lambda+1)_n}{n!\,\nu} z^{\nu}\, {}_2F_2\left(\begin{matrix} -n,\, 1;\ z \\ \lambda+1,\, \nu+1 \end{matrix}\right).$

4. $\displaystyle\sum_{k=0}^{\infty} \frac{\left(\pm\frac{w}{2}\right)^k}{(\lambda+1)_k} \left\{\begin{matrix} J_{\nu+k}(w) \\ I_{\nu+k}(w) \end{matrix}\right\} L_k^{\lambda}(z) = \frac{\left(\frac{w}{2}\right)^{\nu}}{\Gamma(\nu+1)}\, {}_0F_2\left(\begin{matrix} \mp\frac{w^2 z}{4} \\ \lambda+1,\, \nu+1 \end{matrix}\right).$

5. $$\sum_{k=0}^{\infty} (2k-\lambda)\,(-\lambda)_k(-z)^{-k} J_{k+\mu}(w)\, J_{k-\lambda-\mu}(w)\, L_k^{\lambda-2k}(z)$$
$$= -\frac{\lambda\left(\frac{2}{w}\right)^{\lambda}}{\Gamma(\mu+1)\Gamma(1-\lambda-\mu)}\, {}_2F_2\left(\begin{matrix} \frac{1-\lambda}{2},\, 1-\frac{\lambda}{2};\ \frac{w^2}{z} \\ \mu+1,\, 1-\lambda-\mu \end{matrix}\right).$$

6. $$\sum_{k=0}^{\infty} t^k P_k(w) L_k(z) = v^{-1/2} e^{t(t-w)z/v} J_0\left(\frac{tz\sqrt{1-w^2}}{v}\right)$$ $\left[v = 1 - 2tw + t^2;\ |t| < 1\right].$

6.14.3. Series containing products of $L_{nk+m}^{\lambda \pm lk}(z)$

1. $$\sum_{k=0}^{\infty} \frac{t^k}{k!} L_m^{\lambda+k}(w) L_n^{\mu+k}(z) = e^t \sum_{k=0}^{m} \frac{t^k}{k!} L_{m-k}^{\lambda+k}(w-t) L_{n-k}^{\mu+k}(z-t) \qquad [m \le n].$$

2. $$\sum_{k=0}^{\infty} \frac{k!}{(\lambda+1)_k} t^k L_k^\lambda(w) L_k^\lambda(z)$$
$$= \Gamma(\lambda+1) \frac{(txy)^{-\lambda/2}}{1-t} \exp\left[\frac{t(w+z)}{t-1}\right] I_\lambda\left(\frac{2\sqrt{txy}}{1-t}\right) \qquad [|t| < 1;\ [14],\ 2.5(17)].$$

3. $$\sum_{k=0}^{\infty} t^k \left[L_k^\lambda(z)\right]^2$$
$$= z^{-\lambda} e^{2z} \int_0^\infty \int_0^\infty (xy)^{\lambda/2} e^{-x-y} J_\lambda(2\sqrt{xz})\, J_\lambda(2\sqrt{yz})\, I_0(2z\sqrt{txy})\, dx$$ $[|t| < 1;\ [34]].$

6.14.4. Series containing products of $L_n^\lambda(kx)$

1. $$\sum_{k=1}^{\infty} \frac{\cos(ky)}{k^{2m-2}} L_n^\lambda(-kx) L_n^\lambda(kx)$$
$$= -\frac{(\lambda+1)_n^2}{(n!)^2} \frac{(2x)^{2m-2}}{2(2m-1)!} \frac{(-n)_{m-1}(\lambda+n+1)_{m-1}\left(\frac{3}{2}\right)_{m-1}}{2(2m-1)!\,(\lambda+1)_{m-1}(\lambda+1)_{2m-2}}$$
$$+ \frac{(\lambda+1)_n^2}{(n!)^2} \sum_{k=0}^{m-2} \frac{x^{2k}}{k!} \frac{(-n)_k(\lambda+n+1)_k}{(\lambda+1)_k(\lambda+1)_{2k}} \left[(-1)^{m-k-1} \frac{\pi y^{2m-2k-3}}{2(2m-2k-3)!}\right.$$
$$\left. + \sum_{j=0}^{m-k-1} (-1)^j \frac{y^{2j}}{(2j)!} \zeta(2m-2j-2k-2)\right]$$ $[m \ge 1;\ 0 < x < \pi;\ x < y < 2\pi - x].$

6.14.5. Series containing $L_{nk+m}^{\lambda \pm lk}(\varphi(k, z))$

1. $$\sum_{k=0}^{\infty} \frac{t^k}{k+1} L_k^{\lambda-k}((k+1)z) = \frac{ze^{-w}}{(\lambda+1)\,w}\left[\left(1+\frac{w}{z}\right)^{\lambda+1} - 1\right]$$ $\left[t = we^w/z;\ |we^{w+1}|,\ |w/z| < 1\right].$

2. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} t^k L_k\left(\frac{z}{k+1}\right) = \frac{e^{-w}}{\sqrt{wz}} I_1(2\sqrt{wz}) \qquad \left[t=-we^{w};\ |we^{w+1}|<1\right].$$

6.14.6. Series containing $L_{nk+m}^{\lambda\pm lk}(\varphi(k,\,z))$ and special functions

1. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-\nu-1}}{(\lambda+1)_k} \gamma\,(k+\nu,\,(k+1)w)\, L_k^{\lambda}\left(\frac{z}{k+1}\right) = \frac{\lambda w^{\nu-1} z^{-1}}{1-\nu}\left[{}_1F_2\left(\begin{matrix}\nu-1;\ -wz\\ \lambda,\ \nu\end{matrix}\right)-1\right].$$

2. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-\lambda-2}}{(\lambda+1)_k} \gamma(k+\lambda+1,\,(k+1)w)\, L_k^{\lambda}\left(\frac{z}{k+1}\right) = w^{\lambda} z^{-1}\left[1-\Gamma(\lambda+1)(wz)^{-\lambda/2} J_{\lambda}(2\sqrt{wz})\right].$$

3. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{(\lambda+1)_k} \left(\frac{w}{2}\right)^k J_{k+\nu}(\sqrt{k+1}\,w) L_k^{\lambda}\left(\frac{z}{k+1}\right) = \frac{\lambda\left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{\Gamma(\nu)}\left[1-{}_0F_2\left(\begin{matrix}-\frac{w^2 z}{4}\\ \lambda,\ \nu\end{matrix}\right)\right].$$

4. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{(\lambda+1)_k} \left(-\frac{w}{2}\right)^k I_{k+\nu}(\sqrt{k+1}\,w) L_k^{\lambda}\left(\frac{z}{k+1}\right) = \frac{\lambda\left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{\Gamma(\nu)}\left[{}_0F_2\left(\begin{matrix}\frac{w^2 z}{4}\\ \lambda,\ \nu\end{matrix}\right)-1\right].$$

6.14.7. Series containing products of $L_{mk+n}^{\lambda\pm lk}(\varphi(k,\,z))$

1. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(\lambda+n+1)_k} t^k L_n^{\lambda+k}((k+1)w) L_k^{\lambda+n}\left(\frac{z}{k+1}\right) = (tz)^{-1}\frac{(\lambda)_{n+1}}{n!}\left[1-\Gamma(\lambda)(wz)^{(1-\lambda)/2} J_{\lambda-1}(2\sqrt{wz})\right] \qquad \left[t=we^{-w}\right].$$

2. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(\mu+1)_k} t^k L_n^{\lambda+k}((k+1)w) L_k^{\mu}\left(\frac{z}{k+1}\right) = (tz)^{-1}\frac{(\lambda)_n \mu}{n!}\left[1-{}_1F_2\left(\begin{matrix}\lambda+n;\ -wz\\ \mu,\ \lambda\end{matrix}\right)\right] \qquad \left[t=we^{-w}\right].$$

6.15. The Gegenbauer Polynomials $C_n^\lambda(z)$

6.15.1. Series containing $C_{nk+m}^{\lambda\pm lk}(z)$

1. $$\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}t^kC_n^{\lambda+k}(z)=(1-t)^{-\lambda-n/2}C_n^\lambda\left(\frac{z}{\sqrt{1-t}}\right)\qquad[|t|<1].$$

2. $$\sum_{k=0}^{\infty}\frac{(1-2\lambda-n)_{2k}}{k!\,(1-\lambda)_k}t^kC_n^{\lambda-k}(z)$$
$$=(1-4t)^{\lambda-1/2}(1-4t+4tz^2)^{n/2}C_n^\lambda\left(\frac{z}{\sqrt{1-4t+4tz^2}}\right)\qquad[|t|<1/4].$$

3. $$\sum_{k=0}^{\infty}(ka+b)^m\frac{(\lambda)_k}{\left(\lambda+m+\frac{1}{2}\right)_k}(1-z^2)^k\,C_{2m}^{\lambda+k}(z)$$
$$=\frac{(2\lambda)_{2m}}{(2m)!}b^m\sum_{k=0}^{m}\binom{m}{k}\left(\frac{a}{b}\right)^k\sum_{j=0}^{k}\sigma_k^j\frac{j!\,(\lambda+m)_j}{\left(\lambda+\frac{1}{2}\right)_j}(1-z^2)^j$$
$$\times\,{}_2F_1\left(\begin{matrix}-m+j+1,\,\lambda+j+m\\ \lambda+j+\frac{1}{2};\ 1-z^2\end{matrix}\right)\qquad[|z|<1].$$

4. $$\sum_{k=0}^{\infty}(ka+b)^m\frac{(\lambda)_k}{\left(\lambda+m+\frac{1}{2}\right)_k}(1-z^2)^k\,C_{2m+1}^{\lambda+k}(z)$$
$$=\frac{(2\lambda)_{2m+1}}{(2m+1)!}b^mz\sum_{k=0}^{m}\binom{m}{k}\left(\frac{a}{b}\right)^k\sum_{j=0}^{k}\sigma_k^j\frac{j!\,(\lambda+m+1)_j}{\left(\lambda+\frac{1}{2}\right)_j}(1-z^2)^j$$
$$\times\,{}_2F_1\left(\begin{matrix}-m+j+1,\,\lambda+j+m+1\\ \lambda+j+\frac{1}{2};\ 1-z^2\end{matrix}\right)\qquad[|z|<1].$$

5. $$\sum_{k=0}^{\infty}\frac{t^k}{(1-\lambda)_k}C_k^{\lambda-k}(z)=\Gamma(1-\lambda)t^\lambda e^{-2tz}I_{-\lambda}(2t).$$

6. $$\sum_{k=0}^{\infty}\frac{t^k}{(1-\lambda)_k}C_{2k}^{\lambda-k}(z)=e^t\,{}_1F_1\left(\begin{matrix}\lambda;\ -tz^2\\ \frac{1}{2}\end{matrix}\right).$$

7. $$\sum_{k=0}^{\infty}\frac{t^k}{(k+n)!}C_{2k}^{-k-n}(z)=\frac{(-1)^n}{(2n)!}e^tH_{2n}\left(iz\sqrt{t}\right).$$

8. $$\sum_{k=0}^{\infty}\frac{(a)_k}{(1-\lambda)_k}t^kC_{2k}^{\lambda-k}(z)=(1-t)^{-a}\,{}_2F_1\left(\begin{matrix}a,\,\lambda\\ \frac{1}{2};\ \frac{tz^2}{t-1}\end{matrix}\right)\qquad[|t|<1].$$

9. $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{(1-\lambda)_k} C_{2k+1}^{\lambda-k}(z) = 2\lambda z e^t \, {}_1F_1\left(\begin{matrix}\lambda+1 \\ \frac{3}{2};\ -tz^2\end{matrix}\right).$

10. $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{(k+n+1)!} C_{2k+1}^{-k-n-1}(z) = \frac{(-1)^n i}{(2n+1)!} t^{-1/2} e^t H_{2n+1}\left(iz\sqrt{t}\right).$

11. $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{\left(\frac{3}{2}\right)_k} C_{2k+1}^{-k-1/2}(z) = -\frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^t \operatorname{erf}\left(\sqrt{t}\, z\right).$

12. $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{\left(\frac{3}{2}\right)_k (k+1)} C_{2k+1}^{-k-1/2}(i) = -\frac{\pi i}{4t} \operatorname{erfi}^2\left(\sqrt{t}\right).$

13. $\displaystyle\sum_{k=0}^{\infty} \binom{k+n}{k} \frac{(1-2\lambda-n)_k}{(1-\lambda)_k} t^k C_{k+n}^{\lambda-k}(z)$

$$= \left[(1+2tz)^2 - 4t^2\right]^{\lambda-1/2} C_n^{\lambda}(z - 2t + 2tz^2).$$

6.15.2. Series containing $C_{nk+m}^{\lambda \pm lk}(z)$ and special functions

1. $\displaystyle\sum_{k=0}^{\infty} (-1)^k (2k+\lambda)\, J_{2k+\lambda}(w) C_{2k}^{\lambda}(z) = \frac{\left(\frac{w}{2}\right)^{\lambda}}{\Gamma(\lambda)} \cos(wz).$

2. $\displaystyle\sum_{k=0}^{\infty} (-1)^k (2k+\lambda+1)\, J_{2k+\lambda+1}(w) C_{2k+1}^{\lambda}(z) = \frac{\left(\frac{w}{2}\right)^{\lambda}}{\Gamma(\lambda)} \sin(wz).$

3. $\displaystyle\sum_{k=0}^{\infty} (2k+\lambda)\, \frac{\left(\frac{1}{2}\right)_k}{\left(\lambda+\frac{1}{2}\right)_k} J_{2k+\lambda}(w) C_{2k}^{\lambda}(z)$

$$= \sqrt{\frac{w}{2}}\, (z^2-1)^{(1-2\lambda)/4} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} I_{\lambda-1/2}\left(w\sqrt{z^2-1}\right).$$

4. $\displaystyle\sum_{k=0}^{\infty} (2k+\lambda+1)\, \frac{\left(\frac{3}{2}\right)_k}{\left(\lambda+\frac{1}{2}\right)_k} J_{2k+\lambda+1}(w) C_{2k+1}^{\lambda}(z)$

$$= \sqrt{\frac{w^3}{2}}\, z(z^2-1)^{(1-2\lambda)/4} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} I_{\lambda-1/2}\left(w\sqrt{z^2-1}\right).$$

5. $\displaystyle\sum_{k=0}^{\infty} (4k-2\lambda+1)\, \frac{\left(\frac{1}{2}-\lambda\right)_k \left(\frac{1}{2}\right)_k}{(1-\lambda)_{2k}} (z^2-1)^{-k} J_{2k-\lambda+1/2}(w) C_{2k}^{\lambda-2k}(z)$

$$= \sqrt{2w}\, (z^2-1)^{-\lambda/2} \frac{\Gamma(1-\lambda)}{\Gamma\left(\frac{1}{2}-\lambda\right)} I_{-\lambda}\left(\frac{w}{\sqrt{z^2-1}}\right).$$

6. $$\sum_{k=0}^{\infty}(4k-2\lambda+1)\,\frac{\left(\frac{1}{2}-\lambda\right)_k\left(\frac{3}{2}\right)_k}{(1-\lambda)_{2k}}\left(z^2-1\right)^{-k}J_{2k-\lambda+1/2}(w)C_{2k+1}^{\lambda-2k}(z)$$
$$=-\sqrt{2w^3}z\left(z^2-1\right)^{-(\lambda+1)/2}\,\frac{\Gamma(1-\lambda)}{\Gamma\left(\frac{1}{2}-\lambda\right)}\,I_{-\lambda-1}\left(\frac{w}{\sqrt{z^2-1}}\right).$$

7. $$\sum_{k=0}^{\infty}(2k+\lambda)\,\frac{\left(\frac{1}{2}\right)_k}{\left(\lambda+\frac{1}{2}\right)_k}\,J_{k+\nu}(w)\,J_{k+\lambda-\nu}(w)C_{2k}^{\lambda}(z)$$
$$=\frac{\lambda\left(\frac{w}{2}\right)^{\lambda}}{\Gamma(\nu+1)\Gamma(\lambda-\nu+1)}\,{}_2F_3\left(\begin{matrix}\frac{\lambda+1}{2},\ \frac{\lambda}{2}+1;\ w^2z^2-w^2\\ \lambda+\frac{1}{2},\ \nu+1,\ \lambda-\nu+1\end{matrix}\right).$$

8. $$\sum_{k=0}^{\infty}(2k+\lambda+1)\,\frac{\left(\frac{3}{2}\right)_k}{\left(\lambda+\frac{1}{2}\right)_k}\,J_{k+\nu}(w)\,J_{k+\lambda-\nu+1}(w)C_{2k+1}^{\lambda}(z)$$
$$=\frac{2^{-\lambda}\lambda(\lambda+1)w^{\lambda+1}z}{\Gamma(\nu+1)\Gamma(\lambda-\nu+2)}\,{}_2F_3\left(\begin{matrix}\frac{\lambda}{2}+1,\ \frac{\lambda+3}{2};\ w^2z^2-w^2\\ \lambda+\frac{1}{2},\ \nu+1,\ \lambda-\nu+2\end{matrix}\right).$$

9. $$\sum_{k=0}^{\infty}(4k-2\lambda+1)\,\frac{\left(\frac{1}{2}-\lambda\right)_k\left(\frac{1}{2}\right)_k}{(1-\lambda)_{2k}}\left(z^2-1\right)^{-k}J_{k+\nu}(w)\,J_{k-\lambda-\nu+1/2}(w)$$
$$\times\, C_{2k}^{\lambda-2k}(z)\ =\frac{(1-2\lambda)\left(\frac{w}{2}\right)^{1/2-\lambda}}{\Gamma(\nu+1)\Gamma\left(\frac{3}{2}-\lambda-\nu\right)}\,{}_2F_3\left(\begin{matrix}\frac{3-2\lambda}{4},\ \frac{5-2\lambda}{4};\ \frac{w^2}{z^2-1}\\ 1-\lambda,\ \nu+1,\ \frac{3}{2}-\lambda-\nu\end{matrix}\right).$$

10. $$\sum_{k=0}^{\infty}(4k-2\lambda+1)\,\frac{\left(\frac{1}{2}-\lambda\right)_k\left(\frac{3}{2}\right)_k}{(-\lambda)_{2k+1}}\left(z^2-1\right)^{-k}J_{k+\nu}(w)\,J_{k-\lambda-\nu+1/2}(w)$$
$$\times\, C_{2k+1}^{\lambda-2k}(z)=\frac{2^{\lambda+1/2}(2\lambda-1)w^{1/2-\lambda}z}{\Gamma(\nu+1)\Gamma\left(\frac{3}{2}-\lambda-\nu\right)}\,{}_2F_3\left(\begin{matrix}\frac{3-2\lambda}{4},\ \frac{5-2\lambda}{4};\ \frac{w^2}{z^2-1}\\ -\lambda,\ \nu+1,\ \frac{3}{2}-\lambda-\nu\end{matrix}\right).$$

11. $$\sum_{k=0}^{\infty}(k+\lambda)\,I_{k+\lambda}(w)C_k^{\lambda}(z)=\frac{\left(\frac{w}{2}\right)^{\lambda}}{\Gamma(\lambda)}\,e^{wz}.$$

12. $$\sum_{k=0}^{\infty}(2k+\lambda)\,I_{2k+\lambda}(w)C_{2k}^{\lambda}(z)=\frac{\left(\frac{w}{2}\right)^{\lambda}}{\Gamma(\lambda)}\,\cosh(wz).$$

13. $$\sum_{k=0}^{\infty}(2k+\lambda+1)\,I_{2k+\lambda+1}(w)C_{2k+1}^{\lambda}(z)=\frac{\left(\frac{w}{2}\right)^{\lambda}}{\Gamma(\lambda)}\,\sinh(wz).$$

14. $$\sum_{k=0}^{\infty}(-1)^k(2k+\lambda)\,\frac{\left(\frac{1}{2}\right)_k}{\left(\lambda+\frac{1}{2}\right)_k}\,I_{2k+\lambda}(w)C_{2k}^{\lambda}(z)$$
$$=\sqrt{\frac{w}{2}}\,(z^2-1)^{(1-2\lambda)/4}\,\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\,J_{\lambda-1/2}\left(w\sqrt{z^2-1}\right).$$

15. $$\sum_{k=0}^{\infty}(-1)^k(2k+\lambda+1)\,\frac{\left(\frac{3}{2}\right)_k}{\left(\lambda+\frac{1}{2}\right)_k}\,I_{2k+\lambda+1}(w)C_{2k+1}^{\lambda}(z)$$
$$=\sqrt{\frac{w^3}{2}}\,z(z^2-1)^{(1-2\lambda)/4}\,\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\,J_{\lambda-1/2}\left(w\sqrt{z^2-1}\right).$$

16. $$\sum_{k=0}^{\infty}(4k-2\lambda+1)\,\frac{\left(\frac{1}{2}-\lambda\right)_k\left(\frac{1}{2}\right)_k}{(1-\lambda)_{2k}}\,(1-z^2)^{-k}\,I_{k+(1-2\lambda)/4}(w)C_{2k}^{\lambda-2k}(z)$$
$$=\frac{2^{(2\lambda+7)/4}}{\Gamma\left(\frac{1-2\lambda}{4}\right)}\,w^{(1-2\lambda)/4}e^{-w}\,{}_1F_1\left(\begin{matrix}\frac{3-2\lambda}{4};\ \frac{2w}{1-z^2}\\ 1-\lambda\end{matrix}\right).$$

17. $$\sum_{k=0}^{\infty}(4k-2\lambda+1)\,\frac{\left(\frac{1}{2}-\lambda\right)_k\left(\frac{3}{2}\right)_k}{(1-\lambda)_{2k}}\,(1-z^2)^{-k}\,I_{k+(1-2\lambda)/4}(w)C_{2k+1}^{\lambda-2k}(z)$$
$$=\frac{2^{(2\lambda+11)/4}}{\Gamma\left(\frac{1-2\lambda}{4}\right)}\,\lambda z w^{(1-2\lambda)/4}e^{-w}\,{}_1F_1\left(\begin{matrix}\frac{3-2\lambda}{4}\\ -\lambda;\ \frac{2w}{1-z^2}\end{matrix}\right).$$

18. $$\sum_{k=0}^{\infty}(k+\lambda)\,I_{(k+\lambda)/2}^2(w)C_k^{\lambda}(z)=2^{2-\lambda}w^{\lambda}$$
$$\times\left[\frac{1}{\lambda\Gamma^2\left(\frac{\lambda}{2}\right)}\,{}_1F_2\left(\begin{matrix}\frac{\lambda+1}{2};\ w^2z^2\\ \frac{\lambda}{2}+1,\ \frac{1}{2}\end{matrix}\right)+\frac{\lambda wz}{(\lambda+1)\Gamma^2\left(\frac{\lambda+1}{2}\right)}\,{}_1F_2\left(\begin{matrix}\frac{\lambda}{2}+1;\ w^2z^2\\ \frac{\lambda+3}{2},\ \frac{3}{2}\end{matrix}\right)\right].$$

6.15.3. Series containing products of $C_{nk+m}^{\lambda\pm lk}(z)$

1. $$\sum_{k=0}^{\infty}\frac{k!\,(1-2\lambda)_k}{(1-\lambda)_k^2}\,t^kC_k^{\lambda-k}(w)\,C_k^{\lambda-k}(z)=(u_-u_+)^{\lambda-1/2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}-\lambda,\ \frac{1}{2}-\lambda\\ 1-2\lambda;\ \frac{16t}{u_-u_+}\end{matrix}\right)$$
$$\left[u_-=1-4t(w+1)(z-1),\ u_+=1-4t(w-1)(z+1);\ |16tu_-^{-1}u_+^{-1}|<1\right].$$

2. $$\sum_{k=0}^{\infty}(-4)^k k!\,(2k+\lambda)\,\frac{\left(\frac{1}{2}\right)_k}{(2\lambda)_{2k}}\,J_{2k+\lambda}(w) \times C_{2k}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)C_{2k}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right) = 2^{\lambda-1}w^{1/2}(1-z^2)^{(1-2\lambda)/4}\,\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} \times \cos\left(\frac{w}{2}\sqrt{1-z^2}\right)J_{\lambda-1/2}\left(\frac{w}{2}\sqrt{1-z^2}\right).$$

3. $$\sum_{k=0}^{\infty}(-4)^k k!\,(2k+\lambda+1)\,\frac{\left(\frac{3}{2}\right)_k}{(2\lambda)_{2k+1}}\,J_{2k+\lambda+1}(w) \times C_{2k+1}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)C_{2k+1}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right) = 2^{\lambda-1}w^{1/2}(1-z^2)^{(1-2\lambda)/4}\,\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} \times \sin\left(\frac{w}{2}\sqrt{1-z^2}\right)J_{\lambda-1/2}\left(\frac{w}{2}\sqrt{1-z^2}\right).$$

4. $$\sum_{k=0}^{\infty}(2k+\lambda)\,\frac{(2k)!}{(2\lambda)_{2k}}\,I_{k+\lambda/2}(w)C_{2k}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)C_{2k}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right) = \frac{2^{1-\lambda/2}}{\Gamma\left(\frac{\lambda}{2}\right)}\,w^{\lambda/2}e^{-w}\,{}_3F_3\left(\begin{matrix}\frac{\lambda}{2},\ \frac{\lambda+1}{2},\ \frac{\lambda+1}{2};\ 2w-2wz^2\\ \frac{1}{2},\ \lambda,\ \lambda+\frac{1}{2}\end{matrix}\right).$$

5. $$\sum_{k=0}^{\infty}(2k+\lambda+1)\,\frac{(2k+1)!}{(2\lambda+1)_{2k}}\,I_{k+(\lambda+1)/2}(w) \times C_{2k+1}^{\lambda}\left(\sqrt{\frac{1+z}{2}}\right)C_{2k+1}^{\lambda}\left(\sqrt{\frac{1-z}{2}}\right) = \frac{2^{(3-\lambda)/2}\lambda^2}{\Gamma\left(\frac{\lambda+1}{2}\right)}\,w^{(\lambda+1)/2}\sqrt{1-z^2}\,e^{-w} \times {}_3F_3\left(\begin{matrix}\frac{\lambda+1}{2},\ \frac{\lambda}{2}+1,\ \frac{\lambda}{2}+1;\ 2w-2wz^2\\ \frac{3}{2},\ \lambda+\frac{1}{2},\ \lambda+1\end{matrix}\right).$$

6.15.4. Series containing $C_{nk+m}^{\lambda\pm lk}(\varphi(k,z))$

1. $$\sum_{k=0}^{\infty}\frac{(k+1)^{-1}}{(1-\lambda)_k}\,t^k C_k^{\lambda-k}(1+(k+1)z) = \frac{2\lambda z e^{-w}}{(2\lambda+1)w}\left[1-{}_1F_1\left(\begin{matrix}-\lambda-\frac{1}{2}\\ -2\lambda;\ -\frac{2w}{z}\end{matrix}\right)\right]$$
$$\left[t=\frac{w}{2z}e^{w};\ |we^{w+1}|<1\right].$$

2. $$\sum_{k=0}^{\infty}\frac{(k+1)^{k-1}}{(1-\lambda)_k}\,t^k C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{e^{-w}}{2(\lambda-1)wz^2}\left[1-{}_1F_1\left(\begin{matrix}\lambda-1\\ -\frac{1}{2};\ wz^2\end{matrix}\right)\right]$$
$$\left[t=-we^{w};\ |we^{w+1}|<1\right].$$

3. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1/2}}{(1-\lambda)_k} t^k C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right) = \frac{e^{-w}}{wz}\left[{}_1F_1\left(\begin{matrix}\lambda;\ wz^2\\ \frac{1}{2}\end{matrix}\right) - 1\right]$$
$$[t = -we^w;\ |we^{w+1}| < 1].$$

4. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-1}}{(1-\lambda)_k} t^k (k+z)^k C_{2k}^{\lambda-k}\left(\sqrt{\frac{z-1}{z+k}}\right)$$
$$= \frac{e^{-w}}{(2\lambda+1)w(z-1)}\left[1 - {}_1F_1\left(\begin{matrix}-\lambda-\frac{1}{2}\\ -\frac{1}{2};\ w(1-z)\end{matrix}\right)\right]$$
$$[t = -we^w;\ |we^{w+1}| < 1].$$

5. $$\sum_{k=0}^{\infty} \frac{(k+z)^{k+1/2}}{(1-\lambda)_k\,(k+1)} t^k C_{2k+1}^{\lambda-k}\left(\sqrt{\frac{z-1}{z+k}}\right)$$
$$= \frac{2\lambda e^{-w}}{(2\lambda+1)w\sqrt{z-1}}\left[{}_1F_1\left(\begin{matrix}-\lambda-\frac{1}{2}\\ \frac{1}{2};\ w(1-z)\end{matrix}\right) - 1\right]$$
$$[t = -we^w;\ |we^{w+1}| < 1].$$

6.15.5. Series containing $C_{nk+m}^{\lambda\pm lk}(\varphi(k,z))$ and special functions

1. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-(k+\nu)/2-1}}{(1-\lambda)_k}\left(-\frac{w}{4z}\right)^k J_{k+\nu}(\sqrt{k+1}\,w) C_k^{\lambda-k}((k+1)z+1)$$
$$= \frac{2\lambda\left(\frac{w}{2}\right)^{\nu-2} z}{(2\lambda+1)\Gamma(\nu)}\left[{}_1F_2\left(\begin{matrix}-\lambda-\frac{1}{2};\ \frac{w^2z^{-1}}{2}\\ -2\lambda,\ \nu\end{matrix}\right) - 1\right].$$

2. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{(1-\lambda)_k}\left(\frac{w}{2}\right)^k J_{k+\nu}(\sqrt{k+1}\,w) C_{2k}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$= \frac{\left(\frac{w}{2}\right)^{\nu-2} z^{-2}}{2(\lambda-1)\Gamma(\nu)}\left[{}_1F_2\left(\begin{matrix}\lambda-1;\ -\frac{w^2z^2}{4}\\ -\frac{1}{2},\ \nu\end{matrix}\right) - 1\right].$$

3. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu-1)/2}}{(1-\lambda)_k}\left(\frac{w}{2}\right)^k J_{k+\nu}(\sqrt{k+1}\,w) C_{2k+1}^{\lambda-k}\left(\frac{z}{\sqrt{k+1}}\right)$$
$$= \frac{\left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{\Gamma(\nu)}\left[1 - {}_1F_2\left(\begin{matrix}\lambda;\ -\frac{w^2z^2}{4}\\ \frac{1}{2},\ \nu\end{matrix}\right)\right].$$

6.16. The Jacobi Polynomials $P_n^{(\rho,\sigma)}(z)$

6.16.1. Series containing $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(z)$

1. $$\sum_{k=0}^{\infty} \frac{t^k}{(\sigma+1)_k} P_k^{(\rho-k,\,\sigma)}(z) = e^{-t}\,{}_1F_1\left(\begin{matrix} \rho+\sigma+1 \\ \sigma+1;\ \frac{t(z+1)}{2} \end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty} \frac{(a)_k}{(\sigma+1)_k} t^k P_k^{(\rho-k,\,\sigma)}(z) = (1+t)^{-a}\,{}_2F_1\left(\begin{matrix} a,\ \rho+\sigma+1 \\ \sigma+1;\ \frac{t(1+z)}{2(t+1)} \end{matrix}\right) \qquad [|t|<1].$$

3. $$\sum_{k=0}^{\infty} (ka+b)^m \frac{(\rho+\sigma+m+1)_k}{(\rho+m+1)_k} \left(\frac{1-z}{2}\right)^k P_m^{(\rho+k,\,\sigma)}(z)$$
$$= \frac{(\rho+1)_m}{m!} b^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a}{b}\right)^k$$
$$\times \sum_{j=0}^{k} \sigma_k^j \frac{j!\,(\rho+\sigma+m+1)_j}{(\rho+1)_j} \left(\frac{1-z}{2}\right)^j {}_2F_1\left(\begin{matrix} -m+j+1,\ \rho+\sigma+j+m+1 \\ \rho+j+1;\ \frac{1-z}{2} \end{matrix}\right)$$
$$[|1-z|<2].$$

4. $$\sum_{k=0}^{\infty} \frac{t^k}{(\rho+1)_k} P_k^{(\rho,\,-\rho-k)}(z) = \rho\left(\frac{2}{tz-t}\right)^{\rho} e^{t(z+1)/2} \gamma\left(\rho,\ \frac{tz-t}{2}\right).$$

5. $$\sum_{k=0}^{\infty} \frac{t^k}{(-\rho-\sigma)_k} P_k^{(\rho-k,\,\sigma-k)}(z) = e^{-t(z+1)/2}\,{}_1F_1\left(\begin{matrix} -\sigma;\ t \\ -\rho-\sigma \end{matrix}\right).$$

6. $$\sum_{k=0}^{\infty} \frac{(a)_k}{(-\rho-\sigma)_k} t^k P_k^{(\rho-k,\,\sigma-k)}(z)$$
$$= \left(1+\frac{tz-t}{2}\right)^{-a} {}_2F_1\left(\begin{matrix} a,\ -\rho \\ -\rho-\sigma;\ \frac{2t}{t-tz-2} \end{matrix}\right) \quad [|t|<1].$$

7. $$\sum_{k=0}^{\infty} \binom{k+n}{k} t^k P_{k+n}^{(\rho-k,\,\sigma-k)}(z)$$
$$= \left(1+\frac{t(z+1)}{2}\right)^{\rho} \left(1+\frac{t(z-1)}{2}\right)^{\sigma} P_n^{(\rho,\,\sigma)}\left(z+\frac{tz^2-t}{2}\right).$$

8. $$\sum_{k=0}^{\infty} \frac{(-\rho-n)_k}{k!} t^k P_n^{(\rho-k,\,\sigma+k)}(z) = (1-t)^{\rho} P_n^{(\rho,\,\sigma)}(t+z-tz) \qquad [|t|<1].$$

9. $$\sum_{k=0}^{\infty} \frac{(a)_k(\frac{1}{2}-a+n)_k(\rho+\sigma+n+1)_k}{k!\,(\rho+n+1)_{2k}} 2^k(1-z)^k\, P_{k+n}^{(\rho+k,\,\sigma-k)}(z)$$
$$= \frac{(\rho+1)_n}{n!}\,{}_3F_2\left(\begin{matrix} 2a-n,\, n-2a+1,\, \rho+\sigma+n+1 \\ n+1,\, \rho+1;\ \frac{1-z}{2} \end{matrix}\right).$$

10. $$\sum_{k=0}^{\infty} \frac{(a)_k\left(\frac{1}{2}-a+n\right)_k(-\rho-n)_k}{k!\,(-\rho-\sigma-n)_{2k}} \left(\frac{4}{z-1}\right)^{2k} P_{k+n}^{(\rho-2k,\,\sigma-k)}(z)$$
$$= \frac{(\rho+\sigma+1)_{2n}}{n!\,(\rho+\sigma+1)_n}\left(\frac{z-1}{2}\right)^n {}_3F_2\left(\begin{matrix} 2a-n,\, n-2a+1,\, -\rho-n \\ n+1,\, -\rho-\sigma-2n;\ \frac{2}{1-z} \end{matrix}\right).$$

6.16.2. Series containing $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(z)$ and special functions

1. $$\sum_{k=0}^{\infty}(2k+\rho+\sigma+1)\,\frac{(\rho+\sigma+1)_k}{(\rho+1)_k} J_{k+a}(w)\, J_{k+\rho+\sigma-a+1}(w) P_k^{(\rho,\,\sigma)}(z)$$
$$= \frac{(\rho+\sigma+1)\left(\frac{w}{2}\right)^{\rho+\sigma+1}}{\Gamma(a+1)\Gamma(\rho+\sigma-a+2)}\,{}_2F_3\left(\begin{matrix} \frac{\rho+\sigma}{2}+1,\, \frac{\rho+\sigma+3}{2};\ \frac{w^2z-w^2}{2} \\ a+1,\, \rho+1,\, \rho+\sigma-a+2 \end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty}(2k-\rho)\,\frac{(-\rho)_k}{(-\rho-\sigma)}\left(\frac{2}{1-z}\right)^k J_{k+a}(w)\, J_{k-\rho-a}(w) P_k^{(\rho-2k,\,\sigma)}(z)$$
$$= -\frac{\rho\left(\frac{2}{w}\right)^{\rho}}{\Gamma(a+1)\Gamma(1-\rho-a)}\,{}_2F_3\left(\begin{matrix} \frac{1-\rho}{2},\, 1-\frac{\rho}{2};\ \frac{2w^2}{z-1} \\ a+1,\, -\rho-\sigma,\, 1-\rho-a \end{matrix}\right).$$

3. $$\sum_{k=0}^{\infty}(2k-\rho)\,\frac{(-\rho)_k}{(-\rho-\sigma)_k}\left(\frac{2}{z-1}\right)^k I_{k-\rho/2}(w) P_k^{(\rho-2k,\,\sigma)}(z)$$
$$= \frac{2}{\Gamma\left(-\frac{\rho}{2}\right)}\left(\frac{2}{w}\right)^{\rho/2} e^{-w}\,{}_1F_1\left(\begin{matrix} \frac{1-\rho}{2};\ \frac{4w}{1-z} \\ -\rho-\sigma \end{matrix}\right).$$

4. $$\sum_{k=0}^{\infty} \frac{(\rho+\sigma+n+1)_k}{(1-\lambda)_k}\left(\frac{1-z}{2}\right)^k C_{2k}^{\lambda-k}(w)\, P_n^{(\rho+k,\,\sigma)}(z)$$
$$= \frac{(\rho+1)_n}{n!}\left(\frac{1+z}{2}\right)^{-\rho-\sigma-n-1} {}_3F_2\left(\begin{matrix} \lambda,\, \rho+n+1,\, \rho+\sigma+n+1 \\ \frac{1}{2},\, \rho+1;\ \frac{w^2(z-1)}{z+1} \end{matrix}\right)$$
$$[|1-z|<2].$$

5. $\displaystyle\sum_{k=0}^{\infty} \frac{(\rho+\sigma+n+1)_k}{(1-\lambda)_k} \left(\frac{1-z}{2}\right)^k C_{2k+1}^{\lambda-k}(w) P_n^{(\rho+k,\,\sigma)}(z)$

$$= 2\lambda w \frac{(\rho+1)_n}{n!} \left(\frac{1+z}{2}\right)^{-\rho-\sigma-n-1} {}_3F_2\left(\begin{matrix}\lambda+1,\, \rho+n+1,\, \rho+\sigma+n+1 \\ \frac{3}{2},\, \rho+1;\; \frac{w^2(z-1)}{z+1}\end{matrix}\right)$$

$[|1-z|<2]$.

6.16.3. Series containing products of $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(z)$

1. $\displaystyle\sum_{k=0}^{\infty} (-1)^k (2k+\rho+\sigma+1) \frac{k!\,(\rho+\sigma+1)_k}{(\rho+1)_k (\sigma+1)_k}$

$$\times J_{k+a}(w)\, J_{k+\rho+\sigma-a+1}(w) P_k^{(\rho,\sigma)}(z) P_k^{(\rho,\sigma)}(-z) = \frac{(\rho+\sigma+1)\left(\frac{w}{2}\right)^{\rho+\sigma+1}}{\Gamma(a+1)\Gamma(\rho+\sigma-a+2)}$$

$$\times\, {}_4F_5\left(\begin{matrix}\frac{\rho+\sigma+1}{2},\, \frac{\rho+\sigma}{2}+1,\, \frac{\rho+\sigma}{2}+1,\, \frac{\rho+\sigma+3}{2};\; w^2z^2-w^2 \\ a+1,\, \rho+1,\, \sigma+1,\, \rho+\sigma+1,\, \rho+\sigma-a+2\end{matrix}\right).$$

6.16.4. Series containing $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(\varphi(k,\,z))$

Notation: $t=-we^w$, $|we^{w+1}|<1$.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{(k+1)^{-1}}{(-\rho-\sigma)_k} \left(-\frac{2t}{z}\right)^k P_k^{(\rho-k,\,\sigma-k)}(1+(k+1)\,z)$

$$= \frac{(\rho+\sigma+1)ze^{-w}}{2(\rho+1)w} \left[1-{}_1F_1\left(\begin{matrix}-\rho-1;\; -\frac{2w}{z} \\ -\rho-\sigma-1\end{matrix}\right)\right].$$

2. $\displaystyle\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!\,(\rho+1)_k} t^k P_k^{(\rho,\,\sigma-k)}\left(1+\frac{z}{k+1}\right)$

$$= \frac{2\rho e^{-w}}{(\rho+\sigma)wz} \left[1-{}_1F_1\left(\begin{matrix}\rho+\sigma;\; -\frac{wz}{2} \\ \rho\end{matrix}\right)\right].$$

6.16.5. Series containing $P_{m\pm nk}^{(\rho\pm pk,\,\sigma\pm qk)}(\varphi(k,\,z))$ and special functions

1. $\displaystyle\sum_{k=0}^{\infty} \frac{(k+1)^{-(k+\nu)/2-1}}{(-\rho-\sigma)_k} \left(-\frac{w}{z}\right)^k J_{k+\nu}(\sqrt{k+1}\,w) P_k^{(\rho-k,\,\sigma-k)}((k+1)z-1)$

$$= \frac{(\rho+\sigma+1)\left(\frac{w}{2}\right)^{\nu-2} z}{2(\sigma+1)\Gamma(\nu)} \left[1-{}_1F_2\left(\begin{matrix}-\sigma-1;\; -\frac{w^2z^{-1}}{2} \\ -\rho-\sigma-1,\, \nu\end{matrix}\right)\right].$$

2. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{(\sigma+1)_k} \left(-\frac{w}{2}\right)^k J_{k+\nu}(\sqrt{k+1}\,w) P_k^{(\rho-k,\,\sigma)}\left(\frac{z}{k+1}-1\right) = \frac{2\sigma \left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{(\rho+\sigma)\Gamma(\nu)} \left[1 - {}_1F_2\left(\begin{matrix}\rho+\sigma;\\ \sigma,\,\nu\end{matrix}\; -\frac{w^2 z}{8}\right)\right].$$

3. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-(k+\nu)/2-1}}{(\sigma+1)_k} \left(\frac{w}{2}\right)^k \times (z-k-1)^k J_{k+\nu}(\sqrt{k+1}\,w) P_k^{(\rho-k,\,\sigma)}\left(\frac{z+k+1}{z-k-1}\right) = \frac{\sigma \left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{(\rho+1)\Gamma(\nu)} \left[{}_1F_2\left(\begin{matrix}-\rho-1;\\ \sigma,\,\nu\end{matrix}\; -\frac{w^2 z}{4}\right) - 1\right].$$

4. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-(k+\nu)/2-1}}{(-\rho-\sigma)_k} \left(\frac{w}{z}\right)^k I_{k+\nu}(\sqrt{k+1}\,w) P_k^{(\rho-k,\,\sigma-k)}((k+1)z-1) = \frac{(\rho+\sigma+1)\left(\frac{w}{2}\right)^{\nu-2} z}{2(\sigma+1)\Gamma(\nu)} \left[{}_1F_2\left(\begin{matrix}-\sigma-1;\\ -\rho-\sigma-1,\,\nu\end{matrix}\; \frac{w^2 z^{-1}}{2}\right) - 1\right].$$

5. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{(\sigma+1)_k} \left(\frac{w}{z}\right)^k I_{k+\nu}(\sqrt{k+1}\,w) P_k^{(\rho-k,\,\sigma)}\left(\frac{z}{k+1}-1\right) = \frac{2\sigma \left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{(\rho+\sigma)\Gamma(\nu)} \left[{}_1F_2\left(\begin{matrix}\rho+\sigma;\\ \sigma,\,\nu\end{matrix}\; \frac{w^2 z}{8}\right) - 1\right].$$

6. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-(k+\nu)/2-1}}{(\sigma+1)_k} \left(-\frac{w}{2}\right)^k (z-k-1)^k I_{k+\nu}(\sqrt{k+1}\,w) \times P_k^{(\rho-k,\,\sigma)}\left(\frac{z+k+1}{z-k-1}\right) = \frac{\sigma \left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{(\rho+1)\Gamma(\nu)} \left[1 - {}_1F_2\left(\begin{matrix}-\rho-1;\\ \sigma,\,\nu\end{matrix}\; \frac{w^2 z}{4}\right)\right].$$

6.17. The Generalized Hypergeometric Function ${}_pF_q((a_p);\ (b_q);\ z)$

6.17.1. Series containing ${}_pF_q((a_p(k));\ (b_q(k));\ z)$

1. $$\sum_{k=0}^{\infty} \frac{t^k}{k!\,(b)_k}\, {}_1F_1\left(\begin{matrix}a;\\ b+k\end{matrix}\; z\right) = \Phi_3(a;\ b;\ z,\, t).$$

2. $$\sum_{k=0}^{\infty} \frac{(-z^2)^k}{k!\,(k+1)!}\, {}_1F_1\left(\begin{matrix}2;\\ k+2\end{matrix}\; z\right) = 1 + 2z\, J_0(2z) - J_1(2z) + \pi z\,[J_1(2z)\,\mathbf{H}_0(2z) - J_0(2z)\,\mathbf{H}_1(2z)].$$

3. $$\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k(b-a)_k\left(b-\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{k!\,(b)_k(2b-1)_{4k}}(2z)^{2k}\,{}_1F_1\left(\begin{matrix}2a+2k;\ z\\ 2b+4k\end{matrix}\right) = \left[{}_1F_1\left(\begin{matrix}a;\ \frac{z}{2}\\ b\end{matrix}\right)\right]^2.$$

4. $$\sum_{k=0}^{\infty}(-1)^k\frac{\left(a+\frac{1}{2}\right)_k\left(b-a+\frac{1}{2}\right)_k\left(b-\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{k!\,(b)_k(2b-1)_{4k}}(2z)^{2k}\,{}_1F_1\left(\begin{matrix}2a+2k;\ z\\ 2b+4k\end{matrix}\right) = {}_1F_1\left(\begin{matrix}a-\frac{1}{2}\\ b;\ \frac{z}{2}\end{matrix}\right){}_1F_1\left(\begin{matrix}a+\frac{1}{2}\\ b;\ \frac{z}{2}\end{matrix}\right).$$

5. $$\sum_{k=0}^{\infty}\frac{(b)_k}{k!\,(c)_k}t^k\,{}_2F_1\left(\begin{matrix}a,\ b+k;\ z\\ c+k\end{matrix}\right) = \Phi_1(b,\ a;\ c;\ z,\ t) \qquad [|z|<1].$$

6. $$\sum_{k=0}^{\infty}\frac{t^k}{k!}\,{}_3F_2\left(\begin{matrix}-k,\ k+1,\ z\\ 1,\ 1;\ 1\end{matrix}\right) = {}_1F_1\left(\begin{matrix}z;\ -t\\ 1\end{matrix}\right){}_1F_1\left(\begin{matrix}1-z\\ 1;\ t\end{matrix}\right).$$

7. $$\sum_{k=0}^{\infty}\frac{z^k}{(k!)^2}\,{}_3F_2\left(\begin{matrix}-k,\ \frac{1}{2},\ 1;\ 1\\ k+1,\ \frac{3}{2}\end{matrix}\right) = \frac{1}{4}\left\{-\pi I_1(4\sqrt{z})\,\mathbf{L}_0(4\sqrt{z}) + [1+I_0(4\sqrt{z})]\,[2+\pi\,\mathbf{L}_1(4\sqrt{z})]\right\}.$$

8. $$\sum_{k=0}^{\infty}\frac{t^{2k+1}}{k!\,(2k+1)}\,{}_3F_2\left(\begin{matrix}-k,\ -k-\frac{1}{2},\ a;\ z\\ \frac{1}{2}-k,\ a+1\end{matrix}\right) = \frac{\sqrt{\pi}\,a}{2(t^2z)^a}\operatorname{erfi}(t)\,\gamma(a,\ t^2z) \qquad [9].$$

9. $$\sum_{k=0}^{\infty}\frac{(-z)^k}{k!\,(a)_k}\,{}_1F_2\left(\begin{matrix}\frac{1}{2};\ z\\ a+k,\ \frac{3}{2}\end{matrix}\right) = \frac{\sqrt{\pi}}{2}\Gamma(a)z^{(1-2a)/4}\,\mathbf{H}_{a-3/2}(2\sqrt{z}).$$

10. $$\sum_{k=1}^{\infty}\frac{k}{(2k+1)!}t^k\,{}_1F_2\left(\begin{matrix}k+\frac{1}{2};\ z\\ k+\frac{3}{2},\ \frac{3}{2}\end{matrix}\right) = \frac{1}{8}\sqrt{\frac{t}{z}}\left[\frac{\sinh(\sqrt{t}+2\sqrt{z})}{\sqrt{t}+2\sqrt{z}} - \frac{\sinh(\sqrt{t}-2\sqrt{z})}{\sqrt{t}-2\sqrt{z}}\right].$$

11. $$\sum_{k=0}^{\infty}\frac{(-z)^k}{(k!)^2}\,{}_1F_2\left(\begin{matrix}a;\ z\\ k+1,\ a+1\end{matrix}\right) = \Gamma(a+1)z^{-a/2}J_a(2\sqrt{z}).$$

12. $$\sum_{k=0}^{\infty}\frac{(-z)^k}{k!\,(a)_k}\,{}_1F_2\left(\begin{matrix}\frac{1}{2};\ z\\ k+a,\ \frac{3}{2}\end{matrix}\right) = \frac{1}{2}\Gamma(a)\sqrt{\pi}\,z^{(1-2a)/4}\,\mathbf{H}_{a-3/2}(2\sqrt{z}).$$

13. $$\sum_{k=1}^{\infty} \frac{2k-1}{(2k)!} t^k \, {}_1F_2\left(\begin{matrix} k;\ z \\ k+1,\ \frac{3}{2} \end{matrix}\right) = \frac{t}{2(4z-t)} \left[2\cosh\sqrt{t}\cosh(2\sqrt{z}) - \sqrt{\frac{t}{z}}\sinh\sqrt{t}\sinh(2\sqrt{z}) - 2\right].$$

14. $$\sum_{k=1}^{\infty} \frac{t^k}{(2k)!} \, {}_1F_2\left(\begin{matrix} k;\ z \\ k+1,\ \frac{1}{2} \end{matrix}\right) = \frac{t}{t-4z} \left[\cosh\sqrt{t}\cosh(2\sqrt{z}) - 2\sqrt{\frac{z}{t}}\sinh\sqrt{t}\sinh(2\sqrt{z}) - 1\right].$$

15. $$\sum_{k=1}^{\infty} \frac{(4z)^k}{(2k)!} \, {}_1F_2\left(\begin{matrix} k;\ z \\ k+1,\ \frac{1}{2} \end{matrix}\right) = \frac{1}{2}\sinh^2(2\sqrt{z}).$$

16. $$\sum_{k=1}^{\infty} \frac{2k-1}{(2k)!} (4z)^k \, {}_1F_2\left(\begin{matrix} k;\ z \\ k+1,\ \frac{3}{2} \end{matrix}\right) = \frac{1}{2}\sinh^2(2\sqrt{z}).$$

17. $$\sum_{k=0}^{\infty} \frac{t^k}{(2k+1)!} \, {}_1F_2\left(\begin{matrix} k+\frac{1}{2};\ z \\ k+\frac{3}{2},\ \frac{1}{2} \end{matrix}\right) = \frac{1}{2}\left[\frac{\sinh(\sqrt{t}-2\sqrt{z})}{\sqrt{t}-2\sqrt{z}} + \frac{\sinh(\sqrt{t}+2\sqrt{z})}{\sqrt{t}+2\sqrt{z}}\right].$$

18. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{(2k+1)!} \, {}_1F_2\left(\begin{matrix} k+\frac{1}{2};\ z \\ k+\frac{3}{2},\ \frac{1}{2} \end{matrix}\right) = \frac{1}{2} + \frac{1}{8\sqrt{z}}\sinh(4\sqrt{z}).$$

19. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+1)!} \, {}_1F_2\left(\begin{matrix} 1;\ z \\ \frac{k}{2}+1,\ \frac{k+3}{2} \end{matrix}\right) = \frac{\sinh(4\sqrt{z})}{4\sqrt{z}}.$$

20. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+2)!} \, {}_1F_2\left(\begin{matrix} 1;\ z \\ \frac{k+3}{2},\ \frac{k}{2}+2 \end{matrix}\right) = \frac{1}{8z}\left[1 - J_0(4\sqrt{z})\right].$$

21. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!\left(\frac{1}{2}\right)_k} \, {}_1F_2\left(\begin{matrix} -\frac{1}{2};\ z \\ \frac{1}{2},\ k+\frac{1}{2} \end{matrix}\right) = 1 - \pi\sqrt{z}\,\mathbf{H}_0(2\sqrt{z}).$$

22. $$\sum_{k=0}^{\infty} \frac{(-16\sqrt{z})^k}{(k!)^2} \, {}_1F_4\left(\begin{matrix} 1;\ -z \\ \frac{k+1}{4},\ \frac{k+2}{4},\ \frac{k+3}{4},\ \frac{k}{4}+1 \end{matrix}\right) = \frac{1}{2}J_0\left(8z^{1/4}\right) + \frac{1}{2}\cos\left(4\sqrt{2}\,z^{1/4}\right).$$

23. $$\sum_{k=0}^{\infty} \frac{(-16\sqrt{z})^k}{k!\,(k+2)!} \, {}_1F_4\left(\begin{matrix} 1;\ z \\ \frac{k+3}{4},\ \frac{k+4}{4},\ \frac{k+5}{4},\ \frac{k+6}{4} \end{matrix}\right) = \frac{\sin^2\left(4z^{1/4}\right)}{32z^{1/2}}.$$

24. $$\sum_{k=0}^{\infty} \frac{(-z^2)^k}{k!\,(k+1)!}\,{}_2F_2\left(\begin{matrix} 2,\ 2;\ z \\ 1,\ k+2 \end{matrix}\right) = 2z + (4z^2+1)\,J_0(2z) - 2z\,J_1(2z) + 2\pi z^2[J_1(2z)\,\mathbf{H}_0(2z) - J_0(2z)\,\mathbf{H}_1(2z)].$$

25. $$\sum_{k=0}^{\infty} \frac{\left(\frac{z}{4}\right)^k}{\left(\frac{3}{2}\right)_k}\,{}_2F_2\left(\begin{matrix} k+\frac{1}{2},\ k+1;\ z \\ k+\frac{3}{2},\ 2k+2 \end{matrix}\right) = \frac{1}{2}\sqrt{\frac{\pi}{z}}\ \operatorname{erfi}(\sqrt{z}).$$

26. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,(k+1)!}\,{}_2F_3\left(\begin{matrix} 1,\ 1;\ z \\ 2,\ 2,\ k+2 \end{matrix}\right) = \frac{\pi}{2z}Y_0(2\sqrt{z}) - \frac{1}{z}\left(\mathbf{C} + \frac{1}{2}\ln z\right) J_0(2\sqrt{z}).$$

27. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,(k+1)!}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\ 1;\ z \\ k+2,\ \frac{3}{2},\ 2 \end{matrix}\right) = \frac{1}{z}\,[J_0(2\sqrt{z}) - \cos(2\sqrt{z})].$$

28. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{(2k+1)!\,(2k+3)}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\ 1;\ z \\ k+\frac{5}{2},\ \frac{3}{2},\ 2 \end{matrix}\right) = \frac{1}{4z^{3/2}}[\sin(2\sqrt{z}) - \pi\sqrt{z}\,\mathbf{H}_{-1}(2\sqrt{z})].$$

29. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(2a-1)_k}\,{}_2F_3\left(\begin{matrix} a,\ 1;\ z \\ a+\frac{k-1}{2},\ a+\frac{k}{2},\ a-1 \end{matrix}\right) = 2^{3-2a}\Gamma(2a-2)z^{3/2-a}I_{2a-3}(4\sqrt{z}).$$

30. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} 1,\ 1;\ z \\ \frac{k+3}{2},\ \frac{k}{2}+2,\ 2 \end{matrix}\right) = \frac{1}{8z}\,\{[\ln(2\sqrt{z}) + \mathbf{C}]\,I_0(4\sqrt{z}) + K_0(4\sqrt{z})\}.$$

31. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} 1,\ 1;\ -z \\ \frac{k+3}{2},\ \frac{k}{2}+2,\ 2 \end{matrix}\right) = \frac{1}{16z}\,\{\pi Y_0(4\sqrt{z}) - 2\,[\ln(2\sqrt{z}) + \mathbf{C}]\,J_0(4\sqrt{z})\}.$$

32. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} 1,\ 1;\ z \\ \frac{k+3}{2},\ \frac{k}{2}+2,\ 2 \end{matrix}\right) = -\frac{1}{2\pi z}\int_0^1 \frac{\cos(4x\sqrt{z})\ln(2x)}{\sqrt{1-x^2}}\,dx.$$

33. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{(k!)^2}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\ 1;\ z \\ \frac{k+1}{2},\ \frac{k}{2}+1,\ \frac{3}{2} \end{matrix}\right) = \frac{1}{4}\,\{\pi J_1(4\sqrt{z})\,\mathbf{H}_0(4\sqrt{z}) + [1 + J_0(4\sqrt{z})]\,[2 - \pi\,\mathbf{H}_1(4\sqrt{z})]\}.$$

34. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+1)!}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\,1;\; z \\ \frac{k}{2}+1,\; \frac{k+3}{2},\; \frac{3}{2} \end{matrix}\right) = \frac{\pi}{8\sqrt{z}}\,\mathbf{H}_0(4\sqrt{z}\,).$$

35. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} 2,\,2;\; z \\ \frac{k+3}{2},\; \frac{k}{2}+2,\; 1 \end{matrix}\right) = \frac{\sinh(4\sqrt{z}\,)}{8\sqrt{z}}.$$

36. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} a,\,1;\; z \\ \frac{k+3}{2},\; \frac{k}{2}+2,\; 3-a \end{matrix}\right) = \frac{a-2}{8(a-1)z}\,I_0(4\sqrt{z}\,) + \frac{2^{a-4}}{a-1}\Gamma(3-a)\,z^{(a-3)/2}I_{1-a}(4\sqrt{z}\,).$$

37. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(2a+1)_k}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\,1;\; z \\ \frac{k+1}{2}+a,\; \frac{k}{2}+a+1,\; \frac{3}{2} \end{matrix}\right)$$
$$= 2^{2-2a}(2a-1)\,\pi z^{-a}\Gamma(2a+1)\int_0^{\pi/2} \frac{1}{x}\,J_{2a-1}(x)\,\mathbf{H}_0(4\sqrt{z}-x)\,dx$$
$[\operatorname{Re} a > 1/2]$.

38. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\,1;\; z \\ \frac{k+3}{2},\; \frac{k}{2}+2,\; \frac{5}{2} \end{matrix}\right) = \frac{3}{32z^{3/2}}\,[4\sqrt{z}\,I_0(4\sqrt{z}\,) - \sinh(4\sqrt{z}\,)].$$

39. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\,1;\; z \\ \frac{k+3}{2},\; \frac{k}{2}+2,\; \frac{5}{2} \end{matrix}\right) = \frac{3}{8z}\left[J_0(4\sqrt{z}\,) + \frac{\pi}{2}\,\mathbf{H}_1(4\sqrt{z}\,) - 1\right].$$

40. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+2)!}\,{}_2F_3\left(\begin{matrix} \frac{1}{2},\,1;\; z \\ \frac{k+3}{2},\; \frac{k}{2}+2,\; \frac{5}{2} \end{matrix}\right) = \frac{3}{32z^{3/2}}\left[4z^{1/2}I_0(4\sqrt{z}\,) - \sinh(4\sqrt{z}\,)\right].$$

41. $$\sum_{k=0}^{\infty} \frac{(16z)^k}{k!\,(k+1)!}\,{}_2F_5\left(\begin{matrix} 1,\,\frac{5}{4};\; z \\ \frac{k+2}{4},\; \frac{k+3}{4},\; \frac{k+4}{4},\; \frac{k+5}{4},\; \frac{1}{4} \end{matrix}\right) = \frac{1}{2} + \frac{1}{2}\,I_0(8\sqrt{z}\,).$$

42. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(a+1)_k}\,{}_3F_4\left(\begin{matrix} a,\,\frac{a}{2}+1,\,b;\; z \\ \frac{k+a+1}{2},\; \frac{k+a+2}{2},\; \frac{a}{2},\; a-b+1 \end{matrix}\right) = 2^{b-a}\Gamma(a-b+1)\,z^{(b-a)/2}I_{a-b}(4\sqrt{z}\,).$$

43. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(2a+2)_k}\, {}_3F_4\left(\begin{matrix} a,\ a+\frac{3}{2},\ 1;\ z \\ \frac{k+2}{2}+a,\ \frac{k+3}{2}+a,\ a+\frac{1}{2},\ a+2 \end{matrix}\right)$$
$$= \frac{2^{-2a-2}z^{-a-1/2}}{2a+1}\Gamma(2a+3)\,J_{2a+1}(4\sqrt{z}) + \frac{a}{2a+1}\,{}_1F_2\left(\begin{matrix} a+1;\ -4z \\ a+2,\ 2a+2 \end{matrix}\right).$$

44. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+2)!}\, {}_3F_4\left(\begin{matrix} \frac{2}{3},\ 1,\ \frac{4}{3};\ z \\ \frac{k+3}{2},\ \frac{k}{2}+2,\ \frac{5}{3},\ \frac{7}{3} \end{matrix}\right) = \frac{1}{z}J_0(4\sqrt{z}) - \frac{1}{z}\,{}_1F_2\left(\begin{matrix} 1;\ -4z \\ \frac{2}{3},\ \frac{4}{3} \end{matrix}\right).$$

45. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+1)!}\, {}_3F_4\left(\begin{matrix} 1,\ \frac{3}{2},\ \frac{3}{2};\ z \\ \frac{k}{2}+1,\ \frac{k+3}{2},\ \frac{1}{2},\ \frac{1}{2} \end{matrix}\right) = \cosh(4\sqrt{z}).$$

46. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+2)!}\, {}_3F_4\left(\begin{matrix} 1-a,\ 1+a,\ 1;\ z \\ \frac{k+3}{2},\ \frac{k}{2}+2,\ 2-a,\ 2+a \end{matrix}\right)$$
$$= \frac{1-a^2}{8a^2z}\left[I_0(4\sqrt{z}) - \frac{2a}{\sin(a\pi)}\int_0^{\pi/2} \frac{\cos(2ax)\cosh(4\sqrt{z}\sin x)}{\sqrt{1-x^2}}\,dx\right].$$

47. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{(k+1)!\,(2k+1)}\, {}_4F_4\left(\begin{matrix} k+\frac{1}{2},\ k+1,\ 1,\ \frac{3}{2};\ z \\ k+\frac{3}{2},\ \frac{k}{2}+1,\ \frac{k+3}{2},\ \frac{1}{2} \end{matrix}\right) = \frac{1}{4}\sqrt{\frac{\pi}{z}}\ \mathrm{erfi}\,(2\sqrt{z}).$$

48. $$\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{\left(a+b+\frac{1}{2}\right)_k}(4z)^k \frac{\prod(a_p)_k}{\prod(b_q)_k}\, {}_{p+1}F_q\left(\begin{matrix} -k,\ (a_p)+k \\ (b_q)+k;\ z \end{matrix}\right)$$
$$= {}_{p+2}F_{q+1}\left(\begin{matrix} (a_p),\ 2a,\ 2b \\ (b_q),\ a+b+\frac{1}{2};\ z \end{matrix}\right).$$

49. $$\sum_{k=0}^{\infty} \frac{(n+a)_k}{k!} t^k\, {}_{p+1}F_{q+1}\left(\begin{matrix} -n-k,\ (a_p) \\ a,\ (b_q);\ z \end{matrix}\right)$$
$$= (1-t)^{-n-a}\sum_{k=0}^{n}\binom{n}{k}\frac{(t-1)^{-k}z^k}{(a)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}\, {}_pF_q\left(\begin{matrix} (a_p)+k;\ \frac{tz}{t-1} \\ (b_q)+k \end{matrix}\right) \quad [|t|<1].$$

50. $$\sum_{k=0}^{\infty} \frac{(1-b)_k}{k!} t^k\, {}_pF_{q+1}\left(\begin{matrix} (a_p);\ z \\ (b_q),\ b-k \end{matrix}\right) = (1-t)^{b-1}\, {}_pF_{q+1}\left(\begin{matrix} (a_p);\ z-tz \\ (b_q),\ b \end{matrix}\right)$$
$$[|t|<1].$$

51. $$\sum_{k=0}^{\infty} \frac{(1-a)_k}{k!} t^k\, {}_{p+1}F_{q+1}\left(\begin{matrix} -n-k,\ (a_p) \\ a-k,\ (b_q);\ z \end{matrix}\right)$$
$$= (1-t)^{a-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(t-1)^{k}z^k}{(a)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}\, {}_pF_q\left(\begin{matrix} (a_p)+k \\ (b_q)+k;\ tz \end{matrix}\right) \quad [|t|<1].$$

52. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{(b)_k} \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k\end{matrix}\right) = {}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\, b-1;\ z\\ (b_q),\, b\end{matrix}\right).$$

53. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,(k+b)} \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k\end{matrix}\right) = \frac{1}{b}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\, 1;\ z\\ (b_q),\, b+1\end{matrix}\right).$$

54. $$\sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k}{k!\left(a+\frac{1}{2}\right)_k} (-z)^k \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{1}{2a}\left[(2a-1)\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\, a;\ z\\ (b_q),\, a-\frac{1}{2}\end{matrix}\right) + {}_{p+1}F_{p+1}\left(\begin{matrix}(a_p),\, a;\ z\\ (b_q),\, a+\frac{1}{2}\end{matrix}\right)\right].$$

55. $$\sum_{k=0}^{\infty} \frac{\sigma_{k+m}^m}{(k+m)!} z^k \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ az\\ (b_q)+k\end{matrix}\right)$$
$$= (-1)^{m(p+q+1)} \frac{z^{-m}}{m!} \frac{\prod(1-(b_q))_m}{\prod(1-(a_p))_m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} {}_pF_q\left(\begin{matrix}(a_p)-m;\ (k+a)\,z\\ (b_q)-m\end{matrix}\right).$$

56. $$\sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} P_k^{(\rho-k,\,1)}(3) \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k\end{matrix}\right)$$
$$= {}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\, \rho+2;\ 2z\\ (b_q),\, 2\end{matrix}\right).$$

57. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_{p+1}F_q\left(\begin{matrix}(a_p)+k,\, 1;\ z\\ (b_q)+k\end{matrix}\right) = {}_{p+1}F_q\left(\begin{matrix}(a_p),\, a+1\\ (b_q);\ z\end{matrix}\right).$$

58. $$\sum_{k=0}^{\infty} (ka+b)^m z^k \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_{p+1}F_q\left(\begin{matrix}(a_p)+k,\, a;\ z\\ (b_q)+k\end{matrix}\right)$$
$$= b^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a}{b}\right)^k \sum_{j=0}^{k} \sigma_k^j j!\, z^j \frac{\prod(a_p)_j}{\prod(b_q)_j} {}_{p+1}F_q\left(\begin{matrix}(a_p)+j,\, a+j+1\\ (b_q)+j;\ z\end{matrix}\right)$$
$$[|z| < 1].$$

59. $$\sum_{k=0}^{\infty} \frac{z^k}{k!\,(a)_k} \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k,\, b\end{matrix}\right)$$
$$= {}_{p+2}F_{q+3}\left(\begin{matrix}(a_p),\, \frac{a+b-1}{2},\, \frac{a+b}{2};\ 4z\\ (b_q),\, a,\, b,\, a+b-1\end{matrix}\right).$$

60. $$\sum_{k=0}^{\infty}\frac{(-z)^k}{k!\,(b)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k,\ b\end{matrix}\right)$$
$$={}_{2p}F_{2q+3}\left(\begin{matrix}\left(\frac{a_p}{2}\right),\ \left(\frac{a_p+1}{2}\right);\ -\frac{z^2}{4^{q-p+1}}\\ \left(\frac{b_q}{2}\right),\ \left(\frac{b_q+1}{2}\right),\ \frac{b}{2},\ \frac{b+1}{2},\ b\end{matrix}\right).$$

61. $$\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)+k,\ a;\ z\\ (b_q)+k,\ b\end{matrix}\right)$$
$$={}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\ b-a;\ -z\\ (b_q),\ b\end{matrix}\right).$$

62. $$\sum_{k=0}^{\infty}\frac{(a)_k}{k!\,(b)_k}(-z)^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)+k,\ a;\ z\\ (b_q)+k,\ b\end{matrix}\right)$$
$$={}_{2p+2}F_{2q+3}\left(\begin{matrix}\frac{(a_p)}{2},\ \frac{(a_p)+1}{2},\ a,\ b-a;\ \frac{z^2}{4^{q-p+1}}\\ \frac{(b_q)}{2},\ \frac{(b_q)+1}{2},\ \frac{b}{2},\ \frac{b+1}{2},\ b\end{matrix}\right).$$

63. $$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!\left(a+b+\frac{1}{2}\right)_k}z^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+2}F_{q+1}\left(\begin{matrix}(a_p)+k,\ a,\ b;\ z\\ (b_q)+k,\ a+b+\frac{1}{2}\end{matrix}\right)$$
$$={}_{p+3}F_{q+2}\left(\begin{matrix}(a_p),\ 2a,\ 2b,\ a+b;\ z\\ (b_q),\ a+b+\frac{1}{2},\ 2a+2b\end{matrix}\right).$$

64. $$\sum_{k=0}^{\infty}\frac{(b)_k\left(\frac{1}{2}-a-b\right)_k}{k!\,(1-a)_k}(4z)^k\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_q\left(\begin{matrix}(a_p)+k,\ a-k\\ (b_q)+k;\ z\end{matrix}\right)$$
$$={}_{p+2}F_{q+1}\left(\begin{matrix}(a_p),\ a+2b,\ 1-a-2b\\ (b_q),\ 1-a;\ z\end{matrix}\right).$$

65. $$\sum_{k=0}^{\infty}\frac{(b)_k}{k!\,(k+a)}\,{}_{p+1}F_{q+1}\left(\begin{matrix}k+a,\ (a_p);\ z\\ k+a+1,\ (b_q)\end{matrix}\right)$$
$$=\mathrm{B}\,(a,\ 1-b)\,{}_{p+1}F_{q+1}\left(\begin{matrix}a,\ (a_p);\ z\\ a-b+1,\ (b_q)\end{matrix}\right)\quad[\operatorname{Re}b<1].$$

66. $$\sum_{k=0}^{\infty}\frac{(-4z)^k}{(k!)^2}\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+2}F_{q+3}\left(\begin{matrix}(a_p)+k,\ \frac{1}{2},\ 1;\ z\\ (b_q)+k,\ \frac{k+1}{2},\ \frac{k+2}{2},\ \frac{3}{2}\end{matrix}\right)$$
$$=\frac{1}{2}+\frac{1}{2}\,{}_{p+1}F_{q+2}\left(\begin{matrix}(a_p),\ \frac{1}{2};\ -4z\\ (b_q),\ 1,\ \frac{3}{2}\end{matrix}\right)-\frac{8z}{3}\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\,{}_{p+1}F_{q+2}\left(\begin{matrix}(a_p)+1,\ 1;\ -4z\\ (b_q)+1,\ \frac{3}{2},\ \frac{5}{2}\end{matrix}\right).$$

67. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(2a)_k} \frac{\prod (a_p)_k}{\prod (b_q)_k}\, {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p)+k,\ 1;\ z \\ (b_q)+k,\ \frac{k}{2}+a,\ \frac{k+1}{2}+a \end{matrix}\right) = {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ a-\frac{1}{2};\ -4z \\ (b_q),\ a+\frac{1}{2},\ 2a-1 \end{matrix}\right).$$

68. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+2)!} \frac{\prod (a_p)_k}{\prod (b_q)_k}\, {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p)+k,\ 1;\ z \\ (b_q)+k,\ \frac{k+3}{2},\ \frac{k}{2}+2 \end{matrix}\right) = \frac{1}{8z} \frac{\prod_{j=1}^{q} (b_j-1)}{\prod_{i=1}^{p} (a_i-1)} \left[1 - {}_{p}F_{q+1}\left(\begin{matrix} (a_p)-1;\ -4z \\ (b_q)-1,\ 1 \end{matrix}\right)\right].$$

69. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(2a+1)_k} \frac{\prod (a_p)_k}{\prod (b_q)_k}\, {}_{p+2}F_{q+3}\left(\begin{matrix} (a_p)+k,\ a+1,\ 2a;\ z \\ (b_q)+k,\ \frac{k+1}{2}+a,\ \frac{k}{2}+a+1,\ a \end{matrix}\right) = 1.$$

70. $$\sum_{k=0}^{\infty} \frac{(4z)^k}{k!\,(k+2)!} \frac{\prod (a_p)_k}{\prod (b_q)_k}\, {}_{p+2}F_{q+3}\left(\begin{matrix} (a_p)+k,\ a,\ 1;\ z \\ (b_q)+k,\ \frac{k+3}{2},\ \frac{k}{2}+2,\ 3-a \end{matrix}\right)$$
$$= \frac{a-2}{8\,(a-1)\,z} \frac{\prod_{j=1}^{q} (b_j-1)}{\prod_{i=1}^{p} (a_i-1)} \left[{}_{p}F_{q+1}\left(\begin{matrix} (a_p)-1;\ 4z \\ (b_q)-1,\ 1 \end{matrix}\right) - {}_{p}F_{q+1}\left(\begin{matrix} (a_p)-1;\ 4z \\ (b_q)-1,\ 2-a \end{matrix}\right)\right].$$

71. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+2)!} \frac{(a_p)_k}{(b_q)_k}\, {}_{p+2}F_{q+3}\left(\begin{matrix} (a_p)+k,\ 2,\ 2;\ z \\ (b_q)+k,\ \frac{k+3}{2},\ \frac{k}{2}+2,\ 1 \end{matrix}\right) = \frac{1}{2}\,.$$

72. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+1)!}\, {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p)+k,\ 1;\ z \\ (b_q)+k,\ \frac{k}{2}+1,\ \frac{k+3}{2} \end{matrix}\right) = {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ \frac{1}{2};\ -4z \\ (b_q),\ 1,\ \frac{3}{2} \end{matrix}\right).$$

73. $$\sum_{k=0}^{\infty} \frac{(-16z)^k}{k!\,(k+1)!\,(a-1)_k (a)_k} \frac{\prod (a_p)_k}{\prod (b_q)_k}$$
$$\times\, {}_{p+2}F_{q+5}\left(\begin{matrix} (a_p)+k,\ 1,\ \frac{3}{2};\ z \\ (b_q)+k,\ \frac{a+k}{2},\ \frac{a+k+1}{2},\ \frac{k+2}{2},\ \frac{k+3}{2},\ \frac{1}{2} \end{matrix}\right) = {}_{p}F_{q+3}\left(\begin{matrix} (a_p);\ -4z \\ (b_q),\ a-1,\ \frac{a}{2},\ \frac{a+1}{2} \end{matrix}\right).$$

74. $$\sum_{k=0}^{\infty}\frac{(16z)^k}{(k!)^2\,(b)_k^2}\frac{\prod(a_p)_k}{\prod(b_q)_k}$$
$$\times\,{}_{p+2}F_{q+5}\left(\begin{matrix}(a_p)+k,\,a,\,1;\;z\\(b_q)+k,\,\frac{k+b}{2},\,\frac{k+b+1}{2},\,\frac{k+1}{2},\,\frac{k+2}{2},\,1-a\end{matrix}\right)$$
$$=\frac{1}{2}\,{}_{p+2}F_{q+5}\left(\begin{matrix}(a_p),\,\frac{b-a}{2},\,\frac{b-a+1}{2};\;16z\\(b_q),\,1-a,\,b,\,\frac{b}{2},\,\frac{b+1}{2},\,b-a\end{matrix}\right)+\frac{1}{2}\,{}_pF_{q+3}\left(\begin{matrix}(a_p);\;16z\\(b_q),\,b,\,b,\,1\end{matrix}\right).$$

75. $$\sum_{k=0}^{\infty}\frac{(-4z)^k}{k!\,(2a-1)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}$$
$$\times\,{}_{p+3}F_{q+4}\left(\begin{matrix}(a_p)+k,\,1,\,a,\,b;\;z\\(b_q)+k,\,\frac{k-1}{2}+a,\,\frac{k}{2}+a,\,a-1,\,2a-b-1\end{matrix}\right)$$
$$={}_pF_{q+1}\left(\begin{matrix}(a_p),\,2a-b-2;\;-4z\\(b_q),\,2a-2,\,2a-b-1\end{matrix}\right).$$

76. $$\sum_{k=0}^{\infty}\frac{(-4z)^k}{k!\,(2a+3)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}$$
$$\times\,{}_{p+3}F_{q+4}\left(\begin{matrix}(a_p)+k,\,a,\,a+2,\,1;\;z\\(b_q)+k,\,\frac{k+3}{2}+a,\,\frac{k}{2}+a+2,\,a+1,\,a+3\end{matrix}\right)$$
$$={}_{p+1}F_{q+2}\left(\begin{matrix}(a_p),\,a+2;\;-4z\\(b_q),\,a+3,\,2a+2\end{matrix}\right).$$

77. $$\sum_{k=0}^{\infty}\frac{(4z)^k}{k!\,(a+1)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}$$
$$\times\,{}_{p+3}F_{q+4}\left(\begin{matrix}(a_p)+k,\,a,\,\frac{a}{2}+1,\,b;\;z\\(b_q)+k,\,\frac{k+a+1}{2},\,\frac{k+a+2}{2},\,\frac{a}{2},\,a-b+1\end{matrix}\right)$$
$$={}_pF_{q+1}\left(\begin{matrix}(a_p);\;4z\\(b_q),\,a-b+1\end{matrix}\right).$$

78. $$\sum_{k=0}^{\infty}\frac{(-4z)^k}{k!\,(2a+1)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}$$
$$\times\,{}_{p+4}F_{q+5}\left(\begin{matrix}(a_p)+k,\,\frac{1}{2},\,2a,\,a+1,\,2a-\frac{1}{2};\;z\\(b_q)+k,\,\frac{k+1}{2}+a,\,\frac{k+2}{2}+a,\,\frac{3}{2},\,a,\,2a+\frac{1}{2}\end{matrix}\right)$$
$$={}_{p+1}F_{q+2}\left(\begin{matrix}(a_p),\,1;\;-4z\\(b_q),\,\frac{3}{2},\,2a+\frac{1}{2}\end{matrix}\right).$$

79. $$\sum_{k=0}^{\infty} \frac{(-4z)^k}{k!\,(k+1)!} \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_{p+4}F_{q+5}\left(\begin{matrix} (a_p)+k,\ \frac{1}{3},\ \frac{2}{3},\ 1,\ \frac{3}{2};\ z \\ (b_q)+k,\ \frac{k+2}{2},\ \frac{k+3}{2},\ \frac{1}{2},\ \frac{4}{3},\ \frac{5}{3} \end{matrix} \right) = {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ 1;\ -4z \\ (b_q),\ \frac{4}{3},\ \frac{5}{3} \end{matrix} \right).$$

80. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,(k+a)} \frac{\prod (2a_p)_{2k}}{\prod (2b_q)_{2k}} {}_{2p}F_{2q}\left(\begin{matrix} (a_p)+k,\ (a_p)+k+\frac{1}{2};\ z \\ (b_q)+k,\ (b_q)+k+\frac{1}{2} \end{matrix} \right) = \frac{1}{a} {}_{2p+1}F_{2q+1}\left(\begin{matrix} (a_p),\ (a_p)+\frac{1}{2},\ 1;\ z \\ (b_q),\ (b_q)+\frac{1}{2},\ a+1 \end{matrix} \right).$$

81. $$\sum_{k=0}^{\infty} \frac{(a)_{2k}}{k!\,(b)_k} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_{p+1}F_q\left(\begin{matrix} (a_p)+k,\ a+2k \\ (b_q)+k;\ z \end{matrix} \right) = {}_{p+2}F_{q+1}\left(\begin{matrix} (a_p),\ a,\ a-b+1 \\ (b_q),\ b;\ -z \end{matrix} \right).$$

82. $$\sum_{k=0}^{\infty} \frac{\left(\frac{a}{2}\right)_k (b)_k}{k!\,(2b)_k} (-4z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_{p+1}F_q\left(\begin{matrix} (a_p)+k,\ a+2k \\ (b_q)+k;\ z \end{matrix} \right) = {}_{2p+2}F_{2q+1}\left(\begin{matrix} \frac{(a_p)}{2},\ \frac{(a_p)+1}{2},\ \frac{a}{2},\ \frac{a+1}{2}-b \\ \frac{(b_q)}{2},\ \frac{(b_q)+1}{2},\ b+\frac{1}{2};\ \frac{z^2}{4^{q-p}} \end{matrix} \right).$$

83. $$\sum_{k=0}^{\infty} \frac{(a)_{2k}(b)_k}{k!\,(c)_k(a+b-c+1)_k} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_{p+1}F_q\left(\begin{matrix} (a_p)+k,\ a+2k \\ (b_q)+k;\ z \end{matrix} \right) = {}_{p+2}F_{q+1}\left(\begin{matrix} (a_p),\ a,\ a-c+1,\ c-b \\ (b_q),\ c,\ a+b-c+1;\ z \end{matrix} \right).$$

84. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_{p+2}F_{q+1}\left(\begin{matrix} (a_p)+k,\ 2a+2k,\ a \\ (b_q)+k,\ 2a;\ z \end{matrix} \right) = {}_{3p+2}F_{3q+1}\left(\begin{matrix} \frac{(a_p)}{3},\ \frac{(a_p)+1}{3},\ \frac{(a_p)+2}{3},\ \frac{a}{2},\ \frac{a+1}{2} \\ \frac{(b_q)}{3},\ \frac{(b_q)+1}{3},\ \frac{(b_q)+2}{3},\ a+\frac{1}{2};\ -\frac{z^3}{27^{q-p}} \end{matrix} \right).$$

85. $$\sum_{k=0}^{\infty} \frac{(a)_{2k}}{k!\,(b)_k} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_{p+1}F_q\left(\begin{matrix} (a_p)+k,\ a+2k \\ (b_q)+k;\ z \end{matrix} \right) = {}_{p+2}F_{q+1}\left(\begin{matrix} (a_p),\ a,\ a-b+1 \\ (b_q),\ b;\ -z \end{matrix} \right).$$

86. $$\sum_{k=1}^{\infty} \frac{(-z)^k}{(a)_{2k}} \frac{\prod (a_p)_k}{\prod (b_q)_k} \, {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ a+2k \end{matrix} \right) = -\frac{z}{a(a+1)} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j} \, {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p)+1,\ \frac{a+1}{2};\ z \\ (b_p)+1,\ \frac{a+3}{2},\ a+1 \end{matrix} \right).$$

87. $$\sum_{k=0}^{\infty} \frac{(2k+a-1)\,(a-1)_k (b)_k (c)_k}{k!\,(a-b)_k (a-c)_k (a-1)_{2k+1}} \times (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} \, {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ a+2k \end{matrix} \right) = {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ a-b-c;\ z \\ (b_q),\ a-b,\ a-c \end{matrix} \right).$$

88. $$\sum_{k=0}^{\infty} \frac{2k+a-1}{(2k+a-2)(2k+a)} \frac{(a-1)_k}{k!\,(a)_{2k}} (-z)^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ a+2k \end{matrix} \right) = \frac{a-1}{a(a-2)} \, {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ 1;\ z \\ (b_q),\ \frac{a}{2},\ \frac{a}{2}+1 \end{matrix} \right).$$

89. $$\sum_{k=0}^{\infty} \frac{(2k+a-1)(a-b)_k}{(a-1)_{2k+1} (b)_k} (-z)^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_pF_{p+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_p)+k,\ a+2k \end{matrix} \right) = {}_{p+1}F_{p+2}\left(\begin{matrix} (a_p),\ b-1;\ z \\ (b_p),\ a-1,\ b \end{matrix} \right).$$

90. $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!\,(b)_{2k}} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} \, {}_pF_{p+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ b+2k \end{matrix} \right) = {}_{p+2}F_{p+3}\left(\begin{matrix} (a_p),\ \frac{b-a}{2},\ \frac{b-a+1}{2};\ z \\ (b_q),\ \frac{b}{2},\ \frac{b+1}{2},\ b-a \end{matrix} \right).$$

91. $$\sum_{k=0}^{\infty} \frac{(a-1)_k (a-b)_k}{k!\,(a)_{2k} (b)_k} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} \, {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ a+2k \end{matrix} \right) = {}_{p+1}F_{p+2}\left(\begin{matrix} (a_p),\ \frac{1-a}{2}+b;\ z \\ (b_q),\ \frac{a+1}{2},\ b \end{matrix} \right).$$

92. $$\sum_{k=0}^{\infty} \frac{2k+a-1}{(2k+a)(2k+a-2)} \frac{(a-1)_k}{k!\,(a)_{2k}} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} \, {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ a+2k \end{matrix} \right) = \frac{a-1}{a(a-2)} \, {}_{p+1}F_{q+2}\left(\begin{matrix} (a_p),\ 1;\ z \\ (b_q),\ \frac{a}{2},\ \frac{a}{2}+1 \end{matrix} \right).$$

93. $$\sum_{k=0}^{\infty} \frac{(2k+b)(b)_k}{k!\,(a)_{2k}} \frac{\prod (a_p)_k}{\prod (b_q)_k} (-z)^k {}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\ (b_p)+k,\ a+2k\end{matrix}\right) = b\,{}_{p+2}F_{q+3}\left(\begin{matrix}(a_p),\ \frac{a-b-1}{2},\ \frac{a-b}{2};\ z\\ (b_p),\ \frac{a}{2},\ \frac{a+1}{2},\ a-b\end{matrix}\right).$$

94. $$\sum_{k=0}^{\infty} \frac{(2k+a-1)(a-1)_k(a-b)_k}{k!\,(a)_{2k}(b)_k} z^k \frac{\prod (a_p)_k}{\prod (b_p)_k} {}_pF_{p+1}\left(\begin{matrix}(a_p)+k;\ z\\ (b_p)+k,\ a+2k\end{matrix}\right) = (a-1)\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ z\\ (b_p),\ b\end{matrix}\right).$$

95. $$\sum_{k=0}^{\infty} \frac{(1-a)_k}{(2k)!\,(a)_k(b)_{2k}} z^k \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_{q+3}\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k,\ b+2k,\ 2k+1,\ b\end{matrix}\right)$$
$$= \frac{1}{2}\,{}_{p+2}F_{q+5}\left(\begin{matrix}(a_p),\ \frac{a+b-1}{2},\ \frac{a+b}{2};\ z\\ (b_q),\ a,\ b,\ \frac{b}{2},\ \frac{b+1}{2},\ a+b-1\end{matrix}\right) + \frac{1}{2}\,{}_pF_{q+3}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ b,\ b,\ 1\end{matrix}\right).$$

96. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{(2k)!\,(b)_{2k}} \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_{q+3}\left(\begin{matrix}(a_p)+k;\ z\\ (b_q)+k,\ b+2k,\ 2k+2,\ b-1\end{matrix}\right) = {}_pF_{q+3}\left(\begin{matrix}(a_p);\ \frac{z}{4}\\ (b_q),\ \frac{b}{2},\ \frac{b+1}{2},\ b-1\end{matrix}\right).$$

97. $$\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!\,(a)_k} \frac{\prod (a_p)_{2k}}{\prod (b_q)_{2k}} {}_pF_q\left(\begin{matrix}(a_p)+2k;\ z\\ (b_q)+2k\end{matrix}\right) = {}_{p+1}F_{q+1}\left(\begin{matrix}(a_p),\ a-\frac{1}{2};\ 2z\\ (b_q),\ 2a-1\end{matrix}\right).$$

98. $$\sum_{k=0}^{\infty} \frac{(-z^2)^k}{k!\,(k+1)!} \frac{\prod (a_p)_{2k}}{\prod (b_q)_{2k}} {}_pF_{q+1}\left(\begin{matrix}(a_p)+2k,\ 2;\ z\\ (b_q)+2k,\ k+2\end{matrix}\right)$$
$$= 1 + z\frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j}\left[2\,{}_{2p}F_{2q+1}\left(\begin{matrix}\frac{(a_p)+1}{2},\ \frac{(a_p)}{2}+1,\ \frac{1}{2};\ -z^2\\ \frac{(b_q)+1}{2},\ \frac{(b_q)}{2}+1,\ 1,\ \frac{3}{2}\end{matrix}\right) - {}_{2p}F_{2q+1}\left(\begin{matrix}\frac{(a_p)+1}{2},\ \frac{(a_p)}{2}+1;\ -z^2\\ \frac{(b_q)+1}{2},\ \frac{(b_q)}{2}+1,\ 2\end{matrix}\right)\right].$$

99. $$\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!\left(a+b+\frac{1}{2}\right)_k} \left(-\frac{z^2}{4}\right)^k \frac{\prod (a_p)_{2k}}{\prod (b_q)_{2k}} {}_{p+1}F_q\left(\begin{matrix}(a_p)+2k,\ a+k\\ (b_q)+2k;\ z\end{matrix}\right) = {}_{p+2}F_{q+1}\left(\begin{matrix}(a_p),\ 2a,\ a+b\\ (b_q),\ 2a+2b;\ z\end{matrix}\right).$$

100. $$\sum_{k=0}^{\infty} \frac{(a)_{3k}}{k!\,(b)_k\left(a-b+\frac{3}{2}\right)_k} \frac{\prod(a_p)_{2k}}{\prod(b_q)_{2k}} \left(\frac{z}{2}\right)^{2k} {}_{p+1}F_q\left(\begin{matrix}(a_p)+2k,\ a+3k\\ (b_q)+2k;\ z\end{matrix}\right)$$
$$= {}_{p+3}F_{q+2}\left(\begin{matrix}(a_p),\ a,\ b-\frac{1}{2},\ a-b+1\\ (b_q),\ 2b-1,\ 2a-2b+2;\ 4z\end{matrix}\right) \quad [9].$$

101. $$\sum_{k=0}^{\infty} \frac{z^{2k}}{(4k+2)!\,(b)_{4k}} \frac{\prod(a_p)_{2k}}{\prod(b_q)_{2k}} {}_pF_{q+3}\left(\begin{matrix}(a_p)+2k;\ z\\ (b_q)+2k,\ b+4k,\ 4k+3,\ b-2\end{matrix}\right)$$
$$= \frac{(b-2)(b-1)}{4z} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_{p+2}F_{q+5}\left(\begin{matrix}(a_p)-1,\ \frac{2b-5}{4},\ \frac{2b-3}{4};\ z\\ (b_q)-1,\ \frac{b}{2}-1,\ \frac{b-1}{2},\ b-\frac{5}{2},\ b-2,\ \frac{1}{2}\end{matrix}\right)\right.$$
$$\left. - {}_pF_{q+3}\left(\begin{matrix}(a_p)-1;\ \frac{z}{4}\\ (b_q)-1,\ \frac{b}{2}-1,\ \frac{b-1}{2},\ b-2\end{matrix}\right)\right].$$

102. $$\sum_{k=0}^{\infty} \frac{(6k+a)\left(\frac{a}{3}\right)_k}{k!\,(a)_{6k+1}} (-z^3)^k \frac{\prod(a_p)_{3k}}{\prod(b_q)_{3k}} {}_pF_{q+1}\left(\begin{matrix}(a_p)+3k;\ z\\ (b_q)+3k,\ a+6k+1\end{matrix}\right)$$
$$= {}_{p+1}F_{q+2}\left(\begin{matrix}(a_p),\ \frac{a}{3};\ \frac{3z}{4}\\ (b_q),\ \frac{a}{2},\ \frac{a+1}{2}\end{matrix}\right).$$

6.17.2. Series containing ${}_pF_q((a_p(k));\ (b_q(k));\ z)$ and trigonometric functions

1. $$\sum_{k=1}^{\infty} \frac{1}{k} \frac{\sin(k\nu\pi)}{\Gamma(\mu+k\nu)\Gamma(\mu-k\nu)} {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+k\nu,\ \mu-k\nu\end{matrix}\right)$$
$$= \frac{\pi(1-\nu)}{2\Gamma^2(\mu)} {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu,\ \mu\end{matrix}\right) \quad [\operatorname{Re}\mu > 1/2;\ 0<\nu<1].$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{\sin(k\nu\pi)}{\Gamma(\mu+k\nu)\Gamma(\mu-k\nu)} {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+k\nu,\ \mu-k\nu\end{matrix}\right)$$
$$= -\frac{\pi\nu}{2\Gamma^2(\mu)} {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu,\ \mu\end{matrix}\right) \quad [\operatorname{Re}\mu > 0;\ 0<\nu<1/2].$$

3. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \frac{\cos[(2k+1)\nu\pi]}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)}$$
$$\times\, {}_pF_{p+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right) = \frac{\pi}{4\Gamma^2(\mu)} {}_pF_{p+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu,\ \mu\end{matrix}\right)$$
$$[\operatorname{Re}\mu > 1/2;\ -1/4<\nu<1/4].$$

4. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \frac{\sin[(2k+1)\nu\pi]}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)}$$
$$\times {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right) = \frac{\pi^2\nu}{4\Gamma^2(\mu)}\,{}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu,\ \mu\end{matrix}\right)$$
$[\operatorname{Re}\mu > 1/2;\ -1/4 < \nu < 1/4]$.

5. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k-1)(2k+3)} \frac{\sin[(2k+1)\nu\pi]}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)}$$
$$\times {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right)$$
$$= -\frac{\pi\sin(2\nu\pi)}{8\Gamma(\mu+2\nu)\Gamma(\mu-2\nu)}\,{}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+2\nu,\ \mu-2\nu\end{matrix}\right)$$
$[\operatorname{Re}\mu > 0;\ -1/4 < \nu < 1/4]$.

6. $$\sum_{k=0}^{\infty} \frac{1}{(2k-1)(2k+3)} \frac{\cos[(2k+1)\nu\pi]}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)}$$
$$\times {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right)$$
$$= -\frac{\pi\sin(2\nu\pi)}{8\Gamma(\mu+2\nu)\Gamma(\mu-2\nu)}\,{}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+2\nu,\ \mu-2\nu\end{matrix}\right)$$
$[\operatorname{Re}\mu > 1/2;\ 0 < \nu < 1/2]$.

7. $$\sum_{k=0}^{\infty} \frac{(2k+1)}{(2k-1)(2k+3)} \frac{\sin[(2k+1)\nu\pi]}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)}$$
$$\times {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right)$$
$$= \frac{\pi\cos(2\nu\pi)}{4\Gamma(\mu+2\nu)\Gamma(\mu-2\nu)}\,{}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+2\nu,\ \mu-2\nu\end{matrix}\right)$$
$[\operatorname{Re}\mu > 1/2;\ 0 < \nu < 1/2]$.

8. $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)}{(2k-1)(2k+3)} \frac{\cos[(2k+1)\nu\pi]}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)}$$
$$\times {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right)$$
$$= -\frac{\pi\cos(2\nu\pi)}{4\Gamma(\mu+2\nu)\Gamma(\mu-2\nu)}\,{}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+2\nu,\ \mu-2\nu\end{matrix}\right)$$
$[\operatorname{Re}\mu > 1/2;\ -1/4 < \nu < 1/4]$.

9. $$\sum_{k=1}^{\infty} \frac{1}{(k^2a^2-b^2)} \frac{\cos(k\nu\pi)}{\Gamma(\mu+k\nu)\Gamma(\mu-k\nu)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+k\nu,\ \mu-k\nu\end{matrix}\right)$$
$$= -\frac{\pi\csc\frac{b\pi}{a}\cos\frac{(1-\nu)b\pi}{a}}{2ab\Gamma\left(\mu+\frac{b\nu}{a}\right)\Gamma\left(\mu-\frac{b\nu}{a}\right)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+\frac{b\nu}{a},\ \mu-\frac{b\nu}{a}\end{matrix}\right)$$
$$+\frac{1}{2\Gamma^2(\mu)b^2}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu,\ \mu\end{matrix}\right) \quad [\operatorname{Re}\mu>0;\ 0<\nu<1].$$

10. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2a^2-b^2} \frac{\cos(k\nu\pi)}{\Gamma(\mu+k\nu)\Gamma(\mu-k\nu)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+k\nu,\ \mu-k\nu\end{matrix}\right)$$
$$= -\frac{\pi\csc\frac{b\pi}{a}\cos\frac{b\nu\pi}{a}}{2ab\Gamma\left(\mu+\frac{b\nu}{a}\right)\Gamma\left(\mu-\frac{b\nu}{a}\right)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+\frac{b\nu}{a},\ \mu-\frac{b\nu}{a}\end{matrix}\right)$$
$$+\frac{1}{2\Gamma^2(\mu)b^2}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu,\ \mu\end{matrix}\right) \quad [\operatorname{Re}\mu>-1/2;\ 0<\nu<1].$$

11. $$\sum_{k=1}^{\infty} \frac{k}{(k^2a^2-b^2)} \frac{\sin(k\nu\pi)}{\Gamma(\mu-k\nu)\Gamma(\mu+k\nu)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+k\nu,\ \mu-k\nu\end{matrix}\right)$$
$$= \frac{\pi\csc\frac{b\pi}{a}\sin\frac{(1-\nu)b\pi}{a}}{2a^2\Gamma\left(\mu+\frac{b\nu}{a}\right)\Gamma\left(\mu-\frac{b\nu}{a}\right)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+\frac{b\nu}{a},\ \mu-\frac{b\nu}{a}\end{matrix}\right)$$
$$[\operatorname{Re}\mu>1/2;\ 0<\nu<1].$$

12. $$\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2a^2-b^2} \frac{\sin(k\nu\pi)}{\Gamma(\mu-k\nu)\Gamma(\mu+k\nu)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+k\nu,\ \mu-k\nu\end{matrix}\right)$$
$$= -\frac{\pi\csc\frac{b\pi}{a}\sin\frac{\nu b\pi}{a}}{2a^2\Gamma\left(\mu+\frac{b\nu}{a}\right)\Gamma\left(\mu-\frac{b\nu}{a}\right)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+\frac{b\nu}{a},\ \mu-\frac{b\nu}{a}\end{matrix}\right)$$
$$[\operatorname{Re}\mu>0;\ 0<\nu<1/2].$$

13. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)^2a^2-b^2]} \frac{\sin[(2k+1)\nu\pi)}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)}$$
$$\times\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right)$$
$$= \frac{\pi\sec\frac{b\pi}{2a}\sin\frac{b\nu\pi}{a}}{4ab\Gamma\left(\mu+\frac{b\nu}{a}\right)\Gamma\left(\mu-\frac{b\nu}{a}\right)}\, {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\ (b_q),\ \mu+\frac{b\nu}{a},\ \mu-\frac{b\nu}{a}\end{matrix}\right)$$
$$[\operatorname{Re}\mu>0;\ -1/4<\nu<1/4].$$

14. $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)}{[(2k+1)^2a^2-b^2]} \frac{\cos[(2k+1)\nu\pi]}{\Gamma(\mu+(2k+1)\nu)\Gamma(\mu-(2k+1)\nu)} \times {}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+(2k+1)\nu,\ \mu-(2k+1)\nu\end{matrix}\right)$$
$$= \frac{\pi\sec\frac{b\pi}{2a}\cos\frac{b\nu\pi}{a}}{4a^2\Gamma\left(\mu+\frac{b\nu}{a}\right)\Gamma\left(\mu-\frac{b\nu}{a}\right)}\,{}_pF_{q+2}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \mu+\frac{b\nu}{a},\ \mu-\frac{b\nu}{a}\end{matrix}\right)$$
$[\operatorname{Re}\mu>1/2;\ -1/4<\nu<1/4]$.

6.17.3. Series containing ${}_pF_q((a_p(k));\ (b_q(k));\ z)$ and special functions

1. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!}\psi(k+a)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_q\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k\end{matrix}\right) = \psi(a) - \frac{z}{a}\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p)+1,\ 1,\ 1;\ z\\(b_q)+1,\ a+1,\ 2\end{matrix}\right).$$

2. $$\sum_{k=1}^{\infty} \frac{(-z)^k}{(a)_k}\psi(k+a)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_q\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k\end{matrix}\right) = -\frac{z}{a^2}\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\left[a\psi(a)\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)+1,\ a;\ z\\(b_q)+1,\ a+1\end{matrix}\right) + {}_{p+2}F_{q+2}\left(\begin{matrix}(a_p)+1,\ a,\ a;\ z\\(b_q)+1,\ a+1,\ a+1\end{matrix}\right)\right].$$

3. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!}\psi(2k+a)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_q\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k\end{matrix}\right) = \psi(a) - z\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\left[\frac{1}{a}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p)+1,\ 1,\ 1;\ z\\(b_q)+1,\ 2,\ \frac{a}{2}+1\end{matrix}\right) + \frac{1}{a+1}\,{}_{p+2}F_{q+2}\left(\begin{matrix}(a_p)+1,\ 1,\ 1;\ z\\(b_q)+1,\ 2,\ \frac{a+3}{2}\end{matrix}\right)\right].$$

4. $$\sum_{k=0}^{\infty} \frac{(16z)^k}{(2k)!}B_{2k}\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k,\ \frac{3}{2}\end{matrix}\right) = {}_pF_{q+1}\left(\begin{matrix}(a_p);\ z\\(b_q),\ \frac{1}{2}\end{matrix}\right).$$

5. $$\sum_{k=0}^{\infty}(2^{2k}-1)\frac{(16z)^k}{(2k)!}B_{2k}\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k,\ \frac{1}{2}\end{matrix}\right)$$
$$=4z\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+1;\ z\\(b_q)+1,\ \frac{3}{2}\end{matrix}\right).$$

6. $$\sum_{k=0}^{\infty}\frac{z^k}{k!}B_k(w)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)+k,\ 1;\ z\\(b_q)+k,\ 2\end{matrix}\right)={}_pF_q\left(\begin{matrix}(a_p);\ wz\\(b_q)\end{matrix}\right).$$

7. $$\sum_{k=0}^{\infty}\frac{z^k}{\left(\frac{1}{2}\right)_{2k}}P_{2k}(w)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k,\ 2k+\frac{3}{2}\end{matrix}\right)$$
$$={}_pF_{q+1}\left(\begin{matrix}(a_p);\ w^2z\\(b_q),\ \frac{1}{2}\end{matrix}\right).$$

8. $$\sum_{k=0}^{\infty}\frac{z^k}{\left(\frac{1}{2}\right)_{2k+1}}P_{2k+1}(w)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k,\ 2k+\frac{5}{2}\end{matrix}\right)$$
$$=2w\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ w^2z\\(b_q),\ \frac{3}{2}\end{matrix}\right).$$

9. $$\sum_{k=0}^{\infty}\frac{z^k}{(2k)!}T_{2k}(w)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k,\ 2k+1\end{matrix}\right)$$
$$=\frac{1}{2}\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ w^2z\\(b_q),\ \frac{1}{2}\end{matrix}\right)+\frac{1}{2}\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ z\\(b_q),\ 1\end{matrix}\right).$$

10. $$\sum_{k=0}^{\infty}\frac{z^k}{(2k+1)!}T_{2k+1}(w)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k,\ 2k+2\end{matrix}\right)$$
$$=w\,{}_pF_{q+1}\left(\begin{matrix}(a_p);\ w^2z\\(b_q),\ \frac{3}{2}\end{matrix}\right).$$

11. $$\sum_{k=0}^{\infty}\frac{z^k}{(2k)!}U_{2k}(w)\frac{\prod(a_p)_k}{\prod(b_q)_k}\,{}_pF_{q+1}\left(\begin{matrix}(a_p)+k;\ z\\(b_q)+k,\ 2k+2\end{matrix}\right)$$
$$={}_pF_{q+1}\left(\begin{matrix}(a_p);\ w^2z\\(b_q),\ \frac{1}{2}\end{matrix}\right).$$

12. $$\sum_{k=0}^{\infty} \frac{z^k}{(2k+1)!} U_{2k+1}(w) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ 2k+3 \end{matrix}\right) = 2w\, {}_pF_{q+1}\left(\begin{matrix} (a_p);\ w^2 z \\ (b_q),\ \frac{3}{2} \end{matrix}\right).$$

13. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{(\lambda+1)_k} L_k^{\lambda}(w) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_{p+1}F_q\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k \end{matrix}\right) = {}_pF_{q+1}\left(\begin{matrix} (a_p);\ wz \\ (b_q),\ \lambda+1 \end{matrix}\right).$$

14. $$\sum_{k=0}^{\infty} \frac{(b)_k (c-a)_k}{(c)_k (\lambda+1)_k} z^k L_k^{\lambda}(w)\, {}_2F_1\left(\begin{matrix} a,\ b+k;\ z \\ c+k \end{matrix}\right) = (1-z)^{-b}\, {}_2F_2\left(\begin{matrix} b,\ c-a;\ \frac{wz}{z-1} \\ c,\ \lambda+1 \end{matrix}\right) \quad [|z|<1].$$

15. $$\sum_{k=0}^{\infty} \frac{z^{k/2}}{(\lambda)_k} C_k^{\lambda}(w) \frac{\prod (a_p)_{k/2}}{\prod (b_q)_{k/2}} {}_pF_{q+1}\left(\begin{matrix} (a_p)+\frac{k}{2};\ z \\ (b_q)+\frac{k}{2},\ \lambda+k+1 \end{matrix}\right)$$
$$= {}_pF_{q+1}\left(\begin{matrix} (a_p);\ w^2 z \\ (b_q),\ \frac{1}{2} \end{matrix}\right) + 2wz^{1/2} \frac{\prod \Gamma[(b_q)]}{\prod \Gamma[(a_p)]} \frac{\prod \Gamma\left[(a_p)+\frac{1}{2}\right]}{\prod \Gamma\left[(b_q)+\frac{1}{2}\right]} \times {}_pF_{q+1}\left(\begin{matrix} (a_p)+\frac{1}{2};\ w^2 z \\ (b_q)+\frac{1}{2},\ \frac{3}{2} \end{matrix}\right).$$

16. $$\sum_{k=0}^{\infty} \frac{z^k}{(\lambda)_{2k}} C_{2k}^{\lambda}(w) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ \lambda+2k+1 \end{matrix}\right) = {}_pF_{q+1}\left(\begin{matrix} (a_p);\ w^2 z \\ (b_q),\ \frac{1}{2} \end{matrix}\right).$$

17. $$\sum_{k=0}^{\infty} \frac{z^k}{(\lambda)_{2k+1}} C_{2k+1}^{\lambda}(w) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_{q+1}\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k,\ \lambda+2k+2 \end{matrix}\right) = 2w\, {}_pF_{q+1}\left(\begin{matrix} (a_p);\ w^2 z \\ (b_q),\ \frac{3}{2} \end{matrix}\right).$$

18. $$\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2w}\right)^k}{(1-\lambda)_k} C_k^{\lambda-k}(w) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_q\left(\begin{matrix} (a_p)+k;\ z \\ (b_q)+k \end{matrix}\right) = {}_pF_{q+1}\left(\begin{matrix} \frac{(a_p)}{2},\ \frac{(a_p)+1}{2};\ \frac{z^2}{4^{q-p+1} w^2} \\ \frac{(b_q)}{2},\ \frac{(b_q)+1}{2},\ 1-\lambda \end{matrix}\right).$$

19. $$\sum_{k=0}^{\infty}\frac{(b)_k(c-a)_k}{(c)_k(1-\lambda)_k}z^kC_{2k}^{\lambda-k}(w)\,{}_2F_1\left(\begin{matrix}a,\ b+k;\ z\\ c+k\end{matrix}\right)$$
$$=(1-z)^{-b}\,{}_3F_2\left(\begin{matrix}b,\ c-a,\ \lambda\\ \frac{1}{2},\ c;\ \frac{w^2z}{z-1}\end{matrix}\right)\quad[|z|<1].$$

20. $$\sum_{k=0}^{\infty}\frac{(b)_k(c-a)_k}{(c)_k(1-\lambda)_k}z^kC_{2k+1}^{\lambda-k}(w)\,{}_2F_1\left(\begin{matrix}a,\ b+k;\ z\\ c+k\end{matrix}\right)$$
$$=2\lambda w(1-z)^{-b}\,{}_3F_2\left(\begin{matrix}b,\ c-a,\ \lambda+1\\ \frac{3}{2},\ c;\ \frac{w^2z}{z-1}\end{matrix}\right)\quad[|z|<1].$$

21. $$\sum_{k=0}^{\infty}\frac{(b)_k(c-a)_k}{(c)_k(\rho+1)_k}z^kP_k^{(\rho,\sigma-k)}(w)\,{}_2F_1\left(\begin{matrix}a,\ b+k;\ z\\ c+k\end{matrix}\right)$$
$$=(1-z)^{-b}\,{}_3F_2\left(\begin{matrix}b,\ c-a,\ \rho+\sigma+1\\ c,\ \rho+1;\ \frac{(w-1)z}{2(1-z)}\end{matrix}\right)\quad[|z|<1].$$

6.17.4. Series containing products of ${}_pF_q((a_p(k));\ (b_q(k));\ z)$

1. $$\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\frac{\prod(c_r)_k}{\prod(d_s)_k}\,{}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ w\end{matrix}\right){}_rF_s\left(\begin{matrix}(c_r)+k;\ z\\ (d_s)+k\end{matrix}\right)$$
$$={}_{p+r}F_{q+s}\left(\begin{matrix}(a_p),\ (c_r);\ wz\\ (b_q),\ (d_s)\end{matrix}\right).$$

2. $$\sum_{k=0}^{\infty}(-1)^{k(r+s+1)}\binom{n}{k}\frac{\left(\frac{w}{z}\right)^k}{(1-a)_k}\frac{\prod(a_p)_k}{\prod(b_q)_k}\frac{\prod(1-d_s)_k}{\prod(1-c_r)_k}$$
$$\times\,{}_{p+1}F_{q+1}\left(\begin{matrix}-n+k,\ (a_p)+k;\ w\\ 2k+a+1,\ (b_q)+k\end{matrix}\right)$$
$$\times\,{}_{r+2}F_{s+1}\left(\begin{matrix}-k,\ a-k,\ (c_r)-k;\ z\\ 1-a-2k,\ (d_s)-k\end{matrix}\right)$$
$$={}_{p+s+1}F_{q+r+1}\left(\begin{matrix}-n,\ (a_p),\ 1-(d_s);\ (-1)^{(r+s)}\ \frac{w}{z}\\ 1-a,\ (b_q),\ 1-(c_r)\end{matrix}\right).$$

6.17.5. Series containing ${}_pF_{p+1}((a_p);\ (b_{p+1});\ \varphi(k,x))$

Notation: $d=\operatorname{Re}\left(\sum_{j=1}^{p+1}b_j-\sum_{i=1}^{p}a_i\right).$

1. $$\sum_{k=1}^{\infty}(-1)^k\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -k^2x\\ (b_{p+1})\end{matrix}\right)=-\frac{1}{2}\quad[\operatorname{Re}b_j>1;\ d>1/2;\ 0<x<\pi^2/4].$$

2. $$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\ -k^2x\\(b_{p+1})\end{matrix}\right) = \frac{(-1)^{n+1}}{(2n+1)!\sqrt{\pi}}$$
$$\times \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_k (2\pi)^k (4x)^{n-k/2} \Gamma\left[\left(n+\frac{3-k}{2}\right)\right]$$
$$\times \frac{\prod \Gamma[(a_p)+n-k/2] \prod \Gamma[(b_{p+1})]}{\prod \Gamma[(a_p)] \prod \Gamma[(b_{p+1})+n-k/2]}$$
$$\left[\operatorname{Re} b_j > -n-1/2;\ \operatorname{Re} a_i > 2n-2+(1\pm 1)/2;\ d > 2n+1/2;\ 0 < x < \pi^2\right].$$

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\ -k^2x\\(b_{p+1})\end{matrix}\right) = \frac{(-1)^{n+1}}{2(2n)!} \pi^{2n-1/2}$$
$$\times \sum_{k=0}^{2n} \binom{2n}{k} 2^k B_k \sum_{j=0}^{2n-k} \binom{2n-k}{j} \pi^{-j} (4x)^{j/2} \Gamma\left(\frac{j+1}{2}\right)$$
$$\times \frac{\prod \Gamma[(a_p)+j/2] \prod \Gamma[(b_{p+1})]}{\prod \Gamma[(a_p)] \prod \Gamma[(b_{p+1})+j/2]}$$
$$\left[\operatorname{Re} b_j > -n-1/2;\ d > 1/2-2n;\ 0 < x < \pi^2/4\right].$$

4. $$\sum_{k=0}^{\infty} \frac{1}{k^2a^2 \pm b^2} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\ -k^2x\\(b_{p+1})\end{matrix}\right)$$
$$= \mp\frac{1}{2b^2} \pm \frac{\pi}{2ab} \left\{\begin{matrix}\coth(b\pi/a)\\ \cot(b\pi/a)\end{matrix}\right\} {}_pF_{p+1}\left(\begin{matrix}(a_p);\ \pm\frac{b^2x}{a^2}\\(b_{p+1})\end{matrix}\right)$$
$$\pm \frac{\sqrt{\pi x}}{a^2} \frac{\prod \Gamma[(a_p)] \prod \Gamma\left[(b_{p+1})+\frac{1}{2}\right]}{\prod \Gamma\left[(a_p)+\frac{1}{2}\right] \prod \Gamma[(b_{p+1})]} \, {}_{p+1}F_{p+2}\left(\begin{matrix}(a_p)+\frac{1}{2},\ 1;\ \pm\frac{b^2x}{a^2}\\(b_{p+1})+\frac{1}{2},\ \frac{3}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re} b_j > 1/2;\ d > -3/2;\ 0 < x < \pi^2\right].$$

5. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2a^2 \pm b^2} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\ -k^2x\\(b_{p+1})\end{matrix}\right)$$
$$= \mp\frac{1}{2b^2} \pm \frac{\pi}{2ab} \left\{\begin{matrix}\operatorname{csch}(b\pi/a)\\ \csc(b\pi/a)\end{matrix}\right\} {}_pF_{p+1}\left(\begin{matrix}(a_p);\ \pm\frac{b^2x}{a^2}\\(b_{p+1})\end{matrix}\right)$$
$$\left[\operatorname{Re} b_j > 1/2;\ d > -3/2;\ 0 < x < \pi^2/4\right].$$

6. $$\sum_{k=2}^{\infty}\frac{(-1)^k}{k^2-1}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -k^2x\\(b_{p+1})\end{matrix}\right)=\frac{1}{2}-\frac{1}{4}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -x\\(b_{p+1})\end{matrix}\right)$$
$$-x\frac{\prod\limits_{i=1}^{p}a_i}{\prod\limits_{j=1}^{p+1}b_j}\,{}_pF_{p+1}\left(\begin{matrix}(a_p)+1;\ -x\\(b_{p+1})+1\end{matrix}\right)\quad\left[\operatorname{Re}b_j>1/2;\ d>-3/2;\ 0<x<\pi^2/4\right].$$

7. $$\sum_{k=1}^{\infty}\frac{\cos(ky)}{k^{2m-2}}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -k^2x\\(b_{p+1})\end{matrix}\right)=(-1)^m\frac{(4x)^{m-1}}{2(m-1)!}\frac{\prod(a_p)_{m-1}}{\prod(b_{p+1})_{m-1}}$$
$$+\sum_{k=0}^{m-2}\frac{x^k}{k!}\frac{\prod(a_p)_k}{\prod(b_{p+1})_k}\left[(-1)^{m-1}\frac{\pi y^{2m-2k-3}}{2(2m-2k-3)!}\right.$$
$$\left.+(-1)^k\sum_{j=0}^{m-k-1}\frac{(-y^2)^j}{(2j)!}\zeta(2m-2j-2k-2)\right]$$
$$\left[\operatorname{Re}b_j>3/2-m;\ d>2m-5/2;\ m\ge1;\ 0<x<\pi^2/4;\ 2\sqrt{x}<y<2\pi-2\sqrt{x}\right].$$

8. $$\sum_{k=1}^{\infty}{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -(k^2+a^2)x\\(b_{p+1})\end{matrix}\right)=-\frac{1}{2}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -a^2x\\(b_{p+1})\end{matrix}\right)$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\frac{\prod\Gamma\left[(a_p)-\frac{1}{2}\right]\prod\Gamma[(b_{p+1})]}{\prod\Gamma[(a_p)]\prod\Gamma\left[(b_{p+1})-\frac{1}{2}\right]}\,{}_pF_{p+1}\left(\begin{matrix}(a_p)-\frac{1}{2};\ -a^2x\\(b_{p+1})-\frac{1}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re}b_j>1/2;\ d>1/2;\ 0<x<\pi^2\right].$$

9. $$\sum_{k=1}^{\infty}(-1)^k\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -(k^2+a^2)x\\(b_{p+1})\end{matrix}\right)=-\frac{1}{2}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -a^2x\\(b_{p+1})\end{matrix}\right)$$
$$\left[\operatorname{Re}b_j>1/2;\ d>1/2;\ 0<x<\pi^2/4\right].$$

10. $$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -(k^2+a^2)x\\(b_{p+1})\end{matrix}\right)$$
$$=-\frac{\pi^2}{12}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -a^2x\\(b_{p+1})\end{matrix}\right)+\frac{x}{2}\frac{\prod\limits_{i=1}^{p}a_i}{\prod\limits_{j=1}^{p}b_j}\,{}_pF_{p+1}\left(\begin{matrix}(a_p)+1;\ -a^2x\\(b_{p+1})+1\end{matrix}\right)$$
$$\left[\operatorname{Re}b_j>-1/2;\ d>-3/2;\ 0<x<\pi^2/4\right].$$

11. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -(k^2+a^2)x \\ (b_{p+1})\end{matrix}\right) = -\frac{\pi^2}{12} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -a^2x \\ (b_{p+1})\end{matrix}\right)$$
$$+ \frac{x}{2} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{p+1} b_j} \, {}_pF_{p+1}\left(\begin{matrix}(a_p)+1;\; -a^2x \\ (b_{p+1})+1\end{matrix}\right) \quad \left[\operatorname{Re} b_j > -1/2;\; d > -3/2;\; 0 < x < \pi^2/4\right].$$

12. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -(2k+1)^2x \\ (b_{p+1})\end{matrix}\right)$$
$$= \frac{\pi^2}{8} - \frac{\sqrt{\pi x}}{2} \frac{\prod \Gamma\left[(a_p)+\frac{1}{2}\right] \prod \Gamma[(b_{p+1})]}{\prod \Gamma[(a_p)] \prod \Gamma\left[(b_{p+1})+\frac{1}{2}\right]}$$
$$\left[\operatorname{Re} b_j > -1/2;\; d > -3/2;\; 0 < x < \pi^2/4\right].$$

13. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -(2k+1)^2x \\ (b_{p+1})\end{matrix}\right)$$
$$= \frac{(-1)^n}{(2n)!} 2^{2n-2} \sqrt{\pi}\, x^n \sum_{k=0}^{2n} \binom{2n}{k} E_k\left(\frac{\pi}{4\sqrt{x}}\right)^k \Gamma\left(n + \frac{1-k}{2}\right)$$
$$\times \frac{\prod \Gamma\left[(a_p)+n-\frac{k}{2}\right] \prod \Gamma[(b_{p+1})]}{\prod \Gamma[(a_p)] \prod \Gamma\left[(b_{p+1})+n-\frac{k}{2}\right]}$$
$$\left[\operatorname{Re} b_j > -n;\; d > -2n-1/2;\; 0 < x < \pi^2/16\right].$$

14. $$\sum_{k=0}^{\infty} \frac{1}{(2k-1)(2k+3)} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -(2k+1)^2x \\ (b_{p+1})\end{matrix}\right)$$
$$= -\frac{\sqrt{\pi x}}{2} \frac{\prod \Gamma\left[(a_p)+\frac{1}{2}\right] \prod \Gamma[(b_{p+1})]}{\prod \Gamma[(a_p)] \prod \Gamma\left[(b_{p+1})+\frac{1}{2}\right]} \, {}_pF_{p+1}\left(\begin{matrix}(a_p)+\frac{1}{2}, 1;\; -4x \\ (b_{p+1})+\frac{1}{2}, \frac{3}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re} b_j > -1/2;\; d > -3/2;\; 0 < x < \pi^2/4\right].$$

15. $$\sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{(2k-1)(2k+3)} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -(2k+1)^2x \\ (b_{p+1})\end{matrix}\right)$$
$$= -\frac{\pi}{4} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -4x \\ (b_{p+1})\end{matrix}\right) \quad \left[\operatorname{Re} b_j > 0;\; d > -1/2;\; 0 < x < \pi^2/16\right].$$

16. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k^2-1)} \, {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -(2k+1)^2x \\ (b_{p+1})\end{matrix}\right)$$
$$= -\frac{\pi}{16}\left[1 + {}_pF_{p+1}\left(\begin{matrix}(a_p);\; -4x \\ (b_{p+1})\end{matrix}\right)\right] \quad \left[\operatorname{Re} b_j > -1;\; d > -1/2;\; |x| < \pi^2/16\right].$$

17. $$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -(2k+1)^2x\\ (b_{p+1})\end{matrix}\right)={}_pF_{p+1}\left(\begin{matrix}(a_p);\ -x\\ (b_{p+1})\end{matrix}\right)-$$
$$-2\sqrt{\pi x}\,\frac{\prod\Gamma\left[(a_p)+\frac{1}{2}\right]\prod\Gamma[(b_{p+1})]}{\prod\Gamma[(a_p)]\prod\Gamma\left[(b_{p+1})+\frac{1}{2}\right]}\,{}_{p+1}F_{p+2}\left(\begin{matrix}(a_p)+\frac{1}{2},\,1;\ -x\\ (b_{p+1})+\frac{1}{2},\,\frac{3}{2}\end{matrix}\right)$$
$$+4x\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{p+1}b_j}\,{}_pF_{p+1}\left(\begin{matrix}(a_p)+1;\ -x\\ (b_{p+1})+1\end{matrix}\right)\qquad\left[\operatorname{Re}b_j>-1/2;\ d>-3/2;\ 0<x<\pi^2/4\right].$$

18. $$\sum_{k=1}^{\infty}(-1)^k\frac{2k+1}{k(k+1)}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -(2k+1)^2x\\ (b_{p+1})\end{matrix}\right)={}_pF_{p+1}\left(\begin{matrix}(a_p);\ -x\\ (b_{p+1})\end{matrix}\right)$$
$$-2\,{}_{p+1}F_{p+2}\left(\begin{matrix}(a_p),\,\frac{3}{2};\ -x\\ (b_{p+1}),\,\frac{1}{2}\end{matrix}\right)\qquad\left[\operatorname{Re}b_j>-n;\ d>-1/2;\ 0<x<\pi^2/16\right].$$

19. $$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2a^2\pm b^2}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -(2k+1)^2x\\ (b_{p+1})\end{matrix}\right)$$
$$=\frac{\pi}{4ab}\left\{\begin{matrix}\tanh[b\pi/(2a)]\\ \tan[b\pi/(2a)]\end{matrix}\right\}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ \pm\frac{b^2x}{a^2}\\ (b_{p+1})\end{matrix}\right)$$
$$-\frac{\sqrt{\pi x}}{2a^2}\,\frac{\prod\Gamma\left[(a_p)+\frac{1}{2}\right]\prod\Gamma[(b_{p+1})]}{\prod\Gamma[(a_p)]\prod\Gamma\left[(b_{p+1})+\frac{1}{2}\right]}\,{}_{p+1}F_{p+2}\left(\begin{matrix}(a_p)+\frac{1}{2},\,1;\ \pm\frac{b^2x}{a^2}\\ (b_{p+1})+\frac{1}{2},\,\frac{3}{2}\end{matrix}\right)$$
$$\left[\operatorname{Re}b_j>-1/2;\ d>-3/2;\ 0\le x\le\pi^2/4\right].$$

20. $$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -((2k+1)^2+a^2)x\\ (b_{p+1})\end{matrix}\right)=\frac{\pi}{4}\,{}_pF_{p+1}\left(\begin{matrix}(a_p);\ -a^2x\\ (b_{p+1})\end{matrix}\right)$$
$$\left[\operatorname{Re}b_j>0;\ d>-1/2;\ 0<x<\pi^2/16\right].$$

6.17.6. Series containing ${}_pF_{p+1}((a_p(k));\ (b_{p+1}(k));\ \varphi(k)z)$

Notation: $\Delta(z)=\left|\dfrac{z^{1/2}}{1+\sqrt{1-z}}\,e^{\sqrt{1-z}}\right|$.

1. $$\sum_{k=1}^{\infty}\frac{k^{2k}}{(2k)!}z^k\frac{\prod(a_p)_k}{\prod(b_p)_k}\,{}_pF_{p+1}\left(\begin{matrix}(a_p)+k;\ -k^2z\\ (b_p)+k,\,2k+1\end{matrix}\right)$$
$$=\frac{1}{2}\,{}_{p+1}F_p\left(\begin{matrix}(a_p),\,1;\ z\\ (b_p)\end{matrix}\right)-\frac{1}{2}\qquad[\Delta(z)<1].$$

2. $$\sum_{k=1}^{\infty} \frac{k^{2k-2}}{(2k)!} z^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_pF_{p+1}\left(\begin{matrix} (a_p)+k;\ -k^2 z \\ (b_p)+k,\ 2k+1 \end{matrix} \right) = \frac{z}{2} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{p} b_j} \qquad [\Delta(z) < 1].$$

3. $$\sum_{k=1}^{\infty} \frac{k^{2k-4}}{(2k)!} z^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_pF_{p+1}\left(\begin{matrix} (a_p)+k;\ -k^2 z \\ (b_p)+k,\ 2k+1 \end{matrix} \right)$$
$$= \frac{z}{2} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{p} b_j} - \frac{z^2}{8} \frac{\prod_{i=1}^{p} a_i(a_i+1)}{\prod_{j=1}^{p} b_j(b_j+1)} \qquad [\Delta(z) < 1].$$

4. $$\sum_{k=1}^{\infty} \frac{k^{2k-6}}{(2k)!} z^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_{p+1}F_{p+2}\left(\begin{matrix} (a_p)+k;\ -k^2 z \\ (b_p)+k,\ 2k+1 \end{matrix} \right)$$
$$= \frac{z}{2} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{p} b_j} - \frac{5z^2}{32} \frac{\prod_{i=1}^{p} a_i(a_i+1)}{\prod_{j=1}^{p} b_j(b_j+1)} + \frac{z^3}{72} \frac{\prod_{i=1}^{p} a_i(a_i+1)(a_i+2)}{\prod_{j=1}^{p} b_j(b_j+1)(b_j+2)} \qquad [\Delta(z) < 1].$$

5. $$\sum_{k=1}^{\infty} \frac{k^{2k}}{(2k)!\,(k^2-a^2)} z^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_pF_{p+1}\left(\begin{matrix} (a_p)+k;\ -k^2 z \\ (b_p)+k,\ 2k+1 \end{matrix} \right)$$
$$= \frac{1}{2a^2} \left[1 - {}_{p+1}F_{p+2}\left(\begin{matrix} (a_p),\ 1;\ -a^2 z \\ (b_p),\ 1-a,\ 1+a \end{matrix} \right) \right] \qquad [\Delta(z) < 1].$$

6. $$\sum_{k=0}^{\infty} \frac{(2k+1)^{2k-3}}{(2k+1)!} z^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_pF_{p+1}\left(\begin{matrix} (a_p)+k;\ -(2k+1)^2 z \\ (b_p)+k,\ 2k+2 \end{matrix} \right)$$
$$= 1 - \frac{4z}{9} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{p} b_j} \qquad [|\Delta(2z)| < 1].$$

7. $$\sum_{k=0}^{\infty} \frac{(2k+1)^{2k-5}}{(2k+1)!} z^k \frac{\prod (a_p)_k}{\prod (b_p)_k} \, {}_pF_{p+1}\left(\begin{matrix} (a_p)+k;\ -(2k+1)^2 z \\ (b_p)+k,\ 2k+2 \end{matrix} \right)$$
$$= 1 - \frac{40z}{81} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{p} b_j} + \frac{16z^2}{225} \frac{\prod_{i=1}^{p} a_i(a_i+1)}{\prod_{j=1}^{p} b_j(b_j+1)} \qquad [|\Delta(2z)| < 1].$$

8. $$\sum_{k=0}^{\infty} \frac{(2k+1)^{2k+1}}{(2k+1)![(2k+1)^2+a^2]} z^k \frac{\prod(a_p)_k}{\prod(b_p)_k} {}_pF_{p+1}\left(\begin{matrix}(a_p)+k;\ -(2k+1)^2 z\\ (b_p)+k,\ 2k+2\end{matrix}\right)$$
$$= \frac{1}{a^2+1} {}_{p+1}F_{p+2}\left(\begin{matrix}(a_p),\ 1;\ a^2 z\\ (b_p),\ \frac{3-ia}{2},\ \frac{3+ia}{2}\end{matrix}\right) \quad [|\Delta(2z)|<1].$$

9. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} t^k {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{z}{k+1}\end{matrix}\right)$$
$$= (tz)^{-1} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_pF_q\left(\begin{matrix}(a_p)-1;\ wz\\ (b_q)-1\end{matrix}\right)-1\right] \quad [t=-we^w;\ |we^{w+1}|<1].$$

6.17.7. Series containing ${}_pF_q((a_p(k));\ (b_q(k));\ \varphi(k)z)$ and special functions

1. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{k!} \left(\frac{w}{2}\right)^k J_{k+\nu}(\sqrt{k+1}\,w)\, {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p);\ \frac{z}{k+1}\\ (b_q)\end{matrix}\right)$$
$$= \frac{\left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{\Gamma(\nu)} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[1-{}_pF_{q+1}\left(\begin{matrix}(a_p)-1;\ -\frac{wz^2}{4}\\ (b_q)-1,\ \nu\end{matrix}\right)\right].$$

2. $$\sum_{k=0}^{\infty} \frac{(k+1)^{(k-\nu)/2-1}}{k!} \left(-\frac{w}{2}\right)^k I_{k+\nu}(\sqrt{k+1}\,w)\, {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p);\ \frac{z}{k+1}\\ (b_q)\end{matrix}\right)$$
$$= \frac{\left(\frac{w}{2}\right)^{\nu-2} z^{-1}}{\Gamma(\nu)} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_pF_{q+1}\left(\begin{matrix}(a_p)-1;\ \frac{wz^2}{4}\\ (b_q)-1,\ \nu\end{matrix}\right)-1\right].$$

3. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(2k)!} z^k H_{2k}\left(\frac{w}{\sqrt{k+1}}\right) \frac{\prod(a_p)_k}{\prod(b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{w^{-2}z^{-1}}{2} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[1-{}_pF_{q+1}\left(\begin{matrix}(a_p)-1;\ w^2 z\\ (b_q)-1,\ -\frac{1}{2}\end{matrix}\right)\right].$$

4. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1/2}}{(2k+1)!} z^k H_{2k+1}\left(\frac{w}{\sqrt{k+1}}\right) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= (wz)^{-1} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_pF_{q+1}\left(\begin{matrix}(a_p)-1;\ w^2z\\ (b_q)-1,\ \frac{1}{2}\end{matrix}\right)-1\right].$$

5. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(\lambda+1)_k} (-z)^k L_k^{\lambda-k}\left(\frac{w}{k+1}\right) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \lambda(wz)^{-1} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_pF_{q+1}\left(\begin{matrix}(a_p)-1;\ wz\\ (b_q)-1,\ \lambda\end{matrix}\right)-1\right].$$

6. $$\sum_{k=0}^{\infty} (k+1)^{-1} \frac{\left(\frac{1}{2}-\lambda\right)_k}{(1-2\lambda)_{2k}} \left(\frac{z}{2w}\right)^k$$
$$\times \frac{\prod (a_p)_k}{\prod (b_q)_k} C_k^{\lambda-k}(1+(k+1)w)\, {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{2\lambda w}{(2\lambda+1)\,z} \frac{\prod_{i=1}^{q}(b_i-1)}{\prod_{i=1}^{p}(a_i-1)} \left[1-{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\ -\lambda-\frac{1}{2};\ -2w^{-1}z\\ (b_q)-1,\ -2\lambda\end{matrix}\right)\right]$$
$$\left[\left|ze^{z+1}\right|<1\right].$$

7. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(1-\lambda)_k} (-z)^k C_{2k}^{\lambda-k}\left(\frac{w}{\sqrt{k+1}}\right) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{w^{-2}z^{-1}}{2(\lambda-1)} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[1-{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\ \lambda-1;\ w^2z\\ (b_q)-1,\ -\frac{1}{2}\end{matrix}\right)\right].$$

8. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1/2}}{(1-\lambda)_k} (-z)^k C_{2k+1}^{\lambda-k}\left(\frac{w}{\sqrt{k+1}}\right) \frac{\prod (a_p)_k}{\prod (b_q)_k} {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= (wz)^{-1} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\ \lambda;\ w^2z\\ (b_q)-1,\ \frac{1}{2}\end{matrix}\right)-1\right].$$

9. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(\sigma+1)_k} z^k P_k^{(\rho-k,\sigma)}\left(\frac{w}{k+1}-1\right) \frac{\prod(a_p)_k}{\prod(b_q)_k}\, {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{2\sigma(wz)^{-1}}{\rho+\sigma} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\, \rho+\sigma\\ (b_q)-1,\, \sigma;\ \frac{wz}{2}\end{matrix}\right)-1\right].$$

10. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-1}}{(-\lambda-\sigma)_k} \left(\frac{2z}{w}\right)^k$$
$$\times P_k^{(\rho-k,\sigma-k)}((k+1)w-1) \frac{\prod(a_p)_k}{\prod(b_q)_k}\, {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{(\rho+\sigma+1)w}{2(\sigma+1)z} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\, -\sigma-1;\ 2w^{-1}z\\ (b_q)-1,\, -\rho-\sigma-1\end{matrix}\right)-1\right].$$

11. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-1}(w-k-1)^k}{(\sigma+1)_k}(-z)^k$$
$$\times P_k^{(\rho-k,\sigma)}\left(\frac{w+k+1}{w-k-1}\right) \frac{\prod(a_p)_k}{\prod(b_q)_k}\, {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{\sigma(wz)^{-1}}{\rho+1} \frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{i=1}^{p}(a_i-1)} \left[1-{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\, -\rho-1;\ wz\\ (b_q)-1,\, \sigma\end{matrix}\right)\right].$$

12. $$\sum_{k=0}^{\infty} \frac{(k+1)^{-1}}{(-\rho-\sigma)_k} \left(\frac{2z}{w}\right)^k P_k^{(\rho-k,\sigma-k)}(1+(k+1)w)$$
$$\times \frac{\prod(a_p)_k}{\prod(b_q)_k}\, {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{\rho+\sigma+1}{\rho+1}\frac{w}{2z} \frac{\prod_{i=1}^{q}(b_i-1)}{\prod_{i=1}^{p}(a_i-1)} \left[1-{}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\, -\rho-1;\ -2w^{-1}z\\ (b_q)-1,\, -\rho-\sigma-1\end{matrix}\right)\right]$$
$$\left[\left|ze^{z+1}\right|<1\right].$$

13. $$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(\rho+1)_k} (-z)^k \frac{\prod (a_p)_k}{\prod (b_q)_k} \times P_k^{(\rho,\,\sigma-k)}\left(1+\frac{w}{k+1}\right) {}_pF_q\left(\begin{matrix}(a_p)+k;\ (k+1)z\\ (b_q)+k\end{matrix}\right)$$
$$= \frac{2\rho}{(\rho+\sigma)\,wz} \frac{\prod_{i=1}^{q}(b_i-1)}{\prod_{i=1}^{p}(a_i-1)} \left[1 - {}_{p+1}F_{q+1}\left(\begin{matrix}(a_p)-1,\ \rho+\sigma\\ (b_q)-1,\ \rho;\ -\frac{wz}{2}\end{matrix}\right)\right] \quad \left[\left|ze^{z+1}\right|<1\right].$$

6.17.8. Series containing products of ${}_pF_q((a_p(k));\ (b_q(k));\ \varphi(k)z)$

1. $$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!} (k+1)^{k-1} \frac{\prod (c_r)_k}{\prod (d_s)_k} \times {}_{p+1}F_q\left(\begin{matrix}-k,\ (a_p)\\ (b_q);\ \frac{w}{k+1}\end{matrix}\right) {}_rF_s\left(\begin{matrix}(c_r)+k;\ (k+1)z\\ (d_s)+k\end{matrix}\right)$$
$$= (wz)^{-1} \frac{\prod_{i=1}^{q}(b_i-1)}{\prod_{i=1}^{p}(a_i-1)} \frac{\prod_{j=1}^{s}(d_j-1)}{\prod_{j=1}^{r}(d_j-1)} \left[{}_{p+r}F_{q+s}\left(\begin{matrix}(a_p)-1,\ (c_r)-1;\ wz\\ (b_q)-1,\ (d_s)-1\end{matrix}\right) - 1\right]$$
$$\left[\left|ze^{z+1}\right|<1\right].$$

Chapter 7

The Connection Formulas

7.1. Elementary Functions

7.1.1. Trigonometric functions

1. $\sin nz = \sin z \sum_{k=0}^{[(n-1)/2]} \frac{(1-n)_{2k}}{k!\,(1-n)_k} 2^{n-2k-1} \cos^{n-2k-1} z.$

2. $\sin 2nz = n \cos z \sum_{k=0}^{n-1} \frac{(1-n)_k (1+n)_k}{(2k+1)!} 2^{2k+1} \sin^{2k+1} z.$

3. $\sin (2n+1)z = (2n+1) \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(2k+1)!} 2^{2k} \sin^{2k+1} z.$

4. $\cos nz = \frac{1}{2} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!(1-n)_k} 2^{n-2k} \cos^{n-2k} z.$

5. $\cos 2nz = \sum_{k=0}^{n} \frac{(-n)_k (n)_k}{(2k)!} 2^{2k} \sin^{2k} z.$

6. $\cos (2n+1)z = \cos z \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(2k)!} 2^{2k} \sin^{2k} z.$

7.2. Special Functions

7.2.1. The psi function $\psi(z)$

1. $\psi'\left(\frac{1}{4}\right) = \pi^2 + 8\mathrm{G}.$

2. $\psi'\left(\frac{3}{4}\right) = \pi^2 - 8\mathrm{G}.$

3. $\psi^{(n)}(z) = (-1)^{n+1} n!\, z^{-n-1} {}_{n+2}F_{n+1}\left(\begin{matrix} 1,\, z,\, z,\, \ldots,\, z;\ 1 \\ z+1,\, z+1,\, \ldots,\, z+1 \end{matrix} \right) \qquad [n \geq 1].$

7.2.2. The incomplete gamma functions $\Gamma(\nu, z)$ and $\gamma(\nu, z)$

1. $\Gamma(\nu - n, z) = \dfrac{(-1)^n}{(1-\nu)_n} \left[\Gamma(\nu, z) - z^{\nu-1} e^{-z} \sum_{k=0}^{n-1} (1-\nu)_k (-z)^{-k} \right].$

2. $\Gamma(\nu + n, z) = (\nu)_n \Gamma(\nu, z) + z^{\nu+n-1} e^{-z} \sum_{k=0}^{n-1} (1-\nu-n)_k (-z)^{-k}.$

3. $$\gamma\left(n + \frac{1}{2}, z\right) = \sqrt{\pi} \left(\frac{1}{2}\right)_n \operatorname{erf}(\sqrt{z}) - \frac{2}{2n+1} z^{n+1/2} e^{-z} \sum_{k=1}^{n} \left(-n - \frac{1}{2}\right)_k (-z)^{-k}.$$

4. $$\gamma\left(\frac{1}{2} - n, z\right) = \frac{(-1)^n}{\left(\frac{1}{2}\right)_n} \sqrt{\pi} \operatorname{erf}(\sqrt{z}) + z^{-n-1/2} e^{-z} \sum_{k=1}^{n} \frac{z^k}{\left(\frac{1}{2} - n\right)_k}.$$

7.2.3. The parabolic cylinder function $D_\nu(z)$

1. $D_{\nu+n}(z) = 2^{-n/2} \sum_{k=0}^{n} \binom{n}{k} 2^{k/2} (-\nu)_k H_{n-k}\left(\dfrac{z}{\sqrt{2}}\right) D_{\nu-k}(z).$

2. $D_{\nu-n}(z) = \dfrac{1}{(-\nu)_n} \left(\dfrac{i}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{2}\, i)^k H_{n-k}\left(\dfrac{iz}{\sqrt{2}}\right) D_{\nu+k}(z).$

3. $D_n(z) = 2^{-n/2} e^{-z^2/4} H_n\left(\dfrac{z}{\sqrt{2}}\right).$

4. $$D_{-n-1}(z) = \frac{2^{(1-n)/2} i^n}{n!} e^{-z^2/4} \sum_{k=1}^{n} \binom{n}{k} (-i)^k H_{k-1}\left(\frac{z}{\sqrt{2}}\right) H_{n-k}\left(\frac{iz}{\sqrt{2}}\right) + \frac{2^{-(n+1)/2}\sqrt{\pi}}{n!} i^n e^{z^2/4} \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right) H_n\left(\frac{iz}{\sqrt{2}}\right).$$

5. $$D_{2n-1/2}(z) = (-2)^n n! \sqrt{\frac{z}{2\pi}} \sum_{k=0}^{n} \frac{\left(-\frac{z^2}{8}\right)^k}{k!} L_{n-k}^{k-1/2}\left(\frac{z^2}{4}\right) \sum_{m=0}^{k} \binom{k}{m} K_{2m-k+1/4}\left(\frac{z^2}{4}\right).$$

6. $$D_{2n-3/2}(z) = (-2)^n n! \sqrt{\frac{z^3}{2\pi}} \sum_{k=0}^{n} \frac{\left(-\frac{z^2}{8}\right)^k}{k!} L_{n-k}^{k-1/2}\left(\frac{z^2}{4}\right) \times \sum_{m=0}^{k} \binom{k}{m} \left[K_{2m-k+3/4}\left(\frac{z^2}{4}\right) - K_{2m-k+1/4}\left(\frac{z^2}{4}\right) \right].$$

7. $D_{-2n-1/2}(z)$

$$= \frac{2^{n-1/2} n!}{\left(\frac{1}{2}\right)_{2n}} \sqrt{\frac{z}{\pi}} \sum_{k=0}^{n} \frac{\left(-\frac{z^2}{8}\right)^k}{k!} L_{n-k}^{k-1/2}\left(-\frac{z^2}{4}\right) \sum_{m=0}^{k} \binom{k}{m} K_{2m-k+1/4}\left(\frac{z^2}{4}\right).$$

8. $$D_{-2n-3/2}(z) = \frac{2^{n-1/2} n!}{\left(\frac{3}{2}\right)_{2n}} \sqrt{\frac{z^3}{\pi}} \sum_{k=0}^{n} \frac{\left(-\frac{z^2}{8}\right)^k}{k!} L_{n-k}^{k+1/2}\left(-\frac{z^2}{4}\right) \times \sum_{m=0}^{k} \binom{k}{m} \left[K_{2m-k+3/4}\left(\frac{z^2}{4}\right) - K_{2m-k+1/4}\left(\frac{z^2}{4}\right)\right].$$

7.2.4. The Bessel functions $J_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, $I_\nu(z)$ and $K_\nu(z)$

1. $$J_{\nu+n}(z) = (\nu)_n \left(\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2};\ -z^2 \\ -n,\ \nu,\ 1-\nu-n \end{matrix}\right) J_\nu(z) - (\nu+1)_{n-1} \left(\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ -z^2 \\ 1-n,\ \nu+1,\ 1-\nu-n \end{matrix}\right) J_{\nu-1}(z)$$
$[n \geq 1;$ [7], 2.7.5.2.(23)].

2. $$J_{\nu-n}(z) = (-\nu)_n \left(-\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2};\ -z^2 \\ -n,\ -\nu,\ \nu-n+1 \end{matrix}\right) J_\nu(z) - (1-\nu)_{n-1} \left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ -z^2 \\ 1-n,\ 1-\nu,\ \nu-n+1 \end{matrix}\right) J_{\nu+1}(z) \quad [n \geq 1].$$

3. $$J_{n+1/2}(z) = (-1)^n \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{k!\,(n-k)!} (2z)^{-k} \sin\left(z + \frac{n-k}{2}\pi\right).$$

4. $$J_{-n-1/2}(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{k!\,(n-k)!} (2z)^{-k} \cos\left(z - \frac{n+k}{2}\pi\right).$$

5. $$J_{n-1/2}(mz) = m^{-n-1/2} \left(\frac{\pi z}{2}\right)^{(m-1)/2} \sum_{k=0}^{[(m-1)/2]} (-1)^{k+n} \binom{m}{2k} \times \sum_{p_1+\ldots+p_m=n} (-1)^{p_{2k+1}+\ldots+p_m} \frac{n!}{p_1!\ldots p_m!} \prod_{i=1}^{2k} J_{1/2-p_i}(z) \prod_{j=2k+1}^{m} J_{p_j-1/2}(z).$$

6. $$J_{n-1/2}(2z) = 2^{-n-1} (\pi z)^{1/2} \sum_{k=0}^{n} \binom{n}{k} [J_{k-1/2}(z)\, J_{n-k-1/2}(z) - (-1)^n J_{1/2-k}(z)\, J_{k-n+1/2}(z)].$$

7. $$J_{1/2-n}(mz) = m^{-n-1/2}\left(\frac{\pi z}{2}\right)^{(m-1)/2} \sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m}{2k+1}$$
$$\times \sum_{p_1+\ldots+p_m=n} (-1)^{p_{2k+2}+\ldots+p_m} \frac{n!}{p_1!\ldots p_m!} \prod_{i=1}^{2k+1} J_{1/2-p_i}(z) \prod_{j=2k+2}^{m} J_{p_j-1/2}(z).$$

8. $$J_{n+1/2}^2(z) + J_{-n-1/2}^2(z) = \frac{4}{\pi^2} K_{n+1/2}(iz)\, K_{n+1/2}(-iz) \qquad [|\arg z| < \pi/2].$$

9. $$= \frac{(n!)^2}{2^{2n-1}\pi} z^{-2n-1} L_n^{-2n-1}(2iz) L_n^{-2n-1}(-2iz).$$

10. $$J_{n-1/2}(x+iy) = \sqrt{\frac{\pi}{2}}\, \frac{x^{n+1/2} y^{1/2}}{(x^2+y^2)^{(2n+1)/4}}$$
$$\times \left\{ \sum_{k=0}^{n} \binom{n}{k} \left(\frac{y}{x}\right)^k \left[(-1)^k \cos\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{n-k-1/2}(x)\, I_{k-1/2}(y) \right.\right.$$
$$\left. - (-1)^n \sin\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{k-n+1/2}(x)\, I_{1/2-k}(y) \right]$$
$$- i\left[(-1)^k \sin\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{n-k-1/2}(x)\, I_{k-1/2}(y) \right.$$
$$\left.\left. + (-1)^n \cos\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{k-n-1/2}(x)\, I_{1/2-k}(y) \right] \right\}.$$

11. $$J_{1/2-n}(x+iy) = \sqrt{\frac{\pi}{2}}\, \frac{x^{n+1/2} y^{1/2}}{(x^2+y^2)^{(2n+1)/4}}$$
$$\times \left\{ \sum_{k=0}^{n} \binom{n}{k} \left(\frac{y}{x}\right)^k \left[\cos\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{k-n+1/2}(x)\, I_{k-1/2}(y) + \right.\right.$$
$$\left. + (-1)^{n-k} \sin\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{n-k-1/2}(x)\, I_{1/2-k}(y) \right]$$
$$+ i\left[-\sin\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{k-n+1/2}(x)\, I_{k-1/2}(y) \right.$$
$$\left.\left. + (-1)^{n-k} \cos\left(\frac{2n+1}{2} \arctan\frac{y}{x}\right) J_{n-k-1/2}(x)\, I_{1/2-k}(y) \right] \right\}.$$

12. $$H_{n-1/2}^{(1)}(\sqrt{z})$$
$$= \sqrt{\frac{2}{\pi}}\, z^{-1/4} e^{i(\sqrt{z}-n\pi/2)} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!\,(n-k-1)!} \frac{1}{(2i\sqrt{z})^k} \qquad [n \ge 1].$$

13. $$H_{n-1/2}^{(2)}(\sqrt{z}) = \sqrt{\frac{2}{\pi}}\, z^{-1/4} e^{-i(\sqrt{z}-n\pi/2)} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!\,(n-k-1)!} \frac{1}{(2i\sqrt{z})^k}$$
$$[n \ge 1].$$

14. $$I_{\nu+n}(z) = (\nu)_n\left(-\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2};\ z^2\\ -n,\ \nu,\ 1-\nu-n\end{matrix}\right)I_\nu(z)$$
$$+ (\nu+1)_{n-1}\left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix}\frac{1-n}{2},\ 1-\frac{n}{2};\ z^2\\ 1-n,\ \nu+1,\ 1-\nu-n\end{matrix}\right)I_{\nu-1}(z) \quad [n \geq 1].$$

15. $$I_{\nu-n}(z) = (-\nu)_n\left(-\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2};\ z^2\\ -n,\ -\nu,\ \nu-n+1\end{matrix}\right)I_\nu(z)$$
$$+ (1-\nu)_{n-1}\left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix}\frac{1-n}{2},\ 1-\frac{n}{2};\ z^2\\ 1-n,\ 1-\nu,\ \nu-n+1\end{matrix}\right)I_{\nu+1}(z) \quad [n \geq 1].$$

16. $$I_{n+1/2}^2(z) - I_{-n-1/2}^2(z) = (-1)^n \frac{4i}{\pi^2} K_{n+1/2}(z)\, K_{n+1/2}(-z)$$
$$[0 < \arg z < \pi].$$

17. $$= -\frac{(n!)^2}{2^{2n-1}\pi} z^{-2n-1} L_n^{-2n-1}(2z) L_n^{-2n-1}(-2z).$$

18. $$K_{\nu+n}(z) = (\nu)_n\left(\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2};\ z^2\\ -n,\ \nu,\ 1-\nu-n\end{matrix}\right)K_\nu(z)$$
$$+ (\nu+1)_{n-1}\left(\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix}\frac{1-n}{2},\ 1-\frac{n}{2};\ z^2\\ 1-n,\ \nu+1,\ 1-\nu-n\end{matrix}\right)K_{\nu-1}(z) \quad [n \geq 1].$$

19. $$K_{\nu-n}(z) = (-\nu)_n\left(\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2};\ z^2\\ -n,\ -\nu,\ \nu-n+1\end{matrix}\right)K_\nu(z)$$
$$+ (1-\nu)_{n-1}\left(\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix}\frac{1-n}{2},\ 1-\frac{n}{2};\ z^2\\ 1-n,\ 1-\nu,\ \nu-n+1\end{matrix}\right)K_{\nu+1}(z) \quad [n \geq 1].$$

20. $$K_{n+1/2}(z) = n!\sqrt{\frac{\pi}{2z}}\,(-2z)^{-n} e^{-z} L_n^{-2n-1}(2z).$$

21. $$K_{n-1/2}(mz)$$
$$= m^{-n-1/2}\left(\frac{2\pi}{z}\right)^{(m-1)/2} \sum_{p_1+\ldots+p_m=n} \frac{n!}{p_1!\ldots p_m!} \prod_{i=1}^m K_{p_i-1/2}(z).$$

22. $$K_{n-1/2}(2z) = \pm(-2)^{-n}\,(\pi z)^{1/2} \sum_{k=0}^n (-1)^k \binom{n}{k} I_{\pm n\mp k\mp 1/2}(z)\, K_{k-1/2}(z).$$

23. $$= 2^{-n}\left(\frac{z}{\pi}\right)^{1/2} \sum_{k=0}^n \binom{n}{k} K_{k-1/2}(z)\, K_{n-k-1/2}(z).$$

24. $$J_{\nu+n}(\sqrt[4]{z})\,K_\nu(\sqrt[4]{z}) = \frac{1}{4\sqrt{\pi}}(\nu)_n\left\{2^{-n/2}\,{}_2F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2};\,-\sqrt{z}\\ -n,\,\nu,\,-\nu-n+1\end{matrix}\right)\right.$$
$$\times G_{04}^{30}\left(\frac{z}{64}\middle|-\frac{n}{4},\,\frac{2-n}{4},\,\frac{2\nu-n}{4},\,-\frac{2\nu+n}{4}\right)$$
$$-\frac{2^n z^{-n/4}}{\nu}(1-\delta_{n,0})\,{}_2F_3\left(\begin{matrix}\frac{1-n}{2},\,1-\frac{n}{2};\,-\sqrt{z}\\ 1-n,\,\nu+1,\,-\nu-n+1\end{matrix}\right)$$
$$\left.\times\left[G_{04}^{30}\left(\frac{z}{64}\middle|0,\,\frac{1}{2},\,\frac{\nu}{2},\,1-\frac{\nu}{2}\right)+G_{04}^{30}\left(\frac{z}{64}\middle|0,\,\frac{1}{2},\,\frac{1+\nu}{2},\,\frac{1-\nu}{2}\right)\right]\right\}.$$

25. $$J_{\nu-n}(\sqrt[4]{z})\,K_\nu(\sqrt[4]{z})$$
$$=(-1)^n\frac{2^{n-2}}{\sqrt{\pi}}(-\nu)_n z^{-n/4}\left\{\left[{}_2F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2};\,-\sqrt{z}\\ -n,\,-\nu,\,\nu-n+1\end{matrix}\right)\right.\right.$$
$$\left.-(1-\delta_{n,0})\,{}_2F_3\left(\begin{matrix}\frac{1-n}{2},\,1-\frac{n}{2};\,-\sqrt{z}\\ 1-n,\,1-\nu,\,\nu-n+1\end{matrix}\right)\right]G_{04}^{30}\left(\frac{z}{64}\middle|0,\,\frac{1}{2},\,\frac{\nu}{2},\,-\frac{\nu}{2}\right)$$
$$+\frac{1-\delta_{n,0}}{\nu}\,{}_2F_3\left(\begin{matrix}\frac{1-n}{2},\,1-\frac{n}{2};\,-\sqrt{z}\\ 1-n,\,1-\nu,\,\nu-n+1\end{matrix}\right)$$
$$\left.\times\left[G_{04}^{30}\left(\frac{z}{64}\middle|0,\,\frac{1}{2},\,\frac{\nu}{2},\,1-\frac{\nu}{2}\right)+G_{04}^{30}\left(\frac{z}{64}\middle|0,\,\frac{1}{2},\,\frac{1+\nu}{2},\,\frac{1-\nu}{2}\right)\right]\right\}.$$

7.2.5. The Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$

1. $$\mathbf{H}_{-n}(z)=(-1)^n\,\mathbf{H}_n(z)-\frac{(-1)^n}{\pi}\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{n-k}}\left(\frac{z}{2}\right)^{n-2k-1}.$$

2. $$\mathbf{H}_{n+1/2}(z)=Y_{n+1/2}(z)+\frac{\left(\frac{z}{2}\right)^{n-1/2}}{n!\sqrt{\pi}}\sum_{k=0}^{n}(-n)_k\left(\frac{1}{2}\right)_k\left(-\frac{4}{z^2}\right)^k.$$

3. $$\mathbf{H}_{-n-1/2}(z)=(-1)^nJ_{n+1/2}(z).$$

4. $$\mathbf{H}_{\nu+n}(z)=(\nu)_n\left(\frac{2}{z}\right)^n{}_2F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2};\,-z^2\\ -n,\,\nu,\,1-\nu-n\end{matrix}\right)\mathbf{H}_\nu(z)$$
$$-(\nu+1)_{n-1}\left(\frac{2}{z}\right)^{n-1}{}_2F_3\left(\begin{matrix}\frac{1-n}{2},\,1-\frac{n}{2};\,-z^2\\ 1-n,\,\nu+1,\,1-\nu-n\end{matrix}\right)\mathbf{H}_{\nu-1}(z)$$
$$+\frac{\left(\frac{z}{2}\right)^{\nu+n-1}}{\sqrt{\pi}}\sum_{k=0}^{n-1}\frac{(1-\nu-n)_k}{\Gamma\left(\nu-k+n+\frac{1}{2}\right)}\left(-\frac{4}{z^2}\right)^k{}_2F_3\left(\begin{matrix}-\frac{k}{2},\,\frac{1-k}{2};\,-z^2\\ -k,\,\nu-k+n,\,1-\nu-n\end{matrix}\right)$$
$$[n\geq 1].$$

5. $$\mathbf{H}_{\nu-n}(z) = (-\nu)_n \left(-\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2};\ -z^2 \\ -n,\ -\nu,\ \nu-n+1 \end{matrix}\right) \mathbf{H}_\nu(z)$$
$$- (1-\nu)_{n-1}\left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ -z^2 \\ 1-n,\ 1-\nu,\ \nu-n+1 \end{matrix}\right) \mathbf{H}_{\nu+1}(z)$$
$$+ \frac{\left(\frac{z}{2}\right)^{\nu-n+1}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(\nu-n+1)_k}{\Gamma\left(\nu+k-n+\frac{5}{2}\right)} {}_2F_3\left(\begin{matrix} -\frac{k}{2}, \frac{1-k}{2};\ -z^2 \\ -k,\ n-k-\nu,\ \nu-n+1 \end{matrix}\right) \quad [n \geq 1].$$

6. $$\mathbf{L}_{-n}(z) = \mathbf{L}_n(z) + \frac{1}{\pi} \sum_{k=0}^{n-1} (-1)^k \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_{n-k}} \left(\frac{z}{2}\right)^{n-2k-1}.$$

7. $$\mathbf{L}_{n+1/2}(z) = I_{-n-1/2}(z) - \frac{\left(\frac{z}{2}\right)^{n-1/2}}{n!\sqrt{\pi}} \sum_{k=0}^{n} (-n)_k \left(\frac{1}{2}\right)_k \left(\frac{4}{z^2}\right)^k.$$

8. $$\mathbf{L}_{-n-1/2}(z) = I_{n+1/2}(z).$$

9. $$\mathbf{L}_{\nu+n}(z) = (\nu)_n \left(-\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2};\ z^2 \\ -n,\ \nu,\ 1-\nu-n \end{matrix}\right) \mathbf{L}_\nu(z)$$
$$+ (\nu+1)_{n-1}\left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ z^2 \\ 1-n,\ \nu+1,\ 1-\nu-n \end{matrix}\right) \mathbf{L}_{\nu-1}(z)$$
$$- \frac{\left(\frac{z}{2}\right)^{\nu+n-1}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(1-\nu-n)_k}{\Gamma\left(\nu-k+n+\frac{1}{2}\right)} \left(\frac{4}{z^2}\right)^k {}_2F_3\left(\begin{matrix} -\frac{k}{2}, \frac{1-k}{2};\ z^2 \\ -k,\ \nu-k+n,\ 1-\nu-n \end{matrix}\right)$$
$$[n \geq 1].$$

10. $$\mathbf{L}_{\nu-n}(z) = (-\nu)_n \left(-\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2};\ z^2 \\ -n,\ -\nu,\ \nu-n+1 \end{matrix}\right) \mathbf{L}_\nu(z)$$
$$+ (1-\nu)_{n-1}\left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ z^2 \\ 1-n,\ 1-\nu,\ \nu-n+1 \end{matrix}\right) \mathbf{L}_{\nu+1}(z)$$
$$+ \frac{\left(\frac{z}{2}\right)^{\nu-n+1}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(\nu-n+1)_k}{\Gamma\left(\nu+k-n+\frac{5}{2}\right)} {}_2F_3\left(\begin{matrix} -\frac{k}{2}, \frac{1-k}{2};\ z^2 \\ -k,\ n-k-\nu,\ \nu-n+1 \end{matrix}\right) \quad [n \geq 1].$$

7.2.6. The Anger $\mathbf{J}_\nu(z)$ and Weber $\mathbf{E}_\nu(z)$ functions

1. $$\mathbf{J}_{-\nu}(z) = \mathbf{J}_\nu(-z).$$

2. $$\mathbf{J}_n(z) = J_n(z).$$

3. $$\mathbf{J}_{\nu+n}(z) = (\nu+1)_{n-1}\left(\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ -z^2 \\ 1-n,\ \nu+1,\ 1-n-\nu \end{matrix}\right)\mathbf{J}_\nu(z)$$
$$- (\nu+2)_{n-2}\left(\frac{2}{z}\right)^{n-2} {}_2F_3\left(\begin{matrix} 1-\frac{n}{2}, \frac{3-n}{2};\ -z^2 \\ 2-n,\ \nu+2,\ 1-n-\nu \end{matrix}\right)\mathbf{J}_{\nu+1}(z)$$
$$+ (-1)^n \frac{\sin(\nu\pi)}{\pi} \sum_{k=0}^{n-2}(1-\nu-n)_k\left(\frac{2}{z}\right)^{k+1} {}_2F_3\left(\begin{matrix} -\frac{k}{2}, \frac{1-k}{2};\ -z^2 \\ -k,\ \nu-k+n,\ 1-n-\nu \end{matrix}\right)$$
$[n \geq 2]$.

4. $$\mathbf{J}_{\nu-n}(z) = (-\nu)_n\left(-\frac{2}{z}\right)^{n} {}_2F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2};\ -z^2 \\ -n,\ -\nu,\ \nu-n+1 \end{matrix}\right)\mathbf{J}_\nu(z)$$
$$- (1-\nu)_{n-1}\left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ -z^2 \\ 1-n,\ 1-\nu,\ \nu-n+1 \end{matrix}\right)\mathbf{J}_{\nu+1}(z)$$
$$- (-1)^n \frac{\sin(\nu\pi)}{\pi} \sum_{k=0}^{n-1}(\nu-n+1)_k\left(-\frac{2}{z}\right)^{k+1} {}_2F_3\left(\begin{matrix} -\frac{k}{2}, \frac{1-k}{2};\ -z^2 \\ k,\ n-k-\nu,\ \nu-n+1 \end{matrix}\right)$$
$[n \geq 1]$.

5. $\mathbf{E}_{-\nu}(z) = -\,\mathbf{E}_\nu(-z)$.

6. $$\mathbf{E}_n(z) = \frac{2z^{n-1}}{(2n-1)!!\,\pi} \sum_{k=0}^{[(n-1)/2]} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-n\right)_k \left(-\frac{4}{z^2}\right)^k - \mathbf{H}_n(z).$$

7. $$\mathbf{E}_{-n}(z) = (-1)^n \frac{2z^{n-1}}{(2n-1)!!\,\pi} \sum_{k=0}^{[(n-1)/2]} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-n\right)_k \left(-\frac{4}{z^2}\right)^k - (-1)^n\,\mathbf{H}_{-n}(z).$$

8. $\mathbf{E}_{n+1/2}(z) = (-1)^n \mathbf{J}_{-n-1/2}(z) = (-1)^n \mathbf{J}_{n+1/2}(-z)$.

9. $\mathbf{E}_{-n+1/2}(z) = (-1)^n \mathbf{J}_{n-1/2}(z) = (-1)^n \mathbf{J}_{-n+1/2}(-z)$.

10. $$\mathbf{E}_{\nu+n}(z) = (\nu+1)_{n-1}\left(\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2}, 1-\frac{n}{2};\ -z^2 \\ 1-n,\ \nu+1,\ 1-n-\nu \end{matrix}\right)\mathbf{E}_\nu(z)$$
$$- (\nu+2)_{n-2}\left(\frac{2}{z}\right)^{n-2} {}_2F_3\left(\begin{matrix} 1-\frac{n}{2}, \frac{3-n}{2};\ -z^2 \\ 2-n,\ \nu+2,\ 1-n-\nu \end{matrix}\right)\mathbf{E}_{\nu+1}(z)$$
$$+ \frac{1}{\pi} \sum_{k=0}^{n-2} \left[1 + (-1)^{k+n}\cos(\nu\pi)\right] (1-\nu-n)_k$$
$$\times \left(-\frac{2}{z}\right)^{k+1} {}_2F_3\left(\begin{matrix} -\frac{k}{2}, \frac{1-k}{2};\ -z^2 \\ -k,\ \nu-k+n,\ 1-n-\nu \end{matrix}\right) \quad [n \geq 2].$$

11. $$\mathbf{E}_{\nu-n}(z) = (-\nu)_n\left(-\frac{2}{z}\right)^n {}_2F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2};\ -z^2 \\ -n,\ -\nu,\ \nu-n+1 \end{matrix}\right)\mathbf{E}_\nu(z)$$
$$- (1-\nu)_{n-1}\left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\begin{matrix} \frac{1-n}{2},\ 1-\frac{n}{2};\ -z^2 \\ 1-n,\ 1-\nu,\ \nu-n+1 \end{matrix}\right)\mathbf{E}_{\nu+1}(z)$$
$$- \frac{1}{\pi}\sum_{k=0}^{n-1}\left[1+(-1)^{k+n}\cos(\nu\pi)\right](\nu-n+1)_k\left(\frac{2}{z}\right)^{k+1}$$
$$\times {}_2F_3\left(\begin{matrix} -\frac{k}{2},\ \frac{1-k}{2};\ -z^2 \\ -k,\ n-k-\nu,\ \nu-n+1 \end{matrix}\right) \quad [n \geq 1].$$

12. $$\mathbf{J}_{n+1/2}(z) = \frac{(-1)^n}{2\pi}\sum_{k=0}^{n-1}(-1)^k\left(\frac{2n-2k+3}{4}\right)_k\left(\frac{2}{z}\right)^{k+1}$$
$$- \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{z}\right)^{k-1}\sum_{m=0}^{k-1}\binom{k-1}{m}\left(\frac{3}{4}\right)_{k-m-1}\left(\frac{z}{2}\right)^m$$
$$\times \{J_{m-1/2}(z)\left[(-1)^{k+n}J_{k-n-1/2}(z) + J_{n-k+1/2}(z)\right]$$
$$- (-1)^m J_{1/2-m}(z)\left[(-1)^{k+n}J_{k-n-1/2}(z) - J_{n-k+1/2}(z)\right]\}$$
$$+ [J_{n+1/2}(z) - (-1)^n J_{-n-1/2}(z)]S(z) + [J_{n+1/2}(z)$$
$$+ (-1)^n J_{-n-1/2}(z)]C(z).$$

13. $$\mathbf{J}_{1/2-n}(z) = \frac{(-1)^n}{2\pi}\sum_{k=0}^{n-1}\left(\frac{2n-2k+1}{4}\right)_k\left(\frac{2}{z}\right)^{k+1}$$
$$- \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{z}\right)^{k-1}\sum_{m=0}^{k-1}\binom{k-1}{m}\left(\frac{3}{4}\right)_{k-m-1}\left(\frac{z}{2}\right)^m$$
$$\times \{J_{m-1/2}(z)\left[(-1)^k J_{1/2-n+k}(z) + (-1)^n J_{n-k-1/2}(z)\right]$$
$$+ (-1)^m J_{1/2-m}(z)\left[(-1)^k J_{1/2-n+k}(z) - (-1)^n J_{n-k-1/2}(z)\right]\}$$
$$+ [J_{1/2-n}(z) - (-1)^n J_{n-1/2}(z)]S(z) + [J_{1/2-n}(z) + (-1)^n J_{n-1/2}(z)]C(z).$$

7.2.7. The Airy functions $\mathrm{Ai}\,(z)$ and $\mathrm{Bi}\,(z)$

1. $$\mathrm{Ai}\left(e^{\pi i/6}z\right) = \frac{1}{\pi}\sqrt{\frac{z}{6}}\left[\mathrm{ker}_{1/3}\left(\frac{2}{3}z^{3/2}\right) - \mathrm{kei}_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right]$$
$$+ \frac{i}{\pi}\sqrt{\frac{z}{6}}\left[\mathrm{ker}_{1/3}\left(\frac{2}{3}z^{3/2}\right) + \mathrm{kei}_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right] \quad [\mathrm{Re}\, z > 0].$$

2. $$\mathrm{Bi}\left(e^{\pi i/6}z\right) = \frac{1}{2}\sqrt{\frac{z}{6}}\left[2\,\mathrm{ber}_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - 2\,\mathrm{bei}_{-1/3}\left(\frac{2}{3}z^{3/2}\right)\right.$$
$$\left. + \left(1+\sqrt{3}\right)\mathrm{ber}_{1/3}\left(\frac{2}{3}z^{3/2}\right) - \left(1-\sqrt{3}\right)\mathrm{bei}_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right]$$
$$+ \frac{i}{2}\sqrt{\frac{z}{6}}\left[2\,\mathrm{ber}_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + 2\,\mathrm{bei}_{-1/3}\left(\frac{2}{3}z^{3/2}\right)\right.$$
$$\left. + \left(1-\sqrt{3}\right)\mathrm{ber}_{1/3}\left(\frac{2}{3}z^{3/2}\right) + \left(1+\sqrt{3}\right)\mathrm{bei}_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right] \quad [\mathrm{Re}\, z > 0].$$

7.2.8. The Kelvin functions $\mathrm{ber}_\nu(z)$, $\mathrm{bei}_\nu(z)$, $\mathrm{ker}_\nu(z)$ and $\mathrm{kei}_\nu(z)$

1. $\mathrm{ber}_{1/2}(z) = \sqrt{\dfrac{2}{\pi z}}\left(\sin\dfrac{3\pi}{8}\sinh\dfrac{z}{\sqrt{2}}\cos\dfrac{z}{\sqrt{2}} - \cos\dfrac{3\pi}{8}\cosh\dfrac{z}{\sqrt{2}}\sin\dfrac{z}{\sqrt{2}}\right).$

2. $\mathrm{bei}_{1/2}(z) = \sqrt{\dfrac{2}{\pi z}}\left(\sin\dfrac{3\pi}{8}\cosh\dfrac{z}{\sqrt{2}}\sin\dfrac{z}{\sqrt{2}} + \cos\dfrac{3\pi}{8}\sinh\dfrac{z}{\sqrt{2}}\cos\dfrac{z}{\sqrt{2}}\right).$

3. $\mathrm{ber}_{-1/2}(z) = \sqrt{\dfrac{2}{\pi z}}\left(\cos\dfrac{3\pi}{8}\cosh\dfrac{z}{\sqrt{2}}\cos\dfrac{z}{\sqrt{2}} + \sin\dfrac{3\pi}{8}\sinh\dfrac{z}{\sqrt{2}}\sin\dfrac{z}{\sqrt{2}}\right).$

4. $\mathrm{bei}_{-1/2}(z) = \sqrt{\dfrac{2}{\pi z}}\left(\cos\dfrac{3\pi}{8}\sinh\dfrac{z}{\sqrt{2}}\sin\dfrac{z}{\sqrt{2}} - \sin\dfrac{3\pi}{8}\cosh\dfrac{z}{\sqrt{2}}\cos\dfrac{z}{\sqrt{2}}\right).$

5. $\mathrm{ker}_{1/2}(z) = \sqrt{\dfrac{\pi}{2z}}\,e^{-z/\sqrt{2}}\cos\left(\dfrac{z}{\sqrt{2}} + \dfrac{3\pi}{8}\right).$

6. $\mathrm{kei}_{1/2}(z) = -\sqrt{\dfrac{\pi}{2z}}\,e^{-z/\sqrt{2}}\sin\left(\dfrac{z}{\sqrt{2}} + \dfrac{3\pi}{8}\right).$

7. $\mathrm{ker}_{-1/2}(z) = \sqrt{\dfrac{\pi}{2z}}\,e^{-z/\sqrt{2}}\cos\left(\dfrac{z}{\sqrt{2}} - \dfrac{\pi}{8}\right).$

8. $\mathrm{kei}_{-1/2}(z) = -\sqrt{\dfrac{\pi}{2z}}\,e^{-z/\sqrt{2}}\sin\left(\dfrac{z}{\sqrt{2}} - \dfrac{\pi}{8}\right).$

9. $\mathrm{ber}_{\nu+n}(z) = (\nu)_n\left(-\dfrac{2}{z}\right)^n$

$$\times\ {}_4F_7\left(\begin{matrix}-\frac{n}{4},\ \frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4};\ -\frac{z^4}{16}\\ -\frac{n}{2},\ \frac{1-n}{2},\ \frac{\nu}{2},\ \frac{1+\nu}{2},\ \frac{1-n-\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{1}{2}\end{matrix}\right)$$
$$\times\left[\cos\frac{n\pi}{4}\,\mathrm{ber}_\nu(z) - \sin\frac{n\pi}{4}\,\mathrm{bei}_\nu(z)\right]$$
$$-\frac{1}{\nu}(\nu)_n\left(-\frac{2}{z}\right)^{n-1}{}_4F_7\left(\begin{matrix}\frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ 1-\frac{n}{2},\ \frac{1+\nu}{4},\ 1+\frac{\nu}{2},\ \frac{1-n-\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{1}{2}\end{matrix}\right)$$
$$\times\left[\sin\frac{(n+1)\pi}{4}\,\mathrm{ber}_{\nu-1}(z) + \cos\frac{(n+1)\pi}{4}\,\mathrm{bei}_{\nu-1}(z)\right]$$
$$+\frac{n-1}{\nu(\nu+n-1)}(\nu)_n\left(-\frac{2}{z}\right)^{n-2}$$
$$\times\ {}_4F_7\left(\begin{matrix}\frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ 1-\frac{n}{2},\ \frac{1+\nu}{2},\ 1+\frac{\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{3-n-\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$
$$\times\left[\sin\frac{n\pi}{4}\,\mathrm{ber}_\nu(z) + \cos\frac{n\pi}{4}\,\mathrm{bei}_\nu(z)\right] - \frac{n-2}{\nu(\nu+1)(\nu+n-1)}(\nu)_n\left(-\frac{2}{z}\right)^{n-3}$$

$$\times\ {}_4F_7\left(\begin{matrix}\frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4},\ \frac{6-n}{4};\ -\frac{z^4}{16}\\ 1-\frac{n}{2},\ \frac{3-n}{2},\ 1+\frac{\nu}{2},\ \frac{3+\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{3-n-\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$
$$\times\left[\cos\frac{(n+1)\pi}{4}\operatorname{ber}_{\nu-1}(z)-\sin\frac{(n+1)\pi}{4}\operatorname{bei}_{\nu-1}(z)\right].$$

10. $\operatorname{bei}_{\nu+n}(z)=(\nu)_n\left(-\frac{2}{z}\right)^n$

$$\times\ {}_4F_7\left(\begin{matrix}-\frac{n}{4},\ \frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4};\ -\frac{z^4}{16}\\ -\frac{n}{2},\ \frac{1-n}{2},\ \frac{\nu}{2},\ \frac{1+\nu}{2},\ \frac{1-n-\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{1}{2}\end{matrix}\right)$$
$$\times\left[\sin\frac{n\pi}{4}\operatorname{ber}_{\nu}(z)+\cos\frac{n\pi}{4}\operatorname{bei}_{\nu}(z)\right]$$
$$+\frac{(\nu+1)_n}{n+\nu}\left(-\frac{2}{z}\right)^{n-1}{}_4F_7\left(\begin{matrix}\frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ 1-\frac{n}{2},\ \frac{1+\nu}{2},\ 1+\frac{\nu}{2},\ \frac{1-n-\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{1}{2}\end{matrix}\right)$$
$$\times\left[\cos\frac{(n+1)\pi}{4}\operatorname{ber}_{\nu-1}(z)-\sin\frac{(n+1)\pi}{4}\operatorname{bei}_{\nu-1}(z)\right]$$
$$+\frac{n-1}{(n+\nu)(n+\nu-1)}(\nu+1)_n\left(-\frac{2}{z}\right)^{n-2}$$
$$\times\ {}_4F_7\left(\begin{matrix}\frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ \frac{3-2n}{4},\ 1-\frac{n}{2},\ \frac{1+\nu}{2},\ 1+\frac{\nu}{2},\ \frac{1-n-\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$
$$\times\left[\cos\frac{n\pi}{4}\operatorname{ber}_{\nu}(z)-\sin\frac{n\pi}{4}\operatorname{bei}_{\nu}(z)\right]$$
$$-\frac{n-2}{(n+\nu)\left[(n+\nu)^2-1\right]}(\nu+2)_n\left(-\frac{2}{z}\right)^{n-3}$$
$$\times\ {}_4F_7\left(\begin{matrix}\frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4},\ \frac{6-n}{4};\ -\frac{z^4}{16}\\ 1-\frac{n}{2},\ \frac{3-n}{2},\ 1+\frac{\nu}{2},\ \frac{3+\nu}{2},\ 1-\frac{n+\nu}{2},\ \frac{3-n-\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$
$$\times\left[\sin\frac{(n+1)\pi}{4}\operatorname{ber}_{\nu-1}(z)+\cos\frac{(n+1)\pi}{4}\operatorname{bei}_{\nu-1}(z)\right].$$

11. $\operatorname{ber}_{\nu-n}(z)=(-\nu)_n\left(\frac{2}{z}\right)^n$

$$\times\ {}_4F_7\left(\begin{matrix}-\frac{n}{4},\ \frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4};\ -\frac{z^4}{16}\\ -\frac{n}{2},\ \frac{1-n}{2},\ -\frac{\nu}{2},\ \frac{1-\nu}{2},\ \frac{1-n+\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{1}{2}\end{matrix}\right)$$
$$\times\left[\cos\frac{n\pi}{4}\operatorname{ber}_{\nu}(z)-\sin\frac{n\pi}{4}\operatorname{bei}_{\nu}(z)\right]$$
$$-\frac{2}{\nu}(-\nu)_n\left(\frac{2}{z}\right)^{n-1}{}_4F_7\left(\begin{matrix}\frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ 1-\frac{n}{2},\ \frac{1-\nu}{2},\ 1-\frac{\nu}{2},\ \frac{1-n+\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{1}{2}\end{matrix}\right)$$

$$\times\left[\sin\frac{(n+1)\pi}{4}\,\mathrm{ber}_{\nu+1}(z)+\cos\frac{(n+1)\pi}{4}\,\mathrm{bei}_{\nu+1}(z)\right]$$

$$+\frac{n-1}{\nu\,(n-\nu-1)}\,(-\nu)_n\left(\frac{2}{z}\right)^{n-2}$$

$$\times\,{}_4F_7\left(\begin{matrix}\frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ 1-\frac{n}{2},\ \frac{1-\nu}{2},\ 1-\frac{\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{3-n+\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$

$$\times\left[\sin\frac{n\pi}{4}\,\mathrm{ber}_{\nu}(z)+\cos\frac{n\pi}{4}\,\mathrm{bei}_{\nu}(z)\right]+\frac{n-2}{\nu(\nu-1)\,(\nu-n+1)}\,(-\nu)_n\left(\frac{2}{z}\right)^{n-3}$$

$$\times\,{}_4F_7\left(\begin{matrix}\frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4},\ \frac{6-n}{4};\ -\frac{z^4}{16}\\ 1-\frac{n}{2},\ \frac{3-n}{2},\ 1-\frac{\nu}{2},\ \frac{3-\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{3-n+\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$

$$\times\left[\cos\frac{(n+1)\pi}{4}\,\mathrm{ber}_{\nu+1}(z)-\sin\frac{(n+1)\pi}{4}\,\mathrm{bei}_{\nu+1}(z)\right].$$

12. $\mathrm{bei}_{\nu-n}(z)=(-\nu)_n\left(\frac{2}{z}\right)^{n}$

$$\times\,{}_4F_7\left(\begin{matrix}-\frac{n}{4},\ \frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4};\ -\frac{z^4}{16}\\ -\frac{n}{2},\ \frac{1-n}{2},\ -\frac{\nu}{2},\ \frac{1-\nu}{2},\ \frac{1-n+\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{1}{2}\end{matrix}\right)$$

$$\times\left[\sin\frac{n\pi}{4}\,\mathrm{ber}_{\nu}(z)+\cos\frac{n\pi}{4}\,\mathrm{bei}_{\nu}(z)\right]$$

$$-\frac{(1-\nu)_n}{\nu-n}\left(\frac{2}{z}\right)^{n-1}{}_4F_7\left(\begin{matrix}\frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ 1-\frac{n}{2},\ \frac{1-\nu}{2},\ 1-\frac{\nu}{2},\ \frac{1-n+\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{1}{2}\end{matrix}\right)$$

$$\times\left[\cos\frac{(n+1)\pi}{4}\,\mathrm{ber}_{\nu+1}(z)-\sin\frac{(n+1)\pi}{4}\,\mathrm{bei}_{\nu+1}(z)\right]$$

$$+\frac{n-1}{(n-\nu)\,(n-\nu-1)}\,(1-\nu)_n\left(\frac{2}{z}\right)^{n-2}$$

$$\times\,{}_4F_7\left(\begin{matrix}\frac{2-n}{4},\ \frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4};\ -\frac{z^4}{16}\\ \frac{1-n}{2},\ 1-\frac{n}{2},\ \frac{1-\nu}{2},\ 1-\frac{\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{3-n+\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$

$$\times\left[\cos\frac{n\pi}{4}\,\mathrm{ber}_{\nu}(z)-\sin\frac{n\pi}{4}\,\mathrm{bei}_{\nu}(z)\right]$$

$$-\frac{n-2}{(n-\nu)\left[(n-\nu)^2-1\right]}\,(2-\nu)_n\left(\frac{2}{z}\right)^{n-3}$$

$$\times\,{}_4F_7\left(\begin{matrix}\frac{3-n}{4},\ 1-\frac{n}{4},\ \frac{5-n}{4},\ \frac{6-n}{4};\ -\frac{z^4}{16}\\ 1-\frac{n}{2},\ \frac{3-n}{2},\ 1-\frac{\nu}{2},\ \frac{3-\nu}{2},\ 1+\frac{\nu-n}{2},\ \frac{3-n+\nu}{2},\ \frac{3}{2}\end{matrix}\right)$$

$$\times\left[\sin\frac{(n+1)\pi}{4}\,\mathrm{ber}_{\nu+1}(z)+\cos\frac{(n+1)\pi}{4}\,\mathrm{bei}_{\nu+1}(z)\right].$$

13. $\mathrm{ker}_{n+1/2}(z)=(-1)^n n!\sqrt{\pi}\,(2z)^{-n-1/2}e^{-z/\sqrt{2}}$

$$\times\left[\cos\left(\frac{z}{\sqrt{2}}+\frac{6n+3}{8}\pi\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k\left(\sqrt{2}\,z\right)^{2k}}{(2k)!}\,L_{n-2k}^{2k-2n-1}\left(\sqrt{2}\,z\right)\right.$$

$$- \sin\left(\frac{z}{\sqrt{2}} + \frac{6n+3}{8}\pi\right) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k \left(\sqrt{2}\,z\right)^{2k+1}}{(2k+1)!} L_{n-2k-1}^{2k-2n}\left(\sqrt{2}\,z\right)\Bigg] \qquad [|\arg z| < \pi].$$

14. $\operatorname{kei}_{n+1/2}(z) = (-1)^{n+1} n! \sqrt{\pi}\,(2z)^{-n-1/2} e^{-z/\sqrt{2}}$

$$\times \Bigg[\sin\left(\frac{z}{\sqrt{2}} + \frac{6n+3}{8}\pi\right) \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\sqrt{2}\,z\right)^{2k}}{(2k)!} L_{n-2k}^{2k-2n-1}\left(\sqrt{2}\,z\right)$$

$$+ \cos\left(\frac{z}{\sqrt{2}} + \frac{6n+3}{8}\pi\right) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k \left(\sqrt{2}\,z\right)^{2k+1}}{(2k+1)!} L_{n-2k-1}^{2k-2n}\left(\sqrt{2}\,z\right)\Bigg] \qquad [|\arg z| < \pi].$$

7.2.9. The Legendre polynomials $P_n(z)$

1. $P_n(z) = (-1)^n P_n(-z)$.

2. $\qquad = C_n^{1/2}(z)$.

3. $\qquad = P_n^{(0,0)}(z)$.

4. $\qquad = \left(\frac{z-1}{2}\right)^n P_n^{(0,-2n-1)}\left(\frac{3+z}{1-z}\right)$.

5. $\qquad = (-2)^n (z^2-1)^{n/2} P_n^{(-n-1/2,-n-1/2)}\left(\frac{z}{\sqrt{z^2-1}}\right)$.

6. $P_{2n}(z) = (z^2-1)^n P_n^{(-1/2,-2n-1/2)}\left(\frac{1+z^2}{1-z^2}\right)$.

7. $\qquad = z^{2n} P_n^{(0,-2n-1/2)}\left(\frac{2}{z^2} - 1\right)$.

8. $P_{2n+1}(z) = z(z^2-1)^n P_n^{(1/2,-2n-3/2)}\left(\frac{1+z^2}{1-z^2}\right)$.

9. $\qquad = z^{2n+1} P_n^{(0,-2n-3/2)}\left(\frac{2}{z^2} - 1\right)$.

7.2.10. The Chebyshev polynomials $T_n(z)$ and $U_n(z)$

1. $T_n(z) = (-1)^n T_n(-z)$.

2. $\qquad = (-1)^n T_{2n}\left(\sqrt{\frac{1-z}{2}}\right)$.

3. $\qquad = T_{2n}\left(\sqrt{\frac{1+z}{2}}\right)$.

4. $\qquad = U_n(z) - zU_{n-1}(z) \qquad [n \geq 1]$.

5. $$= \frac{n}{2} \lim_{\lambda \to 0} \frac{1}{\lambda} C_n^{\lambda}(z).$$

6. $$= \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{(-1/2,\,-1/2)}(z).$$

7. $$T_{2n}(z) = T_n(2z^2 - 1).$$

8. $$= (-1)^n T_{2n}\left(\sqrt{1 - z^2}\right).$$

9. $$= 2T_n^2(z) - 1.$$

10. $$= (-1)^{n+1} + 2T_{2n}\left(\sqrt{\frac{1}{2} + \frac{1}{2}(1 - z^2)^{1/2}}\right) T_{2n}\left(\sqrt{\frac{1}{2} - \frac{1}{2}(1 - z^2)^{1/2}}\right).$$

11. $$= \frac{n!}{\left(n + \frac{1}{2}\right)_n} z^{2n} C_{2n}^{1/2-2n}\left(\frac{\sqrt{z^2 - 1}}{z}\right).$$

12. $$= \frac{n!}{\left(\frac{1}{2}\right)_n} z^{2n} P_n^{(-1/2,\,-2n)}\left(\frac{2}{z^2} - 1\right).$$

13. $$T_{2n+1}(z) = 2T_{2n+1}\left(\sqrt{\frac{1 + \sqrt{1 - z^2}}{2}}\right) T_{2n+1}\left(\sqrt{\frac{1 - \sqrt{1 - z^2}}{2}}\right).$$

14. $$= (-1)^n z U_{2n}\left(\sqrt{1 - z^2}\right).$$

15. $$= \frac{n!}{\left(n + \frac{3}{2}\right)_n} z^{2n+1} C_{2n}^{-1/2-2n}\left(\frac{\sqrt{z^2 - 1}}{z}\right).$$

16. $$= \frac{n!}{\left(\frac{1}{2}\right)_n} z P_n^{(-1/2,\,1/2)}(2z^2 - 1).$$

17. $$= \frac{n!}{\left(\frac{1}{2}\right)_n} z^{2n+1} P_n^{(-1/2,\,-2n-1)}\left(\frac{2}{z^2} - 1\right).$$

18. $$T_n^2(z) = \frac{1}{2}[T_{2n}(z) + 1].$$

19. $$= 1 - (1 - z^2) U_{n-1}^2(z) \qquad [n \geq 1].$$

20. $$U_n(z) = (-1)^n U_n(-z).$$

21. $$= C_n^1(z).$$

22. $$= \frac{(n+1)!}{\left(\frac{3}{2}\right)_n} P_n^{(1/2,\,1/2)}(z).$$

23. $U_{2n}(z) = \frac{(-1)^n}{\sqrt{1-z^2}} T_{2n+1}\left(\sqrt{1-z^2}\right).$

24. $\quad = -\frac{n!\,(2n+1)}{2\left(n+\frac{1}{2}\right)_{n+1}} z^{2n+1}(z^2-1)^{-1/2} C_{2n+1}^{-1/2-2n}\left(\sqrt{1-\frac{1}{z^2}}\right).$

25. $\quad = \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{(1/2,\,-1/2)}(2z^2-1).$

26. $\quad = \frac{(n!)^2}{(2n)!}(2z)^{2n} P_n^{(1/2,\,-2n-1)}\left(\frac{2}{z^2}-1\right).$

27. $U_{2n+1}(z) = 2zU_n(2z^2-1).$

28. $\quad = (-1)^n \frac{z}{\sqrt{1-z^2}} U_{2n+1}\left(\sqrt{1-z^2}\right).$

29. $\quad = -\frac{2(n+1)!}{(4n+3)\left(n+\frac{3}{2}\right)_n} z^{2n+2}(z^2-1)^{-1/2} C_{2n+1}^{-3/2-2n}\left(\sqrt{1-\frac{1}{z^2}}\right).$

30. $\quad = \frac{n!\,(n+1)!}{(2n+1)!}(2z)^{2n+1} P_n^{(1/2,\,-2n-2)}\left(\frac{2}{z^2}-1\right).$

31. $U_n^2(z) = \frac{1}{1-z^2}\left[1-T_{n+1}^2(z)\right].$

32. $$T_{2n}(z)T_{2n+1}\left(\sqrt{1-z^2}\right) + zT_{2n+1}(z)U_{2n-1}\left(\sqrt{1-z^2}\right) = (-1)^n\sqrt{1-z^2} \quad [n \ge 1].$$

7.2.11. The Hermite polynomials $H_n(z)$

1. $H_n(z) = (-1)^n H_n(-z).$

2. $H_{2n}(z) = (-1)^n 2^{2n} n!\, L_n^{-1/2}(z^2).$

3. $H_{2n+1}(z) = (-1)^n 2^{2n+1} n!\, z L_n^{1/2}(z^2).$

4. $$H_{2n}(z_1 z_2 \ldots z_m) = (2n)! \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} \frac{z_1^{2k_1}(z_1^2-1)^{n-k_1}}{(n-k_1)!} \frac{z_2^{2k_2}(z_2^2-1)^{k_1-k_2}}{(k_1-k_2)!} \cdots \times \frac{z_{m-1}^{2k_{m-1}}(z_{m-1}^2-1)^{k_{m-2}-k_{m-1}}}{(k_{m-2}-k_{m-1})!} \frac{1}{(2k_{m-1})!} H_{2k_{m-1}}(z_m).$$

5. $$H_{2n+1}(z_1 z_2 \ldots z_m) = (2n+1)!\, z_1 z_2 \ldots z_{m-1} \times \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} \frac{z_1^{2k_1}(z_1^2-1)^{n-k_1}}{(n-k_1)!} \frac{z_2^{2k_2}(z_2^2-1)^{k_1-k_2}}{(k_1-k_2)!} \cdots \times \frac{z_{m-1}^{2k_{m-1}}(z_{m-1}^2-1)^{k_{m-2}-k_{m-1}}}{(k_{m-2}-k_{m-1})!} \frac{1}{(2k_{m-1}+1)!} H_{2k_{m-1}+1}(z_m).$$

7.2.12. The Laguerre polynomials $L_n^\lambda(z)$

1. $$L_n^{-1/2}(z) = \frac{(-1)^n}{2^{2n} n!} H_{2n}(\sqrt{z}).$$

2. $$L_n^{1/2}(z) = \frac{(-1)^n}{2^{2n+1} n! \sqrt{z}} H_{2n+1}(\sqrt{z}).$$

3. $$L_n^{-m}(z) = \frac{(n-m)!}{n!}(-z)^m L_{n-m}^m(z) \qquad [1 \le m \le n].$$

4. $$L_n^{-n}(z) = \frac{(-z)^n}{n!}.$$

5. $$L_n^{1-n}(z) = \frac{(-1)^n}{n!} z^{n-1}(z-n).$$

6. $$L_n^{-n-1}(z) = \frac{(-1)^n}{n!} e^z \Gamma(n+1, z).$$

7. $$= (-1)^n \sum_{k=0}^{n} \frac{z^k}{k!}.$$

8. $$L_n^{-2n-1}(z) = \frac{(-z)^n}{n!} \sqrt{\frac{z}{\pi}}\, e^{z/2} K_{n+1/2}\left(\frac{z}{2}\right).$$

9. $$L_n^{-2n-1}(z) L_n^{-2n-1}(-z) = \frac{\pi}{4(n!)^2} z^{2n+1} \left[I_{-n-1/2}^2\left(\frac{z}{2}\right) - I_{n+1/2}^2\left(\frac{z}{2}\right)\right].$$

10. $$L_n^\lambda(z_1 + z_2 + \ldots + z_m) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} L_{n-k_1}^{\lambda-\lambda_1-1}(z_1) L_{k_1-k_2}^{\lambda_1-\lambda_2-1}(z_2) \ldots \times L_{k_{m-2}-k_{m-1}}^{\lambda_{m-2}-\lambda_{m-1}-1}(z_{m-1}) L_{k_{m-1}}^{\lambda_{m-1}}(z_m).$$

11. $$L_n^\lambda(z_1 z_2 \ldots z_m) = (\lambda+1)_n \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} \frac{z_1^{k_1}(1-z_1)^{n-k_1}}{(n-k_1)!} \frac{z_2^{k_2}(1-z_2)^{k_1-k_2}}{(k_1-k_2)!} \cdots \times \frac{z_{m-1}^{k_{m-1}}(1-z_{m-1})^{k_{m-1}}}{(k_{m-2}-k_{m-1})!} \frac{1}{(\lambda+1)_{k_{m-1}}} L_{k_{m-1}}^\lambda(z_m).$$

7.2.13. The Gegenbauer polynomials $C_n^\lambda(z)$

1. $C_n^\lambda(z) = (-1)^n C_n^\lambda(-z)$.

2. $$= (-2)^{-n} \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} (z^2-1)^{n/2} C_n^{1/2-\lambda-n}\left(\frac{z}{\sqrt{z^2-1}}\right).$$

3. $$= \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\,\lambda-1/2)}(z).$$

4. $$= 2^n \frac{(\lambda)_n}{(2\lambda+n)_n} (z+1)^n P_n^{(\lambda-1/2,\,-2\lambda-2n)}\left(\frac{3-z}{1+z}\right).$$

5. $$= \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} \left(\frac{z+1}{2}\right)^n P_n^{(\lambda-1/2,\,-2\lambda-2n)}\left(\frac{3-z}{1+z}\right).$$

6. $$= (2z^2+2z\sqrt{z^2-1}-1)^{-n/2} P_n^{(2\lambda-1,\,-\lambda-n)}\left(4z^2+4z\sqrt{z^2-1}-3\right).$$

7. $$= (-2)^n (z^2-1)^{n/2} P_n^{(-\lambda-n,\,-\lambda-n)}\left(\frac{z}{\sqrt{z^2-1}}\right).$$

8. $$C_{2n}^\lambda(z) = (-1)^n \frac{(\lambda)_n}{\left(\frac{1}{2}\right)_n} P_n^{(-1/2,\,\lambda-1/2)}(1-2z^2).$$

9. $$= \frac{(\lambda)_n}{\left(\frac{1}{2}\right)_n} (z^2-1)^n P_n^{(-1/2,\,-\lambda-2n)}\left(\frac{1+z^2}{1-z^2}\right).$$

10. $$= \frac{n!}{(2n)!} (\lambda)_n (2z)^{2n} P_n^{(\lambda-1/2,\,-\lambda-2n)}\left(\frac{2}{z^2}-1\right).$$

11. $$C_{2n+1}^\lambda(z) = 2(-1)^n \frac{(\lambda)_{n+1}}{\left(\frac{3}{2}\right)_n} z P_n^{(1/2,\,\lambda-1/2)}(1-2z^2).$$

12. $$= \frac{2(\lambda)_{n+1}}{\left(\frac{3}{2}\right)_n} z(z^2-1)^n P_n^{(1/2,\,-\lambda-2n-1)}\left(\frac{1+z^2}{1-z^2}\right).$$

13. $$= \frac{n!}{(2n+1)!} (\lambda)_{n+1} (2z)^{2n+1} P_n^{(\lambda-1/2,\,-\lambda-2n-1)}\left(\frac{2}{z^2}-1\right).$$

14. $C_{2n}^{-m-n}(z) = C_{2m}^{-m-n}(z)$.

15. $C_{2n+1}^{-m-n}(z) = C_{2m-1}^{-m-n}(z) \qquad [m \geq 1].$

16. $\lim\limits_{\lambda\to 0} \frac{1}{\lambda} C_n^\lambda(z) = \frac{2}{n} T_n(z) \qquad [n \geq 1].$

17. $C_n^{1/2}(z) = P_n(z)$.

18. $C_n^{-1/2}(z) = \dfrac{1}{n-1}\left[zP_{n-1}(z) - P_n(z)\right]$ $[n \geq 2]$.

19. $C_n^{1}(z) = U_n(z)$.

20. $C_n^{1/2-n}(z) = (-2)^n \dfrac{\left(\frac{1}{2}\right)_n}{n!}(z^2-1)^{n/2} T_n\left(\dfrac{z}{\sqrt{z^2-1}}\right)$.

21. $C_n^{-n-1/2}(z) = 2^{n-1} \dfrac{\left(\frac{3}{2}\right)_n}{(n+1)!}\left[(1-z)^{n+1} + (-1)^n(1+z)^{n+1}\right]$.

22. $C_n^{-n}(z) = (-2)^n (z^2-1)^{n/2} P_n\left(\dfrac{z}{\sqrt{z^2-1}}\right)$ $[n \geq 1]$.

23. $C_{2n}^{1/2-n}(z) = \dfrac{\left(\frac{1}{2}\right)_n}{n!}(1-z^2)^n$.

24. $C_{2n+1}^{1/2-n}(z) = \dfrac{\left(\frac{1}{2}\right)_n}{n!} z(1-z^2)^n$.

25. $C_{2n}^{1/2-2n}(z) = \dfrac{\left(n+\frac{1}{2}\right)_n}{n!}(1-z^2)^n T_{2n}\left(\dfrac{1}{\sqrt{1-z^2}}\right)$.

26. $\qquad = \dfrac{\left(n+\frac{1}{2}\right)_n}{n!}(1-z^2)^n T_n\left(\dfrac{1+z^2}{1-z^2}\right)$.

27. $C_{2n}^{-1/2-2n}(z) = 2^{2n-1} \dfrac{\left(\frac{3}{2}\right)_{2n}}{(2n+1)!}\left[(1-z)^{2n+1} + (1+z)^{2n+1}\right]$.

28. $C_{2n}^{m-n+1/2}(z) = \dfrac{m!\left(\frac{1}{2}-m\right)_n}{n!\left(\frac{1}{2}-n\right)_m}\left(1-z^2\right)^{n-m} C_{2m}^{n-m+1/2}(z)$.

29. $C_{2n+1}^{m-n+1/2}(z) = \dfrac{m!\left(-\frac{1}{2}-m\right)_{n+1}}{n!\left(-\frac{1}{2}-n\right)_{m+1}}\left(1-z^2\right)^{n-m} C_{2m+1}^{n-m+1/2}(z)$.

30. $C_{2n}^{1/2-2n}(z) = \dfrac{\left(\frac{1}{2}\right)_{2n}}{\left(\frac{1}{2}\right)_n^2}(-z)^n C_n^{1/2-n}\left(\dfrac{1+z^2}{2z}\right)$.

31. $C_{2n+1}^{-1/2-2n}(z) = -\dfrac{2\left(n+\frac{1}{2}\right)_{n+1}}{n!\,(2n+1)} z(1-z^2)^n U_{2n}\left(\dfrac{1}{\sqrt{1-z^2}}\right)$.

32. $C_{2n+1}^{-3/2-2n}(z) = 2^{2n} \dfrac{\left(\frac{3}{2}\right)_{2n+1}}{(2n+2)!}\left[(1-z)^{2n+2} - (1+z)^{2n+2}\right]$.

33. $C_{2n}^{1/4-n}(z) = \left(\frac{z}{2}\right)^{2n} C_n^{1/4-n}\left(1 - \frac{2}{z^2}\right).$

34. $C_{2n+1}^{-n-1/4}(z) = -\left(\frac{z}{2}\right)^{2n+1} C_n^{-n-1/4}\left(1 - \frac{2}{z^2}\right).$

35. $\left[C_n^{-n-1/2}(z)\right]^2 = 2^{2n-1} \frac{\left(\frac{3}{2}\right)_n^2}{[(n+1)!]^2} (z^2-1)^{n+1} \left[T_{n+1}\left(\frac{z^2+1}{z^2-1}\right) - 1\right].$

36. $C_n^\lambda(z_1 + z_2 + \ldots + z_m)$

$$= 2^n (\lambda)_n \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} \frac{z_1^{n-k_1}}{(n-k_1)!} \frac{z_2^{k_1-k_2}}{(k_1-k_2)!} \cdots$$
$$\times \frac{z_{m-1}^{k_{m-2}-k_{m-1}}}{(k_{m-2}-k_{m-1})!} \frac{(-2)^{-k_{m-1}}}{(1-\lambda-n)_{k_{m-1}}} C_{k_{m-1}}^{\lambda+n-k_{m-1}}(z_m).$$

37. $C_{2n}^\lambda(z_1 z_2 \ldots z_m) = (-1)^n (\lambda)_n$

$$\times \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} \frac{z_1^{2k_1}(1-z_1^2)^{n-k_1}}{(n-k_1)!} \frac{z_2^{2k_2}(1-z_2^2)^{k_1-k_2}}{(k_1-k_2)!} \cdots$$
$$\times \frac{z_{m-1}^{2k_{m-1}}(1-z_{m-1}^2)^{k_{m-2}-k_{m-1}}}{(k_{m-2}-k_{m-1})!} \frac{1}{(1-\lambda-n)_{k_{m-1}}} C_{2k_{m-1}}^{\lambda+n-k_{m-1}}(z_m).$$

38. $C_{2n+1}^\lambda(z_1 z_2 \ldots z_m) = (-1)^n (\lambda)_n z_1 z_2 \ldots z_{m-1}$

$$\times \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-1}=0}^{k_{m-2}} \frac{z_1^{2k_1}(1-z_1^2)^{n-k_1}}{(n-k_1)!} \frac{z_2^{2k_2}(1-z_2^2)^{k_1-k_2}}{(k_1-k_2)!} \cdots$$
$$\times \frac{z_{m-1}^{2k_{m-1}}(1-z_{m-1}^2)^{k_{m-2}-k_{m-1}}}{(k_{m-2}-k_{m-1})!} \frac{1}{(1-\lambda-n)_{k_{m-1}}} C_{2k_{m-1}+1}^{\lambda+n-k_{m-1}}(z_m).$$

7.2.14. The Jacobi polynomials $P_n^{(\rho,\sigma)}(z)$

1. $P_n^{(\rho,\sigma)}(z) = (-1)^n P_n^{(\sigma,\rho)}(-z).$

2. $\quad = \left(\frac{1-z}{2}\right)^n P_n^{(-\rho-\sigma-2n-1,\,\sigma)}\left(\frac{z+3}{z-1}\right).$

3. $\quad = (-1)^n \left(\frac{1+z}{2}\right)^n P_n^{(-\rho-\sigma-2n-1,\,\rho)}\left(\frac{z-3}{z+1}\right).$

4. $P_n^{(\rho,\rho)}(z) = \frac{(\rho+1)_n}{(2\rho+1)_n} C_n^{\rho+1/2}(z).$

5. $\quad = (-1)^n \left(\frac{z^2-1}{4}\right)^{n/2} C_n^{-\rho-n}\left(\frac{z}{\sqrt{z^2-1}}\right).$

6. $P_n^{(1/2,\sigma)}(z) = \dfrac{(-1)^n}{\sqrt{2(1-z)}} \dfrac{\left(\frac{3}{2}\right)_n}{\left(\sigma+\frac{1}{2}\right)_{n+1}} C_{2n+1}^{\sigma+1/2}\left(\sqrt{\dfrac{1-z}{2}}\right).$

7. $\quad = -2^{-n-1} \dfrac{\left(\frac{3}{2}\right)_n}{(\sigma+n+1)_{n+1}} \dfrac{(z+1)^{n+1/2}}{(z-1)^{1/2}} C_{2n+1}^{-\sigma-2n-1}\left(\sqrt{\dfrac{z-1}{z+1}}\right).$

8. $P_n^{(-1/2,\sigma)}(z) = (-1)^n \dfrac{\left(\frac{1}{2}\right)_n}{(\sigma+1/2)_n} C_{2n}^{\sigma+1/2}\left(\sqrt{\dfrac{1-z}{2}}\right).$

9. $\quad = \dfrac{\left(\frac{1}{2}\right)_n}{(\sigma+n+1)_n} \left(\dfrac{z+1}{2}\right)^n C_{2n}^{-\sigma-2n}\left(\sqrt{\dfrac{z-1}{z+1}}\right).$

10. $P_n^{(\rho,-\rho-2n-1/2)}(z) = \dfrac{(2n)!\,(\rho+1)_n}{n!\,(2\rho+1)_{2n}} \left(\dfrac{z+1}{2}\right)^n C_{2n}^{\rho+1/2}\left(\sqrt{\dfrac{2}{z+1}}\right).$

11. $\quad = \dfrac{(2n)!\,(\rho+1)_n}{n!\,(\rho+1)_{2n}} \left(\dfrac{1-z}{8}\right)^n C_{2n}^{-\rho-2n}\left(\sqrt{\dfrac{2}{1-z}}\right).$

12. $P_n^{(\rho,-\rho-2n-3/2)}(z) = \dfrac{(2n+1)!\,(\rho+1)_n}{n!\,(2\rho+1)_{2n+1}} \left(\dfrac{z+1}{2}\right)^{n+1/2} C_{2n+1}^{\rho+1/2}\left(\sqrt{\dfrac{2}{z+1}}\right).$

13. $\quad = -\dfrac{(2n+1)!\,(\rho+1)_n}{n!\,(\rho+1)_{2n+1}} \left(\dfrac{1-z}{8}\right)^{n+1/2} C_{2n+1}^{-\rho-2n-1}\left(\sqrt{\dfrac{2}{1-z}}\right).$

14. $P_n^{(\rho,-2\rho-2n-1)}(z) = \dfrac{(\rho+1)_n}{(2\rho+1)_n} \left(\dfrac{z+1}{2}\right)^n C_n^{\rho+1/2}\left(\dfrac{3-z}{1+z}\right).$

15. $\quad = (-1)^n \left(\dfrac{1-z}{2}\right)^{n/2} C_n^{-\rho-n}\left(\dfrac{3-z}{2^{3/2}\sqrt{1-z}}\right).$

16. $P_n^{(\rho,-n-(\rho+1)/2)}(z) = 2^{-3n} \dfrac{(\rho+1)_n}{\left(\frac{\rho}{2}+1\right)_n} (1-z)^n C_n^{-n-\rho/2}\left(\dfrac{z+3}{z-1}\right).$

17. $P_n^{(\rho,m-n)}(z) = \dfrac{m!}{n!} \dfrac{\Gamma(n+\rho+1)}{\Gamma(m+\rho+1)} \left(\dfrac{z+1}{2}\right)^{n-m} P_m^{(\rho,n-m)}(z).$

18. $P_n^{(\rho,-\rho-m-n)}(z) = \dfrac{(m-1)!}{n!} \dfrac{\Gamma(n+\rho+1)}{\Gamma(m+\rho)} P_{m-1}^{(\rho,-\rho-m-n)}(z) \qquad [m \geq 1].$

19. $P_n^{(\rho,m)}(z) = \dfrac{(m+n)!}{n!\,(\rho+n+1)_m} \left(\dfrac{2}{z+1}\right)^m P_{m+n}^{(\rho,-m)}(z).$

20. $P_n^{(-n,\rho)}(z) = \dfrac{(\rho+1)_n}{n!} \left(\dfrac{z-1}{2}\right)^n.$

21. $P_n^{(0,0)}(z) = P_n(z).$

22. $P_n^{(0,-1)}(z) = \frac{1}{2}\left[P_{n-1}(z) + P_n(z)\right]$ $\quad [n \geq 1].$

23. $P_n^{(0,1)}(z) = \frac{1}{1-z}\left[P_n(z) - P_{n+1}(z)\right].$

24. $P_n^{(0,1/2)}(z) = \left(\frac{2}{z+1}\right)^{1/2} P_{2n+1}\left(\sqrt{\frac{z+1}{2}}\right).$

25. $P_n^{(0,-1/2)}(z) = P_{2n}\left(\sqrt{\frac{z+1}{2}}\right).$

26. $P_n^{(1/2,1/2)}(z) = \frac{\left(\frac{3}{2}\right)_n}{(n+1)!} U_n(z).$

27. $P_n^{(-1/2,-1/2)}(z) = \frac{\left(\frac{1}{2}\right)_n}{n!} T_n(z).$

28. $P_n^{(-1/2,1/2)}(z) = (-1)^n P_n^{(1/2,-1/2)}(-z).$

29. $\qquad = (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} U_{2n}\left(\sqrt{\frac{1-z}{2}}\right).$

30. $\qquad = \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{2}{z+1}\right)^{1/2} T_{2n+1}\left(\sqrt{\frac{z+1}{2}}\right).$

31. $P_n^{(0,-n-1/2)}(z) = \left(\frac{z+1}{2}\right)^{n/2} P_n\left(\frac{z+3}{2^{3/2}\sqrt{z+1}}\right).$

32. $P_n^{(0,-2n-1)}(z) = (-1)^n \left(\frac{z+1}{2}\right)^n P_n\left(\frac{z-3}{z+1}\right).$

33. $P_n^{(0,-2n-1/2)}(z) = \left(\frac{z+1}{2}\right)^n P_{2n}\left(\sqrt{\frac{2}{z+1}}\right).$

34. $P_n^{(0,-2n-3/2)}(z) = \left(\frac{z+1}{2}\right)^{n+1/2} P_{2n+1}\left(\sqrt{\frac{2}{z+1}}\right).$

35. $P_n^{(1/2,-2n-1)}(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z+1}{8}\right)^n U_{2n}\left(\sqrt{\frac{2}{z+1}}\right).$

36. $P_n^{(1/2,-2n-3/2)}(z) = (-2)^{-n} \frac{(-z-1)^{n+1/2}}{(1-z)^{1/2}} P_{2n+1}\left(\sqrt{\frac{z-1}{z+1}}\right).$

37. $P_n^{(1/2,-2n-2)}(z) = \frac{(2n+1)!}{n!\,(n+1)!} \left(\frac{z+1}{8}\right)^{n+1/2} U_{2n+1}\left(\sqrt{\frac{2}{z+1}}\right).$

38. $$= (-1)^n \frac{\left(\frac{3}{2}\right)_n}{(n+1)!} \left(\frac{z+1}{2}\right)^n U_n\left(\frac{z-3}{z+1}\right).$$

39. $$P_n^{(-1/2,\,-2n)}(z) = (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{z+1}{2}\right)^n T_n\left(\frac{z-3}{z+1}\right).$$

40. $$= \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{z+1}{2}\right)^n T_{2n}\left(\sqrt{\frac{2}{z+1}}\right).$$

41. $$P_n^{(-1/2,\,-2n-1/2)}(z) = (-1)^n \left(\frac{z+1}{2}\right)^n P_{2n}\left(\sqrt{\frac{z-1}{z+1}}\right).$$

42. $$P_n^{(-1/2,\,-2n-1)}(z) = (-1)^n \frac{\left(\frac{3}{2}\right)_n}{n!\,(2n+1)} \left(\frac{z+1}{2}\right)^n U_{2n}\left(\sqrt{\frac{z-1}{z+1}}\right).$$

43. $$= \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{z+1}{2}\right)^{n+1/2} T_{2n+1}\left(\sqrt{\frac{2}{z+1}}\right).$$

44. $$P_n^{(-n-1,\,-n-1)}(z) = \left(-\frac{1}{2}\right)^n (z^2-1)^{n/2}\, U_n\left(\frac{z}{\sqrt{z^2-1}}\right).$$

45. $$P_n^{(-n-1/2,\,-n-1/2)}(z) = \left(\frac{1}{2}\right)^n (z^2-1)^{n/2} P_n\left(\frac{z}{\sqrt{z^2-1}}\right).$$

46. $$P_{2n}^{(-2n-1,\,-2n-1)}(z) = 2^{-2n}(1-z^2)^{n+1/2} T_{2n+1}\left(\frac{1}{\sqrt{1-z^2}}\right).$$

47. $$P_{2n+1}^{(-2n-2,\,-2n-2)}(z) = -2^{-2n-1} z(1-z^2)^{n+1/2} U_{2n+1}\left(\frac{1}{\sqrt{1-z^2}}\right).$$

48. $$(1+z)P_n^{(\rho,\,-\rho)}(z)P_n^{(-\rho-1,\,\rho+1)}(z) + (1-z)P_n^{(-\rho,\,\rho)}(z)P_n^{(\rho+1,\,-\rho-1)}(z) = \frac{2(-\rho)_n(\rho+1)_n}{(n!)^2}.$$

49. $$P_n^{(\rho,\sigma)}(z_1+z_2+\ldots+z_m) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{m-1}=0}^{k_{m-2}} \frac{\left(\frac{z_1}{2}\right)^{n-k_1}}{(n-k_1)!} \frac{\left(\frac{z_2}{2}\right)^{k_1-k_2}}{(k_1-k_2)!}\cdots\frac{\left(\frac{z_{m-1}}{2}\right)^{k_{m-2}-k_{m-1}}}{(k_{m-2}-k_{m-1})!} \times (\rho+\sigma+n+1)_{n-k_{m-1}} P_{k_{m-1}}^{(\rho+n-k_{m-1},\,\sigma+n-k_{m-1})}(z_m).$$

7.2.15. The polynomials of the imaginary argument

1. $$P_n(iz) = \left(-\frac{i}{2}\right)^n (1+z^2)^{n/2} C_n^{-n}\left(\frac{z}{\sqrt{1+z^2}}\right) \qquad [n \geq 1].$$

2. $$T_{2n}(iz) = (-1)^n T_{2n}\left(\sqrt{1+z^2}\right).$$

3. $T_{2n+1}(iz) = (-1)^n iz U_{2n}\left(\sqrt{1+z^2}\right).$

4. $U_{2n}(iz) = \dfrac{(-1)^n}{\sqrt{1+z^2}} T_{2n+1}\left(\sqrt{1+z^2}\right).$

5. $U_{2n+1}(iz) = (-1)^n \dfrac{iz}{\sqrt{1+z^2}} U_{2n+1}\left(\sqrt{1+z^2}\right).$

6. $H_{2n}(iz) = (-4)^n n!\, L_n^{-1/2}(-z^2).$

7. $H_{2n+1}(iz) = (-1)^n 2^{2n+1} n!\, iz L_n^{1/2}(-z^2).$

8. $C_n^\lambda(iz) = (-2i)^n \dfrac{(\lambda)_n}{(2\lambda+n)_n}(1+z^2)^{n/2} C_n^{1/2-\lambda-n}\left(\dfrac{z}{\sqrt{1+z^2}}\right).$

7.2.16. The complete elliptic integral $\mathbf{K}(z)$

1. $\mathbf{K}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{4\sqrt{\pi}}\Gamma^2\left(\dfrac{1}{4}\right).$

2. $\mathbf{K}\left(\sqrt{\dfrac{1-\sqrt{2}}{2}}\right) = \dfrac{2^{-11/4}}{\pi^{1/2}}\Gamma\left(\dfrac{1}{8}\right)\Gamma\left(\dfrac{3}{8}\right).$

3. $\mathbf{K}\left(\sqrt{\dfrac{2-\sqrt{3}}{4}}\right) = \dfrac{3^{-1/4}\pi^{-1/2}}{4}\Gamma\left(\dfrac{1}{6}\right)\Gamma\left(\dfrac{1}{3}\right).$

4. $\mathbf{K}\left(\sqrt{\dfrac{4-3\sqrt{2}}{8}}\right) = \dfrac{2^{-9/4}}{\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right).$

5. $\mathbf{K}\left(\sqrt{2}-1\right) = \dfrac{\left(1+\sqrt{2}\right)^{1/2}}{2^{13/4}\pi^{1/2}}\Gamma\left(\dfrac{1}{8}\right)\Gamma\left(\dfrac{3}{8}\right)$ [48].

6. $\mathbf{K}\left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right) = \dfrac{3^{1/4}}{2^{7/3}\pi}\Gamma^3\left(\dfrac{1}{3}\right)$ [48].

7. $\mathbf{K}\left(3-2\sqrt{2}\right) = \dfrac{1+\sqrt{2}}{2^{7/2}\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right)$ [48].

8. $\mathbf{K}\left(\sqrt{2\sqrt{2}-2}\right) = \dfrac{\left(2+\sqrt{2}\right)^{1/2}}{8\pi^{1/2}}\Gamma\left(\dfrac{1}{8}\right)\Gamma\left(\dfrac{3}{8}\right).$

9. $\mathbf{K}\left(\sqrt{3-2\sqrt{2}}\right) = \dfrac{\left(2+\sqrt{2}\right)^{1/2}}{2^{7/2}\pi^{1/2}}\Gamma\left(\dfrac{1}{8}\right)\Gamma\left(\dfrac{3}{8}\right).$

10. $\mathbf{K}\left(2\sqrt{3\sqrt{2}-4}\right) = \dfrac{\left(3+2\sqrt{2}\right)^{1/2}}{2^{5/2}\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right).$

11. $\mathbf{K}\left(\sqrt{12\sqrt{2}-16}\right) = \dfrac{2+\sqrt{2}}{8\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right).$

12. $\mathbf{K}\left(\sqrt{17-12\sqrt{2}}\right) = \dfrac{\left(3+2\sqrt{2}\right)^{1/2}}{2^{7/2}\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right).$

13. $\mathbf{K}\left(\left(\sqrt{3}-\sqrt{2}\right)\left(2-\sqrt{3}\right)\right)$
$$= \frac{\left[\left(2-\sqrt{2}\right)\left(3+\sqrt{6}\right)\left(1+\sqrt{3}\right)\right]^{1/2}}{48\pi^{1/2}}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right) \quad [48].$$

14. $\mathbf{K}\left(\dfrac{3-\sqrt{7}}{4\sqrt{2}}\right) = \dfrac{7^{-1/4}}{4\pi}\Gamma\left(\dfrac{1}{7}\right)\Gamma\left(\dfrac{2}{7}\right)\Gamma\left(\dfrac{4}{7}\right)$ [48].

15. $\mathbf{K}\left(\dfrac{\left(\sqrt{3}-1\right)\left(\sqrt{2}-\sqrt[4]{3}\right)}{2}\right) = \dfrac{3^{-3/4}\left(1+\sqrt{3}\right)}{2^{5/2}\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right)$ [48].

16. $\mathbf{K}\left(\dfrac{3-2\sqrt{2}}{\left(1+\sqrt[4]{2}\right)^4}\right) = \dfrac{\left(1+\sqrt[4]{2}\right)^2}{2^{9/2}\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right)$ [48].

17. $\mathbf{K}\left(\dfrac{\left(3-2\sqrt[4]{5}\right)\left(\sqrt{5}-2\right)}{\sqrt{2}}\right) = \dfrac{\left(2+\sqrt{5}\right)}{20\pi^{1/2}}\Gamma^2\left(\dfrac{1}{4}\right)$ [48].

18. $\mathbf{K}\left(\left(5-2\sqrt{6}\right)\left(3-2\sqrt{2}\right)\right) = \dfrac{3^{1/4}\left(\sqrt{3}+2\sqrt{2}+1\right)}{2^{29/6}\pi}\Gamma^3\left(\dfrac{1}{3}\right)$ [49].

19. $\mathbf{K}\left(\left(\sqrt{2}-1\right)^3\left(2-\sqrt{3}\right)^2\right)$
$$= \frac{2^{-13/4}\left(1+\sqrt{2}\right)^{1/2}\left(\sqrt{6}+\sqrt{2}-1\right)}{3\pi^{1/2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right) \quad [49].$$

20. $\mathbf{K}\left(\dfrac{\left(2-\sqrt{3}\right)\left(3-\sqrt{5}\right)\left(\sqrt{3}-\sqrt{5}\right)}{8\sqrt{2}}\right)$
$$= \frac{3^{-1/4}5^{-7/12}}{4\pi}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{2}{3}\right) \quad [49].$$

21. $\mathbf{K}\left(\sqrt{\dfrac{5\sqrt{6}-3\sqrt{2}-8}{5\sqrt{6}-3\sqrt{2}+8}}\right)$
$$= \frac{3^{-1/4}}{16\pi^{1/2}}\left(2\sqrt{2}+\sqrt{3}+2\sqrt{6}+6\right)^{1/2}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right).$$

7.2.17. The complete elliptic integral $\mathbf{E}(z)$

1. $\mathbf{E}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{8\sqrt{\pi}}\left[\Gamma^2\left(\dfrac{1}{4}\right)+4\Gamma^2\left(\dfrac{3}{4}\right)\right].$

2. $\mathbf{E}\left(3-2\sqrt{2}\right)=\frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma^2\left(\frac{1}{4}\right)+4\left(\sqrt{2}-1\right)\Gamma^2\left(\frac{3}{4}\right)\right].$

3. $\mathbf{E}\left(\sqrt{3-2\sqrt{2}}\right)=\frac{\left(\sqrt{2}-1\right)^{1/2}}{2^{15/4}\pi^{1/2}}\left[(1+\sqrt{2})\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)+8\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)\right].$

4. $\mathbf{E}\left(\sqrt{2\sqrt{2}-2}\right)=\frac{\left(2-\sqrt{2}\right)^{1/2}}{2^{7/2}\pi^{1/2}}\left[\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)+8\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)\right].$

5. $\mathbf{E}\left(2\sqrt{3\sqrt{2}-4}\right)=\frac{2-\sqrt{2}}{8\pi^{1/2}}\left[\Gamma^2\left(\frac{1}{4}\right)+8\Gamma^2\left(\frac{3}{4}\right)\right].$

6. $\mathbf{E}\left(\sqrt{\frac{1-\sqrt{2}}{2}}\right)=\frac{2^{-5/4}}{\pi^{1/2}}\left[\frac{1+\sqrt{2}}{8}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)+\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)\right].$

7. $\mathbf{E}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)=\frac{3^{-3/4}\left(1+\sqrt{3}\right)}{8\pi^{1/2}}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)+\frac{3^{1/4}}{4\pi^{1/2}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right).$

8. $\mathbf{E}\left(\sqrt{\frac{4-3\sqrt{2}}{8}}\right)=\frac{2^{-15/4}}{\pi^{1/2}}\left[\left(1+\sqrt{2}\right)\Gamma^2\left(\frac{1}{4}\right)+4\Gamma^2\left(\frac{3}{4}\right)\right].$

9. $$\mathbf{E}\left(\frac{3-\sqrt{7}}{4\sqrt{2}}\right)=\frac{7^{-3/4}\left(2+\sqrt{7}\right)}{8\pi}\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)+\frac{7^{1/4}}{8\pi}\Gamma\left(\frac{3}{7}\right)\Gamma\left(\frac{5}{7}\right)\Gamma\left(\frac{6}{7}\right).$$

7.2.18. The Legendre function $P_\nu^\mu(z)$

1. $$P_{n-1/2}^{\mu}(z)=2^n\left(\frac{3-2\mu-2n}{4}\right)_n(1-z^2)^{n/2}\sum_{k=0}^{n}\binom{n}{k}\frac{(-z^2)^k}{\left(\frac{2\mu-2n+1}{4}\right)_k}\left[\delta_{k,0}P_{-1/2}^{\mu-n}(z)+(2z)^{-k}(1-z^2)^{-k/2}\right.$$
$$\left.\times\sum_{p=0}^{k-1}\frac{(k+p-1)!}{p!\,(k-p-1)!}(2z)^{-p}(1-z^2)^{p/2}P_{-1/2}^{\mu+k-n-p}(z)\right].$$

2. $$=2^n\frac{\left(\frac{3+2\mu-2n}{4}\right)_n}{\left(\frac{1}{2}-\mu\right)_n\left(\frac{1}{2}+\mu\right)_n}(1-z^2)^{n/2}\sum_{k=0}^{n}\binom{n}{k}\frac{z^{2k}}{\left(\frac{1-2\mu-2n}{4}\right)_k}\left[\delta_{k,0}P_{-1/2}^{\mu+n}(z)+(-2z)^{-k}(1-z^2)^{-k/2}\right.$$
$$\left.\times\sum_{p=0}^{k-1}\frac{(k+p-1)!}{p!\,(k-p-1)!}\left(\frac{1}{2}-\mu-n\right)_{k-p}^2(2z)^{-p}(1-z^2)^{p/2}P_{-1/2}^{\mu-k+n+p}(z)\right].$$

3. $P_{-n-1/2}^{\mu}(z)=P_{n-1/2}^{\mu}(z).$

Chapter 8

Representations of Hypergeometric Functions and of the Meijer G Function

8.1. The Hypergeometric Functions

8.1.1. The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$

1. $${}_2F_1\left(\begin{matrix} a-n,\, b \\ c;\ z \end{matrix}\right) = \sum_{k=0}^{n} (-z)^k \binom{n}{k} \frac{(b)_k}{(c)_k}\, {}_2F_1\left(\begin{matrix} a,\, b+k \\ c+k;\ z \end{matrix}\right).$$

2. $${}_2F_1\left(\begin{matrix} a+j,\, b+m \\ c+n;\ z \end{matrix}\right) = \frac{(-1)^m (c)_n (1-z)^{c-a-b-j+n}}{(a)_j (b)_m (c-a)_n (c-b)_n}$$
$$\times \sum_{k=0}^{m} (-1)^k \binom{m}{k} (j+k)!\, (a-b-m)_{m-k} \sum_{p=0}^{j+k} \frac{z^p}{p!} (1-z)^{p-k}$$
$$\times P_{j+k-p}^{(a-k+p-1,\, c-a-b-j-k+n+p)}(1-2z)\, \mathrm{D}^{n+p}\left[(1-z)^{a+b-c}\, {}_2F_1\left(\begin{matrix} a,\, b \\ c;\ z \end{matrix}\right)\right].$$

3. $${}_2F_1\left(\begin{matrix} a+j,\, b-m \\ c+n;\ z \end{matrix}\right) = \frac{m!\, (c)_n (1-z)^{c-a-b-j+n}}{(a)_j (c-b+n)_m (c-a)_n (c-b)_n}$$
$$\times \sum_{k=0}^{m} \frac{(j+k)!}{k!} P_{m-k}^{(c-a-b+k+n,\, a+b-c+j+k-m-n)}(1-2z) \sum_{p=0}^{j+k} \frac{z^p (1-z)^p}{p!}$$
$$\times P_{j+k-p}^{(a-k+p-1,\, c-a-b-j-k+n+p)}(1-2z)\, \mathrm{D}^{n+p}\left[(1-z)^{a+b-c}\, {}_2F_1\left(\begin{matrix} a,\, b \\ c;\ z \end{matrix}\right)\right].$$

4. $${}_2F_1\left(\begin{matrix} a-j,\, b-m \\ c+n;\ z \end{matrix}\right) = \frac{(-1)^{j+m} (c)_n (1-z)^{c-a-b+j+m+n}}{(c-a+n)_j (c-b+n)_m (c-a)_n (c-b)_n}$$
$$\times \sum_{k=0}^{m} \binom{m}{k} (b-a-m)_{m-k} \sum_{p=0}^{j+k} \binom{j+k}{p} (a-c-j-n+1)_{j+k-p} (-z)^p$$
$$\times \mathrm{D}^{n+p}\left[(1-z)^{a+b-c}\, {}_2F_1\left(\begin{matrix} a,\, b \\ c;\ z \end{matrix}\right)\right].$$

5. $${}_2F_1\left(\begin{matrix} a+j,\, b+m \\ c-n;\; z \end{matrix}\right) = \frac{(-1)^{j+m+n} z^{n-c+1}}{(a)_j (b)_m (1-c)_n} \sum_{k=0}^{m} \binom{m}{k} (a-b-m)_{m-k}$$
$$\times \sum_{p=0}^{j+k} \binom{j+k}{p} (c-a-j-n)_{j+k-p} (-z)^p \, \mathrm{D}^{n+p} \left[z^{c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ c;\; z \end{matrix}\right)\right].$$

6. $${}_2F_1\left(\begin{matrix} a+j,\, b-m \\ c-n;\; z \end{matrix}\right) = \frac{(-1)^{j+n} m!\, z^{n-c+1}}{(a)_j (c-b-n)_m (1-c)_n}$$
$$\times \sum_{k=0}^{m} \frac{(z-1)^k}{k!} P_{m-k}^{(c-a-b+k-n,\, a+b-c+j+k-m+n)}(1-2z)$$
$$\times \sum_{p=0}^{j+k} (-z)^p \binom{j+k}{p} (c-a-j-n)_{j+k-p} \, \mathrm{D}^{n+p} \left[z^{c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ c;\; z \end{matrix}\right)\right].$$

7. $${}_2F_1\left(\begin{matrix} a-j,\, b-m \\ c-n;\; z \end{matrix}\right) = \frac{(-1)^{m+n} z^{n-c+1} (1-z)^m}{(c-a-n)_j (c-b-n)_m (1-c)_n}$$
$$\times \sum_{k=0}^{m} (z-1)^{-k} \binom{m}{k} (j+k)!\, (b-a-m)_{m-k} \sum_{p=0}^{j+k} \frac{z^p (1-z)^p}{p!}$$
$$\times P_{j+k-p}^{(p-a-k,\, a+b-c-j-k+n+p)}(1-2z) \, \mathrm{D}^{n+p} \left[z^{c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ c;\; z \end{matrix}\right)\right].$$

8. $${}_2F_1\left(\begin{matrix} a,\, a+m+\frac{1}{2} \\ n+\frac{1}{2};\; z \end{matrix}\right) = \frac{2^{n-1} m! \left(\frac{1}{2}\right)_n z^{-n/2} (1-z)^{n-m-2a}}{\left(a+\frac{1}{2}\right)_m (-2a)_{2n}}$$
$$\times \sum_{k=0}^{m} \frac{z^k (1-z)^k}{k!} P_{m-k}^{(k+a-1/2,\, n+k-m-2a)}(1-2z)$$
$$\times \Bigg[(-1)^{k+n} (-2a)_{k+n} (2\sqrt{z})^{-k} (1+\sqrt{z})^{2a-k-n}$$
$$+ (-1)^{k+n} \sum_{p=1}^{k+n-1} \frac{(k+n+p-1)!}{p!\,(k+n-p-1)!} (-2a)_{k+n-p}$$
$$\times (2\sqrt{z})^{-k-p} (1+\sqrt{z})^{2a-k-n+p} + (-2a)_{k+n} (2\sqrt{z})^{-k} (1-\sqrt{z})^{2a-k-n}$$
$$+ \sum_{p=1}^{k+n-1} (-1)^p \frac{(k+n+p-1)!}{p!\,(k+n-p-1)!} (-2a)_{k+n-p} (2\sqrt{z})^{-k-p} (1-\sqrt{z})^{2a-k-n+p} \Bigg].$$

9. $${}_2F_1\left(\begin{matrix} a,\, a+m+\frac{1}{2} \\ \frac{1}{2}-n;\; z \end{matrix}\right) = \frac{(-1)^m}{2\left(a+\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n}$$
$$\times \sum_{k=0}^{m} \binom{m}{k} (-m-n-a)_{m-k} \sum_{p=0}^{k+n} z^p \binom{k+n}{p} \left(\frac{1}{2}\right)_{k+n-p}$$

$$\times\Bigg[(2a)_p(2\sqrt{z})^{-p}(1+\sqrt{z})^{-p-2a}$$

$$+\sum_{r=1}^{p-1}\frac{(p+r-1)!}{r!\,(p-r-1)!}(2a)_{p-r}(2\sqrt{z})^{(-p-r)/2}(1+\sqrt{z})^{r-p-2a}$$

$$+(-1)^p(2a)_p(2\sqrt{z})^{-p}(1-\sqrt{z})^{-p-2a}$$

$$+(-1)^p\sum_{r=1}^{p-1}(-1)^r\frac{(p+r-1)!}{r!\,(p-r-1)!}(2a)_{p-r}(2\sqrt{z})^{(-p-r)/2}(1-\sqrt{z})^{r-p-2a}\Bigg].$$

10. $${}_2F_1\left(\begin{matrix}a,\ a+m+\frac{1}{2}\\ 2a+n+1;\ z\end{matrix}\right)=\frac{(-1)^m 2^{2a}z^{-n}n!\,(2a+1)_n}{\left(a+\frac{1}{2}\right)_m(a+1)_n\left(a-m+\frac{1}{2}\right)_n}$$

$$\times\sum_{k=0}^{n}\frac{(z-1)^k}{k!}P_{n-k}^{(k-n-a+1/2,\,k+m-n-1/2)}(1-2z)\sum_{p=0}^{k+m}\binom{k+m}{p}$$

$$\times\left(\frac{1}{2}-a-m\right)_{k+m-p}(-z)^p\Bigg[(2a)_p\left(2\sqrt{1-z}\right)^{-p}\left(1+\sqrt{1-z}\right)^{-2a-p}$$

$$+\sum_{r=1}^{p-1}\frac{(p+r-1)!}{r!\,(p-r-1)!}(2a)_{p-r}\left(2\sqrt{1-z}\right)^{-p-r}\left(1+\sqrt{1-z}\right)^{-2a-p+r}\Bigg].$$

11. $${}_2F_1\left(\begin{matrix}a,\ a+m+\frac{1}{2}\\ 2a-n+1;\ z\end{matrix}\right)=\frac{(-1)^m 2^{2a}}{\left(a+\frac{1}{2}\right)_m(-2a)_n}$$

$$\times\sum_{k=0}^{n}\binom{n}{k}\left(-a-\frac{1}{2}\right)_{n-k}\sum_{p=0}^{k+m}\binom{k+m}{p}$$

$$\times\left(\frac{1}{2}-a-m\right)_{k+m-p}(-z)^p\Bigg[(2a)_p\left(2\sqrt{1-z}\right)^{-p}\left(1+\sqrt{1-z}\right)^{-2a-p}$$

$$+\sum_{r=1}^{p-1}\frac{(p+r-1)!}{r!\,(p-r-1)!}(2a)_{p-r}\left(2\sqrt{1-z}\right)^{-p-r}\left(1+\sqrt{1-z}\right)^{-2a-p+r}\Bigg].$$

12. $${}_2F_1\left(\begin{matrix}j+1,\ m+1\\ n+2;\ z\end{matrix}\right)=\frac{(-1)^{n+1}(n+1)z^{-n-1}(1-z)^n}{j!\,n!}\sum_{k=0}^{m}\binom{m}{k}\frac{(j+k)!}{k!}$$

$$\times\sum_{p=0}^{j+k}(-1)^p\frac{(n+p)!}{p!}(1-z)^{p-k-j}P_{j+k-p}^{(p-k,\,n-j-k+p)}(1-2z)$$

$$\times\left[\ln(1-z)-\sum_{r=1}^{n+p}\frac{1}{r}\left(\frac{z}{z-1}\right)^r\right].$$

13. $$ {}_2F_1\left(\begin{matrix} j+1,\ m+1 \\ n+\frac{3}{2};\ z \end{matrix}\right) = \frac{(-1)^n (2n+1) z^{-n-1/2} (1-z)^{n-j-1/2}}{j! \left(\frac{1}{2}\right)_n} \sum_{k=0}^{m} \binom{m}{k} \frac{(j+k)!}{k!} (1-z)^{-k} \times \sum_{p=0}^{j+k} \frac{(z-1)^p}{p!} P_{j+k-p}^{(p-k,\, n-j-k+p-1/2)}(1-2z) \left[\left(\frac{1}{2}\right)_{n+p} \arcsin\sqrt{z} - (n+p)! \frac{i}{2} \sum_{r=1}^{n+p} \frac{\left(\frac{1}{2}\right)_{n+p-r}}{(n+p-r)!\, r} \left(\frac{z}{z-1}\right)^{r/2} P_{r-1}\left(\frac{2z-1}{2\sqrt{z(z-1)}}\right)\right]. $$

14. $$ {}_2F_1\left(\begin{matrix} j+1,\ m+1 \\ n+\frac{3}{2};\ -z \end{matrix}\right) = \frac{(2n+1) z^{-n-1/2} (1+z)^{n-j-1/2}}{j! \left(\frac{1}{2}\right)_n} \sum_{k=0}^{m} \binom{m}{k} \frac{(j+k)!}{k!} (1+z)^{-k} \times \sum_{p=0}^{j+k} \frac{(-1)^p}{p!} (1+z)^p P_{j+k-p}^{(p-k,\, n-j-k+p-1/2)}(1+2z) \times \left[\left(\frac{1}{2}\right)_{n+p} \ln\left(\sqrt{z}+\sqrt{1+z}\right) - \frac{(n+p)!}{2} \sum_{r=1}^{n+p} \frac{\left(\frac{1}{2}\right)_{n+p-r}}{(n+p-r)!\, r} \left(\frac{z}{1+z}\right)^{r/2} \times P_{r-1}\left(\frac{2z+1}{2\sqrt{z(1+z)}}\right)\right]. $$

15. $$ {}_2F_1\left(\begin{matrix} j+1,\ m+1 \\ \frac{3}{2}-n;\ z \end{matrix}\right) = \frac{(-1)^{j+m+n} z^{n-1/2} (1-z)^{-n-1/2}}{j!\, m! \left(-\frac{1}{2}\right)_n} \times \sum_{k=0}^{m} \binom{m}{k} (-m)_{m-k} \sum_{p=0}^{j+k} \binom{j+k}{p} \left(\frac{1}{2}-j-n\right)_{j+k-p} \left(\frac{z}{z-1}\right)^p \times \left[\left(\frac{1}{2}\right)_{n+p} \arcsin\sqrt{z} + \frac{1}{2} \sum_{r=1}^{n+p} \binom{n+p}{r} (r-1)! \left(\frac{1}{2}\right)_{n+p-r} i^{r-1} \left(\frac{z}{1-z}\right)^{-r/2} \times P_{r-1}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right)\right]. $$

16. $${}_2F_1\left(\begin{matrix} j+1,\ m+1 \\ \frac{3}{2}-n;\ -z \end{matrix}\right) = \frac{(-1)^j z^{n-1/2}(1+z)^{-n-1/2}}{j!\left(-\frac{1}{2}\right)_n}$$
$$\times \sum_{k=0}^{m} \frac{(-1)^k}{k!} \sum_{p=0}^{j+k} \binom{j+k}{p} \left(\frac{1}{2}-j-n\right)_{j+k-p} \left(\frac{z}{1+z}\right)^p$$
$$\times \left[\left(\frac{1}{2}\right)_{n+p} \ln\left(\sqrt{z}+\sqrt{1+z}\right) - \frac{(n+p)!}{2} \sum_{r=1}^{n+p} \frac{\left(\frac{1}{2}\right)_{n+p-r}}{(n+p-r)!\,r} \left(\frac{z}{1+z}\right)^{-r/2}\right.$$
$$\left.\times P_{r-1}\left(\frac{2z+1}{2\sqrt{z(1+z)}}\right)\right].$$

17. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\ m+1 \\ n+2;\ z \end{matrix}\right) = \frac{2(-1)^m(n+1)z^{-n-1}}{m!\left(\frac{1}{2}\right)_j\left(\frac{3}{2}\right)_n} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (j+k)!$$
$$\times \left(-m-\frac{1}{2}\right)_{m-k} \sum_{p=0}^{k+j} \frac{(n+p)!}{p!} (1-z)^{-j-k} P_{j+k-p}^{(p-k-1/2,\, n-j-k+p+1/2)}(1-2z)$$
$$\times \left[P_{n+p}^{(-n-p-1,\, -n-p-1/2)}(1-2z) - (-1)^{n+p}(1-z)^{n+p+1/2}\right].$$

18. $${}_2F_1\left(\begin{matrix} \frac{1}{2}-j,\ m+1 \\ n+2;\ z \end{matrix}\right) = \frac{2(n+1)j!\,z^{-n-1}(1-z)^{-m}}{m!\left(\frac{3}{2}\right)_{j+n}}$$
$$\times \sum_{k=0}^{j} \frac{(k+m)!}{k!} P_{j-k}^{(k+n+1/2,\, k-j+m-n-1/2)}(1-2z)$$
$$\times \sum_{p=0}^{k+m} \frac{(n+p)!}{p!} (1-z)^{-j-k} P_{k+m-p}^{(p-k,\, n-k-m+p+1/2)}(1-2z)$$
$$\times \left[P_{n+p}^{(-n-p-1,\, -n-p-1/2)}(1-2z) - (-1)^{n+p}(1-z)^{n+p+1/2}\right].$$

19. $${}_2F_1\left(\begin{matrix} j+1,\ m+1 \\ \frac{3}{2}-n;\ -z \end{matrix}\right) = \frac{(-1)^j z^{n-1/2}(1+z)^{-n-1/2}}{j!\left(-\frac{1}{2}\right)_n}$$
$$\times \sum_{k=0}^{m} \frac{(-1)^k}{k!} \sum_{p=0}^{j+k} \binom{j+k}{p} \left(\frac{1}{2}-j-n\right)_{j+k-p} \left(\frac{z}{1+z}\right)^p$$
$$\times \left[\left(\frac{1}{2}\right)_{n+p} \ln\left(\sqrt{z}+\sqrt{1+z}\right) - \frac{(n+p)!}{2} \sum_{r=1}^{n+p} \frac{\left(\frac{1}{2}\right)_{n+p-r}}{(n+p-r)!\,r} \left(\frac{z}{1+z}\right)^{-r/2}\right.$$
$$\left.\times P_{r-1}\left(\frac{2z+1}{2\sqrt{z(1+z)}}\right)\right].$$

20. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\, m+1 \\ n+\frac{3}{2};\ -z \end{matrix}\right)$$
$$= \frac{(2n+1)\left(\frac{3}{2}\right)_m z^{-n}(1+z)^{n-j}}{n!\left(\frac{1}{2}\right)_j} \sum_{k=0}^{m} \frac{(j+k)!}{(m-k)!\,(2k+1)!}\left(\frac{4}{1+z}\right)^k$$
$$\times \sum_{p=0}^{j+k}(-1)^p \frac{(1+z)^p}{p!} P_{j+k-p}^{(p-k-1/2,\, n-j-k+p)}(1+2z)\left[\left(\frac{1}{2}\right)_{n+p} z^{-1/2}\arctan\sqrt{z}\right.$$
$$\left. + \frac{(n+p)!}{2}\sum_{r=1}^{n+p}(-1)^r \frac{\left(\frac{1}{2}\right)_{n+p-r}}{(n+p-r)!\,r}(z+1)^{-r} P_{r-1}^{(1/2-r,\,-r)}(1+2z)\right].$$

21. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\, m+1 \\ \frac{3}{2}-n;\ z \end{matrix}\right) = \frac{(-1)^{j+n}\left(\frac{3}{2}\right)_m}{2\left(\frac{1}{2}\right)_j\left(-\frac{1}{2}\right)_n}(1-z)^{-n}$$
$$\times \sum_{k=0}^{m}\frac{(-4)^k}{(m-k)!\,(2k+1)!}\sum_{p=0}^{j+k}\binom{j+k}{p}(n+p-1)!\,(1-j-n)_{j+k-p}(z-1)^{-p}$$
$$\times P_{n+p-1}^{(1/2-n-p,\,-n-p)}(1-2z).$$

22. $${}_2F_1\left(\begin{matrix} \frac{1}{2}-j,\, m+1 \\ \frac{3}{2}+n;\ z \end{matrix}\right) = \frac{(-1)^{j+n}(2n+1)}{2(j+n)!} z^{-n}(1-z)^{j-m+n}$$
$$\times \sum_{k=0}^{m}\frac{(z-1)^k}{k!} P_{m-k}^{(k-n,\,j+k-m+n)}(1-2z)\sum_{p=0}^{j+k}\binom{j+k}{p}(-j-n)_{j+k-p}$$
$$\times\left[\left(\frac{1}{2}\right)_{n+p} z^{-1/2}\ln\frac{1+\sqrt{z}}{1-\sqrt{z}} + \sum_{r=1}^{n+p}\binom{n+p}{r}(r-1)!\left(\frac{1}{2}\right)_{n+p-r}(z-1)^{-r}\right.$$
$$\left.\times P_{r-1}^{(1/2-r,\,-r)}(1-2z)\right].$$

23. $${}_2F_1\left(\begin{matrix} \frac{1}{2}-j,\, m+1 \\ \frac{3}{2}+n;\ -z \end{matrix}\right) \frac{(-1)^j(2n+1)}{(j+n)!} z^{-n}(1+z)^{j-m+n}$$
$$\times \sum_{k=0}^{m}\frac{(-1)^k}{k!}(z+1)^k P_{m-k}^{(k-n,\,j+k-m+n)}(1+2z)\sum_{p=0}^{j+k}\binom{j+k}{p}(-j-n)_{j+k-p}$$
$$\times\left[\left(\frac{1}{2}\right)_{n+p} z^{-1/2}\arctan\sqrt{z} + \frac{(n+p)!}{2}\sum_{r=1}^{n+p}(-1)^r\frac{(1+z)^{-r}}{(n+p-r)!\,r}\left(\frac{1}{2}\right)_{n+p-r}\right.$$
$$\left.\times P_{r-1}^{(1/2-r,\,-r)}(1+2z)\right].$$

24. $${}_2F_1\left(\begin{matrix}\frac{1}{2}-j,\ m+1\\ \frac{3}{2}-n;\ z\end{matrix}\right)=\frac{(-1)^j(1-z)^{j-m}}{2\,(j!\,m!)}\sum_{k=0}^{n}\binom{n}{k}\frac{(k+m)!}{\left(\frac{3}{2}-n\right)_k}(1-z)^{-k}$$
$$\times\sum_{p=0}^{k+m}\frac{(z-1)^p}{p!}P_{k+m-p}^{(p-k,\,j-k-m+p)}(1-2z)\left[\left(\frac{1}{2}-j\right)_{j+p}z^{-1/2}\ln\frac{1+\sqrt{z}}{1-\sqrt{z}}\right.$$
$$\left.+(j+p)!\sum_{r=1}^{j+p}\frac{\left(\frac{1}{2}-j\right)_{j+p-r}}{(j+p-r)!\,r}(z-1)^{-r}P_{r-1}^{(1/2-r,\,-r)}(1-2z)\right].$$

25. $${}_2F_1\left(\begin{matrix}\frac{1}{2}-j,\ m+1\\ \frac{3}{2}-n;\ -z\end{matrix}\right)=\frac{(-1)^j(1+z)^{j-m}}{j!\,m!}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}(k+m)!\,(1+z)^{-k}\sum_{p=0}^{k+m}\frac{(-1)^p}{p!}(1+z)^pP_{k+m-p}^{(p-k,\,j-k-m+p)}(1+2z)$$
$$\times$$
$$\times\left[\left(\frac{1}{2}-j\right)_{j+p}z^{-1/2}\arctan\sqrt{z}+\frac{(j+p)!}{2}\sum_{r=1}^{m+p}(-1)^r\frac{\left(\frac{1}{2}-j\right)_{j+p-r}}{(j+p-r)!\,r}(z+1)^{-r}\right.$$
$$\left.\times P_{r-1}^{(1/2-r,\,-r)}(1+2z)\right].$$

26. $${}_2F_1\left(\begin{matrix}j+\frac{1}{2},\ m+\frac{1}{2}\\ n+\frac{3}{2};\ z\end{matrix}\right)=\frac{(-1)^j\left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_m(n!)^2}z^{-(n+1)/2}(z-1)^{-j+n/2}$$
$$\times\sum_{k=0}^{m}\binom{m}{k}\frac{(j+k)!}{k!}(1-z)^{-k}\sum_{p=0}^{j+k}\frac{(p+n)!}{p!}(z^2-z)^{p/2}$$
$$\times P_{j+k-p}^{(p-k-1/2,\,n-j-k+p+1/2)}(1-2z)\left[\arcsin\sqrt{z}\,P_{n+p}\left(\frac{2z-1}{2\sqrt{z^2-z}}\right)\right.$$
$$\left.-\frac{i}{2}\sum_{r=1}^{n+p}\frac{1}{r}P_{n+p-r}\left(\frac{2z-1}{2\sqrt{z^2-z}}\right)P_{r-1}\left(\frac{2z-1}{2\sqrt{z^2-z}}\right)\right].$$

27. $${}_2F_1\left(\begin{matrix}j+\frac{1}{2},\ m+\frac{1}{2}\\ n+\frac{3}{2};\ -z\end{matrix}\right)=\frac{(-1)^{m+n}\left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_m(n!)^2}z^{-(n+1)/2}(1+z)^{-j+n/2}$$
$$\times\sum_{k=0}^{m}\binom{m}{k}\frac{(j+k)!}{k!}(1+z)^{-k}\sum_{p=0}^{j+k}(-1)^p\frac{(p+n)!}{p!}$$
$$\times(z^2+z)^{p/2}P_{j+k-p}^{(p-k-1/2,\,n-j-k+p+1/2)}(1+2z)$$

$$\times \left[-\ln\left(\sqrt{z} + \sqrt{1+z}\right) P_{n+p}\left(\frac{2z+1}{2\sqrt{z^2+z}}\right) \right.$$

$$\left. + \frac{1}{2} \sum_{r=1}^{n+p} \frac{1}{r} P_{n+p-r}\left(\frac{2z+1}{2\sqrt{z^2+z}}\right) P_{r-1}\left(\frac{2z+1}{2\sqrt{z^2+z}}\right) \right].$$

28. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\ \frac{1}{2}-m \\ n+\frac{3}{2};\ z \end{matrix}\right) = \frac{m!\left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_j (n+1)_m (n!)^2} z^{-(n+1)/2}(1-z)^{-j+n/2}$$

$$\times \sum_{k=0}^{m} \frac{(j+k)!}{k!} P_{m-k}^{(k+n+1/2,\, j+k-m-n-1/2)}(1-2z) \sum_{p=0}^{j+k} \frac{(p+n)!}{p!}$$

$$\times i^{p+n}(z-z^2)^{p/2} P_{j+k-p}^{(p-k-1/2,\, n-j-k+p+1/2)}(1-2z)$$

$$\times \left[\arcsin\sqrt{z}\, P_{n+p}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right) - \frac{i}{2} \sum_{r=1}^{n+p} \frac{1}{r} P_{n+p-r}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right) \right.$$

$$\left. \times P_{r-1}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right) \right].$$

29. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\ \frac{1}{2}-m \\ n+\frac{3}{2};\ -z \end{matrix}\right) = \frac{m!\left(\frac{3}{2}\right)_n z^{-(n+1)/2}(1+z)^{-j+n/2}}{\left(\frac{1}{2}\right)_j (n+1)_m (n!)^2} \sum_{k=0}^{m} \frac{(j+k)!}{k!}$$

$$\times P_{m-k}^{(k+n+1/2,\, j+k-m-n-1/2)}(1+2z) \sum_{p=0}^{j+k} (-1)^p \frac{(n+p)!}{p!}$$

$$\times (z+z^2)^{p/2} P_{j+k-p}^{(p-k-1/2,\, n-j-k+p+1/2)}(1+2z) \left[\ln\left(\sqrt{z} + \sqrt{1+z}\right) \right.$$

$$\left. \times P_{n+p}\left(\frac{2z+1}{2\sqrt{z+z^2}}\right) - \frac{1}{2} \sum_{r=1}^{n+p} \frac{1}{r} P_{n+p-r}\left(\frac{2z+1}{2\sqrt{z+z^2}}\right) P_{r-1}\left(\frac{2z+1}{2\sqrt{z+z^2}}\right) \right].$$

30. $${}_2F_1\left(\begin{matrix} j-\frac{1}{2},\ m-\frac{1}{2} \\ n+\frac{3}{2};\ z \end{matrix}\right) = \frac{(-1)^{j+m}\left(\frac{3}{2}\right)_n}{(j+n)!\,(m+n)!}$$

$$\times z^{-(n+1)/2}(1-z)^{j+m+n/2} \sum_{k=0}^{m-k} \binom{m}{k} (-m)_{m-k}$$

$$\times \sum_{p=0}^{j+k} (-1)^p \binom{j+k}{p} (n+p)!\,(-j-n)_{j+k-p}\, i^{p+n} \left(\frac{z}{1-z}\right)^{-p/2}$$

$$\times\left[\arcsin\sqrt{z}\,P_{n+p}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right)-\frac{i}{2}\sum_{r=1}^{n+p}\frac{1}{r}\,P_{n+p-r}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right)\right.$$
$$\left.\times P_{r-1}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right)\right].$$

31. $${}_2F_1\left(\begin{matrix} j-\frac{1}{2},\ m-\frac{1}{2}\\ n+\frac{3}{2};\ -z\end{matrix}\right)=\frac{(-1)^j m!\left(\frac{3}{2}\right)_n}{(j+n)!\,(m+n)!}\,z^{-(n+1)/2}(1+z)^{j+m+n/2}$$
$$\times\sum_{k=0}^{m}(-1)^k\binom{m}{k}\sum_{p=0}^{j+k}\binom{j+k}{p}(n+p)!\,(-j-n)_{j+k-p}\left(\frac{z}{1+z}\right)^{p/2}$$
$$\times\left[\ln\left(\sqrt{z}+\sqrt{1+z}\right)P_{n+p}\left(\frac{2z+1}{2\sqrt{z^2+z}}\right)-\frac{1}{2}\sum_{r=1}^{n+p}\frac{1}{r}\,P_{n+p-r}\left(\frac{2z+1}{2\sqrt{z^2+z}}\right)\right.$$
$$\left.\times P_{r-1}\left(\frac{2z+1}{2\sqrt{z^2+z}}\right)\right].$$

32. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\ m+\frac{1}{2}\\ \frac{3}{2}-n;\ z\end{matrix}\right)=\frac{(-1)^{j+1}m!}{2\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_m\left(-\frac{1}{2}\right)_n}(-z)^{-1/2}\left(\frac{z}{z-1}\right)^{n/2}$$
$$\times\sum_{k=0}^{m}\frac{(-1)^k}{k!}\binom{m}{k}\sum_{p=0}^{j+k}i^{n+p-1}\binom{j+k}{p}(n+p-1)!\,(1-j-n)_{j+k-p}$$
$$\times\left(\frac{z}{z-1}\right)^{p/2}P_{n+p-1}\left(\frac{1-2z}{2\sqrt{z^2-z}}\right).$$

33. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\ m+\frac{1}{2}\\ \frac{3}{2}-n;\ -z\end{matrix}\right)=\frac{(-1)^{j+1}m!\,z^{(n-1)/2}(1+z)^{-n/2}}{2\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_m\left(-\frac{1}{2}\right)_n}\sum_{k=0}^{m}\frac{(-1)^k}{k!}\binom{m}{k}$$
$$\times\sum_{p=0}^{j+k}\binom{j+k}{p}(n+p-1)!\,(1-j-n)_{j+k-p}\left(\frac{z}{1+z}\right)^{p/2}P_{n+p-1}\left(\frac{1+2z}{2\sqrt{z^2+z}}\right).$$

34. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\ \frac{1}{2}-m\\ \frac{3}{2}-n;\ z\end{matrix}\right)=\frac{(-1)^{n+1}z^{-1/2}(1-z)^{-j}}{m!\left(\frac{1}{2}\right)_j\left(-\frac{1}{2}\right)_n}$$
$$\times\sum_{k=n-1}^{n}\binom{n}{k}(j+k)!\,(1-z)^{-k}\sum_{p=0}^{j+k}\frac{(m+p)!}{p!}\,P_{j+k-p}^{(p-k-1/2,\,m-j-k+p+1/2)}(1-2z)$$
$$\times\left[\arcsin\sqrt{z}\,P_{m+p}^{(-p-1/2,\,-m-p-1/2)}(1-2z)+\frac{1}{2}\sum_{r=1}^{m+p}\frac{i^{r+1}}{r}(z-z^2)^{r/2}\right.$$
$$\left.\times P_{m+p-r}^{(r-p-1/2,\,r-m-p-1/2)}(1-2z)P_{r-1}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right)\right].$$

35. $${}_2F_1\left(\begin{matrix} j+\frac{1}{2},\ \frac{1}{2}-m \\ \frac{3}{2}-n;\ -z \end{matrix}\right) = \frac{(-1)^{n+1} z^{-1/2}(1-z)^{-j}}{m!\left(\frac{1}{2}\right)_j\left(-\frac{1}{2}\right)_n}$$
$$\times \sum_{k=n-1}^{n} \binom{n}{k}(j+k)! \sum_{p=0}^{j+k} \frac{(m+p)!}{p!}(1+z)^{-k} P_{j+k-p}^{(p-k-1/2,\, m-j-k+p+1/2)}(1+2z)$$
$$\times \left[\ln\left(\sqrt{z}+\sqrt{1+z}\right) P_{m+p}^{(-p-1/2,\,-m-p-1/2)}(1+2z)\right.$$
$$+\frac{1}{2}\sum_{r=1}^{m+p}\frac{(-1)^r}{r}(z+z^2)^{r/2} P_{m+p-r}^{(r-p-1/2,\,r-m-p-1/2)}(1+2z)$$
$$\left.\times P_{r-1}\left(\frac{1+2z}{2\sqrt{z(1+z)}}\right)\right].$$

36. $${}_2F_1\left(\begin{matrix} \frac{1}{2}-j,\ \frac{1}{2}-m \\ \frac{3}{2}-n;\ z \end{matrix}\right) = \frac{(-1)^{m+n} n!\, z^{-1/2}(1-z)^{m-n}}{j!\, m!\left(-\frac{1}{2}\right)_n}$$
$$\times \sum_{k=0}^{n} \frac{(1-z)^k}{k!} P_{n-k}^{(k-n+1/2,\, j+k+m-n+1/2)}(1-2z)$$
$$\times \sum_{p=0}^{k+m} (-1)^{k-p}\binom{k+m}{p}(j+p)!\,(-m)_{k+m-p}(1-z)^{-p}$$
$$\times \left[\arcsin\sqrt{z}\, P_{j+p}^{(-p-1/2,\,-j-p-1/2)}(1-2z) + \frac{1}{2}\sum_{r=1}^{j+p}\frac{i^{r-1}}{r}(z-z^2)^{r/2}\right.$$
$$\left.\times P_{j+p-r}^{(r-p-1/2,\,r-j-p-1/2)}(1-2z) P_{r-1}\left(\frac{2z-1}{2i\sqrt{z(1-z)}}\right)\right].$$

37. $${}_2F_1\left(\begin{matrix} \frac{1}{2}-j,\ \frac{1}{2}-m \\ \frac{3}{2}-n;\ -z \end{matrix}\right) = \frac{(-1)^{m+n} n!\, z^{-1/2}(1+z)^{m-n}}{j!\, m!\left(-\frac{1}{2}\right)_n}$$
$$\times \sum_{k=0}^{n} \frac{(1+z)^k}{k!} P_{n-k}^{(k-n+1/2,\, j+k+m-n+1/2)}(1+2z)$$
$$\times \sum_{p=0}^{k+m} (-1)^{k-p}\binom{k+m}{p}(j+p)!\,(-m)_{k+m-p}(1+z)^{-p}$$
$$\times \left[\ln\left(\sqrt{z}+\sqrt{1+z}\right) P_{j+p}^{(-p-1/2,\,-j-p-1/2)}(1+2z)\right.$$

$$+\frac{1}{2}\sum_{r=1}^{j+p}\frac{(-1)^{r-1}}{r}(z+z^2)^{r/2}P_{j+p-r}^{(r-p-1/2,\,r-j-p-1/2)}(1+2z)$$
$$\times P_{r-1}\left(\frac{2z+1}{2\sqrt{z(1+z)}}\right)\Bigg].$$

38. $${}_2F_1\left(\begin{matrix}a,\,b\\ \frac{1}{2};\ z\end{matrix}\right)=\frac{1}{2\pi^{1/2}}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)}{\Gamma\left(a+b+\frac{1}{2}\right)}\times\left[{}_2F_1\left(\begin{matrix}2a,\,2b\\ a+b+\frac{1}{2};\ \frac{1+\sqrt{z}}{2}\end{matrix}\right)+{}_2F_1\left(\begin{matrix}2a,\,2b\\ a+b+\frac{1}{2};\ \frac{1-\sqrt{z}}{2}\end{matrix}\right)\right].$$

39. $${}_2F_1\left(\begin{matrix}a,\,b\\ \frac{3}{2};\ z\end{matrix}\right)=\frac{1}{4(\pi z)^{1/2}}\frac{\Gamma\left(a-\frac{1}{2}\right)\Gamma\left(b-\frac{1}{2}\right)}{\Gamma\left(a+b-\frac{1}{2}\right)}\times\left[{}_2F_1\left(\begin{matrix}2a-1,\,2b-1\\ a+b-\frac{1}{2};\ \frac{1+\sqrt{z}}{2}\end{matrix}\right)-\left(\begin{matrix}2a-1,\,2b-1\\ a+b-\frac{1}{2};\ \frac{1-\sqrt{z}}{2}\end{matrix}\right)\right].$$

40. $${}_2F_1\left(\begin{matrix}a,\,1-a\\ b;\ z\end{matrix}\right)=(1-z)^{b-1}\,{}_2F_1\left(\begin{matrix}\frac{b-a}{2},\,\frac{b+a-1}{2}\\ c;\ 4z(1-z)\end{matrix}\right)$$ [[64], (3.31)].

41. $${}_2F_1\left(\begin{matrix}a,\,b\\ 2b;\ -\frac{4z}{(1-z)^2}\end{matrix}\right)=(1-z)^{2a}\,{}_2F_1\left(\begin{matrix}a,\,a-b+\frac{1}{2}\\ b+\frac{1}{2};\ z^2\end{matrix}\right)$$ [[18], (4.10)].

42. $${}_2F_1\left(\begin{matrix}a,\,b;\ \frac{z^2}{4(z-1)}\\ a+b+\frac{1}{2}\end{matrix}\right)=(1-z)^{a}\,{}_2F_1\left(\begin{matrix}2a,\,a+b;\ z\\ 2a+2b\end{matrix}\right).$$

43. $${}_2F_1\left(\begin{matrix}a,\,b;\ 4z(1-z)\\ a+b+\frac{1}{2}\end{matrix}\right)=(1-z)^{1/2-a-b}\,{}_2F_1\left(\begin{matrix}a-b+\frac{1}{2},\,b-a+\frac{1}{2}\\ a+b+\frac{1}{2};\ z\end{matrix}\right)$$ [[64], (3.31)].

44. $${}_2F_1\left(\begin{matrix}-n,\,a\\ 2a;\ z\end{matrix}\right)=(-1)^n\frac{n!}{(2a)_n}(1-z)^{n/2}C_n^a\left(\frac{z-2}{2\sqrt{1-z}}\right).$$

45. $$\lim_{a\to 0}\,{}_2F_1\left(\begin{matrix}-n,\,a\\ 2a;\ z\end{matrix}\right)=(-1)^n(1-z)^{n/2}T_n\left(\frac{z-2}{2\sqrt{1-z}}\right).$$

46. $${}_2F_1\left(\begin{matrix}-n,\,1\\ 2;\ z\end{matrix}\right)=\frac{2}{(n+1)z}\left[1+(-1)^n(1-z)^{(n+1)/2}T_{n+1}\left(\frac{z-2}{2\sqrt{1-z}}\right)\right].$$

47. $$=\frac{(-1)^n}{n+1}(1-z)^{n/2}U_n\left(\frac{z-2}{2\sqrt{1-z}}\right).$$

48. $${}_2F_1\left(\begin{matrix}-n,\ b\\ \frac{1}{2};\ z\end{matrix}\right)=\frac{n!}{\left(b+\frac{1}{2}\right)_n}(1-z)^n C_{2n}^{1/2-b-n}\left(\sqrt{\frac{z}{z-1}}\right).$$

49. $${}_2F_1\left(\begin{matrix}-n,\ b\\ \frac{3}{2};\ z\end{matrix}\right)$$
$$=(-1)^n\frac{n!}{2\left(b-\frac{1}{2}\right)_{n+1}}z^{-1/2}(z-1)^{n+1/2}C_{2n+1}^{1/2-b-n}\left(\sqrt{\frac{z}{z-1}}\right).$$

50. $${}_2F_1\left(\begin{matrix}-n,\ n\\ 1;\ z\end{matrix}\right)=\frac{1}{2}\left[P_n(1-2z)+P_{n-1}(1-2z)\right]\qquad [n\geq 1].$$

51. $${}_2F_1\left(\begin{matrix}-n,\ n\\ \frac{3}{2};\ z\end{matrix}\right)=\frac{(-1)^n}{1-4n^2}\left[T_{2n}(\sqrt{z})+\frac{2n(1-z)}{\sqrt{z}}U_{2n-1}(\sqrt{z})\right]\qquad [n\geq 1].$$

52. $${}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ \frac{1}{2};\ z\end{matrix}\right)=(z-1)^n T_{2n}\left(\sqrt{\frac{z}{z-1}}\right).$$

53. $${}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ 1;\ z\end{matrix}\right)=(1-z)^n P_{2n}\left(\frac{1}{\sqrt{1-z}}\right).$$

54. $${}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ \frac{3}{2};\ z\end{matrix}\right)=\frac{1}{2n+1}z^{-1/2}(z-1)^{n+1/2}T_{2n+1}\left(\sqrt{\frac{z}{z-1}}\right).$$

55. $${}_2F_1\left(\begin{matrix}-n,\ -n-\frac{1}{2}\\ \frac{1}{2};\ z\end{matrix}\right)=(z-1)^n U_{2n}\left(\sqrt{\frac{z}{z-1}}\right).$$

56. $${}_2F_1\left(\begin{matrix}-n,\ -n-\frac{1}{2}\\ \frac{3}{2};\ z\end{matrix}\right)=\frac{1}{2(n+1)}z^{-1/2}(z-1)^{n+1/2}U_{2n+1}\left(\sqrt{\frac{z}{z-1}}\right).$$

57. $${}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ \frac{1}{2}-2n;\ z\end{matrix}\right)=\frac{(2n)!}{\left(\frac{1}{2}\right)_{2n}}\left(\frac{z}{4}\right)^n P_{2n}\left(\frac{1}{\sqrt{z}}\right).$$

58. $${}_2F_1\left(\begin{matrix}-n,\ -n-\frac{1}{2}\\ -\frac{1}{2}-2n;\ z\end{matrix}\right)=2^{-2n}\frac{(2n+1)!}{\left(\frac{3}{2}\right)_{2n}}z^{n+1/2}P_{2n+1}\left(\frac{1}{\sqrt{z}}\right).$$

59. $${}_2F_1\left(\begin{matrix}-n,\ a;\ z\\ \frac{a-n+1}{2}\end{matrix}\right)=\frac{(-1)^n n!}{\left(\frac{1-a-n}{2}\right)_n}(z^2-z)^{n/2}C_n^{(1-a-n)/2}\left(\frac{2z-1}{2\sqrt{z^2-z}}\right).$$

60. $${}_2F_1\left(\begin{matrix}-n,\ -n\\ \frac{1}{2}-n;\ z\end{matrix}\right)=\frac{(-1)^n n!}{\left(\frac{1}{2}\right)_n}(z^2-z)^{n/2}P_n\left(\frac{2z-1}{2\sqrt{z^2-z}}\right).$$

61. $${}_2F_1\left(\begin{matrix}-n,\ -n-\frac{1}{2}\\ -2n;\ z\end{matrix}\right)=\left(-\frac{z}{4}\right)^n U_{2n}\left(\sqrt{1-\frac{1}{z}}\right).$$

62. $${}_2F_1\left(\begin{matrix}-\frac{1}{4},\ \frac{1}{4}\\ 1;\ z\end{matrix}\right)=\frac{2}{\pi}\left(\sqrt{z}+1\right)^{1/2}\mathbf{E}\left(\sqrt{\frac{2z^{1/2}}{z^{1/2}+1}}\right).$$

63. $${}_2F_1\left(\begin{matrix}\frac{1}{8},\ \frac{1}{8}\\ \frac{3}{4};\ -z\end{matrix}\right)=\frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}}\left(\sqrt{z}+\sqrt{z+1}\right)^{-1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+\sqrt{z+1})^{1/2}}}\right)+\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+\sqrt{z+1})^{1/2}}}\right)\right].$$

64. $${}_2F_1\left(\begin{matrix}\frac{1}{8},\ \frac{3}{8}\\ \frac{1}{2};\ z\end{matrix}\right)=\frac{2^{1/4}}{\pi^{3/2}}\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)$$
$$\times\left\{\left[2^{1/2}+(1+\sqrt{z})^{1/2}\right]^{-1/2}\mathbf{K}\left(\frac{2^{1/2}(1+\sqrt{z})^{1/4}}{\sqrt{2^{1/2}+(1+\sqrt{z})^{1/2}}}\right)\right.$$
$$\left.+\left[2^{1/2}+(1-\sqrt{z})^{1/2}\right]^{-1/2}\mathbf{K}\left(\frac{2^{1/2}(1-\sqrt{z})^{1/4}}{\sqrt{2^{1/2}+(1-\sqrt{z})^{1/2}}}\right)\right\}.$$

65. $${}_2F_1\left(\begin{matrix}\frac{1}{8},\ \frac{3}{8}\\ 1;\ z\end{matrix}\right)$$
$$=\frac{2^{5/4}}{\pi}\left(\sqrt{2}+\sqrt{1-\sqrt{1-z}}\right)^{-1/2}\mathbf{K}\left(\frac{2^{1/2}\left(1-\sqrt{1-z}\right)^{1/4}}{\left(\sqrt{2}+\sqrt{1-\sqrt{1-z}}\right)^{1/2}}\right).$$

66. $${}_2F_1\left(\begin{matrix}\frac{1}{8},\ \frac{3}{8}\\ 1;\ -z\end{matrix}\right)$$
$$=\frac{2^{5/4}}{\pi}\left(z+2+2\sqrt{z+1}\right)^{-1/8}\mathbf{K}\left(\sqrt{\frac{1}{2}-2^{-1/2}\left(\frac{1+\sqrt{z+1}}{z+2+2\sqrt{z+1}}\right)^{1/2}}\right).$$

67. $${}_2F_1\left(\begin{matrix}\frac{1}{8},\ \frac{5}{8}\\ \frac{3}{4};\ z\end{matrix}\right)=\frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}}\left(\sqrt{z}+1\right)^{-1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+1)^{1/2}}}\right)+\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+1)^{1/2}}}\right)\right].$$

68. $$= \frac{2^{1/4}}{\pi^{3/2} z^{1/8}} \Gamma^2\left(\frac{3}{4}\right) u^{1/2} \left[\mathbf{K}\left(\sqrt{\frac{1}{2}+u}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2}-u}\right)\right]$$
$$\left[u = z^{1/4}\left[\left(1+\sqrt{1-z}\right)^{1/2} + \left(1-\sqrt{1-z}\right)^{1/2}\right]^{-1}\right].$$

69. $${}_2F_1\left(\begin{matrix}\frac{1}{8}, \frac{5}{8} \\ 1;\ z\end{matrix}\right) = \frac{2^{5/4}}{\pi}\left(1+\sqrt{1-z}\right)^{-1/4} \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{2^{-1/2}(1-z)^{1/4}}{\left(1+\sqrt{1-z}\right)^{1/2}}}\right).$$

70. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{1}{2};\ z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}}\left[\mathbf{K}\left(\sqrt{\frac{1+\sqrt{z}}{2}}\right) + \mathbf{K}\left(\sqrt{\frac{1-\sqrt{z}}{2}}\right)\right].$$

71. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{1}{2};\ -z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}}(z+1)^{-1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{z}{z+1}}}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)\right].$$

72. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{3}{4};\ z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}}\left(2\sqrt{z^2-z} - 2z + 1\right)^{-1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{(z^2-z)^{1/4}}{\left(2\sqrt{z^2-z}-2z+1\right)^{1/2}}}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{(z^2-z)^{1/4}}{\left(2\sqrt{z^2-z}-2z+1\right)^{1/2}}}\right)\right]$$
$$[\operatorname{Re} z < 1/2].$$

73. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{3}{4};\ -z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}}\left(\sqrt{z} + \sqrt{z+1}\right)^{-1/2}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{(z^2+z)^{1/4}}{\sqrt{z}+\sqrt{z+1}}}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{(z^2+z)^{1/4}}{\sqrt{z}+\sqrt{z+1}}}\right)\right] \quad [\operatorname{Re} z > -1/2].$$

74. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ 1;\ z\end{matrix}\right) = \frac{2}{\pi}\mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{1}{2}(1-z)^{1/2}}\right).$$

75. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ 1;\ -z\end{matrix}\right) = \frac{2^{3/2}}{\pi}(1+\sqrt{z+1})^{-1/2}\mathbf{K}\left(\frac{\sqrt{z}}{1+\sqrt{z+1}}\right).$$

76. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{2} \\ \frac{3}{4};\ z\end{matrix}\right) = \frac{\pi^{-3/2}\Gamma^2\left(\frac{3}{4}\right)}{(\sqrt{z}+1)^{1/2}}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{z^{1/4}}{z^{1/2}+1}}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{z^{1/4}}{z^{1/2}+1}}\right)\right] \quad [0 < z < 1].$$

77. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{2}\\ 1;\ z\end{matrix}\right) = \frac{2^{3/2}}{\pi\left(\sqrt{1-z}+1\right)^{1/2}}\mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{(1-z)^{1/4}}{z} + \frac{(1-z)^{3/4}}{z}}\right).$$

78. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{2}\\ 1;\ -z\end{matrix}\right) = \frac{8}{\pi u_+}\mathbf{K}\left(\frac{u_-}{u_+}\right) \qquad \left[u_\pm = 1 + \sqrt[4]{z+1} \pm \sqrt{2}\left(1+\sqrt{z+1}\right)^{1/2}\right].$$

79. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}\\ 1;\ z\end{matrix}\right) = \frac{2}{\pi}(1+\sqrt{z})^{-1/2}\,\mathbf{K}\left(\frac{2^{1/2}z^{1/4}}{(\sqrt{z}+1)^{1/2}}\right).$$

80. $$= \frac{2}{\pi}(1-z)^{-1/4}\,\mathbf{K}\left(\sqrt{\frac{1-(1-z)^{-1/2}}{2}}\right).$$

81. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}\\ 1;\ -z\end{matrix}\right) = \frac{2}{\pi}(z+1)^{-1/4}\,\mathbf{K}\left(\sqrt{\frac{\sqrt{z+1}-1}{2\sqrt{z+1}}}\right).$$

82. $${}_2F_1\left(\begin{matrix}\frac{3}{8}, \frac{3}{8}\\ \frac{5}{4};\ -z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{(2\pi)^{3/2}z^{1/4}}\left(\sqrt{z}+\sqrt{z+1}\right)^{-1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+\sqrt{z+1})^{1/2}}}\right) - \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+\sqrt{z+1})^{1/2}}}\right)\right].$$

83. $${}_2F_1\left(\begin{matrix}\frac{3}{8}, \frac{7}{8}\\ 1;\ z\end{matrix}\right) = \frac{2^{5/4}}{\pi}(1-z)^{-1/4}\left(1+\sqrt{1-z}\right)^{-1/4}$$
$$\times\,\mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{2^{-1/2}(1-z)^{1/2}}{\left(1-z+\sqrt{1-z}\right)^{1/2}}}\right).$$

84. $${}_2F_1\left(\begin{matrix}\frac{3}{8}, \frac{7}{8}\\ \frac{5}{4};\ z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{(2\pi)^{3/2}z^{1/4}}(\sqrt{z}+1)^{-1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+1)^{1/2}}}\right) - \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+1)^{1/2}}}\right)\right].$$

85. $${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ \frac{3}{4};\ -z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}}(z+1)^{-1/4}\left(\sqrt{z}+\sqrt{z+1}\right)^{-1/2}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{\left(z^2+z\right)^{1/4}}{\sqrt{z}+\sqrt{z+1}}}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{\left(z^2+z\right)^{1/4}}{\sqrt{z}+\sqrt{z+1}}}\right)\right]$$
$$[\operatorname{Re} z > -1/2].$$

86. $${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1;\ z\end{matrix}\right) = \frac{2}{\pi}\,\mathbf{K}\left(\sqrt{z}\right).$$

87. $${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1;\ -z\end{matrix}\right) = \frac{4}{\pi\left(\sqrt{z+1}+1\right)}\,\mathbf{K}\left(\frac{\sqrt{z+1}-1}{\sqrt{z+1}+1}\right).$$

88. $${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{3}{4}\\ 1;\ z\end{matrix}\right) = \frac{2^{3/2}}{\pi}(1-z)^{-1/4}\left(1+\sqrt{1-z}\right)^{-1/2} \times \mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{(1-z)^{1/4}}{1+\sqrt{1-z}}}\right).$$

89. $${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{3}{4}\\ 1;\ -z\end{matrix}\right) = \frac{8}{\pi u_+\sqrt{z+1}}\,\mathbf{K}\left(\frac{u_-}{u_+}\right) \qquad \left[u_\pm = 1+\sqrt[4]{z+1}\pm\sqrt{2}\left(1+\sqrt{z+1}\right)^{1/2}\right].$$

90. $${}_2F_1\left(\begin{matrix}\frac{5}{8}, \frac{7}{8}\\ 1;\ z\end{matrix}\right) = \frac{2^{5/4}}{\pi}(1-z)^{-1/2}\left[2^{1/2}+\left(1-\sqrt{1-z}\right)^{1/2}\right]^{-1/2} \times \mathbf{K}\left(\frac{2^{1/2}(1-\sqrt{1-z})^{1/4}}{\sqrt{2^{1/2}+\left(1-\sqrt{1-z}\right)^{1/2}}}\right).$$

91. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4}\\ \frac{1}{2};\ -z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\pi^{3/2}}(z+1)^{-3/4}$$
$$\times\left[2\,\mathbf{E}\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)+2\,\mathbf{E}\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right) - \mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)-\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)\right] \qquad [|\arg z|<\pi].$$

92. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4}\\ 1;\ z\end{matrix}\right) = \frac{2}{\pi\sqrt{1-z}}\,\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-z}}\right).$$

93. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4}\\ 1;\ -z\end{matrix}\right) = \frac{2^{3/2}}{\pi\sqrt{z+1}\left(\sqrt{z+1}+1\right)^{1/2}}\,\mathbf{K}\left(\frac{\sqrt{z}}{\sqrt{z+1}+1}\right).$$

94. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4}\\ \frac{5}{4};\ -z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\pi^{3/2}(z^2+z)^{1/4}}\left(\sqrt{z}+\sqrt{z+1}\right)^{-1/2}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{(z^2+z)^{1/4}}{\sqrt{z}+\sqrt{z+1}}}\right)-\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{(z^2+z)^{1/4}}{\sqrt{z}+\sqrt{z+1}}}\right)\right] \qquad [\operatorname{Re} z>0].$$

95. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4}\\ \frac{3}{2};\ z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi^{3/2}z^{1/2}}\left[\mathbf{K}\left(\sqrt{\frac{1+\sqrt{z}}{2}}\right)-\mathbf{K}\left(\sqrt{\frac{1-\sqrt{z}}{2}}\right)\right].$$

96. $${}_2F_1\left(\begin{matrix}\frac{3}{4},\ \frac{3}{4}\\ \frac{3}{2};\ -z\end{matrix}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi^{3/2}z^{1/2}}(z+1)^{-1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)-\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)\right] \quad [|\arg z|<\pi].$$

97. $${}_2F_1\left(\begin{matrix}a,\ b\\ c;\ -1\end{matrix}\right) = 2^{-a}\,{}_2F_1\left(\begin{matrix}a,\ c-b\\ c;\ \frac{1}{2}\end{matrix}\right).$$

98. $${}_2F_1\left(\begin{matrix}a,\ a+b+n+1\\ a+\frac{b}{2}+n+1;\ -1\end{matrix}\right) = 2^{-a}\,{}_2F_1\left(\begin{matrix}a,\ -\frac{b}{2};\ \frac{1}{2}\\ a+\frac{b}{2}+n+1\end{matrix}\right).$$

99. $${}_2F_1\left(\begin{matrix}m,\ a\\ a+n;\ -1\end{matrix}\right) = \frac{2^{-m}}{(m-1)!}\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{(n)_k(1-n)_{m-k-1}}{\mathrm{B}\,(k+n,\ a-k)}$$
$$\times\sum_{p=0}^{k+n-1}(-1)^p\binom{k+n-1}{p}\left[\psi\left(\frac{a-k+p+1}{2}\right)-\psi\left(\frac{a-k+p}{2}\right)\right] \quad [m\ge 1].$$

100. $${}_2F_1\left(\begin{matrix}a,\ \frac{a-2-\sqrt{2-a}}{2}\\ \frac{a+4-\sqrt{2-a}}{2};\ -1\end{matrix}\right) = 2^{-a-1}\left[2+a\left(3+\sqrt{2-a}\right)\right].$$

101. $${}_2F_1\left(\begin{matrix}a,\ \frac{a-3-\sqrt{7-3a}}{2}\\ \frac{a+5-\sqrt{7-3a}}{2};\ -1\end{matrix}\right) = \frac{2^{-a-1}}{3}\left[6+a\left(15-a+4\sqrt{7-3a}\right)\right].$$

102. $${}_2F_1\left(\begin{matrix}\frac{1}{4},\ \frac{1}{4}\\ 1;\ -1\end{matrix}\right) = 2^{-7/4}\pi^{-3/2}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

103. $${}_2F_1\left(\begin{matrix}-n,\ -2n-\frac{2}{3}\\ \frac{4}{3};\ -8\end{matrix}\right) = (-27)^n\frac{\left(\frac{5}{6}\right)_n}{\left(\frac{3}{2}\right)_n} \quad [[38],\ (3.7)].$$

104. $${}_2F_1\left(\begin{matrix}-n,\ -2n+\frac{2}{3}\\ \frac{2}{3};\ -8\end{matrix}\right) = \frac{(-4)^n(2n)_n}{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}\left[\left(\frac{1}{6}\right)_n+\frac{1}{2}\left(\frac{1}{2}\right)_n\right]$$
$$[n\ge 1;\ [38],\ (3.8)].$$

105. $${}_2F_1\left(\begin{matrix}-n,\ -2n+\frac{4}{3}\\ \frac{1}{3};\ -8\end{matrix}\right) = \frac{(-4)^n(2n-1)_n}{\left(-\frac{1}{3}\right)_n\left(\frac{1}{3}\right)_n}\left[\left(-\frac{1}{6}\right)_n+\frac{1}{2}\left(-\frac{1}{2}\right)_n\right]$$
$$[n\ge 1;\ [38],\ (3.9)].$$

106. $${}_2F_1\left(\begin{matrix}-n,\ \frac{1-3n}{6}\\ \frac{2}{3};\ -8\end{matrix}\right)=(-1)^{(n+1)/2}3^{(3n-1)/2}\delta_{1,n-2[n/2]}$$
$$+(-1)^{n/2}3^{3n/2}\delta_{0,n-2[n/2]}\quad [[38],\ (3.12)].$$

107. $${}_2F_1\left(\begin{matrix}-n,\ \frac{2-3n}{6}\\ \frac{1}{3};\ -8\end{matrix}\right)=(-1)^{(n+1)/2}3^{(3n-1)/2}\delta_{1,n-2[n/2]}$$
$$+(-1)^{n/2}3^{3n/2-1}\left[1+2\frac{\left(\frac{1}{2}\right)_{n/2}}{\left(\frac{1}{6}\right)_{n/2}}\right]\delta_{0,n-2[n/2]}\quad [n\geq 2;\ [38],\ (3.13)].$$

108. $${}_2F_1\left(\begin{matrix}-2n,\ -n-\frac{1}{6}\\ \frac{4}{3};\ -8\end{matrix}\right)=(-1)^n\frac{3^{3n}}{2n+1}\quad [[38],\ (3.14)].$$

109. $${}_2F_1\left(\begin{matrix}-2n-1,\ -n-\frac{5}{6}\\ \frac{5}{3};\ -8\end{matrix}\right)=(-1)^{n+1}\frac{3^{3n+2}}{2n+3}\quad [[38],\ (3.15)].$$

110. $${}_2F_1\left(\begin{matrix}-2n-1,\ -n-\frac{7}{6}\\ \frac{7}{3};\ -8\end{matrix}\right)=(-1)^{n+1}\frac{5\cdot 3^{3n+2}}{(2n+3)(2n+5)}\quad [[38],\ (3.16)].$$

111. $${}_2F_1\left(\begin{matrix}\frac{1}{4},\ \frac{1}{4}\\ \frac{3}{4};\ -\frac{1}{8}\end{matrix}\right)=\frac{3^{3/4}\left(1+\sqrt{3}\right)}{2^{5/4}\pi^2}\Gamma\left(\frac{1}{3}\right)\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{7}{6}\right).$$

112. $${}_2F_1\left(\begin{matrix}\frac{1}{4},\ \frac{1}{4}\\ 1;\ -\frac{1}{8}\end{matrix}\right)=2^{-5/4}\pi^{-3/2}\Gamma^2\left(\frac{1}{4}\right).$$

113. $${}_2F_1\left(\begin{matrix}\frac{3}{4},\ \frac{3}{4}\\ 1;\ -\frac{1}{8}\end{matrix}\right)=\frac{2^{1/4}}{3\pi^{3/2}}\Gamma^2\left(\frac{1}{4}\right).$$

114. $${}_2F_1\left(\begin{matrix}\frac{3}{4},\ \frac{3}{4}\\ \frac{5}{4};\ -\frac{1}{8}\end{matrix}\right)=\frac{\sqrt{3}-1}{2^{37/12}3^{1/4}\pi^{5/2}}\Gamma^2\left(\frac{1}{4}\right)\Gamma^3\left(\frac{1}{3}\right).$$

115. $${}_2F_1\left(\begin{matrix}a,\ a+\frac{1}{2}\\ \frac{4a+5}{6};\ \frac{1}{9}\end{matrix}\right)=\sqrt{\pi}\left(\frac{3}{4}\right)^a\frac{\Gamma\left(\frac{4a+5}{6}\right)}{\Gamma\left(\frac{2a+3}{6}\right)\Gamma\left(\frac{2a+5}{6}\right)}\quad [[51],\ (1.1)].$$

116. $${}_2F_1\left(\begin{matrix}a,\ \frac{1-a}{2}\\ \frac{3a+5}{6};\ \frac{1}{9}\end{matrix}\right)=\left(\frac{3}{4}\right)^{a/2}\frac{\Gamma\left(\frac{3a+5}{6}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{3a+4}{6}\right)\Gamma\left(\frac{5}{6}\right)}\quad [[51],\ (1.2)].$$

117. $${}_2F_1\left(\begin{matrix} a,\, 1-\frac{a}{2} \\ \frac{3a+4}{6};\; \frac{1}{9}\end{matrix}\right) = \sqrt{\pi}\left(\frac{3}{4}\right)^{a/2} \frac{\Gamma\left(\frac{3a+4}{6}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{2}{3}\right)} \qquad [[51], (1.3)].$$

118. $${}_2F_1\left(\begin{matrix} a,\, a+\frac{1}{4} \\ \frac{5}{4}-2a;\; \frac{1}{9}\end{matrix}\right) = \frac{3^{5a}\Gamma\left(\frac{5}{4}-2a\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{13}{12}\right)}{2^{6a}\Gamma\left(\frac{2}{3}-a\right)\Gamma\left(\frac{13}{12}-a\right)\Gamma\left(\frac{5}{4}\right)} \qquad [[51], (1.4)].$$

119. $${}_2F_1\left(\begin{matrix} a,\, a+\frac{1}{4} \\ \frac{9}{4}-2a;\; \frac{1}{9}\end{matrix}\right) = \frac{3^{5a}\Gamma\left(\frac{9}{4}-2a\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{17}{12}\right)}{2^{6a}\Gamma\left(\frac{4}{3}-a\right)\Gamma\left(\frac{17}{12}-a\right)\Gamma\left(\frac{9}{4}\right)} \qquad [[51], (1.5)].$$

120. $${}_2F_1\left(\begin{matrix} -n,\, -n+\frac{1}{4} \\ 2n+\frac{5}{4};\; \frac{1}{9}\end{matrix}\right) = \frac{\left(\frac{5}{4}\right)_{2n}}{\left(\frac{2}{3}\right)_n\left(\frac{13}{12}\right)_n}\left(\frac{2^6}{3^5}\right)^n \qquad [[38], (6.5)].$$

121. $${}_2F_1\left(\begin{matrix} -n,\, -n+\frac{1}{4} \\ 2n+\frac{9}{4};\; \frac{1}{9}\end{matrix}\right) = \frac{\left(\frac{9}{4}\right)_{2n}}{\left(\frac{4}{3}\right)_n\left(\frac{17}{12}\right)_n}\left(\frac{2^6}{3^5}\right)^n \qquad [[38], (6.6)].$$

122. $${}_2F_1\left(\begin{matrix} -\frac{1}{4},\, \frac{1}{4} \\ 1;\; \frac{1}{9}\end{matrix}\right) = \frac{3^{-1/2}}{2\pi^{3/2}}\left[\Gamma^2\left(\frac{1}{4}\right)+4\Gamma^2\left(\frac{3}{4}\right)\right].$$

123. $${}_2F_1\left(\begin{matrix} \frac{1}{4},\, \frac{1}{2} \\ \frac{3}{4};\; \frac{1}{9}\end{matrix}\right) = \frac{3^{1/4}(1+\sqrt{3})}{8\pi^2}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\Gamma^2\left(\frac{3}{4}\right) \qquad [[49], (7.4)].$$

124. $${}_2F_1\left(\begin{matrix} \frac{1}{4},\, \frac{3}{4} \\ 1;\; \frac{1}{9}\end{matrix}\right) = \frac{3^{1/2}}{4\pi^{3/2}}\Gamma^2\left(\frac{1}{4}\right).$$

125. $$\begin{aligned} {}_2F_1&\left(\begin{matrix} a,\, \frac{1}{2};\; \frac{1}{4} \\ -2a\pm n+\frac{3}{2}\end{matrix}\right) \\ &= \frac{2^{\pm n+3/2}}{3^{\pm n+1}}\Gamma\left[\begin{matrix} \frac{1}{2}-a,\, \frac{\pm 2n+3}{4}-a,\, \frac{\pm 2n+5}{4}-a \\ \frac{\pm 2n+3}{6}-a,\, \frac{\pm 2n+5}{6}-a,\, \frac{\pm 2n+7}{6}-a\end{matrix}\right] R_1(\pm n) \\ &\qquad - 2^{3/2}(-3)^{\pm n-2}\Gamma\left[\begin{matrix} 1-a,\, \frac{\pm 2n+3}{4}-a,\, \frac{\pm 2n+5}{4}-a \\ \frac{\pm n+1}{2}-a,\, \frac{\pm n}{2}-a+1,\, \frac{3}{2}-a\end{matrix}\right] R_2(\pm n), \end{aligned}$$

$R_1(1) = R_2(0) = 0,\ R_1(0) = R_2(1) = 1,$

$$R_1(n) = (-1)^n n \sum_{k=[n/3]}^{[n/2]}\left(\frac{3}{2}\right)^{4k}\frac{(k-1)!}{(n-2k)!\,(3k-n)!}\frac{\left(\frac{1}{2}-a\right)_k}{(1-a)_k} \qquad [n>1],$$

$$R_2(n) = {}_4F_3\left(\begin{matrix} -\frac{n-1}{3},\, -\frac{n-2}{3},\, -\frac{n}{3}+1,\, 1-a \\ -\frac{n}{2}+1,\, -\frac{n-3}{2},\, \frac{3}{2}-a;\; 1\end{matrix}\right) \qquad [n>1],$$

$$R_1(-n) = {}_4F_3\left(\begin{matrix} -\frac{n}{3}, -\frac{n-1}{3}, -\frac{n-2}{3}, a \\ -\frac{n-1}{2}, 1-\frac{n}{2}, \frac{1}{2}-a;\ 1 \end{matrix}\right) \qquad [n \geq 1],$$

$$R_2(-n) = (-1)^n (n+1) \sum_{k=[(n+1)/3]}^{[(n+1)/2]} \left(\frac{3}{2}\right)^{4k} \frac{(k-1)!}{(n-2k+1)!\,(3k-n-1)!} \frac{\left(a-\frac{1}{2}\right)_k}{(a)_k}$$

$$[n \geq 1;\ [83]].$$

126. $${}_2F_1\left(\begin{matrix} m+\frac{1}{2}, a;\ \frac{1}{4} \\ -2a+n+\frac{3}{2} \end{matrix}\right) = \left(\frac{4}{3}\right)^m \frac{m!}{(1/2)_m} \sum_{k=0}^{m} \frac{2^{-2k}}{k!} P_{m-k}^{(k-1/2,\,-3a-m+k+n+1)}\left(\frac{1}{2}\right)$$
$$\times \frac{(-2a+n+1)_k \left(-3a+n+\frac{3}{2}\right)_k}{\left(-2a+n+\frac{3}{2}\right)_k} {}_2F_1\left(\begin{matrix} \frac{1}{2}, a;\ \frac{1}{4} \\ -2a+k+n+\frac{3}{2} \end{matrix}\right).$$

127. $${}_2F_1\left(\begin{matrix} \frac{1}{2}-m, a;\ \frac{1}{4} \\ -2a+n+\frac{3}{2} \end{matrix}\right) = \frac{2^{-2m}(-3)^m}{(-2a+n+1)_m} \sum_{k=0}^{m} \binom{m}{k} (-3)^{-k}$$
$$\times \frac{(-2a+n+1)_k (2a-m-n)_{m-k} \left(-3a+n+\frac{3}{2}\right)_k}{\left(-2a+n+\frac{3}{2}\right)_k} {}_2F_1\left(\begin{matrix} \frac{1}{2}, a;\ \frac{1}{4} \\ -2a+k+n+\frac{3}{2} \end{matrix}\right).$$

128. $${}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{1}{2} \\ \frac{5}{6};\ \frac{1}{4} \end{matrix}\right) = \frac{2^{5/3}}{3} \qquad [[66],\ (\text{A7})].$$

129. $${}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{2}{3} \\ \frac{7}{6};\ \frac{1}{4} \end{matrix}\right) = \frac{2^{-8/3}}{\sqrt{3}\,\pi^3} \Gamma^6\left(\frac{1}{3}\right) \qquad [[66],\ (\text{A8})].$$

130. $${}_2F_1\left(\begin{matrix} a, b;\ \frac{1}{2} \\ \frac{a+b+n}{2} \end{matrix}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{a+b+n}{2}\right)}{\left(\frac{b-a-n}{2}\right)_n} \sum_{k=0}^{n} \binom{n}{k} (-2)^{-k} (a)_k$$
$$\times \left[\frac{1}{\Gamma\left(\frac{a+k+1}{2}\right)\Gamma\left(\frac{b+k-n}{2}\right)} + \frac{1}{\Gamma\left(\frac{a+k}{2}\right)\Gamma\left(\frac{b+k-n}{2}+1\right)}\right].$$

131. $${}_2F_1\left(\begin{matrix} a,\ n-a \\ b;\ \frac{1}{2}\end{matrix}\right) = 2^{n-b}\sqrt{\pi}\,\frac{\Gamma(b)}{(-a)_n}\sum_{k=0}^{n}\binom{n}{k}(-2)^{-k}(a+b-n)_k$$
$$\times\left[\frac{1}{\Gamma\left(\frac{a+b+k-n}{2}\right)\Gamma\left(\frac{b-a+k-n+1}{2}\right)} + \frac{1}{\Gamma\left(\frac{a+b+k-n+1}{2}\right)\Gamma\left(\frac{b-a+k-n}{2}\right)}\right].$$

132. $${}_2F_1\left(\begin{matrix} a,\ -n-a \\ b;\ \frac{1}{2}\end{matrix}\right) = 2^{-b}\sqrt{\pi}\,n!\,\frac{\Gamma(b)}{(a+b)_n}\sum_{k=0}^{n}\frac{1}{k!}P_{n-k}^{(a+k,\,k-b-n)}(0)$$
$$\times\left[\frac{1}{\Gamma\left(\frac{a+b-k}{2}\right)\Gamma\left(\frac{b-a-k+1}{2}\right)} + \frac{1}{\Gamma\left(\frac{a+b-k+1}{2}\right)\Gamma\left(\frac{b-a-k}{2}\right)}\right].$$

133. $${}_2F_1\left(\begin{matrix} a,\ a-2b+m \\ a-b+n;\ \frac{1}{2}\end{matrix}\right) = \frac{\sqrt{\pi}\,\Gamma(a-b+n)}{(-b)_n(b-m)_n(a-2b)_m}$$
$$\times\sum_{k=0}^{n}2^{-k}\binom{n}{k}(a)_k(a-2b)_k(b-a-m)_{n-k}$$
$$\times\sum_{p=0}^{m}2^{-p}\binom{m}{p}(a+k)_p(a-2b+k)_m$$
$$\times\left[\frac{1}{\Gamma\left(\frac{k+p+a}{2}-b\right)\Gamma\left(\frac{k+p+a+1}{2}\right)} + \frac{1}{\Gamma\left(\frac{k+p+a+1}{2}-b\right)\Gamma\left(\frac{k+p+a}{2}\right)}\right].$$

134. $${}_2F_1\left(\begin{matrix} a,\ a-2b-n \\ a-b;\ \frac{1}{2}\end{matrix}\right)$$
$$= n!\sqrt{\pi}\,\frac{\Gamma(a-b)}{(b)_n}\sum_{k=0}^{n}\frac{2^{-2k}}{k!}(a)_k(a-2b)_k P_{n-k}^{(b+k-1,\,a-b+k-n)}(0)$$
$$\times\left[\frac{1}{\Gamma\left(\frac{k+a+1}{2}\right)\Gamma\left(\frac{k+a}{2}-b\right)} + \frac{1}{\Gamma\left(\frac{k+a}{2}\right)\Gamma\left(\frac{k+a+1}{2}-b\right)}\right].$$

135. $${}_2F_1\left(\begin{matrix} m+1,\ n+1 \\ \frac{s}{t}+r;\ \frac{1}{2}\end{matrix}\right) = (-1)^{r-1}2^{m+n+1}\left(\frac{s}{t}+r-1\right)$$
$$\times\sum_{j=0}^{m}\frac{(-2)^{-j}}{j!}\left(\frac{s}{t}+r-n-1\right)_j P_{m-j}^{(j,\,s/t+j+r-m-n-2)}(0)$$
$$\times\sum_{k=0}^{n}\frac{(-2)^{-k}}{k!}\left(\frac{s}{t}+j+r-1\right)_k P_{n-k}^{(k,\,s/t+j+k+r-n-2)}(0)$$

$$\times\left\{\frac{\pi}{2}\csc\frac{s\pi}{t}-2\sum_{q=0}^{[t/2]-1}\cos\left[(2q+1)\frac{s\pi}{t}\right]\ln\sin\left[(2q+1)\frac{s\pi}{2t}\right]\right.$$
$$\left.-\sum_{p=0}^{j+k+r-2}\frac{(-1)^p}{p+s/t}\right\}\quad [s,\, t=1,2,\ldots;\ m=1,2,\ldots,n-1;\ n=2,3,\ldots].$$

136. $${}_2F_1\left(\begin{matrix}-\frac{1}{4},\ \frac{1}{4}\\ 1;\ \frac{1}{2}\end{matrix}\right)=\frac{2^{-5/2}}{\pi^{3/2}}\left[\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)+8\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)\right].$$

137. $${}_2F_1\left(\begin{matrix}\frac{1}{6},\ \frac{1}{3}\\ \frac{1}{2};\ \frac{1}{2}\end{matrix}\right)=\frac{\left[\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}+2\right)\right]^{1/2}}{4}\,\frac{\Gamma^2\left(\frac{1}{24}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma^2\left(\frac{1}{6}\right)}\qquad [[49],\ (5.19)].$$

138. $${}_2F_1\left(\begin{matrix}\frac{2}{3},\ \frac{5}{6}\\ \frac{3}{2};\ \frac{1}{2}\end{matrix}\right)=\frac{\left[\left(\sqrt{6}-2\right)\left(5\sqrt{2}-7\right)\right]^{1/2}}{2^{13/4}\pi}\,\frac{\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{13}{24}\right)}\qquad [[49],\ (6.11)].$$

139. $${}_2F_1\left(\begin{matrix}\frac{1}{4},\ \frac{1}{4}\\ \frac{1}{2};\ \frac{3}{4}\end{matrix}\right)=\frac{3^{3/4}\left(1+\sqrt{3}\right)}{2\pi^2}\Gamma\left(\frac{1}{3}\right)\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{7}{6}\right)\qquad [[49],\ (3.12)].$$

140. $${}_2F_1\left(\begin{matrix}\frac{1}{4},\ \frac{1}{2}\\ \frac{3}{4};\ \frac{3}{4}\end{matrix}\right)=\left(\frac{4}{3}\right)^{3/4}\qquad [[49],\ (7.6)].$$

141. $${}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{3}{4}\\ \frac{5}{4};\ \frac{3}{4}\end{matrix}\right)=\frac{1}{12\pi^2}\Gamma^4\left(\frac{1}{4}\right)\qquad [[49],\ (7.9)].$$

142. $${}_2F_1\left(\begin{matrix}\frac{3}{4},\ \frac{3}{4}\\ \frac{3}{2};\ \frac{3}{4}\end{matrix}\right)=\frac{\sqrt{3}-1}{2^{7/3}3^{1/4}\pi^{5/2}}\Gamma^2\left(\frac{1}{4}\right)\Gamma^3\left(\frac{1}{3}\right)\qquad [[49],\ (6.5)].$$

143. $${}_2F_1\left(\begin{matrix}a,\ a+\frac{1}{2}\\ \frac{4a+2}{3};\ \frac{8}{9}\end{matrix}\right)=\sqrt{\pi}\,\frac{3^a\Gamma\left(\frac{4a+5}{6}\right)}{\Gamma\left(\frac{2a+3}{6}\right)\Gamma\left(\frac{2a+5}{6}\right)}\qquad [[51],\ (3.1)].$$

144. $${}_2F_1\left(\begin{matrix}a,\ 1-2a\\ \frac{2}{3};\ \frac{8}{9}\end{matrix}\right)=\frac{2}{3^a}\sin\left(\frac{5\pi}{6}-\pi a\right)\qquad [[51],\ (3.2)].$$

145. $${}_2F_1\left(\begin{matrix}a,\ 2-2a\\ \frac{4}{3};\ \frac{8}{9}\end{matrix}\right)=\sqrt{\pi}\,\frac{3^{-a}\Gamma\left(\frac{1}{6}\right)}{2\Gamma\left(a+\frac{1}{6}\right)\Gamma\left(\frac{3}{2}-a\right)}\qquad [[51],\ (3.3)].$$

146. $${}_2F_1\left(\begin{matrix}a,\ a+\frac{1}{4}\\ \frac{4a+1}{3};\ \frac{8}{9}\end{matrix}\right)=\frac{108^{a/3}\Gamma\left(\frac{4a+7}{12}\right)\Gamma\left(\frac{2a+5}{6}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{4a+9}{12}\right)\Gamma\left(\frac{a+2}{3}\right)\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{5}{6}\right)}\qquad [[51],\ (3.4)].$$

147. $${}_2F_1\left(\begin{matrix} a,\, a-\frac{1}{4} \\ \frac{4a+1}{3};\ \frac{8}{9} \end{matrix}\right) = \frac{108^{a/3}\Gamma\left(\frac{4a+1}{12}\right)\Gamma\left(\frac{2a+5}{6}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4a+3}{12}\right)\Gamma\left(\frac{a+2}{3}\right)\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{6}\right)} \qquad [[51], (3.5)].$$

148. $${}_2F_1\left(\begin{matrix} -\frac{1}{4},\, \frac{1}{4} \\ 1;\ \frac{8}{9} \end{matrix}\right) = \frac{3^{-1/2}}{(2\pi)^{3/2}}\left[\Gamma^2\left(\frac{1}{4}\right) + 8\Gamma^2\left(\frac{3}{4}\right)\right].$$

149. $${}_2F_1\left(\begin{matrix} \frac{1}{8},\, \frac{3}{8} \\ \frac{1}{2};\ \frac{8}{9} \end{matrix}\right) = \frac{\left(\sqrt{2}-1\right)^{3/2}\left(\sqrt{3}-1\right)\left(\sqrt{6}+\sqrt{2}-1\right)^{1/2}}{2^{11/3}3^{1/2}\pi^3} \times \Gamma^2\left(\frac{1}{24}\right)\Gamma\left(\frac{1}{4}\right)\Gamma^2\left(\frac{7}{8}\right)\Gamma\left(\frac{11}{12}\right) \qquad [[49], (4.12)].$$

150. $${}_2F_1\left(\begin{matrix} \frac{1}{4},\, \frac{3}{4} \\ 1;\ \frac{8}{9} \end{matrix}\right) = \frac{3^{1/2}}{(2\pi)^{3/2}}\Gamma^2\left(\frac{1}{4}\right).$$

151. $${}_2F_1\left(\begin{matrix} \frac{5}{8},\, \frac{7}{8} \\ \frac{3}{2};\ \frac{8}{9} \end{matrix}\right) = \frac{3^{1/2}\left(2\sqrt{3}-\sqrt{6}-1\right)^{1/2}}{8\pi^2} \times \cos\frac{\pi}{24}\cos\frac{\pi}{8}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)\Gamma\left(\frac{11}{24}\right) \qquad [[49], (6.8)].$$

152. $${}_2F_1\left(\begin{matrix} \frac{1}{6},\, \frac{1}{3} \\ \frac{1}{2};\ \frac{25}{27} \end{matrix}\right) = \frac{3^{3/2}}{4} \qquad [[49], (5.15)].$$

153. $${}_2F_1\left(\begin{matrix} \frac{2}{3},\, \frac{5}{6} \\ \frac{3}{2};\ \frac{25}{27} \end{matrix}\right) = \frac{3^{5/2}}{160\pi^2}\Gamma^2\left(\frac{1}{6}\right)\Gamma^2\left(\frac{1}{3}\right) \qquad [[49], (6.12)].$$

154. $${}_2F_1\left(\begin{matrix} \frac{3}{8},\, \frac{7}{8} \\ \frac{5}{4};\ \frac{48}{49} \end{matrix}\right) = \frac{2^{-7/2}7^{3/4}}{3\pi^2}\Gamma^4\left(\frac{1}{4}\right).$$

155. $${}_2F_1\left(\begin{matrix} \frac{1}{4},\, \frac{1}{4} \\ \frac{1}{2};\ \frac{63}{64} \end{matrix}\right) = \frac{\left(4+\sqrt{7}\right)^{1/2}}{2^{3/2}7^{1/4}\pi^{5/2}}\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)\Gamma^2\left(\frac{3}{4}\right) \qquad [[49], (3.13)].$$

156. $${}_2F_1\left(\begin{matrix} \frac{3}{4},\, \frac{3}{4} \\ \frac{3}{2};\ \frac{63}{64} \end{matrix}\right) = \frac{\sqrt{7}-1}{3\cdot 7^{3/4}\pi^{5/2}}\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)\Gamma^2\left(\frac{1}{4}\right) \qquad [[49], (6.6)].$$

157. $${}_2F_1\left(\begin{matrix} \frac{1}{8},\, \frac{3}{8} \\ \frac{1}{2};\ \frac{49}{81} \end{matrix}\right) = \frac{8\sqrt{3}\left(1+\sqrt{2}\right)^{1/2}}{\pi^{5/2}}\Gamma^2\left(\frac{7}{8}\right)\Gamma^3\left(\frac{5}{4}\right) \qquad [[49], (4.11)].$$

158. $${}_2F_1\left(\begin{matrix} \frac{1}{4},\, \frac{1}{2} \\ \frac{3}{4};\ \frac{49}{81} \end{matrix}\right) = \frac{3\left(1+\sqrt{7}\right)}{2^4 7^{1/4}\pi^{5/2}}\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)\Gamma^2\left(\frac{3}{4}\right) \qquad [[49], (7.5)].$$

159. $${}_2F_1\left(\begin{matrix}\frac{5}{8}, \frac{7}{8} \\ \frac{3}{2}; \frac{49}{81}\end{matrix}\right) = \frac{3^{5/2}\left(\sqrt{2}-1\right)^{3/2}}{112\pi^{3/2}}\Gamma^2\left(\frac{1}{8}\right)\Gamma\left(\frac{1}{4}\right)$$ [[49], (6.7)].

160. $${}_2F_1\left(\begin{matrix}\frac{1}{8}, \frac{3}{8} \\ \frac{1}{2}; \frac{80}{81}\end{matrix}\right) = \frac{3^{1/2}\left(3\sqrt{5}+4\sqrt{2}+1\right)^{1/2}}{2^{3/10}5^{1/4}\pi^3} \times \sin^2\frac{3\pi}{40}\cos\frac{3\pi}{20}\Gamma^2\left(\frac{7}{40}\right)\Gamma\left(\frac{3}{20}\right)\Gamma\left(\frac{13}{20}\right)\Gamma^2\left(\frac{37}{40}\right)$$ [[49], (4.13)].

161. $${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{2} \\ \frac{3}{4}; \frac{80}{81}\end{matrix}\right) = \frac{9}{5}$$ [[49], (7.7)].

162. $${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{3}{4} \\ \frac{5}{4}; \frac{80}{81}\end{matrix}\right) = \frac{9}{8\cdot 5^{5/4}\pi^2}\Gamma^4\left(\frac{1}{4}\right)$$ [[49], (7.10)].

163. $${}_2F_1\left(\begin{matrix}\frac{5}{8}, \frac{7}{8} \\ \frac{3}{2}; \frac{80}{81}\end{matrix}\right) = \frac{3^{5/2}\left(3\sqrt{5}-4\sqrt{2}-1\right)^{1/2}}{2^{7/2}5^{5/4}\pi^2} \times \cos\frac{\pi}{40}\cos\frac{9\pi}{40}\Gamma\left(\frac{1}{40}\right)\Gamma\left(\frac{9}{40}\right)\Gamma\left(\frac{11}{40}\right)\Gamma\left(\frac{19}{40}\right)$$ [[49], (6.9)].

164. $${}_2F_1\left(\begin{matrix}\frac{1}{12}, \frac{5}{12} \\ \frac{1}{2}; \frac{98}{125}\end{matrix}\right) = \frac{1+\sqrt{2}}{2^{11/4}\pi^2}\left(\frac{5}{3}\right)^{1/4}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)\Gamma^2\left(\frac{3}{4}\right)$$ [48].

165. $${}_2F_1\left(\begin{matrix}\frac{1}{12}, \frac{5}{12} \\ \frac{1}{2}; \frac{121}{125}\end{matrix}\right) = \frac{\sqrt[4]{15}\left(1+\sqrt{3}\right)}{2^{17/6}\pi^{5/2}}\Gamma^3\left(\frac{1}{3}\right)\Gamma^2\left(\frac{3}{4}\right)$$ [48].

166. $${}_2F_1\left(\begin{matrix}\frac{1}{6}, \frac{1}{3} \\ \frac{1}{2}; \frac{121}{125}\end{matrix}\right) = \frac{3^{1/2}\left(\sqrt{5}-1\right)^{1/2}}{2^{11/2}5^{1/4}\pi^4} \times \Gamma\left(\frac{1}{30}\right)\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{19}{30}\right)\Gamma^2\left(\frac{2}{3}\right)\Gamma^2\left(\frac{5}{6}\right)$$ [[49], (5.20)].

167. $${}_2F_1\left(\begin{matrix}\frac{2}{3}, \frac{5}{6} \\ \frac{3}{2}; \frac{121}{125}\end{matrix}\right) = \frac{5^{5/4}\left(5\sqrt{5}-11\right)^{1/2}}{11\cdot 2^{9/2}\pi^2}\Gamma\left(\frac{1}{30}\right)\Gamma\left(\frac{19}{30}\right)\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{4}{15}\right)$$ [[49], (6.13)].

168. $${}_2F_1\left(\begin{matrix}\frac{1}{12}, \frac{5}{12} \\ \frac{1}{2}; \frac{1323}{1331}\end{matrix}\right) = \frac{3\sqrt[4]{11}}{4}$$ [48].

169. $${}_2F_1\left(\begin{matrix}\frac{1}{8}, \frac{3}{8} \\ \frac{1}{2}; \frac{2400}{2401}\end{matrix}\right) = \frac{2\sqrt{7}}{3}$$ [[49], (4.10)].

170. $${}_2F_1\left(\begin{matrix}\frac{5}{8},\ \frac{7}{8}\\ \frac{3}{2};\ \frac{2400}{2401}\end{matrix}\right)=\frac{49}{480\pi^2}\sqrt{\frac{7}{3}}\,\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)$$ [[49], (6.10)].

171. $${}_2F_1\left(\begin{matrix}\frac{3}{8},\ \frac{7}{8}\\ \frac{5}{4};\ \frac{25920}{25921}\end{matrix}\right)=\frac{5^{-5/4}161^{3/4}}{24\pi^2}\Gamma^4\left(\frac{1}{4}\right){}_2F_1\left(\begin{matrix}a,\ \frac{4a+1}{6}\\ \frac{2a+5}{6};\ 17-12\sqrt{2}\end{matrix}\right)$$
$$=48^{-a/2}(3+2\sqrt{2})^a\frac{\sqrt{\pi}\,\Gamma\left(\frac{2a+5}{6}\right)}{\Gamma\left(\frac{a+3}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}.$$

172. $${}_2F_1\left(\begin{matrix}a,\ \frac{4a+1}{6}\\ \frac{4a+1}{3};\ 12\sqrt{2}-16\end{matrix}\right)=3^{-a/2}(3+2\sqrt{2})^a\frac{\sqrt{\pi}\,\Gamma\left(\frac{2a+5}{6}\right)}{\Gamma\left(\frac{a+3}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}.$$

173. $${}_2F_1\left(\begin{matrix}-\frac{1}{2},\ -\frac{1}{2}\\ 1;\ 3-2\sqrt{2}\end{matrix}\right)=\frac{2^{-7/4}}{\pi^{3/2}}\left(\sqrt{2}-1\right)^{1/2}\left[\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)+8\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)\right].$$

174. $${}_2F_1\left(\begin{matrix}-\frac{1}{4},\ \frac{1}{4}\\ 1;\ \frac{57-40\sqrt{2}}{49}\end{matrix}\right)$$
$$=\frac{\left(3+\sqrt{2}\right)^{-1/2}}{2^{9/4}\pi^{3/2}}\left[(1+\sqrt{2})\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)+8\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)\right].$$

175. $${}_2F_1\left(\begin{matrix}-\frac{1}{4},\ \frac{1}{4}\\ 1;\ \frac{249-176\sqrt{2}}{441}\end{matrix}\right)$$
$$=\frac{3^{-1/2}}{2\pi^{3/2}}\left(1+2\sqrt{2}\right)^{-1/2}\left[(1+\sqrt{2})\Gamma^2\left(\frac{1}{4}\right)+4\Gamma^2\left(\frac{3}{4}\right)\right].$$

176. $${}_2F_1\left(\begin{matrix}\frac{1}{12},\ \frac{5}{12}\\ \frac{1}{2};\ \frac{3514+988\sqrt{2}}{17^3}\end{matrix}\right)$$
$$=\frac{3^{-3/4}\left(1+\sqrt{6}\right)}{8\pi^2}\left(\frac{5+2\sqrt{2}}{2+\sqrt{3}}\right)^{1/4}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\Gamma^2\left(\frac{3}{4}\right)$$ [48].

177. $${}_2F_1\left(\begin{matrix}\frac{1}{8},\ \frac{3}{8}\\ \frac{1}{2};\ \frac{25}{2401}\left(16\sqrt{2}-13\right)^2\end{matrix}\right)=\frac{3}{8}\left(6\sqrt{2}+4\right)^{1/2}$$ [[49], (4.9)].

178. $${}_2F_1\left(\begin{matrix}\frac{1}{8},\ \frac{3}{8}\\ 1;\ \frac{32}{2401}(325\sqrt{2}-457)\end{matrix}\right)=\frac{\left(3+\sqrt{2}\right)^{1/2}}{2^{11/4}\pi^{3/2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

179. $${}_2F_1\left(\begin{matrix} \frac{1}{6}, \frac{1}{3} \\ \frac{1}{2}; \ \frac{3^4\left(17\sqrt{6}-22\right)^2}{2\cdot 5^6} \end{matrix}\right) = \frac{5}{8}\left(\sqrt{6}+1\right)$$ [[49], (5.17)].

180. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}; \ -96+56\sqrt{3} \end{matrix}\right) = \frac{\sqrt{6}+\sqrt{2}}{3^{3/4}}$$ [[49], (3.9)].

181. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1; \ -56-40\sqrt{2} \end{matrix}\right) = \frac{\left(2-\sqrt{2}\right)^{1/2}}{4\pi^{3/2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

182. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1; \ 40\sqrt{2}-56 \end{matrix}\right) = \frac{1}{2^{9/4}\pi^{3/2}}\left(1+\sqrt{2}\right)^{1/2}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

183. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1; \ -16(26-15\sqrt{3}) \end{matrix}\right) = \frac{3^{-1/4}}{4\pi^{3/2}}\left(2+\sqrt{3}\right)^{1/2}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right).$$

184. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1; \ \frac{7-5\sqrt{2}}{8} \end{matrix}\right) = \frac{\left(1+\sqrt{2}\right)^{1/4}}{4\pi^{3/2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

185. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1; \ \frac{15\sqrt{3}-26}{16} \end{matrix}\right) = \frac{3^{-1/4}}{(2\pi)^{3/2}}\left(2+\sqrt{3}\right)^{1/4}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right).$$

186. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1; \ \frac{140-99\sqrt{2}}{32} \end{matrix}\right) = \frac{1}{2^{15/8}\pi^{3/2}}\left(1+\sqrt{2}\right)^{1/2}\Gamma^2\left(\frac{1}{4}\right).$$

187. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1; \ \frac{8}{49}(5\sqrt{2}-1) \end{matrix}\right) = \frac{\left(2+3\sqrt{2}\right)^{1/2}}{4\pi^{3/2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

188. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1; \ \frac{16}{441}(12-11\sqrt{2}) \end{matrix}\right) = \frac{\left(12-3\sqrt{2}\right)^{1/2}}{2^{11/4}\pi^{3/2}}\Gamma^2\left(\frac{1}{4}\right).$$

189. $${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1; \ \frac{249-176\sqrt{2}}{441} \end{matrix}\right) = \frac{3^{1/2}}{8\pi^{3/2}}\left(1+2\sqrt{2}\right)^{1/2}\Gamma^2\left(\frac{1}{4}\right).$$

190. $${}_2F_1\left(\begin{matrix} \frac{3}{8}, \frac{7}{8} \\ 1; \ \frac{64}{13225}(153\sqrt{3}-266) \end{matrix}\right)$$
$$= \frac{2^{-5/2}}{11\pi^{3/2}}\left(\frac{5}{3}\right)^{3/4}\left(29861+18884\sqrt{3}\right)^{1/4}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right).$$

191. $${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1;\ 17-12\sqrt{2}\end{matrix}\right) = \frac{\left(3+2\sqrt{2}\right)^{1/2}}{2^{5/2}\pi^{3/2}}\Gamma^2\left(\frac{1}{4}\right).$$

192. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4} \\ 1;\ 40\sqrt{2}-56\end{matrix}\right) = \frac{2^{-9/4}}{7\pi^{3/2}}\left(137+97\sqrt{2}\right)^{1/2}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

193. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4} \\ 1;\ \frac{7-5\sqrt{2}}{8}\end{matrix}\right) = \frac{2^{-1/2}}{7\pi^{3/2}}\left(31+41\sqrt{2}\right)^{1/4}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right).$$

194. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4} \\ 1;\ \frac{15\sqrt{3}-26}{16}\end{matrix}\right) = \frac{2^{1/2}3^{-3/4}}{11\pi^{3/2}}\left(962+551\sqrt{3}\right)^{1/4}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right).$$

195. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4} \\ 1;\ \frac{140-99\sqrt{2}}{32}\end{matrix}\right) = \frac{2^{1/2}}{21\pi^{3/2}}\left(782+\frac{1107}{\sqrt{2}}\right)^{1/4}\Gamma^2\left(\frac{1}{4}\right).$$

196. $${}_2F_1\left(\begin{matrix}\frac{3}{4}, \frac{3}{4} \\ \frac{3}{2};\ 56\sqrt{3}-96\end{matrix}\right) = \frac{2^{-7/2}}{3\pi^2}\left(7+4\sqrt{3}\right)^{3/4}\Gamma^4\left(\frac{1}{4}\right).$$

8.1.2. The hypergeometric function ${}_3F_2(a_1, a_2, a_3;\ b_1, b_2;\ z)$

1. $${}_3F_2\left(\begin{matrix}a, b, 1-b;\ z \\ \frac{a+b}{2}, \frac{a-b+1}{2}\end{matrix}\right) = \frac{1+8z}{3(1-4z)^a}\,{}_3F_2\left(\begin{matrix}\frac{a}{3}, \frac{a+1}{3}, \frac{a+2}{3};\ -\frac{27z}{(1-4z)^3} \\ \frac{a+b}{2}, \frac{a-b+1}{2}\end{matrix}\right)$$
$$+\frac{2}{3(1-4z)^{a-1}}\,{}_3F_2\left(\begin{matrix}\frac{a-1}{3}, \frac{a}{3}, \frac{a+1}{3};\ -\frac{27z}{(1-4z)^3} \\ \frac{a+b}{2}, \frac{a-b+1}{2}\end{matrix}\right) \quad [-1/8 < z < 1/4;\ [52], (6)].$$

2. $${}_3F_2\left(\begin{matrix}a, b, 3-b;\ z \\ \frac{a+b+1}{2}, \frac{a-b}{2}+2\end{matrix}\right) = \frac{(a+b-1)(b-a-2)}{12(b-1)(b-2)z(1-4z)^{a-1}}$$
$$\times\left[\frac{1+8z}{1-4z}\,{}_3F_2\left(\begin{matrix}\frac{a}{3}, \frac{a+1}{3}, \frac{a+2}{3};\ -\frac{27z}{(1-4z)^3} \\ \frac{a+b-1}{2}, \frac{a-b}{2}+1\end{matrix}\right)\right.$$
$$\left.-\,{}_3F_2\left(\begin{matrix}\frac{a-1}{3}, \frac{a}{3}, \frac{a+1}{3};\ -\frac{27z}{(1-4z)^3} \\ \frac{a+b-1}{2}, \frac{a-b}{2}+1\end{matrix}\right)\right] \quad [-1/8 < z < 1/4;\ [52], (6)].$$

3. $${}_3F_2\left(\begin{matrix}a+1, -a, b \\ a, c;\ z\end{matrix}\right) = (1-z)^{-b}\,{}_3F_2\left(\begin{matrix}b, a+c-1, \frac{a+c+1}{2} \\ c, \frac{a+c-1}{2};\ \frac{z}{z-1}\end{matrix}\right)$$
$$[|\arg(1-z)| < \pi].$$

4. $${}_3F_2\left(\begin{matrix} a,\, a+\frac{1}{2},\, b;\; -\frac{4z}{(1-z)^2} \\ c,\, 2a+b-c+1 \end{matrix}\right) = (1-z)^{2a}\; {}_3F_2\left(\begin{matrix} 2a,\, 2a-c+1,\, c-b;\; z \\ c,\, 2a+b-c+1 \end{matrix}\right).$$

5. $${}_3F_2\left(\begin{matrix} a,\, a+\frac{1}{3},\, a+\frac{2}{3};\; \frac{27z^2}{4(1-z)^3} \\ b,\, 3a-b+\frac{3}{2} \end{matrix}\right)$$
$$= (1-z)^{3a}\; {}_3F_2\left(\begin{matrix} 3a,\, b-\frac{1}{2},\, 3a-b+1;\; 4z \\ 2b-1,\, 6a-2b+2 \end{matrix}\right).$$

6. $${}_3F_2\left(\begin{matrix} -n,\, -n,\, -n;\; z \\ \frac{1}{2}-n,\, -2n \end{matrix}\right) = \left(-\frac{z}{4}\right)^n \left[\frac{n!}{\left(\frac{1}{2}\right)_n} P_n\left(\sqrt{1-\frac{1}{z}}\right)\right]^2.$$

7. $${}_3F_2\left(\begin{matrix} \frac{1}{4},\, \frac{1}{4},\, \frac{1}{4} \\ \frac{1}{2},\, \frac{3}{4};\; -z \end{matrix}\right) = \frac{\Gamma^4\left(\frac{3}{4}\right)}{\pi^3} \left(\sqrt{z}+\sqrt{z+1}\right)^{-1/2}$$
$$\times \left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{2^{-1/2}z^{1/4}}{\left(\sqrt{z}+\sqrt{z+1}\right)^{1/2}}}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{2^{-1/2}z^{1/4}}{\left(\sqrt{z}+\sqrt{z+1}\right)^{1/2}}}\right)\right]^2.$$

8. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ 1,\, 1;\; z \end{matrix}\right) = \frac{4}{\pi^2}\mathbf{K}^2\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right).$$

9. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ 1,\, 1;\; -z \end{matrix}\right) = \frac{8}{\pi^2\left(\sqrt{z+1}+1\right)}\mathbf{K}^2\left(\frac{\sqrt{z}}{\sqrt{z+1}+1}\right).$$

10. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1 \\ \frac{3}{2},\, \frac{3}{2};\; z \end{matrix}\right) = \frac{1}{2\sqrt{z}}\left[\mathrm{Li}_2(\sqrt{z}) - \mathrm{Li}_2(-\sqrt{z})\right].$$

11. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, 1 \\ \frac{3}{2},\, \frac{3}{2};\; -z \end{matrix}\right) = \frac{1}{2\sqrt{z}}\left[2\arctan\sqrt{z}\,\ln\sqrt{z}\right.$$
$$\left. + \mathrm{Cl}_2(2\arctan\sqrt{z}) + \mathrm{Cl}_2(\pi - 2\arctan\sqrt{z})\right].$$

12. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{2}{3},\, \frac{4}{3};\; -\frac{27}{4}z(1-z)^{-3} \\ \frac{3}{2},\, \frac{3}{2} \end{matrix}\right) = \frac{(1-z)^{3/2}}{\sqrt{z}}\arcsin\sqrt{z}$$
$$\left[|z|,\, |27z(1-z)^{-3}/4| < 1\right].$$

13. $${}_3F_2\left(\begin{matrix}\frac{1}{2},\,1,\,1\\ \frac{3}{4},\,\frac{5}{4};\,-z\end{matrix}\right)=\frac{1}{z^{1/4}(z+1)^{1/4}}\left\{\cos\frac{\pi+2\arctan\sqrt{z}}{4}\ln\left(z_{+}+z_{-}\right)\right.$$
$$\left.+\sin\frac{\pi+2\arctan\sqrt{z}}{4}\arcsin\frac{\left(\sqrt{z}-\sqrt{z+1}+1\right)^{1/2}}{\sqrt{2}}\right\}$$
$$\left[z_{\pm}=2^{-1/2}\left(\sqrt{z}+\sqrt{z+1}\pm1\right)^{1/2}\right].$$

14. $${}_3F_2\left(\begin{matrix}\frac{1}{2},\,1,\,1;\ z\\ \frac{3}{2},\,\frac{3}{2}\end{matrix}\right)=\frac{1}{4\sqrt{z}}\left[i\pi^2+4\arcsin\sqrt{z}\,\ln\frac{1-e^{i\arcsin\sqrt{z}}}{1+e^{i\arcsin\sqrt{z}}}\right.$$
$$\left.+4i\operatorname{Li}_2\left(-e^{i\arcsin\sqrt{z}}\right)-4i\operatorname{Li}_2\left(e^{i\arcsin\sqrt{z}}\right)\right].$$

15. $${}_3F_2\left(\begin{matrix}\frac{1}{2},\,1,\,1\\ \frac{3}{2},\,\frac{3}{2};\,-z\end{matrix}\right)=\frac{1}{4\sqrt{z}}\left[\pi^2+4\ln\left(\sqrt{z}+\sqrt{z+1}\right)\ln\frac{\sqrt{z}+\sqrt{z+1}-1}{\sqrt{z}+\sqrt{z+1}+1}\right.$$
$$\left.+4\operatorname{Li}_2\left(-\frac{1}{\sqrt{z}+\sqrt{z+1}}\right)-4\operatorname{Li}_2\left(\frac{1}{\sqrt{z}+\sqrt{z+1}}\right)\right].$$

16. $${}_3F_2\left(\begin{matrix}1,\,1,\,\frac{3}{2}\\ \frac{5}{4},\,\frac{7}{4};\,-z\end{matrix}\right)=\frac{3}{2z^{3/4}(z+1)^{1/4}}\left\{\sin\frac{\pi+2\arctan\sqrt{z}}{4}\ln\left(z_{+}+z_{-}\right)\right.$$
$$\left.-\cos\frac{\pi+2\arctan\sqrt{z}}{4}\arcsin\frac{\left(\sqrt{z}-\sqrt{z+1}+1\right)^{1/2}}{\sqrt{2}}\right\}$$
$$\left[z_{\pm}=2^{-1/2}\left(\sqrt{z}+\sqrt{z+1}\pm1\right)^{1/2}\right].$$

17. $$=\frac{\Gamma^4\left(\frac{3}{4}\right)}{\pi^3}\left(1+2z-2\sqrt{z^2+z}\right)^{1/4}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{1}{2^{1/2}}\left(\sqrt{z^2+z}-z\right)^{1/2}}\right)\right.$$
$$\left.+\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2^{1/2}}\left(\sqrt{z^2+z}-z\right)^{1/2}}\right)\right]^2.$$

18. $${}_3F_2\left(\begin{matrix}\frac{3}{4},\,\frac{3}{4},\,\frac{3}{4}\\ \frac{5}{4},\,\frac{3}{2};\,-z\end{matrix}\right)=\frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi^3z^{1/2}}\left(\sqrt{z}+\sqrt{z+1}\right)^{-1/2}$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{2^{-1/2}z^{1/4}}{\left(\sqrt{z}+\sqrt{z+1}\right)^{1/2}}}\right)-\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{2^{-1/2}z^{1/4}}{\left(\sqrt{z}+\sqrt{z+1}\right)^{1/2}}}\right)\right]^2.$$

19. $$= \frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi^3} \frac{(1+2z-2\sqrt{z^2+z})^{3/4}}{\sqrt{z^2+z}-z} \times \left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{1}{2^{1/2}}\left(\sqrt{z^2+z}-z\right)^{1/2}}\right) - \mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2^{1/2}}\left(\sqrt{z^2+z}-z\right)^{1/2}}\right)\right]^2.$$

20. $${}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1;\ z\end{matrix}\right) = \frac{2^{5/2}}{\pi^2}\left(2-z+2\sqrt{1-z}\right)^{-1/4} \times \mathbf{K}^2\left(\sqrt{\frac{1}{2}-\left(\frac{1-\sqrt{1-z}}{2z}\right)^{1/2}}\right).$$

21. $${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, \frac{5}{4};\ -z\end{matrix}\right) = \frac{1}{2^{1/2}\pi z^{1/4}}\left(\sqrt{z+1}+\sqrt{z}\right)^{-1/2} \times \left[\mathbf{K}^2\left(\sqrt{\frac{1}{2}+\frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+\sqrt{z+1})^{1/2}}}\right) - \mathbf{K}^2\left(\sqrt{\frac{1}{2}-\frac{2^{-1/2}z^{1/4}}{(\sqrt{z}+\sqrt{z+1})^{1/2}}}\right)\right].$$

22. $${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2};\ z\end{matrix}\right) = \frac{2}{\pi\sqrt{z}} \int_0^{\sqrt{z}} \mathbf{K}(x)\, dx.$$

23. $${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2};\ -z\end{matrix}\right) = \frac{2}{\pi\sqrt{z}} \int_0^{\sqrt{z/(z+1)}} \frac{\mathbf{K}(x)}{1-x^2}\, dx.$$

24. $${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2};\ z\end{matrix}\right) = \frac{1}{2\sqrt{z}}\operatorname{Cl}_2(2\arcsin\sqrt{z}) + \frac{\arcsin\sqrt{z}}{\sqrt{z}}\ln(2\sqrt{z}).$$

25. $${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2};\ -z\end{matrix}\right) = \frac{1}{\sqrt{z}}\left[\frac{\pi^2}{12} - \frac{1}{2}\ln^2(\sqrt{z}+\sqrt{1+z}) + \ln(\sqrt{z}+\sqrt{1+z})\ln(1+\sqrt{z}+\sqrt{1+z}) + \operatorname{Li}_2(-\sqrt{z}-\sqrt{1+z}) - \operatorname{Li}_2(1-\sqrt{z}-\sqrt{1+z})\right].$$

26. $${}_3F_2\left(\begin{matrix}1, 1, 1 \\ \frac{3}{2}, \frac{3}{2};\ z\end{matrix}\right) = \frac{1}{\sqrt{z}} \int_0^{\arcsin\sqrt{z}} \frac{x\,dx}{\sqrt{z-\sin^2 x}}.$$

27. $${}_3F_2\left(\begin{matrix}1,\,1,\,1\\ \frac{3}{2},\,\frac{3}{2};\,-z\end{matrix}\right)=\frac{1}{\sqrt{z}}\int\limits_0^{\ln(\sqrt{z}+\sqrt{z+1})}\frac{x\,dx}{\sqrt{z-\sinh^2 x}}.$$

28. $${}_3F_2\left(\begin{matrix}a,\,b,\,c\\ d,\,e;\,1\end{matrix}\right)=\frac{\Gamma(d)\,\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(c)\,\Gamma(d+e-a-c)\,\Gamma(d+e-b-c)}$$
$$\times\,{}_3F_2\left(\begin{matrix}d-c,\,e-c,\,d+e-a-b-c\\ d+e-a-c,\,d+e-b-c;\,1\end{matrix}\right)\qquad[\operatorname{Re}c,\,\operatorname{Re}(d+e-a-b-c)>0].$$

29. $${}_3F_2\left(\begin{matrix}a-n,\,b-n,\,c-n\\ d-n,\,e-n;\,1\end{matrix}\right)$$
$$=\frac{1}{(1-d)_n(1-e)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1-d)_k(e-c)_k$$
$$\times\,(c-n)_{n-k}(d-a-b)_{n-k}\,{}_3F_2\left(\begin{matrix}a,\,b,\,c-k;\,1\\ d-k,\,e\end{matrix}\right).$$

30. $${}_3F_2\left(\begin{matrix}a,\,b,\,c;\,1\\ 2a+b+1,\,a+2b+1\end{matrix}\right)=\frac{2(a+b)-c}{2(a+b)}\,{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c;\,1\\ 2a+b+1,\,a+2b+1\end{matrix}\right)$$
$$[\operatorname{Re}(2a+2b-c)>0;\ [56]].$$

31. $${}_3F_2\left(\begin{matrix}a,\,b,\,c;\,1\\ a+1,\,2a-b-c+\frac{bc}{a+1}+3\end{matrix}\right)$$
$$=\frac{(a+1)(a-b+2)(a-c+2)}{(a+2)(a^2+3a-b-ab-c-ac+bc+2)}$$
$$\times\,{}_3F_2\left(\begin{matrix}a+2,\,b,\,c;\,1\\ a+3,\,2a-b-c+\frac{bc}{a+1}+3\end{matrix}\right)\quad\left[\operatorname{Re}\left(2a-2b-2c+\frac{b}{a+1}\right)>-4;\ [56]\right].$$

32. $${}_3F_2\left(\begin{matrix}a,\,b,\,c;\,1\\ a+1,\,c+\frac{a\,(a-c+1)}{b-1}+1\end{matrix}\right)=\frac{(a-b+2)(a^2+a-c-ac+bc)}{(a+1)(a^2+2a-ab-c-ac+bc)}$$
$$\times\,{}_3F_2\left(\begin{matrix}a+1,\,b-1,\,c;\,1\\ a+2,\,c+\frac{a\,(a-c+1)}{b-1}\end{matrix}\right)\quad[\operatorname{Re}(a\,(a-c+1)/(b-1)+b)<2;\ [56]].$$

33. $${}_3F_2\left(\begin{matrix}a,\,b,\,c;\,1\\ a+1,\,\frac{bc}{-a+b+c}\end{matrix}\right)$$
$$=\frac{a^2+b+c+bc-a(b+c+1)}{-a+b+c+bc}\,{}_3F_2\left(\begin{matrix}a,\,b+1,\,c+1;\,1\\ a+1,\,\frac{bc}{-a+b+c}+2\end{matrix}\right)$$
$$[\operatorname{Re}[bc/(-a+b+c)+b+c]<2;\ [56]].$$

34. $${}_3F_2\left(\begin{matrix}a,\, b,\, c;\; 1\\ a+1,\, d\end{matrix}\right) = \frac{(a-b+1)(a-c+1)(a-c+2)(d-1)}{(a+1)(c-1)(a-d+1)(a-d+2)}\, {}_3F_2\left(\begin{matrix}a+1,\, b,\, c-1\\ a+2,\, d-1;\; 1\end{matrix}\right) - \frac{a(a-b+1)+(c-1)(b-d+1)}{(c-1)(a-d+1)(a-d+2)}\, \frac{\Gamma(d)\Gamma(d-b-c+1)}{\Gamma(d-b)\Gamma(d-c)}$$
$[\operatorname{Re}[d-b-c] > -1;\; [56]]$.

35. $${}_3F_2\left(\begin{matrix}1,\, a,\, 1-a\\ \frac{3}{2},\, b;\; 1\end{matrix}\right) = \frac{2^{2-2b}\cos[(a-b)\pi]\,\Gamma(2b-1)\,\Gamma(a-b+1)}{(2a-1)\Gamma(a+b-1)} + \frac{2^{3-2b}(b-1)}{(2a-1)(a-b+1)}\, {}_2F_1\left(\begin{matrix}3-2b,\, a-b+1\\ a-b+2;\; -1\end{matrix}\right)$$
$[\operatorname{Re} b > 1/2]$.

36. $${}_3F_2\left(\begin{matrix}a,\, a,\, 1\\ 2a,\, 2a;\; 1\end{matrix}\right) = \frac{2^{6a-6}\sqrt{\pi}\,(2a-1)^2\csc^2(a\pi)}{21a}\, \frac{\Gamma^3\left(a-\frac{1}{2}\right)}{\Gamma^3(a)} + \frac{2a-1}{a-1}\, {}_3F_2\left(\begin{matrix}a,\, a,\, 1;\; 1\\ 2a,\, 2-a\end{matrix}\right)$$
$[1/2 < \operatorname{Re} a < 1;\; [46],\, (3,\, 4)]$.

37. $${}_3F_2\left(\begin{matrix}a,\, a+\frac{1}{2},\, 1;\; 1\\ 2a+\frac{1}{2},\, 4a\end{matrix}\right) = \frac{2^{18a-7}3^{3/2-6a}\sqrt{\pi}\,(4a-1)^2\csc(2a\pi)}{2a-1}\, \frac{\Gamma^3\left(2a-\frac{1}{2}\right)}{\Gamma\left(2a-\frac{1}{3}\right)\Gamma(2a)\,\Gamma\left(2a+\frac{1}{3}\right)} + \frac{4a-1}{2a-1}\, {}_3F_2\left(\begin{matrix}2a,\, 6a-1,\, 1;\; \frac{1}{2}\\ 4a,\, 2-2a\end{matrix}\right)$$
$[\operatorname{Re} a > 1/4;\; [46],\, (7,\, 8)]$.

38. $${}_3F_2\left(\begin{matrix}a,\, \frac{1}{2},\, 1;\; 1\\ a+\frac{1}{2},\, 2a\end{matrix}\right) = \frac{2^{4a-2}\csc(a\pi)}{2a-1}\, \frac{\Gamma^3\left(a+\frac{1}{2}\right)}{\Gamma(a)\,\Gamma\left(2a-\frac{1}{2}\right)} + \frac{3(2a-1)}{4(a-1)}\, {}_3F_2\left(\begin{matrix}a,\, 2a-\frac{1}{2},\, 1;\; \frac{1}{4}\\ a+\frac{1}{2},\, 2-a\end{matrix}\right)$$
$[\operatorname{Re} a > 1/2;\; [46],\, (7,\, 8)]$.

39. $${}_3F_2\left(\begin{matrix}\frac{1}{2},\, \frac{1}{2},\, 1\\ a,\, a;\; 1\end{matrix}\right) = 2^{2a-3}\sqrt{\pi}\,\sec^2(a\pi)\, \frac{\Gamma^2(a)\,\Gamma(a-1)}{\Gamma^3\left(a-\frac{1}{2}\right)} + \frac{2(a-1)}{2a-3}\, {}_3F_2\left(\begin{matrix}\frac{1}{2},\, 1,\, a-\frac{1}{2}\\ a,\, \frac{5}{2}-a;\; 1\end{matrix}\right)$$
$[1 < \operatorname{Re} a < 3/2;\; [46],\, (1,\, 2)]$.

40. $${}_3F_2\left(\begin{matrix}-n,\, a,\, a+\frac{1}{2}\\ b,\, b+\frac{1}{2};\; 1\end{matrix}\right) = \frac{n!\,(2b-2a)_n}{(2b)_{2n}}\, P_n^{(2b+n-1,\, 2a-2b-2n)}(3).$$

41. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\; 1\\ \frac{a+b+1}{2},\, c\end{matrix}\right) = \frac{\left(\frac{1-a-b}{2}+c\right)_n}{(c)_n}\, {}_3F_2\left(\begin{matrix}-n,\, \frac{a-b+1}{2},\, \frac{b-a+1}{2};\; 1\\ \frac{a+b+1}{2},\, \frac{1-a-b}{2}+c\end{matrix}\right).$$

42. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\; 1\\ 2a,\, b-a-n+1\end{matrix}\right) = \frac{(a)_n(2a-b)_n}{(2a)_n(a-b)_n}.$$

43. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\; 1\\ a-n+\frac{1}{2},\, b+\frac{1}{2}\end{matrix}\right) = \frac{\left(b-a+\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{1}{2}-a\right)_n\left(b+\frac{1}{2}\right)_n}.$$

44. $${}_3F_2\left(\begin{matrix}-n,\, a,\, \frac{1}{2}-a-n;\; 1\\ b,\, b+\frac{1}{2}\end{matrix}\right) = \frac{(2b-2a)_{2n}}{(2b)_{2n}}\, {}_3F_2\left(\begin{matrix}-n,\, a,\, 1-2b-2n;\; 1\\ a-b-n+\frac{1}{2},\, a-b-n+1\end{matrix}\right).$$

45. $${}_3F_2\left(\begin{matrix}-n,\, \frac{1}{2}-n,\, a;\; 1\\ b,\, b+\frac{1}{2}\end{matrix}\right) = \frac{(2n)!}{(2a-2b+1)(2b)_{2n}} \times \left[2aP_{2n}^{(2b-a-1,\, 2a-2b-2n+1)}(3) - (2n+1)P_{2n+1}^{(2b-a-2,\, 2a-2b-2n)}(3)\right]$$

[[55], (3.18)].

46. $${}_3F_2\left(\begin{matrix}-m-n,\, a,\, b;\; 1\\ c-m,\, a+b-c\end{matrix}\right) = \frac{(a-c)_{m+1}(b-c)_{m+1}}{(-c)_{m+1}(a+b-c)_{m+1}}\, {}_3F_2\left(\begin{matrix}-n+1,\, a,\, b;\; 1\\ c+1,\, a+b-c+m+1\end{matrix}\right) \quad [n \geq 1].$$

47. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\; 1\\ \frac{a-n}{2},\, \frac{a-n+1}{2}\end{matrix}\right) = \frac{(-2)^n n!}{(1-a)_n} P_n^{(-n-2b,\, 2b-a)}(0).$$

48. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\; 1\\ \frac{b-n}{2},\, a+\frac{b-n}{2}+1\end{matrix}\right) = \frac{(b+n)\left(a-\frac{b+n}{2}+1\right)_n}{(b-n)\left(-a-\frac{b+n}{2}\right)_n}.$$

49. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\; 1\\ \frac{a-n}{2},\, \frac{a-n+1}{2}\end{matrix}\right) = \frac{n!}{(1-a)_n}\left[P_n^{(2b-a-1,\, a-n-1)}(3) + 2P_{n-1}^{(2b-a,\, a-n)}(3)\right] \quad [n \geq 1].$$

50. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\; 1\\ \frac{b-n+1}{2},\, \frac{b-n}{2}+1\end{matrix}\right) = \frac{n!\, b}{(b+n)(-b)_n} P_n^{(2a-b-1,\, b-n-1)}(3).$$

51. $${}_3F_2\left(\begin{matrix}-n,\, a,\, b;\ 1\\ b+2,\, \frac{a-b+an-bn+ab-n-1}{b}\end{matrix}\right) = \frac{(n+1)(b+1)\left(\frac{b-a-an+n+1}{b}\right)_n}{(b+n+1)\left(\frac{2b-a-ab-an+n+1}{b}\right)_n} \quad [[38], (1.9)].$$

52. $${}_3F_2\left(\begin{matrix}-n,\, \frac{1}{4},\, \frac{1}{2};\ 1\\ \frac{5-4n}{8},\, \frac{9-4n}{8}\end{matrix}\right) = \frac{1-4n}{1+4n}.$$

53. $${}_3F_2\left(\begin{matrix}-2n,\, 2n+1,\, \frac{1}{2}\\ 1-2a,\, 1+2a;\ 1\end{matrix}\right) = {}_3F_2\left(\begin{matrix}-n,\, n+\frac{1}{2},\, \frac{1}{2}\\ 1-a,\, 1+a;\ 1\end{matrix}\right).$$

54. $${}_3F_2\left(\begin{matrix}-2n,\, a,\, b;\ 1\\ 2a,\, \frac{b}{2}-n\end{matrix}\right) = \frac{\left(1+\frac{b}{2}\right)_n}{\left(1-\frac{b}{2}\right)_n}\, {}_3F_2\left(\begin{matrix}-n,\, a+n,\, \frac{b+1}{2};\ 1\\ a+\frac{1}{2},\, \frac{b}{2}+1\end{matrix}\right).$$

55. $${}_3F_2\left(\begin{matrix}-2n,\, a,\, b;\ 1\\ 2a,\, \frac{b}{2}-n\end{matrix}\right) = \frac{\left(a-\frac{b}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n\left(1-\frac{b}{2}\right)_n}.$$

56. $${}_3F_2\left(\begin{matrix}-2n,\, a,\, b;\ 1\\ 2a,\, \frac{b+1}{2}-n\end{matrix}\right) = \frac{\left(a+\frac{1-b}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n\left(\frac{1-b}{2}\right)_n}.$$

57. $${}_3F_2\left(\begin{matrix}-2n-1,\, a,\, b;\ 1\\ 2a,\, \frac{b}{2}-n\end{matrix}\right) = 0.$$

58. $${}_3F_2\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a\\ a+\frac{1}{2},\, b;\ 1\end{matrix}\right) = \frac{(a)_n}{(2a)_n}\, {}_3F_2\left(\begin{matrix}-n,\, a,\, 1-b-n\\ 1-a-n,\, b;\ -1\end{matrix}\right) \quad [[55], (3.14)].$$

59. $${}_3F_2\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a\\ b,\, b+\frac{1}{2};\ 1\end{matrix}\right) = \frac{n!}{(2b)_n}\left[P_n^{(2b-a-1,\, 2a-2b-n-1)}(3) + P_{n-1}^{(2b-a,\, 2a-2b-n)}(3)\right] \quad [n \geq 1].$$

60. $${}_3F_2\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a;\ 1\\ b,\, a-b-n+\frac{3}{2}\end{matrix}\right) = \frac{\left(b-\frac{1}{2}\right)_n(2b-2a-1)_n}{(2b-1)_n\left(b-a-\frac{1}{2}\right)_n}.$$

61. $${}_3F_2\left(\begin{matrix}-\frac{2n}{3},\, \frac{1-2n}{3},\, \frac{2-2n}{3}\\ a,\, \frac{1}{2}-n;\ 1\end{matrix}\right) = \frac{n!}{(a)_n}\left(\frac{2}{3}\right)^{2n} P_n^{(n+2a-2,\, 2-3a-3n)}\left(\frac{1}{2}\right).$$

62. $${}_3F_2\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ a;\ 1\\ \frac{a-n+1}{3},\ \frac{a-n+2}{3}\end{matrix}\right)=\frac{2^{2n}n!}{(1-a)_n}P_n^{(-a/3-2n/3,\,a-n-1)}\left(\frac{5}{4}\right).$$

63. $${}_3F_2\left(\begin{matrix}-n,\ \frac{n}{2}+a,\ \frac{n+1}{2}+a\\ \frac{2a+1}{3},\ \frac{2a+2}{3};\ 1\end{matrix}\right)=\frac{2^n n!}{(2a)_n}P_n^{(-4a/3-n,\,2a-1)}\left(\frac{5}{4}\right)\qquad [[38],\ (3.6)].$$

64. $${}_3F_2\left(\begin{matrix}-\frac{2n}{3},\ \frac{1-2n}{3},\ \frac{2-2n}{3}\\ \frac{1}{2}-n,\ a;\ 1\end{matrix}\right)=\left(\frac{2}{3}\right)^{2n}\frac{n!}{(a)_n}P_n^{(2a+n-2,\,2-3a-3n)}\left(\frac{1}{2}\right)$$

[[38], (3.17)].

65. $${}_3F_2\left(\begin{matrix}-\frac{n}{3},\ \frac{1-n}{3},\ \frac{2-n}{3}\\ a,\ 2-2a-n;\ 1\end{matrix}\right)=\left(-\frac{4}{3}\right)^n\frac{n!}{(2a-1)_n}P_n^{(1/2-a-n,\,3a-5/2)}\left(\frac{1}{2}\right)$$

[[38], (3.18)].

66. $${}_3F_2\left(\begin{matrix}-\frac{n}{3},\ \frac{1-n}{3},\ \frac{2-n}{3};\ 1\\ a,\ a+\frac{1}{3}\end{matrix}\right)=\frac{3^{n/2}n!}{(3a-1)_n}C_n^{a-1/3}\left(\frac{\sqrt{3}}{2}\right)\qquad [[38],\ (3.19)].$$

67. $${}_3F_2\left(\begin{matrix}-n,\ \frac{2n+1}{4},\ \frac{2n+3}{4}\\ \frac{1}{2},\ \frac{5}{6};\ 1\end{matrix}\right)=\frac{(-2)^n n!}{\left(\frac{5}{6}\right)_n}C_{2n}^{1/6-n}\left(\frac{3}{2\sqrt{2}}\right).$$

68. $${}_3F_2\left(\begin{matrix}-\frac{2n}{3},\ \frac{1-2n}{3},\ \frac{2-2n}{3}\\ \frac{1}{2}-n,\ \frac{1}{2};\ 1\end{matrix}\right)=\frac{3^{-n}n!\left(-\frac{1}{2}\right)_n}{\left(-\frac{1}{2}\right)_{2n}}C_{2n}^{n-1/2}\left(\frac{2}{\sqrt{3}}\right).$$

69. $${}_3F_2\left(\begin{matrix}-\frac{2n}{3},\ \frac{1-2n}{3},\ \frac{2-2n}{3}\\ \frac{1}{2}-n,\ \frac{3}{2};\ 1\end{matrix}\right)=\frac{3^{-n-1/2}n!\left(\frac{3}{2}\right)_n}{2\left(\frac{5}{2}\right)_{2n}}C_{2n+1}^{n+3/2}\left(\frac{2}{\sqrt{3}}\right).$$

70. $${}_3F_2\left(\begin{matrix}-\frac{n}{3},\ \frac{1-n}{3},\ \frac{2-n}{3}\\ \frac{5}{6}-n,\ \frac{7}{6}-n;\ 1\end{matrix}\right)=(-i)^n\frac{3^{n/2}\left(\frac{1}{2}\right)_n\left(-\frac{1}{2}\right)_{2n}}{\left(-\frac{1}{2}\right)_{3n}}T_n\left(i\sqrt{3}\right).$$

71. $${}_3F_2\left(\begin{matrix}-mn,\ mn,\ 1;\ 1\\ 1-ma,\ 1+ma\end{matrix}\right)={}_3F_2\left(\begin{matrix}-n,\ n,\ 1;\ 1\\ 1-a,\ 1+a\end{matrix}\right)\qquad [m=1,\ 2,\ \ldots].$$

72. $${}_3F_2\left(\begin{matrix}-n,\ a,\ b;\ 1\\ \frac{b-n+1}{2},\ a+\frac{b-n+1}{2}\end{matrix}\right)=(-1)^n\frac{\left(a+\frac{1-b-n}{2}\right)_n}{\left(a+\frac{1+b-n}{2}\right)_n}.$$

73. $${}_3F_2\left(\begin{matrix}-n,\ a,\ b;\ 1\\ \frac{a+b+1}{2},\ \frac{a+b+1}{2}-n\end{matrix}\right)=\frac{\left(\frac{a-b+1}{2}\right)_n\left(\frac{b-a+1}{2}\right)_n}{\left(\frac{a+b+1}{2}\right)_n\left(\frac{1-a-b}{2}\right)_n}.$$

74. $${}_3F_2\left(\begin{matrix} a,\, 2a-\frac{1}{2},\, \frac{1}{2} \\ a+\frac{1}{2},\, 2a;\; -1 \end{matrix}\right) = \frac{2^{1/2-2a}\pi\Gamma^2(a+\frac{1}{2})}{\Gamma^2\left(\frac{4a+3}{8}\right)\Gamma^2\left(\frac{4a+5}{8}\right)}.$$

75. $${}_3F_2\left(\begin{matrix} \frac{1}{2}+a,\, \frac{1}{2}-a,\, \frac{1}{2} \\ 1+a,\, 1-a;\; -1 \end{matrix}\right) = \frac{2^{-1/2}\pi^2 a\csc(a\pi)}{\Gamma\left(\frac{5+4a}{8}\right)\Gamma\left(\frac{5-4a}{8}\right)\Gamma\left(\frac{7+4a}{8}\right)\Gamma\left(\frac{7-4a}{8}\right)}.$$

76. $${}_3F_2\left(\begin{matrix} -n,\, a,\, b;\; -1 \\ 1-a-n,\, 1-b-n \end{matrix}\right) = \frac{(2a)_n}{(a)_n}\, {}_3F_2\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2},\, a;\; 1 \\ a+\frac{1}{2},\, 1-b-n \end{matrix}\right).$$

77. $${}_3F_2\left(\begin{matrix} -n,\, 2a,\, 1+a \\ a,\, b;\; -1 \end{matrix}\right) = \frac{(b-2a)_n}{(b)_n}\, {}_3F_2\left(\begin{matrix} -n,\, a+\frac{1}{2},\, 2a-b+1;\; 1 \\ a+\frac{1-b-n}{2},\, a-\frac{b+n}{2}+1 \end{matrix}\right).$$

78. $${}_3F_2\left(\begin{matrix} \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2} \\ 2,\, 2;\; -1 \end{matrix}\right) = \frac{1}{4\sqrt{2}\,\pi^3}\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right) - \frac{4}{\pi}.$$

79. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{4} \end{matrix}\right) = \frac{\pi^2}{10}.$$

80. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{8} \end{matrix}\right) = \frac{1}{6\sqrt{2}}\left(\pi^2 - 3\ln^2 2\right).$$

81. $${}_3F_2\left(\begin{matrix} \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2} \\ 2,\, 2;\; -\frac{1}{8} \end{matrix}\right) = -\frac{64\sqrt{2}}{3\pi} + \frac{4\sqrt{2}}{3\pi^3}\Gamma^4\left(\frac{1}{4}\right).$$

82. $${}_3F_2\left(\begin{matrix} \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2} \\ 2,\, 2;\; \frac{1}{64} \end{matrix}\right) = \frac{4096}{21\pi} - \frac{320}{21\sqrt{7}\,\pi^4}\Gamma^2\left(\frac{1}{7}\right)\Gamma^2\left(\frac{2}{7}\right)\Gamma^2\left(\frac{4}{7}\right).$$

83. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, 1,\, 1 \\ \frac{1}{4},\, \frac{3}{4};\; \frac{1}{16} \end{matrix}\right) = \frac{16}{15} + \frac{\pi\sqrt{3}}{27} - \frac{2\sqrt{5}}{25}\ln\frac{1+\sqrt{5}}{2}.$$

84. $${}_3F_2\left(\begin{matrix} a,\, b,\, 1-b;\; \frac{1}{4} \\ \frac{a+b}{2},\, \frac{a-b+1}{2} \end{matrix}\right) = \frac{2^{2a/3}\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{a-b+1}{2}\right)}{3\Gamma(a)}$$
$$\times\left[\frac{\Gamma\left(\frac{a}{3}\right)}{\Gamma\left(\frac{a+3b}{6}\right)\Gamma\left(\frac{a-3b+3}{6}\right)} + \frac{2^{1/3}\Gamma\left(\frac{a+2}{3}\right)}{\Gamma\left(\frac{a+3b+2}{6}\right)\Gamma\left(\frac{a-3b+5}{6}\right)}\right]$$ [[52], (ii)].

85. $${}_3F_2\left(\begin{matrix} a,\, b,\, 1-b;\ \frac{1}{4} \\ \frac{a+b+1}{2},\ \frac{a-b}{2}+1 \end{matrix}\right) = \frac{2^{2a/3}\Gamma\left(\frac{a}{3}+1\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a-b}{2}+1\right)}{\Gamma\left(\frac{a+3b+3}{6}\right)\Gamma\left(\frac{a-3b}{6}+1\right)\Gamma(a+1)}$$ [[52], (i)].

86. $${}_3F_2\left(\begin{matrix} a,\, b,\, 3-b;\ \frac{1}{4} \\ \frac{a+b+1}{2},\ \frac{a-b}{2}+2 \end{matrix}\right) = \frac{2^{2(a+2)/3}\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a-b}{2}+2\right)}{3(b-1)(b-2)\Gamma(a)}$$
$$\times\left[-\frac{2^{2/3}\Gamma\left(\frac{a}{3}\right)}{\Gamma\left(\frac{a+3b-3}{6}\right)\Gamma\left(\frac{a-3b+6}{6}\right)} + \frac{\Gamma\left(\frac{a+2}{3}\right)}{\Gamma\left(\frac{a+3b-1}{6}\right)\Gamma\left(\frac{a-3b+8}{6}\right)}\right]$$ [[52], (iii)].

87. $${}_3F_2\left(\begin{matrix} -3n-m,\, a,\, 1-a;\ \frac{1}{4} \\ \frac{a-3n-m+1}{2},\ 1-\frac{a+3n+m}{2} \end{matrix}\right) = \frac{(3n)!\,\delta_{0,m}}{2^{2n} n!\left(\frac{a+n}{2}\right)_n\left(\frac{n-a+1}{2}\right)_n}$$ [$m = 0, 1, 2$; [52], (iv)].

88. $${}_3F_2\left(\begin{matrix} -3n-m,\, a,\, 1-a;\ \frac{1}{4} \\ \frac{a-3n-m}{2},\ \frac{1-a-3n-m}{2} \end{matrix}\right) = \frac{(3n)!\,\delta_{0,m}}{2^{2n} n!\left(\frac{a+n+1}{2}\right)_n\left(\frac{n-a+2}{2}\right)_n}$$
$$+\frac{(3n+2)!\,\delta_{2,m}}{2^{2n+1} n!\left(\frac{a+n+1}{2}\right)_{n+1}\left(\frac{n-a}{2}+1\right)_{n+1}}$$ [$m = 0, 1, 2$; [52], (v)].

89. $${}_3F_2\left(\begin{matrix} a,\, \frac{4}{3}-a,\, 1-3a;\ \frac{1}{4} \\ \frac{1}{3}-a,\ \frac{3}{2}-2a \end{matrix}\right) = \frac{3^{3a}\Gamma\left(\frac{3}{2}-2a\right)\Gamma\left(\frac{5}{3}\right)}{2^{4a-1}\sqrt{\pi}\,\Gamma\left(\frac{5}{3}-2a\right)}$$ [[52], (vi)].

90. $${}_3F_2\left(\begin{matrix} a,\, 1,\, 2;\ \frac{1}{4} \\ \frac{a}{2}+1,\ \frac{a+3}{2} \end{matrix}\right)$$
$$= \frac{a\,(a+1)}{6}\left[\psi\left(\frac{a+3}{6}\right)-\psi\left(\frac{a}{6}\right)+\psi\left(\frac{a+2}{6}\right)-\psi\left(\frac{a+5}{6}\right)\right]$$ [[52], (vii)].

91. $${}_3F_2\left(\begin{matrix} a,\, 2a-\frac{1}{2},\, 1;\ \frac{1}{4} \\ a+\frac{1}{2},\ 2-a \end{matrix}\right) = -\frac{2^{4a}(a-1)\csc(a\pi)}{3(2a-1)^2}\,\frac{\Gamma^3\left(a+\frac{1}{2}\right)}{\Gamma(a)\,\Gamma\left(2a-\frac{1}{2}\right)}$$
$$+\frac{4(a-1)}{3(2a-1)}\,{}_3F_2\left(\begin{matrix} a,\, \frac{1}{2},\, 1;\ 1 \\ a+\frac{1}{2},\ 2a \end{matrix}\right)$$ [$\operatorname{Re} a > 1/2$; [46], (7, 8)].

92. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, 1,\, 1;\ \frac{1}{4} \\ \frac{3}{2},\ \frac{3}{2} \end{matrix}\right) = \frac{8}{3}\mathrm{G} - \frac{\pi}{3}\ln\left(2+\sqrt{3}\right).$$

93. $${}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ \frac{3}{2},\ \frac{3}{2};\ \frac{1}{4} \end{matrix}\right) = \frac{1}{16\sqrt{3}}\left[\zeta\left(2,\,\frac{1}{6}\right)-\zeta\left(2,\,\frac{1}{3}\right)+\zeta\left(2,\,\frac{2}{3}\right)-\zeta\left(2,\,\frac{5}{6}\right)\right].$$

94. $${}_3F_2\left(\begin{matrix}\frac{1}{2},1,1\\ \frac{3}{2},\frac{3}{2};\ \frac{1}{4}\end{matrix}\right)=\frac{1}{3}\left[8\mathbf{G}-\pi\ln\left(2+\sqrt{3}\right)\right]$$ [[52], (viii)].

95. $${}_3F_2\left(\begin{matrix}1,1,1\\ \frac{3}{2},2;\ \frac{1}{4}\end{matrix}\right)=\frac{\pi^2}{9}.$$

96. $${}_3F_2\left(\begin{matrix}\frac{3}{2},\frac{3}{2},\frac{3}{2}\\ 2,2;\ \frac{1}{4}\end{matrix}\right)=\frac{64}{3\pi}-\frac{4}{3\sqrt{3}\pi^3}\Gamma^2\left(\frac{1}{6}\right)\Gamma^2\left(\frac{1}{3}\right).$$

97. $${}_3F_2\left(\begin{matrix}a,1,1;\ \frac{1}{2}\\ \frac{a}{2}+1,2\end{matrix}\right)=\frac{a}{2(a-1)}\left[\psi\left(\frac{a}{2}\right)+2\ln 2+\mathbf{C}\right].$$

98. $${}_3F_2\left(\begin{matrix}1,1,1\\ 2,a;\ \frac{1}{2}\end{matrix}\right)=\frac{1-a}{4}$$
$$\times\left\{\left[\psi\left(\frac{a-1}{2}\right)-\psi\left(\frac{a}{2}\right)\right]^2-\zeta\left(2,\ \frac{a-1}{2}\right)-\zeta\left(2,\ \frac{a}{2}\right)\right\}$$ [[28], (4.18)].

99. $${}_3F_2\left(\begin{matrix}a,3a-1,1;\ \frac{1}{2}\\ 2a,2-a\end{matrix}\right)=-2^{9a-7}3^{3/2-3a}\sqrt{\pi}\,\frac{\csc(a\pi)\,(2a-1)\Gamma^3\left(a-\frac{1}{2}\right)}{\Gamma\left(a-\frac{1}{3}\right)\Gamma(a-1)\,\Gamma\left(a+\frac{1}{3}\right)}$$
$$+\frac{a-1}{2a-1}\,{}_3F_2\left(\begin{matrix}\frac{a}{2},\frac{a+1}{2},1;\ 1\\ 2a,a+\frac{1}{2}\end{matrix}\right)$$ [$\operatorname{Re} a>1/4$; [46], (7, 8)].

100. $${}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2};\ \frac{1}{2}\end{matrix}\right)=\frac{\pi}{2^{5/2}}\ln 2+\frac{\mathbf{G}}{\sqrt{2}}.$$

101. $${}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2};\ \frac{3}{4}\end{matrix}\right)=\frac{\pi}{3\sqrt{3}}\ln 3$$
$$+\frac{1}{72}\left[\zeta\left(2,\ \frac{1}{6}\right)-\zeta\left(2,\ \frac{1}{3}\right)+\zeta\left(2,\ \frac{2}{3}\right)-\zeta\left(2,\ \frac{5}{6}\right)\right].$$

102. $${}_3F_2\left(\begin{matrix}-3n-1,a,-a-3n-\frac{1}{2}\\ 2a,-2a-6n-1;\ 4\end{matrix}\right)=0$$ [[27], (2.8)].

103. $${}_3F_2\left(\begin{matrix}-3n-2,a,-a-3n-\frac{3}{2}\\ 2a,-2a-6n-3;\ 4\end{matrix}\right)=0$$ [[27], (2.7)].

104. $${}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{1}{3},\frac{2}{3}\\ \frac{1}{2},\frac{5}{4};\ 27(17-12\sqrt{2})\end{matrix}\right)=\frac{1+\sqrt{2}}{16\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right).$$

8.1.3. The hypergeometric function ${}_4F_3(a_1, a_2, a_3, a_4;\, b_1, b_2, b_3;\, z)$

1. $${}_4F_3\left(\begin{matrix} a,\, a+\frac{1}{3},\, a+\frac{2}{3},\, b \\ \frac{3a}{2},\, \frac{3a+1}{2},\, b+1;\; -\frac{27z}{4(1-z)^3} \end{matrix}\right) = (1-z)^{3a}\, {}_2F_1\left(\begin{matrix} 1,\, 3a-2b \\ b+1;\; z \end{matrix}\right)$$ $$\left[|z|,\, |27z(1-z)^{-3}/4| < 1;\; [[38], (5.13)]\right].$$

2. $${}_4F_3\left(\begin{matrix} a,\, a+\frac{1}{3},\, a+\frac{2}{3},\, a-n \\ \frac{3a}{2},\, \frac{3a+1}{2},\, a-n+1;\; -\frac{27z}{4(1-z)^3} \end{matrix}\right)$$ $$= -\frac{(3n-1)!\,a}{(-a)_n (a)_{2n}} z^{3n-1} (1-z)^{3a-3n} P_{3n-1}^{(-3n,\,1-a-2n)}\left(\frac{2}{z}-1\right)$$ $$\left[n \geq 1;\; |z|,\, |27z(1-z)^{-3}/4| < 1\right].$$

3. $${}_4F_3\left(\begin{matrix} n+\frac{1}{2},\, n+\frac{5}{6},\, n+\frac{7}{6},\, \frac{1}{2} \\ \frac{6n+3}{4},\, \frac{6n+5}{4},\, \frac{3}{2};\; -\frac{27z}{4(1-z)^3} \end{matrix}\right)$$ $$= -\frac{(3n-1)!}{\left(\frac{3}{2}\right)_{3n-1}\sqrt{z}} (1-z)^{3/2} C_{6n-1}^{1/2-3n}(\sqrt{z}) \quad \left[n \geq 1;\; |z|,\, |27z(1-z)^{-3}/4| < 1\right].$$

4. $${}_4F_3\left(\begin{matrix} n-\frac{1}{2},\, n-\frac{1}{6},\, n+\frac{1}{6},\, -\frac{1}{2} \\ \frac{6n-3}{4},\, \frac{6n-1}{4},\, \frac{1}{2};\; -\frac{27z}{4(1-z)^3} \end{matrix}\right) = \frac{(3n-1)!}{\left(\frac{3}{2}\right)_{3n-1}} (1-z)^{-3/2} C_{6n-2}^{1/2-3n}(\sqrt{z})$$ $$\left[n \geq 1;\; |z|,\, |27z(1-z)^{-3}/4| < 1\right].$$

5. $${}_4F_3\left(\begin{matrix} -n,\, -n,\, -n,\, \frac{1}{2}-n \\ -2n,\, -2n,\, 1;\; z \end{matrix}\right)$$ $$= \frac{(n!)^2}{(2n)!}\left(\frac{z}{4}\right)^n P_n\left(\frac{\sqrt{1-z}-3}{\sqrt{1-z}+1}\right) P_n\left(\frac{\sqrt{1-z}+3}{\sqrt{1-z}-1}\right).$$

6. $${}_4F_3\left(\begin{matrix} -n,\, -n,\, -n,\, \frac{1}{2}-n \\ -2n,\, \frac{1}{2}-2n,\, \frac{1}{2};\; z \end{matrix}\right)$$ $$= \frac{(n!)^2}{\left(\frac{1}{2}\right)_{2n}} \left(\frac{z}{4}\right)^n P_{2n}\left(\sqrt{\frac{\sqrt{1-z}+1}{\sqrt{1-z}-1}}\right) P_{2n}\left(\sqrt{\frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}}\right).$$

7. $${}_4F_3\left(\begin{matrix} -n,\, n+\frac{1}{2},\, \frac{1}{4},\, \frac{3}{4} \\ \frac{1}{2},\, \frac{1}{2},\, 1;\; z \end{matrix}\right)$$ $$= (-1)^n \frac{n!}{\left(\frac{1}{2}\right)_n} P_{2n}\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right) P_{2n}\left(\sqrt{\frac{1+\sqrt{1-z}}{2}}\right).$$

8. $${}_4F_3\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{3}{4},\, \frac{5}{4} \\ 1,\, \frac{3}{2},\, \frac{3}{2};\; z \end{matrix}\right)$$
$$= (-1)^n \frac{n!2}{\left(\frac{3}{2}\right)_n \sqrt{z}} P_{2n+1}\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right) P_{2n+1}\left(\sqrt{\frac{1+\sqrt{1-z}}{2}}\right).$$

9. $${}_4F_3\left(\begin{matrix} -n,\, \frac{1}{4}-n,\, \frac{1}{2}-n,\, \frac{3}{4}-n \\ \frac{1}{2}-2n,\, \frac{1}{2}-2n,\, 1;\; z \end{matrix}\right)$$
$$= \frac{(2n)!}{\left(\frac{1}{2}\right)_{2n}} \left(\frac{z}{16}\right)^n P_{2n}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right) P_{2n}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right).$$

10. $${}_4F_3\left(\begin{matrix} -n,\, -\frac{1}{2}-n,\, -\frac{1}{4}-n,\, \frac{1}{4}-n \\ -\frac{1}{2}-2n,\, -\frac{1}{2}-2n,\, 1;\; z \end{matrix}\right)$$
$$= \frac{n!}{\left(n+\frac{3}{2}\right)_n} \left(\frac{z}{4}\right)^{n+1/2} P_{2n+1}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right) P_{2n+1}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right).$$

11. $${}_4F_3\left(\begin{matrix} -n,\, \frac{1-2n}{4},\, \frac{3-2n}{4},\, \frac{1}{2} \\ \frac{1}{2}-n,\, \frac{1}{2}-n,\, 1;\; z \end{matrix}\right)$$
$$= \frac{n!}{\left(\frac{1}{2}\right)_n} \left(\frac{z}{4}\right)^{n/2} P_n\left(\frac{3+\sqrt{1-z}}{2^{3/2}\sqrt{1+\sqrt{1-z}}}\right) P_n\left(\frac{3-\sqrt{1-z}}{2^{3/2}\sqrt{1-\sqrt{1-z}}}\right).$$

12. $${}_4F_3\left(\begin{matrix} -n,\, \frac{1}{4}-n,\, \frac{1}{2}-n,\, \frac{3}{4}-n \\ -2n,\, \frac{1}{2}-2n,\, \frac{3}{2};\; z \end{matrix}\right)$$
$$= \frac{1}{2n+1} \left(\frac{z}{16}\right)^n U_{2n}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right) U_{2n}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right).$$

13. $${}_4F_3\left(\begin{matrix} -n,\, -\frac{1}{2}-n,\, -\frac{1}{4}-n,\, \frac{1}{4}-n \\ -\frac{1}{2}-2n,\, -1-2n,\, \frac{3}{2};\; z \end{matrix}\right)$$
$$= \frac{1}{2n+2} \left(\frac{z}{16}\right)^{n+1/2} U_{2n+1}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right) U_{2n+1}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right).$$

14. $${}_4F_3\left(\begin{matrix} -n,\, \frac{1}{4}-n,\, \frac{1}{2}-n,\, \frac{3}{4}-n \\ \frac{1}{2}-2n,\, 1-2n,\, \frac{1}{2};\; z \end{matrix}\right)$$
$$= 2\left(\frac{z}{16}\right)^n T_n\left(\frac{3-\sqrt{1-z}}{1+\sqrt{1-z}}\right) T_n\left(\frac{3+\sqrt{1-z}}{1-\sqrt{1-z}}\right) \quad [n \geq 1].$$

15. $${}_4F_3\left(\begin{matrix} -n,\ -\frac{1}{4}-n,\ -\frac{1}{2}-n,\ \frac{1}{4}-n \\ -\frac{1}{2}-2n,\ -2n,\ \frac{1}{2};\ z \end{matrix}\right) = 2^{-4n-1} z^{n+1/2} T_{2n+1}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right) T_{2n+1}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right).$$

16. $${}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ a,\ a+\frac{1}{2} \\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4};\ z \end{matrix}\right) = \frac{n!}{2(1-2a)_n} \times \left[C_{2n}^{2a-n}\left(z^{1/4}\right) + (-4)^n \frac{(2a-n)_{2n}}{(4a)_{2n}} (1+z^{1/2})^n \times C_{2n}^{1/2-2a-n}\left(\frac{z^{1/4}}{\sqrt{1+z^{1/2}}}\right)\right].$$

17. $${}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{n}{2},\ \frac{n+1}{2};\ z \\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4} \end{matrix}\right) = \frac{1}{2}\left[(-1)^n T_{2n}\left(z^{1/4}\right) + T_{2n}\left(\sqrt{1+z^{1/2}}\right)\right].$$

18. $${}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{n}{2}+1,\ \frac{n+3}{2} \\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2};\ z \end{matrix}\right) = \frac{z^{-1/2}}{4(n+1)^2}\left[(-1)^n T_{2n+2}\left(z^{1/4}\right) + T_{2n+2}\left(\sqrt{1+z^{1/2}}\right)\right].$$

19. $${}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{n+1}{2},\ \frac{n}{2}+1 \\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4};\ z \end{matrix}\right) = \frac{1}{2(2n+1)}\left[(-1)^n z^{-1/4} T_{2n+1}\left(z^{1/4}\right) + U_{2n}\left(\sqrt{1+z^{1/2}}\right)\right].$$

20. $${}_4F_3\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ \frac{n+3}{2},\ \frac{n}{2}+2 \\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4};\ z \end{matrix}\right) = \frac{3z^{-3/4}}{4(n+1)(n+2)(2n+3)}\left[(-1)^n T_{2n+3}\left(z^{1/4}\right) + z^{1/4} U_{2n+2}\left(\sqrt{1+z^{1/2}}\right)\right].$$

21. $${}_4F_3\left(\begin{matrix} \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4} \\ \frac{1}{2},\ 1,\ 1;\ z \end{matrix}\right) = \frac{4}{\pi^2}\,\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{k}{2}}\right)\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2k}}\right) \qquad \left[k = 1-2z-2\sqrt{z^2-z}\right].$$

22. $${}_4F_3\left(\begin{matrix} \frac{1}{2},\ \frac{1}{2},\ \frac{5}{6},\ \frac{7}{6} \\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2};\ -\frac{27z}{4(1-z)^3} \end{matrix}\right) = \frac{(1-z)^{3/2}}{2\sqrt{z}} \ln\frac{1+\sqrt{z}}{1-\sqrt{z}} \qquad [-1/2 < z < 1].$$

23. $${}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, 1, 1\\ \frac{3}{4}, \frac{5}{4}, \frac{3}{2};\ -z\end{matrix}\right) = \frac{\left(1+2z-2\sqrt{z^2+z}\right)^{1/2}}{4\left(\sqrt{z^2+z}-z\right)} \times \arcsin\sqrt{2\sqrt{z^2+z}+2z}\,\ln\frac{1+\sqrt{2\sqrt{z^2+z}-2z}}{1-\sqrt{2\sqrt{z^2+z}-2z}}.$$

24. $${}_4F_3\left(\begin{matrix}\frac{3}{4}, \frac{3}{4}, \frac{13}{12}, \frac{17}{12}\\ \frac{9}{8}, \frac{13}{8}, \frac{7}{4};\ -\frac{27z}{4(1-z)^3}\end{matrix}\right) = \frac{3(1-z)^{9/4}}{4z^{3/4}}\left(\ln\frac{1+z^{1/4}}{1-z^{1/4}} - 2\arctan z^{1/4}\right)$$
$$\left[|z|,\ |27z(1-z)^{-3}/4| < 1\right].$$

25. $${}_4F_3\left(\begin{matrix}1, 1, 1, 1\\ \frac{3}{2}, 2, 2;\ z\end{matrix}\right)$$
$$= \frac{i}{3z}\left[2\arcsin^3\sqrt{z} - 6i\arcsin^2\sqrt{z}\,\ln\left(1-e^{-2i\arcsin\sqrt{z}}\right) + 6\arcsin\sqrt{z}\,\mathrm{Li}_2\left(e^{-2i\arcsin\sqrt{z}}\right) - 3i\,\mathrm{Li}_3\left(e^{-2i\arcsin\sqrt{z}}\right) + 3i\zeta(3)\right].$$

26. $${}_4F_3\left(\begin{matrix}1, 1, 1, 1\\ \frac{3}{2}, 2, 2;\ -z\end{matrix}\right) = \frac{1}{3z}\Big\{-2\ln^3\left(\sqrt{z}+\sqrt{z+1}\right) + 6\ln^2\left(\sqrt{z}+\sqrt{z+1}\right)\left[i\pi + \ln\left(2z+2\sqrt{z+z^2}\right)\right] + 6\ln\left(\sqrt{z}+\sqrt{z+1}\right)\mathrm{Li}_2\left(1+2z+2\sqrt{z+z^2}\right) - 3\,\mathrm{Li}_3\left(1+2z+2\sqrt{z+z^2}\right) + 3\zeta(3)\Big\}.$$

27. $${}_4F_3\left(\begin{matrix}1, 1, 1, \frac{3}{2}\\ \frac{5}{4}, \frac{7}{4}, 2;\ -z\end{matrix}\right) = -\frac{3}{z}\arcsin^2\frac{\left(\sqrt{z}-\sqrt{z+1}+1\right)^{1/2}}{\sqrt{2}} + \frac{3}{z}\ln^2\left[\left(\sqrt{z}+\sqrt{z+1}+\sqrt{2}\sqrt{z+\sqrt{z}\sqrt{z+1}}\right)^{1/2}\right].$$

28. $${}_4F_3\left(\begin{matrix}1, 1, \frac{4}{3}, \frac{5}{3}\\ \frac{3}{2}, 2, 2;\ -\frac{27z}{4(1-z)^3}\end{matrix}\right) = -\frac{(1-z)^3}{z}\ln(1-z)$$
$$\left[|z|,\ |27z(1-z)^{-3}/4| < 1\right].$$

29. $$ {}_4F_3\left(\begin{matrix} a,\ a+\frac{1}{2},\ b,\ b+\frac{1}{2};\ 1 \\ a-b+\frac{1}{2},\ a-b+1,\ \frac{1}{2} \end{matrix}\right) = \frac{\Gamma(2a-2b+1)}{2}\left[\frac{2^{-4b}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}-2b\right)}{\Gamma(2a-4b+1)} + \frac{2^{-2a}\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)\Gamma(a-2b+1)}\right] $$

[Re $b < 1/4$; [31], (2.2)].

30. $$ {}_4F_3\left(\begin{matrix} a,\ a+\frac{1}{2},\ b,\ b+\frac{1}{2};\ 1 \\ a-b+1,\ a-b+\frac{3}{2},\ \frac{3}{2} \end{matrix}\right) = \frac{\Gamma(2a-2b+2)}{(2a-1)(2b-1)}\left[\frac{2^{1-4b}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}-2b\right)}{\Gamma(2a-4b+2)} - \frac{2^{-2a}\sqrt{\pi}}{\Gamma(a)\Gamma\left(a-2b+\frac{3}{2}\right)}\right] $$

[Re $b < 1/4$; [31], (2.3)].

31. $$ {}_4F_3\left(\begin{matrix} a,\ b,\ 1,\ 1;\ 1 \\ \frac{a}{2}+1,\ 2b-1,\ 2 \end{matrix}\right) = \frac{a}{2(a-1)} \times\left[2\ln 2 + \mathbf{C} + \psi\left(\frac{a}{2}\right) + \psi\left(b-\frac{1}{2}\right) - \psi\left(b-\frac{a}{2}\right)\right] $$

[Re $(2b-a) > 0$].

32. $$ {}_4F_3\left(\begin{matrix} a,\ b,\ c,\ 2c-2;\ 1 \\ c-1,\ 2c-b-1,\ d \end{matrix}\right) = \frac{\Gamma(d)\Gamma(d-a-2b)}{\Gamma(d-a)\Gamma(d-2b)}\,{}_3F_2\left(\begin{matrix} a,\ -b,\ 2c-2b-2;\ 1 \\ 2c-b-1,\ d-2b \end{matrix}\right). $$

33. $$ {}_4F_3\left(\begin{matrix} a,\ 1,\ 1,\ 1;\ 1 \\ 2a-1,\ \frac{3}{2},\ 2 \end{matrix}\right) = \frac{\pi^2}{8} + \frac{1}{4}\psi'\left(a-\frac{1}{2}\right) $$

[Re $a > 1/2$].

34. $$ {}_4F_3\left(\begin{matrix} a,\ b,\ 1,\ 1;\ 1 \\ \frac{a}{2}+1,\ 2b-1,\ 2 \end{matrix}\right) = \frac{a}{2(a-1)}\left[\psi\left(\frac{a}{2}\right) + \psi\left(b-\frac{1}{2}\right) - \psi\left(b-\frac{a}{2}\right) + 2\ln 2 + \mathbf{C}\right]. $$

35. $$ {}_4F_3\left(\begin{matrix} a,\ 1,\ 1,\ 1;\ 1 \\ 3-a,\ 2,\ 2 \end{matrix}\right) = \frac{a-2}{2(a-1)}\left[\frac{\pi^2}{6} - \psi'(2-a)\right] $$

[$a < 2$].

36. $$ = \zeta(3) $$

[$a = 1$].

37. $$ {}_4F_3\left(\begin{matrix} -n,\ a,\ b,\ b+\frac{1}{2};\ 1 \\ c,\ \frac{a-n}{2},\ \frac{a-n+1}{2} \end{matrix}\right) = \frac{(2b-a+1)_n}{(1-a)_n}\,{}_3F_2\left(\begin{matrix} -n,\ 2b,\ 2b-c+1;\ -1 \\ 2b-a+1,\ c \end{matrix}\right) $$

[[55], (3.30)].

38. $$ {}_4F_3\left(\begin{matrix} -n,\ a,\ b,\ b+\frac{1}{2};\ 1 \\ a-n+\frac{1}{2},\ c,\ 2b-c+1 \end{matrix}\right) = \frac{(c-2b)_n}{(c)_n}\,{}_3F_2\left(\begin{matrix} -2n,\ a-n,\ 2b;\ 1 \\ 2a-2n,\ 2b-c-n+1 \end{matrix}\right) $$

[[55], (3.14)].

39. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, c;\; 1\\ 2a,\, \frac{b+c}{2},\, \frac{b+c+1}{2}\end{matrix}\right) = \frac{(c)_{2n}}{(b+c)_{2n}}\,{}_4F_3\left(\begin{matrix}-n,\, \frac{1}{2}-a-n,\, \frac{b}{2},\, \frac{b+1}{2};\; 1\\ a+\frac{1}{2},\, \frac{1-c}{2}-n,\, 1-\frac{c}{2}-n\end{matrix}\right)$$ [[55], (3.12)].

40. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, c;\; 1\\ a-n+\frac{1}{2},\, \frac{b+c}{2},\, \frac{b+c+1}{2}\end{matrix}\right) = {}_3F_2\left(\begin{matrix}-2n,\, 2a,\, c;\; 1\\ a-n+\frac{1}{2},\, b+c\end{matrix}\right)$$ [[55], (3.12)].

41. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, b+\frac{1}{2};\; 1\\ 1-a-n,\, \frac{1}{2}-b-n,\, 1-b-n\end{matrix}\right) = \frac{(4b)_{2n}}{(2b)_{2n}}\,{}_4F_3\left(\begin{matrix}-n,\, \frac{1}{2}-a-n,\, 2b;\; 1\\ 1-2a-2n,\, 2b+\frac{1}{2}\end{matrix}\right)$$ [[55], (3.12)].

42. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, c;\; 1\\ a+b+\frac{1}{2},\, \frac{c-n}{2},\, \frac{c-n+1}{2}\end{matrix}\right) = \frac{\left(a+b-c+\frac{1}{2}\right)_n}{(1-a)_n}\,{}_3F_2\left(\begin{matrix}-n,\, a-b+\frac{1}{2},\, b-a+\frac{1}{2};\; 1\\ a+b+\frac{1}{2},\, c-a-b-n+\frac{1}{2}\end{matrix}\right)$$ [[55], (3.16)].

43. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, b+\frac{1}{2};\; 1\\ 1-a-n,\, 1-b-n,\, \frac{3}{2}-b-n\end{matrix}\right) = \frac{(4b-1)_{2n}}{(2b-1)_{2n}}\,{}_4F_3\left(\begin{matrix}-n,\, \frac{1}{2}-a-n,\, 2b-1;\; 1\\ 1-2a-2n,\, 2b-\frac{1}{2}\end{matrix}\right)$$ [[55], (3.12)].

44. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, 1-b\\ -b,\, c,\, d;\; 1\end{matrix}\right) = \frac{(d-a)_n}{(d)_n}\,{}_4F_3\left(\begin{matrix}-n,\, a,\, c-b-1,\, \frac{c-b+1}{2}\\ c,\, \frac{c-b-1}{2},\, a-d-n+1;\; 1\end{matrix}\right).$$

45. $${}_4F_3\left(\begin{matrix}-n,\, 1,\, 1,\, 1;\; 1\\ b,\, 2,\, 2\end{matrix}\right) = \frac{1-b}{n+1}\left\{[\psi(n+2)+\mathbf{C}]\,\psi(b-1) - \sum_{k=0}^{n}\frac{1}{k+1}\psi(k+b)\right\}.$$

46. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, b+\frac{1}{2};\; 1\\ \frac{a-n}{2},\, \frac{a-n+1}{2},\, c\end{matrix}\right) = \frac{(2b-a+1)_n}{(1-a)_n}\,{}_3F_2\left(\begin{matrix}-n,\, 2b,\, 2b-c+1;\; -1\\ 2b-a+1,\, c\end{matrix}\right)$$ [[55], (3.30)].

47. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, a+b+\frac{1}{2};\; 1\\ \frac{a-n}{2},\, \frac{a-n+1}{2},\, a+2b+1\end{matrix}\right) = \frac{(a+2b+1)_n}{(1-a)_n}\,{}_4F_3\left(\begin{matrix}-n,\, a,\, a+b+1,\, 2a+2b;\; 1\\ a+b,\, a+2b+1,\, a+2b+1\end{matrix}\right)$$ [[55], (3.20)].

48. $${}_4F_3\left(\begin{matrix}-n,\, a,\, a+\frac{1}{2},\, b;\; 1\\ \frac{b-n+1}{2},\, \frac{b-n}{2}+1,\, c\end{matrix}\right) = \frac{b(2a-b)_n}{(b+n)(-b)_n}\,{}_4F_3\left(\begin{matrix}-n,\, a+\frac{1}{2},\, 2a-1,\, 2a-c;\; -1\\ a-\frac{1}{2},\, 2a-b,\, c\end{matrix}\right)$$ [[55], (3.18)].

49. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, c;\; 1\\ a+b+\frac{1}{2},\, \frac{c-n}{2},\, \frac{c-n+1}{2}\end{matrix}\right) = \frac{\left(a+b-c+\frac{1}{2}\right)_n}{(1-c)_n}\,{}_3F_2\left(\begin{matrix}-n,\, a-b+\frac{1}{2},\, b-a+\frac{1}{2};\; 1\\ a+b+\frac{1}{2},\, c-a-b-n+\frac{1}{2}\end{matrix}\right)$$ [[55], (3.6)].

50. $${}_4F_3\left(\begin{matrix}-n,\, a,\, b,\, c;\; 1\\ a+b+\frac{1}{2},\, \frac{c-n}{2},\, \frac{c-n+1}{2}\end{matrix}\right) = {}_3F_2\left(\begin{matrix}-n,\, 2a,\, 2b;\; 1\\ a+b+\frac{1}{2},\, c-n\end{matrix}\right)$$ [[55], (3.6)].

51. $${}_4F_3\left(\begin{matrix}-n,\, \frac{1}{2}-n,\, a,\, b;\; 1\\ \frac{a+b+1}{2},\, \frac{a+b}{2}+1,\, c\end{matrix}\right) = \frac{a(a+1)_{2n}}{(a-b)(a+b+1)_{2n}}\,{}_4F_3\left(\begin{matrix}-2n-1,\, \frac{1}{2}-n,\, b,\, -c-2n;\; -1\\ -n-\frac{1}{2},\, -a-2n,\, c\end{matrix}\right)$$ [[55], (3.18)].

52. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a,\, b;\; 1\\ \frac{a+b}{2},\, \frac{a+b+1}{2},\, c\end{matrix}\right) = \frac{(b)_n}{(a+b)_n}\,{}_3F_2\left(\begin{matrix}-n,\, a,\, 1-c-n;\; -1\\ c,\, 1-b-n\end{matrix}\right)$$ [[55], (3.30)].

53. $${}_4F_3\left(\begin{matrix}-n,\, \frac{1}{2}-n,\, \frac{1}{6},\, \frac{1}{2};\; 1\\ \frac{1}{3}-n,\, \frac{2}{3}-n,\, \frac{7}{6}\end{matrix}\right) = \frac{(2n)!\left(\frac{3}{2}\right)_{3n}}{(3n)!\left(\frac{3}{2}\right)_{2n}}\,{}_3F_2\left(\begin{matrix}-n,\, \frac{1}{6},\, \frac{1}{2};\; \frac{1}{4}\\ 2n+\frac{3}{2},\, \frac{7}{6}\end{matrix}\right).$$

54. $${}_4F_3\left(\begin{matrix}-n,\, \frac{1}{6},\, \frac{1}{2},\, \frac{2}{3};\; 1\\ \frac{4-3n}{9},\, \frac{7-3n}{9},\, \frac{10-3n}{9}\end{matrix}\right) = \frac{n!}{\left(-\frac{1}{3}\right)_n(6n+1)}\,P_n^{(1/6,\,-n-2/3)}\left(\frac{1}{2}\right).$$

55. $${}_4F_3\left(\begin{matrix}-n,\, \frac{1}{6},\, \frac{1}{2},\, \frac{5}{3};\; 1\\ \frac{7-3n}{9},\, \frac{10-3n}{9},\, \frac{13-3n}{9}\end{matrix}\right) = -\frac{n!}{4\left(-\frac{4}{3}\right)_n(6n-5)(6n+1)} \times\left[20P_n^{(1/6,\,-n-2/3)}\left(\frac{1}{2}\right)+3P_{n-1}^{(7/6,\,-n+1/3)}\left(\frac{1}{2}\right)\right]$$ $[n\geq 1]$.

56. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a,\, b;\; 1\\ a-b+1,\, b-\frac{n}{2},\, b+\frac{1-n}{2}\end{matrix}\right) = \frac{(2a-2b+1)_n}{(1-2b)_n}\,{}_3F_2\left(\begin{matrix}-n,\, a,\, 2a-2b+n+1;\; 1\\ a-b+1,\, 2a-2b+1\end{matrix}\right)$$ [[55], (3.12)].

57. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,a,\,-a;\;1\\ b,\,1-b-n,\,\frac{1}{2}\end{matrix}\right)=\frac{1}{2(b)_n}\left[(a+b)_n+(b-a)_n\right].$$

58. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,a,\,b;\;1\\ a-b+1,\,b+\frac{1-n}{2},\,b-\frac{n}{2}+1\end{matrix}\right)=\frac{(2a-2b)_n}{(-2b)_n}\,{}_3F_2\left(\begin{matrix}-n,\,a,\,2a-2b+n;\;1\\ a-b,\,2a-2b+1\end{matrix}\right)\quad [[55],\,(3.12)].$$

59. $${}_4F_3\left(\begin{matrix}-2n,\,2n+2,\,\frac{1}{2},\,1;\;1\\ 1-2a,\,1+2a,\,\frac{3}{2}\end{matrix}\right)={}_4F_3\left(\begin{matrix}-n,\,n+1,\,\frac{1}{2},\,1;\;1\\ 1-a,\,1+a,\,\frac{3}{2}\end{matrix}\right).$$

60. $${}_4F_3\left(\begin{matrix}-2n,\,2n+2a,\,\frac{1}{2},\,1;\;1\\ a+\frac{1}{2},\,1-2i,\,1+2i\end{matrix}\right)={}_4F_3\left(\begin{matrix}-n,\,n+a,\,\frac{1}{2},\,1;\;1\\ a+\frac{1}{2},\,1-i,\,1+i\end{matrix}\right).$$

61. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,a,\,a+\frac{1}{2};\;1\\ b,\,c,\,c+\frac{1}{2}\end{matrix}\right)=\frac{(2c-2a)_n}{(2c)_n}\,{}_3F_2\left(\begin{matrix}-n,\,2a,\,b-\frac{1}{2};\;2\\ 2b-1,\,2a-2c-n+1\end{matrix}\right)\quad [[55],\,(3.4)].$$

62. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,a,\,a+n;\;1\\ 1-n,\,a+\frac{n+1}{2},\,a+\frac{n}{2}+1\end{matrix}\right)=\frac{2(a+n)_n}{(2a+n+1)_n}.$$

63. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,a,\,b;\;1\\ a+b+\frac{1}{2},\,c,\,1-c-n\end{matrix}\right)=\frac{(a+c)_n}{(c)_n}\,{}_3F_2\left(\begin{matrix}-n,\,2a,\,a+b;\;1\\ 2a+2b,\,a+c\end{matrix}\right)$$

[[55], (3.8)].

64. $${}_4F_3\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,a,\,b;\;1\\ \frac{a+b+1}{2},\,\frac{a+3b}{2},\,1-b-n\end{matrix}\right)=\frac{(b)_n}{(a+b)_n}\,{}_4F_3\left(\begin{matrix}-n,\,a,\,b,\,1-\frac{a+3b}{2}-n;\;1\\ \frac{a+3b}{2},\,1-b-n,\,1-b-n\end{matrix}\right)\quad [[55],\,(3.20)].$$

65. $${}_4F_3\left(\begin{matrix}-\frac{n}{3},\,\frac{1-n}{3},\,\frac{2-n}{3},\,a;\;1\\ \frac{n+1}{3}+a,\,\frac{n+2}{3}+a,\,\frac{n}{3}+a+1\end{matrix}\right)=\frac{(2n)!\,(3a+1)_n}{(3a+1)_{2n}\left(\frac{1}{2}\right)_n}\,P_n^{(a-1/2,\,-n-1/2)}\left(\frac{1}{2}\right).$$

66. $${}_4F_3\left(\begin{matrix} -\frac{n}{3}, \frac{1-n}{3}, \frac{2-n}{3}, a;\ 1 \\ \frac{2+a-n}{3}, \frac{3+a-n}{3}, \frac{4+a-n}{3} \end{matrix}\right) = \frac{(2a-1)_n}{(-a-1)_n}\, {}_2F_1\left(\begin{matrix} -n,\, 2a+n-1;\ \frac{3}{4} \\ a-\frac{1}{2} \end{matrix}\right).$$

67. $$= \frac{(-1)^n n!}{2(-a-1)_n}\left[3C_{n-1}^{a}\left(\frac{1}{2}\right) + \frac{2a+n-2}{a-1} C_n^{a-1}\left(\frac{1}{2}\right)\right] \qquad [n \geq 1].$$

68. $${}_4F_3\left(\begin{matrix} -\frac{n}{3}, \frac{1-n}{3}, \frac{2-n}{3}, \frac{1}{2};\ 1 \\ \frac{3-2n}{6}, \frac{5-2n}{6}, \frac{7-2n}{6} \end{matrix}\right) = \frac{(-1)^n n!}{\left(-\frac{1}{2}\right)_n} P_n\left(\frac{1}{2}\right).$$

69. $${}_4F_3\left(\begin{matrix} -\frac{n}{3}, \frac{1-n}{3}, \frac{2-n}{3}, \frac{1}{2};\ 1 \\ \frac{2n+5}{6}, \frac{2n+7}{6}, \frac{2n+9}{6} \end{matrix}\right) = \frac{12^{n/2} n! \left(\frac{5}{2}\right)_n}{\left(\frac{5}{2}\right)_{2n}} P_n\left(\frac{7}{4\sqrt{3}}\right).$$

70. $${}_4F_3\left(\begin{matrix} -\frac{n}{3}, \frac{1-n}{3}, \frac{2-n}{3}, \frac{3}{2};\ 1 \\ \frac{7-2n}{6}, \frac{9-2n}{6}, \frac{11-2n}{6} \end{matrix}\right) = \frac{(-1)^n n!}{2\left(-\frac{5}{2}\right)_n}\left[3C_{n-1}^{3/2}\left(\frac{1}{2}\right) + 2(n+1)P_n\left(\frac{1}{2}\right)\right] \qquad [n \geq 1].$$

71. $${}_4F_3\left(\begin{matrix} -\frac{n}{3}, \frac{1-n}{3}, \frac{2-n}{3}, -n;\ 1 \\ \frac{1-2n}{3}, \frac{2-2n}{3}, 1-\frac{2n}{3} \end{matrix}\right) = \frac{(-i)^n 2^{1-2n} 3^{n/2} n!}{\left(\frac{1}{2}\right)_n} P_n\left(\frac{i}{\sqrt{3}}\right) \qquad [n \geq 1].$$

72. $${}_4F_3\left(\begin{matrix} -\frac{1+2n}{3}, -\frac{2n}{3}, \frac{1-2n}{3}, -\frac{1}{2}-2n \\ -\frac{1+8n}{6}, \frac{1-8n}{6}, \frac{3-8n}{6};\ 1 \end{matrix}\right) = \frac{3^n \left(\frac{3}{2}\right)_{2n}^2}{(4n+1)\left(\frac{3}{2}\right)_{4n}} U_{2n}\left(\frac{2}{\sqrt{3}}\right).$$

73. $${}_4F_3\left(\begin{matrix} -\frac{1+2n}{3}, -\frac{2n}{3}, \frac{1-2n}{3}, -\frac{3}{2}-2n \\ -\frac{3+8n}{6}, -\frac{1+8n}{6}, \frac{1-8n}{6};\ 1 \end{matrix}\right) = \frac{3^{n-1}(4n+3)\left(\frac{3}{2}\right)_{2n}^2}{(n+1)\left(\frac{5}{2}\right)_{4n}} U_n\left(\frac{5}{3}\right).$$

74. $${}_4F_3\left(\begin{matrix} \frac{1}{2}, 1, 1, 1 \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2};\ 1 \end{matrix}\right) = \frac{7}{2}\zeta(3) - \pi\mathbf{G}.$$

75. $${}_4F_3\left(\begin{matrix} 1, 1, 1, 1 \\ \frac{3}{2}, \frac{3}{2}, 2;\ 1 \end{matrix}\right) = 2\pi\mathbf{G} - \frac{7}{2}\zeta(3).$$

76. $${}_4F_3\left(\begin{matrix} 1, 1, 1, 1 \\ \frac{3}{2}, 2, 2;\ 1 \end{matrix}\right) = \frac{\pi^2}{2}\ln 2 - \frac{7}{4}\zeta(3).$$

77. $${}_4F_3\left(\begin{matrix}1,\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2}\\ 2,\ \frac{5}{2},\ \frac{5}{2};\ 1\end{matrix}\right) = 9\pi\ln 2 - 18.$$

78. $${}_4F_3\left(\begin{matrix}a,\ b,\ c,\ 2c-2;\ -1\\ c-1,\ 2c-a-1,\ 2c-b-1\end{matrix}\right) = \frac{\Gamma(2c-a-1)\,\Gamma(2c-b-1)}{\Gamma(2c-1)\,\Gamma(2c-a-b-1)}$$ $[\operatorname{Re}(c-a-b) > 1/2]$.

79. $${}_4F_3\left(\begin{matrix}a,\ a,\ a,\ \frac{a}{2}+1\\ \frac{a}{2},\ 1,\ 1;\ -1\end{matrix}\right) = \frac{\sin(a\pi)}{a\pi}.$$

80. $${}_4F_3\left(\begin{matrix}a,\ \frac{1}{2},\ \frac{1}{2},\ \frac{5}{4};\ -1\\ \frac{3}{2}-a,\ \frac{1}{4},\ 1\end{matrix}\right) = \frac{2\Gamma\left(\frac{3}{2}-a\right)}{\sqrt{\pi}\,\Gamma(1-a)}$$ $[\operatorname{Re} a < 3/4]$.

81. $${}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{5}{4}\\ \frac{1}{4},\ 1,\ 1;\ -1\end{matrix}\right) = \frac{2}{\pi}.$$

82. $${}_4F_3\left(\begin{matrix}\frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{6}{5}\\ \frac{1}{5},\ 1,\ 1;\ -\frac{9}{16}\end{matrix}\right) = \frac{4}{\sqrt{3}\,\pi}.$$

83. $${}_4F_3\left(\begin{matrix}a,\ a+\frac{1}{2},\ \frac{6-a}{5},\ \frac{1}{2}\\ \frac{1-a}{5},\ \frac{3}{2}-2a,\ 1;\ -\frac{1}{4}\end{matrix}\right) = \frac{2\Gamma\left(\frac{3}{2}-2a\right)}{\sqrt{\pi}\,(1-a)\Gamma(1-2a)}$$ [[45], (3.1)].

84. $${}_4F_3\left(\begin{matrix}\frac{1}{8},\ \frac{1}{2},\ \frac{5}{8},\ \frac{47}{40}\\ \frac{7}{40},\ 1,\ \frac{5}{4};\ -\frac{1}{4}\end{matrix}\right) = \frac{2\sqrt{2}}{7\pi^{3/2}}\Gamma^2\left(\frac{1}{4}\right).$$

85. $${}_4F_3\left(\begin{matrix}\frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ \frac{23}{20}\\ \frac{3}{20},\ 1,\ 1;\ -\frac{1}{4}\end{matrix}\right) = \frac{8}{3\pi}$$ [[46], (5, 6)].

86. $${}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ 1\\ \frac{3}{2},\ 2,\ 2;\ -\frac{1}{4}\end{matrix}\right) = \frac{4}{5}\zeta(3).$$

87. $${}_4F_3\left(\begin{matrix}a+1,\ 3a,\ 3a,\ 1-3a\\ a,\ 3a+\frac{1}{2},\ 1;\ -\frac{1}{8}\end{matrix}\right) = \frac{2^{3a}\Gamma\left(3a+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(3a+1)}$$ [[45], (2.3)].

88. $${}_4F_3\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{7}{6}\\ \frac{1}{6},\ 1,\ 1;\ -\frac{1}{8}\end{matrix}\right) = \frac{2\sqrt{2}}{\pi}$$ [[43], (17)].

89. $${}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ 1\\ \frac{3}{2},\ 2,\ 2;\ -\frac{1}{8}\end{matrix}\right) = \zeta(3) - \frac{2}{3}\ln^3 2.$$

90. $${}_4F_3\left(\begin{matrix}\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{58}{51}\\ \frac{7}{51},1,1;\ -\frac{1}{16}\end{matrix}\right)=\frac{12\sqrt{3}}{7\pi}$$ [[43], (9)].

91. $${}_4F_3\left(\begin{matrix}\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{10}{9}\\ \frac{1}{9},1,1;\ -\frac{1}{80}\end{matrix}\right)=\frac{4}{\pi}\sqrt{\frac{3}{5}}$$ [[43], (10)].

92. $${}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{31}{28}\\ \frac{3}{28},1,1;\ -\frac{1}{48}\end{matrix}\right)=\frac{16}{3\sqrt{3}\,\pi}$$ [[43], (2)].

93. $${}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{283}{260}\\ \frac{23}{260},1,1;\ -\frac{1}{324}\end{matrix}\right)=\frac{72}{23\pi}$$ [[43], (3)].

94. $${}_4F_3\left(\begin{matrix}\frac{1}{6},\frac{1}{2},\frac{5}{6},\frac{5681}{5418}\\ \frac{263}{5418},1,1;\ -\frac{1}{512000}\end{matrix}\right)=\frac{640}{263\pi}\sqrt{\frac{5}{3}}$$ [[43], (6)].

95. $${}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{299}{280}\\ \frac{19}{280},1,1;\ \frac{1}{9801}\end{matrix}\right)=\frac{18\sqrt{11}}{19\pi}$$ [[43], (25)].

96. $${}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{43}{40}\\ \frac{3}{40},1,1;\ \frac{1}{2401}\end{matrix}\right)=\frac{49}{9\sqrt{3}\,\pi}$$ [[43], (24)].

97. $${}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{11}{10}\\ \frac{1}{10},1,1;\ \frac{1}{81}\end{matrix}\right)=\frac{9}{2\sqrt{2}\,\pi}$$ [[43], (23)].

98. $${}_4F_3\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{47}{42}\\ \frac{5}{42},1,1;\ \frac{1}{64}\end{matrix}\right)=\frac{16}{5\pi}$$ [[43], (21)].

99. $${}_4F_3\left(\begin{matrix}a,\ a+\frac{1}{2},\ 1-2a,\ \frac{5-2a}{4}\\ \frac{1-2a}{4},\ \frac{3}{2}-2a,\ 1;\ \frac{1}{9}\end{matrix}\right)=\frac{3^{2a}\Gamma\left(\frac{3}{2}-2a\right)}{2^{4a-1}\sqrt{\pi}\,\Gamma(2-2a)}$$ [[45], (3.2)].

100. $${}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{9}{8}\\ \frac{1}{8},1,1;\ \frac{1}{9}\end{matrix}\right)=\frac{2\sqrt{3}}{\pi}$$ [[43], (22)].

101. $${}_4F_3\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\ a+1\\ a,\ \frac{6a+3}{4},\ \frac{6a+3}{4};\ \frac{1}{4}\end{matrix}\right)=\frac{2\Gamma^2\left(\frac{6a+3}{4}\right)}{3a\,\Gamma^2\left(\frac{6a+1}{4}\right)}$$ [[45], (2.1)].

102. $${}_4F_3\left(\begin{matrix}a+1,\ 3a,\ 1-3a,\ \frac{1}{2}\\ a,\ 3a+\frac{1}{2},\ 1;\ \frac{1}{4}\end{matrix}\right)=\frac{2\Gamma\left(3a+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(3a+1)}$$ [[45], (2.2)].

103. $${}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; \frac{1}{4}\end{matrix}\right) = \frac{7\pi^3}{216}.$$

104. $${}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{6} \\ \frac{1}{6}, 1, 1; \frac{1}{4}\end{matrix}\right) = \frac{4}{\pi}$$ [[43], (20)].

105. $${}_4F_3\left(\begin{matrix}1, 1, 1, 1 \\ \frac{3}{2}, 2, 2; \frac{1}{4}\end{matrix}\right)$$
$$= -\frac{8}{3}\zeta(3) + \frac{\pi}{12\sqrt{3}}\left[\zeta\left(2, \frac{1}{6}\right) - \zeta\left(2, \frac{1}{3}\right) + \zeta\left(2, \frac{2}{3}\right) - \zeta\left(2, \frac{5}{6}\right)\right].$$

106. $${}_4F_3\left(\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{8}{7} \\ \frac{1}{7}, 1, 1; \frac{32}{81}\end{matrix}\right) = \frac{9}{2\pi}$$ [[43], (29)].

107. $${}_4F_3\left(\begin{matrix}1, 1, 1, 1 \\ \frac{3}{2}, 2, 2; \frac{1}{2}\end{matrix}\right) = \frac{\pi^2}{8}\ln 2 - \frac{35}{16}\zeta(3) + \pi\mathbf{G}.$$

108. $${}_4F_3\left(\begin{matrix}1, 1, 1, 1 \\ \frac{3}{2}, 2, 2; \frac{3}{4}\end{matrix}\right)$$
$$= \frac{4\pi^2}{27}\ln 3 - \frac{52}{27}\zeta(3) + \frac{\pi}{27\sqrt{3}}\left[\zeta\left(2, \frac{1}{6}\right) - \zeta\left(2, \frac{1}{3}\right) + \zeta\left(2, \frac{2}{3}\right) - \zeta\left(2, \frac{5}{6}\right)\right].$$

109. $${}_4F_3\left(\begin{matrix}-n, -n, -\frac{2n}{3}+1, \frac{1}{2} \\ -\frac{2n}{3}, 1, 1; 4\end{matrix}\right) = 0$$ $[n \neq 3n]$.

110. $${}_4F_3\left(\begin{matrix}\frac{1}{6}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9} \\ \frac{1}{3}, \frac{5}{6}, \frac{7}{6}; 108(56\sqrt{3}-97)\end{matrix}\right) = \frac{(2+\sqrt{3})^{2/3}}{2}\sqrt{\frac{3}{\pi}}\,\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{7}{6}\right).$$

8.1.4. The hypergeometric function ${}_5F_4((a_1, \ldots, a_5); (b_1, \ldots, b_4); z)$

1. $${}_5F_4\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, 1, 1, 1 \\ \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}; -z\end{matrix}\right) = \frac{2}{\sqrt{z}}\arcsin\frac{\left(\sqrt{z}-\sqrt{z+1}+1\right)^{1/2}}{\sqrt{2}}$$
$$\times \ln\left[\left(\sqrt{z}+\sqrt{z+1}+\sqrt{2}\sqrt{z+\sqrt{z}\sqrt{z+1}}\right)^{1/2}\right].$$

2. $${}_5F_4\left(\begin{matrix} a,\, b,\, c,\, 2c-2,\, -\frac{1}{2};\ 1 \\ c-1,\, 2c-\frac{1}{2},\, 2c-a-1,\, 2c-b-1 \end{matrix}\right) = \frac{\Gamma\left(2c-\frac{1}{2}\right)\Gamma(2c-a-1)\Gamma(2c-b-1)\Gamma\left(2c-a-b-\frac{1}{2}\right)}{\Gamma(2c-1)\,\Gamma\left(2c-a-\frac{1}{2}\right)\Gamma\left(2c-b-\frac{1}{2}\right)\Gamma(2c-a-b-1)}$$
$[\mathrm{Re}\,(2c-a-b) > 1/2].$

3. $${}_5F_4\left(\begin{matrix} a,\, b,\, b+1,\, a-b-1,\, 2a-2;\ 1 \\ a-1,\, a+b,\, 2a-b-2,\, 2a-b-1 \end{matrix}\right) = \frac{4^{b-a+1}\sqrt{\pi}\,\Gamma(a+b)\,\Gamma(2a-b-2)\Gamma(2a-b-1)}{\Gamma^2(a)\,\Gamma\left(a-b-\frac{1}{2}\right)\Gamma(2a-2)} \quad [\mathrm{Re}\,(a-b) > 1].$$

4. $${}_5F_4\left(\begin{matrix} a,\, a,\, b,\, b,\, 2a+2b-1;\ 1 \\ a+2b,\, a+2b,\, 2a+b,\, 2a+b \end{matrix}\right) = \frac{\Gamma^2(a+2b)\Gamma^2(2a+b)}{\Gamma^2(a+b)\Gamma^2(2a+2b)}\, {}_3F_2\left(\begin{matrix} 2a,\, 2b,\, a+b;\ 1 \\ 2a+2b,\, 2a+2b \end{matrix}\right) \quad [\mathrm{Re}\,(a+b) > 0].$$

5. $${}_5F_4\left(\begin{matrix} a,\, b,\, c,\, 2c-2,\, \frac{1}{2};\ 1 \\ c-1,\, 2c-\frac{3}{2},\, 2c-a-1,\, 2c-b-1 \end{matrix}\right) = \frac{\Gamma\left(2c-\frac{3}{2}\right)\Gamma(2c-a-1)\Gamma(2c-b-1)\Gamma\left(2c-a-b-\frac{3}{2}\right)}{\Gamma(2c-1)\,\Gamma\left(2c-a-\frac{3}{2}\right)\Gamma\left(2c-b-\frac{3}{2}\right)\Gamma(2c-a-b-1)}$$
$[\mathrm{Re}\,(2c-a-b) > 3/2].$

6. $${}_5F_4\left(\begin{matrix} -n,\, a,\, b,\, c,\, c+\frac{1}{2};\ 1 \\ \frac{a-n}{2},\, \frac{a-n+1}{2},\, a+b,\, 2c-a+1 \end{matrix}\right) = \frac{(2c-a-1)_n}{(1-a)_n}\, {}_4F_3\left(\begin{matrix} -n,\, a,\, 2c,\, 2c-a-b+1;\ 1 \\ a+b,\, 2c-a+1,\, 2c-a+1 \end{matrix}\right) \quad [[55],\ (3.20)].$$

7. $${}_5F_4\left(\begin{matrix} -n,\, a,\, b,\, a+b+\frac{1}{2};\ 1 \\ \frac{a-n}{2},\, \frac{a-n+1}{2},\, a+2b+1 \end{matrix}\right) = \frac{(a+2b+1)_n}{(1-a)_n}\, {}_4F_3\left(\begin{matrix} -n,\, a,\, a+b+1,\, 2a+2b;\ 1 \\ a+b,\, a+2b+1,\, a+2b+1 \end{matrix}\right) \quad [[55],\ (3.20)].$$

8. $${}_5F_4\left(\begin{matrix} -n,\, a,\, a+\frac{1}{2},\, \frac{1}{6},\, \frac{1}{2};\ 1 \\ \frac{7}{6},\, \frac{2a-n}{3},\, \frac{2a-n+1}{3},\, \frac{2a-n+2}{3} \end{matrix}\right) = \frac{\left(\frac{3}{2}-2a\right)_n}{(1-2a)_n}\, {}_3F_2\left(\begin{matrix} -n,\, \frac{1}{6},\, \frac{1}{2};\ \frac{1}{4} \\ \frac{7}{6},\, \frac{3}{2}-2a \end{matrix}\right).$$

9. $${}_5F_4\left(\begin{matrix}-n,\, a,\, a+\frac{1}{2},\, b,\, b+\frac{1}{3};\; 1\\ \frac{2a-n}{3},\, \frac{2a-n+1}{3},\, \frac{2a-n+2}{3},\, 2b+\frac{5}{6}\end{matrix}\right) = \frac{(3b-2a+1)_n}{(1-2a)_n}\,{}_3F_2\left(\begin{matrix}-n,\, \frac{1}{3}-b,\, 3b;\; \frac{1}{4}\\ 2b+\frac{5}{6},\, 3b-a+1\end{matrix}\right) \quad [[55], (3.26)].$$

10. $${}_5F_4\left(\begin{matrix}-n,\, a,\, b,\, b+\frac{1}{3},\, b+\frac{2}{3};\; 1\\ \frac{2a-n-1}{3},\, \frac{2a-n}{3},\, \frac{2a-n+1}{3},\, 3b-a+2\end{matrix}\right) = \frac{(3b-2a+2)_n}{(2-2a)_n}\,{}_4F_3\left(\begin{matrix}-n,\, 3b,\, 3b-2a+3,\, 2a-3b-2;\; \frac{1}{4}\\ a-\frac{1}{2},\, 3b-a+2,\, c,\, 3b-2a+2\end{matrix}\right) \quad [[55], (3.26)].$$

11. $${}_5F_4\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a,\, b,\, c;\; 1\\ 1-a-n,\, a+b,\, \frac{a+c}{2},\, \frac{a+c+1}{2}\end{matrix}\right) = \frac{(a)_n}{(a+c)_n}\,{}_4F_3\left(\begin{matrix}-n,\, a,\, 1-a-b,\, c;\; 1\\ 1-a-n,\, 1-a-n,\, a+b\end{matrix}\right) \quad [[55], (3.20)].$$

12. $${}_5F_4\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a,\, b+\frac{1}{3},\, b+\frac{2}{3};\; 1\\ \frac{a-n}{3},\, \frac{a-n+1}{3},\, \frac{a-n+2}{3},\, 2b+\frac{3}{2}\end{matrix}\right) = \frac{(3b-a+1)_n}{(1-a)_n}\,{}_4F_3\left(\begin{matrix}-n,\, b-\frac{1}{2},\, 2b+1,\, 3b;\; 4\\ 2b-1,\, 3b-a+1,\, 4b+2\end{matrix}\right) \quad [[55], (3.28)].$$

13. $${}_5F_4\left(\begin{matrix}-\frac{n}{2},\, \frac{1-n}{2},\, a,\, b,\, b+\frac{1}{3};\; 1\\ \frac{a-n}{3},\, \frac{a-n+1}{3},\, \frac{a-n+2}{3},\, 2b+\frac{5}{6}\end{matrix}\right) = \frac{(3b-a)_n}{(1-a)_n}\,{}_4F_3\left(\begin{matrix}-n,\, b-\frac{5}{6},\, 2b+\frac{1}{3},\, 3b-1;\; 4\\ 2b-\frac{5}{3},\, 3b-a,\, 4b+\frac{2}{3}\end{matrix}\right) \quad [[55], (3.28)].$$

14. $${}_5F_4\left(\begin{matrix}-2n,\, -2n,\, -2n,\, -2n,\, 3n+1\\ -5n,\, 1,\, 1,\, 1;\; 1\end{matrix}\right) = (-1)^n\,\frac{[(3n)!]^3\,(4n)!}{[n!]^4[(2n)!]^2\,(5n)!}.$$

15. $${}_5F_4\left(\begin{matrix}a,\, b,\, c,\, d,\, \frac{d}{2}+1;\; -1\\ \frac{d}{2},\, d-b+1,\, d-c+1,\, d-a+1\end{matrix}\right) = \frac{\Gamma(d-c+1)\Gamma(d-a+1)}{\Gamma(d+1)\Gamma(d-c-a+1)} \times {}_3F_2\left(\begin{matrix}a,\, c,\, \frac{d+1}{2}-b;\; 1\\ \frac{d+1}{2},\, d-b+1\end{matrix}\right) \quad [\operatorname{Re}(d-a-c) > -1].$$

16. $${}_5F_4\left(\begin{matrix}\frac{5}{6},\, 1,\, 1,\, \frac{7}{6},\, \frac{17}{10}\\ \frac{7}{10},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{9}{16}\end{matrix}\right) = \frac{8}{21}\ln\frac{27}{4} \quad [[43], (30)].$$

17. $${}_5F_4\left(\begin{matrix} a+1,\, a+\frac{1}{10},\, a+\frac{7}{20},\, a+\frac{3}{5},\, 1 \\ a,\, a+\frac{17}{20},\, a+\frac{17}{20},\, a+\frac{17}{20};\; -\frac{1}{4}\end{matrix}\right) = \frac{20a-3}{25a} \times {}_3F_2\left(\begin{matrix} a+\frac{7}{20},\, \frac{1}{2},\, 1;\; 1 \\ a+\frac{17}{20},\, 2a+\frac{7}{10}\end{matrix}\right)$$ [Re $a > 3/20$; [46], (7, 8)].

18. $${}_5F_4\left(\begin{matrix} \frac{1}{2},\, \frac{3}{4},\, 1,\, 1,\, \frac{7}{5} \\ \frac{2}{5},\, \frac{5}{4},\, \frac{5}{4},\, \frac{5}{4};\; -\frac{1}{4}\end{matrix}\right) = \frac{1}{64\pi}\Gamma^4\left(\frac{1}{4}\right)$$ [[43]].

19. $${}_5F_4\left(\begin{matrix} \frac{3}{4},\, 1,\, 1,\, \frac{5}{4},\, \frac{33}{20} \\ \frac{13}{20},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{4}\end{matrix}\right) = \frac{16}{13}\ln 2$$ [[43]].

20. $${}_5F_4\left(\begin{matrix} a+1,\, a+\frac{1}{3},\, a+\frac{1}{3},\, a+\frac{1}{3},\, 1 \\ a,\, a+\frac{5}{6},\, a+\frac{5}{6},\, a+\frac{5}{6};\; -\frac{1}{8}\end{matrix}\right) = \frac{6a-1}{9a} \times {}_3F_2\left(\begin{matrix} \frac{3a+1}{6},\, \frac{3a+4}{6},\, 1 \\ a+\frac{5}{6},\, a+\frac{5}{6},\, a+\frac{5}{6};\; 1\end{matrix}\right)$$ [Re $a > 1/6$; [46], (5, 6)].

21. $${}_5F_4\left(\begin{matrix} 1,\, 1,\, 1,\, 1,\, \frac{5}{3} \\ \frac{2}{3},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{8}\end{matrix}\right) = \mathbf{G}.$$

22. $${}_5F_4\left(\begin{matrix} a+1,\, a+\frac{16}{231},\, a+\frac{31}{77},\, a+\frac{170}{231},\, 1 \\ a,\, a+\frac{139}{154},\, a+\frac{139}{154},\, a+\frac{139}{154};\; -\frac{27}{512}\end{matrix}\right) = \frac{32(154a-15)}{5929a} \times {}_3F_2\left(\begin{matrix} \frac{a}{2}+\frac{31}{154},\, \frac{a}{2}+\frac{54}{77},\, 1;\; 1 \\ a+\frac{139}{154},\, 2a+\frac{62}{77}\end{matrix}\right)$$ [Re $a > 15/154$; [46], (7, 8)].

23. $${}_5F_4\left(\begin{matrix} \frac{1}{6},\, \frac{1}{2},\, \frac{5}{6},\, \frac{169}{154} \\ \frac{15}{154},\, 1,\, 1;\; -\frac{27}{512}\end{matrix}\right) = \frac{32\sqrt{2}}{15\pi}$$ [[46], (9, 10)].

24. $${}_5F_4\left(\begin{matrix} \frac{2}{3},\, 1,\, 1,\, \frac{4}{3},\, \frac{123}{77} \\ \frac{46}{77},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{27}{512}\end{matrix}\right) = \frac{128}{92}\ln 2$$ [[46], (9, 10)].

25. $${}_5F_4\left(\begin{matrix} \frac{5}{6},\, 1,\, 1,\, \frac{7}{6},\, \frac{167}{102} \\ \frac{65}{102},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{16}\end{matrix}\right) = \frac{216}{65}\ln\frac{4}{3}$$ [[43], (9)].

26. $${}_5F_4\left(\begin{matrix} a+1,\, a+\frac{1}{7},\, a+\frac{11}{28},\, a+\frac{9}{14},\, 1 \\ a,\, a+\frac{25}{28},\, a+\frac{25}{28},\, a+\frac{25}{28};\; -\frac{1}{48}\end{matrix}\right)$$
$$= 2^{6a+19/14}3^{a-17/28}\sec\left(\frac{28a-3}{28}\pi\right)\frac{\Gamma^3\left(a+\frac{25}{28}\right)}{7a\,\Gamma\left(a+\frac{11}{28}\right)\Gamma\left(2a+\frac{2}{7}\right)}$$
$$+\frac{3(28a-3)^2}{49a(28a-17)}\,{}_3F_2\left(\begin{matrix} a+\frac{11}{28},\, 2a+\frac{2}{7},\, 1 \\ \frac{45}{28}-a,\, 2a+\frac{11}{14};\; \frac{3}{4}\end{matrix}\right) \quad [[46], (12)].$$

27. $${}_5F_4\left(\begin{matrix} \frac{3}{4},\, 1,\, 1,\, \frac{5}{4},\, \frac{45}{28} \\ \frac{17}{28},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{48}\end{matrix}\right) = \frac{32}{17}\ln\frac{27}{16} \quad [[43], (2)].$$

28. $${}_5F_4\left(\begin{matrix} \frac{1}{2},\, \frac{3}{4},\, 1,\, 1,\, \frac{19}{14} \\ \frac{5}{14},\, \frac{5}{4},\, \frac{5}{4},\, \frac{5}{4};\; -\frac{1}{48}\end{matrix}\right) = \frac{3^{-5/4}}{10\sqrt{2}\,\pi}\Gamma^4\left(\frac{1}{4}\right).$$

29. $${}_5F_4\left(\begin{matrix} \frac{5}{6},\, 1,\, 1,\, \frac{7}{6},\, \frac{29}{18} \\ \frac{11}{18},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{80}\end{matrix}\right) = \frac{24}{11}\ln\frac{3^9}{2^2 5^5} \quad [[43], (10)].$$

30. $${}_5F_4\left(\begin{matrix} \frac{3}{4},\, 1,\, 1,\, \frac{5}{4},\, \frac{413}{260} \\ \frac{153}{260},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{324}\end{matrix}\right) = \frac{144}{17}\ln\frac{9}{8} \quad [[43], (3)].$$

31. $${}_5F_4\left(\begin{matrix} \frac{3}{4},\, 1,\, 1,\, \frac{5}{4},\, \frac{1007}{644} \\ \frac{363}{644},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; -\frac{1}{25920}\end{matrix}\right) = \frac{3456}{121}\ln\frac{2^{18}}{3^4 5^5} \quad [[43], (40)].$$

32. $${}_5F_4\left(\begin{matrix} \frac{3}{4},\, 1,\, 1,\, \frac{5}{4},\, \frac{439}{280} \\ \frac{159}{280},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; \frac{1}{9801}\end{matrix}\right) = \frac{594\sqrt{11}}{53}\left(-\frac{\pi}{2}+4\arcsin\frac{7}{18}\right) \quad [[43], (25)].$$

33. $${}_5F_4\left(\begin{matrix} \frac{3}{4},\, 1,\, 1,\, \frac{5}{4},\, \frac{63}{40} \\ \frac{23}{40},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; \frac{1}{2401}\end{matrix}\right) = \frac{2401}{69\sqrt{3}}\left(-\frac{\pi}{6}+4\arcsin\frac{1}{7}\right) \quad [[43], (24)].$$

34. $${}_5F_4\left(\begin{matrix} \frac{3}{4},\, 1,\, 1,\, \frac{5}{4},\, \frac{8}{5} \\ \frac{3}{5},\, \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2};\; \frac{1}{81}\end{matrix}\right) = \frac{27}{4\sqrt{2}}\left(\frac{\pi}{2}-4\arcsin\frac{1}{3}\right) \quad [[43], (23)].$$

35. $${}_5F_4\left(\begin{matrix} a+1,\, a+\frac{8}{21},\, a+\frac{8}{21},\, a+\frac{8}{21},\, 1 \\ a,\, a+\frac{37}{42},\, a+\frac{37}{42},\, a+\frac{37}{42};\; \frac{1}{64}\end{matrix}\right) = \frac{8(42a-5)}{441a}$$
$$\times\,{}_3F_2\left(\begin{matrix} a+\frac{8}{21},\, a+\frac{8}{21},\, 1;\; 1 \\ 2a+\frac{16}{21},\, 2a+\frac{16}{21}\end{matrix}\right) \quad [\operatorname{Re} a > 5/42;\ [46], (3, 4)].$$

36. $${}_5F_4\left(\begin{matrix}1,\,1,\,1,\,1,\,\frac{34}{21}\\ \frac{13}{21},\,\frac{3}{2},\,\frac{3}{2},\,\frac{3}{2};\;\frac{1}{64}\end{matrix}\right)=\frac{4\pi^2}{39}\qquad [[43],\,(21)].$$

37. $${}_5F_4\left(\begin{matrix}\frac{3}{4},\,1,\,1,\,\frac{5}{4},\,\frac{13}{8}\\ \frac{5}{8},\,\frac{3}{2},\,\frac{3}{2},\,\frac{3}{2};\;\frac{1}{9}\end{matrix}\right)=\frac{\sqrt{3}\,\pi}{5}\qquad [[46],\,(11)].$$

38. $${}_5F_4\left(\begin{matrix}a+1,\,a+\frac{1}{8},\,a+\frac{3}{8},\,a+\frac{5}{8},\,1\\ a,\,a+\frac{7}{8},\,a+\frac{7}{8},\,a+\frac{7}{8};\;\frac{1}{9}\end{matrix}\right)$$
$$=2^{8a-5}3^{2a+1/4}\csc\left(\frac{8a+1}{4}\pi\right)\frac{\Gamma\left(a-\frac{1}{8}\right)\Gamma^3\left(a+\frac{7}{8}\right)}{a\pi\Gamma\left(4a-\frac{1}{2}\right)}$$
$$+\frac{9(8a-1)^2}{64a(8a-3)}\,{}_3F_2\left(\begin{matrix}a+\frac{3}{8},\,\frac{1}{2},\,1;\;\frac{3}{4}\\ a+\frac{7}{8},\,\frac{7}{4}-2a\end{matrix}\right)\qquad [[46],\,(11)].$$

39. $${}_5F_4\left(\begin{matrix}a+1,\,a+\frac{1}{3},\,a+\frac{1}{3},\,a+\frac{1}{3},\,1\\ a,\,a+\frac{5}{6},\,a+\frac{5}{6},\,a+\frac{5}{6};\;\frac{1}{4}\end{matrix}\right)=\frac{2(6a-1)}{9a}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,1;\;1\\ a+\frac{5}{6},\,a+\frac{5}{6}\end{matrix}\right)$$
$$[\operatorname{Re}a>1/6;\ [46],\,(1,\,2)].$$

40. $${}_5F_4\left(\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2}\\ \frac{3}{2},\,\frac{3}{2},\,\frac{3}{2},\,\frac{3}{2};\;\frac{1}{4}\end{matrix}\right)$$
$$=\frac{\pi}{12}\zeta(3)+\frac{1}{1024\sqrt{3}}\left[\zeta\left(4,\,\frac{1}{6}\right)-\zeta\left(4,\,\frac{1}{3}\right)+\zeta\left(4,\,\frac{2}{3}\right)-\zeta\left(4,\,\frac{5}{6}\right)\right].$$

41. $${}_5F_4\left(\begin{matrix}1,\,1,\,1,\,1,\,1\\ \frac{3}{2},\,2,\,2,\,2;\;\frac{1}{4}\end{matrix}\right)=\frac{17\pi^4}{1620}.$$

42. $${}_5F_4\left(\begin{matrix}1,\,1,\,1,\,1,\,\frac{5}{3}\\ \frac{2}{3},\,\frac{3}{2},\,\frac{3}{2},\,\frac{3}{2};\;\frac{1}{4}\end{matrix}\right)=\frac{\pi^2}{8}\qquad [[43]].$$

43. $${}_5F_4\left(\begin{matrix}\frac{3}{4},\,1,\,1,\,\frac{5}{4},\,\frac{23}{14}\\ \frac{9}{14},\,\frac{3}{2},\,\frac{3}{2},\,\frac{3}{2};\;\frac{32}{81}\end{matrix}\right)=\frac{9}{4\sqrt{2}}\left(\frac{\pi}{2}-2\arcsin\frac{1}{3}\right)\qquad [[43],\,(29)].$$

44. $${}_5F_4\left(\begin{matrix}-n,\,a,\,a+\frac{1}{2},\,\frac{a}{2}+1,\,\frac{2a+1}{3}+n\\ -3n,\,\frac{a}{2},\,2a+3n+1,\,\frac{1}{2};\;9\end{matrix}\right)=\frac{\left(\frac{2a+2}{3}\right)_n\left(\frac{2a}{3}+1\right)_n}{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}$$
$$[[38],\,(6.3)].$$

8.1.5. The hypergeometric function ${}_6F_5(a_1, \ldots, a_6;\, b_1, \ldots, b_5;\, z)$

1. $${}_6F_5\left(\begin{matrix} a,\, a+\frac{1}{2},\, b,\, b+\frac{1}{2},\, \frac{a+b}{2},\, \frac{a+b+1}{2};\; z \\ \frac{1}{2},\, a+b,\, a+b+\frac{1}{2},\, \frac{2a+2b+1}{4},\, \frac{2a+2b+3}{4} \end{matrix}\right) = {}_4F_3^2\left(\begin{matrix} \frac{a}{2},\, \frac{a+1}{2},\, \frac{b}{2},\, \frac{b+1}{2};\; z \\ \frac{2a+2b+1}{4},\, \frac{2a+2b+3}{4},\, \frac{1}{2} \end{matrix}\right) + \frac{4a^2b^2z}{(2a+2b+1)^2} \times {}_4F_3^2\left(\begin{matrix} \frac{a+1}{2},\, \frac{a}{2}+1,\, \frac{b+1}{2},\, \frac{b}{2}+1;\; z \\ \frac{2a+2b+3}{4},\, \frac{2a+2b+5}{4},\, \frac{3}{2} \end{matrix}\right) \quad [42].$$

2. $${}_6F_5\left(\begin{matrix} -\frac{n}{3},\, \frac{1-n}{3},\, \frac{2-n}{3},\, a,\, b,\, b+\frac{1}{2};\; 1 \\ \frac{a+2b}{3},\, \frac{a+2b+1}{3},\, \frac{a+2b+2}{3},\, c,\, \frac{3}{2}-c-n \end{matrix}\right) = \frac{(2b)_n}{(a+2b)_n}\, {}_4F_3\left(\begin{matrix} -n,\, a,\, 2c+n-1,\, 2-2c-n;\; \frac{1}{4} \\ 1-2b-n,\, c,\, \frac{3}{2}-c-n \end{matrix}\right) \quad [[55], (3.26)].$$

3. $${}_6F_5\left(\begin{matrix} a,\, b,\, c,\, \frac{c+3}{2},\, 1,\, 1;\; 1 \\ \frac{c+1}{2},\, c+1,\, c-a+2,\, c-b+2,\, 2 \end{matrix}\right) = \frac{c(c-a+1)(c-b+1)}{(a-1)(b-1)(c+1)} \times [-\psi(c) + \psi(c-a+1) + \psi(c-b+1) - \psi(c-a-b+2)].$$

4. $${}_6F_5\left(\begin{matrix} -n,\, a,\, a+\frac{1}{2},\, b,\, b+\frac{1}{3},\, b+\frac{2}{3};\; 1 \\ \frac{2a-n}{3},\, \frac{2a-n+1}{3},\, \frac{2a-n+2}{3},\, c,\, 3b-c+\frac{3}{2} \end{matrix}\right) = \frac{(3b-2a+1)_n}{(1-2a)_n}\, {}_4F_3\left(\begin{matrix} -n,\, 3b,\, 3b-2c+2,\, 2c-3b-1;\; \frac{1}{4} \\ 3b-2a+1,\, c,\, 3b-c+\frac{3}{2} \end{matrix}\right) \quad [[55], (3.26)].$$

5. $${}_6F_5\left(\begin{matrix} -n,\, \frac{1}{3}-n,\, \frac{2}{3}-n,\, a,\, b,\, b+\frac{1}{2};\; 1 \\ \frac{a+2b}{3},\, \frac{a+2b+1}{3},\, \frac{a+2b+2}{3},\, c,\, \frac{3}{2}-c-3n \end{matrix}\right) = \frac{(a)_{3n}}{(a+2b)_{3n}}\, {}_4F_3\left(\begin{matrix} -3n,\, 2b,\, c-\frac{1}{2},\, 1-c-3n;\; 4 \\ 1-a-3n,\, 2c-1,\, 2-2c-6n \end{matrix}\right) \quad [[55], (3.28)].$$

6. $${}_6F_5\left(\begin{matrix} a,\, b,\, c,\, d,\, e,\, \frac{e}{2}+1;\; -1 \\ \frac{e}{2},\, e-a+1,\, e-b+1,\, e-c+1,\, e-d+1 \end{matrix}\right) = \frac{\Gamma(e-a+1)\Gamma(e-d+1)}{\Gamma(e+1)\Gamma(e-d-a+1)}\, {}_3F_2\left(\begin{matrix} a,\, d,\, e-b-c+1;\; 1 \\ e-b+1,\, e-c+1 \end{matrix}\right)$$
$$[\mathrm{Re}\,(3e-2b-2c-2d-2a+3),\ \mathrm{Re}\,(e-d-a+1) > 0].$$

7. $${}_6F_5\left(\begin{matrix} a,\, b,\, c,\, d,\, e,\, 2e-2;\; -1 \\ e-1,\, 2e-a-1,\, 2e-b-1,\, 2e-c-1,\, 2e-d-1 \end{matrix}\right) = \frac{\Gamma(2e-c-1)\Gamma(2e-d-1)}{\Gamma(2e-1)\Gamma(2e-c-d-1)}\, {}_3F_2\left(\begin{matrix} c,\, d,\, 2e-a-b-1;\; 1 \\ 2e-a-1,\, 2e-b-1 \end{matrix}\right)$$
$[\operatorname{Re}(2e-c-d) > 1;\; \operatorname{Re}(3e-a-b-c-d) > 3/2].$

8. $${}_6F_5\left(\begin{matrix} a,\, a+\frac{1}{2},\, 2a,\, 1-2a,\, \frac{9a+16-\sqrt{25a^2+8a+4}}{14},\, \frac{9a+16+\sqrt{25a^2+8a+4}}{14} \\ a+\frac{3}{4},\, a+\frac{5}{4},\, \frac{9a+2-\sqrt{25a^2+8a+4}}{14},\, \frac{9a+2+\sqrt{25a^2+8a+4}}{14};\; -\frac{1}{48} \end{matrix}\right) = \frac{2^{4a+1}\Gamma\left(2a+\frac{3}{2}\right)}{3^{2a}\sqrt{\pi}\,\Gamma(2a+2)}$$
[[45], (3.3)].

9. $${}_6F_5\left(\begin{matrix} \frac{1}{4},\, \frac{1}{2},\, \frac{3}{4},\, \frac{3}{4},\, \frac{39-2\sqrt{11}}{28},\, \frac{39+2\sqrt{11}}{28} \\ 1,\, \frac{9}{8},\, \frac{13}{8},\, \frac{11-2\sqrt{11}}{28},\, \frac{11+2\sqrt{11}}{28};\; \frac{1}{64} \end{matrix}\right) = \frac{10\sqrt{2}}{33\pi^{3/2}}\Gamma^2\left(\frac{1}{4}\right)$$
[42].

10. $${}_6F_5\left(\begin{matrix} a,\, a,\, 1-a,\, \frac{1}{2},\, \frac{7a+24-\sqrt{28a^2+9}}{21},\, \frac{7a+24+\sqrt{28a^2+9}}{21} \\ 1,\, \frac{2a+3}{4},\, \frac{2a+5}{4},\, \frac{7a+3-\sqrt{28a^2+9}}{21},\, \frac{7a+3+\sqrt{28a^2+9}}{21};\; \frac{1}{64} \end{matrix}\right) = \frac{4\Gamma\left(a+\frac{3}{2}\right)}{\sqrt{\pi}\,(a+2)\Gamma(a+1)}$$
[[45], (2.4)].

11. $${}_6F_5\left(\begin{matrix} \frac{1}{4},\, \frac{1}{4},\, \frac{3}{4},\, \frac{3}{4},\, \frac{16-\sqrt{7}}{12},\, \frac{16+\sqrt{7}}{12} \\ 1,\, 1,\, \frac{3}{2},\, \frac{4-\sqrt{7}}{12},\, \frac{4+\sqrt{7}}{12};\; \frac{1}{4} \end{matrix}\right) = \frac{16}{3\sqrt{2}\,\pi}$$
[45].

12. $${}_6F_5\left(\begin{matrix} 1,\, 1,\, 1,\, 1,\, 1,\, 1 \\ \frac{3}{2},\, 2,\, 2,\, 2,\, 2;\; \frac{1}{4} \end{matrix}\right)$$
$$= \frac{2\pi^2}{9}\zeta(3) - \frac{38}{3}\zeta(5) + \frac{\pi}{192\sqrt{3}}\left[\zeta\left(4,\, \frac{1}{6}\right) - \zeta\left(4,\, \frac{1}{3}\right) + \zeta\left(4,\, \frac{2}{3}\right) - \zeta\left(4,\, \frac{5}{6}\right)\right].$$

8.1.6. The hypergeometric function ${}_7F_6(a_1, \ldots, a_7;\; b_1, \ldots, b_6;\; z)$

1. $${}_7F_6\left(\begin{matrix} -n,\, a,\, b,\, c,\, \frac{a}{3}+1,\, 1-b,\, a-c+n+\frac{1}{2};\; 1 \\ \frac{a}{3},\, \frac{a-b}{2}+1,\, \frac{a+b+1}{2},\, a-2c+1,\, a+2n+1,\, 2c-a-2n \end{matrix}\right) = \frac{(a+1)_{2n}\left(\frac{a+b+1}{2}-c\right)_n\left(\frac{a-b}{2}-c+1\right)_n}{\left(\frac{a-b}{2}+1\right)_n\left(\frac{a+b+1}{2}\right)_n(a-2c+1)_{2n}}$$
[[38], (1.7)].

2. $${}_7F_6\left(\begin{matrix} -2n-m,\, a,\, b,\, c,\, \frac{2a}{3}+1,\, a-b+\frac{1}{2},\, 2a-c+2n+m+1;\ 1 \\ \frac{2a}{3},\, 2b,\, 2a-2b+1,\, a-\frac{c}{2}+1,\, \frac{c+1}{2}-n-\frac{m}{2},\, 1+a+n+\frac{m}{2} \end{matrix}\right)$$
$$= 2^{-2n}\frac{(2n)!\,(a+1)_n\left(b+\frac{1-c}{2}\right)_n(a-b-\frac{c}{2}+1)_n}{n!\left(b+\frac{1}{2}\right)_n(a-b+1)_n\left(a-\frac{c}{2}+1\right)_n\left(\frac{1-c}{2}\right)_n}\delta_{0,m}$$
$[m=0,\,1;$ [38], (1.8)].

3. $${}_7F_6\left(\begin{matrix} a,\, 1-a,\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{6a+21-\sqrt{36a^2-28a+1}}{20},\, \frac{6a+21+\sqrt{36a^2-28a+1}}{20} \\ a+\frac{1}{2},\, a+\frac{1}{2},\, 1,\, 1,\, \frac{6a+1-\sqrt{36a^2-28a+1}}{20},\, \frac{6a+1+\sqrt{36a^2-28a+1}}{20};\ -\frac{1}{4} \end{matrix}\right)$$
$$= \frac{4\Gamma^2\left(a+\frac{1}{2}\right)}{\pi a\,\Gamma^2(a)}$$
[[45], (5.1)].

4. $${}_7F_6\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{12-i}{10},\, \frac{12+i}{10} \\ 1,\, 1,\, 1,\, 1,\, \frac{2-i}{10},\, \frac{2+i}{10};\ -\frac{1}{4} \end{matrix}\right) = \frac{8}{\pi^2}$$
[[45], (5.4)].

5. $${}_7F_6\left(\begin{matrix} \frac{1}{4},\, \frac{1}{3},\, \frac{1}{2},\, \frac{2}{3},\, \frac{3}{4},\, \frac{9}{8}-\frac{i}{24}\sqrt{\frac{17}{7}},\, \frac{9}{8}+\frac{i}{24}\sqrt{\frac{17}{7}} \\ 1,\, 1,\, 1,\, 1,\, \frac{1}{8}-\frac{i}{24}\sqrt{\frac{17}{7}},\, \frac{1}{8}+\frac{i}{24}\sqrt{\frac{17}{7}};\ -\frac{1}{48} \end{matrix}\right) = \frac{48}{5\pi^2}$$
[[43], (15)].

6. $${}_7F_6\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{91}{82}-\frac{4i}{41}\sqrt{\frac{2}{5}},\, \frac{91}{82}+\frac{4i}{41}\sqrt{\frac{2}{5}} \\ 1,\, 1,\, 1,\, 1,\, \frac{9}{82}-\frac{4i}{41}\sqrt{\frac{2}{5}},\, \frac{9}{82}+\frac{4i}{41}\sqrt{\frac{2}{5}};\ -\frac{1}{1024} \end{matrix}\right) = \frac{128}{13\pi^2}$$
[[44], (1–1)].

7. $${}_7F_6\left(\begin{matrix} \frac{1}{6},\, \frac{1}{4},\, \frac{1}{2},\, \frac{3}{4},\, \frac{5}{6},\, \frac{1779-i\sqrt{5279}}{1640},\, \frac{1779+i\sqrt{5279}}{1640} \\ 1,\, 1,\, 1,\, 1,\, \frac{139-i\sqrt{5279}}{1640},\, \frac{139+i\sqrt{5279}}{1640};\ -\frac{1}{1024} \end{matrix}\right) = \frac{256}{15\sqrt{3}\,\pi^2}$$
[[43], (14)].

8. $${}_7F_6\left(\begin{matrix} \frac{1}{8},\, \frac{3}{8},\, \frac{1}{2},\, \frac{5}{8},\, \frac{7}{8},\, \frac{259-i\sqrt{89}}{240},\, \frac{259+i\sqrt{89}}{240} \\ 1,\, 1,\, 1,\, 1,\, \frac{19-i\sqrt{89}}{240},\, \frac{19+i\sqrt{89}}{240};\ \frac{1}{2401} \end{matrix}\right) = \frac{56\sqrt{7}}{15\pi^2}$$
[[44], (2–5)].

9. $${}_7F_6\left(\begin{matrix} \frac{1}{8},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{5}{8},\, \frac{59}{48}-\frac{1}{48}\sqrt{\frac{77}{5}},\, \frac{59}{48}-\frac{1}{48}\sqrt{\frac{77}{5}} \\ 1,\, 1,\, \frac{5}{4},\, \frac{5}{4},\, \frac{11}{48}-\frac{1}{48}\sqrt{\frac{77}{5}},\, \frac{11}{48}+\frac{1}{48}\sqrt{\frac{77}{5}};\ \frac{1}{16} \end{matrix}\right) = \frac{2}{11\pi^3}\Gamma^2\left(\frac{1}{4}\right)$$
[[43], (20)].

10. $${}_7F_6\left(\begin{matrix}\frac{1}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}, \frac{295-\sqrt{385}}{240}, \frac{295+\sqrt{385}}{240} \\ 1, 1, \frac{5}{4}, \frac{5}{4}, \frac{55-\sqrt{385}}{240}, \frac{55+\sqrt{385}}{240};\ \frac{1}{16}\end{matrix}\right) = \frac{2}{11\pi^3}\Gamma^4\left(\frac{1}{4}\right) \quad [42].$$

11. $${}_7F_6\left(\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{137-i\sqrt{71}}{120}, \frac{137+i\sqrt{71}}{120} \\ 1, 1, 1, 1, \frac{17-i\sqrt{71}}{120}, \frac{17+i\sqrt{71}}{120};\ \frac{1}{16}\end{matrix}\right) = \frac{32}{3\pi^2} \quad [[43], (27)].$$

8.1.7. The hypergeometric function ${}_8F_7(a_1, \ldots, a_8;\ b_1, \ldots, b_7;\ z)$

1. $${}_8F_7\left(\begin{matrix}-n, -n-\frac{1}{2}, a, a+\frac{1}{2}, b, b+\frac{1}{2}, -a-b-n-\frac{1}{4}, -a-b-n+\frac{1}{4};\ 1 \\ \frac{1}{2}, -a-n, \frac{1}{2}-a-n, -b-n, \frac{1}{2}-b-n, a+b+\frac{1}{4}, a+b+\frac{3}{4}\end{matrix}\right)$$
$$= \frac{(4a+1)_{2n}(4b+1)_{2n}(2a+2b+1)_{2n}}{(2a+1)_{2n}(2b+1)_{2n}(4a+4b+1)_{2n}}.$$

2. $${}_8F_7\left(\begin{matrix}1, 1, 1, 1, \frac{3}{2}, \frac{9}{4}, \frac{9-i\sqrt{3}}{4}, \frac{9+i\sqrt{3}}{4} \\ \frac{5}{4}, 2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5-i\sqrt{3}}{4}, \frac{5+i\sqrt{3}}{4};\ 1\end{matrix}\right) = \frac{27}{10}\zeta(3) - \frac{54}{35}.$$

3. $${}_8F_7\left(\begin{matrix}\frac{a}{6}+1, \frac{a}{3}, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}, \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3};\ -1 \\ \frac{a}{6}, \frac{a-b+1}{3}, \frac{a-b+2}{3}, \frac{a-b}{3}+1, \frac{a-c+1}{3}, \frac{a-c+2}{3}, \frac{a-c}{3}+1\end{matrix}\right)$$
$$= \frac{\Gamma(a-b+1)\Gamma(a-c+1)}{\Gamma(a+1)\Gamma(a-b-c+1)}\,{}_3F_2\left(\begin{matrix}\frac{a}{3}, b, c;\ \frac{3}{4} \\ \frac{a}{2}, \frac{a+1}{2}\end{matrix}\right) \quad [\operatorname{Re}(5a-6b-6c) > -3].$$

4. $${}_8F_7\left(\begin{matrix}\frac{3}{4}, 1, 1, 1, 1, \frac{5}{4}, \frac{197-i\sqrt{71}}{120}, \frac{197+i\sqrt{71}}{120} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{77-i\sqrt{71}}{120}, \frac{77+I\sqrt{71}}{120};\ -\frac{1}{4}\end{matrix}\right) = \frac{8\pi^2}{75} \quad [43].$$

5. $${}_8F_7\left(\begin{matrix}1, 1, 1, 1, 1, 1, \frac{17-i}{10}, \frac{17+i}{10} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{7-i}{10}, \frac{7+i}{10};\ -\frac{1}{4}\end{matrix}\right) = \frac{7}{10}\zeta(3).$$

8.1.8. The hypergeometric function ${}_{10}F_9(a_1, \ldots, a_{10};\ b_1, \ldots, b_9;\ z)$

1. $${}_{10}F_9\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{6}, \frac{16-i\sqrt{3}}{14}, \frac{16+i\sqrt{3}}{14} \\ \frac{1}{6}, 1, 1, 1, 1, 1, 1, \frac{2-i\sqrt{3}}{14}, \frac{2+i\sqrt{3}}{14};\ \frac{1}{64}\end{matrix}\right) = \frac{32}{\pi^3} \quad [[44], (4–1)].$$

8.1.9. The Kummer confluent hypergeometric function ${}_1F_1(a;\ b;\ z)$

1. $${}_1F_1\left(\begin{matrix} a+m;\ z \\ b+n \end{matrix}\right) = \frac{(-1)^n m!\,(b)_n}{(a)_m (b-a)_n} e^z \sum_{k=0}^{m} \frac{z^k}{k!} L_{m-k}^{a+k-1}(-z)\, \mathrm{D}_z^{k+n} \left[e^{-z}\,{}_1F_1\left(\begin{matrix} a;\ z \\ b \end{matrix}\right)\right].$$

2. $${}_1F_1\left(\begin{matrix} a-m;\ z \\ b+n \end{matrix}\right) = \frac{(-1)^{m+n}(b)_n}{(b-a+n)_m (b-a)_n} e^z \times \sum_{k=0}^{m} (-z)^k \binom{m}{k} (a-b-m-n+1)_{m-k}\, \mathrm{D}^{k+n} \left[e^{-z}\,{}_1F_1\left(\begin{matrix} a;\ z \\ b \end{matrix}\right)\right].$$

3. $${}_1F_1\left(\begin{matrix} a+m;\ z \\ b-n \end{matrix}\right) = \frac{z^{1-a}}{(a)_m (1-b)_n} \times \sum_{k=0}^{n} \binom{n}{k} (a-b)_{n-k} (-z)^k\, \mathrm{D}^{k+m} \left[z^{a+m-1}\,{}_1F_1\left(\begin{matrix} a;\ z \\ b \end{matrix}\right)\right].$$

4. $${}_1F_1\left(\begin{matrix} a-m;\ z \\ b-n \end{matrix}\right) = \frac{(-1)^n n!\, z^{a-b+1}}{(b-a)_m (1-b)_n} e^z \sum_{k=0}^{n} \frac{z^k}{k!} L_{n-k}^{a+k-n}(-z)\, \mathrm{D}^{k+m} \left[z^{b-a+m-1} e^{-z}\,{}_1F_1\left(\begin{matrix} a;\ z \\ b \end{matrix}\right)\right].$$

5. $${}_1F_1\left(\begin{matrix} a;\ z \\ 2a+n \end{matrix}\right) = \Gamma\left(a-\frac{1}{2}\right) \left(\frac{z}{4}\right)^{1/2-a} e^{z/2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \times \frac{(2a-1)_k}{(2a+n)_k} \left(a+k-\frac{1}{2}\right) I_{a+k-1/2}\left(\frac{z}{2}\right).$$

6. $${}_1F_1\left(\begin{matrix} a;\ z \\ 2a-n \end{matrix}\right) = \Gamma\left(a-n-\frac{1}{2}\right) \left(\frac{z}{4}\right)^{n-a+1/2} e^{z/2} \sum_{k=0}^{n} \binom{n}{k} \times \frac{(2a-2n-1)_k}{(2a-n)_k} \left(a+k-n-\frac{1}{2}\right) I_{a+k-n-1/2}\left(\frac{z}{2}\right).$$

7. $${}_1F_1\left(\begin{matrix} m+1 \\ b;\ z \end{matrix}\right) = (b-1) z^{1-b} e^z \sum_{k=0}^{m} \frac{(-1)^k}{k!} L_{m-k}^{k}(-z) \gamma(b+k-1,\, z).$$

8. $${}_1F_1\left(\begin{matrix} m+1;\ z \\ n+2 \end{matrix}\right) = (-1)^n (n+1) z^{-n-1} \sum_{k=0}^{m} \frac{(k+n)!}{k!} L_{m-k}^{k}(-z) [(-1)^{k+n} e^z - L_{k+n}^{-k-n-1}(z)].$$

9. $${}_1F_1\left(\begin{matrix} \frac{1}{6};\ z \\ \frac{1}{3} \end{matrix}\right) = \frac{3^{1/6}}{2} \Gamma\left(\frac{2}{3}\right) e^{z/2} \left[\sqrt{3}\, \mathrm{Ai}\left(\left(\frac{3z}{4}\right)^{2/3}\right) + \mathrm{Bi}\left(\left(\frac{3z}{4}\right)^{2/3}\right)\right].$$

10. $${}_1F_1\left(\begin{matrix}\frac{5}{6};\ z\\ \frac{2}{3}\end{matrix}\right) = 2^{-5/3}3^{5/6}\Gamma\left(\frac{4}{3}\right)e^{z/2}$$
$$\times\left[-3^{5/6}z^{1/3}\operatorname{Ai}\left(\left(\frac{3z}{4}\right)^{2/3}\right) - 2^{2/3}\sqrt{3}\operatorname{Ai}'\left(\left(\frac{3z}{4}\right)^{2/3}\right)\right.$$
$$\left. + (3z)^{1/3}\operatorname{Bi}\left(\left(\frac{3z}{4}\right)^{2/3}\right) + 2^{2/3}\operatorname{Bi}'\left(\left(\frac{3z}{4}\right)^{2/3}\right)\right].$$

8.1.10. The Tricomi confluent hypergeometric function $\Psi(a;\ b;\ z)$

1. $$\Psi\left(\begin{matrix}a+m;\ z\\ b+n\end{matrix}\right) = \frac{(-1)^n n!\,z^{-a-n+1}}{(a)_m(a-b+1)_m}$$
$$\times\sum_{k=0}^{n}\frac{z^k}{k!}L_{n-k}^{k-a-n+1}(z)\,\mathrm{D}^{k+m}\left[z^{a+m-1}\Psi\left(\begin{matrix}a;\ z\\ b\end{matrix}\right)\right].$$

2. $$\Psi\left(\begin{matrix}a-m;\ z\\ b+n\end{matrix}\right) = (-1)^{m+n}(b-a+n)_m e^z$$
$$\times\sum_{k=0}^{m}\binom{m}{k}\frac{z^k}{(b-a+n)_k}\,\mathrm{D}^{k+n}\left[e^{-z}\Psi\left(\begin{matrix}a;\ z\\ b\end{matrix}\right)\right].$$

3. $$\Psi\left(\begin{matrix}a+m;\ z\\ b-n\end{matrix}\right) = \frac{(a-b)_n z^{-a+1}}{(a)_m(a-b+1)_m(a-b+m+1)_n}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}\frac{z^k}{(b-a-n+1)_k}\,\mathrm{D}^{k+m}\left[z^{a+m-1}\Psi\left(\begin{matrix}a;\ z\\ b\end{matrix}\right)\right].$$

4. $$\Psi\left(\begin{matrix}a-m;\ z\\ b-n\end{matrix}\right) = \frac{(-1)^{m+n}m!\,z^{n-b+1}}{(a-b+1)_n}\sum_{k=0}^{m}\frac{z^k}{k!}L_{m-k}^{k-a}(z)\,\mathrm{D}^{k+n}\left[z^{b-1}\Psi\left(\begin{matrix}a;\ z\\ b\end{matrix}\right)\right].$$

5. $$\Psi\left(\begin{matrix}a;\ z\\ a+n\end{matrix}\right) = (-1)^{n-1}(n-1)!z^{1-a-n}L_{n-1}^{1-a-n}(z) \qquad [n \geq 1].$$

6. $$\Psi\left(\begin{matrix}a;\ z\\ a-n\end{matrix}\right) = (-1)^n$$
$$\times\left[e^z\Gamma(1-a,\,z)L_n^{a-n-1}(-z) - z^{1-a}\sum_{k=1}^{n}\frac{1}{k}L_{n-k}^{a+k-n-1}(-z)L_{k-1}^{1-a-k}(z)\right].$$

7. $$\Psi\left(\begin{matrix}a;\ z\\ 2a+n\end{matrix}\right) = \frac{(-1)^n}{\sqrt{\pi}}n!\,z^{1/2-a-n}e^{z/2}$$
$$\times\sum_{k=0}^{n}L_{n-k}^{k-n-a+1}(z)\sum_{p=0}^{k}\frac{(-z/4)^p}{p!}L_{k-p}^{p-k-1/2}\left(-\frac{z}{2}\right)\sum_{r=0}^{p}\binom{p}{r}K_{a-p+2r-1/2}\left(\frac{z}{2}\right).$$

8. $$\Psi\binom{a;\ z}{2a-n} = \frac{(-1)^n n!}{\sqrt{\pi}\,(1-a)_n} z^{1/2-a} e^{z/2} \times \sum_{k=0}^{n} \frac{(-z/4)^k}{k!} L_{n-k}^{a+k-n-1/2}\left(-\frac{z}{2}\right) \sum_{p=0}^{k} \binom{k}{p} K_{a-k+2p-1/2}\left(\frac{z}{2}\right).$$

9. $$\Psi\left(\begin{matrix} a;\ z \\ n+\frac{1}{2} \end{matrix}\right) = 2^a n!\,(-z)^{-n} e^{z/2} \times \sum_{k=0}^{n} \frac{(-1)^k}{k!} L_{n-k}^{k-n-a+1}(z) \sum_{p=0}^{k} \binom{k}{p} (1-a)_{k-p} z^p \times \left[(-1)^p \delta_{p,0} D_{-2a}\left(\sqrt{2z}\right) + (2z)^{-p/2} \sum_{r=0}^{p-1} \frac{\Gamma(p+r)}{r!\,\Gamma(p-r)} (8z)^{-r/2} (2a)_{p-r} \times D_{r-p-2a}\left(\sqrt{2z}\right)\right].$$

10. $$\Psi\left(\begin{matrix} a;\ z \\ \frac{1}{2}-n \end{matrix}\right) = \frac{2^a e^{z/2}}{\left(a+\frac{1}{2}\right)_n} \sum_{p=0}^{n} \binom{n}{p} \left(\frac{1}{2}\right)_{n-p} z^p \times \left[(-1)^p \delta_{p,0} D_{-2a}(\sqrt{2z}) + (2z)^{-p/2} \sum_{r=0}^{p-1} \frac{\Gamma(p+r)}{r!\,\Gamma(p-r)} (8z)^{-r/2} (2a)_{p-r} \times D_{r-p-2a}\left(\sqrt{2z}\right)\right].$$

11. $$\Psi\binom{m+1}{b;\ z} = \frac{1}{(2-b)_m} \left[z^{1-b} e^z \Gamma(b-1,\, z) L_m^{1-b}(-z) - \sum_{k=1}^{m} \frac{1}{k} L_{m-k}^{k-b+1}(-z) L_{k-1}^{b-k-1}(z)\right].$$

12. $$\Psi\binom{m+1;\ z}{n+1} = -\frac{1}{(m!)^2} \sum_{k=0}^{n} \binom{n}{k} (k+m)!\,(-z)^{-k} \times \left[e^z\, \mathrm{Ei}\,(-z) L_{k+m}^{-k}(-z) + \sum_{p=1}^{k+m} \frac{(-z)^p}{p} L_{k+m-p}^{p-k}(-z) L_{p-1}^{-p}(z)\right] \quad [m>n].$$

13. $$\Psi\left(\begin{matrix} m+\frac{1}{2};\ z \\ n+\frac{1}{2} \end{matrix}\right) = \frac{2^{2m} n!\,(-z)^{-n}}{(2m)!} \sum_{k=0}^{n} \frac{(k+m)!}{k!} L_{n-k}^{k-n+1/2}(z) \times \left[\sqrt{\pi}\, e^z \operatorname{erfc}(\sqrt{z})\, L_{k+m}^{-k-1/2}(-z) + 2 \sum_{p=1}^{k+m} \frac{(-z)^p}{p!} L_{k+m-p}^{p-k-1/2}(-z) \times \sum_{r=0}^{p-1} \frac{\Gamma(p+r)}{r!\,\Gamma(p-r)} (2\sqrt{z})^{-p-r} H_{p-r-1}(\sqrt{z})\right] \quad [m > n].$$

14. $$\Psi\left(\begin{matrix} m+\frac{1}{2};\ z \\ \frac{1}{2}-n \end{matrix}\right) = \frac{1}{\left(\frac{1}{2}-n\right)_{m+n}} \left[e^z \Gamma\left(n+\frac{1}{2},\ z\right) L_{m+n}^{-n-1/2}(-z) - z^{n+1/2} \sum_{k=1}^{m+n} \frac{1}{k} L_{m+n-k}^{k-n-1/2}(-z) L_{k-1}^{n-k+1/2}(z)\right].$$

15. $$\Psi\left(\begin{matrix} \frac{1}{2}-m;\ z \\ n+\frac{1}{2} \end{matrix}\right) = (-1)^m 2^{1-n} (n)_m z^{-n/2} \sum_{k=0}^{m} \binom{m}{k} \frac{\left(-\frac{\sqrt{z}}{2}\right)^k}{(n)_k} \times \sum_{r=0}^{k+n-1} \frac{\Gamma(k+n+r)}{r!\,\Gamma(k+n-r)} (2\sqrt{z})^{-r} H_{k+n-r-1}(\sqrt{z}) \quad [n \geq 1].$$

16. $$\Psi\left(\begin{matrix} \frac{1}{2}-m;\ z \\ \frac{1}{2}-n \end{matrix}\right) = \frac{(-1)^{m+n} m!}{n!} \sum_{k=0}^{m} \frac{(k+n)!}{k!} L_{m-k}^{k-1/2}(z) [\sqrt{\pi}\, e^z \operatorname{erfc}(\sqrt{z})\, L_{k+n}^{-k-n-1/2}(-z) + 2 \sum_{p=1}^{k+n} \frac{(-z)^p}{p!} L_{k+n-p}^{p-k-n-1/2}(-z) \sum_{r=0}^{p-1} \frac{\Gamma(p+r)}{r!\,\Gamma(p-r)} (2\sqrt{z})^{-p-r} H_{p-r-1}(\sqrt{z})] \quad [m > n].$$

17. $$\Psi\left(\begin{matrix} \frac{1}{6};\ z \\ \frac{1}{3} \end{matrix}\right) = 2^{2/3} 3^{1/6} \sqrt{\pi}\, e^{z/2} \operatorname{Ai}\left(\left(\frac{3z}{4}\right)^{2/3}\right).$$

8.1.11. The hypergeometric function ${}_1F_2(a_1;\ b_1, b_2;\ z)$

1. $${}_1F_2\left(\begin{matrix} 1;\ z \\ b,\ b+\frac{1}{2} \end{matrix}\right) = \frac{2b-1}{2\sqrt{z}} \sinh(2\sqrt{z})\, {}_1F_2\left(\begin{matrix} b-1;\ z \\ b,\ \frac{1}{2} \end{matrix}\right) - 2(b-1)\cosh(2\sqrt{z})\, {}_1F_2\left(\begin{matrix} b-\frac{1}{2};\ z \\ b+\frac{1}{2},\ \frac{3}{2} \end{matrix}\right).$$

2. $$= \frac{1-2b}{b}\sqrt{z}\sinh(2\sqrt{z})\,{}_1F_2\left(\begin{matrix} b;\ z \\ b+1,\ \frac{3}{2}\end{matrix}\right) + \cosh(2\sqrt{z})\,{}_1F_2\left(\begin{matrix} b-\frac{1}{2};\ z \\ b+\frac{1}{2},\ \frac{1}{2}\end{matrix}\right).$$

3. $$= \frac{2^{-2b}e^{-b\pi i}z^{1/2-b}}{\Gamma(1-2b)} \times \left\{\Gamma(3-2b)\left[e^{-2\sqrt{z}-b\pi i}\Gamma(2b-2,\,-2\sqrt{z}) - e^{2\sqrt{z}+b\pi i}\Gamma(2b-2,\,2\sqrt{z})\right] + \pi[\csc(b\pi)\sinh(2\sqrt{z}) + i\sec(b\pi)\cosh(2\sqrt{z})]\right\} \quad [z>0].$$

4. $${}_1F_2\left(\begin{matrix} a;\ z \\ a+1,\ \frac{1}{2}\end{matrix}\right) = \cosh(2\sqrt{z})\,{}_1F_2\left(\begin{matrix} 1;\ z \\ a+\frac{1}{2},\ a+1\end{matrix}\right) - \frac{2\sqrt{z}}{2a+1}\sinh(2\sqrt{z})\,{}_1F_2\left(\begin{matrix} 1;\ z \\ a+1,\ a+\frac{3}{2}\end{matrix}\right).$$

5. $${}_1F_2\left(\begin{matrix} a;\ z \\ a+1,\ \frac{3}{2}\end{matrix}\right) = \frac{1}{2a-1}\left[\frac{a}{\sqrt{z}}\sinh(2\sqrt{z})\,{}_1F_2\left(\begin{matrix} 1;\ z \\ a,\ a+\frac{1}{2}\end{matrix}\right) - \cosh(2\sqrt{z})\,{}_1F_2\left(\begin{matrix} 1;\ z \\ a+\frac{1}{2},\ a+1\end{matrix}\right)\right].$$

6. $${}_1F_2\left(\begin{matrix} j+1;\ \pm z \\ \frac{3}{2}-m,\ c\end{matrix}\right) = (-1)^j\frac{\sqrt{\pi}}{2}\frac{\Gamma(c)}{j!\left(-\frac{1}{2}\right)_m}\sum_{i=0}^{j}\binom{j}{i}\left(\frac{1}{2}-j-m\right)_{j-i} \times \sum_{k=0}^{i+m}(-1)^k\binom{i+m}{k}\left(c-\frac{3}{2}\right)_{i-k+m} z^{(2k-2c+1)/4}\left\{\begin{matrix}\mathbf{L}_{c-k-3/2}(2\sqrt{z}) \\ \mathbf{H}_{c-k-3/2}(2\sqrt{z})\end{matrix}\right\}.$$

7. $${}_1F_2\left(\begin{matrix} j+1;\ \pm z \\ m+\frac{5}{2},\ c\end{matrix}\right) = \frac{\left(\frac{3}{2}\right)_{m+1}}{j!\,m!}\Gamma(c)\sum_{k=0}^{m}(-1)^k\binom{m}{k}\sum_{i=0}^{k}(-k)_i \times \left[\frac{\sqrt{\pi}}{2}\sum_{p=0}^{j}(-1)^{j-p}\binom{j}{p}(c-j-1)_{j-p}z^{(2p-2c+2i-1)/4}\left\{\begin{matrix}\mathbf{L}_{c-i-p-5/2}(2\sqrt{z}) \\ \mathbf{H}_{c-i-p-5/2}(2\sqrt{z})\end{matrix}\right\} - \frac{(-1)^j(i-j+1)_j}{(2k-2i+1)\Gamma(c-i-1)}z^{-i-1}\right].$$

8. $${}_1F_2\left(\begin{matrix} n-\frac{1}{2};\ \pm z \\ n+1,\ 2n+1\end{matrix}\right) = \frac{2^{2n+1}(n!)^2}{2n+1}z^{1/2-n}\left[\sqrt{z}\left\{\begin{matrix} I_{n-1}(\sqrt{z}) \\ J_{n-1}(\sqrt{z})\end{matrix}\right\}^2 - \left\{\begin{matrix} I_{n-1}(\sqrt{z})\,I_n(\sqrt{z}) \\ J_{n-1}(\sqrt{z})\,J_n(\sqrt{z})\end{matrix}\right\} - \sqrt{z}\left\{\begin{matrix} I_{n-2}(\sqrt{z})\,I_n(\sqrt{z}) \\ J_{n-2}(\sqrt{z})\,J_n(\sqrt{z})\end{matrix}\right\} + \frac{1-2n}{2\sqrt{z}}\left\{\begin{matrix} I_n(\sqrt{z}) \\ J_n(\sqrt{z})\end{matrix}\right\}^2\right].$$

9. $$ {}_1F_2\left(\begin{matrix} j+\frac{1}{2};\ \pm z \\ j+m+\frac{3}{2},\ n+1 \end{matrix}\right) = \frac{n!}{m!}\left(j+\frac{1}{2}\right)_{m+1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{z^{-j-k}}{\left(\frac{1}{2}-j-k\right)_n} $$
$$ \times\left[\sum_{p=0}^{2j+2k-1}(\pm 1)^{j+k}\left(\frac{1}{2}-j-k\right)_p z^{(2j+2k-p-1)/2}\left\{\begin{matrix} I_{p+1}(2\sqrt{z}) \\ J_{p+1}(2\sqrt{z}) \end{matrix}\right\}\right. $$
$$ +(-1)^{j+k}\left(\frac{1}{2}-j-k\right)_{2j+2k}\left(2\left\{\begin{matrix} I_0(2\sqrt{z}) \\ J_0(2\sqrt{z}) \end{matrix}\right\}\right. $$
$$ +\pi\left\{\begin{matrix} I_0(2\sqrt{z})\,\mathbf{L}_1(2\sqrt{z})-I_1(2\sqrt{z})\,\mathbf{L}_0(2\sqrt{z}) \\ J_0(2\sqrt{z})\,\mathbf{H}_1(2\sqrt{z})+J_1(2\sqrt{z})\,\mathbf{H}_0(2\sqrt{z}) \end{matrix}\right\} $$
$$ \left.-\frac{2}{\sqrt{z}}\sum_{r=0}^{j+k-1}(\mp 1)^r\left\{\begin{matrix} I_{2r+1}(2\sqrt{z}) \\ J_{2r+1}(2\sqrt{z}) \end{matrix}\right\}\right) $$
$$ \left.-\sum_{i=0}^{n-1}(\pm 1)^{j+k}\left(\frac{1}{2}-j-k\right)_i z^{(2j+2k-i-1)/2}\left\{\begin{matrix} I_{i+1}(2\sqrt{z}) \\ J_{i+1}(2\sqrt{z}) \end{matrix}\right\}\right]. $$

10. $$ {}_1F_2\left(\begin{matrix} j+\frac{1}{4};\ \pm z \\ j+m+\frac{5}{4},\ n+\frac{1}{2} \end{matrix}\right) = \frac{\left(j+\frac{1}{4}\right)_{m+1}}{m!}\Gamma\left(n+\frac{1}{2}\right)\sum_{k=0}^{m}(-1)^k\binom{m}{k} $$
$$ \times\frac{1}{\left(\frac{1}{4}-j-k\right)_n}\left[(\pm 1)^{j+k}(4j+4k-1)!!\,2^{-4j-4k-1/2}z^{-j-k-1/4}\right. $$
$$ \times\left\{\begin{matrix} \operatorname{erf}(\sqrt{2}\,z^{1/4})+\operatorname{erfi}(\sqrt{2}\,z^{1/4}) \\ 2^{3/2}C(2\sqrt{z}) \end{matrix}\right\}+\frac{1}{\sqrt{\pi z}}\left\{\begin{matrix} \sinh(2\sqrt{z}) \\ \sin(2\sqrt{z}) \end{matrix}\right\} $$
$$ \times\sum_{p=0}^{j+k-1}\frac{(4j+4k-1)!!}{(4j+4k-4p-1)!!}(\pm 16z)^{-p} $$
$$ -\frac{4}{\sqrt{\pi}}\left\{\begin{matrix} \cosh(2\sqrt{z}) \\ \cos(2\sqrt{z}) \end{matrix}\right\}\sum_{p=0}^{j+k-1}\frac{(4j+4k-1)!!}{(4j+4k-4p-3)!!}(\pm 16z)^{-p-1} $$
$$ \left.-\sum_{k=0}^{n-1}\left(\frac{1}{4}-j-k\right)_i z^{-(2i+1)/4}\left\{\begin{matrix} I_{i+1/2}(2\sqrt{z}) \\ J_{i+1/2}(2\sqrt{z}) \end{matrix}\right\}\right]. $$

11. $$ {}_1F_2\left(\begin{matrix} j+\frac{3}{4};\ \pm z \\ j+m+\frac{7}{4},\ n+\frac{1}{2} \end{matrix}\right) = \frac{\left(j+\frac{3}{4}\right)_{m+1}}{m!}\Gamma\left(n+\frac{1}{2}\right)\sum_{k=0}^{m}(-1)^k\binom{m}{k} $$
$$ \times\frac{1}{\left(-j-k-\frac{1}{4}\right)_n}\left[(\pm 1)^{j+k}(4j+4k+1)!!\,2^{-4j-4k-5/2}z^{-j-k-3/4}\right. $$
$$ \times\left\{\begin{matrix} \operatorname{erf}(\sqrt{2}\,z^{1/4})-\operatorname{erfi}(\sqrt{2}\,z^{1/4}) \\ -2^{3/2}\,S(2\sqrt{z}) \end{matrix}\right\}+\frac{1}{\sqrt{\pi z}}\left\{\begin{matrix} \sinh(2\sqrt{z}) \\ \sin(2\sqrt{z}) \end{matrix}\right\} $$

$$\times \sum_{p=0}^{j+k} \frac{(4j+4k+1)!!}{(4j+4k-4p+1)!!} (\pm 16z)^{-p}$$

$$- \frac{4}{\sqrt{\pi}} \left\{ \begin{matrix} \cosh(2\sqrt{z}) \\ \cos(2\sqrt{z}) \end{matrix} \right\} \sum_{p=0}^{j+k-1} \frac{(4j+4k+1)!!}{(4j+4k-4p-1)!!} (\pm 16z)^{-p-1}$$

$$\left. - \sum_{i=0}^{n-1} \left(-j-k-\frac{1}{4}\right)_i z^{-(2i+1)/4} \left\{ \begin{matrix} I_{i+1/2}(2\sqrt{z}) \\ J_{i+1/2}(2\sqrt{z}) \end{matrix} \right\} \right].$$

12. $${}_1F_2\left(\begin{matrix} j+\frac{1}{4};\ \pm z \\ j+m+\frac{5}{4},\ \frac{1}{2}-n \end{matrix} \right) = \frac{\left(j+\frac{1}{4}\right)_{m+1}}{m!} \Gamma\left(\frac{1}{2}-n\right) \sum_{k=0}^{m} (-1)^k \binom{m}{k}$$

$$\times \left[\sum_{i=0}^{n-1} \left(\frac{1}{4}-j-k-n\right)_i z^{(2n-2i-1)/4} \left\{ \begin{matrix} I_{i-n+1/2}(2\sqrt{z}) \\ J_{i-n+1/2}(2\sqrt{z}) \end{matrix} \right\} \right.$$

$$+ (-1)^n \left(j+k+\frac{3}{4}\right)_n$$

$$\times \left(\frac{1}{\sqrt{\pi z}} \left\{ \begin{matrix} \sinh(2\sqrt{z}) \\ \sin(2\sqrt{z}) \end{matrix} \right\} \sum_{p=0}^{j+k-1} \frac{(4j+4k-1)!!}{(4j+4k-4p-1)!!} (\pm 16z)^{-p} \right.$$

$$- \frac{4}{\sqrt{\pi}} \left\{ \begin{matrix} \cosh(2\sqrt{z}) \\ \cos(2\sqrt{z}) \end{matrix} \right\} \sum_{p=0}^{j+k-1} \frac{(4j+4k-1)!!}{(4j+4k-4p-3)!!} (\pm 16z)^{-p-1}$$

$$+ (\pm 1)^{j+k} (4j+4k-1)!!\, 2^{-4j-4k-1/2} z^{-j-k-1/4}$$

$$\left.\left. \times \left\{ \begin{matrix} \operatorname{erf}(\sqrt{2}\, z^{1/4}) + \operatorname{erfi}(\sqrt{2}\, z^{1/4}) \\ 2^{3/2} C(2\sqrt{z}) \end{matrix} \right\} \right) \right].$$

13. $${}_1F_2\left(\begin{matrix} j+\frac{3}{4};\ \pm z \\ j+m+\frac{7}{4},\ \frac{1}{2}-n \end{matrix} \right) = \frac{\left(j+\frac{3}{4}\right)_{m+1}}{m!} \Gamma\left(\frac{1}{2}-n\right) \sum_{k=0}^{m} (-1)^k \binom{m}{k}$$

$$\times \left[\sum_{i=0}^{n-1} \left(-\frac{1}{4}-j-k-n\right)_i z^{(2n-2i-1)/4} \left\{ \begin{matrix} I_{i-n+1/2}(2\sqrt{z}) \\ J_{i-n+1/2}(2\sqrt{z}) \end{matrix} \right\} \right.$$

$$+ (-1)^n \left(j+k+\frac{5}{4}\right)_n$$

$$\times \left(\frac{1}{\sqrt{\pi z}} \left\{ \begin{matrix} \sinh(2\sqrt{z}) \\ \sin(2\sqrt{z}) \end{matrix} \right\} \sum_{p=0}^{j+k} \frac{(4j+4k+1)!!}{(4j+4k-4p+1)!!} (\pm 16z)^{-p} \right.$$

$$- \frac{4}{\sqrt{\pi}} \left\{ \begin{matrix} \cosh(2\sqrt{z}) \\ \cos(2\sqrt{z}) \end{matrix} \right\} \sum_{p=0}^{j+k-1} \frac{(4j+4k+1)!!}{(4j+4k-4p-1)!!} (\pm 16z)^{-p-1}$$

$$+ (\pm 1)^{j+k} (4j+4k+1)!!\, 2^{-4j-4k-5/2} z^{-j-k-3/4}$$

$$\left.\left. \times \left\{ \begin{matrix} \operatorname{erf}(\sqrt{2}\, z^{1/4}) - \operatorname{erfi}(\sqrt{2}\, z^{1/4}) \\ -2^{3/2} S(2\sqrt{z}) \end{matrix} \right\} \right) \right].$$

14. $${}_1F_2\left(\begin{matrix} a;\ \pm z \\ a-m+\frac{1}{2},\ 2a-n \end{matrix}\right) = \frac{\Gamma^2\left(a+\frac{1}{2}\right)}{\left(\frac{1}{2}-a\right)_m (1-2a)_n} \left(\frac{2}{\sqrt{z}}\right)^{2a-1}$$
$$\times \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}-a-m\right)_{n-k} \sum_{i=0}^{k+m} (\mp 1)^i \binom{k+m}{i} \left(\frac{1}{2}-a\right)_{k+m-i} \left(\frac{\sqrt{z}}{2}\right)^i$$
$$\times \sum_{p=0}^{i} \binom{i}{p} \left\{ \begin{matrix} I_{a+p-1/2}(\sqrt{z})\, I_{a+i-p-1/2}(\sqrt{z}) \\ J_{a+p-1/2}(\sqrt{z})\, J_{a+i-p-1/2}(\sqrt{z}) \end{matrix} \right\}.$$

15. $${}_1F_2\left(\begin{matrix} m+\frac{1}{2};\ \pm z \\ a-n,\ 2-a \end{matrix}\right) = (-1)^{m+n} \frac{\pi(1-a)\csc(a\pi)}{\left(\frac{1}{2}\right)_m (1-a)_n} \left(\frac{\sqrt{z}}{2}\right)^n$$
$$\times \sum_{k=0}^{m} (\mp 1)^k \binom{m}{k} \left(a-m-n-\frac{1}{2}\right)_{m-k} \left(\frac{\sqrt{z}}{2}\right)^k \sum_{i=0}^{k+n} (\pm 1)^{i+n} \binom{k+n}{i}$$
$$\times \left\{ \begin{matrix} I_{n+k-i-a+1}(\sqrt{z})\, I_{a-i-1}(\sqrt{z}) \\ J_{n+k-i-a+1}(\sqrt{z})\, J_{a-i-1}(\sqrt{z}) \end{matrix} \right\}.$$

16. $${}_1F_2\left(\begin{matrix} a;\ \pm z \\ a-n,\ b \end{matrix}\right) = \Gamma(b) z^{(1-b)/2} \sum_{k=0}^{n} (\pm 1) \binom{n}{k}^k \frac{z^{k/2}}{(a-n)_k} \left\{ \begin{matrix} I_{b+k-1}(2\sqrt{z}) \\ J_{b+k-1}(2\sqrt{z}) \end{matrix} \right\}.$$

8.1.12. The hypergeometric function ${}_2F_2(a_1, a_2;\ b_1, b_2;\ z)$

1. $${}_2F_2\left(\begin{matrix} a+1,\ 2a \\ a,\ b;\ z \end{matrix}\right) = e^z\, {}_2F_2\left(\begin{matrix} 2a-b+2,\ b-2a-1 \\ 2a-b+1,\ b;\ -z \end{matrix}\right)$$ [[30], (12)].

2. $${}_2F_2\left(\begin{matrix} \frac{n+1}{2},\ \frac{n}{2}+1 \\ n+1,\ 1;\ z \end{matrix}\right) = e^{z/2} \sum_{k=0}^{n} \binom{n}{k} I_k\left(\frac{z}{2}\right).$$

8.1.13. The hypergeometric function ${}_2F_3(a_1, a_2;\ b_1, b_2, b_3;\ z)$

1. $${}_2F_3\left(\begin{matrix} a,\ a+\frac{1}{2};\ z \\ b,\ b+\frac{1}{2},\ \frac{1}{2} \end{matrix}\right) = \cosh(2\sqrt{z})\, {}_2F_3\left(\begin{matrix} b-a,\ b-a+\frac{1}{2};\ z \\ b,\ b+\frac{1}{2},\ \frac{1}{2} \end{matrix}\right)$$
$$+ \frac{a-b}{b} 2\sqrt{z} \sinh(2\sqrt{z})\, {}_2F_3\left(\begin{matrix} b-a+\frac{1}{2},\ b-a+1;\ z \\ b+\frac{1}{2},\ b+1,\ \frac{3}{2} \end{matrix}\right)$$ [[25], (12)].

2. $$= \frac{1}{2}\left[{}_1F_1\left(\begin{matrix} 2a;\ -2\sqrt{z} \\ 2b \end{matrix}\right) + {}_1F_1\left(\begin{matrix} 2a;\ 2\sqrt{z} \\ 2b \end{matrix}\right) \right].$$

3. $${}_2F_3\left(\begin{matrix} a,\, a+\frac{1}{2};\; z \\ b,\, b+\frac{1}{2},\, \frac{3}{2} \end{matrix}\right) = \frac{2b-1}{2\,(2a-1)\sqrt{z}} \sinh\left(2\sqrt{z}\right) {}_2F_3\left(\begin{matrix} b-a,\, b-a+\frac{1}{2};\; z \\ b-\frac{1}{2},\, b,\, \frac{1}{2} \end{matrix}\right)$$
$$+ \frac{2\,(a-b)}{2a-1} \cosh\left(2\sqrt{z}\right)\, {}_2F_3\left(\begin{matrix} b-a+\frac{1}{2},\, b-a+1;\; z \\ b,\, b+\frac{1}{2},\, \frac{3}{2} \end{matrix}\right) \quad [[25],\ (12)].$$

4. $$= \frac{2b-1}{4(2a-1)\sqrt{z}} \left[{}_1F_1\left(\begin{matrix} 2a-1;\; 2\sqrt{z} \\ 2b-1 \end{matrix}\right) - {}_1F_1\left(\begin{matrix} 2a-1;\; -2\sqrt{z} \\ 2b-1 \end{matrix}\right) \right].$$

5. $${}_2F_3\left(\begin{matrix} -n,\, n+\frac{1}{2};\; z \\ \frac{1}{4},\, \frac{1}{2},\, \frac{3}{4} \end{matrix}\right) = \left[\frac{n!}{(2n)!}\right]^2 H_{2n}\left(\sqrt[4]{4z}\right) H_{2n}\left(i\sqrt[4]{4z}\right).$$

6. $${}_2F_3\left(\begin{matrix} -n,\, n+\frac{3}{2};\; z \\ \frac{3}{4},\, \frac{5}{4},\, \frac{3}{2} \end{matrix}\right) = \left[\frac{n!}{(2n+1)!}\right]^2 \frac{1}{8i\sqrt{z}} H_{2n+1}\left(\sqrt[4]{4z}\right) H_{2n+1}\left(i\sqrt[4]{4z}\right).$$

7. $${}_2F_3\left(\begin{matrix} n,\, n+\frac{1}{2};\; z \\ n+1,\, n+1,\, 2n+1 \end{matrix}\right)$$
$$= (n!)^2\left(-\frac{4}{z}\right)^n \left[1 + I_0^2(\sqrt{z}) + (-1)^n I_n^2(\sqrt{z}) - 2\sum_{k=0}^{n}(-1)^k I_k^2(\sqrt{z})\right].$$

8. $${}_2F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2};\; z \\ 1,\, 1,\, 1 \end{matrix}\right) = \frac{1}{\pi}\int_0^z \frac{1}{\sqrt{x(z-x)}} I_0^2(\sqrt{x})\, dx.$$

9. $${}_2F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2};\; z \\ 1,\, 1,\, \frac{3}{2} \end{matrix}\right) = \frac{1}{2\sqrt{z}}\int_0^z \frac{1}{\sqrt{x}} I_0^2(\sqrt{x})\, dx.$$

10. $${}_2F_3\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2};\; z \\ \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2} \end{matrix}\right) = \frac{1}{4\sqrt{z}}\int_0^z \frac{\operatorname{shi}\left(2\sqrt{x}\right)}{x}\, dx.$$

11. $${}_2F_3\left(\begin{matrix} \frac{1}{2},\, 1;\; z \\ \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2} \end{matrix}\right) = \frac{\pi}{8\sqrt{z}}\int_0^z \frac{1}{x} \mathbf{L}_0(2\sqrt{x})\, dx.$$

12. $${}_2F_3\left(\begin{matrix} \frac{1}{2},\, 1;\; z \\ \frac{3}{2},\, \frac{3}{2},\, 2 \end{matrix}\right) = \frac{2\sqrt{z}\operatorname{shi}\left(2\sqrt{z}\right) - \cosh(2\sqrt{z}) + 1}{2z}.$$

13. $${}_2F_3\left(\begin{matrix} \frac{1}{2},\, 1;\; z \\ \frac{3}{2},\, 2,\, 2 \end{matrix}\right) = \frac{1}{z}[1 + (4z-1)\, I_0(2\sqrt{z}) - 2\sqrt{z}\, I_1(2\sqrt{z})]$$
$$+ 2\pi[I_0(2\sqrt{z})\, \mathbf{L}_1(2\sqrt{z}) - I_1(2\sqrt{z})\, \mathbf{L}_0(2\sqrt{z})].$$

14. $${}_2F_3\left(\begin{matrix}1,\,1;\ z\\ 2,\,2,\,2\end{matrix}\right) = \frac{1}{z}\int_0^z \frac{I_0(2\sqrt{x})-1}{x}\,dx.$$

15. $${}_2F_3\left(\begin{matrix}1,\,1;\ z\\ \frac{3}{2},\,\frac{3}{2},\,\frac{3}{2}\end{matrix}\right) = \frac{\pi}{8\sqrt{z}}\int_0^z \frac{1}{\sqrt{x(z-x)}}\,\mathbf{L}_0(2\sqrt{x})\,dx.$$

16. $${}_2F_3\left(\begin{matrix}1,\,1;\ z\\ \frac{3}{2},\,\frac{3}{2},\,2\end{matrix}\right) = \frac{1}{4\sqrt{z}}\int_0^z \frac{\cosh(2\sqrt{x})-1}{x\sqrt{z-x}}\,dx.$$

17. $${}_2F_3\left(\begin{matrix}1,\,1;\ z\\ \frac{3}{2},\,2,\,2\end{matrix}\right) = \frac{\operatorname{chi}(2\sqrt{z})-\ln(2\sqrt{z})-\mathbf{C}}{z}.$$

8.1.14. The hypergeometric function ${}_3F_0(a_1, a_2, a_3;\ z)$

1. $${}_3F_0\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ z\end{matrix}\right) = n!\sqrt{\pi}\,z^{(2n-1)/4}\left[I_{-n-1/2}\left(\frac{2}{\sqrt{z}}\right) - \mathbf{L}_{n+1/2}\left(\frac{2}{\sqrt{z}}\right)\right].$$

2. $${}_3F_0\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ -z\end{matrix}\right) = n!\sqrt{\pi}\,z^{(2n-1)/4}\left[\mathbf{H}_{n+1/2}\left(\frac{2}{\sqrt{z}}\right) - Y_{n+1/2}\left(\frac{2}{\sqrt{z}}\right)\right].$$

8.1.15. The hypergeometric function ${}_5F_0(a_1, a_2, \ldots, a_5;\ z)$

1. $${}_5F_0\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,\frac{1}{4},\,\frac{3}{4},\,1\\ z\end{matrix}\right) = 2^{-n-1/2}n!\sqrt{\pi}\,z^{(2n-1)/8}$$
$$\times\left[\mathbf{H}_{n+1/2}\left(4z^{-1/4}\right) - \mathbf{L}_{n+1/2}\left(4z^{-1/4}\right) + I_{-n-1/2}\left(4z^{-1/4}\right) - Y_{n+1/2}\left(4z^{-1/4}\right)\right].$$

2. $${}_5F_0\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2},\,\frac{3}{4},\,1,\,\frac{5}{4}\\ z\end{matrix}\right) = n!\sqrt{\pi}\left(\frac{z}{16}\right)^{(2n-3)/8}$$
$$\times\left[\mathbf{H}_{n+3/2}\left(4z^{-1/4}\right) + \mathbf{L}_{n+3/2}\left(4z^{-1/4}\right) - Y_{n+3/2}\left(4z^{-1/4}\right) - I_{-n-3/2}\left(4z^{-1/4}\right)\right].$$

8.1.16. The hypergeometric function ${}_4F_1(a_1, \ldots, a_4;\ b_1;\ z)$

1. $${}_4F_1\left(\begin{matrix}-n,\,n+1,\,a,\,1-a\\ \frac{1}{2};\ z\end{matrix}\right)$$
$$= \frac{1}{\sqrt{z}}\left[I_{a+n}\left(\frac{1}{\sqrt{z}}\right)K_{a-n-1}\left(\frac{1}{\sqrt{z}}\right) + I_{a-n-1}\left(\frac{1}{\sqrt{z}}\right)K_{a+n}\left(\frac{1}{\sqrt{z}}\right)\right].$$

2. $${}_4F_1\left(\begin{matrix}-n,\,n+1,\,a,\,1-a\\ \frac{1}{2};\ -z\end{matrix}\right)$$
$$= (-1)^n\frac{\pi}{2\sqrt{z}}\left[J_{a+n}\left(\frac{1}{\sqrt{z}}\right)Y_{a-n-1}\left(\frac{1}{\sqrt{z}}\right) - J_{a-n-1}\left(\frac{1}{\sqrt{z}}\right)Y_{a+n}\left(\frac{1}{\sqrt{z}}\right)\right].$$

3. $${}_4F_1\left(\begin{matrix}-n,\ \frac{1}{4}-n,\ \frac{1}{2}-n,\ \frac{3}{4}-n\\ \frac{1}{2}-2n;\ z\end{matrix}\right)=\left(-\frac{z}{64}\right)^n H_{2n}\left(\sqrt[4]{\frac{4}{z}}\right)H_{2n}\left(i\sqrt[4]{\frac{4}{z}}\right).$$

4. $${}_4F_1\left(\begin{matrix}-n,\ -\frac{1}{2}-n,\ -\frac{1}{4}-n,\ \frac{1}{4}-n\\ -\frac{1}{2}-2n;\ z\end{matrix}\right)$$
$$=(-1)^{n+1}i\left(\frac{z}{64}\right)^{n+1/2}H_{2n+1}\left(\sqrt[4]{\frac{4}{z}}\right)H_{2n+1}\left(i\sqrt[4]{\frac{4}{z}}\right).$$

8.1.17. The hypergeometric function ${}_6F_1(a_1, \ldots, a_6;\ b_1;\ z)$

1. $${}_6F_1\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{n+1}{2},\ \frac{n}{2}+1,\ \frac{1}{4},\ \frac{3}{4}\\ \frac{1}{2};\ -z\end{matrix}\right)$$
$$=\frac{\pi}{\sqrt{2}\,z^{1/4}}\Big[\operatorname{bei}^2_{n+1/2}\big(2z^{-1/4}\big)-2\operatorname{ber}_{n+1/2}\big(2z^{-1/4}\big)\operatorname{bei}_{n+1/2}\big(2z^{-1/4}\big)$$
$$-\operatorname{ber}^2_{n+1/2}\big(2z^{-1/4}\big)+\operatorname{bei}^2_{-n-1/2}\big(2z^{-1/4}\big)$$
$$-2\operatorname{ber}_{-n-1/2}\big(2z^{-1/4}\big)\operatorname{bei}_{-n-1/2}\big(2z^{-1/4}\big)-\operatorname{ber}^2_{-n-1/2}\big(2z^{-1/4}\big)\Big].$$

2. $${}_6F_1\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2},\ \frac{n+3}{2},\ \frac{n}{2}+2,\ \frac{3}{4},\ \frac{5}{4}\\ \frac{3}{2};\ -z\end{matrix}\right)=-\frac{4\sqrt{2}\,\pi}{(n+1)(n+2)z^{3/4}}$$
$$\times\Big[\operatorname{bei}^2_{n+3/2}\big(2z^{-1/4}\big)+2\operatorname{ber}_{n+3/2}\big(2z^{-1/4}\big)\operatorname{bei}_{n+3/2}\big(2z^{-1/4}\big)$$
$$-\operatorname{ber}^2_{n+3/2}\big(2z^{-1/4}\big)+\operatorname{bei}^2_{-n-3/2}\big(2z^{-1/4}\big)$$
$$+2\operatorname{ber}_{-n-3/2}\big(2z^{-1/4}\big)\operatorname{bei}_{-n-3/2}\big(2z^{-1/4}\big)-\operatorname{ber}^2_{-n-3/2}\big(2z^{-1/4}\big)\Big].$$

8.1.18. The hypergeometric function ${}_8F_3(a_1, \ldots, a_8;\ b_1, b_2, b_3;\ z)$

1. $${}_8F_3\left(\begin{matrix}-\frac{n}{4},\ \frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4},\ \frac{n+1}{4},\ \frac{n+2}{4},\ \frac{n+3}{4},\ \frac{n}{4}+1\\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4};\ -z\end{matrix}\right)=\frac{\sqrt{\pi}}{z^{1/8}}$$
$$\times\left\{\left[\sin\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right.\right.$$
$$\left.+\cos\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right]\operatorname{ber}_{-n-1/2}\big(2z^{-1/4}\big)$$
$$+\left[\cos\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right.$$
$$\left.-\sin\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right]\operatorname{bei}_{-n-1/2}\big(2z^{-1/4}\big)$$
$$-\left[\cos\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\right.$$
$$\left.+\sin\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\right]\operatorname{ber}_{n+1/2}\big(2z^{-1/4}\big)$$

$$+\left[\sin\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\right.$$
$$\left.\left.-\cos\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\operatorname{bei}_{n+1/2}\left(2z^{-1/4}\right)\right]\right\}.$$

2. $${}_8F_3\left(\begin{matrix}-\frac{n}{4},\ \frac{1-n}{4},\ \frac{2-n}{4},\ \frac{3-n}{4},\ \frac{n+5}{4},\ \frac{n+6}{4},\ \frac{n+7}{4},\ \frac{n}{4}+2\\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2};\ -z\end{matrix}\right)$$
$$=\frac{32\sqrt{\pi}}{(n+1)(n+2)(n+3)(n+4)z^{5/8}}$$
$$\times\left\{\left[\sin\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right.\right.$$
$$\left.-\cos\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right]\operatorname{ber}_{-n-5/2}\left(2z^{-1/4}\right)$$
$$+\left[\cos\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right.$$
$$\left.+\sin\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}-\sqrt{2}\,z^{-1/4}\right)\right]\operatorname{bei}_{-n-5/2}\left(2z^{-1/4}\right)$$
$$+\left[\cos\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\right.$$
$$\left.-\sin\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\right]\operatorname{ber}_{n+5/2}\left(2z^{-1/4}\right)$$
$$-\left[\sin\frac{3\pi}{8}\sinh\left(\sqrt{2}\,z^{-1/4}\right)\cos\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\right.$$
$$\left.\left.+\cos\frac{3\pi}{8}\cosh\left(\sqrt{2}\,z^{-1/4}\right)\sin\left(\frac{n\pi}{2}+\sqrt{2}\,z^{-1/4}\right)\operatorname{bei}_{n+5/2}\left(2z^{-1/4}\right)\right]\right\}.$$

8.1.19. The hypergeometric function ${}_0F_3(b_1, b_2, b_3;\ z)$

1. $${}_0F_3\left(\begin{matrix}z\\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\end{matrix}\right)$$
$$=z^{1/4}\operatorname{ber}_1(2^{3/2}z^{1/4})\left[2\ker_0(2^{3/2}z^{1/4})-2\operatorname{kei}_0(2^{3/2}z^{1/4})-\pi\operatorname{bei}_0(2^{3/2}z^{1/4})\right]$$
$$+z^{1/4}\operatorname{bei}_1(2^{3/2}z^{1/4})\left[2\ker_0(2^{3/2}z^{1/4})+2\operatorname{kei}_0(2^{3/2}z^{1/4})+\pi\operatorname{ber}_0(2^{3/2}z^{1/4})\right]$$
$$+z^{1/4}\operatorname{bei}_0(2^{3/2}z^{1/4})\left[2\ker_1(2^{3/2}z^{1/4})-2\operatorname{kei}_1(2^{3/2}z^{1/4})-\pi\operatorname{bei}_1(2^{3/2}z^{1/4})\right]$$
$$+z^{1/4}\operatorname{ber}_0(2^{3/2}z^{1/4})\left[2\ker_1(2^{3/2}z^{1/4})+2\operatorname{kei}_1(2^{3/2}z^{1/4})+\pi\operatorname{ber}_1(2^{3/2}z^{1/4})\right].$$

2. $${}_0F_3\left(\begin{matrix}-z\\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\end{matrix}\right)$$
$$=-\left(\frac{z}{4}\right)^{1/4}\left[2J_1\left(2^{3/2}z^{1/4}\right)K_0\left(2^{3/2}z^{1/4}\right)-2J_0\left(2^{3/2}z^{1/4}\right)K_1\left(2^{3/2}z^{1/4}\right)\right.$$
$$\left.+\pi Y_0\left(2^{3/2}z^{1/4}\right)I_1\left(2^{3/2}z^{1/4}\right)+\pi Y_1\left(2^{3/2}z^{1/4}\right)I_0\left(2^{3/2}z^{1/4}\right)\right].$$

3. $${}_0F_3\left(\begin{matrix}-z\\ \frac{1}{2},\ \frac{1}{2},\ \frac{3}{2}\end{matrix}\right)$$
$$=\frac{1}{2}\left[2J_1\left(2^{3/2}z^{1/4}\right)K_1\left(2^{3/2}z^{1/4}\right)-\pi Y_1\left(2^{3/2}z^{1/4}\right)I_1\left(2^{3/2}z^{1/4}\right)\right].$$

4. $${}_0F_3\left(\begin{matrix} -z \\ \frac{1}{2},\, \frac{5}{6},\, \frac{7}{6} \end{matrix}\right) = \frac{2^{-4/3}\pi}{3^{2/3}z^{1/6}}\left[\mathrm{Ai}\left(-2^{1/3}3^{2/3}z^{1/6}\right)\mathrm{Bi}\left(2^{1/3}3^{2/3}z^{1/6}\right) - \mathrm{Ai}\left(2^{1/3}3^{2/3}z^{1/6}\right)\mathrm{Bi}\left(-2^{1/3}3^{2/3}z^{1/6}\right)\right].$$

5. $${}_0F_3\left(\begin{matrix} -z \\ \frac{2}{3},\, \frac{7}{6},\, \frac{4}{3} \end{matrix}\right) = \frac{3^{1/3}\Gamma^2\left(\frac{4}{3}\right)}{2^{8/3}z^{1/3}}\left[\mathrm{Bi}\left(2^{1/3}3^{2/3}z^{1/6}\right) - \sqrt{3}\,\mathrm{Ai}\left(2^{1/3}3^{2/3}z^{1/6}\right)\right] \times\left[\sqrt{3}\,\mathrm{Ai}\left(-2^{1/3}3^{2/3}z^{1/6}\right) - \mathrm{Bi}\left(-2^{1/3}3^{2/3}z^{1/6}\right)\right].$$

6. $${}_0F_3\left(\begin{matrix} z \\ \frac{4}{3},\, \frac{3}{2},\, \frac{5}{3} \end{matrix}\right) = \frac{4\pi}{3^{5/2}z^{1/2}}\left[\mathrm{ber}_{-1/3}(2^{3/2}z^{1/4})\,\mathrm{ber}_{1/3}(2^{3/2}z^{1/4}) + \mathrm{bei}_{-1/3}(2^{3/2}z^{1/4})\,\mathrm{bei}_{1/3}(2^{3/2}z^{1/4})\right].$$

7. $${}_0F_3\left(\begin{matrix} -z \\ \frac{4}{3},\, \frac{3}{2},\, \frac{5}{3} \end{matrix}\right) = \frac{2^{-1/3}\pi}{3^{13/6}z^{2/3}}\left[3\,\mathrm{Ai}\left(2^{1/3}3^{2/3}z^{1/6}\right)\mathrm{Ai}\left(-2^{1/3}3^{2/3}z^{1/6}\right) - \mathrm{Bi}\left(2^{1/3}3^{2/3}z^{1/6}\right)\mathrm{Bi}\left(-2^{1/3}3^{2/3}z^{1/6}\right)\right].$$

8. $${}_0F_3\left(\begin{matrix} -z \\ \frac{3}{2},\, \frac{3}{2},\, \frac{3}{2} \end{matrix}\right) = \frac{1}{8\sqrt{z}}\left[2J_0\left(2^{3/2}z^{1/4}\right)K_0\left(2^{3/2}z^{1/4}\right) + \pi Y_0\left(2^{3/2}z^{1/4}\right)I_0\left(2^{3/2}z^{1/4}\right)\right].$$

8.1.20. The hypergeometric function ${}_0F_7(b_1, \ldots, b_7;\, z)$

1. $${}_0F_7\left(\begin{matrix} -z \\ \frac{1}{8},\, \frac{1}{4},\, \frac{3}{8},\, \frac{1}{2},\, \frac{5}{8},\, \frac{3}{4},\, \frac{7}{8} \end{matrix}\right) = \sinh\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right)\sinh\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right) \times\sin\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right)\sin\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right) + \cosh\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right)\cosh\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right) \times\cos\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right)\cos\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right).$$

2. $${}_0F_7\left(\begin{matrix} -z \\ \frac{5}{8},\, \frac{3}{4},\, \frac{7}{8},\, \frac{9}{8},\, \frac{5}{4},\, \frac{11}{8},\, \frac{3}{2} \end{matrix}\right) = \frac{3}{512\sqrt{z}} \times\left\{\sinh\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right)\cosh\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right) \times\sin\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right)\cos\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right) - \sinh\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right)\cosh\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right) \times\sin\left(4\sqrt{2}\,z^{1/8}\sin\frac{\pi}{8}\right)\cos\left(4\sqrt{2}\,z^{1/8}\cos\frac{\pi}{8}\right)\right\}.$$

8.1.21. The hypergeometric function ${}_2F_5(a_1, a_2;\ b_1, \ldots, b_5;\ z)$

1. $${}_2F_5\left(\begin{matrix} a,\ a+\frac{1}{2};\ z \\ a+\frac{1}{4},\ a+\frac{3}{4},\ 2a-\frac{1}{2},\ 2a,\ \frac{1}{2}\end{matrix}\right) = \frac{1}{2}\Gamma\left(2a-\frac{1}{2}\right)\Gamma\left(2a+\frac{1}{2}\right) z^{1/2-a}$$
$$\times \left[J_{2a-3/2}\left(2z^{1/4}\right) J_{2a-1/2}\left(2z^{1/4}\right) + I_{2a-3/2}\left(2z^{1/4}\right) I_{2a-1/2}\left(2z^{1/4}\right)\right].$$

2. $${}_2F_5\left(\begin{matrix} a,\ a+\frac{1}{2};\ -z \\ a+\frac{1}{4},\ a+\frac{3}{4},\ 2a-\frac{1}{2},\ 2a,\ \frac{1}{2}\end{matrix}\right) = \Gamma\left(2a-\frac{1}{2}\right)\Gamma\left(2a+\frac{1}{2}\right) z^{1/2-a}$$
$$\times \left\{\sin(3a\pi)\left[\operatorname{bei}_{2a-3/2}\left(2z^{1/4}\right)\operatorname{bei}_{2a-1/2}\left(2z^{1/4}\right) - \operatorname{ber}_{2a-3/2}\left(2z^{1/4}\right)\operatorname{ber}_{2a-1/2}\left(2z^{1/4}\right)\right]\right.$$
$$\left. + \cos(3a\pi)\left[\operatorname{ber}_{2a-3/2}\left(2z^{1/4}\right)\operatorname{bei}_{2a-1/2}\left(2z^{1/4}\right) + \operatorname{bei}_{2a-3/2}\left(2z^{1/4}\right)\operatorname{ber}_{2a-1/2}\left(2z^{1/4}\right)\right]\right\}$$ [[81], (13)].

3. $${}_2F_5\left(\begin{matrix} a,\ a+\frac{1}{2};\ z \\ a+\frac{1}{4},\ a+\frac{3}{4},\ 2a-1,\ 2a-\frac{1}{2},\ \frac{3}{2}\end{matrix}\right) = \frac{1}{4(2a-1)}\Gamma\left(2a-\frac{1}{2}\right)\Gamma\left(2a+\frac{1}{2}\right) z^{1/2-a}$$
$$\times \left[I_{2a-5/2}\left(2z^{1/4}\right) I_{2a-3/2}\left(2z^{1/4}\right) - J_{2a-5/2}\left(2z^{1/4}\right) J_{2a-3/2}\left(2z^{1/4}\right)\right].$$

4. $${}_2F_5\left(\begin{matrix} a,\ a+\frac{1}{2};\ -z \\ a+\frac{1}{4},\ a+\frac{3}{4},\ 2a-1,\ 2a-\frac{1}{2},\ \frac{3}{2}\end{matrix}\right) = \frac{1}{2(2a-1)}\Gamma\left(2a-\frac{1}{2}\right)\Gamma\left(2a+\frac{1}{2}\right) z^{1/2-a}$$
$$\times \left\{\sin(3a\pi)\left[\operatorname{ber}_{2a-5/2}\left(2z^{1/4}\right)\operatorname{ber}_{2a-3/2}\left(2z^{1/4}\right) - \operatorname{bei}_{2a-5/2}\left(2z^{1/4}\right)\operatorname{bei}_{2a-3/2}\left(2z^{1/4}\right)\right]\right.$$
$$\left. - \cos(3a\pi)\left[\operatorname{ber}_{2a-5/2}\left(2z^{1/4}\right)\operatorname{bei}_{2a-3/2}\left(2z^{1/4}\right) + \operatorname{bei}_{2a-5/2}\left(2z^{1/4}\right)\operatorname{ber}_{2a-3/2}\left(2z^{1/4}\right)\right]\right\}$$ [[81], (14)].

5. $${}_2F_5\left(\begin{matrix} \frac{1}{4},\ \frac{3}{4};\ -z \\ \frac{1}{2}-a,\ 1-a,\ a+\frac{1}{2},\ a+1,\ \frac{1}{2}\end{matrix}\right) = 2a\pi\csc(2a\pi)$$
$$\times \left[\operatorname{ber}_{2a}\left(2z^{1/4}\right)\operatorname{ber}_{-2a}\left(2z^{1/4}\right) - \operatorname{bei}_{2a}\left(2z^{1/4}\right)\operatorname{bei}_{-2a}\left(2z^{1/4}\right)\right].$$

6. $${}_2F_5\left(\begin{matrix} \frac{3}{4},\ \frac{5}{4};\ -z \\ \frac{3}{2}-a,\ 1-a,\ a+1,\ a+\frac{3}{2},\ \frac{3}{2}\end{matrix}\right) = a\left(1-4a^2\right)\pi\csc(2a\pi) z^{-1/2}$$
$$\times \left[\operatorname{ber}_{2a}\left(2z^{1/4}\right)\operatorname{bei}_{-2a}\left(2z^{1/4}\right) + \operatorname{ber}_{-2a}\left(2z^{1/4}\right)\operatorname{bei}_{2a}\left(2z^{1/4}\right)\right].$$

8.1.22. The hypergeometric function ${}_4F_7(a_1, \ldots, a_4;\; b_1, \ldots, b_7;\; z)$

1. $${}_4F_7\left(\begin{matrix} a,\, a+\frac{1}{4},\, a+\frac{1}{2},\, a+\frac{3}{4};\; z \\ 2a,\, 2a+\frac{1}{2},\, b,\, b+\frac{1}{2},\, 2a-b+\frac{1}{2},\, 2a-b+1,\, \frac{1}{2} \end{matrix}\right) = \frac{1}{2}\Gamma(2b)\,\Gamma(4a-2b+1)\, z^{1/4-a}$$
$$\times \left[J_{2b-1}\left(2z^{1/4}\right) J_{4a-2b}\left(2z^{1/4}\right) + I_{2b-1}\left(2z^{1/4}\right) I_{4a-2b}\left(2z^{1/4}\right)\right] \quad \text{[[81], (11)].}$$

2. $${}_4F_7\left(\begin{matrix} a,\, a+\frac{1}{4},\, a+\frac{1}{2},\, a+\frac{3}{4};\; -z \\ 2a,\, 2a+\frac{1}{2},\, b,\, b+\frac{1}{2},\, 2a-b+\frac{1}{2},\, 2a-b+1,\, \frac{1}{2} \end{matrix}\right) = \Gamma(2b)\,\Gamma(4a-2b+1)\, z^{1/4-a}$$
$$\times \left\{\cos\frac{3(4a-1)\pi}{4}\left[\operatorname{ber}_{2b-1}\left(2z^{1/4}\right) \operatorname{ber}_{4a-2b}\left(2z^{1/4}\right) - \operatorname{bei}_{2b-1}\left(2z^{1/4}\right) \operatorname{bei}_{4a-2b}\left(2z^{1/4}\right)\right]\right.$$
$$\left. + \sin\frac{3(4a-1)\pi}{4}\left[\operatorname{ber}_{2b-1}\left(2z^{1/4}\right) \operatorname{bei}_{4a-2b}\left(2z^{1/4}\right) + \operatorname{bei}_{2b-1}\left(2z^{1/4}\right) \operatorname{ber}_{4a-2b}\left(2z^{1/4}\right)\right]\right\} \quad \text{[[81], (11)].}$$

3. $${}_4F_7\left(\begin{matrix} a,\, a+\frac{1}{4},\, a+\frac{1}{2},\, a+\frac{3}{4};\; z \\ 2a,\, 2a-\frac{1}{2},\, b,\, b+\frac{1}{2},\, 2a-b+\frac{1}{2},\, 2a-b+1,\, \frac{3}{2} \end{matrix}\right) = \frac{\Gamma(2b)\,\Gamma(4a-2b+1)}{2(4a-1)}\, z^{1/4-a}$$
$$\times \left[I_{2b-2}\left(2z^{1/4}\right) I_{4a-2b-1}\left(2z^{1/4}\right) - J_{2b-2}\left(2z^{1/4}\right) J_{4a-2b-1}\left(2z^{1/4}\right)\right]$$
[[81], (12)].

4. $${}_4F_7\left(\begin{matrix} a,\, a+\frac{1}{4},\, a+\frac{1}{2},\, a+\frac{3}{4};\; -z \\ 2a,\, 2a-\frac{1}{2},\, b,\, b+\frac{1}{2},\, 2a-b+\frac{1}{2},\, 2a-b+1,\, \frac{3}{2} \end{matrix}\right) = \frac{\Gamma(2b)\,\Gamma(4a-2b+1)}{4a-1}\, z^{1/4-a}$$
$$\times \left\{\sin\frac{3(4a-3)\pi}{4}\left[\operatorname{bei}_{2b-2}\left(2z^{1/4}\right) \operatorname{bei}_{4a-2b-1}\left(2z^{1/4}\right) - \operatorname{ber}_{2b-2}\left(2z^{1/4}\right) \operatorname{ber}_{4a-2b-1}\left(2z^{1/4}\right)\right]\right.$$
$$\left. + \cos\frac{3(4a-3)\pi}{4}\left[\operatorname{ber}_{2b-2}\left(2z^{1/4}\right) \operatorname{bei}_{4a-2b-1}\left(2z^{1/4}\right) + \operatorname{bei}_{2b-2}\left(2z^{1/4}\right) \operatorname{ber}_{4a-2b-1}\left(2z^{1/4}\right)\right]\right\} \quad \text{[[81], (12)].}$$

8.1.23. The generalized hypergeometric function $_pF_q((a_p);\ (b_q);\ z)$

Notation: $K_{n-1} = k_1 + k_2 + \ldots + k_{n-1}$.

1. $${}_pF_q\left(\begin{matrix}(a_{p-n}),\ (a_n);\ z\\ (b_{q-n}),\ (a_n+m_n)\end{matrix}\right) = \frac{1}{\prod\limits_{i=0}^{n} \mathrm{B}\,(a_i,\ m_i)} \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_n=0}^{m_n-1} \prod_{i=0}^{n} \frac{(-1)^{k_i}}{a_i+k_i} \binom{m_i-1}{k_i}$$
$$\times\ {}_pF_q\left(\begin{matrix}(a_{p-n}),\ (a_n)+k_i;\ z\\ (b_{q-n}),\ (a_n)+k_i+1\end{matrix}\right) \quad [m_i = 1,\ 2,\ \ldots].$$

2. $${}_{p+1}F_q\left(\begin{matrix}(a_p),\ 1\\ (b_q);\ z\end{matrix}\right) = z^{-1}\frac{\prod\limits_{j=0}^{q}(b_j-1)}{\prod\limits_{i=0}^{p}(a_i-1)}\left[{}_{p+1}F_q\left(\begin{matrix}(a_p)-1,\ 1\\ (b_q)-1;\ z\end{matrix}\right)-1\right].$$

3. $${}_{p+1}F_p\left(\begin{matrix}(a)_{p+1}\,;\ z\\ (b)_p\end{matrix}\right) = \frac{\prod\limits_{j=1}^{p}\Gamma(b_j)}{\prod\limits_{h=1}^{p+1}\Gamma(a_h)}\sum_{k=1}^{p+1}\left(e^{\pi i}z^{-1}\right)^{a_k}$$
$$\times\ \frac{\Gamma(a_k)}{\prod\limits_{j=1}^{p}\Gamma(b_h-a_k)}\prod_{\substack{h=1\\ h\neq k}}^{p+1}\Gamma(a_h-a_k)_{p+1}\ F_p\left(\begin{matrix}1+a_k-(b)_p\,,\ a_k;\ z\\ 1+a_k-(a)'_{p+1}\end{matrix}\right)$$
$$\left[(a)'_{p+1} = (a_1,\ a_2,\ \ldots,\ a_{k-1},\ a_{k+1},\ \ldots,\ a_{p+1})\,;\ 0 < \arg z < 2\pi\right].$$

4. $${}_pF_q\left(\begin{matrix}(a_p);\ z_1+z_2+\ldots+z_n\\ (b_q)\end{matrix}\right) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\left(\prod_{i=1}^{n-1}\frac{z_i^{k_i}}{k_i!}\right)\frac{\prod(a_p)_{K_{n-1}}}{\prod(b_q)_{K_{n-1}}}\,{}_pF_q\left(\begin{matrix}(a_p)+K_{n-1};\ z_n\\ (b_q)+K_{n-1}\end{matrix}\right)$$
$$\left[\begin{matrix}K_{n-1} = k_1+k_2+\ldots+k_{n-1};\ |z_1|+|z_2|+\ldots+|z_n| < 1,\\ \text{if } p = q+1 \text{ and } a_i \neq 0,\ -1,\ -2,\ \ldots \text{ for } i = 1,\ \ldots,\ n\end{matrix}\right].$$

5. $${}_{p+1}F_q\left(\begin{matrix}-m,\ (a_p);\ z_1+z_2+\ldots+z_n\\ (b_q)\end{matrix}\right) = m!\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_{n-1}=0}^{n-K_{n-2}}\left(\prod_{i=1}^{n-1}\frac{(-z_i)^{k_i}}{k_i!}\right)\frac{\prod(a_p)_{K_{n-1}}}{\prod(b_q)_{K_{n-1}}}\frac{1}{(m-K_{n-1})!}$$
$$\times\ {}_pF_q\left(\begin{matrix}-m+K_{n-1},\ (a_p)+K_{n-1}\\ (b_q)+K_{n-1};\ z_n\end{matrix}\right).$$

6. $${}_pF_q\left(\begin{matrix}(a_p);\ z_1z_2\ldots z_n\\ (b_q)\end{matrix}\right)=e^{-z_1-z_2-\ldots-z_{n-1}}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\left(\prod_{i=1}^{n-1}\frac{z_i^{k_i}}{k_i!}\right)$$
$$\times\ {}_{p+n-1}F_q\left(\begin{matrix}-k_1,\ -k_2,\ \ldots,\ -k_{n-1},\ (a_p)\\ (b_q);\ (-1)^{n-1}z_n\end{matrix}\right).$$

7. $${}_{p+1}F_q\left(\begin{matrix}-m,\ (a_p);\ z_1z_2\ldots z_n\\ (b_q)\end{matrix}\right)=m!\,(z_1z_2\ldots z_{n-1})^m$$
$$\times\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_{n-1}=0}^{n-k_1-\ldots-k_{n-1}}\sum_{k_{n-1}=0}^{n-K_{n-2}}\left(\prod_{i=1}^{n-1}\frac{z_i^{-K_i}(1-z_i)^{k_i}}{k_i!}\right)\frac{1}{(m-K_{n-1})!}$$
$$\times\ {}_pF_q\left(\begin{matrix}-m+K_{n-1},\ (a_p)\\ (b_q);\ z_n\end{matrix}\right).$$

8. $${}_{p+2}F_{p+1}\left(\begin{matrix}-n,\ a,\ 2,\ \ldots,\ 2;\ 1\\ b,\ 1,\ \ldots,\ 1\end{matrix}\right)=\frac{n!}{(b)_n}\sum_{k=0}^{p}(-1)^k\frac{(a)_k(b-a)_{n-k}}{(n-k)!}\mathcal{S}_{p+1}^{k+1}.$$

9. $${}_pF_p\left(\begin{matrix}a+1,\ a+1,\ \ldots,\ a+1\\ a,\ a,\ \ldots,\ a;\ \ z\end{matrix}\right)=\sum_{k=0}^{p}\binom{p}{k}\frac{1}{a^k}\,\mathrm{D}_t^k\left[e^{ze^t}\right]\Big|_{t=0}.$$

8.2. The Meijer Function $G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)$

8.2.1. General formulas

1. $$G_{p+1,q+2}^{m+1,n+1}\left(z\,\middle|\,\begin{matrix}a,\ (a_p)\\ a,\ (b_q),\ b\end{matrix}\right)=(-1)^{a-b}G_{p,q+1}^{m+1,n}\left(z\,\middle|\,\begin{matrix}(a_p)\\ b,\ (b_q)\end{matrix}\right)$$
$$-(-1)^{a-b}\sum_{k=1}^{a-b}\operatorname*{Res}_{s=-a+k}\left[\frac{\Gamma(b+s)\prod_{i=1}^{m}\Gamma(b_i+s)\prod_{j=1}^{n}\Gamma(1-a_i-s)}{\prod_{i=n+1}^{p}\Gamma(a_i+s)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)}z^{-s}\right]$$
$$\left[\begin{matrix}q\ge 1;\ 0\le n\le p\le q;\ 0\le m\le q;\ a-b=1,\,2,\,\ldots;\\ a_k-b\ne 1,\,2,\,\ldots\ \text{for}\ k=1,\,\ldots,\,n;\ [15],\ (2.1)\end{matrix}\right].$$

2. $$G_{p+1,q+2}^{m+1,n+1}\left(z\,\middle|\,\begin{matrix}a,\ (a_p)\\ a,\ (b_q),\ b\end{matrix}\right)=(-1)^{a-b}G_{p,q+1}^{m+1,n}\left(z\,\middle|\,\begin{matrix}(a_p)\\ b,\ (b_q)\end{matrix}\right)$$
$$\left[\begin{matrix}q\ge 1;\ 0\le n\le p\le q;\ 0\le m\le q;\ a-b=0,\,-1,\,-2,\,\ldots;\\ a_k-b\ne 1,\,2,\,\ldots\ \text{for}\ k=1,\,\ldots,\,n;\ [15],\ (2.2)\end{matrix}\right].$$

3. $$G_{p+2,q+1}^{m,n+1}\left(z\,\middle|\,\begin{matrix}a,\ (a_p),\ b\\ (b_q),\ b\end{matrix}\right)=(-1)^{a-b}G_{p,q+1}^{m+1,n}\left(z\,\middle|\,\begin{matrix}(a_p),\ a\\ (b_q)\end{matrix}\right)$$
$$\left[\begin{matrix}q\ge 1,\ 0\le n\le p\le q,\ 0\le m\le q;\ a-b=0,\,-1,\,-2,\,\ldots;\\ a_k-b\ne 1,\,2,\,\ldots\ \text{for}\ k=1,\,\ldots,\,n;\ [15],\ (2.4)\end{matrix}\right].$$

4. $$G_{2p,\,2q+2}^{2m+1,\,2n}\left(z\,\middle|\,\begin{matrix}(a_p),\,(a_p)+\frac{1}{2}\\0,\,(b_q),\,(b_q)+\frac{1}{2},\,\frac{1}{2}\end{matrix}\right) = 2^{-\tau}\pi^{-\rho}G_{p,\,q+1}^{m+1,\,n}\left(2^{\delta}e^{\pi i/2}z^{1/2}\,\middle|\,\begin{matrix}(2a_p)\\0,\,(2b_q)\end{matrix}\right) + 2^{-\tau}\pi^{-\rho}G_{p,\,q+1}^{m+1,\,n}\left(2^{\delta}e^{-\pi i/2}z^{1/2}\,\middle|\,\begin{matrix}(2a_p)\\0,\,(2b_q)\end{matrix}\right)$$
$$\left[\begin{matrix}\delta=q-p+1,\ \rho=(p+q+1)/2-m-n,\\ \tau=2\sum_{j=1}^{q}b_j-2\sum_{i=1}^{a}a_i+p-m-n+1;\ [25],\ (21)\end{matrix}\right].$$

5. $$G_{2p,\,2q+2}^{2m+1,\,2n}\left(z\,\middle|\,\begin{matrix}(a_p),\,(a_p)+\frac{1}{2}\\1/2,\,(b_q),\,(b_q)+\frac{1}{2},\,0\end{matrix}\right) = 2^{-\tau}\pi^{-\rho}iG_{p,\,q+1}^{m+1,\,n}\left(2^{\delta}e^{\pi i/2}z^{1/2}\,\middle|\,\begin{matrix}(2a_p)\\0,\,(2b_q)\end{matrix}\right) - 2^{-\tau}\pi^{-\rho}iG_{p,\,q+1}^{m+1,\,n}\left(2^{\delta}e^{-\pi i/2}z^{1/2}\,\middle|\,\begin{matrix}(2a_p)\\0,\,(2b_q)\end{matrix}\right)$$
$$\left[\begin{matrix}\delta=q-p+1;\ \rho=(p+q+1)/2-m-n;\\ \tau=2\sum_{j=1}^{q}b_j-2\sum_{i=1}^{a}a_i+p-m-n+1;\ [25],\ (21)\end{matrix}\right].$$

6. $$G_{p,\,q+1}^{m+1,\,n}\left(z\,\middle|\,\begin{matrix}(a_p)\\0,\,(b_q)\end{matrix}\right) = 2^{\sigma}\pi^{\rho}G_{2p,\,2q+2}^{2m+1,\,2n}\left(\frac{e^{\pm\pi i}z^2}{4^{\delta}}\,\middle|\,\begin{matrix}(a_p)/2,\,(a_p)/2+1/2\\0,\,(b_q)/2,\,(b_q)/2+1/2,\,1/2\end{matrix}\right) - 2^{\sigma-\delta}\pi^{\rho}zG_{2p,\,2q+2}^{2m+1,\,2n}\left(\frac{e^{\pm\pi i}z^2}{4^{\delta}}\,\middle|\,\begin{matrix}(a_p)/2,\,(a_p)/2-1/2\\0,\,(b_q)/2,\,(b_q)/2+1/2,\,-1/2\end{matrix}\right)$$
$$\left[\begin{matrix}\delta=q-p+1;\ \rho=(p+q+1)/2-m-n;\\ \sigma=\sum_{j=1}^{q}b_j-\sum_{i=1}^{a}a_i+p-m-n;\ [20],\ (19)\end{matrix}\right].$$

8.2.2. Various Meijer G functions

1. $$G_{22}^{12}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\,\frac{3}{4}\\\frac{1}{2},\,0\end{matrix}\right) = \frac{2}{(z+1)^{1/4}} \times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)-\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{z}{z+1}}}\right)\right] \quad [|\arg(z+1)|<\pi].$$

2. $$G_{48}^{12}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4},\,\frac{1}{6},\,\frac{5}{6}\\ 0,\,\frac{1}{6},\,\frac{5}{6},\,-a,\,\frac{1}{2},\,a,\,\frac{1}{2}-a,\,a+\frac{1}{2}\end{matrix}\right)$$
$$=\frac{1}{2\sqrt{2}\,\pi^{5/2}}\left[\operatorname{ber}_{2a}\left(2\sqrt[4]{z}\right)\operatorname{ber}_{-2a}\left(2\sqrt[4]{z}\right)-\operatorname{bei}_{2a}\left(2\sqrt[4]{z}\right)\operatorname{bei}_{-2a}\left(2\sqrt[4]{z}\right)\right].$$

3. $$G_{33}^{12}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\,\frac{3}{4},\,\frac{5}{4}\\ 0,\,0,\,\frac{5}{4}\end{matrix}\right)=-\frac{\sqrt{2}}{\pi^2}\Gamma^2\left(\frac{1}{4}\right)\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-z}}\right)\qquad[|z|<1].$$

4. $$G_{13}^{20}\left(z\,\middle|\,\begin{matrix}\frac{3}{2}\\ 1,\,1,\,0\end{matrix}\right)=-\sqrt{\pi}\,z[J_0(\sqrt{z})\,Y_0(\sqrt{z})+J_1(\sqrt{z})\,Y_1(\sqrt{z})].$$

5. $$G_{22}^{20}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\,\frac{3}{4}\\ 0,\,0\end{matrix}\right)=\frac{i\Gamma^2\left(\frac{1}{4}\right)}{\pi^2 z^{1/4}}\theta(1-|z|)\left[\delta_{\operatorname{Im}z,0}+\operatorname{sgn}(\operatorname{Im}z)\right]$$
$$\times\left[\mathbf{K}\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{z-1}{z}}}\right)-\mathbf{K}\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{z-1}{z}}}\right)\right]\qquad[z\notin(-1,\,0)].$$

6. $$G_{15}^{20}\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\,0,\,\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)=\frac{4}{\pi^2}\operatorname{ker}\left(4z^{1/4}\right)-\frac{1}{\pi}\operatorname{bei}\left(2z^{1/4}\right).$$

7. $$G_{24}^{20}\left(z\,\middle|\,\begin{matrix}a,\,\frac{1}{2}\\ -a-\frac{1}{2},\,a+\frac{1}{2},\,a,\,\frac{1}{2}\end{matrix}\right)=\frac{1}{\pi}\cos(a\pi)\,I_{-2a-1}(2\sqrt{z})$$
$$+\frac{2}{\pi^2}\sin(a\pi)\,K_{2a+1}(2\sqrt{z}).$$

8. $$G_{24}^{20}\left(z\,\middle|\,\begin{matrix}a,\,0\\ -a-\frac{1}{2},\,a+\frac{1}{2},\,a,\,0\end{matrix}\right)=\frac{1}{\pi}\sin(a\pi)\,I_{-2a-1}(2\sqrt{z})$$
$$+\frac{2}{\pi^2}\cos(a\pi)\,K_{-2a-1}(2\sqrt{z}).$$

9. $$G_{24}^{20}\left(z\,\middle|\,\begin{matrix}a,\,\frac{1}{2}\\ \frac{1}{2}-a,\,a-\frac{1}{2},\,a,\,0\end{matrix}\right)=\frac{1}{\pi^{5/2}}\left[\pi^2\cos(a\pi)\,I_{1/2-a}^2(\sqrt{z})\right.$$
$$\left.+2\pi I_{1/2-a}(\sqrt{z})\,K_{1/2-a}(\sqrt{z})+2\cos(a\pi)\,K_{1/2-a}^2(\sqrt{z})\right].$$

10. $$G_{26}^{20}\left(z\,\middle|\,\begin{matrix}a,\,a+\frac{1}{2}\\ \frac{1}{2},\,a+\frac{1}{4},\,0,\,-a-\frac{1}{4},\,a,\,a+\frac{1}{2}\end{matrix}\right)$$
$$=\frac{1}{\pi^{5/2}}\operatorname{ber}_{2a+1/2}\left(2^{3/2}z^{1/4}\right)\left[\pi\sin\frac{(12a+1)\pi}{4}\operatorname{ber}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)\right.$$
$$+\pi\cos\frac{(12a+3)\pi}{4}\operatorname{bei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$

$$+ 2\sin\frac{(4a+1)\pi}{4}\,\mathrm{ker}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$- 2\cos\frac{(4a+1)\pi}{4}\,\mathrm{kei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)\Big]$$
$$+ \frac{1}{\pi^{5/2}}\,\mathrm{bei}_{2a+1/2}\left(2^{3/2}z^{1/4}\right)\Big[\pi\sin\frac{(12a+1)\pi}{4}\,\mathrm{bei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$- \pi\cos\frac{(12a+1)\pi}{4}\,\mathrm{ber}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$+ 2\cos\frac{(4a+1)\pi}{4}\,\mathrm{ker}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$+ 2\sin\frac{(4a+1)\pi}{4}\,\mathrm{kei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)\Big].$$

11. $$G_{26}^{20}\left(z\,\middle|\,\begin{matrix}a,\ a+\frac{1}{2}\\ 0,\ a+\frac{1}{4},\ \frac{1}{2},\ -a-\frac{1}{4},\ a,\ a+\frac{1}{2}\end{matrix}\right)$$
$$= \frac{1}{\pi^{5/2}}\,\mathrm{ber}_{2a+1/2}\left(2^{3/2}z^{1/4}\right)\Big[\pi\cos\frac{(12a+1)\pi}{4}\,\mathrm{ber}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$- \pi\sin\frac{(12a+1)\pi}{4}\,\mathrm{bei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$- 2\cos\frac{(4a+1)\pi}{4}\,\mathrm{ker}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$- 2\sin\frac{(4a+1)\pi}{4}\,\mathrm{kei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)\Big]$$
$$+ \frac{1}{\pi^{5/2}}\,\mathrm{bei}_{2a+1/2}\left(2^{3/2}z^{1/4}\right)\Big[\pi\sin\frac{(12a+1)\pi}{4}\,\mathrm{ber}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$+ \pi\cos\frac{(12a+1)\pi}{4}\,\mathrm{bei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$+ 2\sin\frac{(4a+1)\pi}{4}\,\mathrm{ker}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)$$
$$- 2\cos\frac{(4a+1)\pi}{4}\,\mathrm{kei}_{-2a-1/2}\left(2^{3/2}z^{1/4}\right)\Big].$$

12. $$G_{35}^{20}\left(z\,\middle|\,\begin{matrix}\frac{1}{6},\ \frac{1}{2},\ \frac{5}{6}\\ 0,\ \frac{1}{3},\ \frac{1}{6},\ \frac{1}{2},\ \frac{5}{6}\end{matrix}\right) = \frac{3^{1/6}}{2\pi^2}\,\mathrm{Ai}\left(-\sqrt[3]{9z}\right).$$

13. $$G_{35}^{20}\left(z\,\middle|\,\begin{matrix}\frac{1}{6},\ \frac{1}{2},\ \frac{5}{6}\\ 0,\ \frac{2}{3},\ \frac{1}{6},\ \frac{1}{3},\ \frac{1}{2}\end{matrix}\right) = \frac{1}{\pi^{3/2}}\left(\frac{3}{2}\right)^{1/3}\mathrm{Ai}\left(-\sqrt[3]{\frac{9z}{4}}\right)\mathrm{Bi}\left(-\sqrt[3]{\frac{9z}{4}}\right).$$

14. $$G_{37}^{20}\left(z\,\middle|\,\begin{matrix}0,\ 0,\ \frac{1}{2}\\ a,\ a+\frac{1}{2},\ 0,\ 0,\ \frac{1}{2},\ \frac{1}{2}-a,\ -a\end{matrix}\right)$$
$$= \frac{\sin(4a\pi)}{4\pi^3}\left[\mathrm{bei}_{4a}\left(4z^{1/4}\right) - \tan(2a\pi)\,\mathrm{ber}_{4a}\left(4z^{1/4}\right)\right].$$

15. $G_{13}^{21}\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\,n,\,0\end{matrix}\right) = (-1)^n\frac{\sqrt{\pi}}{2^{n-1}}\,z^{n/2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}I_{n-k}(\sqrt{z})\,K_k(\sqrt{z}).$

16. $G_{13}^{21}\left(z\,\middle|\,\begin{matrix}1\\ \frac{1}{2}-n,\,n+\frac{1}{2},\,0\end{matrix}\right)$
$$= 2(-1)^n\pi\sqrt{z}\,[K_0(2\sqrt{z})\,\mathbf{L}_{-1}(2\sqrt{z}) + K_1(2\sqrt{z})\,\mathbf{L}_0(2\sqrt{z})] + 4(-1)^n\sum_{k=0}^{n-1}(-1)^k K_{2k+1}(2\sqrt{z}).$$

17. $G_{13}^{21}\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\ \frac{1}{2},\,\frac{1}{2},\,0\end{matrix}\right) = \frac{1}{\sqrt{\pi}}\left[e^{-2\sqrt{z}}\,\mathrm{Ei}\,(2\sqrt{z}) - e^{2\sqrt{z}}\,\mathrm{Ei}\,(-2\sqrt{z})\right] \quad [z>0].$

18. $G_{13}^{21}\left(z\,\middle|\,\begin{matrix}\frac{5}{6}\\ 0,\,\frac{2}{3},\,\frac{1}{3}\end{matrix}\right) = 2^{5/3}3^{1/3}\pi^{3/2}\,\mathrm{Ai}\left(\sqrt[3]{\frac{9z}{4}}\right)\mathrm{Bi}\left(\sqrt[3]{\frac{9z}{4}}\right).$

19. $G_{13}^{21}\left(z\,\middle|\,\begin{matrix}\frac{3}{2}\\ \frac{1}{3},\,\frac{2}{3},\,0\end{matrix}\right)$
$$= 8\pi^{3/2}\left\{-3\left(\frac{3z^2}{2}\right)^{1/3}\mathrm{Ai}\left(\left(\frac{9z}{4}\right)^{1/3}\right)\mathrm{Bi}\left(\left(\frac{9z}{4}\right)^{1/3}\right) + \mathrm{Ai}'\left(\left(\frac{9z}{4}\right)^{1/3}\right)\right.$$
$$\left.\times\,\mathrm{Bi}\left(\left(\frac{9z}{4}\right)^{1/3}\right) + (18z)^{1/3}\,\mathrm{Ai}'\left(\left(\frac{9z}{4}\right)^{1/3}\right)\mathrm{Bi}'\left(\left(\frac{9z}{4}\right)^{1/3}\right)\right\} + 4\sqrt{\pi}.$$

20. $G_{13}^{21}\left(z\,\middle|\,\begin{matrix}\frac{3}{2}\\ \frac{1}{3},\,\frac{2}{3},\,0\end{matrix}\right)$
$$= 8\pi^{3/2}\left\{-3\left(\frac{3z^2}{2}\right)^{1/3}\mathrm{Ai}\left(\left(\frac{9z}{4}\right)^{1/3}\right)\mathrm{Bi}\left(\left(\frac{9z}{4}\right)^{1/3}\right) + \mathrm{Ai}\left(\left(\frac{9z}{4}\right)^{1/3}\right)\right.$$
$$\left.\times\,\mathrm{Bi}'\left(\left(\frac{9z}{4}\right)^{1/3}\right) + (18z)^{1/3}\,\mathrm{Ai}'\left(\left(\frac{9z}{4}\right)^{1/3}\right)\mathrm{Bi}'\left(\left(\frac{9z}{4}\right)^{1/3}\right)\right\} - 4\sqrt{\pi}.$$

21. $G_{23}^{21}\left(z\,\middle|\,\begin{matrix}1,\,n+\frac{3}{2}\\ 1,\,2n+1,\,0\end{matrix}\right) = \frac{z^{n+1}e^{-z/2}}{(2n+1)\sqrt{\pi}}\left[K_n\left(\frac{z}{2}\right) - K_{n+1}\left(\frac{z}{2}\right)\right] + \frac{2^{2n+1}n!}{(2n+1)\sqrt{\pi}}.$

22. $G_{23}^{21}\left(z\,\middle|\,\begin{matrix}a+1,\,a\\ 0,\,b,\,a\end{matrix}\right) = \frac{e^{-z}}{a}\left[z\Psi\left(\begin{matrix}a-b+1\\ 2-b;\;z\end{matrix}\right) - \Psi\left(\begin{matrix}a-b\\ 1-b;\;z\end{matrix}\right)\right].$

23. $G_{23}^{21}\left(z \,\middle|\, \begin{matrix}1,\, a\\ 1,\, b,\, 0\end{matrix}\right)$

$$= \frac{\Gamma(b)}{\Gamma(a)} + \frac{e^{-z}}{a-1}\left[(b-a)z\Psi\left(\begin{matrix}a-b+1\\ 3-b;\ z\end{matrix}\right) - (b-1)\Psi\left(\begin{matrix}a-b;\ z\\ 2-b\end{matrix}\right)\right].$$

24. $G_{24}^{21}\left(z \,\middle|\, \begin{matrix}1,\, -n\\ \frac{1}{2}-n,\, n+\frac{1}{2},\, 0,\, -n\end{matrix}\right) = \pi\sqrt{z}Y_1(2\sqrt{z})\,\mathbf{H}_0(2\sqrt{z})$

$$-2\sqrt{z}Y_2(2\sqrt{z}) + \pi\left[\sqrt{z}Y_2(2\sqrt{z}) - Y_1(2\sqrt{z})\right]\mathbf{H}_1(2\sqrt{z}) - 2\sum_{k=1}^{n-1} Y_{2k+1}(2\sqrt{z})$$

$[n \geq 1]$.

25. $G_{33}^{21}\left(z \,\middle|\, \begin{matrix}1,\, 0,\, 1\\ \frac{1}{2},\, \frac{1}{2},\, 0\end{matrix}\right) = -\theta(1-z)\,\frac{2\sqrt{z}}{\pi}\,\mathbf{K}\left(\sqrt{1-z}\right).$

26. $G_{35}^{21}\left(z \,\middle|\, \begin{matrix}\frac{1}{2},\, 0,\, 1\\ \frac{1}{2},\, \frac{1}{2},\, 0,\, 0,\, 1\end{matrix}\right)$

$$= \frac{2}{\pi^{5/2}}\left[\sinh(2\sqrt{z})\,\mathrm{chi}\,(2\sqrt{z}) - \cosh(2\sqrt{z})\,\mathrm{shi}\,(2\sqrt{z})\right].$$

27. $G_{22}^{22}\left(z \,\middle|\, \begin{matrix}\frac{3}{4},\, \frac{3}{4}\\ 0,\, 0\end{matrix}\right)$

$$= 2\Gamma^2\left(\frac{1}{4}\right) z^{-1/4}\left[\mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{1}{z}}}\right) + \mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{1}{z}}}\right)\right].$$

28. $G_{23}^{22}\left(z \,\middle|\, \begin{matrix}1,\, n+\frac{3}{2}\\ 1,\, 2n+1,\, 0\end{matrix}\right) = (-1)^n\,\frac{\sqrt{\pi}\,z^{n+1}e^{z/2}}{2n+1}\left[K_n\left(\frac{z}{2}\right) + K_{n+1}\left(\frac{z}{2}\right)\right]$

$$-(-1)^n\,\frac{2^{2n+1}n!\,\sqrt{\pi}}{2n+1}.$$

29. $G_{23}^{22}\left(z \,\middle|\, \begin{matrix}1,\, n+1\\ n+1,\, n+1,\, 0\end{matrix}\right)$

$$= \left[\frac{z^n}{n} + (-1)^n n!\,(\mathbf{C} + \ln z) + n!\sum_{k=1}^{n-1}(-1)^k\,\frac{z^{n-k}}{(n-k)!\,(n-k)}\right]$$

$$-e^z\left[z^n + n!\sum_{k=1}^{n}(-1)^k\,\frac{z^{n-k}}{(n-k)!}\right]\mathrm{Ei}\,(-z) \qquad [z>0].$$

30. $G_{23}^{22}\left(z \,\middle|\, \begin{matrix}1,\, a\\ 1,\, b,\, 0\end{matrix}\right) = \Gamma(1-a)\Gamma(b) - \Gamma(1-a)\Gamma(b-a+1)$

$$\times\left[(b+z-1)\Psi\left(\begin{matrix}2-a;\ z\\ 2-b\end{matrix}\right) - (a-2)z\Psi\left(\begin{matrix}3-a;\ z\\ 3-b\end{matrix}\right)\right].$$

31. $G_{34}^{22}\left(z \,\middle|\, \begin{matrix} 1,\, 1,\, \frac{1}{2} \\ 1,\, 1,\, 0,\, \frac{1}{2} \end{matrix}\right) = \frac{1}{\pi}\left[\mathrm{C} + \ln z - e^{-z}\,\mathrm{Ei}\,(z)\right] \qquad [|\arg z| < \pi].$

32. $$G_{46}^{22}\left(z \,\middle|\, \begin{matrix} \frac{1}{4},\, \frac{3}{4},\, 0,\, a \\ -a-\frac{1}{2},\, a+\frac{1}{2},\, 0,\, -a,\, a,\, a+1 \end{matrix}\right) = \frac{\sqrt{2}}{\pi^2}\cosh\,(\sqrt{z})\,[\pi \sin\,(a\pi)\, I_{-2a-1}(\sqrt{z}) + 2\cos\,(a\pi)\, K_{2a+1}(\sqrt{z})].$$

33. $$G_{46}^{22}\left(z \,\middle|\, \begin{matrix} \frac{1}{4},\, \frac{3}{4},\, 0,\, a \\ \frac{1}{2}-a,\, a+\frac{1}{2},\, 0,\, -a,\, a,\, a \end{matrix}\right) = \frac{\sqrt{2}}{\pi^2}\sinh\,(\sqrt{z})\,[2\cos\,(a\pi)\, K_{2a}(\sqrt{z}) - \pi \sin\,(a\pi)\, I_{-2a}(\sqrt{z})].$$

34. $$G_{46}^{22}\left(z \,\middle|\, \begin{matrix} \frac{1}{4},\, \frac{3}{4},\, \frac{1}{2},\, a \\ \frac{1}{2}-a,\, a+\frac{1}{2},\, \frac{1}{2},\, -a,\, a,\, a \end{matrix}\right) = \frac{\sqrt{2}}{\pi^2}\sinh\,(\sqrt{z})\,[2\sin\,(a\pi)\, K_{2a}(\sqrt{z}) - \pi \cos\,(a\pi)\, I_{-2a}(\sqrt{z})].$$

35. $$G_{46}^{22}\left(z \,\middle|\, \begin{matrix} \frac{1}{4},\, \frac{3}{4},\, \frac{1}{2},\, a \\ -a-\frac{1}{2},\, a+\frac{1}{2},\, \frac{1}{2},\, -a,\, a,\, a+1 \end{matrix}\right) = \frac{\sqrt{2}}{\pi^2}\cosh\,(\sqrt{z})\,[\pi \cos\,(a\pi)\, I_{-2a-1}(\sqrt{z}) + 2\sin\,(a\pi)\, K_{2a+1}(\sqrt{z})].$$

36. $G_{33}^{23}\left(z \,\middle|\, \begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ 0,\, 0,\, 0 \end{matrix}\right) = 4\sqrt{\pi}\,\mathbf{K}\left(\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1-z}}\right)\mathbf{K}\left(\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1-z}}\right).$

37. $G_{04}^{30}\left(z \,\middle|\, 0,\, \frac{1}{6},\, \frac{1}{3},\, \frac{2}{3}\right) = 2^{5/3}3^{1/3}\pi^{3/2}\,\mathrm{Ai}\left(\sqrt[6]{324z}\right)\mathrm{Bi}\left(-\sqrt[6]{324z}\right).$

38. $G_{13}^{30}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0,\, 0,\, n \end{matrix}\right) = \frac{2^{1-n}}{\sqrt{\pi}}\, z^{n/2} \sum_{k=0}^{n} \binom{n}{k} K_k(\sqrt{z})\, K_{n-k}(\sqrt{z}).$

39. $$G_{13}^{30}\left(z \,\middle|\, \begin{matrix} 1 \\ -n,\, 0,\, n+1 \end{matrix}\right) = -2(-1)^n K_0(2\sqrt{z}) + 4(-1)^n \sum_{k=1}^{n} (-1)^k K_{2k}(2\sqrt{z}).$$

40. $$G_{15}^{30}\left(z\,\middle|\,\begin{matrix}\frac{1}{4}\\0,\ \frac{1}{2},\ 1,\ 0,\ \frac{1}{4}\end{matrix}\right) = \frac{2^{3/2}}{\sqrt{\pi}}\,z^{1/4}$$
$$\times\left[\operatorname{ber}_1\left(2^{3/2}z^{1/4}\right)\operatorname{kei}_0\left(2^{3/2}z^{1/4}\right) - \operatorname{ber}_0\left(2^{3/2}z^{1/4}\right)\operatorname{ker}_1\left(2^{3/2}z^{1/4}\right)\right.$$
$$\left.- \operatorname{bei}_0\left(2^{3/2}z^{1/4}\right)\operatorname{kei}_1\left(2^{3/2}z^{1/4}\right) - \operatorname{bei}_1\left(2^{3/2}z^{1/4}\right)\operatorname{ker}_0\left(2^{3/2}z^{1/4}\right)\right].$$

41. $$G_{15}^{30}\left(z\,\middle|\,\begin{matrix}\frac{3}{4}\\0,\ \frac{1}{2},\ 1,\ 0,\ \frac{3}{4}\end{matrix}\right) = \frac{2^{3/2}z^{1/4}}{\sqrt{\pi}}$$
$$\times\left[\operatorname{bei}_0\left(2^{3/2}z^{1/4}\right)\operatorname{ker}_1\left(2^{3/2}z^{1/4}\right) - \operatorname{ber}_1\left(2^{3/2}z^{1/4}\right)\operatorname{ker}_0\left(2^{3/2}z^{1/4}\right)\right.$$
$$\left.- \operatorname{ber}_0\left(2^{3/2}z^{1/4}\right)\operatorname{kei}_1\left(2^{3/2}z^{1/4}\right) - \operatorname{bei}_1\left(2^{3/2}z^{1/4}\right)\operatorname{kei}_0\left(2^{3/2}z^{1/4}\right)\right].$$

42. $$G_{23}^{30}\left(z\,\middle|\,\begin{matrix}1,\ 1\\0,\ 0,\ a\end{matrix}\right) = \Gamma(a)\left[\psi(a) - \ln z\right] + \frac{z^a}{a^2}\,{}_2F_2\left(\begin{matrix}a,\ a;\ -z\\a+1,\ a+1\end{matrix}\right).$$

43. $$G_{15}^{30}\left(z\,\middle|\,\begin{matrix}\frac{1}{4}\\0,\ \frac{1}{2},\ 1,\ 0,\ \frac{3}{4}\end{matrix}\right) = \frac{2^{3/2}}{\sqrt{\pi}}\,z^{1/4}$$
$$\times\left[\operatorname{ber}_1\left(2^{3/2}z^{1/4}\right)\operatorname{ker}\left(2^{3/2}z^{1/4}\right) - \operatorname{bei}\left(2^{3/2}z^{1/4}\right)\operatorname{ker}_1\left(2^{3/2}z^{1/4}\right)\right.$$
$$\left.- \operatorname{ber}\left(2^{3/2}z^{1/4}\right)\operatorname{kei}_1\left(2^{3/2}z^{1/4}\right) - \operatorname{bei}_1\left(2^{3/2}z^{1/4}\right)\operatorname{kei}\left(2^{3/2}z^{1/4}\right)\right].$$

44. $$G_{15}^{30}\left(z\,\middle|\,\begin{matrix}\frac{3}{4}\\0,\ \frac{1}{2},\ 1,\ 0,\ \frac{3}{4}\end{matrix}\right) = \frac{2^{3/2}}{\sqrt{\pi}}\,z^{1/4}$$
$$\times\left[\operatorname{bei}\left(2^{3/2}z^{1/4}\right)\operatorname{ker}_1\left(2^{3/2}z^{1/4}\right) - \operatorname{ber}\left(2^{3/2}z^{1/4}\right)\operatorname{kei}_1\left(2^{3/2}z^{1/4}\right)\right.$$
$$\left.- \operatorname{ber}_1\left(2^{3/2}z^{1/4}\right)\operatorname{ker}\left(2^{3/2}z^{1/4}\right) - \operatorname{bei}_1\left(2^{3/2}z^{1/4}\right)\operatorname{kei}\left(2^{3/2}z^{1/4}\right)\right].$$

45. $$G_{26}^{30}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\ \frac{5}{4}\\\frac{1}{2},\ \frac{1}{2},\ 1,\ 0,\ \frac{1}{2},\ 1\end{matrix}\right)$$
$$= \frac{2^{5/2}z^{1/2}}{\pi^{3/2}}\left\{\operatorname{ber}_0\left(2z^{1/4}\right)\operatorname{ker}_0\left(2z^{1/4}\right) + \operatorname{ber}_1\left(2z^{1/4}\right)\operatorname{ker}_1\left(2z^{1/4}\right)\right.$$
$$- \operatorname{bei}_0\left(2z^{1/4}\right)\left[\pi\operatorname{ber}_0\left(2z^{1/4}\right) + \operatorname{kei}_0\left(2z^{1/4}\right)\right]$$
$$\left.- \operatorname{bei}_1\left(2z^{1/4}\right)\left[\pi\operatorname{ber}_1\left(2z^{1/4}\right) + \operatorname{kei}_1\left(2z^{1/4}\right)\right]\right\}.$$

46. $$G_{35}^{30}\left(z\,\middle|\,\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\0,\ -a,\ a,\ \frac{1}{2},\ \frac{1}{2}\end{matrix}\right) = \frac{1}{\pi^{5/2}}\left\{2K_a^2(\sqrt{z}) - \pi^2 I_{-a}(\sqrt{z})\,I_a(\sqrt{z})\right.$$
$$\left.+ \pi\sin(a\pi)\left[I_a(\sqrt{z}) - I_{-a}(\sqrt{z})\right]K_a(\sqrt{z})\right\}.$$

47. $$G_{35}^{30}\left(z\,\middle|\,\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,a\\ 0,\,\frac{1}{2}-a,\,a-\frac{1}{2},\,\frac{1}{2},\,a\end{matrix}\right)=\frac{\sin(a\pi)}{\pi^{5/2}}\left[2K_{1/2-a}^2(\sqrt{z})-\pi^2I_{1/2-a}^2(\sqrt{z})\right]\quad[a\neq 0,1,2,\ldots].$$

48. $$G_{15}^{31}\left(z\,\middle|\,\begin{matrix}\frac{3}{4}\\ \frac{1}{4},\,\frac{3}{4},\,\frac{3}{4},\,0,\,\frac{1}{2}\end{matrix}\right)=-\sqrt{\frac{2}{\pi}}\,e^{-7\pi i/4}\Big[\frac{i\pi}{2}\sin\left(4e^{\pi i/4}z^{1/4}\right)$$
$$-\sin\left(4e^{\pi i/4}z^{1/4}\right)\operatorname{ci}\left(4e^{\pi i/4}z^{1/4}\right)+\sinh\left(4e^{\pi i/4}z^{1/4}\right)\operatorname{chi}\left(4e^{\pi i/4}z^{1/4}\right)$$
$$-\cosh\left(4e^{\pi i/4}z^{1/4}\right)\operatorname{shi}\left(4e^{\pi i/4}z^{1/4}\right)+\cos\left(4e^{\pi i/4}z^{1/4}\right)\operatorname{Si}\left(4e^{\pi i/4}z^{1/4}\right)\Big].$$

49. $$G_{24}^{31}\left(z\,\middle|\,\begin{matrix}1,\,\frac{3}{2}\\ 1,\,1,\,1,\,0\end{matrix}\right)=\frac{2z}{\sqrt{\pi}}\left[K_0^2(\sqrt{z})-K_1^2(\sqrt{z})\right]+\frac{2}{\sqrt{\pi}}.$$

50. $$G_{24}^{31}\left(z\,\middle|\,\begin{matrix}0,\,1\\ 0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)=\frac{2}{\sqrt{\pi}}\left[\operatorname{chi}^2(2\sqrt{z})-\operatorname{shi}^2(2\sqrt{z})\right].$$

51. $$G_{24}^{31}\left(z\,\middle|\,\begin{matrix}\frac{3}{2},\,\frac{1}{2}\\ 0,\,\frac{1}{3},\,\frac{2}{3},\,\frac{1}{2}\end{matrix}\right)=8\pi^{3/2}\left\{-\frac{3^{4/3}}{2^{1/3}}z^{2/3}\operatorname{Ai}^2\left(\left(\frac{9z}{4}\right)^{1/3}\right)\right.$$
$$\left.+\operatorname{Ai}\left(\left(\frac{9z}{4}\right)^{1/3}\right)\operatorname{Ai}'\left(\left(\frac{9z}{4}\right)^{1/3}\right)+(18z)^{1/3}\left[\operatorname{Ai}'\left(\left(\frac{9z}{4}\right)^{1/3}\right)\right]^2\right\}.$$

52. $$G_{26}^{32}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4}\\ 0,\,0,\,\frac{1}{2},\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)=4\sqrt{2\pi}\left[\operatorname{ber}\left(2z^{1/4}\right)\operatorname{ker}\left(2z^{1/4}\right)-\operatorname{bei}\left(2z^{1/4}\right)\operatorname{kei}\left(2z^{1/4}\right)\right].$$

53. $$G_{26}^{32}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4}\\ 0,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,\,\frac{1}{2}\end{matrix}\right)=-4\sqrt{2\pi}\left[\operatorname{ber}\left(2z^{1/4}\right)\operatorname{kei}\left(2z^{1/4}\right)+\operatorname{bei}\left(2z^{1/4}\right)\operatorname{ker}\left(2z^{1/4}\right)\right].$$

54. $$G_{33}^{32}\left(z\,\middle|\,\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2}\\ 0,\,0,\,0\end{matrix}\right)=2\sqrt{\pi}\left[\mathbf{K}^2\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-z}}\right)+\mathbf{K}^2\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-z}}\right)\right].$$

55. $$G_{24}^{40}\left(z\,\middle|\,\begin{matrix}\frac{1}{2},\,1\\ a+b,\,a-b,\,b-a,\,-a-b\end{matrix}\right)=\frac{2}{(a+b)\sqrt{\pi}}K_{2a}(\sqrt{z})\,K_{2b}(\sqrt{z})$$
$$+\frac{\sqrt{z}}{(a^2-b^2)\sqrt{\pi}}\left[K_{2a-1}(\sqrt{z})\,K_{2b}(\sqrt{z})-K_{2a}(\sqrt{z})\,K_{2b-1}(\sqrt{z})\right].$$

56. $$G_{35}^{40}\left(z\,\middle|\,\begin{matrix}0,\,\frac{1}{6},\,\frac{2}{3}\\ \frac{1}{3}-a,\,-a,\,a,\,a+\frac{1}{3},\,0\end{matrix}\right)$$
$$=\frac{3^{1/6}\sqrt{\pi}}{2^{2/3}}\left[e^{ia\pi}H_{2a}^{(1)}(\sqrt{z})\operatorname{Ai}\left(\frac{3^{2/3}e^{-i\pi/3}}{2^{2/3}}z^{1/3}\right)\right.$$
$$\left.+\,e^{-ia\pi}H_{2a}^{(2)}(\sqrt{z})\operatorname{Ai}\left(\frac{3^{2/3}e^{i\pi/3}}{2^{2/3}}z^{1/3}\right)\right].$$

57. $$G_{35}^{40}\left(z\,\middle|\,\begin{matrix}\frac{1}{6},\,\frac{1}{2},\,\frac{2}{3}\\ \frac{1}{3}-a,\,-a,\,a,\,a+\frac{1}{3},\,\frac{1}{2}\end{matrix}\right)$$
$$=\frac{3^{1/6}\sqrt{\pi}\,i}{2^{2/3}}\left[e^{ia\pi}H_{2a}^{(1)}(\sqrt{z})\operatorname{Ai}\left(\frac{3^{2/3}e^{-i\pi/3}}{2^{2/3}}z^{1/3}\right)\right.$$
$$\left.-\,e^{-ia\pi}H_{2a}^{(2)}(\sqrt{z})\operatorname{Ai}\left(\frac{3^{2/3}e^{i\pi/3}}{2^{2/3}}z^{1/3}\right)\right].$$

58. $$G_{37}^{40}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{1}{2},\,\frac{3}{4}\\ \frac{1}{2}-a,\,-a,\,a,\,a+\frac{1}{2},\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$
$$=\frac{\sqrt{2}}{\pi^{3/2}}\left\{\operatorname{ber}_{2a}\left(2z^{1/4}\right)\left[\cos(4a\pi)\operatorname{ker}_{-2a}\left(2z^{1/4}\right)+\sin(4a\pi)\operatorname{kei}_{-2a}\left(2z^{1/4}\right)\right]\right.$$
$$-\operatorname{bei}_{2a}\left(2z^{1/4}\right)\left[\pi\operatorname{ber}_{-2a}\left(2z^{1/4}\right)-\sin(4a\pi)\operatorname{ker}_{-2a}\left(2z^{1/4}\right)\right.$$
$$\left.+\cos(4a\pi)\operatorname{kei}_{-2a}\left(2z^{1/4}\right)\right]+\operatorname{ber}_{-2a}\left(2z^{1/4}\right)$$
$$\times\left[\cos(4a\pi)\operatorname{ker}_{2a}\left(2z^{1/4}\right)-\sin(4a\pi)\operatorname{kei}_{2a}\left(2z^{1/4}\right)\right]-\operatorname{bei}_{-2a}\left(2z^{1/4}\right)$$
$$\left.\times\left[\pi\operatorname{ber}_{2a}\left(2z^{1/4}\right)+\sin(4a\pi)\operatorname{ker}_{2a}\left(2z^{1/4}\right)+\cos(4a\pi)\operatorname{kei}_{2a}\left(2z^{1/4}\right)\right]\right\}.$$

59. $$G_{37}^{40}\left(z\,\middle|\,\begin{matrix}0,\,\frac{1}{4},\,\frac{3}{4}\\ \frac{1}{2}-a,\,-a,\,a,\,a+\frac{1}{2},\,0,\,0,\,\frac{1}{2}\end{matrix}\right)$$
$$=\frac{\sqrt{2}}{\pi^{3/2}}\left\{\operatorname{ber}_{2a}\left(2z^{1/4}\right)\left[\cos(4a\pi)\operatorname{kei}_{-2a}\left(2z^{1/4}\right)-\sin(4a\pi)\operatorname{ker}_{-2a}\left(2z^{1/4}\right)\right]\right.$$
$$+\operatorname{bei}_{2a}\left(2z^{1/4}\right)\left[\cos(4a\pi)\operatorname{ker}_{-2a}\left(2z^{1/4}\right)+\sin(4a\pi)\operatorname{kei}_{-2a}\left(2z^{1/4}\right)\right]$$
$$+\operatorname{ber}_{-2a}\left(2z^{1/4}\right)\left[\pi\operatorname{ber}_{2a}\left(2z^{1/4}\right)+\sin(4a\pi)\operatorname{ker}_{2a}\left(2z^{1/4}\right)\right.$$
$$\left.+\cos(4a\pi)\operatorname{kei}_{2a}\left(2z^{1/4}\right)\right]-\operatorname{bei}_{-2a}\left(2z^{1/4}\right)$$
$$\left.\times\left[\pi\operatorname{bei}_{2a}\left(2z^{1/4}\right)-\cos(4a\pi)\operatorname{ker}_{2a}\left(2z^{1/4}\right)+\sin(4a\pi)\operatorname{kei}_{2a}\left(2z^{1/4}\right)\right]\right\}.$$

60. $$G_{59}^{40}\left(z\,\middle|\,\begin{matrix}0,\,\frac{1}{2},\,\frac{1}{2},\,a,\,a+\frac{1}{2}\\ \frac{1}{2}-b,\,-b,\,b,\,b+\frac{1}{2},\,0,\,\frac{1}{2},\,\frac{1}{2},\,a,\,a+\frac{1}{2}\end{matrix}\right)$$
$$=\frac{1}{4\pi^3}\left\{\sin\left[2\left(a-b\right)\pi\right]\operatorname{ber}_{4b}\left(4z^{1/4}\right)+\sin\left[2\left(a+b\right)\pi\right]\operatorname{ber}_{-4b}\left(4z^{1/4}\right)\right.$$
$$\left.+\left(\sin\left[2\left(a-b\right)\pi\right]\operatorname{bei}_{4b}\left(4z^{1/4}\right)-\sin\left[2\left(a+b\right)\pi\right]\operatorname{bei}_{-4b}\left(4z^{1/4}\right)\right)\tan(2b\pi)\right\}.$$

61. $$G_{59}^{40}\left(z\,\middle|\,\begin{matrix}0,\,0,\,\frac{1}{2},\,a,\,a+\frac{1}{2}\\ \frac{1}{2}-b,\,-b,\,b,\,b+\frac{1}{2},\,0,\,0,\,\frac{1}{2},\,a,\,a+\frac{1}{2}\end{matrix}\right)$$
$$=\frac{1}{4\pi^3}\left\{\left(\sin\left[2\left(a-b\right)\pi\right]\operatorname{ber}_{4b}\left(4z^{1/4}\right)\right.\right.$$
$$\left.-\sin\left[2\left(a+b\right)\pi\right]\operatorname{ber}_{-4b}\left(4z^{1/4}\right)\right)\tan(2b\pi)$$
$$\left.-\sin\left[2\left(a-b\right)\pi\right]\operatorname{bei}_{4b}\left(4z^{1/4}\right)-\sin\left[2\left(a+b\right)\pi\right]\operatorname{bei}_{-4b}\left(4z^{1/4}\right)\right\}.$$

62. $$G_{24}^{41}\left(z\,\middle|\,\begin{matrix}0,\,1\\ 0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)=2\sqrt{\pi}\,[\operatorname{si}^2(2\sqrt{z})-\operatorname{ci}^2(2\sqrt{z})].$$

63. $$G_{46}^{42}\left(z\,\middle|\,\begin{matrix}1,\,1,\,\frac{1}{4},\,\frac{3}{4}\\ \frac{1}{2},\,\frac{1}{2},\,1,\,1,\,\frac{1}{4},\,\frac{3}{4}\end{matrix}\right)=\frac{1}{2\pi^2}G_{24}^{42}\left(z\,\middle|\,\begin{matrix}1,\,1\\ \frac{1}{2},\,\frac{1}{2},\,1,\,1\end{matrix}\right)-2\sqrt{z}\,K_0(2\sqrt{z}).$$

64. $$G_{6,10}^{43}\left(z\,\middle|\,\begin{matrix}0,\,\frac{1}{4},\,\frac{3}{4},\,\frac{1}{2},\,a,\,a+\frac{1}{2}\\ a-\frac{1}{4},\,a+\frac{1}{4},\,b,\,b+\frac{1}{2},\,\frac{1}{4}-a,\,\frac{3}{4}-a,\,a,\,a+\frac{1}{2},\,\frac{1}{2}-b,\,-b\end{matrix}\right)$$
$$=-\frac{\sqrt{2}}{\pi^{3/2}}\left\{\operatorname{ber}_{2a+2b-1/2}\left(2z^{1/4}\right)\left[\pi\cos\left(4b\pi\right)\operatorname{bei}_{2b-2a+1/2}\left(2z^{1/4}\right)\right.\right.$$
$$-\pi\sin\left(4a\pi\right)\operatorname{ber}_{2b-2a+1/2}\left(2z^{1/4}\right)$$
$$\left.+2\cos\left(4a\pi\right)\operatorname{ker}_{2b-2a+1/2}\left(2z^{1/4}\right)+2\sin\left(4a\pi\right)\operatorname{kei}_{2b-2a+1/2}\left(2z^{1/4}\right)\right]$$
$$\operatorname{bei}_{2a+2b-1/2}\left(2z^{1/4}\right)\left[\pi\cos\left(4b\pi\right)\operatorname{ber}_{2b-2a+1/2}\left(2z^{1/4}\right)\right.$$
$$+\pi\sin\left(4b\pi\right)\operatorname{bei}_{2b-2a+1/2}\left(2z^{1/4}\right)$$
$$\left.\left.+2\sin\left(4a\pi\right)\operatorname{ker}_{2b-2a+1/2}\left(2z^{1/4}\right)-2\cos\left(4a\pi\right)\operatorname{kei}_{2b-2a+1/2}\left(2z^{1/4}\right)\right]\right\}.$$

65. $$G_{6,10}^{43}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{1}{2},\,\frac{3}{4},\,0,\,a,\,a+\frac{1}{2}\\ a-\frac{1}{4},\,a+\frac{1}{4},\,b,\,b+\frac{1}{2},\,\frac{1}{4}-a,\,\frac{3}{4}-a,\,a,\,a+\frac{1}{2},\,\frac{1}{2}-b,\,-b\end{matrix}\right)$$
$$=\frac{\sqrt{2}}{\pi^{3/2}}\left\{\operatorname{ber}_{2a+2b-1/2}\left(2z^{1/4}\right)\left[\cos\left(4a\pi\right)\operatorname{kei}_{2b-2a+1/2}\left(2z^{1/4}\right)\right.\right.$$
$$-\sin\left(4a\pi\right)\operatorname{ker}_{2b-2a+1/2}\left(2z^{1/4}\right)\;-2\operatorname{bei}_{2a+2b-1/2}\left(2z^{1/4}\right)$$
$$\times\left[\cos\left(4a\pi\right)\operatorname{ker}_{2b-2a+1/2}\left(2z^{1/4}\right)+\sin\left(4a\pi\right)\operatorname{kei}_{2b-2a+1/2}\left(2z^{1/4}\right)\right.$$
$$+\pi\operatorname{ber}_{2b-2a+1/2}\left(2z^{1/4}\right)\left[\cos\left(4b\pi\right)\operatorname{ber}_{2a+2b-1/2}\left(2z^{1/4}\right)\right.$$

$$+\sin(4b\pi)\operatorname{bei}_{2a+2b-1/2}\left(2z^{1/4}\right)+\pi\operatorname{bei}_{2b-2a+1/2}\left(2z^{1/4}\right)$$
$$\times\left[\sin(4b\pi)\operatorname{ber}_{2a+2b-1/2}\left(2z^{1/4}\right)-\cos(4b\pi)\operatorname{bei}_{2a+2b-1/2}\left(2z^{1/4}\right)\right\}.$$

66. $$G_{48}^{50}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4},\,a,\,a+\frac{1}{2}\\ \frac{1}{2},\,\frac{1}{4}-a,\,\frac{3}{4}-a,\,a-\frac{1}{4},\,a+\frac{1}{4},\,0,\,a,\,a+\frac{1}{2}\end{matrix}\right)$$
$$=\frac{\sqrt{2}}{\pi^{5/2}}\left\{2\pi\operatorname{bei}_{1/2-2a}\left(2z^{1/4}\right)\left[\pi\cos(4a\pi)\operatorname{ber}_{1/2-2a}\left(2z^{1/4}\right)\right.\right.$$
$$\left.-\operatorname{kei}_{1/2-2a}\left(2z^{1/4}\right)\right]+2\operatorname{ker}_{1/2-2a}\left(2z^{1/4}\right)\left[\pi\operatorname{ber}_{1/2-2a}\left(2z^{1/4}\right)\right.$$
$$\left.-2\cos(4a\pi)\operatorname{kei}_{1/2-2a}\left(2z^{1/4}\right)\right]+\sin(4a\pi)\left[\pi^2\operatorname{ber}^2_{1/2-2a}\left(2z^{1/4}\right)\right.$$
$$\left.\left.-\pi^2\operatorname{bei}^2_{1/2-2a}\left(2z^{1/4}\right)+2\operatorname{ker}^2_{1/2-2a}\left(2z^{1/4}\right)-2\operatorname{kei}^2_{1/2-2a}\left(2z^{1/4}\right)\right]\right\}.$$

67. $$G_{26}^{52}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4}\\ 0,\,\frac{1}{2}-a,\,-a,\,a,\,a+\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$
$$=8\sqrt{2}\,\pi^{3/2}\sec(2a\pi)\left\{\cos(4a\pi)\left[\operatorname{ker}^2_{2a}\left(2z^{1/4}\right)-\operatorname{kei}^2_{2a}\left(2z^{1/4}\right)\right]\right.$$
$$-2\sin(4a\pi)\operatorname{ker}_{2a}\left(2z^{1/4}\right)\operatorname{kei}_{2a}\left(2z^{1/4}\right)$$
$$\left.-\pi\left[\operatorname{ber}_{2a}\left(2z^{1/4}\right)\operatorname{kei}_{2a}\left(2z^{1/4}\right)+\operatorname{bei}_{2a}\left(2z^{1/4}\right)\operatorname{ker}_{2a}\left(2z^{1/4}\right)\right]\right\}.$$

68. $$G_{26}^{52}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4}\\ \frac{1}{2},\,\frac{1}{2}-a,\,-a,\,a,\,a+\frac{1}{2},\,0\end{matrix}\right)$$
$$=8\sqrt{2}\,\pi^{3/2}\sec(2a\pi)\left\{\sin(4a\pi)\left[\operatorname{ker}^2_{2a}\left(2z^{1/4}\right)-\operatorname{kei}^2_{2a}\left(2z^{1/4}\right)\right]\right.$$
$$+2\cos(4a\pi)\operatorname{ker}_{2a}\left(2z^{1/4}\right)\operatorname{kei}_{2a}\left(2z^{1/4}\right)$$
$$\left.+\pi\left[\operatorname{ber}_{2a}\left(2z^{1/4}\right)\operatorname{ker}_{2a}\left(2z^{1/4}\right)-\operatorname{bei}_{2a}\left(2z^{1/4}\right)\operatorname{kei}_{2a}\left(2z^{1/4}\right)\right]\right\}.$$

69. $$G_{37}^{60}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4},\,a\\ 0,\,\frac{1}{2},\,\frac{1-a}{2},\,-\frac{a}{2},\,\frac{a}{2},\,\frac{a+1}{2},\,a\end{matrix}\right)$$
$$=-8\sqrt{\frac{2}{\pi}}\operatorname{ker}_a\left(2z^{1/4}\right)\operatorname{kei}_a\left(2z^{1/4}\right).$$

70. $$G_{37}^{60}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{3}{4},\,a\\ 0,\,\frac{1}{2},\,\frac{1-2a}{4},\,\frac{3-a}{4},\,\frac{2a+1}{4},\,\frac{2a+1}{4},\,a\end{matrix}\right)$$
$$=4\sqrt{\frac{2}{\pi}}\left[\operatorname{ker}^2_{a-1/2}\left(2z^{1/4}\right)-\operatorname{kei}^2_a\left(2z^{1/4}\right)\right].$$

71. $$G_{59}^{60}\left(z\,\middle|\,\begin{matrix}0,\,0,\,\frac{1}{4},\,\frac{1}{2},\,\frac{3}{4}\\ a,\,a+\frac{1}{2},\,\frac{1}{2}-b,\,-b,\,b,\,b+\frac{1}{2},\,0,\,\frac{1}{2}-a,\,-a\end{matrix}\right)$$
$$=\frac{\sqrt{2}}{\pi^{3/2}}\left\{\operatorname{ber}_{2a+2b}\left(2z^{1/4}\right)\left[\pi\cos(4a\pi)\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)\right.\right.$$
$$+\pi\sin(4a\pi)\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)-\sin(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)$$
$$\left.+\cos(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]+\operatorname{bei}_{2a+2b}\left(2z^{1/4}\right)$$
$$\times\left[\pi\sin(4a\pi)\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)-\pi\cos(4a\pi)\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)\right.$$
$$\left.+\cos(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)+\sin(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]$$
$$+\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)\left[\sin(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)+\cos(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)\right]$$
$$\left.+\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)\left[\cos(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)-\sin(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)\right]\right\}.$$

72. $$G_{59}^{60}\left(z\,\middle|\,\begin{matrix}0,\,\frac{1}{4},\,\frac{1}{2},\,\frac{1}{2},\,\frac{3}{4}\\ a,\,a+\frac{1}{2},\,\frac{1}{2}-b,\,-b,\,b,\,b+\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2}-a,\,-a\end{matrix}\right)$$
$$=\frac{\sqrt{2}}{\pi^{3/2}}\left\{\operatorname{ber}_{2a+2b}\left(2z^{1/4}\right)\left[\pi\sin(4a\pi)\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)\right.\right.$$
$$-\pi\cos(4a\pi)\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)+\cos(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)$$
$$\left.+\sin(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]-\operatorname{bei}_{2a+2b}\left(2z^{1/4}\right)$$
$$\times\left[\pi\cos(4a\pi)\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)+\pi\sin(4a\pi)\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)\right.$$
$$\left.-\sin(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)+\cos(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]$$
$$+\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)\left[\cos(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)-\sin(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)\right]$$
$$\left.-\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)\left[\sin(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)+\cos(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)\right]\right\}.$$

73. $$G_{48}^{63}\left(z\,\middle|\,\begin{matrix}0,\,\frac{1}{4},\,\frac{3}{4},\,\frac{1}{2}\\ a,\,a+\frac{1}{2},\,\frac{1}{2}-b,\,-b,\,b,\,b+\frac{1}{2},\,\frac{1}{2}-a,\,-a\end{matrix}\right)=$$
$$-8\sqrt{2}\,\pi^{5/2}\csc(4b\pi)\left\{\operatorname{ber}_{2a+2b}\left(2z^{1/4}\right)\right.$$
$$\times\left[\cos(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)+\sin(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]$$
$$+\operatorname{bei}_{2a+2b}\left(2z^{1/4}\right)\left[\sin(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)-\cos(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]$$
$$+\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)\left[\sin(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)-\cos(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)\right]$$
$$\left.+\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)\left[\sin(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)+\cos(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)\right]\right\}.$$

74. $$G_{48}^{63}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\,\frac{1}{2},\,\frac{3}{4},\,0\\ a,\,a+\frac{1}{2},\,\frac{1}{2}-b,\,-b,\,b,\,b+\frac{1}{2},\,\frac{1}{2}-a,\,-a\end{matrix}\right)$$
$$=8\sqrt{2}\,\pi^{5/2}\csc(4b\pi)\left\{\operatorname{ber}_{2a+2b}\left(2z^{1/4}\right)\right.$$
$$\times\left[\sin(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)-\cos(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]$$
$$-\operatorname{bei}_{2a+2b}\left(2z^{1/4}\right)\left[\cos(4b\pi)\operatorname{ker}_{2a-2b}\left(2z^{1/4}\right)+\sin(4b\pi)\operatorname{kei}_{2a-2b}\left(2z^{1/4}\right)\right]$$

$$+\operatorname{ber}_{2a-2b}\left(2z^{1/4}\right)\left[\sin(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)+\cos(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)\right]$$
$$+\operatorname{bei}_{2a-2b}\left(2z^{1/4}\right)\left[\cos(4b\pi)\operatorname{ker}_{2a+2b}\left(2z^{1/4}\right)-\sin(4b\pi)\operatorname{kei}_{2a+2b}\left(2z^{1/4}\right)\right]\Big\}.$$

75. $$G_{59}^{80}\left(z\,\middle|\,\begin{matrix}0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ a\\ \frac{1-a}{2},\ -\frac{a}{2},\ \frac{a}{2},\ \frac{a+1}{2},\ \frac{1}{2}-b,\ -b,\ b,\ b+\frac{1}{2},\ a\end{matrix}\right)=$$
$$-4\sqrt{\frac{2}{\pi}}\left[\operatorname{ker}_{a+2b}\left(2z^{1/4}\right)\operatorname{kei}_{a-2b}\left(2z^{1/4}\right)+\operatorname{kei}_{a+2b}\left(2z^{1/4}\right)\operatorname{ker}_{a-2b}\left(2z^{1/4}\right)\right].$$

76. $$G_{59}^{80}\left(z\,\middle|\,\begin{matrix}0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ a\\ \frac{1-2a}{4},\ \frac{3-2a}{4},\ \frac{2a-1}{4},\ \frac{2a+1}{4},\ \frac{1}{2}-b,\ -b,\ b,\ b+\frac{1}{2},\ a\end{matrix}\right)$$
$$=4\sqrt{\frac{2}{\pi}}\left[\operatorname{ker}_{a+2b-1/2}\left(2z^{1/4}\right)\operatorname{ker}_{a-2b-1/2}\left(2z^{1/4}\right)\right.$$
$$\left.-\operatorname{kei}_{a+2b-1/2}\left(2z^{1/4}\right)\operatorname{kei}_{a-2b-1/2}\left(2z^{1/4}\right)\right].$$

77. $$G_{59}^{80}\left(z\,\middle|\,\begin{matrix}\frac{1}{8},\ \frac{3}{8},\ \frac{5}{8},\ \frac{7}{8},\ a\\ \frac{1-2a}{4},\ \frac{3-2a}{4},\ \frac{1-a}{2},\ -\frac{a}{2},\ \frac{a}{2},\ \frac{a+1}{2},\ \frac{2a+1}{4},\ \frac{2a+3}{4},\ a\end{matrix}\right)$$
$$=2\sqrt{2}\,e^{-\sqrt{2}\,z^{1/4}}\left[\sin\left(\sqrt{2}\,z^{1/4}\right)\operatorname{ker}_{2a}\left(2z^{1/4}\right)-\cos\left(\sqrt{2}\,z^{1/4}\right)\operatorname{kei}_{2a}\left(2z^{1/4}\right)\right].$$

78. $$G_{59}^{80}\left(z\,\middle|\,\begin{matrix}\frac{1}{8},\ \frac{3}{8},\ \frac{5}{8},\ \frac{7}{8},\ a\\ \frac{1-2a}{4},\ \frac{3-2a}{4},\ \frac{1-a}{2},\ 1-\frac{a}{2},\ \frac{a}{2},\ \frac{a+1}{2},\ \frac{2a-1}{4},\ \frac{2a+1}{4},\ a\end{matrix}\right)$$
$$=2\sqrt{2}\,e^{-\sqrt{2}\,z^{1/4}}\left[\cos\left(\sqrt{2}\,z^{1/4}\right)\operatorname{ker}_{2a-1}\left(2z^{1/4}\right)\right.$$
$$\left.+\sin\left(\sqrt{2}\,z^{1/4}\right)\operatorname{kei}_{2a-1}\left(2z^{1/4}\right)\right].$$

79. $$G_{59}^{84}\left(z\,\middle|\,\begin{matrix}\frac{1}{8},\ \frac{3}{8},\ \frac{5}{8},\ \frac{7}{8},\ a\\ \frac{1-2a}{4},\ \frac{3-2a}{4},\ \frac{1-a}{2},\ -\frac{a}{2},\ \frac{a}{2},\ \frac{a+1}{2},\ \frac{2a+1}{4},\ \frac{2a+3}{4},\ a\end{matrix}\right)=$$
$$-16\sqrt{2}\,\pi^4 e^{\sqrt{2}\,z^{1/4}}\sec(2a\pi)\left[\sin\left(\sqrt{2}\,z^{1/4}\right)\operatorname{ker}_{2a}\left(2z^{1/4}\right)\right.$$
$$\left.+\cos\left(\sqrt{2}\,z^{1/4}\right)\operatorname{kei}_{2a}\left(2z^{1/4}\right)\right].$$

80. $$G_{59}^{84}\left(z\,\middle|\,\begin{matrix}\frac{1}{8},\ \frac{3}{8},\ \frac{5}{8},\ \frac{7}{8},\ a\\ \frac{1-2a}{4},\ \frac{3-2a}{4},\ \frac{1-a}{2},\ 1-\frac{a}{2},\ \frac{a}{2},\ \frac{a+1}{2},\ \frac{2a-1}{4},\ \frac{2a+1}{4},\ a\end{matrix}\right)$$
$$=16\sqrt{2}\,\pi^4 e^{\sqrt{2}\,z^{1/4}}\sec(2a\pi)\left[\sin\left(\sqrt{2}\,z^{1/4}\right)\operatorname{kei}_{2a-1}\left(2z^{1/4}\right)\right.$$
$$\left.-\cos\left(\sqrt{2}\,z^{1/4}\right)\operatorname{ker}_{2a-1}\left(2z^{1/4}\right)\right].$$

8.3. Representation in Terms of Hypergeometric Functions

8.3.1. Elementary functions

1. $\dfrac{1}{1+\sqrt{1+z}} = \dfrac{1}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\ 1\\ 2;\ -z\end{matrix}\right).$

2. $\left(1+\sqrt{1+z}\right)^{-1/2} = 2^{-1/2}\,{}_2F_1\left(\begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ \frac{3}{2};\ -z\end{matrix}\right).$

3. $(1-\sqrt{z})^{-1/2} + (1+\sqrt{z})^{-1/2} = 2\,{}_2F_1\left(\begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ \frac{1}{2};\ z\end{matrix}\right).$

4. $(1-\sqrt{z})^{-3/2} + (1+\sqrt{z})^{-3/2} = 2\,{}_2F_1\left(\begin{matrix}\frac{3}{4},\ \frac{5}{4}\\ \frac{1}{2};\ z\end{matrix}\right).$

5. $(1+z)^{\nu} = {}_1F_0(-\nu;\ z).$

6. $\sinh z = z\,{}_3F_2\left(\begin{matrix}\frac{iz}{\pi},\ \frac{iz}{\pi},\ \frac{iz}{\pi}\\ 1,\ 1;\ -1\end{matrix}\right) + \dfrac{2z^3}{\pi^2}\,{}_3F_2\left(\begin{matrix}\frac{iz}{\pi}+1,\ \frac{iz}{\pi}+1,\ \frac{iz}{\pi}+1\\ 2,\ 2;\ -1\end{matrix}\right)$ $[\operatorname{Re}(iz) < 2\pi/3].$

7. $\cosh z = -\left(iz - \dfrac{\pi}{2}\right) z\,{}_3F_2\left(\begin{matrix}\frac{iz}{\pi}-\frac{1}{2},\ \frac{iz}{\pi}-\frac{1}{2},\ \frac{iz}{\pi}-\frac{1}{2}\\ 1,\ 1;\ -1\end{matrix}\right)$

$$+\frac{2}{\pi^2}\left(iz-\frac{\pi}{2}\right)^3 {}_3F_2\left(\begin{matrix}\frac{iz}{\pi}+\frac{1}{2},\ \frac{iz}{\pi}+\frac{1}{2},\ \frac{iz}{\pi}+\frac{1}{2}\\ 2,\ 2;\ -1\end{matrix}\right) \quad [\operatorname{Re}(iz) < 7\pi/6].$$

8. $\sinh\sqrt{z} = \sqrt{z}\,{}_0F_1\left(\frac{3}{2};\ \frac{z}{4}\right).$

9. $\cosh\sqrt{z} = {}_0F_1\left(\frac{1}{2};\ \frac{z}{4}\right).$

10. $\sinh\sqrt[4]{z} = z^{1/4}\,{}_0F_3\left(\begin{matrix}\frac{z}{256}\\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4}\end{matrix}\right) + \dfrac{z^{3/4}}{6}\,{}_0F_3\left(\begin{matrix}\frac{z}{256}\\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4}\end{matrix}\right).$

11. $\cosh\sqrt[4]{z} = {}_0F_3\left(\begin{matrix}\frac{z}{256}\\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}\end{matrix}\right) + \dfrac{z^{1/2}}{2}\,{}_0F_3\left(\begin{matrix}\frac{z}{256}\\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2}\end{matrix}\right).$

12. $\operatorname{sech} z = \dfrac{4\pi}{\pi^2+4z^2}\,{}_4F_3\left(\begin{matrix}1,\ \frac{3}{2},\ \frac{1}{2}-\frac{iz}{\pi},\ \frac{1}{2}+\frac{iz}{\pi}\\ \frac{1}{2},\ \frac{3}{2}-\frac{iz}{\pi},\ \frac{3}{2}+\frac{iz}{\pi};\ -1\end{matrix}\right).$

13. $\operatorname{csch} z = \frac{2}{z}\, {}_3F_2\left(\begin{matrix} 1,\, -\frac{iz}{\pi},\, \frac{iz}{\pi};\; -1 \\ 1-\frac{iz}{\pi},\, 1+\frac{iz}{\pi} \end{matrix}\right) - \frac{1}{z}.$

14. $\tanh z = \frac{8z}{\pi^2+4z^2}\, {}_3F_2\left(\begin{matrix} 1,\, \frac{1}{2}-\frac{iz}{\pi},\, \frac{1}{2}+\frac{iz}{\pi} \\ \frac{3}{2}-\frac{iz}{\pi},\, \frac{3}{2}+\frac{iz}{\pi};\; 1 \end{matrix}\right).$

15. $\coth z = \frac{2}{z}\, {}_3F_2\left(\begin{matrix} 1,\, -\frac{iz}{\pi},\, \frac{iz}{\pi} \\ 1-\frac{iz}{\pi},\, 1+\frac{iz}{\pi};\; 1 \end{matrix}\right) - \frac{1}{z}.$

16. $\sin z = z\, {}_3F_2\left(\begin{matrix} \frac{z}{\pi},\, \frac{z}{\pi},\, \frac{z}{\pi} \\ 1,\, 1;\; -1 \end{matrix}\right) - \frac{2z^3}{\pi^2}\, {}_3F_2\left(\begin{matrix} \frac{z}{\pi}+1,\, \frac{z}{\pi}+1,\, \frac{z}{\pi}+1 \\ 2,\, 2;\; -1 \end{matrix}\right) \quad [\operatorname{Re} z < 2\pi/3].$

17. $\cos z = -\left(z-\frac{\pi}{2}\right) {}_3F_2\left(\begin{matrix} \frac{z}{\pi}-\frac{1}{2},\, \frac{z}{\pi}-\frac{1}{2},\, \frac{z}{\pi}-\frac{1}{2} \\ 1,\, 1;\; -1 \end{matrix}\right)$

$- \frac{2}{\pi^2}\left(z-\frac{\pi}{2}\right)^3 {}_3F_2\left(\begin{matrix} \frac{z}{\pi}+\frac{1}{2},\, \frac{z}{\pi}+\frac{1}{2},\, \frac{z}{\pi}+\frac{1}{2} \\ 2,\, 2;\; -1 \end{matrix}\right) \quad [\operatorname{Re} z < 7\pi/6].$

18. $\sin\sqrt{z} = \sqrt{z}\, {}_0F_1\left(\frac{3}{2};\, -\frac{z}{4}\right).$

19. $\cos\sqrt{z} = {}_0F_1\left(\frac{1}{2};\, -\frac{z}{4}\right).$

20. $\sin\sqrt[4]{z} = z^{1/4}\, {}_0F_3\left(\begin{matrix} \frac{z}{256} \\ \frac{1}{2},\, \frac{3}{4},\, \frac{5}{4} \end{matrix}\right) - \frac{z^{3/4}}{6}\, {}_0F_3\left(\begin{matrix} \frac{z}{256} \\ \frac{5}{4},\, \frac{3}{2},\, \frac{7}{4} \end{matrix}\right).$

21. $\cos\sqrt[4]{z} = {}_0F_3\left(\begin{matrix} \frac{z}{256} \\ \frac{1}{4},\, \frac{1}{2},\, \frac{3}{4} \end{matrix}\right) - \frac{z^{1/2}}{2}\, {}_0F_3\left(\begin{matrix} \frac{z}{256} \\ \frac{3}{4},\, \frac{5}{4},\, \frac{3}{2} \end{matrix}\right).$

22. $\sinh\sqrt[4]{z}\, \sin\sqrt[4]{z} = z^{1/2}\, {}_0F_3\left(\begin{matrix} -\frac{z}{64} \\ \frac{3}{4},\, \frac{5}{4},\, \frac{3}{2} \end{matrix}\right).$

23. $\cosh\sqrt[4]{z}\, \cos\sqrt[4]{z} = {}_0F_3\left(\begin{matrix} -\frac{z}{64} \\ \frac{1}{4},\, \frac{1}{2},\, \frac{3}{4} \end{matrix}\right).$

24. $\left\{\begin{matrix} \sinh\sqrt[4]{z}\, \cos\sqrt[4]{z} \\ \cosh\sqrt[4]{z}\, \sin\sqrt[4]{z} \end{matrix}\right\} = z^{1/4}\, {}_0F_3\left(\begin{matrix} -\frac{z}{64} \\ \frac{1}{2},\, \frac{3}{4},\, \frac{5}{4} \end{matrix}\right) \mp \frac{z^{3/4}}{3}\, {}_0F_3\left(\begin{matrix} -\frac{z}{64} \\ \frac{5}{4},\, \frac{3}{2},\, \frac{7}{4} \end{matrix}\right).$

25. $\sec z = \frac{4\pi}{\pi^2-4z^2}\, {}_4F_3\left(\begin{matrix} 1,\, \frac{3}{2},\, \frac{1}{2}-\frac{z}{\pi},\, \frac{1}{2}+\frac{z}{\pi} \\ \frac{1}{2},\, \frac{3}{2}-\frac{z}{\pi},\, \frac{3}{2}+\frac{z}{\pi};\; -1 \end{matrix}\right).$

26. $\csc z = \frac{2}{z}\,{}_3F_2\left(\begin{matrix}1, -\frac{z}{\pi}, \frac{z}{\pi};\ -1\\ 1-\frac{z}{\pi}, 1+\frac{z}{\pi}\end{matrix}\right) - \frac{1}{z}.$

27. $\tan z = \frac{8z}{\pi^2 - 4z^2}\,{}_3F_2\left(\begin{matrix}1, \frac{1}{2}-\frac{z}{\pi}, \frac{1}{2}+\frac{z}{\pi}\\ \frac{3}{2}-\frac{z}{\pi}, \frac{3}{2}+\frac{z}{\pi};\ 1\end{matrix}\right).$

28. $\cot z = \frac{2}{z}\,{}_3F_2\left(\begin{matrix}1, -\frac{z}{\pi}, \frac{z}{\pi}\\ 1-\frac{z}{\pi}, 1+\frac{z}{\pi};\ 1\end{matrix}\right) - \frac{1}{z}.$

29. $\ln(1+z) = z\,{}_2F_1\left(\begin{matrix}1, 1\\ 2;\ -z\end{matrix}\right).$

30. $\ln\left(\sqrt{z} + \sqrt{1+z}\right) = \sqrt{z}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ \frac{3}{2};\ -z\end{matrix}\right).$

31. $\frac{1}{\sqrt{1+z}}\ln\left(\sqrt{z} + \sqrt{1+z}\right) = \sqrt{z}\,{}_2F_1\left(\begin{matrix}1, 1\\ \frac{3}{2};\ -z\end{matrix}\right).$

32. $\ln^2\left(\sqrt{z} + \sqrt{1+z}\right) = z\,{}_3F_2\left(\begin{matrix}1, 1, 1;\ -z\\ \frac{3}{2}, 2\end{matrix}\right).$

33. $\ln\frac{1+\sqrt{z}}{1-\sqrt{z}} = 2\sqrt{z}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, 1\\ \frac{3}{2};\ z\end{matrix}\right).$

34. $\arcsin\sqrt{z} = \sqrt{z}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ \frac{3}{2};\ z\end{matrix}\right).$

35. $\frac{1}{\sqrt{1-z}}\arcsin\sqrt{z} = \sqrt{z}\,{}_2F_1\left(\begin{matrix}1, 1\\ \frac{3}{2};\ z\end{matrix}\right).$

36. $\arcsin^2\sqrt{z} = z\,{}_3F_2\left(\begin{matrix}1, 1, 1;\ z\\ \frac{3}{2}, 2\end{matrix}\right).$

37. $\arctan\sqrt{z} = \sqrt{z}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, 1\\ \frac{3}{2};\ -z\end{matrix}\right).$

38. $\sin\left(\nu\arcsin\sqrt{z}\right) = \nu\sqrt{z}\,{}_2F_1\left(\begin{matrix}\frac{1-\nu}{2}, \frac{1+\nu}{2}\\ \frac{3}{2};\ z\end{matrix}\right).$

39. $\cos\left(\nu \arcsin\sqrt{z}\right) = {}_2F_1\left(\begin{matrix} -\frac{\nu}{2}, \frac{\nu}{2} \\ \frac{1}{2};\ z \end{matrix}\right).$

40. $\dfrac{1}{\sqrt{1-z}} \sin\left(\nu \arcsin\sqrt{z}\right) = \nu\sqrt{z}\, {}_2F_1\left(\begin{matrix} 1-\frac{\nu}{2}, 1+\frac{\nu}{2} \\ \frac{3}{2};\ z \end{matrix}\right).$

41. $\dfrac{1}{\sqrt{1-z}} \cos\left(\nu \arcsin\sqrt{z}\right) = {}_2F_1\left(\begin{matrix} \frac{1-\nu}{2}, \frac{1+\nu}{2} \\ \frac{1}{2};\ z \end{matrix}\right).$

42. $(1+z)^{-\nu/2} \sin\left(\nu \arctan\sqrt{z}\right) = \nu\sqrt{z}\, {}_2F_1\left(\begin{matrix} \frac{1+\nu}{2}, 1+\frac{\nu}{2} \\ \frac{3}{2};\ -z \end{matrix}\right).$

43. $(1+z)^{-\nu/2} \cos\left(\nu \arctan\sqrt{z}\right) = {}_2F_1\left(\begin{matrix} \frac{\nu}{2}, \frac{1+\nu}{2} \\ \frac{1}{2};\ -z \end{matrix}\right).$

8.3.2. Special functions

1. $\psi^{(n)}(z) = n!\,(-z)^{-n-1}\ {}_{n+2}F_{n+1}\left(\begin{matrix} 1, z, z, \ldots, z;\ 1 \\ z+1, z+1, \ldots, z+1 \end{matrix}\right).$

2. $\Phi(z, n, a) = a^{-n}\ {}_{n+1}F_n\left(\begin{matrix} 1, a, a, \ldots, a;\ z \\ a+1, a+1, \ldots, a+1 \end{matrix}\right).$

3. $\mathrm{Li}_n(z) = z\ {}_{n+1}F_n\left(\begin{matrix} 1, 1, \ldots, 1;\ z \\ 2, 2, \ldots, 2 \end{matrix}\right).$

4. $\mathrm{Si}\left(\sqrt{z}\right) = z^{1/2}\ {}_1F_2\left(\begin{matrix} \frac{1}{2};\ -\frac{z}{4} \\ \frac{3}{2}, \frac{3}{2} \end{matrix}\right).$

5. $\mathrm{ci}\left(\sqrt{z}\right) - \ln\sqrt{z} - \mathbf{C} = -\frac{z}{4}\, {}_2F_3\left(\begin{matrix} 1, 1;\ -\frac{z}{4} \\ \frac{3}{2}, 2, 2 \end{matrix}\right).$

6. $\mathrm{Ei}\,(-z) - \ln z - \mathbf{C} = -z\, {}_2F_2\left(\begin{matrix} 1, 1;\ -z \\ 2, 2 \end{matrix}\right).$

7. $e^z\, \mathrm{Ei}\,(-z) = -\Psi\left(\begin{matrix} 1;\ z \\ 1 \end{matrix}\right).$

8. $\mathrm{erf}\left(\sqrt{z}\right) = \dfrac{2z^{1/2}}{\sqrt{\pi}}\, {}_1F_1\left(\begin{matrix} \frac{1}{2};\ -z \\ \frac{3}{2} \end{matrix}\right).$

9. $e^z\, \mathrm{erf}\left(\sqrt{z}\right) = \dfrac{2z^{1/2}}{\sqrt{\pi}}\, {}_1F_1\left(\begin{matrix} 1;\ z \\ \frac{3}{2} \end{matrix}\right).$

10. $\left\{\begin{matrix} \operatorname{erf}(\sqrt[4]{z}) \\ \operatorname{erfi}(\sqrt[4]{z}) \end{matrix}\right\} = \frac{2z^{1/4}}{\sqrt{\pi}}\, {}_1F_2\left(\begin{matrix} \frac{1}{4};\ \frac{z}{4} \\ \frac{1}{2}, \frac{5}{4} \end{matrix}\right) \mp \frac{2z^{3/4}}{3\sqrt{\pi}}\, {}_1F_2\left(\begin{matrix} \frac{3}{4};\ \frac{z}{4} \\ \frac{3}{2}, \frac{7}{4} \end{matrix}\right).$

11. $\left\{\begin{matrix} e^{\sqrt{z}} \operatorname{erf}(\sqrt[4]{z}) \\ e^{-\sqrt{z}} \operatorname{erfi}(\sqrt[4]{z}) \end{matrix}\right\} = \frac{2}{\sqrt{\pi}} z^{1/4}\, {}_1F_2\left(\begin{matrix} 1;\ \frac{z}{4} \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\right) \pm \frac{4}{3\sqrt{\pi}} z^{3/4}\, {}_1F_2\left(\begin{matrix} 1;\ \frac{z}{4} \\ \frac{5}{4}, \frac{7}{4} \end{matrix}\right).$

12. $e^z \operatorname{erfc}(\sqrt{z}) = \frac{1}{\sqrt{\pi}} \Psi\left(\begin{matrix} \frac{1}{2};\ z \\ \frac{1}{2} \end{matrix}\right).$

13. $\operatorname{erf}(\sqrt[4]{z}) \operatorname{erfi}(\sqrt[4]{z}) = \frac{4z^{1/2}}{\pi}\, {}_2F_3\left(\begin{matrix} \frac{1}{2}, 1;\ \frac{z}{4} \\ \frac{3}{4}, \frac{5}{4}, \frac{3}{2} \end{matrix}\right).$

14. $S(\sqrt{z}) = \frac{1}{3}\sqrt{\frac{2}{\pi}}\, z^{3/4}\, {}_1F_2\left(\begin{matrix} \frac{3}{4};\ -\frac{z}{4} \\ \frac{3}{2}, \frac{7}{4} \end{matrix}\right).$

15. $C(\sqrt{z}) = \sqrt{\frac{2}{\pi}}\, z^{1/4}\, {}_1F_2\left(\begin{matrix} \frac{1}{4};\ -\frac{z}{4} \\ \frac{1}{2}, \frac{5}{4} \end{matrix}\right).$

16. $\sin\sqrt{z}\, S(\sqrt{z}) + \cos\sqrt{z}\, C(\sqrt{z}) = \sqrt{\frac{2}{\pi}}\, z^{1/4}\, {}_1F_2\left(\begin{matrix} 1;\ -\frac{z}{4} \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\right).$

17. $\sin\sqrt{z}\, C(\sqrt{z}) - \cos\sqrt{z}\, S(\sqrt{z}) = \frac{2}{3}\sqrt{\frac{2}{\pi}}\, z^{3/4}\, {}_1F_2\left(\begin{matrix} 1;\ -\frac{z}{4} \\ \frac{5}{4}, \frac{7}{4} \end{matrix}\right).$

18. $S(\sqrt{z}, \nu) = \Gamma(\nu) \sin\frac{\nu\pi}{2} - \frac{z^{(\nu+1)/2}}{\nu+1}\, {}_1F_2\left(\begin{matrix} \frac{\nu+1}{2};\ -\frac{z}{4} \\ \frac{\nu+3}{2}, \frac{3}{2} \end{matrix}\right).$

19. $C(\sqrt{z}, \nu) = \Gamma(\nu) \cos\frac{\nu\pi}{2} - \frac{z^{\nu/2}}{\nu}\, {}_1F_2\left(\begin{matrix} \frac{\nu}{2};\ -\frac{z}{4} \\ \frac{\nu}{2}+1, \frac{1}{2} \end{matrix}\right).$

20. $\gamma(\nu, z) = \frac{z^\nu}{\nu}\, {}_1F_1\left(\begin{matrix} \nu;\ -z \\ \nu+1 \end{matrix}\right).$

21. $e^z \gamma(\nu, z) = \frac{z^\nu}{\nu}\, {}_1F_1\left(\begin{matrix} 1;\ z \\ \nu+1 \end{matrix}\right).$

22. $e^z \Gamma(\nu, z) = z^\nu \Psi\left(\begin{matrix} 1;\ z \\ \nu+1 \end{matrix}\right).$

23. $\qquad = \Psi\left(\begin{matrix} 1-\nu;\ z \\ 1-\nu \end{matrix}\right).$

24. $D_\nu(\sqrt{z}) = 2^{\nu/2} e^{-z/4} \Psi\begin{pmatrix} -\frac{\nu}{2};\ \frac{z}{2} \\ \frac{1}{2} \end{pmatrix}.$

25. $\left\{\begin{matrix} J_\nu(\sqrt{z}) \\ I_\nu(\sqrt{z}) \end{matrix}\right\} = \frac{\left(\frac{\sqrt{z}}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\nu+1;\ \mp\frac{z}{4}\right).$

26. $\left\{\begin{matrix} J_\nu(\sqrt[4]{z}) \\ I_\nu(\sqrt[4]{z}) \end{matrix}\right\} = \frac{z^{\nu/4}}{2^\nu \Gamma(\nu+1)}\, {}_0F_3\begin{pmatrix} \frac{z}{256} \\ \frac{1}{2},\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1 \end{pmatrix}$

$$\mp \frac{z^{(\nu+2)/4}}{2^{\nu+2}\Gamma(\nu+2)}\, {}_0F_3\begin{pmatrix} \frac{z}{256} \\ \frac{3}{2},\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2} \end{pmatrix}.$$

27. $e^z I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_1F_1\begin{pmatrix} \nu+\frac{1}{2};\ 2z \\ 2\nu+1 \end{pmatrix}.$

28. $e^{iz} J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_1F_1\begin{pmatrix} \nu+\frac{1}{2};\ 2iz \\ 2\nu+1 \end{pmatrix}.$

29. $\left\{\begin{matrix} \sin\sqrt{z}\, J_\nu(\sqrt{z}) \\ \sinh\sqrt{z}\, I_\nu(\sqrt{z}) \end{matrix}\right\} = \frac{z^{(\nu+1)/2}}{2^\nu \Gamma(\nu+1)}\, {}_2F_3\begin{pmatrix} \frac{2\nu+3}{4},\ \frac{2\nu+5}{4};\ \mp z \\ \frac{3}{2},\ \nu+1,\ \nu+\frac{3}{2} \end{pmatrix}.$

30. $\left\{\begin{matrix} \cos\sqrt{z}\, J_\nu(\sqrt{z}) \\ \cosh\sqrt{z}\, I_\nu(\sqrt{z}) \end{matrix}\right\} = \frac{z^{\nu/2}}{2^\nu \Gamma(\nu+1)}\, {}_2F_3\begin{pmatrix} \frac{2\nu+1}{4},\ \frac{2\nu+3}{4};\ \mp z \\ \frac{1}{2},\ \nu+\frac{1}{2},\ \nu+1 \end{pmatrix}.$

31. $\left\{\begin{matrix} J_\nu^2(\sqrt{z}) \\ I_\nu^2(\sqrt{z}) \end{matrix}\right\} = \frac{\left(\frac{z}{4}\right)^\nu}{\Gamma^2(\nu+1)}\, {}_1F_2\begin{pmatrix} \nu+\frac{1}{2};\ \mp z \\ \nu+1,\ 2\nu+1 \end{pmatrix}.$

32. $\left\{\begin{matrix} J_\mu(\sqrt{z})\, J_\nu(\sqrt{z}) \\ I_\mu(\sqrt{z})\, I_\nu(\sqrt{z}) \end{matrix}\right\} = \frac{\left(\frac{\sqrt{z}}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\, {}_2F_3\begin{pmatrix} \frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1;\ \mp z \\ \mu+1,\ \nu+1,\ \mu+\nu+1 \end{pmatrix}.$

33. $J_\nu(\sqrt[4]{z})\, I_\nu(\sqrt[4]{z}) = \frac{\left(\frac{\sqrt{z}}{4}\right)^\nu}{\Gamma^2(\nu+1)}\, {}_0F_3\begin{pmatrix} -\frac{z}{64} \\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ \nu+1 \end{pmatrix}.$

34. $J_\nu(\sqrt[4]{z})\, I_{-\nu}(\sqrt[4]{z}) = \frac{\sin\nu\pi}{\nu\pi}\, {}_0F_3\begin{pmatrix} -\frac{z}{64} \\ 1-\frac{\nu}{2},\ 1+\frac{\nu}{2},\ \frac{1}{2} \end{pmatrix}$

$$+ \frac{\sin\nu\pi}{2(1-\nu^2)\pi} z^{1/2}\, {}_0F_3\begin{pmatrix} -\frac{z}{64} \\ \frac{3-\nu}{2},\ \frac{3+\nu}{2},\ \frac{3}{2} \end{pmatrix}.$$

35. $J_{n+1/2}^{2}\left(\frac{1}{\sqrt{z}}\right)+Y_{n+1/2}^{2}\left(\frac{1}{\sqrt{z}}\right)=\frac{2\sqrt{z}}{\pi}\,{}_3F_0\left(\begin{matrix}-n,\,n+1,\,\frac{1}{2}\\ -z\end{matrix}\right).$

36. $I_{-n-1/2}^{2}\left(\frac{1}{\sqrt{z}}\right)-I_{n+1/2}^{2}\left(\frac{1}{\sqrt{z}}\right)=(-1)^{n}\frac{2\sqrt{z}}{\pi}\,{}_3F_0\left(\begin{matrix}-n,\,n+1,\,\frac{1}{2}\\ z\end{matrix}\right).$

37. $e^{z}K_{n+1/2}(z)=\frac{(2n)!\sqrt{\pi}}{n!}(2z)^{-n-1/2}\,{}_1F_1\left(\begin{matrix}-n;\,2z\\ -2n\end{matrix}\right).$

38. $\left\{\begin{matrix}\mathbf{H}_{\nu}(\sqrt{z})\\ \mathbf{L}_{\nu}(\sqrt{z})\end{matrix}\right\}=\frac{2^{-\nu}z^{(\nu+1)/2}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_1F_2\left(\begin{matrix}1;\,\mp\frac{z}{4}\\ \nu+\frac{3}{2},\,\frac{3}{2}\end{matrix}\right).$

39. $\mathbf{H}_{n+1/2}(\sqrt{z})-Y_{n+1/2}(\sqrt{z})=\frac{\left(\frac{\sqrt{z}}{2}\right)^{n-1/2}}{n!\sqrt{\pi}}\,{}_3F_0\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ -\frac{4}{z}\end{matrix}\right).$

40. $\mathbf{L}_{n+1/2}(\sqrt{z})-I_{-n-1/2}(\sqrt{z})=\frac{\left(\frac{\sqrt{z}}{2}\right)^{n-1/2}}{n!\sqrt{\pi}}\,{}_3F_0\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{4}{z}\end{matrix}\right).$

41. $\operatorname{Ai}(z)=\frac{3^{-2/3}}{\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left(\frac{2}{3};\,\frac{z^{3}}{9}\right)-\frac{3^{-1/3}z}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left(\frac{4}{3};\,\frac{z^{3}}{9}\right).$

42. $\operatorname{Bi}(z)=\frac{3^{-1/6}}{\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left(\frac{2}{3};\,\frac{z^{3}}{9}\right)+\frac{3^{1/6}z}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left(\frac{4}{3};\,\frac{z^{3}}{9}\right).$

43. $\operatorname{Ai}'(z)=\frac{3^{-2/3}z^{2}}{2\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left(\frac{5}{3};\,\frac{z^{3}}{9}\right)-\frac{3^{-1/3}}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left(\frac{1}{3};\,\frac{z^{3}}{9}\right).$

44. $\operatorname{Bi}'(z)=\frac{3^{-1/6}z^{2}}{2\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left(\frac{5}{3};\,\frac{z^{3}}{9}\right)+\frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left(\frac{1}{3};\,\frac{z^{3}}{9}\right).$

45. $$\operatorname{Ai}^{2}(z)=\frac{\Gamma\left(\frac{1}{6}\right)}{2^{5/3}3^{5/6}\pi^{3/2}}\,{}_1F_2\left(\begin{matrix}\frac{1}{6};\,\frac{4z^{3}}{9}\\ \frac{1}{3},\,\frac{2}{3}\end{matrix}\right)-\frac{z}{\sqrt{3}\,\pi}\,{}_1F_2\left(\begin{matrix}\frac{1}{2};\,\frac{4z^{3}}{9}\\ \frac{2}{3},\,\frac{4}{3}\end{matrix}\right)+\frac{\Gamma\left(\frac{5}{6}\right)z^{2}}{2^{4/3}3^{1/6}\pi^{3/2}}\,{}_1F_2\left(\begin{matrix}\frac{5}{6};\,\frac{4z^{3}}{9}\\ \frac{4}{3},\,\frac{5}{3}\end{matrix}\right).$$

46. $$\operatorname{Bi}^{2}(z)=\frac{3^{1/6}}{2^{5/3}\pi^{3/2}}\Gamma\left(\frac{1}{6}\right)\,{}_1F_2\left(\begin{matrix}\frac{1}{6};\,\frac{4z^{3}}{9}\\ \frac{1}{3},\,\frac{2}{3}\end{matrix}\right)+\frac{3^{1/2}z}{\pi}\,{}_1F_2\left(\begin{matrix}\frac{1}{2};\,\frac{4z^{3}}{9}\\ \frac{2}{3},\,\frac{4}{3}\end{matrix}\right)+\frac{3^{5/6}z^{2}}{2^{4/3}\pi^{3/2}}\Gamma\left(\frac{5}{6}\right)\,{}_1F_2\left(\begin{matrix}\frac{5}{6};\,\frac{4z^{3}}{9}\\ \frac{4}{3},\,\frac{5}{3}\end{matrix}\right).$$

47. $$\operatorname{Ai}(z)\operatorname{Ai}(-z) = \frac{3^{-4/3}}{\Gamma^2\left(\frac{2}{3}\right)}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{1}{3},\ \frac{2}{3},\ \frac{5}{6}\end{matrix}\right) - \frac{3^{-8/3}z^2}{\Gamma^2\left(\frac{4}{3}\right)}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{2}{3},\ \frac{7}{6},\ \frac{4}{3}\end{matrix}\right) + \frac{3^{-3/2}z^4}{4\pi}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{4}{3},\ \frac{3}{2},\ \frac{5}{3}\end{matrix}\right).$$

48. $$\operatorname{Bi}(z)\operatorname{Bi}(-z) = \frac{3^{-1/3}}{\Gamma^2\left(\frac{2}{3}\right)}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{1}{3},\ \frac{2}{3},\ \frac{5}{6}\end{matrix}\right) - \frac{3^{-5/3}z^2}{\Gamma^2\left(\frac{4}{3}\right)}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{2}{3},\ \frac{7}{6},\ \frac{4}{3}\end{matrix}\right) - \frac{3^{-1/2}z^4}{4\pi}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{4}{3},\ \frac{3}{2},\ \frac{5}{3}\end{matrix}\right).$$

49. $$\operatorname{Ai}(z)\operatorname{Bi}(-z) = \frac{3^{-5/6}}{\Gamma^2\left(\frac{2}{3}\right)}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{1}{3},\ \frac{2}{3},\ \frac{5}{6}\end{matrix}\right) + \frac{3^{-13/6}z^2}{\Gamma^2\left(\frac{4}{3}\right)}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{2}{3},\ \frac{7}{6},\ \frac{4}{3}\end{matrix}\right) - \frac{z}{\pi}\,{}_0F_3\left(\begin{matrix} -\frac{z^6}{324} \\ \frac{1}{2},\ \frac{5}{6},\ \frac{7}{6}\end{matrix}\right).$$

50. $$\operatorname{ber}_\nu(z) = \cos\frac{3\nu\pi}{4}\,\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_3\left(\begin{matrix} -\frac{z^4}{256} \\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ \frac{1}{2}\end{matrix}\right) - \sin\frac{3\nu\pi}{4}\,\frac{\left(\frac{z}{2}\right)^{\nu+2}}{\Gamma(\nu+2)}\,{}_0F_3\left(\begin{matrix} -\frac{z^4}{256} \\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2},\ \frac{3}{2}\end{matrix}\right).$$

51. $$\operatorname{bei}_\nu(z) = \sin\frac{3\nu\pi}{4}\,\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_3\left(\begin{matrix} -\frac{z^4}{256} \\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ \frac{1}{2}\end{matrix}\right) + \cos\frac{3\nu\pi}{4}\,\frac{\left(\frac{z}{2}\right)^{\nu+2}}{\Gamma(\nu+2)}\,{}_0F_3\left(\begin{matrix} -\frac{z^4}{256} \\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2},\ \frac{3}{2}\end{matrix}\right).$$

52. $$\operatorname{ber}_\nu^2(z) = \frac{z^{2\nu}}{2^{2\nu+1}\Gamma^2(\nu+1)} \times\left[\cos\frac{3\nu\pi}{2}\,{}_2F_5\left(\begin{matrix} \frac{2\nu+1}{4},\ \frac{2\nu+3}{4};\ -\frac{z^4}{16} \\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ \nu+\frac{1}{2},\ \nu+1,\ \frac{1}{2}\end{matrix}\right) - \frac{z^2}{2(\nu+1)}\sin\frac{3\nu\pi}{2} \times {}_2F_5\left(\begin{matrix} \frac{2\nu+3}{4},\ \frac{2\nu+5}{4};\ -\frac{z^4}{16} \\ \frac{\nu}{2}+1,\ \frac{\nu+3}{2},\ \nu+1,\ \nu+\frac{3}{2},\ \frac{3}{2}\end{matrix}\right) + {}_0F_3\left(\begin{matrix} \frac{z^4}{64} \\ \frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ \nu+1\end{matrix}\right)\right].$$

53. $$\operatorname{bei}_\nu^2(z) = \frac{z^{2\nu}}{2^{2\nu+1}\Gamma^2(\nu+1)} \times \left[-\cos\frac{3\nu\pi}{2}\, {}_2F_5\left(\begin{matrix} \frac{2\nu+1}{4}, \frac{2\nu+3}{4};\ -\frac{z^4}{16} \\ \frac{\nu+1}{2}, \frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1, \frac{1}{2} \end{matrix} \right) + \frac{z^2}{2(\nu+1)} \sin\frac{3\nu\pi}{2} \times {}_2F_5\left(\begin{matrix} \frac{2\nu+3}{4}, \frac{2\nu+5}{4};\ -\frac{z^4}{16} \\ \frac{\nu}{2}+1, \frac{\nu+3}{2}, \nu+1, \nu+\frac{3}{2}, \frac{3}{2} \end{matrix} \right) + {}_0F_3\left(\begin{matrix} \frac{z^4}{64} \\ \frac{\nu+1}{2}, \frac{\nu}{2}+1, \nu+1 \end{matrix} \right) \right].$$

54. $$\operatorname{ber}_\nu(z)\operatorname{bei}_\nu(z) = \frac{z^{2\nu}}{2^{2\nu+1}\Gamma^2(\nu+1)} \times \left[\sin\frac{3\nu\pi}{2}\, {}_2F_5\left(\begin{matrix} \frac{2\nu+1}{4}, \frac{2\nu+3}{4};\ -\frac{z^4}{16} \\ \frac{\nu+1}{2}, \frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1, \frac{1}{2} \end{matrix} \right) + \frac{z^2}{2(\nu+1)} \cos\frac{3\nu\pi}{2}\, {}_2F_5\left(\begin{matrix} \frac{2\nu+3}{4}, \frac{2\nu+5}{4};\ -\frac{z^4}{16} \\ \frac{\nu}{2}+1, \frac{\nu+3}{2}, \nu+1, \nu+\frac{3}{2}, \frac{3}{2} \end{matrix} \right) \right].$$

55. $$\mathbf{E}_\nu(\sqrt{z}) = \frac{1}{\nu\pi}(1-\cos\nu\pi)\, {}_1F_2\left(\begin{matrix} 1;\ -\frac{z}{4} \\ 1-\frac{\nu}{2}, 1+\frac{\nu}{2} \end{matrix} \right) - \frac{\sqrt{z}}{\pi(1-\nu^2)}(1+\cos\nu\pi)\, {}_1F_2\left(\begin{matrix} 1;\ -\frac{z}{4} \\ \frac{3-\nu}{2}, \frac{3+\nu}{2} \end{matrix} \right).$$

56. $$\mathbf{J}_\nu(\sqrt{z}) = \frac{\sin\nu\pi}{\nu\pi}\, {}_1F_2\left(\begin{matrix} 1;\ -\frac{z}{4} \\ 1-\frac{\nu}{2}, 1+\frac{\nu}{2} \end{matrix} \right) + \frac{\sin\nu\pi}{\pi(1-\nu^2)} \sqrt{z}\, {}_1F_2\left(\begin{matrix} 1;\ -\frac{z}{4} \\ \frac{3-\nu}{2}, \frac{3+\nu}{2} \end{matrix} \right).$$

57. $$P_{2n}(\sqrt{z}) = (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\frac{1}{2} \\ \frac{1}{2};\ z \end{matrix} \right).$$

58. $$P_{2n+1}(\sqrt{z}) = (-1)^n \frac{\left(\frac{3}{2}\right)_n}{n!} \sqrt{z}\, {}_2F_1\left(\begin{matrix} -n, n+\frac{3}{2} \\ \frac{3}{2};\ z \end{matrix} \right).$$

59. $$P_n\left(\frac{1}{\sqrt{z}}\right) = \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{2}{\sqrt{z}}\right)^n {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ \frac{1}{2}-n;\ z \end{matrix} \right).$$

60. $$P_n(1+z) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1;\ -\frac{z}{2} \end{matrix} \right).$$

61. $$P_n\left(1+\frac{1}{z}\right) = \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{2}{z}\right)^n {}_2F_1\left(\begin{matrix} -n, -n \\ -2n;\ -2z \end{matrix} \right).$$

62. $(1-z)^n P_n\left(\frac{1+z}{1-z}\right) = {}_2F_1\left(\begin{matrix}-n,\ -n\\ 1;\ z\end{matrix}\right).$

63. $P_{2n}\left(\sqrt{1+z}\right) = {}_2F_1\left(\begin{matrix}-n,\ n+\frac{1}{2}\\ 1;\ -z\end{matrix}\right).$

64. $\frac{1}{\sqrt{1+z}}\, P_{2n+1}\left(\sqrt{1+z}\right) = {}_2F_1\left(\begin{matrix}-n,\ n+\frac{3}{2}\\ 1;\ -z\end{matrix}\right).$

65. $P_n\left(\frac{1+z}{2\sqrt{z}}\right) = \frac{(2n)!}{(n!)^2}\,(4\sqrt{z})^{-n}\ {}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}\\ \frac{1}{2}-n;\ z\end{matrix}\right).$

66. $(1+z)^{n/2} P_n\left(\frac{2+z}{2\sqrt{1+z}}\right) = {}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}\\ 1;\ -z\end{matrix}\right).$

67. $P_{2n}\left(\sqrt{1+\frac{1}{z}}\right) = \frac{\left(\frac{1}{2}\right)_{2n}}{(2n)!}\left(\frac{4}{z}\right)^n\ {}_2F_1\left(\begin{matrix}-n,\ -n\\ \frac{1}{2}-2n;\ -z\end{matrix}\right).$

68. $\frac{1}{\sqrt{1+z}}\, P_{2n+1}\left(\sqrt{1+\frac{1}{z}}\right) = \frac{\left(\frac{1}{2}\right)_{2n+1}}{(2n+1)!}\left(\frac{4}{z}\right)^{n+1/2}\ {}_2F_1\left(\begin{matrix}-n,\ -n\\ -\frac{1}{2}-2n;\ -z\end{matrix}\right).$

69. $(z-1)^n P_{2n}\left(\sqrt{\frac{z}{z-1}}\right) = \frac{\left(\frac{1}{2}\right)_n}{n!}\ {}_2F_1\left(\begin{matrix}-n,\ -n\\ \frac{1}{2};\ z\end{matrix}\right).$

70. $(z-1)^{n+1/2} P_{2n+1}\left(\sqrt{\frac{z}{z-1}}\right) = \frac{\left(\frac{3}{2}\right)_n}{n!}\sqrt{z}\ {}_2F_1\left(\begin{matrix}-n,\ -n\\ \frac{3}{2};\ z\end{matrix}\right).$

71. $(1+z)^{n/2} P_n\left(\frac{1}{\sqrt{1+z}}\right) = {}_2F_1\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2}\\ 1;\ -z\end{matrix}\right).$

72. $\left[P_n\left(\sqrt{1+z}\right)\right]^2 = {}_3F_2\left(\begin{matrix}-n,\ n+1,\ \frac{1}{2}\\ 1,\ 1;\ -z\end{matrix}\right).$

73. $\left[P_n\left(\sqrt{1+\frac{1}{z}}\right)\right]^2 = \left[\frac{\left(\frac{1}{2}\right)_n}{n!}\right]^2\left(\frac{4}{z}\right)^n\ {}_3F_2\left(\begin{matrix}-n,\ -n,\ -n\\ \frac{1}{2}-n,\ -2n;\ -z\end{matrix}\right).$

74. $P_{2n}\left(\sqrt{\frac{1+\sqrt{1-z}}{2}}\right) P_{2n}\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right)$

$$= (-1)^n\,\frac{\left(\frac{1}{2}\right)_n}{n!}\ {}_4F_3\left(\begin{matrix}-n,\ n+\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4}\\ \frac{1}{2},\ \frac{1}{2},\ 1;\ z\end{matrix}\right).$$

75. $$P_{2n+1}\left(\sqrt{\frac{1+\sqrt{1-z}}{2}}\right) P_{2n+1}\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right) = (-1)^n \frac{\left(\frac{3}{2}\right)_n \sqrt{z}}{2\,(n!)}\, {}_4F_3\left(\begin{matrix} -n,\, n+\frac{3}{2},\, \frac{3}{4},\, \frac{5}{4} \\ 1,\, \frac{3}{2},\, \frac{3}{2};\; z \end{matrix}\right).$$

76. $$P_{2n+1}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right) P_{2n+1}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right) = (-1)^n \frac{\left(n+\frac{3}{2}\right)_n}{n!} \left(\frac{4}{z}\right)^{n+1/2} {}_4F_3\left(\begin{matrix} -n,\, -\frac{1}{2}-n,\, -\frac{1}{4}-n,\, \frac{1}{4}-n \\ -\frac{1}{2}-2n,\, -\frac{1}{2}-2n,\, 1;\; z \end{matrix}\right).$$

77. $$P_{2n}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right) P_{2n}\left(\frac{2}{1-\sqrt{1-z}}\right) = \frac{\left(n+\frac{1}{2}\right)_n}{n!} \left(\frac{4}{z}\right)^n {}_4F_3\left(\begin{matrix} -n,\, \frac{1}{4}-n,\, \frac{1}{2}-n,\, \frac{3}{4}-n \\ \frac{1}{2}-2n,\, \frac{1}{2}-2n,\, 1;\; z \end{matrix}\right).$$

78. $$P_n\left(\frac{3+\sqrt{1-z}}{2^{3/2}\sqrt{1+\sqrt{1-z}}}\right) P_n\left(\frac{3-\sqrt{1-z}}{2^{3/2}\sqrt{1-\sqrt{1-z}}}\right) = \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{4}{z}\right)^{n/2} {}_4F_3\left(\begin{matrix} -n,\, \frac{1-2n}{4},\, \frac{3-2n}{4},\, \frac{1}{2};\; z \\ \frac{1}{2}-n,\, \frac{1}{2}-n,\, 1 \end{matrix}\right).$$

79. $$T_{2n}(\sqrt{z}) = (-1)^n\, {}_2F_1\left(\begin{matrix} -n,\, n \\ \frac{1}{2};\; z \end{matrix}\right).$$

80. $$T_{2n+1}(\sqrt{z}) = (-1)^n (2n+1)\sqrt{z}\, {}_2F_1\left(\begin{matrix} -n,\, n+1 \\ \frac{3}{2};\; z \end{matrix}\right).$$

81. $$T_n\left(\frac{1}{\sqrt{z}}\right) = 2^{n-1} z^{-n/2}\, {}_2F_1\left(\begin{matrix} -\frac{n}{2},\, \frac{1-n}{2} \\ 1-n;\; z \end{matrix}\right).$$

82. $$T_n(1+z) = {}_2F_1\left(\begin{matrix} -n,\, n \\ \frac{1}{2};\; -\frac{z}{2} \end{matrix}\right).$$

83. $$T_n\left(1+\frac{1}{z}\right) = 2^{n-1} z^{-n}\, {}_2F_1\left(\begin{matrix} -n,\, \frac{1}{2}-n \\ 1-2n;\; -2z \end{matrix}\right).$$

84. $$(1-z)^n T_n\left(\frac{1+z}{1-z}\right) = {}_2F_1\left(\begin{matrix} -n,\, \frac{1}{2}-n \\ \frac{1}{2};\; z \end{matrix}\right).$$

85. $T_{2n}\left(\sqrt{1+z}\right) = {}_2F_1\left(\begin{matrix}-n,\ n\\ \frac{1}{2};\ -z\end{matrix}\right).$

86. $\dfrac{1}{\sqrt{1+z}}T_{2n+1}\left(\sqrt{1+z}\right) = {}_2F_1\left(\begin{matrix}-n,\ n+1\\ \frac{1}{2};\ -z\end{matrix}\right).$

87. $T_{2n}\left(\sqrt{1+\frac{1}{z}}\right) = 2^{2n-1}z^{-n}\,{}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ 1-2n;\ -z\end{matrix}\right).$

88. $\dfrac{1}{\sqrt{1+z}}T_{2n+1}\left(\sqrt{1+\frac{1}{z}}\right) = 2^{2n}z^{-n-1/2}\,{}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ -2n;\ -z\end{matrix}\right).$

89. $(z-1)^nT_{2n}\left(\sqrt{\dfrac{z}{z-1}}\right) = {}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ \frac{1}{2};\ z\end{matrix}\right).$

90. $(z-1)^{n+1/2}T_{2n+1}\left(\sqrt{\dfrac{z}{z-1}}\right) = (2n+1)\sqrt{z}\,{}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2}-n\\ \frac{3}{2};\ z\end{matrix}\right).$

91. $(1+z)^{n/2}T_n\left(\dfrac{1}{\sqrt{1+z}}\right) = {}_2F_1\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2}\\ \frac{1}{2};\ -z\end{matrix}\right).$

92. $\left[T_n\left(\sqrt{1+z}\right)\right]^2 = \dfrac{1}{2}\left[1+{}_2F_1\left(\begin{matrix}-n,\ n\\ \frac{1}{2};\ -z\end{matrix}\right)\right].$

93. $$T_n\left(\frac{3-\sqrt{1-z}}{1+\sqrt{1-z}}\right)T_n\left(\frac{3+\sqrt{1-z}}{1-\sqrt{1-z}}\right)$$
$$= 2^{4n-1}z^{-n}\,{}_4F_3\left(\begin{matrix}-n,\ \frac{1}{4}-n,\ \frac{1}{2}-n,\ \frac{3}{4}-n\\ \frac{1}{2}-2n,\ 1-2n,\ \frac{1}{2};\ z\end{matrix}\right) \quad [n\ge 1].$$

94. $U_{2n}(\sqrt{z}) = (-1)^n\,{}_2F_1\left(\begin{matrix}-n,\ n+1\\ \frac{1}{2};\ z\end{matrix}\right).$

95. $U_{2n+1}(\sqrt{z}) = (-1)^n2(n+1)\sqrt{z}\,{}_2F_1\left(\begin{matrix}-n,\ n+2\\ \frac{3}{2};\ z\end{matrix}\right).$

96. $U_n\left(\dfrac{1}{\sqrt{z}}\right) = 2^nz^{-n/2}\,{}_2F_1\left(\begin{matrix}-\frac{n}{2},\ \frac{1-n}{2}\\ -n;\ z\end{matrix}\right).$

97. $U_n(1+z) = (n+1)\,{}_2F_1\left(\begin{matrix}n,\ n+2\\ \frac{3}{2};\ -\frac{z}{2}\end{matrix}\right).$

98. $U_n\left(1+\frac{1}{z}\right)=\left(\frac{2}{z}\right)^n {}_2F_1\left(\begin{matrix}-n,\,-n-\frac{1}{2}\\-2n-1;\;-2z\end{matrix}\right).$

99. $(1-z)^n U_n\left(\frac{1+z}{1-z}\right)=(n+1)\,{}_2F_1\left(\begin{matrix}-n,\,-n-\frac{1}{2}\\\frac{3}{2};\;z\end{matrix}\right).$

100. $U_{2n}\left(\sqrt{1+z}\right)=(2n+1)\,{}_2F_1\left(\begin{matrix}-n,\,n+1\\\frac{3}{2};\;-z\end{matrix}\right).$

101. $\frac{1}{\sqrt{1+z}}U_{2n+1}\left(\sqrt{1+z}\right)=2(n+1)\,{}_2F_1\left(\begin{matrix}-n,\,n+2\\\frac{3}{2};\;-z\end{matrix}\right).$

102. $U_{2n}\left(\sqrt{1+\frac{1}{z}}\right)=2^{2n}z^{-n}\,{}_2F_1\left(\begin{matrix}-n,\,-n-\frac{1}{2}\\-2n;\;-z\end{matrix}\right).$

103. $\frac{1}{\sqrt{1+z}}U_{2n+1}\left(\sqrt{1+\frac{1}{z}}\right)=2^{2n+1}z^{-n-1/2}\,{}_2F_1\left(\begin{matrix}-n,\,-n-\frac{1}{2}\\-2n-1;\;-z\end{matrix}\right).$

104. $(z-1)^n U_{2n}\left(\sqrt{\frac{z}{z-1}}\right)={}_2F_1\left(\begin{matrix}-n,\,-n-\frac{1}{2}\\\frac{1}{2};\;z\end{matrix}\right).$

105. $(z-1)^{n+1/2}U_{2n+1}\left(\sqrt{\frac{z}{z-1}}\right)=2(n+1)\sqrt{z}\,{}_2F_1\left(\begin{matrix}-n,\,-n-\frac{1}{2}\\\frac{3}{2};\;z\end{matrix}\right).$

106. $(1+z)^{n/2}U_n\left(\frac{1}{\sqrt{1+z}}\right)=(n+1)\,{}_2F_1\left(\begin{matrix}-\frac{n}{2},\,\frac{1-n}{2}\\\frac{3}{2};\;-z\end{matrix}\right).$

107. $\left[U_n\left(\sqrt{1+z}\right)\right]^2=(n+1)^2\,{}_3F_2\left(\begin{matrix}-n,\,n+2,\,1\\\frac{3}{2},\,2;\;-z\end{matrix}\right).$

108. $U_{2n}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right)U_{2n}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right)$

$$=(2n+1)\left(\frac{z}{16}\right)^{-n}{}_4F_3\left(\begin{matrix}-n,\,\frac{1}{4}-n,\,\frac{1}{2}-n,\,\frac{3}{4}-n\\-2n,\,\frac{1}{2}-2n,\,\frac{3}{2};\;z\end{matrix}\right).$$

109. $U_{2n+1}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}}\right)U_{2n+1}\left(\sqrt{\frac{2}{1-\sqrt{1-z}}}\right)$

$$=2(n+1)\left(\frac{z}{16}\right)^{-n-1/2}{}_4F_3\left(\begin{matrix}-n,\,-\frac{1}{2}-n,\,-\frac{1}{4}-n,\,\frac{1}{4}-n\\-\frac{1}{2}-2n,\,-1-2n,\,\frac{3}{2};\;z\end{matrix}\right).$$

110. $H_n(\sqrt{z}) = 2^n \sqrt{z}\, \Psi\left(\begin{matrix} \frac{1-n}{2};\ z \\ \frac{3}{2} \end{matrix}\right).$

111. $H_{2n}(\sqrt{z}) = (-1)^n \frac{(2n)!}{n!}\, {}_1F_1\left(\begin{matrix} -n;\ z \\ \frac{1}{2} \end{matrix}\right).$

112. $H_{2n+1}(\sqrt{z}) = (-1)^n \frac{(2n+1)!}{n!} 2\sqrt{z}\, {}_1F_1\left(\begin{matrix} -n;\ z \\ \frac{3}{2} \end{matrix}\right).$

113. $H_n\left(\frac{1}{\sqrt{z}}\right) = 2^n z^{-n/2}\, {}_2F_0\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ -z \end{matrix}\right).$

114. $e^{-z} H_{2n}(\sqrt{z}) = (-1)^n 2^{2n} \left(\frac{1}{2}\right)_n\, {}_1F_1\left(\begin{matrix} n+\frac{1}{2} \\ \frac{1}{2};\ -z \end{matrix}\right).$

115. $e^{-z} H_{2n+1}(\sqrt{z}) = (-1)^n 2^{2n+1} \left(\frac{3}{2}\right)_n \sqrt{z}\, {}_1F_1\left(\begin{matrix} n+\frac{3}{2} \\ \frac{3}{2};\ -z \end{matrix}\right).$

116. $H_{2n}\left(\sqrt[4]{z}\right) H_{2n}\left(i\sqrt[4]{z}\right) = \left[\frac{(2n)!}{n!}\right]^2 {}_2F_3\left(\begin{matrix} -n, n+\frac{1}{2};\ \frac{z}{4} \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \end{matrix}\right).$

117. $H_{2n+1}\left(\sqrt[4]{z}\right) H_{2n+1}\left(i\sqrt[4]{z}\right) = 4i \left[\frac{(2n+1)!}{n!}\right]^2 \sqrt{z}\, {}_2F_3\left(\begin{matrix} -n, n+\frac{3}{2} \\ \frac{3}{4}, \frac{5}{4}, \frac{3}{2};\ \frac{z}{4} \end{matrix}\right).$

118. $H_{2n}\left(z^{-1/4}\right) H_{2n}\left(iz^{-1/4}\right) = \left(-\frac{16}{z}\right)^n {}_4F_1\left(\begin{matrix} -n, \frac{1}{4}-n, \frac{1}{2}-n, \frac{3}{4}-n \\ \frac{1}{2}-2n;\ 4z \end{matrix}\right).$

119. $H_{2n+1}\left(z^{-1/4}\right) H_{2n+1}\left(iz^{-1/4}\right)$

$$= (-1)^n 2^{4n+2} i z^{-n-1/2}\, {}_4F_1\left(\begin{matrix} -n, -\frac{1}{2}-n, -\frac{1}{4}-n, \frac{1}{4}-n \\ -\frac{1}{2}-2n;\ 4z \end{matrix}\right).$$

120. $L_n^\lambda(z) = \frac{(\lambda+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} -n;\ z \\ \lambda+1 \end{matrix}\right).$

121. $\quad = \frac{(-1)^n}{n!} \Psi\left(\begin{matrix} -n;\ z \\ \lambda+1 \end{matrix}\right).$

122. $L_n^\lambda\left(\frac{1}{z}\right) = \frac{(-z)^{-n}}{n!}\, {}_2F_0\left(\begin{matrix} -n, -\lambda-n \\ -z \end{matrix}\right).$

123. $e^{-z} L_n^\lambda(z) = \frac{(\lambda+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} \lambda+n+1 \\ \lambda+1;\ -z \end{matrix}\right).$

124. $L_n^\lambda(\sqrt{z})\, L_n^\lambda(-\sqrt{z}) = \left[\frac{(\lambda+1)_n}{n!}\right]^2 {}_2F_3\left(\begin{matrix} -n,\ \lambda+n+1;\ \frac{z}{4} \\ \frac{\lambda+1}{2},\ \frac{\lambda}{2}+1,\ \lambda+1 \end{matrix}\right).$

125. $L_n^\lambda\left(\frac{1}{\sqrt{z}}\right) L_n^\lambda\left(-\frac{1}{\sqrt{z}}\right) = \frac{(-z)^{-n}}{(n!)^2}\, {}_4F_1\left(\begin{matrix} -n,\ -\frac{\lambda}{2}-n,\ \frac{1-\lambda}{2}-n,\ -\lambda-n \\ -\lambda-2n;\ 4z \end{matrix}\right).$

126. $C_{2n}^\lambda(\sqrt{z}) = (-1)^n \frac{(\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n,\ \lambda+n \\ \frac{1}{2};\ z \end{matrix}\right).$

127. $C_{2n+1}^\lambda(\sqrt{z}) = (-1)^n \frac{(\lambda)_{n+1}}{n!} 2\sqrt{z}\, {}_2F_1\left(\begin{matrix} -n,\ \lambda+n+1 \\ -\frac{3}{2};\ z \end{matrix}\right).$

128. $C_n^\lambda\left(\frac{1}{\sqrt{z}}\right) = \frac{(\lambda)_n}{n!} 2^n z^{-n/2}\, {}_2F_1\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2} \\ 1-\lambda-n;\ z \end{matrix}\right).$

129. $C_n^\lambda(1+z) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n,\ 2\lambda+n \\ \lambda+\frac{1}{2};\ -\frac{z}{2} \end{matrix}\right).$

130. $C_n^\lambda\left(1+\frac{1}{z}\right) = \frac{(\lambda)_n}{n!}\left(\frac{2}{z}\right)^n {}_2F_1\left(\begin{matrix} -n,\ \frac{1}{2}-\lambda-n \\ 1-2\lambda-2n;\ -2z \end{matrix}\right).$

131. $(1-z)^n C_n^\lambda\left(\frac{1+z}{1-z}\right) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n,\ \frac{1}{2}-\lambda-n \\ \lambda+\frac{1}{2};\ z \end{matrix}\right).$

132. $C_{2n}^\lambda\left(\sqrt{1+z}\right) = \frac{(2\lambda)_{2n}}{(2n)!}\, {}_2F_1\left(\begin{matrix} -n,\ \lambda+n \\ \lambda+\frac{1}{2};\ -z \end{matrix}\right).$

133. $\frac{1}{\sqrt{1+z}} C_{2n+1}^\lambda\left(\sqrt{1+z}\right) = \frac{(2\lambda)_{2n+1}}{(2n+1)!}\, {}_2F_1\left(\begin{matrix} -n,\ \lambda+n+1 \\ \lambda+\frac{1}{2};\ -z \end{matrix}\right).$

134. $C_n^\lambda\left(\frac{z+1}{2\sqrt{z}}\right) = \frac{(\lambda)_n}{n!} z^{-n/2}\, {}_2F_1\left(\begin{matrix} -n,\ \lambda \\ 1-\lambda-n;\ z \end{matrix}\right).$

135. $(1+z)^{n/2} C_n^\lambda\left(\frac{1}{\sqrt{1+z}}\right) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2} \\ \lambda+\frac{1}{2};\ -z \end{matrix}\right).$

136. $C_{2n}^\lambda\left(\sqrt{1+\frac{1}{z}}\right) = \frac{(\lambda)_{2n}}{(2n)!} 2^{2n} z^{-n}\, {}_2F_1\left(\begin{matrix} -n,\ \frac{1}{2}-\lambda-n \\ 1-\lambda-2n;\ -z \end{matrix}\right).$

137. $$\frac{1}{\sqrt{1+z}} C_{2n+1}^{\lambda}\left(\sqrt{1+\frac{1}{z}}\right) = \frac{(\lambda)_{2n+1}}{(2n+1)!} 2^{2n+1} z^{-n-1/2} {}_2F_1\left(\begin{matrix} -n,\ \frac{1}{2}-\lambda-n \\ -\lambda-2n;\ -z \end{matrix}\right).$$

138. $$\left[C_n^{\lambda}\left(\sqrt{1+z}\right)\right]^2 = \left[\frac{(2\lambda)_n}{n!}\right]^2 {}_3F_2\left(\begin{matrix} -n,\ \lambda,\ 2\lambda+n;\ -z \\ \lambda+\frac{1}{2},\ 2\lambda \end{matrix}\right).$$

139. $$P_n^{(\rho,\sigma)}(1+z) = \frac{(\rho+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n,\ \rho+\sigma+n+1 \\ \rho+1;\ -\frac{z}{2} \end{matrix}\right).$$

140. $$P_n^{(\rho,\sigma)}\left(1+\frac{1}{z}\right) = \frac{(\rho+\sigma+n+1)_n}{n!} (2z)^{-n} {}_2F_1\left(\begin{matrix} -n,\ -\rho-n;\ -2z \\ -\rho-\sigma-2n \end{matrix}\right).$$

141. $$(1-z)^n P_n^{(\rho,\sigma)}\left(\frac{1+z}{1-z}\right) = \frac{(\rho+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n,\ -\sigma-n \\ \rho+1;\ z \end{matrix}\right).$$

142. $$P_n^{(\rho,\sigma)}\left(\sqrt{1+z}\right) P_n^{(\rho,\sigma)}\left(-\sqrt{1+z}\right)$$
$$= (-1)^n \frac{(\rho+1)_n(\sigma+1)_n}{(n!)^2} {}_4F_3\left(\begin{matrix} -n,\ \frac{\rho+\sigma+1}{2},\ \frac{\rho+\sigma}{2}+1,\ \rho+\sigma+n+1;\ -z \\ \rho+1,\ \sigma+1,\ \rho+\sigma+1 \end{matrix}\right).$$

143. $$\left\{\begin{matrix} \mathbf{K}(\sqrt{z}) \\ \mathbf{E}(\sqrt{z}) \end{matrix}\right\} = \frac{\pi}{2} {}_2F_1\left(\begin{matrix} \pm\frac{1}{2},\ \frac{1}{2} \\ 1;\ z \end{matrix}\right).$$

144. $$\frac{1}{1-z} \mathbf{E}(\sqrt{z}) = \frac{\pi}{2} {}_2F_1\left(\begin{matrix} \frac{1}{2},\ \frac{3}{2} \\ 1;\ z \end{matrix}\right).$$

145. $$\mathbf{D}(\sqrt{z}) = \frac{\pi}{4} {}_2F_1\left(\begin{matrix} \frac{1}{2},\ \frac{3}{2} \\ 2;\ z \end{matrix}\right).$$

References

Monographs

1. Prudnikov A. P., Brychkov Yu. A., Marichev O. I., *Integrals and Series. V. 1. Elementary Functions*, New York–London: Gordon & Breach, 1986.
2. —, —, —, *Integrals and Series. V. 2. Special Functions*, New York–London: Gordon & Breach, 1986.
3. —, —, —, *Integrals and Series. V. 3. More Special Functions*, New York–London: Gordon & Breach, 1990.
4. —, —, —, *Integrals and Series. V. 4. Laplace Transforms*, New York–London: Gordon & Breach, 1992.
5. —, —, —, *Integrals and Series. V. 5. Inverse Laplace Transforms*, New York–London: Gordon & Breach, 1992.
6. Bateman H., Erdélyi A., *Higher Transcendental Functions, V. 1*, New York–Toronto: London, McGraw-Hill, 1953.
7. —, —, *Higher Transcendental Functions, V. 2*, New York–Toronto: London, McGrow-Hill, 1953.
8. Abramowitz M., Stegun I. A. (ed.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover, 1992.
9. Khadzhi P. I., *The Error Function*, Institute of Applied Physics, Acad. Sci. Mold. SSR, Kishinev, 1971.
10. Luke Y. L., *The Special Functions and their Approximations, V. 1*, New York: Academic Press, 1969.
11. Magnus W., Oberhettinger F., Soni R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics*, New York–Berlin: Springer-Verlag, 1966.
12. Rainville E. D., *Special Functions*, New York: Macmillan, 1963.
13. Schwatt I. J., *An Introduction to the Operations with Series*, New York: Chelsea, 1924.
14. Srivastava H. M., Manocha H. L., *A Treatise on Generating Functions*, New York: Halsted Press, 1984.

Journal papers

15. Adamchik V., The evaluation of integrals of Bessel functions via G-function identities, *J. Comput. Appl. Math.*, 64 (1995), N 3, 283–290.
16. —, A class of logarithmic integrals, In: *Proceedings of ISSAC*, 1997, New York: Academic Press, 2001, 1–8.
17. Al-Saqabi B. N., Kalla S. L., Srivastava H. M., A certain family of of infinite series associated with digamma functions, *J. Math. Anal. Appl.*, 159 (1991), N 1, 361–372.

18. Bailey W. N., Products of generalized hypergeometric series, *Proc. London Math. Soc.*, (2) 28(1928), 242–254.

19. Berndt B., Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan, *J. Reine Angew. Math.*, 303/304 (1978), 332–335.

20. Brown B. M., Eastham M. S. P., A note on the Dixon formula for a finite hypergeometric series, *J. Comp. Appl. Math.*, 194 (2006), N 1, 173–175.

21. Brychkov Yu. A., Evaluation of some classes of definite and indefinite integrals, *Integral Transforms Spec. Func.*, 13 (2002), N 2, 163–167.

22. —, On some new series of special functions, *Appl. Math. Comput.*, 187 (2007), N 1, 101–104.

23. —, On some formulas for Weber $\mathbf{E}_\nu(z)$ and Anger $\mathbf{J}_\nu(z)$ functions, *Integral Transforms Spec. Func.*, 18 (2007), N 1–2, 15–26.

24. —, Geddes K.O., Differentiation of hypergeometric functions with respect to parameters. In: *Proc. Int. Conf. on Abstract and Applied Analysis, Hanoi, Vietnam*, 13–17 Aug. 2002, Singapore: World Scientific P. C., 2004, 15–28.

25. Carlitz L., The product of two ultraspherical polynomials, *Proc. Glasgow Math. Assoc.*, 5 (1961), 76–79.

26. Carlson B. C., Some extensions of Lardner's relations between ${}_0F_3$ and Bessel functions, *SIAM J. Math. Anal.*, 1 (1970), N 2, 232–242.

27. Chen Kung-Yu, Liu Shuoh-Jung, Srivastava H. M., Some double-series identities and associated generating function relationships, *Appl. Math. Lett.*, 19 (2006), N 9, 887–893.

28. Choi Junesang, Srivastava H. M., Explicit evaluation of Euler and related sums, *The Ramanujan J.*, 10 (2005), N 1, 51–70.

29. De Doelder P. J., On some series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y, *J. Comput. Appl. Math.*, 37 (1991), N 1–3, 125–141.

30. Exton H., On the reducibility of the Kampé de Fériet function, *J. Comp. Appl. Math.*, 83 (1997), N 1, 119–121.

31. —, Some new summation formula for the generalized hypergeometric function of higher order, *J. Comp. Appl. Math.*, 79 (1997), N 2, 183–187.

32. —, New hypergeometric transformations, *J. Comp. Appl. Math.*, 92 (1998), N 2, 135–137.

33. Feldheim E., Alcuni risultati sulle funzioni di Whittaker e del cilindro parabolico, *Atti Acc. Sci. Torino*, 76 (1941), 541–555.

34. —, Relations entre les polynomes de Jacobi, Laguerre et Hermite, *Acta Math.*, 75 (1943), 117–138.

35. Fox C., The expression of hypergeometric series in terms of similar series, *Proc. London Math. Soc.*, 26 (1927), 201–210.

36. Froehlich J., Parameter derivatives of the Jacobi polynomials and the Gaussian hypergeometric function, *Integral Transforms Spec. Func.*, 2 (1994), N 4, 252–266.

37. Furtlehner C., Ouvry S., Integrals involving four Macdonald functions and their relation to $7\zeta(3)/2$, http://arXiv:math-ph/0306004 v2 1 Apr. 2004.

38. Gessel I., Stanton D., Strange evaluation of hypergeometric series, *SIAM J. Math. Anal.*, 13 (1982), N 2, 295–307.

39. Gosper R. W., Ismail M. E. H., Zhang R., On some strange summation formulas, *Illinois J. Math.*, 37 (1993), N 2, 240–277.

40. Grosjean C. C., Some new integrals arising from mathematical physics. III, *Simon Stevin*, 43 (1969), 3–46.

41. —, Some new integrals arising from mathematical physics. IV, *Simon Stevin*, 45 (1971/72), N 3–4, 321–383.
42. Guillera J., Some binomial series obtained by the WZ-method, *Adv. Appl. Math.*, 29 (2002), 599–603.
43. —, Series closely related to Ramanujan formulas for pi, 2002, 1–8, http://personal.auna.com/jguillera.
44. —, About a new kind of Ramanujan-type series, *Exp. Math.*, 12 (2003), N 4, 507–510.
45. —, Generators of some Ramanujan formulas, *Ramanujan J.*, 11 (2006), N 1, 41–48.
46. —, Hypergeometric identities for 10 extended Ramanujan-type series, *The Ramanujan J.*, 15 (2008), N 2, 219–234.
47. Gustavsson J., Some sums of Legendre and Jacobi polynomials, *Math. Bohem.*, 126 (2001), N 1, 141–149.
48. Joyce G. S., Zucker I. J., Special values of the hypergeometric functions, *Math. Proc. Cambridge Philos. Soc.*, 109 (1991), N 2, 257–261.
49. —, —, Special values of the hypergeometric functions. II, *Math. Proc. Cambridge Philos. Soc.*, 131 (2001), N 2, 309–319.
50. Kanemitsu Sh., Kumagai H., Yoshimoto M., Sums involving Hurwitz Zeta function, *Ramanujan J.*, 5 (2001), N 1, 5–19.
51. Karlsson P. W., On two hypergeometric summation formulas conjectured by Gosper, *Simon Stevin*, 60 (1966), N 4, 329–337.
52. —, Clausen hypergeometric series with variable 1/4, *J. Math. Anal. Appl.*, 196 (1995), N 1, 172–180.
53. Kelber Ju., Schnittler Ch., Übensee H., Integralformeln für Airyfunktionen, *Wiss. Z. TH Ilmenau*, 31 (1985), N 4, 147–156.
54. Koepf W., Identities for families of orthogonal polynomials and special functions, *Integral Transforms Spec. Func.*, 5 (1997), N 1–2, 69–102.
55. Krattenthaler C., Srinivasa Rao K., Automatic generation of hypergeometric identities by the beta integral method, *J. Comp. Appl. Math.*, 160 (2003), N 1–2, 159–173.
56. —, Rivoal T., How can we escape Thomae's relations, *J. Math. Soc. Japan*, 58 (2006), 183–210.
57. Laurenzi B. J., Moment integrals of power of Airy functions, *Z. Angew. Math. Phys.*, 44 (1993), N 5, 891–908.
58. Miller A. R., Summations for certain series containing the digamma function, *J. Phys. A: Math. Gen.*, 39 (2006), 3011–3020.
59. Miller J., Adamchik V. S., Derivatives of the Hurwitz zeta function for rational arguments, *J. Comput. Appl. Math.*, 100 (1998), N 2, 201–206.
60. Muzaffar H., Williams K. S., A restricted Epstein zeta function and the evaluation of some definite integrals, *Acta Arithm.*, 104 (2002), N 1, 23–66.
61. Ogreid O. M., Osland P., Summing one- and two-dimensional series related to the Euler series, *J. Comput. Appl. Math.*, 98 (1998), N 2, 245–271.
62. —, —, More series related to the Euler series, *J. Comput. Appl. Math.*, 136 (2001), N 1–2, 389–403.
63. —, —, Some infinite series related to Feynman diagrams, *J. Comput. Appl. Math.*, 140 (2002), N 1–2, 659–671.
64. Rahman M., Verma A., Quadratic transformation formulas for basic hypergeometric series, *Trans. Amer. Math. Soc.*, 335 (1993), 277–302.

65. Reid W.H., Integral representations for products of Airy functions, *ZAMP*, 46 (1995), N 2, 159–170.
66. —, Integral representations for products of Airy functions. Part 2. Cubic products, *ZAMP*, 48 (1997), N 4, 646–655.
67. —, Integral representations for products of Airy functions. Part 3. Quartic products, *ZAMP*, 48 (1997), N 4, 656–664.
68. Rottbrand K., Finite-sum rules for MacDonald's function and Hankel's symbols, *Integral Transform. Spec. Func.*, 10 (2000), N 2, 115–124.
69. Sánchez-Ruiz J., Linearization and connection formulae involving squares of Gegenbauer polynomials, *Appl. Math. Letters*, 14 (2001), N 3, 261–267.
70. Lin Shy-Der, Srivastava H.M., Wang Pin-Yu, Some families of hypergeometric transformations and generating functions, *Mathematical and computer modelling*, 36 (2002), 445–459.
71. Srivastava H.M., Remarks on some series expansions associated with certain products of the incomplete gamma functions, *Comput. Math. Appl.*, 46 (2003), N 12, 1749–1759.
72. Stanković M.S., Vidanović M.V., Tričković S.B., Some series over the product of two trigonometric functions und series involving Bessel functions, *Z. Anal. Anwendungen*, 20 (2001), N 1, 235–246.
73. Szmytkowski R., On the derivative of the Legendre function of the first kind with respect to its degree, *J. Phys. A: Math. Gen.*, 39 (2006), 15147–15172.
74. Toscano L., Sulle derivate dei polinomi di Laguerre e del tipo ultrasferico rispetto al parametro, *Boll. Un. Mat. Ital.*, 8 (1953), 193–195.
75. —, Le funzioni del cilindro parabolico come caso limite delle funzioni ipergeometriche, *Boll. Un. Mat. Ital.* (3), 9 (1954), 29–38.
76. —, Su una formula limite tra funzioni ipergeometriche di Kummer e funzioni del cilindro parabolico, *Boll. Un. Mat. Ital.*, 10 (1955), 239–243.
77. —, Contributo alla formazione del formulario dei polinomi ultrasferici, *Giornale di matematiche di Battaglini* (5), 89 (1961), 85–109.
78. —, Formule di derivazione su particolari funzioni ipergeometriche di Gauss i di Kummer, *Matematiche* (Catania), 20 (1965), 156–167.
79. —, Formule di derivazione per le funzioni ipergeometriche di Gauss, *Riv. Mat. Univ. Parma* (2), 8 (1967), 163–180.
80. —, Sulle funzioni del cilindro parabolico, *Matematiche* (Catania), 26 (1971), 104–126.
81. Tremblay R., Fugère B.J., Products of two restricted hypergeometric functions, *J. Math. Anal. Appl.*, 198 (1996), N 3, 844–852.
82. Tricomi F., Sulla formula d'inversione di Widder, *Rend. R. Acc. Naz. Lincei*, 25 (1936).
83. Vidunas R., A generalization of Kummer's identity. http://arXiv.org/abs/math.CA/0005095.
84. Zucker I.J., Some infinite series of exponential and hyperbolic functions, *SIAM J. Math. Anal.*, 15 (1984), N 2, 406–413.
85. —, Joyce G.S., Delves R.T., On the evaluation of the integral $\int\limits_0^{\pi/4} \ln\left(\cos^{m/n}\theta + \sin^{m/n}\theta\right) d\theta$, *Ramanujan J.*, 2 (1998), N 3, 317–326.

Index of Notations for Functions and Constants

$\mathrm{Ai}\,(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}}\,K_{1/3}\left(\frac{2}{3}\,z^{3/2}\right)$ is the Airy function

$\arccos z$, $\operatorname{arccot} z$, $\arcsin z$, $\arctan z$ are inverse trigonometric functions

B_n are the Bernoulli numbers

$B_n(z)$ are the Bernoulli polynomials

$\mathrm{bei}_\nu(z)$, $\mathrm{ber}_\nu(z)$, $\mathrm{bei}\,(z) \equiv \mathrm{bei}_0(z)$, $\mathrm{ber}\,(z) \equiv \mathrm{ber}_0(z)$ are the Kelvin functions

$\mathrm{Bi}\,(z) = \sqrt{\frac{z}{3}}\,\left[I_{-1/3}\left(\frac{2}{3}\,z^{3/2}\right) + I_{1/3}\left(\frac{2}{3}\,z^{3/2}\right)\right]$ is the Airy function

$\mathbf{C} = -\psi(1) = 0,5772156649\ldots$ is the Euler constant

$C(z) = \frac{1}{\sqrt{2\pi}}\int\limits_0^z \frac{\cos t}{\sqrt{t}}\,dt$ is the Fresnel cosine integral

$C(z,\,\nu) = \int\limits_z^\infty t^{\nu-1}\cos t\,dt \quad [\mathrm{Re}\,\nu < 1]$ is the generalized Fresnel cosine integral

$C_n^\lambda(z) = \frac{(2\lambda)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\,n+2\lambda\\ \lambda+\frac{1}{2};\ \frac{1-z}{2}\end{matrix}\right)$ are the Gegenbauer polynomials

$\mathrm{chi}\,(z) = \mathbf{C} + \ln z + \int\limits_0^z \frac{\cosh t - 1}{t}\,dt$ is the hyperbolic cosine integral

$\mathrm{ci}\,(z) = -\int\limits_z^\infty \frac{\cos t}{t}\,dt$ is the cosine integral

$\mathrm{Cl}_2(z) = -\int\limits_0^z \ln\left(2\sin\frac{t}{2}\right)dt$ is the Clausen integral

$\mathrm{D} = \frac{d}{dz}$, $\mathrm{D}_a = \frac{d}{da}$

$\mathbf{D}(k) = \int\limits_0^{\pi/2} \frac{\sin^2 t\,dt}{\sqrt{1-k^2\sin^2 t}}$ is the complete elliptic integral

$D_\nu(z) = 2^{\nu/2} e^{-z^2/4} \Psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right)$ is the parabolic cylinder function

$\mathbf{E}(k) = \int\limits_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt$ is the complete elliptic integral of the second kind

E_n are the Euler numbers

$E_n(z)$ are the Euler polynomials

$\mathbf{E}_\nu(z) = \frac{1}{\pi} \int\limits_0^{\pi} \sin(\nu t - z \sin t)\, dt$ is the Weber function

$\mathrm{Ei}\,(z) = \int\limits_{-\infty}^{z} \frac{e^t}{t}\, dt$ is the exponential integral

$\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int\limits_0^{z} e^{-t^2}\, dt$ is the error function

$\mathrm{erfc}\,(z) = 1 - \mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int\limits_z^{\infty} e^{-t^2}\, dt$ is the complementary error function

$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}} \int\limits_0^{z} e^{t^2}\, dt$ is the error function of imaginary argument

$${}_2F_1\left(\begin{matrix} a,\ b;\ z \\ c \end{matrix}\right) \equiv {}_2F_1\left(\begin{matrix} a,\ b \\ c;\ z \end{matrix}\right) \equiv {}_2F_1(a,\ b;\ c;\ z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \qquad [|z| < 1],$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \ [\mathrm{Re}\, c > \mathrm{Re}\, b > 0;\ |\arg(1-z)| < \pi]$$

is the Gauss hypergeometric function

$${}_pF_q\left(\begin{matrix} (a_p);\ z \\ (b_q) \end{matrix}\right) \equiv {}_pF_q\left(\begin{matrix} (a_p) \\ (b_q);\ z \end{matrix}\right) \equiv {}_pF_q((a_p);\ (b_q);\ z)$$

$$\equiv {}_pF_q(a_1, \ldots, a_p;\ b_1, \ldots, b_q;\ z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{z^k}{k!}$$

is the generalized hypergeometric function

${}_1F_1\left(\begin{matrix} a;\ z \\ b \end{matrix}\right) \equiv {}_1F_1\left(\begin{matrix} a \\ b;\ z \end{matrix}\right) \equiv {}_1F_1(a;\ b;\ z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}$ is the Kummer confluent hypergeometric function

$\mathbf{G} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0{,}9159655942\ldots$ is the Catalan constant

$$G_{pq}^{mn}\left(z \middle| \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right) \equiv G_{pq}^{mn}\left(z \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$= \frac{1}{2\pi i} \int\limits_L \frac{\Gamma(b_1+s)\ldots\Gamma(b_m+s)\Gamma(1-a_1-s)\ldots\Gamma(1-a_n-s)}{\Gamma(a_{n+1}+s)\ldots\Gamma(a_p+s)\Gamma(1-b_{m+1}-s)\ldots\Gamma(1-b_q-s)} z^{-s}\, ds,$$

$L = L_{\pm\infty},\ L_{i\infty},$ is the Meijer G function

$\mathbf{H}_\nu(z) = \dfrac{2\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_1F_2\left(\begin{matrix}1;\ -\frac{z^2}{4}\\ \frac{3}{2},\ \nu+\frac{3}{2}\end{matrix}\right)$ is the Struve function

$H_\nu^{(1)}(z) = J_\nu(z) + i\,Y_\nu(z)$, $H_\nu^{(2)}(z) = J_\nu(z) - i\,Y_\nu(z)$ are the Hankel functions of the first and second kind (the Bessel functions of the third kind)

$H_n(z) = (-1)^n e^{z^2}\dfrac{d^n}{dz^n}e^{-z^2}$ are the Hermite polynomials

$I_\nu(z) = \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{z}{2}\right)^\nu {}_0F_1\left(\nu+1;\ \dfrac{z^2}{4}\right) = e^{-\nu\pi i/2}J_\nu\left(e^{\pi i/2}z\right)$ is the modified Bessel function of the first kind

$J_\nu(z) = \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{z}{2}\right)^\nu {}_0F_1\left(\nu+1;\ -\dfrac{z^2}{4}\right)$ is the Bessel function of the first kind

$\mathbf{J}_\nu(z) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(\nu t - z\sin t)\,dt$ is the Anger function

$\mathbf{K}(k) = \displaystyle\int_0^{\pi/2}\dfrac{dt}{\sqrt{1-k^2\sin^2 t}}$ is the complete elliptic integral of the first kind

$K_\nu(z) = \dfrac{\pi[I_{-\nu}(z) - I_\nu(z)]}{2\sin\nu\pi}$ $[\nu \neq n]$, $K_n(z) = \lim\limits_{\nu\to n} K_\nu(z)$ $[n = 0, \pm1, \pm2, \ldots]$ is the Macdonald function (the modified Bessel function of the third kind)

$\mathrm{kei}_\nu(z)$, $\mathrm{ker}_\nu(z)$, $\mathrm{kei}\,(z) = \mathrm{kei}_0(z)$, $\mathrm{ker}\,(z) = \mathrm{ker}_0(z)$ are the Kelvin functions

$\mathbf{L}_\nu(z) = e^{-(\nu+1)\pi i/2}\mathbf{H}_\nu\left(e^{\pi i/2}z\right)$ is the modified Struve function

$L_n(z) = L_n^0(z)$ are the Laguerre polynomials

$L_n^\lambda(z) = \dfrac{z^{-\lambda}e^z}{n!}\dfrac{d^n}{dz^n}\left(z^{n+\lambda}e^{-z}\right)$ are the generalized Laguerre polynomials

$\mathrm{Li}_\nu(z) = \displaystyle\sum_{k=1}^\infty \frac{z^k}{k^\nu}$ $[|z| < 1]$,

$= \dfrac{z}{\Gamma(\nu)}\displaystyle\int_0^\infty \frac{t^{\nu-1}\,dt}{e^t - z}$ $[\mathrm{Re}\,\nu > 0;\ |\arg(1-z)| < \pi]$ is the polylogarithm of the order ν

$\mathrm{Li}_2(z)$ is the Euler dilogarithm

$M_{\varkappa,\mu}(z) = z^{\mu+1/2}e^{-z/2}\,{}_1F_1\left(\begin{matrix}\mu-\varkappa+\frac{1}{2}\\ 2\mu+1;\ z\end{matrix}\right)$ is the Whittaker confluent hypergeometric function

$P_n(z) = \dfrac{2^{-n}}{n!}\dfrac{d^n}{dz^n}(z^2-1)^n$ are the Legendre polynomials

$P_\nu(z) \equiv P_\nu^0(z) = {}_2F_1\left(\begin{matrix}-\nu,\ 1+\nu\\ 1;\ \frac{1-z}{2}\end{matrix}\right)$ $[|\arg(1+z)| < \pi]$ is the Legendre function of the first kind

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(\begin{matrix} -\nu,\ \nu+1 \\ 1-\mu;\ \frac{1-z}{2}\end{matrix}\right)$$

$[|\arg(z \pm 1)| < \pi;\ \mu \neq m;\ m = 1, 2, \dots]$

$$P_\nu^m(z) = (z^2-1)^{m/2}\left(\frac{d}{dz}\right)^m P_\nu(z) \qquad [|\arg(z-1)| < \pi;\ m = 1, 2, \dots]$$

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(\begin{matrix} -\nu,\ \nu+1 \\ 1-\mu;\ \frac{1-x}{2}\end{matrix}\right)$$

$[-1 < x < 1;\ \mu \neq m;\ m = 1, 2, \dots]$

$$P_\nu^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_\nu(x) \qquad [-1 < x < 1;\ m = 1, 2, \dots]$$

is the associated Legendre function of the first kind

$$P_n^{(\rho,\sigma)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\rho}(1+z)^{-\sigma}\frac{d^n}{dz^n}\left[(1-z)^{\rho+n}(1+z)^{\sigma+n}\right]$$

$$= \frac{(\rho+1)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n,\ \rho+\sigma+n+1 \\ \rho+1;\ \frac{1-z}{2}\end{matrix}\right)$$ are the Jacobi polynomials

$Q_\nu(z) \equiv Q_\nu^0(z)$ is the Legendre function of the second kind

$$Q_\nu^\mu(z) = \frac{e^{i\mu\pi}\sqrt{\pi}}{2^{\nu+1}}\Gamma\left[\begin{matrix}\mu+\nu+1\\ \nu+3/2\end{matrix}\right] z^{-\mu-\nu-1}(z^2-1)^{\mu/2}\,{}_2F_1\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu}{2}+1\\ \nu+\frac{3}{2};\ \frac{1}{z^2}\end{matrix}\right)$$

$[|\arg z|, |\arg(z \pm 1)| < \pi;\ \nu + 1/2,\ \mu+\nu \neq -1, -2, -3, \dots]$

$$Q_{-n-3/2}^\mu(z) = \frac{e^{i\mu\pi}\sqrt{\pi}\,\Gamma(\mu+n+3/2)}{2^{n+3/2}(n+1)!} \times z^{-\mu-n-3/2}(z^2-1)^{\mu/2}\,{}_2F_1\left(\begin{matrix}\frac{2\mu+2n+3}{4},\ \frac{2\mu+2n+5}{4}\\ n+2;\ \frac{1}{z^2}\end{matrix}\right)$$

$[|\arg(z \pm 1)|, |\arg z| < \pi;\ \mu+\nu \neq -1, -2, -3, \dots]$

$$Q_\nu^\mu(x) = \frac{e^{-i\mu\pi}}{2}\left[e^{-\mu\pi/2}Q_\nu^\mu(x+i0) + e^{i\mu\pi/2}Q_\nu^\mu(x-i0)\right]$$

$$= \frac{\pi}{2\sin\mu\pi}\left[P_\nu^\mu(x)\cos\mu\pi - \Gamma\left[\begin{matrix}\nu+\mu+1\\ \nu-\mu+1\end{matrix}\right]P_\nu^{-\mu}(x)\right]$$

$[-1 < x < 1;\ \mu \neq \pm m;\ \mu+\nu \neq -1, -2, -3, \dots],$

$$= (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m Q_\nu(x) \qquad [\mu = m;\ \nu \neq -m-1, -m-2, \dots],$$

$$= (-1)^m\Gamma\left[\begin{matrix}\nu-m+1\\ \mu+m+1\end{matrix}\right]Q_\nu^m(x) \qquad [\mu = -m;\ \nu \neq -m-1, -m-2, \dots]$$

is the associated Legendre function of the second kind

$S(z) = \frac{1}{\sqrt{2\pi}}\int_0^z \frac{\sin t}{\sqrt{t}}\,dt$ is the Fresnel cosine integral

$S(z, \nu) = \int_z^\infty t^{\nu-1}\sin t\,dt \quad [\operatorname{Re}\nu < 1]$ is the generalized Fresnel sine integral

$S_n^{(m)} = \sum_{k=0}^{n-m}(-1)^k\binom{n+k-1}{n-m+k}\binom{2n-m}{n-m-k}\sigma_{n-m+k}^k$ are the Stirling numbers of the first kind

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0 \end{cases}$$

$\operatorname{shi}(z) = \int\limits_0^z \frac{\sinh t}{t}\,dt = -i\operatorname{Si}(iz)$ is the hyperbolic sine integral

$\operatorname{Si}(z) = \int\limits_0^z \frac{\sin t}{t}\,dt$ is the sine integral

$\operatorname{si}(z) = \operatorname{Si}(z) - \frac{\pi}{2} = -\int\limits_z^\infty \frac{\sin t}{t}\,dt$ is the sine integral

$T_n(z) = \cos(n \arccos z) = {}_2F_1\left(\begin{matrix} -n,\, n \\ \frac{1}{2};\ \frac{1-z}{2} \end{matrix}\right)$ are the Chebyshev polynomials of the first kind

$U_n(z) = \frac{\sin[(n+1)\arccos z]}{\sqrt{1-z^2}} = (n+1)\,{}_2F_1\left(\begin{matrix} -n,\, n+2 \\ \frac{3}{2};\ \frac{1-z}{2} \end{matrix}\right)$ are the Chebyshev polynomials of the second kind

$W_{\varkappa,\mu}(z) = z^{\mu+1/2}e^{-z/2}\Psi\left(\begin{matrix} \mu - \varkappa + \frac{1}{2} \\ 2\mu+1;\ z \end{matrix}\right)$ is the Whittaker confluent hypergeometric function

$Y_\nu(z) = \frac{\cos\nu\pi J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi}\quad [\nu \neq n],\quad Y_n(z) = \lim\limits_{\nu\to n} Y_\nu(z)\quad [n = 0, \pm1, \pm2, \ldots]$ is the Neumann function (the Bessel function of the second kind)

$\mathrm{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the beta function

$\mathrm{B}_z(\alpha, \beta) = \int\limits_0^z t^{\alpha-1}(1-t)^{\beta-1}\,dt\quad [\operatorname{Re}\alpha > 1;\ z < 1]$ is the incomplete beta function

$\Gamma(z) = \int\limits_0^\infty t^{z-1}e^{-t}dt\quad [\operatorname{Re} z > 0]$ is the gamma function

$\Gamma(\nu, z) = \int\limits_z^\infty t^{\nu-1}e^{-t}dt$ is the complementary incomplete gamma function

$\gamma(\nu, z) = \Gamma(\nu) - \Gamma(\nu, z) = \int\limits_0^z t^{\nu-1}e^{-t}dt\quad [\operatorname{Re}\nu > 0]$ is the incomplete gamma function

$$\Gamma[(a_p)] = \prod_{k=1}^p \Gamma(a_k)$$

$$\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}$$

$$\Delta(k, (a_p)) = \frac{(a_p)}{k}, \frac{(a_p)+1}{k}, \ldots, \frac{(a_p)+k-1}{k}$$

$\delta_{m,n} = \begin{cases} 0, & m \neq n, \\ 1, & m = n \end{cases}$ is the Kronecker symbol

$\zeta(z)=\sum_{k=1}^{\infty}\frac{1}{k^z}$ [Re $z>1$] is the Riemann zeta function

$\zeta(z,v)=\sum_{k=0}^{\infty}\frac{1}{(v+k)^z}$ [Re $z>1;\ v\neq 0,-1,-2,\dots$] is the Hurwitz zeta function

$\theta(x)=\begin{cases}1, & x\geq 0,\\ 0, & x<0\end{cases}$ is the Heaviside function

$$\Xi_1(a,a',b;\ c;\ w,z)=\sum_{k,l=0}^{\infty}\frac{(a)_k(a')_l(b)_k}{(c)_{k+l}}\frac{w^k z^l}{k!\,l!}\qquad [|w|<1]$$

$$\Xi_2(a,b;\ c;\ w,z)=\sum_{k,l=0}^{\infty}\frac{(a)_k(b)_k}{(c)_{k+l}}\frac{w^k z^l}{k!\,l!}\qquad [|w|<1]$$

$\sigma_n^m=\frac{1}{m!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n$ are the Stirling numbers of the second kind

$$\Phi(z,s,v)=\sum_{k=0}^{\infty}\frac{z^k}{(v+k)^s}\qquad [|z|<1;\ v\neq 0,-1,-2,\dots]$$

$$\Phi_1(a,b;\ c;\ w,z)=\sum_{k,l=0}^{\infty}\frac{(a)_{k+l}(b)_k}{(c)_{k+l}}\frac{w^k z^l}{k!\,l!}\qquad [|w|<1]$$

$$\Phi_2(b,b';\ c;\ w,z)=\sum_{k,l=0}^{\infty}\frac{(b)_k(b')_l}{(c)_{k+l}}\frac{w^k z^l}{k!\,l!},$$

$$\Phi_3(b;\ c;\ w,z)=\sum_{k,l=0}^{\infty}\frac{(b)_k}{(c)_{k+l}}\frac{w^k z^l}{k!\,l!}$$

$$\Psi\left(\begin{matrix}a;\ z\\ b\end{matrix}\right)\equiv\Psi\left(\begin{matrix}a\\ b;\ z\end{matrix}\right)\equiv\Psi(a;\ b;\ z)$$
$$=\frac{\Gamma(1-b)}{\Gamma(1+a-b)}\,{}_1F_1\left(\begin{matrix}a;\ z\\ b\end{matrix}\right)+\frac{\Gamma(b-1)}{\Gamma(a)}\,z^{1-b}\,{}_1F_1\left(\begin{matrix}1+a-b\\ 2-b;\ z\end{matrix}\right)$$

is the Tricomi confluent hypergeometric function

$$\psi_1(z)=\frac{4}{\pi^2}\,\mathbf{K}^2(x),$$

$$\psi_2(z)=\frac{4}{\pi^2}\left\{\mathbf{K}^2(x)-\frac{x^2}{1-x^2}\left[\mathbf{K}(x)-\mathbf{D}(x)\right]^2\right\},$$

$$\psi_3(z)=\frac{4}{\pi^2}\left\{3\,\mathbf{K}^2(x)-\frac{4\left(1-x^2+x^4\right)}{\left(1-x^2\right)^2}\left[\mathbf{K}(x)-\mathbf{D}(x)\right]^2\right.$$
$$\left.+\frac{1}{\left(1-x^2\right)^2}\left[2\left(1-2x^2\right)\mathbf{D}(x)-\left(1-3x^2\right)\mathbf{K}(x)\right]^2\right\}\qquad\left[x=\left(\frac{1-\sqrt{1-z}}{2}\right)^{1/2}\right]$$

$$\Psi_1(a,b;\ c,c';\ w,z)=\sum_{k,l=0}^{\infty}\frac{(a)_{k+l}(b)_k}{(c)_k(c')_l}\frac{w^k z^l}{k!\,l!}\qquad [|w|<1]$$

$$\Psi_2(a;\ c,c';\ w,z)=\sum_{k,l=0}^{\infty}\frac{(a)_{k+l}}{(c)_k(c')_l}\frac{w^k z^l}{k!\,l!}$$

$\psi(z)=[\ln\Gamma(z)]'=\frac{\Gamma'(z)}{\Gamma(z)}$ is the psi function

Index of Notations for Symbols

$(a) = a_1, a_2, \ldots, a_A$; $\quad (a_p) = a_1, a_2, \ldots, a_p$

$(a_p - b_p) = a_1 - b_1, a_2 - b_2, \ldots, a_p - b_p$

$(a) + s = a_1 + s, a_2 + s, \ldots, a_A + s$; $\quad (a_p) + s = a_1 + s, \ldots, a_p + s$

$(a)' - a_j = a_1 - a_j, \ldots, a_{j-1} - a_j, a_{j+1} - a_j, \ldots, a_A - a_j \qquad [1 \le j \le A]$

$(a_p)' - a_j = a_1 - a_j, \ldots, a_{j-1} - a_j, a_{j+1} - a_j, \ldots, a_p - a_j \qquad [1 \le j \le p]$

$(a)_k = a(a+1)\ldots(a+k-1) \quad [k = 1, 2, 3, \ldots]$, $\quad (a)_0 = 1$ is the Pochhammer symbol

$n! = 1 \cdot 2 \cdot 3 \ldots (n-1)n = (1)_n, \quad 0! = 1! = (-1)! = 1$

$(2n)!! = 2 \cdot 4 \cdot 6 \ldots (2n-2)2n = 2^n n!$

$$(2n+1)!! = 1 \cdot 3 \cdot 5 \ldots (2n+1) = \frac{2^{n+1}}{\sqrt{\pi}} \Gamma\left(n + \frac{3}{2}\right) = \left(\frac{3}{2}\right)_n 2^n$$

$$n!! = \begin{cases} (2k)!!, & n = 2k, \\ (2k+1)!!, & n = 2k+1, \end{cases} \qquad 0!! = (-1)!! = 1$$

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} = \frac{(-1)^k(-n)_k}{k!}, \qquad \binom{n}{0} = 1$$

$\operatorname{Re} a, \operatorname{Re} b > c$ means $\operatorname{Re} a > c$ and $\operatorname{Re} b > c$

$[x] = n\,[n \le x < n+1,\; n = 0, \pm 1, \pm 2, \ldots]$ is the integer part of x

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0 \end{cases}$$

$$\prod (a_p)_k = \prod_{j=1}^{p} (a_j)_k$$

$$\prod \left((a_p) + b\right)_k = \prod_{j=1}^{p} (a_j + b)_k$$

$$\prod_{k=m}^{n} a_k = a_m a_{m+1} \ldots a_n \qquad [n \ge m],$$
$$= 1 \qquad [n < m]$$

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_n \qquad [n \ge m],$$
$$= 0 \qquad [n < m]$$

$$\prod (a_p)_k = \prod_{j=1}^{p} (a_j)_k$$

$$\prod ((a_p) + b)_k = \prod_{j=1}^{p} (a_j + b)_k$$

$$\begin{aligned} \prod_{k=m}^{n} a_k &= a_m a_{m+1} \ldots a_n \quad [n \geq m], \\ &= 1 \quad [n < m] \end{aligned}$$

$$\prod_{k=1}^{\infty} a_k(z) = \lim_{n\to\infty} \prod_{k=1}^{n} a_k(z)$$

$$\begin{aligned} \sum_{k=m}^{n} a_k &= a_m + a_{m+1} + \ldots + a_n \quad [n \geq m], \\ &= 0 \quad [n < m] \end{aligned}$$

$$\sum_{k=1}^{\infty} a_k(z) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k(z)$$