ELECTRIC CIRCUITS
A PRIMER

JC Olivier

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Contents

Preface

This is a book on circuit theory at an introductory level. Why another book on electric circuits? The author believes there is a cohort of professional workers and students new to circuit theory who need a primer on the subject. For this cohort, working through a modern detailed textbook is a daunting prospect, and the sheer volume of material presented in such textbooks is often prohibitive. What is required is a terse volume that explains the key concepts directly and clearly, to the point, making use of worked examples. This is what this book professes to do.

Emphasis is placed on the use of models of the real world. The limitations and basis of these models are discussed, and it is shown that models enable the design of systems without detailed knowledge of the underlying physics. There are three parts to the book:

1. Part 1 presents circuits with sources that are static (meaning they don't change over time). For these circuits, we are able to reduce any circuit to a system of linear equations. The solution of the system of linear equations is based on MATLAB® and yields all the potentials and currents in the circuit. Armed with that knowledge, we can compute any quantity of interest.

2. Part 2 deals with sources that are dynamic. These types of circuits exhibit transient phenomena and require the solution of linear differential equations. The solutions yield all the voltages and currents (as a function of time) in the circuit. Again MATLAB is used to solve some key differential equations.

3. Part 3 presents circuits with sinusoidal sources, that is the AC paradigm. Here the mathematical tool we deploy is known as a *phasor*. These are mathematical objects based on complex number theory[1]. Much emphasis is placed

[1] These are not the type of phasors Captain Kirk and Mr. Spock deployed in *Star Trek* during the 1960s.

on computing power, interpreting power and energy, and compensating electrical systems if the power factor is too low.

Each chapter focusses on a specific topic, while building on previous chapters. Besides explaining the physics behind circuits, it also explains the models we use and each concept is illuminated via worked examples. In some cases additional solutions based on MATLAB are also provided.

The reader will notice that Laplace and Fourier transforms are not included in the text. That omission is not accidental. The author believes that the readers of this book will best understand and learn the material directly on the time domain. This is especially so as we are able to use direct time domain numerical solutions of linear systems of equations and differential equations using modern computers. These solutions are explained and highlighted in the text, several examples of direct numerical solution are provided, and the MATLAB source code is provided.

Circuit simulation using software such as SPICE is discussed but not used in this text. The author believes that a thorough understanding of the fundamentals of circuits and solutions using MATLAB is the key to understanding the material. References to circuit simulators and their history are provided, along with several URLs where additional information can be obtained in the bibliography. This is meant as resources not only for the reader but also for the instructor who may use this book as a basis for teaching.

The author wishes to acknowledge the support he received from staff and students at the University of Tasmania in Hobart (Australia), and he wishes to thank the anonymous reviewers of the draft manuscript (in the United States). Finally he is indebted to his children, Andre and Melissa, for their support during the writing of this book.

Part I

Static Circuits and Kirchhoff's Laws

Chapter 1

Static Fields, Energy, and Power

1.1 Potential Energy Concepts

There are four forces in nature: the gravitational force, the electromagnetic force, the nuclear force, and the weak force. The first two forces are probably the forces the reader is most familiar with, and can be directly observed. These are also long-range forces, while the nuclear and weak forces are short range forces responsible for the stability of atoms and radioactive decay, respectively.

All matter (atoms) known to humankind feel the effects of gravity. Gravity is well understood by physicists through the general theory of relativity (GR) [1]. GR is a complex theory requiring the use of sophisticated mathematics, and many engineers are not familiar with the theory. Yet they design systems deploying gravity to our advantage — consider hydroelectricity as an example. This is possible through the use of models[1]. A model is developed that is a good approximation to physical reality. As soon as the model has been verified to hold under the expected operating conditions, systems can be designed based on the model.

[1] Von Neumann commented as follows [2]: "The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work — that is, correctly to describe phenomena from a reasonably wide area."

The gravitational force is conservative, and we make use of the idea of potential energy. Consider a system located in a gravitational field induced by Earth, as shown in Figure 1.1. The system contains two identical ladders and a mass M, which is shown to be located at a height h_1 meters above Earth on step one of ladder 1. By applying a force the mass can be moved and occupy a different step on the ladder. The mass M is said to have a potential energy (measured in joules). As shown in the figure, the potential energy at step one of ladder 1 relative to Earth is given by

$$P_{1,E} = Mgh_1 \tag{1.1}$$

where g is the gravitational field strength on the surface of the Earth (it is approximately $10\ \mathrm{m/s^2}$, and on other planets it would be different). *An equipotential or isopotential refers to a region in space where every point in it is at the same potential.* This is indicated by the dotted lines in Figure 1.1.

Consider the closed path indicated in Figure 1.1. If the mass is moved along this path, each move (indicated by arrows in Figure 1.1) either requires or provides

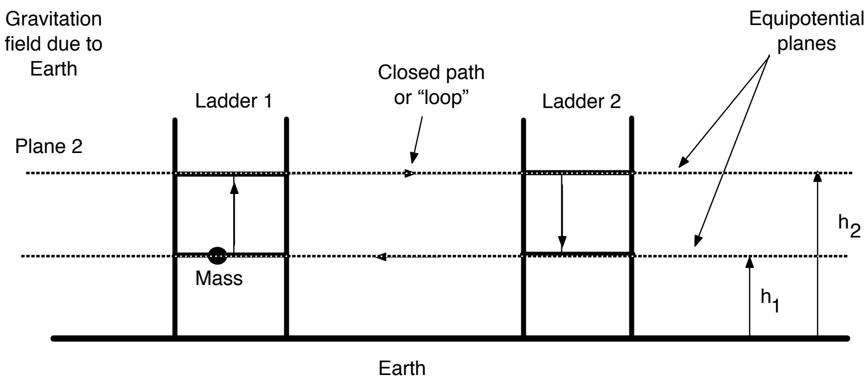

Figure 1.1 A system making use of two ladders and a mass M in the Earth's gravitational field. This serves to illustrate that the energy gained/lost in a closed loop sums to zero. This holds true for a conservative field where energy is conserved.

potential energy. Let us denote $\Delta h = h_2 - h_1$ then we find that

$$\begin{aligned} P_{\text{move}\,1} + P_{\text{move}\,2} + P_{\text{move}\,3} + P_{\text{move}\,4} &= \\ Mg\Delta h + Mg(h_2 - h_2) - Mg\Delta h + Mg(h_1 - h_1) &= \\ Mg\Delta h - Mg\Delta h &= 0. \end{aligned} \tag{1.2}$$

This physically means that if the mass M is moved (by applying a force) in a closed loop, no energy is lost or gained — the total energy required is zero. This observation is known as path independence, and is based on the law of the conservation of energy, as the gravitational field is conservative. This is an important observation, one that will also be shown to hold true in an electric field, which is also conservative.

1.2 An Electric Field: Potential Source Model

Current and voltage are the two basic variables in an electrical system and can be defined based on an understanding of the electromotive force and the effect this force has on charged particles.

The electromagnetic force or electric field is also a force of nature. It is a long-range force and the field is conservative like the gravitational field, but it differs in some key respects: This force only affects matter (particles) that possess a property called *charge*, unlike gravity that affects all matter. This means only charged particles feel the electric field or force. Charge comes in two flavors, positive and negative, unlike gravity. Where in gravity two particles always attract, two identically charged particles repel each other under the action of the electric field. However oppositely charged particles attract each other. Any charged particle, or collection of charged particles creates an electric field.

It is known from elementary physics that all matter is made of fundamental building blocks known as atoms, and that an atom consists of electrons, protons, and neutrons. It is also known that the charge e on an electron is negative and equal in magnitude to 1.602×10^{-19} coulombs (which is the unit of measurement of charge). A proton carries a positive charge of the same magnitude as the electron although it is much heavier. The presence of equal numbers of protons and electrons leaves an atom neutrally charged. If an atom loses an electron it becomes charged and is known as an ion. Atoms also contain neutrons, these particles have almost the same mass as protons but have no charge and do not feel the electromagnetic force.

Quantum electrodynamics (QED) [3] is a rather sophisticated theory that is able to explain these observed phenomena. The theory has been tested in every imaginable configuration, and is famous for its accuracy and versatility. Many electrical engineers are not proficient in applying QED to real-world systems. Yet the reader probably knows that electrical systems work well, as designs are based on models of electrical phenomena. Once these models have been verified to hold, engineers use them to design.

The electromagnetic force model we will make use of says that between two charged objects (with charges q_1 and q_2) there is a force on each object that is proportional to the square of the distance (r) between them, given by

$$F_{\text{electric}} = k\frac{q_1\, q_2}{r^2} \tag{1.3}$$

where k is a universal constant. This model is known as Coulomb's law. If the charges have the same sign the force is repulsive and if they are oppositely charged the force is attracting. It has been shown that the model is very accurate under normal practical conditions. If the objects are free to move, the force between the charged objects causes the objects to accelerate according to Newton's laws.

The force between the charges creates a *field*, in this case an electric field. If the charges are prevented from accelerating (i.e., are held in position) then the field between the charges would be sustained indefinitely. Any other charged particle that is mobile and finds itself in this field would be accelerated by the electromagnetic force.

1.2.1 A Conductor

A conductor contains atoms that are willing to have their electrons move around inside the material. We deploy a model of this process: *it requires no energy to move electrons in the conductor.* Thus we model a conductor as an equipotential plane. By now the reader should have an understanding what this means: the potential difference at a point A in a conductor and another point B in the same conductor is zero. It does not matter how many twists and turns the conductor makes, as long as its all one single uninterrupted conductor containing points A and B, this model will be effective. Examples of good conductors are the metals, such as copper, silver, and gold. Copper is not the best conductor, but is cheaper than silver or gold and that explains its use in electrical systems.

The model explained above for conductors thus assumes that there is no electric field inside a conductor (or else it cannot be an equipotential plane). Sophisticated experiments have confirmed this to be the case. For example, inside a closed copper dome (known as a Faraday cage) no electric field is measured, regardless of the intensity of the field outside the dome. This is shown in Figure 1.2, where the occupant inside a conducting dome is unable to perceive the intense field outside the dome. This is a vivid demonstration of the validity of the model presented above.

1.2.2 Definition of a Voltage

Consider the system shown in Figure 1.3, where an electric field is present between two charged planes. Also shown is a test charge located at a height h_1 meters above

Figure 1.2 Experimental verification of the validity of the model of a conductor such as copper. Inside a conductor the potential difference between any two points is zero — thus there is no field present. (Source: Wikipedia — Printed in black and white under license http://creativecommons.org/licenses/by-sa/3.0.)

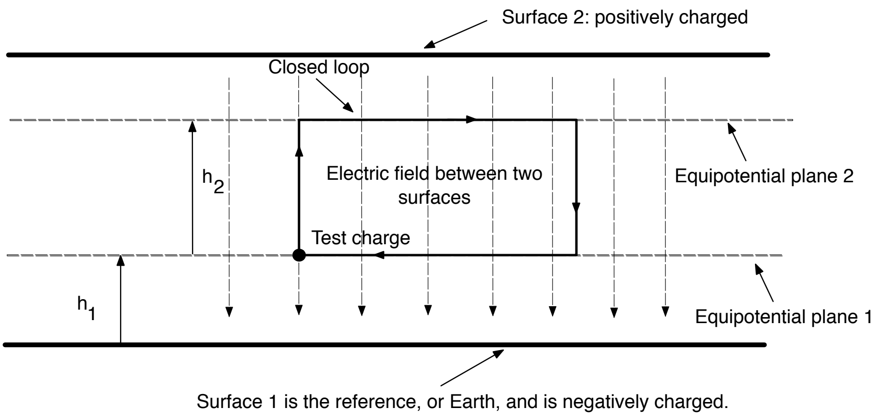

Figure 1.3 A system making use of two charged surfaces to set up an electric field as indicated. The idea of path independence is shown: the energy gained or lost in any closed loop is zero.

the reference plane. Since the test particle is charged it feels the force due to the electric field. In fact the test charge can be made to have a charge of exactly one coulomb[2]. Now consider the case where the test charge is moved directly upwards (against the electric field that has a direction from the positive surface towards the negative surface) by h_2 meters as indicated. The potential energy gained by the test charge is equal to the potential difference between the two equipotential planes as indicated. This potential difference is measured in volts (V) with units joules per coulomb. This experiment is in fact the formal definition of electric field intensity and the unit volt/meter used to measure its intensity:

The voltage between points A and B is the electric potential difference between those two points.

The voltage is the energy required to move a charge of one coulomb between points A and B. A voltage is thus a potential energy difference, which is a function of the electric field intensity between two points (A and B), regardless of the technology used to create and maintain the electric field.

An electrical potential source is shown in Figure 1.4. The source creates an electric field with a potential difference $V_{a,b}$ volts between nodes a and b. Thus the

[2] Charge is measured in coulombs. An electron has a charge of -1.602×10^{-19} coulombs.

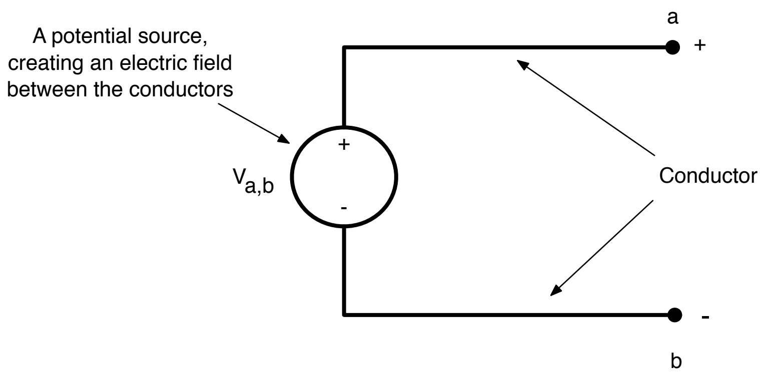

Figure 1.4 A potential source maintains an electric field between nodes a and b with potential difference $V_{a,b}$ (volts or joules/coulomb) between the nodes. The conductor indicated is an equipotential surface, where the potential remains constant.

potential at node a relative to node b is $V_{a,b}$ volts. In this case the electric field is static (i.e., it does not vary in time) and that is why the positive and negative signs can be used to indicate the direction of the electric field: from positive to negative.

1.2.3 Path Independence

Consider Figure 1.3, where a test charge and a closed loop is indicated. If we move the test charge in the closed loop as indicated, there are four sections to cover:

- Move 1: vertically up h_2 meters, requiring J_1 joules.
- Move 2: horizontally along equipotential plane 2, requiring no energy.
- Move 3: vertically down h_2 meters, providing J_1 joules.
- Move 4: horizontally along equipotential plane 1, requiring no energy.

Summing these yield

$$\text{Total energy} = -J_1 + 0 + J_1 + 0 = 0 \text{ joules.} \tag{1.4}$$

Thus the total energy required to move a charge in a closed loop is 0 joules. This is the basis of Kirchhoff's loop law that will be used extensively in this text:

In *any* closed loop in a circuit all the voltages sum to zero.

1.2.4 Charged Particles in Motion: Current

In the previous section the reader was introduced to electric fields, which can be modeled by measuring the potential energy difference between two points per unit charge, measured in volts (joules/coulomb). The reader may know from personal experience that in a gravitational field a mass M left to its own devices (i.e., not kept at point A by a structure) will accelerate: its potential energy will be converted to kinetic energy. The same concept applies in an electric field. If such a field is present in a vacuum, then an isolated charged particle (or a collection of charged particles) will feel the electromotive force due to the presence of the field, and will accelerate. The motion of the charges create what is called an *electrical current*, and it will be shown in the next section that such a current can do work (heat or move objects)[3].

Electrical current thus result in response to an electric field or electromotive force (similarly a mass moves in response to an applied force according to Newton's laws). The law of conservation of charge states that charge can neither be created nor destroyed, only transferred[4]. Thus the algebraic sum of the electric charges in a system does not change.

Clearly we are in need of a proper definition of electrical current. Current is denoted by the symbol i and is defined as the rate of change of the charge q. Put differently, if charge moves from node a to node b then there is current between node a and b. This can be expressed mathematically as [4]

$$i = \frac{dq}{dt} \tag{1.5}$$

where t denotes time. Current thus has units coulombs/second or *ampere*.

Current can exist anywhere where there are free charged particles that are placed inside an electric field — the particles feel the electromotive force and thus accelerate. As explained before a conductor is an equipotential plane where it does not require energy (or work) to move the charged particles[5]. Also remember that the field only penetrates a very thin layer of the conductor, in theory the layer is infinitely thin. The field is located outside the conductor where the free charges feel the effect of the electromotive force.

[3] The reader may have seen or even experimented with vacuum tubes or field effect transistors (FET). These devices make use of an electrical current in response to an electric field present inside the device.

[4] This is the so-called classical model, and it excludes the possibility of pair creation (electron and positron) allowed by quantum mechanics.

[5] This is a model, metals at room temperature are not actually perfect conductors. The model is very good though.

1.2.5 Current Source

A device or hardware that directly induces a current exists, and is referred to as a current source. The symbol and notation for a current source is shown in Figure 1.5. Such a current source guarantees the flow of i amperes between nodes a and b (as shown), regardless of the conditions where its deployed. Similarly a voltage source maintains a potential difference between nodes a and b (as shown in Figure 1.4).

The reader should note that the models for the sources discussed above are not perfect. A physical potential source will have a small internal resistance, and thus the potential difference will be affected by a large current flowing through the source. A physical current source does not have infinite internal resistance and thus will be effected by a large potential across its terminals. But it is remarkable how good and useful these source models are even when we assume the sources are ideal.

1.2.6 Power and Energy

Current and voltage are the two basic variables in an electrical system, and the previous sections explained what they mean physically. For practical applications, we need to be able to compute how much power an electrical device absorbs, or how much power an electrical device delivers.

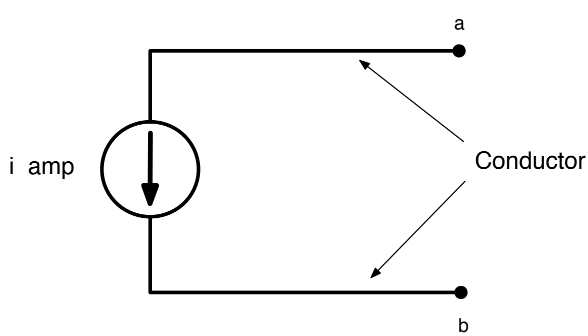

Figure 1.5 A current source maintains i amperes between nodes a and b, regardless of conditions. There is no fixed relation between the current i and the voltage at node a relative to node b.

When the reader pays his/her electric utility bill, he/she is paying for the electrical energy consumed over a certain period of time — measured in joules. To relate power and energy to voltage and current, recall that power means the rate of expending or absorbing energy, measured in watts (W) or joules/second. Thus denoting by p the power in watts (W) and w the energy in joules (J) and t (time) in seconds (sec), then the following must be true:

$$p = \frac{dw}{dt}. \tag{1.6}$$

Clearly the chain rule can be applied so that

$$p = \frac{dw}{dq}\frac{dq}{dt}. \tag{1.7}$$

The term $\frac{dw}{dq}$ literally means the energy change with respect to a change in charge which is a potential energy difference measured in volts, so that

$$\frac{dw}{dq} = v \tag{1.8}$$

while the second term relates the change in charge per time change which is how current was defined. Thus it must be true that

$$\frac{dq}{dt} = i \tag{1.9}$$

which is the current. Thus it can be concluded that the instantaneous power at time t is given by

$$p(t) = v(t)\, i(t). \tag{1.10}$$

The instantaneous power $p(t)$ (delivered or absorbed) between two nodes, a and b, is the product of the potential difference between the two nodes a and b, $v(t)$, multiplied by the current between the two nodes a and b, $i(t)$ [6].

1.2.7 Passive Sign Convention

The passive sign convention says that current contains positive charges, and when a device *absorbs* power, charge flows from a higher potential (a positive terminal) towards a lower potential (a negative terminal), and the instantaneous absorbed power is given by $v(t)\, i(t)$.

[6] Nodes will be explained in the next chapter, for now it can be considered as a single point indicated with a black dot.

When a device *delivers* or provides power, charge flows from a lower potential (a negative terminal) towards a higher potential (a positive terminal), and the instantaneous delivered power is given by $-v(t)\,i(t)$.

These concepts are applied in the example below in Section 1.3.

1.2.8 Conservation of Power

The law of conservation of power must hold in any electrical system. For this reason, the algebraic sum of power in a system, at any instant of time, must be zero. This confirms the fact that the total power supplied to a system must balance the total power absorbed. Thus if the instantaneous power of all devices in a system is summed at any point in time, the sum is zero.

1.3 Example: Conservation of Energy and Power

Consider the circuit shown in Figure 1.6. Based on the conservation laws for charge and potential differences we can make the following statements:

1. Conservation of charge: The electrical current is indicated moving clockwise through the circuit. The current is conserved as the current entering and leaving a source, a load or a node is always identical.
2. Conservation of energy: The potentials in the circuit (which is a closed loop) sums to zero, that is, $-V_1 + V_2 - V_3 + V_4 + V_5 + V_6 = 0$.
3. Conservation of energy: $-V_{1,2} + V_5 + V_6 = 0$.
4. Conservation of energy: $-V_{1,2} + V_1 - V_2 - V_4 + V_3 = 0$.
5. Conservation of energy: $-V_{3,2} - V_2 + V_1 = 0$.

To demonstrate how power is conserved in the circuit under the passive sign convention, we can make the following statements:

1. Source 1 has charge flowing from the negative terminal towards the positive terminal (lower potential to a higher potential) – thus according to the passive sign convention this source has an instantaneous power $p_{\text{source }1}(t) = -V_1 I$ watts and thus it is delivering power to the circuit.
2. Source 2 has charge flowing from the positive terminal towards the negative terminal, thus according to the passive sign convention this source has an instantaneous power $p_{\text{source }2}(t) = V_2 I$ watts and thus it is absorbing power.

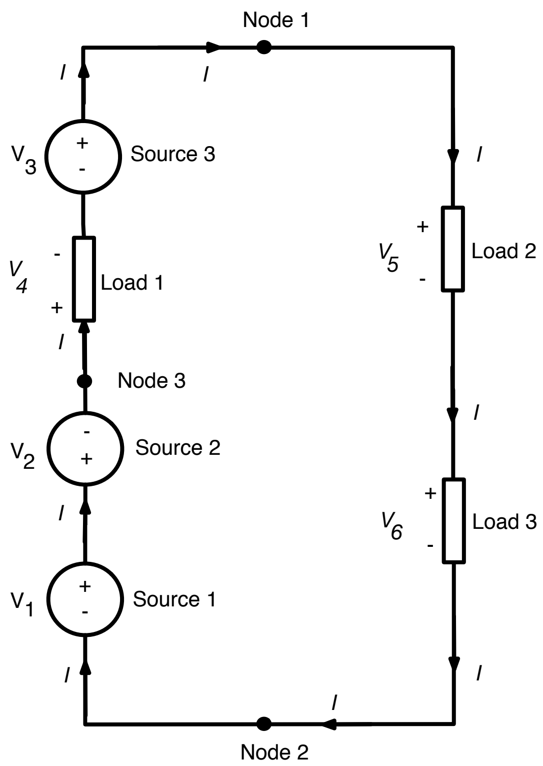

Figure 1.6 A circuit with three loads and three sources. Power is conserved in the circuit.

3. Source 3 has charge flowing from the negative terminal towards the positive terminal and thus is delivering $p_{\text{source 3}}(t) = -V_3 I$ watts to the circuit.

4. Load 1 has the charge flowing from the positive terminal to the negative terminal (higher potential to a lower potential) and thus has an instantaneous power $p_{\text{load 1}}(t) = V_4 I$ watts and thus it is absorbing power in the circuit.

5. Load 2 has the charge flowing from the positive terminal to the negative terminal (higher potential to a lower potential) and thus has an instantaneous power $p_{\text{load 2}}(t) = V_5 I$ watts and thus it is absorbing power in the circuit.

6. Load 3 has the charge flowing from the positive terminal to the negative terminal (higher potential to a lower potential) and thus has an instantaneous power $p_{\text{load }3}(t) = V_6 I$ watts and thus it is absorbing power in the circuit.

1.3.1 Demonstration of Conservation of Power

As instantaneous power must be conserved at any point in time the following statement must be true:

$$p_{\text{source }1}(t)+p_{\text{source }2}(t)+p_{\text{source }3}(t)+p_{\text{load }1}(t)+p_{\text{load }2}(t)+p_{\text{load }3}(t) = 0. \quad (1.11)$$

Hence we find that

$$-V_1 I + V_2 I - V_3 I + V_4 I + V_5 I + V_6 I = 0 \quad (1.12)$$

and thus it must be true that

$$-V_1 + V_2 - V_3 + V_4 + V_5 + V_6 = 0. \quad (1.13)$$

This equation is identical to the first equation derived in this section based on conservation of energy. Thus we have shown that under the passive sign convention conservation of power and conservation of energy are consistent.

1.4 Discussion

In this chapter we introduced several key concepts that will be used throughout this text. We showed that since the electric field is conservative the energy gained or lost by a charged particle in any closed loop must be zero. This is known as the principle of conservation of energy, and is the basis of Kirchhoff's loop law that will be used extensively in this text.

We showed that current is charge in motion, and since charge cannot be created or destroyed, current that enter and exit a node must have the same value. This will lead to Kirchhoff's current law.

We introduced potential and current sources, and we defined voltage as a potential energy difference. These sources are able to provide power and energy to a circuit. We showed that the passive sign convention is a key concept to make the conservation of power and energy in circuits a reality.

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[2] John von Neumann, http://www.azquotes.com/quote/565786, accessed December 26, 2017.

[3] http://www.feynmanlectures.caltech.edu/

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Chapter 2

Electrical Circuits and Circuit Elements

So far we presented voltage and current sources as examples of circuit elements, and below more elements will be presented and discussed. An electrical system or circuit is an interconnection of different elements using conductors.

2.1 The Resistor

One of the most often used elements in a circuit is known as the resistor as shown in Figure 2.1, and it is an element able to absorb electrical energy and convert it to heat (thermal energy). In this text we consider linear resistors only, which have a very simple model: *the potential difference between nodes a and b (containing the resistor) is proportional to the current through it.* This is depicted in Figure 2.1 under the passive sign convention, where charge flow (current) from a higher potential to a lower potential. The constant of proportionality is called the resistance and denoted by the symbol R, so that we can write

$$v = i\,R \implies R = \frac{v}{i}. \tag{2.1}$$

This is known as Ohm's law [1, 2] and resistance is measured in ohms with the symbol Ω. It is important that the reader understand that the potential difference is defined as the potential of one of the resistor nodes minus the potential of the other

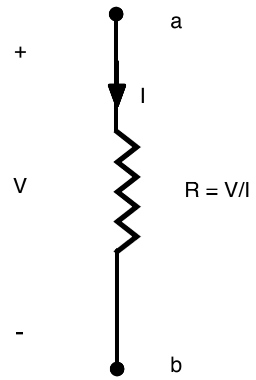

Figure 2.1 The definition of the resistor and Ohm's law. According to the passive sign convention charge flows from positive to negative, that is, a higher to a lower potential. The potential difference between nodes a and b is V volts, and it equals $V = I \times R$.

node of the resistor. It is thus the potential difference across the resistor terminals or nodes that is proportional to the current through the resistor.

Resistors are manufactured from materials that allow electrons to flow when an electric field is applied, but the material resists the flow of electrons and energy is required to compensate for thermal energy loss. Materials with this property are quite common (such as carbon). When charge flows (current) through a resistor, the moving charges collide or interact with the atoms in the resistive material that do not allow the electrons to move freely. These collisions cause the atoms to vibrate which increases the thermal energy of the resistor (it heats up). This requires energy and explains why resistors convert electrical energy to thermal energy which is then lost.

2.2 Concept of a Node

Previously it was stated that nodes define points where the potential (relative to some point of reference) can be defined. Now this definition needs to be refined and extended. Since a conductor is an equipotential surface, it means that anywhere on a conductor the potential remains the same. Thus all parts of a conductor is in fact the same node. A node thus includes a conductor, and the node thus connect two or more circuit elements.

To be specific, consider the circuit shown in Figure 2.2, which contain a potential (voltage) source, a current source and two resistors. This circuit has all elements connected by conductors (such as copper). Each section of conductor is a node as indicated. There are three nodes in the circuit: node 0, node 1 and node 2. All three nodes will have a different potential relative to any point of reference. Node 0 is chosen as the point of reference, and thus has a potential of zero volts (by definition). The potential difference between node 1 and node 2 means the potential of node 1 relative to node 2, and can be denoted as $V_{1,2}$ (measured in volts). The potential between node 1 and node 0 means the potential of node 1 relative to node 0, and it is denoted as $V_{1,0}$. However since node 0 is also the reference node, this can be denoted as V_1 also. In other words if the relative point is not specified, it is implicit that the reference node is used.

Any node can be selected as the reference node, but the convention is to choose the common node. The reader should be able to see that $V_{1,2} \neq V_1$ and $V_1 = V$.

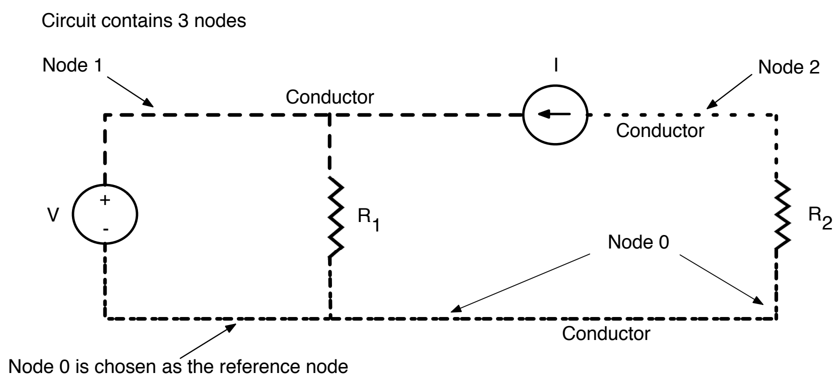

Figure 2.2 A circuit is a connection of a number of circuit elements containing several nodes. A node is an equipotential plane so that the potential anywhere on a node remains the same, and it requires no energy to move a charge along a node. Thus a node is a single section of conducting material as indicated.

2.3 Dependent Voltage and Current Sources

In the previous chapter independent sources were presented and studied. It is also possible to construct special hardware that produce voltage and current sources that are dependent upon voltages or currents in other parts of the electrical system. Figure 2.3 show how the models for such sources look. Chiefly the circles are replaced with a diamond symbol.

In the next two sections the use of dependent sources are illustrated through application in modeling active devices.

2.3.1 A Model for the Bipolar Junction Transistor

The invention of the bipolar junction transistor (BJT) arguably was a turning point in the development of electronics and the miniaturization of circuits that is continuing to this day [3]. Examples of the BJT device is shown in Figure 2.4. The BJT has a circuit symbol as indicated in Figure 2.5, and is a nonlinear and rather complex device. The physics that are able to explain the operation of the BJT are known as semiconductor physics. As explained in this text, it is often the case that engineers are able to design systems based on a model of the device, even if they do not understand the physics at work. If the BJT is properly biased in its linear region, then it turns out that it can be modeled using an exceptionally simple

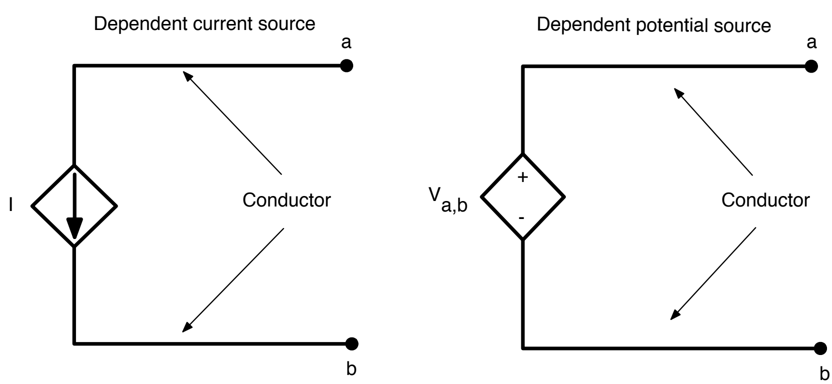

Figure 2.3 A dependent source. The potential or current that is delivered is dependent on (or controlled by) a voltage or a current elsewhere in the circuit.

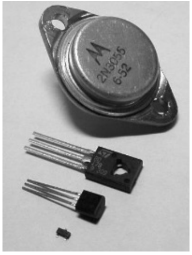

Figure 2.4 Examples of a BJT transistor which is a three terminal device. (Published with permission under the GNU Free Documentation License.)

model as shown in Figure 2.5. The model is based on a dependent current source, with a current gain represented by the symbol β. The value of β can be measured with the BJT under operating conditions. As such the model is very accurate and enables engineers to design complex circuits based on the BJT. The model shows that the current flowing into the collector node is β times the current flowing into the base node. The current exiting at the emitter node is thus $(\beta + 1)\, i_b$, based on the conservation of charge.

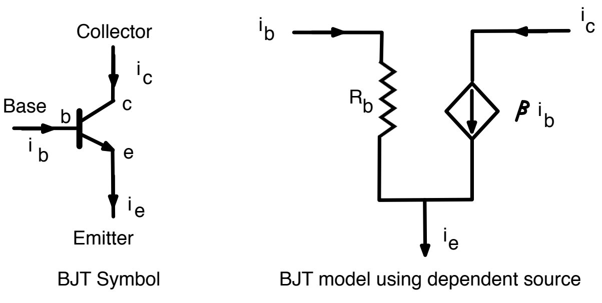

Figure 2.5 The BJT transistor symbol and a model based on a dependent current source.

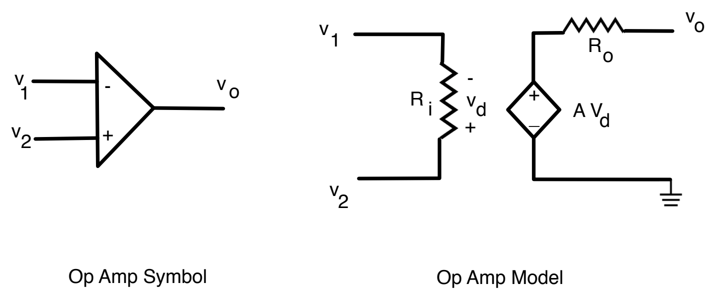

Figure 2.6 The operational amplifier symbol and its model deploying a dependent potential source.

2.3.2 The Operational Amplifier

Another key development was that of the operational amplifier (Op Amp) as shown in Figure 2.6. The input resistance R_i is very large such that little current is drawn from the source, yet the voltage gain A (amplification) achieved at the output is very large. These are very useful properties in circuit design. A simple model is based on the dependent voltage source as shown in Figure 2.6.

2.4 Discussion

This chapter formally introduced the concept of a node, and showed how connected nodes make circuits possible. We also introduced Ohm's law for the resistor, a fundamental law that will be used extensively in this text.

The concept of a controlled or dependent source was also introduced, and it was shown how these elements make it possible to model complex devices such as transistors and operational amplifiers.

References

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

[2] Svoboda, J.A. and R.C. Dorf, *Introduction to Electric Circuits*, 9th Edition, 2015, Wiley.

[3] https://en.wikipedia.org/wiki/Bipolar_junction_transistor

Chapter 3

Kirchhoff's Loop and Current Laws

Up this this point we covered models of the electric field and the electromotive force. Central to this theme is the concept of potential difference between two nodes, and the current (moving charged particles) from one node to another. As conductors connect different nodes, current enters a node, as well as exits a node.

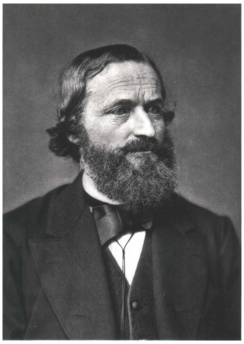

Figure 3.1 Gustav Robert Kirchhoff, whose fundamental laws are presented in this chapter. (Source: Wikipedia — Printed with permission [PD-1923].)

There are two conservation laws that are important and will be presented first in this chapter. These laws were discovered by the German physicist Gustav Kirchhoff (see Figure 3.1), and his conservation laws make possible the analysis of electrical circuits.

The reader will be led through an analysis of two special circuits using Kirchhoff's laws — these circuits we will use often in practice. The analysis will show how the models and ideas of the previous chapters are used to compute potential differences and currents anywhere in a circuit.

3.1 Concept of a Loop: Kirchhoff's Loop Law

A loop is any closed path starting from a node, traversing other nodes and terminating at the starting node. Consider Figure 3.2, which is shown below. Starting at node 1, moving across the current source to node 2, then moving to node 0, then back to node 1 (where the loop started), is a closed loop. Since the electric field is conservative, the conservation of energy says that all the potential differences along any closed loop must sum to zero. In electrical engineering this is known as Kirchhoff's loop law [1, 2]. Let us apply Kirchhoff's loop law for this circuit:

- The potential (relative to node 0, the reference node) at node 1 is $V_1 = V$ volts, at node 2 it's V_2, and at node 0 (which is the reference node) it's zero by definition.

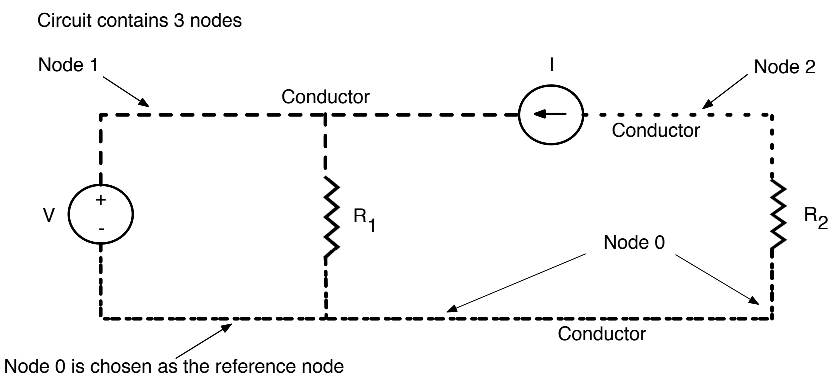

Figure 3.2 A circuit used to demonstrate Kirchhoff's laws.

- Place a test charge of 1 coulomb at node 0.
- Move the test charge towards node 1 — this requires energy as node 1 is at a higher potential. The potential difference that this move requires is $-V_1$ joules since the unit charge is moved against the field and energy is required.
- It is assumed that potential at node 1 is higher than at node 2. If an assumption such as this is wrong, then an analysis of the circuit will find that the potential difference $V_{1,2}$ is negative. Thus such assumptions are harmless. Now move the test charge towards node 2, the energy released is equal to the potential difference between the two nodes, that is, $V_{1,2}$, which means the potential of node 1 relative to node 2.
- Now move the test charge back to node 0 (where the loop started) and this will release energy equal to the potential difference $V_{2,0} = V_2$.
- Now apply Kirchhoff's law, requiring that potential differences in any closed loop sums to zero:

  $$-V_{1,0} + V_{1,2} + V_{2,0} = 0 \tag{3.1}$$

  which is equivalent to

  $$-V_1 + V_{1,2} + V_2 = 0 \implies V_{1,2} = V_1 - V_2. \tag{3.2}$$

  This law thus provides an equation that relates the potential differences in any closed loop.

Kirchhoff's loop law will be used many times in the rest of this text.

3.2 Series and Parallel Circuit Elements

Two elements are in series if they *exclusively* share a single node and consequently carry the same current.

Two or more elements are in parallel if they are connected to the same two nodes and consequently have the same potential difference across them.

As an example in Figure 3.2, the two resistors are not in parallel as they are not connected to the same nodes, and consequently don't have the same potential difference across their nodes.

In Figure 3.2 the current source and resistor R_2 are in series as they exclusively share a node (in this case node 2).

3.3 Conservation of Charge: Kirchhoff's Nodal or Current Law

As previously explained, charge cannot be destroyed or disappear — it can only be transported from one node to another. *Hence all the current entering a node, and all the current exiting the node, must be equal.* This is known as Kirchhoff's current law. The use of this law will become clear when it will be applied to analyze circuits.

3.4 Series Resistors and Voltage Division

Consider the circuit in Figure 3.3, containing a single independent potential (voltage) source and two resistors that are connected in series. This can be seen by the fact that they exclusively share only one node (node 2). The potential at node 1 relative to node 0 is denoted as $V_{1,0}$ and the potential at node 2 relative to node 0 is denoted as $V_{2,0}$. Node 0 is designated as the reference node in this circuit.

Resistor one is connected between nodes 1 and 2. What is the potential difference across the nodes of resistor one? In other words, what is the potential

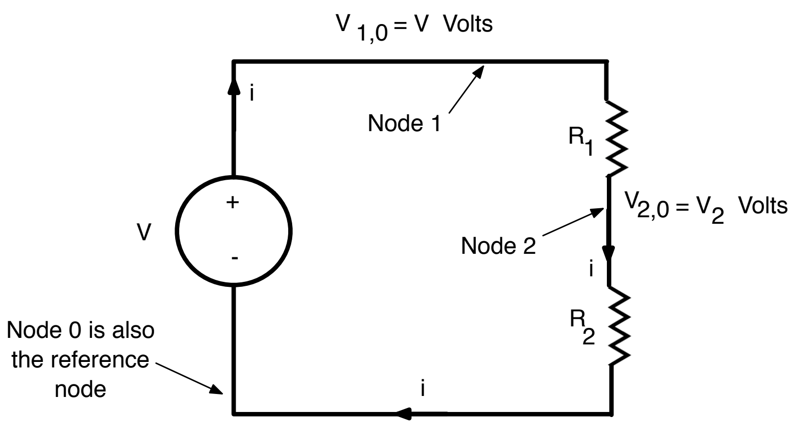

Figure 3.3 A circuit where two resistors are in series, connected to a single independent potential source. Node 0 is the reference node.

difference or voltage of node 1 relative to node 2? It is given by

$$V_{1,2} = V_{1,0} - V_{2,0} = V - V_2 \tag{3.3}$$

as the potential difference across the nodes of resistor one is $V_{1,2}$. Similarly the two nodes that resistor two are connected to are nodes 2 and 0, so the potential difference is given by $V_{2,0} = V_2$.

To remind the reader of Ohm's law, consider Figure 2.1 depicting Ohm's law for a resistor. It says that the potential difference between the two nodes containing the resistor and the current through it has a ratio given by the resistance, so that

$$\frac{V}{I} = \text{Resistance}. \tag{3.4}$$

Thus exploiting this law (Ohm's law) the following statements have to be true:

$$V_{1,2} = iR_1, \tag{3.5}$$

and

$$V_2 = iR_2. \tag{3.6}$$

Now apply Kirchhoff's loop law to the circuit. The loop starts at node 0, then moves to node 1, then to node 2 and then back to node 0. Kirchhoff's loop law says that all potential differences along this loop sum to zero. So for this circuit it must be true that

$$-V_{1,0} + V_{1,2} + V_{2,0} = 0. \tag{3.7}$$

However clearly $V_{1,0} = V$ as the ideal independent potential source maintains a voltage V between node 1 and the reference node. So combining the above equation and Equations (3.5) and (3.6) it must be true that

$$V = V_{1,2} + V_2 = iR_1 + iR_2 = i(R_1 + R_2). \tag{3.8}$$

Hence based on the above equation it can be seen that if the two connected resistors are viewed as a single resistive component connected between node 1 and node 0 (denoted as R) then

$$\frac{V}{i} = R = (R_1 + R_2), \tag{3.9}$$

which is Ohm's law for the series combined two resistors. So we have shown quite formally that resistors in series add up, that is

$$R = R_1 + R_2. \tag{3.10}$$

This holds for an arbitrary number of resistors in series — the total resistance is the sum of the resistors in series. So for N resistors in series the total resistance R is given by

$$R = R_1 + R_2 + \cdots + R_N. \tag{3.11}$$

The last item on the agenda here is the voltage at node 2, that is V_2. From the previous equation it can be seen that

$$\frac{V}{R_1 + R_2} = i. \tag{3.12}$$

From Ohm's law it follows that

$$V_2 = iR_2, \tag{3.13}$$

so that by combining the above two equations

$$V_2 = \frac{V}{R_1 + R_2} R_2 = V \left(\frac{R_2}{R_1 + R_2} \right). \tag{3.14}$$

This shows that the potential at node two is the source voltage V multiplied by the resistor ratio as shown — hence this circuit is often referred to as a voltage divider. Note that this only holds true if the voltage divider node is isolated — not shared by anything else except the two resistors.

3.5 Parallel Resistors and Current Division

Consider a parallel connection of two resistors, as shown in Figure 3.4. Can we combine these two resistors into a single resistor, denoted as R? To answer this question we need to find an expression for $R = \frac{V}{i}$ which is what Ohm's law requires the combined resistor to satisfy (as it is connected to nodes 1 and 0). Kirchhoff's current law says that the current entering and exiting a node must be the same. Hence its clear the in the circuit shown the following statement must be true:

$$i = i_1 + i_2. \tag{3.15}$$

This will guarantee that the current flowing from the independent potential source to node 1, then to the two resistors and onwards to node 0 is the same. This is known as Kirchhoff's current law [1, 2] . Ohm's law (see Figure 2.1) requires the following two equations to hold for the two resistors:

$$V_{1,0} = V_1 = V = i_1 R_1 \tag{3.16}$$

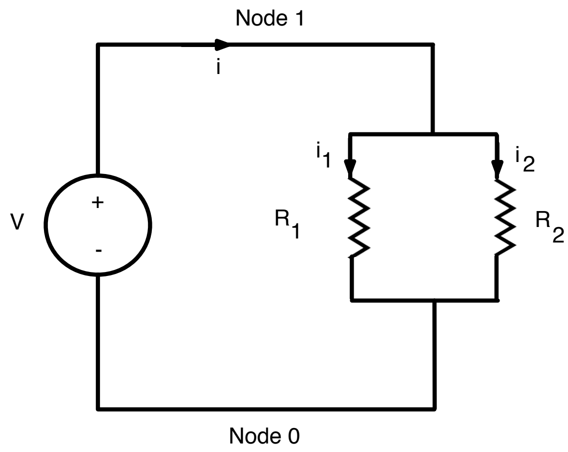

Figure 3.4 A circuit where two resistors are in parallel, connected to a single independent potential source. Node 0 is the reference node.

and

$$V_{1,0} = V_1 = V = i_2 R_2. \tag{3.17}$$

Using the fact that $R = \frac{V}{i}$ and the equations above yields

$$R = \frac{V}{i} = \frac{V}{i_1 + i_2} = \frac{V}{\frac{V}{R_1} + \frac{V}{R_2}}, \tag{3.18}$$

which can be simplified to yield

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \tag{3.19}$$

which is equivalent to

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{3.20}$$

This is the required formula to compute the resistance formed by two resistors in parallel. The same procedure can be repeated with any number of resistors in parallel, thus for N resistors in parallel the expression is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}. \tag{3.21}$$

Also notice that the parallel resistors divide the current, which is why it is often referred to as a current divider.

3.6 Discussion

The two examples above introduced the rules for how series resistors and parallel resistors are combined to yield a single resistor. It also provided an opportunity to show the reader how Kirchhoff's laws are used to compute potentials and currents in the circuit. To summarize:

1. Ohm's law: The potential across a resistor is proportional to the current through it. The constant of proportionality is the resistance R measured in Ω.
2. Kirchhoff's loop law: The potentials in any closed loop sum to zero. This is required to guarantee conservation of energy.
3. Kirchhoff's current law: The total current entering a node and the total current exiting the node must be equal. This is required to guarantee that charge is conserved.

The reader may have noticed that these three laws were the only ones used to do all the calculations. This is a pattern that will repeat itself in Part 1 of this text on circuit theory. We will use these three laws to compute any quantity in a circuit that is required. The mathematical methods and procedures will be formalized in chapters to follow, but the basic concepts and ideas remain the same.

References

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

[2] Svoboda, J.A. and R.C. Dorf, *Introduction to Electric Circuits*, 9th Edition, 2015, Wiley.

Chapter 4

The Nodal Method of Circuit Analysis

In this chapter circuit analysis techniques are formalized based on the theory presented in the previous chapters. This chapter considers the nodal analysis method. Consider the circuit shown in Figure 4.1. The circuit has three nodes, and the reference node is indicated as node 0. The objective of our circuit analysis is to compute all the currents and potential differences in the circuit. Then it will be shown that power is conserved in the circuit, that is some elements absorb power, and others deliver that same power.

4.1 Choosing the Potentials in the Circuit

The choice of the potentials to use are straightforward when it comes to the potentials of the nodes: it is assumed that nodes 1 and 2 are at a higher potential than node 0. If this assumption is wrong, then the results will indicate the potential as negative. Thus there is no risk to make this assumption, it is arbitrary. Hence, as shown in Figure 4.1, the choices made show the potential difference between node 1 and node 0 as V_1 volts, while the potential difference between node 2 and node 0 is V_2 volts.

The potential at node two is assumed to be higher than at node one — this can be seen by the assumed current direction (according to the passive sign convention). That means we assume that $V_{2,1}$ (i.e., the potential of node 2 relative

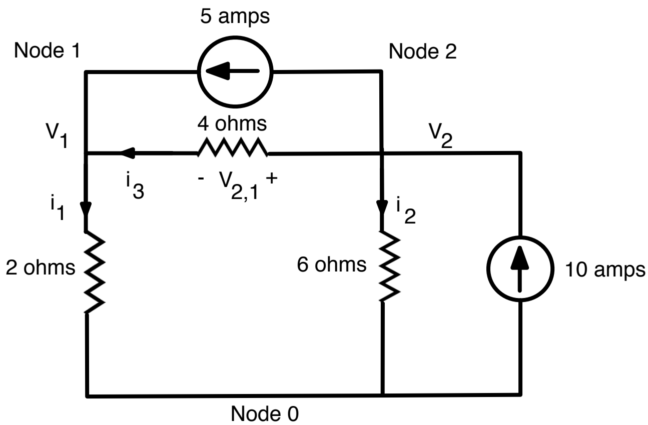

Figure 4.1 A circuit containing 3 nodes and 2 independent current sources. The objective is to use Kirchhoff's and Ohm's laws to determine the node potentials. The reference node is node 0. (This figure is a reworking of Figure 3.3 from [1].)

to node 1) is positive. Again, if this assumption turns out to be wrong then the results will have it as negative, and there is no risk when making this assumption. This choice is thus also completely arbitrary as long as the passive sign convention is adhered to.

To summarize then, the reader is encouraged to answer these questions before looking at the answers that will follow:

1. What is the potential difference across the 6-ohm resistor?
2. What is the potential difference across the 2-ohm resistor?
3. What is the potential difference across the 4-ohm resistor?
4. What is the potential difference across the 5-amp current source ?
5. What is the potential difference across the 10-amp current source ?

Here are the answers: (1) V_2, (2) V_1, (3) $V_{2,1}$, which would be positive if the potential at node 2 or V_2 (relative to node 0) is higher than at node 1 or V_1 (relative to node 0), (4) $V_{2,1}$ and (5) V_2. It is important that the reader is able to derive these same answers, as it goes to the core of many of the ideas that have been explored so far in the text.

4.2 Choosing the Currents

Once the choices for the potentials in the circuit have been made, the currents through the resistors are not arbitrary. These have to satisfy the passive sign convention as shown in Figure 2.1, and thus satisfy Ohm's law. Ohm's law says that the charge (through a resistor) flows from a higher potential to a lower potential. Since it has been assumed that $V_1 > 0$ the current through the 2-ohm resistor has to flow from node 1 to node 0. Similarly the current through the 6-ohm resistor has to flow from node 2 to node 0.

Now the current through the 4-ohm resistor has to be decided. It was assumed that $V_{2,1}$ is positive (as indicated in the figure). Thus Ohm's law requires the flow from right to left as indicated in the figure.

- It is a key requirement in circuit analysis that the currents are chosen consistently with the passive sign convention, else all results obtained will be flawed.
- It's important to notice that Ohm's law applies to resistors only, there is no law relating the potential difference across a current source to the current through it. The circuit sets this potential.
- The potential across a potential source is fixed by the electric field created by the source — the current through it does not influence it. This is a model of course, in practice there are limits to the charge that can flow without affecting the potential.

4.3 Applying the Kirchhoff Loop Law, Kirchhoff Current Law, and Ohm's Law

4.3.1 The Kirchhoff Loop Law

First of all, Kirchhoff's loop law can be applied to the circuit shown in Figure 4.1. This law states that the potential differences in any closed loop must sum to zero.

- Place a test charge of 1 coulomb at node 1. If the test charge is moved to node 0, energy will be released which we designate with a positive sign. Hence the energy required for this step is given by V_1, as this is the potential at node 1 relative to node 0.

- The test charge is now located at node 0, and it is moved to node 2, which will require energy and thus negative, as the charge is moved to a higher potential. In this case it is equal to $-V_2$.
- The test charge now located at node 2 is moved back to node 1 to complete or close the loop. For this move there is energy released as the potential at node 2 was assumed to be higher than at node 1. Thus the energy for this move is given by $V_{2,1}$.

Kirchhoff's loop law says all these potential differences must sum to zero, so that we obtain an equation given by

$$V_1 - V_2 + V_{2,1} = 0. \tag{4.1}$$

This equation will guarantee the conservation of energy in the circuit.

4.3.2 Ohm's Law

There are three resistors in the circuit, and thus its possible to use Ohm's law three times, given by

$$\begin{aligned} V_1 &= i_1\, 2 && (4.2)\\ V_{2,1} &= i_3\, 4 && (4.3)\\ V_2 &= i_2\, 6. && (4.4) \end{aligned}$$

This provides the relations between the currents and the potentials in the circuit.

4.3.3 The Kirchhoff Current Law

The Kirchhoff current law states that the current entering a node must equal the current exiting the same node. This is due to the fact that current (moving charges) cannot be destroyed, it can just be transported from one node to another node. There are two nodes where Kirchhoff's current law can be deployed, namely nodes 1 and 2, and thus the following statements must be true:

$$\begin{aligned} 5 + i_3 &= i_1 && (4.5)\\ i_2 + i_3 + 5 &= 10. && (4.6) \end{aligned}$$

4.3.4 Mathematical Solution

The equations that we derived above can be solved making use of matrix theory as follows. First, the three equations due to Ohm's law are substituted into Equation (4.1) to yield

$$i_1 2 + i_3 4 - i_2 6 = 0. \tag{4.7}$$

Now there are three equations available for the currents, which can be written as

$$\begin{aligned} 2\,i_1 - 6\,i_2 + 4\,i_3 &= 0 && (4.8)\\ -i_1 + 0\,i_2 + i_3 &= -5 && (4.9)\\ 0\,i_1 + i_2 + i_3 &= 5. && (4.10)\end{aligned}$$

In general if we have n equations and n unknowns then we can write these in matrix form. In this case there are three equations and unknowns. The unknowns are placed in a vector given by

$$\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}. \tag{4.11}$$

A 3 by 3 matrix **A** can be defined as such that when multiplied with the vector above it will yield the left-hand side of Equations (4.8), (4.9), and (4.10). The matrix is the coefficients (of the currents) on the left-hand side of Equations (4.8), (4.9), and (4.10). Thus we can write the matrix as

$$\mathbf{A} = \begin{bmatrix} 2 & -6 & 4 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{4.12}$$

The right-hand side of Equations (4.8), (4.9), and (4.10) can be written as a vector

$$\mathbf{c} = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} \tag{4.13}$$

and thus the system in matrix form is given by

$$\mathbf{Ai} = \mathbf{c} \tag{4.14}$$

which represents

$$\begin{bmatrix} 2 & -6 & 4 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix}. \tag{4.15}$$

The first term above on the left is a matrix, designated as $\mathbf{A}$. This matrix is of full rank, which means its determinant is not zero[1], and thus the system can be solved. The solution involves the matrix inverse $\mathbf{A}^{-1}$ and is given by $\mathbf{i} = \mathbf{A}^{-1}\,\mathbf{c}$.

4.3.4.1 Numerical Results Based on MATLAB

The solution based on the matrix inverse is shown below using MATLAB:[2]

```
clear all
A = [2   -6   4
     -1   0   1
      0   1   1]; %matrix A
c = [ 0 -5 5]'; % vector c
i = inv(A)*c % solution of currents & display solution
v1 = i(1)*2 % now calculate the potential of nodes
v2 = i(2)*6
```

The node voltages (potentials) are also computed based on application of Ohm's law as indicated above. This provides the actual potential differences as follows:

$$V_1 = 13.333 \text{ volts}, \quad V_2 = 20 \text{ volts}. \tag{4.16}$$

The currents are given by

$$i_1 = 6.6667 \text{ amps}, \; i_2 = 3.3333 \text{ amps}, \; i_3 = 1.6667 \text{ amps}. \tag{4.17}$$

Clearly the potential differences were chosen correctly in this case as no potential difference turned out to be negative. If the solution provides a negative value for the current, then the direction of the current reverses. Again, in this case the currents indicated in the figure has correct directions as shown, since all currents have a positive sign.

4.3.4.2 Verifying the Solution

After circuit analysis has been completed, it is always possible to verify that the solution is correct. The process of verification is quite straightforward. Verify the following:

[1] MATLAB provides the $det(\,)$ function which provides the determinant. If its greater than zero, the inverse exists and the system has a solution.

[2] Also see the section below where MATLAB is formally introduced.

1. That at each node Kirchhoff's current law is satisfied by the solution.
2. That all the potential differences (based on the solution) in any closed loop sums to zero.

If the solution satisfies these two tests, then the solution is correct. There cannot be more than one solution satisfying Kirchhoff's laws, the solution is unique. In the present case it possible to verify that these two tests are satisfied and that the solution is correct.

4.3.5 Power Consumed and Delivered in the Circuit

The power absorbed by the resistors are as follows:

- Resistor 2 ohms: $V_1 \times i_1 = 88.8889$ watts
- Resistor 6 ohms: $V_2 \times i_2 = 66.6667$ watts
- Resistor 4 ohms: $(V_2 - V_1) \times i_3 = 11.1111$ watts

Thus total power absorbed by resistors is 166.6667 watts. The next step is to compute the power absorbed/delivered by the sources:

- The charge flowing in the 10-amp source is flowing from a lower to a higher potential, so according to the passive sign convention the power is $-V_2 \times 10 = -200$ watts, which means that source is *delivering* power to the circuit.
- The charge flowing in the 5-amp source is flowing from a higher to a lower potential, so according to the passive sign convention the power is $(V_2 - V_1) \times 5$ watts, which means that source is *absorbing* power.

The total power delivered by the sources thus is $P = -V_2\, 10 + (V_2 - V_1)\, 5 = -166.6667$ watts as was expected. Thus the power absorbed by the resistors and the 5-amp source equals the power delivered by the 10-amp source. The circuit thus satisfies the law of the conservation of power.

4.4 Software for Circuit Analysis

MATLAB is a proprietary programming language developed by TheMathWorks corporation, and is a sophisticated numerical computing environment [2, 3]. MATLAB is the result of a long history of development in numerical analysis and linear algebra. It makes use of a scripting language and command line interface based on matrix representation, and has a rich variety of toolboxes and powerful graphics for plotting and visualization.

The need for solving systems of linear equations goes back to the beginning of the development of the first computers [4, 5], and has a rich history of development to the present day. After World War II, work continued at the National Physical Laboratory (NPL) in the UK, and the Institute for Numerical Analysis (INA), a branch of the National Bureau of Standards in Los Angeles. Later work continued at Universities in California after the INA closed.

The continued need to solve systems of equations led to the development of MATLAB in the late 1970s. The idea was to make the use of packages for solving such systems (known as LINPACK and EISPACK) possible without learning the FORTRAN programming language. The concept of MATLAB and its scripting language quickly found support within the applied mathematics and engineering communities. This led to the commercial development of MATLAB after TheMathworks was founded in 1984.

4.4.1 Octave

The GNU Project is a free-software, mass-collaboration project. The project was conceived in 1983 by Richard Stallman at MIT in Boston. Software developed under the GNU license allow users to run the software, share and study it, as well as modify it. GNU software is therefore free.

GNU Octave was developed as part of the GNU Project and started in 1992. It's a programming language, intended for numerical computations. Octave uses a language that is mostly compatible with MATLAB, and it is free software under the terms of the GNU General Public License [6].

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[6] https://en.wikipedia.org/wiki/GNU_Octave

Chapter 5

Combining Independent and Dependent Sources

The previous chapter presented the formal solution of a circuit using the so-called nodal analysis method. The general idea is to apply the Kirchhoff loop law, the Kirchhoff current law, and Ohm's law to reduce the circuit to a system of n linear equations with n unknowns. Such a system represents a solution and will yield the currents and node potentials of the circuit as was shown. This assumes that the equations are not linearly dependent, since such a system will produce a rank deficient matrix (with a determinant of zero) which cannot be inverted.

5.1 A Worked Example

This chapter will also apply the nodal analysis method, but the circuit contains an independent source, as well as a dependent (controlled) source. The circuit is shown in Figure 5.1. Notice that the choices made for the currents imply that under the passive sign convention the following statements must be true:

$$V_1 > V_3, \quad V_1 > V_2, \quad \text{and} \quad V_3 > V_2. \tag{5.1}$$

If any of these choices (assumptions) are wrong, then the solution to the circuit will correct these choices by changing the sign from positive to negative. Thus these choices are in fact arbitrary, as long as the passive sign convention is adhered to.

Note the dependent current source, which forces a current from node 2 to node 0 equal to twice the current through the 2-Ω resistor.

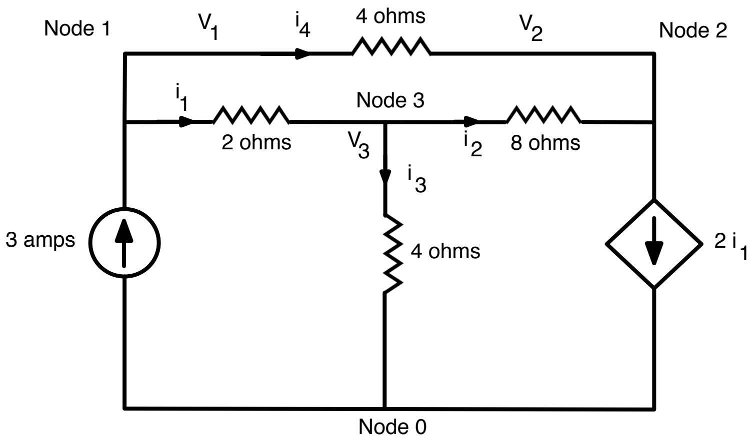

Figure 5.1 A circuit containing 4 nodes, an independent current source, as well as a dependent source. Node 0 is the reference node. (This figure is a reworking of Figure 3.5 from [1].)

There are several ways to apply the nodal method, in this case we will apply Kirchhoff's current law first, then Ohm's law, and finally Kirchhoff's loop law to bind the former two. Hence starting with Kirchhoff's current law the following three equations must be true:

$$3 = i_1 + i_4 \tag{5.2}$$

$$i_1 = i_3 + i_2 \tag{5.3}$$

$$2\, i_1 = i_4 + i_2. \tag{5.4}$$

Applying Ohm's law to the resistors the following equations must be true:

$$V_{1,3} = 2\, i_1 \tag{5.5}$$

$$V_{3,2} = 8\, i_2 \tag{5.6}$$

$$V_3 = 4\, i_3 \tag{5.7}$$

$$V_{12} = 4\, i_4. \tag{5.8}$$

Note that there is no law that binds the potentials at nodes 1 and 2 and the current sources (current values) — these potentials are set by the circuit. If any of these current sources were a potential source, then the node potential would be fixed by the potential source.

Applying the Kirchhoff loop law to the top loop in the circuit (i.e., not including the sources) yields

$$V_{1,3} + V_{3,2} - V_{1,2} = 0. \tag{5.9}$$

Substituting the equations from Ohm's law into Kirchhoff's loop law yields

$$2i_1 + 8i_2 - 4i_4 = 0. \tag{5.10}$$

Thus there are now four equations involving the four currents, and hence a matrix equation follows as

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 2 & 8 & 0 & -4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.11}$$

Using MATLAB (source code provided below) the system can be solved to yield the currents. The node potential follows from knowledge of the currents as

$$V_3 = 4\,i_3 \tag{5.12}$$

by Ohm's law. The potential difference across the 2-ohm resistor is given by $V_1 - V_3 = V_{1,3} = 2\,i_3$, thus by substituting the known currents and potentials into this equation V_1 can be obtained. Similarly the potential difference across the 8-ohm resistor is given by $V_3 - V_2 = 8\,i_2$, and substituting the current and potentials means that V_2 can be obtained. The final solution is as follows:

$$i_1 = 1.2 \text{ amps} \tag{5.13}$$
$$i_2 = 0.6 \text{ amps} \tag{5.14}$$
$$i_3 = 0.6 \text{ amps} \tag{5.15}$$
$$i_4 = 1.8 \text{ amps} \tag{5.16}$$
$$V_1 = 4.8 \text{ volts} \tag{5.17}$$
$$V_2 = -2.4 \text{ volts} \tag{5.18}$$
$$V_3 = 2.4 \text{ volts} \tag{5.19}$$

The MATLAB code used to compute these results is given below.

```
clear all

% Define matrix A and vector c
A = [1 0 0 1
    -1 1 1 0
    -2 1 0 1
    2 8 0 -4];

c = [3
    0
    0
    0];

% compute solution by inverting the matrix
i = inv(A)*c

% compute potentials
v3 = 4*i(3)
v1 = 2*i(1)+v3
v2 = v3 - 8*i(2)
```

5.2 Discussion

This chapter introduced circuits where independent and controlled sources are combined. We also showed that the voltages at the nodes of a current source is set by the circuit. There are no laws that are able to directly relate the voltage at the terminals to the current through the terminals.

The example offered showed that the solution to the static circuit follows from inverting the circuit matrix. The matrix is non-singular as can be verified with the MATLAB code that was provided.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 6

Fixed Potential Between Nodes: Supernode

The last example of nodal analysis to be considered contains a fixed potential between two nodes where both are not the reference node. Normally nodes with potential sources are relative to the reference node, and this was the case in previous examples. However circuits in practice can have a fixed potential between two nodes where neither are the reference node.

6.1 A Circuit Containing a Supernode

An example of such a case is shown in Figure 6.1. Here the potential difference between nodes 2 and 3 is fixed to 5 volts — specifically the potential at node 3 is always 5 volts higher than the potential at node 2. This potential source can also be a dependent source, the same arguments will hold. If Kirchhoff's loop law is applied to a loop from node 3 to node 0, then from node 0 to node 2, then back to node 3 an equation for the potentials at nodes 2 and 3 can be obtained as

$$V_3 - V_2 - V_{3,2} = 0. \tag{6.1}$$

However it is known what the value of $V_{3,2}$ is, since its fixed by the 5-volt potential source! So it can be concluded that

$$V_3 - V_2 = 5 \text{ volts}. \tag{6.2}$$

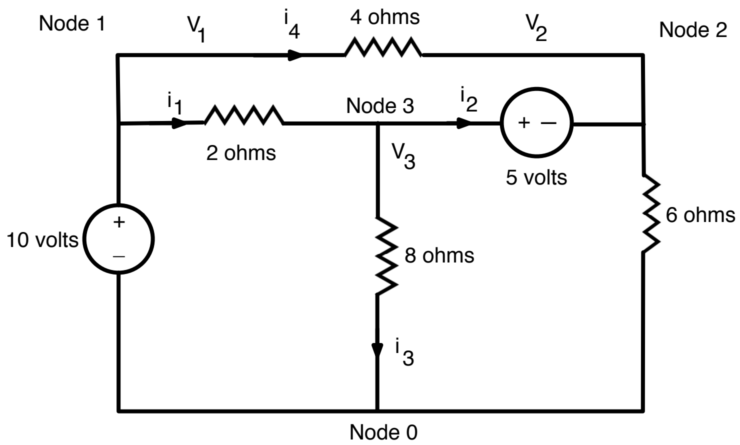

Figure 6.1 A circuit containing a supernode. The potential difference between two non-reference nodes is fixed by a potential source. The reference node is node 0. (This figure is a reworking of Figure 3.7 from [1].)

This equation will be used to complete the system of equations using nodal analysis. To apply the nodal analysis method, the following equations result when applying Kirchhoff's loop law, Ohm's law, and Kirchhoff's nodal law, in that order:

$$10 - V_3 - V_{1,3} = 0 \tag{6.3}$$
$$10 - V_2 - V_{1,2} = 0 \tag{6.4}$$
$$V_{1,3} = 2\,i_1 \tag{6.5}$$
$$V_{1,2} = 4\,i_4 \tag{6.6}$$
$$V_3 = 8\,i_3 \tag{6.7}$$
$$V_2 = 6\,(i_2 + i_4) \tag{6.8}$$
$$i_1 = i_2 + i_3. \tag{6.9}$$

This system of equations can be simplified through substitution to yield

$$\begin{aligned} 10 - 8i_3 - 2i_1 &= 0 \qquad (6.10)\\ 10 - 6\,(i_2 + i_4) - 4i_4 &= 0 \qquad (6.11)\\ i_2 + i_3 &= i_1 \qquad (6.12)\\ 8i_3 - 6(i_2 + i_4) &= 5. \qquad (6.13) \end{aligned}$$

These equations can be written in matrix form as

$$\begin{bmatrix} 2 & 0 & 8 & 0 \\ 0 & 6 & 0 & 10 \\ 1 & -1 & -1 & 0 \\ 0 & -6 & 8 & -6 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 0 \\ 5 \end{bmatrix}. \qquad (6.14)$$

The solution yields the current values as

$$\begin{aligned} i_1 &= 0.4000 \qquad (6.15)\\ i_2 &= -0.7500 \qquad (6.16)\\ i_3 &= 1.1500 \qquad (6.17)\\ i_4 &= 1.4500. \qquad (6.18) \end{aligned}$$

To verify that the solution is correct, the Kirchhoff loop and current laws are used to confirm the solution is correct. The solution and the verification are shown in the MATLAB code below.

```
clear all

A = [2   0   8   0
     0   6   0   10
     1   -1  -1   0
     0   -6   8  -6];
 c = [10
      10
      0
      5];
 i = inv(A)*c  % solution, i.e. currents

 % confirm solutions are correct
 10-8*i(3) - 2*i(1) % must be zero, and it is
 10-6*(i(2)+i(4))-5-2*i(1) % must be zero, and it is
 10-6*(i(2)+i(4))-4*i(4) % must be zero, and it is
```

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 7

The Mesh Method for Circuit Analysis

In previous chapters the nodal circuit analysis method was presented and several examples of its application were provided. The nodal method remains the most general method of circuit analysis, and if correctly applied, it will always provide the reader with the currents and node potentials of the circuit.

There are certain circuits known as planar circuits, and these contain so-called meshes. A mesh is a loop which does not contain any other loops within it. Such circuits can be solved by assuming *mesh currents*, which then induce voltages in circuit elements according to Ohm's law. Circuits suitable to the mesh method generally lead to smaller matrices to solve. It should be realized that all circuits can be solved by the nodal method, and the use of the mesh method is completely optional. It's up to the reader to decide which method to deploy.

7.1 An Example of the Application of the Mesh Method

To show how the mesh method is used, consider the circuit shown in Figure 7.1. There are two mesh currents indicated, i_1 and i_2. These are assumed to be rotating clockwise through the two meshes where they are assumed to be located. The direction of the mesh current is arbitrary, because if the choice of direction is wrong the final result will correct it by having a negative sign. The voltage induced across

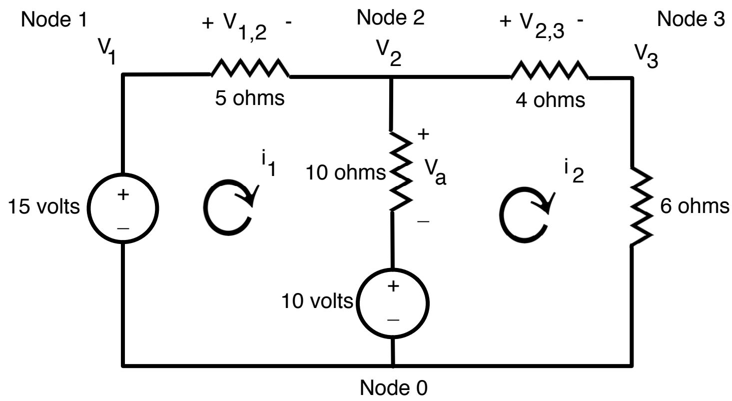

Figure 7.1 A circuit analyzed using the mesh method. This circuit contains two meshes, and two independent sources. Node 0 is the reference node. (This figure is a reworking of Figure 3.18 from [1].)

the 5-ohm resistor is clearly a result of the mesh current i_1, and according to Ohm's law, is given by

$$V_{1,2} = 5\,i_1 \tag{7.1}$$

while the voltage induced across the 4-ohm resistor is clearly a result of the mesh current i_2, and is given by

$$V_{2,3} = 4\,i_2. \tag{7.2}$$

The potential at node 1 is $V_1 = 15$ volts (fixed by the source), and the potential at node 3 is given by

$$V_3 = 6\,i_2. \tag{7.3}$$

What potential is induced (according to Ohm's law) across the 10-ohm resistor? There are two currents flowing through the 10-ohm resistor in opposite directions. Hence according to the passive sign convention the potential across the 10-ohm resistor node must be

$$V_a = 10(i_1 - i_2). \tag{7.4}$$

What is the potential at node 2 (i.e., V_2)? To determine this potential, perform Kirchhoff's loop law as follows: start at node 2, move to the reference node, then across the 10-volt potential source, then across the 10-ohm resistor back to node 2. The potential differences along that closed loop must sum to zero. Hence the following statement must be true:

$$V_2 - 10 - V_a = V_2 - 10 - 10(i_1 - i_2) = 0. \tag{7.5}$$

Hence V_2 is given by

$$V_2 = 10 + 10(i_1 - i_2). \tag{7.6}$$

It is thus clear that all the node potentials will be known once the two mesh currents have been calculated, as shown above. To determine the two mesh currents, apply Kirchhoff's loop law to the two meshes given by

$$15 - 10 - 10(i_1 - i_2) - 5i_1 \quad = \quad 0 \tag{7.7}$$

$$10(i_1 - i_2) + 10 - 4i_2 - 6i_2 \quad = \quad 0. \tag{7.8}$$

This can be written in matrix form as

$$\begin{bmatrix} 15 & -10 \\ 10 & -20 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}. \tag{7.9}$$

The solution shows that

$$i_1 \quad = \quad 1 \tag{7.10}$$

$$i_2 \quad = \quad 1. \tag{7.11}$$

This enables all potential differences to be computed, and the currents through the resistors are now known. To verify that the solution is correct, perform Kirchhoff's loop law in the two meshes. Hence for mesh 1 we find that

$$15 - 10 - 10(i_1 - i_2) - 5i_1 = 5 - 5 = 0 \tag{7.12}$$

and for mesh 2 we find

$$10(i_1 - i_2) + 10 - 4i_2 - 6i_2 = 0 + 10 - 4 - 6 = 0. \tag{7.13}$$

Since both loops (meshes) satisfy Kirchhoff's laws, the currents that were computed must be correct.

7.2 Supermesh: Current Source in Common

A supermesh results when two meshes have a (dependent or independent) current source in common. An example of the supermesh is shown in Figure 7.2.

The issue is that the potential across the current source is not known in advance. The potential across the terminals of a current source is determined by the circuit. There are solutions for this case in the literature [1, 2], but the solution presented here is probably one of the simplest. We introduce an unknown potential across the current source, as shown in Figure 7.2. The introduction of this potential causes the number of unknowns to be solved for to increase with one, but it is offset by the fact that the presence of the source provides an additional equation given by

$$i_2 - i_1 = 6. \tag{7.14}$$

Thus the circuit can be reduced to three equations by applying Kirchhoff's loop law with the equation above as show below:

$$\begin{aligned} 20 - V_{\text{source}} - 2(i_1 - i_2) - 6i_1 &= 0 & (7.15)\\ 2(i_1 - i_2) + V_{\text{source}} - 4i_2 - 10i_2 &= 0 & (7.16)\\ i_2 - i_1 &= 6. & (7.17) \end{aligned}$$

The first two equations above can be summed to produce a single equation containing only currents, as the potential V_{source} cancels, thus

$$\begin{aligned} 20 - 6i_1 - 4i_2 - 10i_2 &= 0 & (7.18)\\ i_2 - i_1 &= 6. & (7.19) \end{aligned}$$

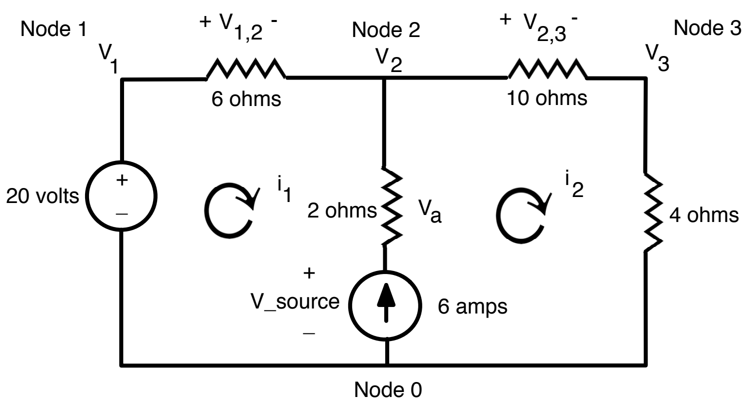

Figure 7.2 A circuit analyzed using the mesh method containing a supermesh. This means that the current source fixes the current through the 2-ohm resistor that is common to both meshes. The reference node is node 0. (This figure is a reworking of Figure 3.23 from [1].)

The two equations can be written in matrix form given by

$$\begin{bmatrix} 6 & 14 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \end{bmatrix}. \tag{7.20}$$

which yields a solution for the currents given by

$$i_1 = -3.2 \tag{7.21}$$

$$i_2 = 2.8. \tag{7.22}$$

7.3 Discussion

The mesh method generally is straightforward to apply, and leads to smaller system of equations to solve. Thus if a circuit is suitable to mesh analysis, then normally it would provide an efficient solution.

However the nodal method remains the most versatile and generally applicable method of analysis. Any linear circuit can be analyzed using the nodal method, and it forms the basis of most commercial software for circuit analysis [1]

References

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

[2] Svoboda, J.A. and R.C. Dorf, *Introduction to Electric Circuits*, 9th Edition, 2015, Wiley.

[1] Circuit analysis software will be discussed in a later chapter.

Chapter 8

Linearity, Superposition, and Equivalence

With the advent of powerful computers it is in fact possible to analyze any circuit using the nodal or mesh methods introduced in the previous chapters. But historically this has not been the case, and hence tools were developed to simplify circuits for analysis. Certainly these methods are still relevant today when simplifying complex circuits for simulation. Generally the objective is reducing the size of the simultaneous linear equations that needs to be solved. One such technique is that of superposition. This method can be applied to any linear circuit. A linear circuit can be defined as follows: *A linear circuit has an output that is directly proportional to its input.*

For linear circuits the potential differences between nodes and currents through nodes is a linear combination of the currents and potentials induced by each independent source. This means that in order to analyze a circuit it is possible to turn on each independent source one at a time (with the others off). The currents and potential in the circuit due to each source is computed in turn (which typically requires smaller matrices to be solved). If all currents and potentials obtained are summed then the actual currents and potentials in the circuit is obtained.

It should be understood that dependent (controlled) sources cannot be turned off. Superposition applies only to independent sources. The following two points are important for the reader to understand:

1. It should be understood that an ideal potential source has no internal resistance. It maintains a potential of V volts at its terminals regardless of the current through it. Thus if an independent potential source is turned off ($V = 0$ volts), it has zero potential across its terminals (nodes) and hence it becomes an ideal conductor, which acts as a short circuit.
2. An ideal independent current source has an infinite internal resistance, so that regardless of the voltage across its terminals it maintains a fixed current. Hence when its current is turned off, it becomes an infinite resistance. In circuits this is modeled by removing the source from the circuit.

8.1 Using Superposition: Analyze a Circuit

As an example of using superposition consider the circuit shown in Figure 7.2, reproduced in Figure 8.1 for convenience. It has two independent sources, thus its possible to analyze the circuit using superposition, as will be shown below. It is left to the reader to judge whether the method made things simpler in this instance.

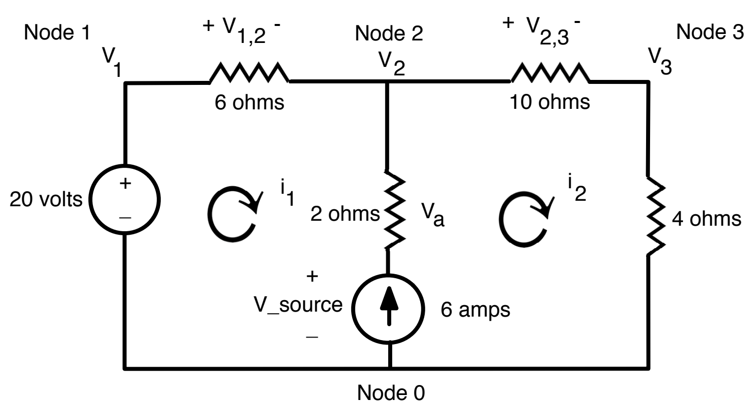

Figure 8.1 A circuit to be analyzed using superposition. The reference node is node 0. (This figure is a reworking of Figure 3.23 from [1].)

8.1.1 Turn off the Current Source (Source with 0 Amps)

The circuit of Figure 8.1 when the current source is turned off is shown in Figure 8.2. The Kirchhoff loop law yields

$$20 - 4i_1 - 10i_1 - 6i_1 = 0. \tag{8.1}$$

Hence $i_1 = \frac{20}{20} = 1$ amp. Thus the currents and potential differences due to the potential source (without the effect of the current source) is now known.

8.1.2 Turn off the Potential Source (Source with 0 Volts)

The circuit of Figure 8.1 when the current source is turned off so that $V_1 = 0$ volts, is shown in Figure 8.3. The solution yields $i_1 = -4.2$ and $i_2 = 1.8$ amps.

8.1.3 Combine Two Separate Solutions

The solution to the original circuit can now be obtained by superposition, which literally means solutions are superimposed or added together. Consider the 6-ohm resistor. The potential source induced a current of 1 amp, and the current source induced a current of -4.2 amps through the 6-ohm resistor. Thus the mesh current i_1 in the original circuit must be $i_1 = -4.2 + 1 = -3.2$ amps. Similarly for

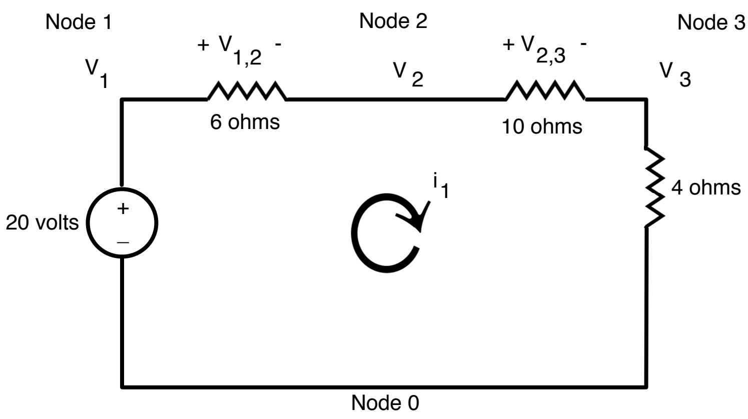

Figure 8.2 The circuit of Figure 8.1 with the current source turned off.

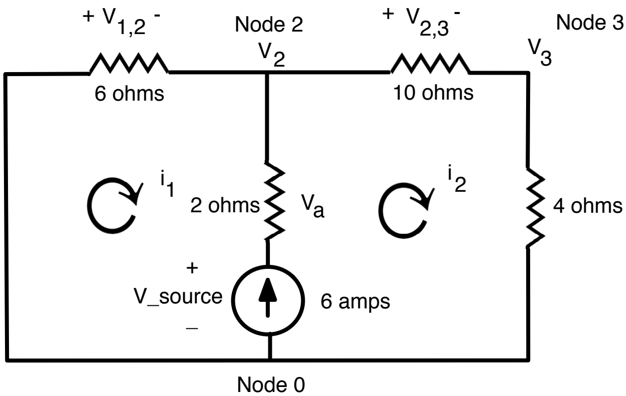

Figure 8.3 The circuit of Figure 8.1 with the potential source turned off.

the 4-ohm resistor the mesh current i_2 of the original circuit must therefore be $i_2 = 1.8 + 1 = 2.8$ amps.

This clearly is the solution that was obtained previously, and the question now is whether superposition made the calculations simpler. In this particular case it appears it did not, but certainly there are examples where it does and the application of the method thus is dependent on the circuit.

8.2 Equivalence

An important concept in engineering (not just in electrical engineering) is that of equivalence. Two different systems are said to be equivalent if their properties viewed externally are the same, yet their inner workings may be different. For circuits it can be best explained via an example.

Consider the two circuits shown in Figure 8.4. Imagine the two circuits being encased in identical black boxes. The only objects protruding from each black box are two copper wires (also known as the terminals). These terminals are referred to as terminal *a* and terminal *b*. Based on tests the that reader can perform using these two terminals (assuming the reader does not know which box contains which circuit), can the reader devise a test that would be able to tell which black box contains which circuit?

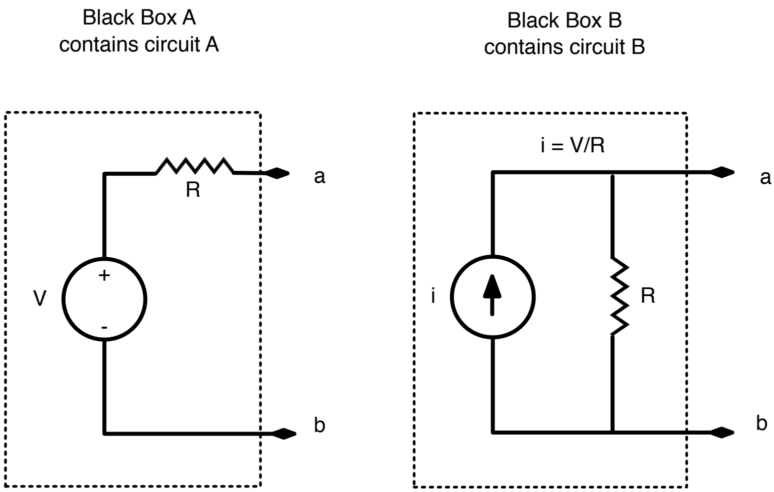

Figure 8.4 Can the reader perform a test based only on measurements at terminals a and b that will enable him/her to tell which box contains which circuit?

The equivalence theorem provides a proof of the following statement: if the two circuits have values for the sources and resistors as shown in the figure then the answer is no. There is no test the reader can devise that would let him/her determine which box contains which circuit. To provide some evidence why this is the case, let's consider two cases.

1. Short circuit the terminals, that is, connect a piece of conductor (such as copper) between nodes a and b. For the black box on the left, what current will result? Ohm's law says it will be $i_{a,b} = \frac{V}{R}$. For the black box on the right, what current will result if the terminals are shorted? It will be $i_{a,b} = i = \frac{V}{R}$ as the current divider between a short circuit and a resistor R shows that all current will flow through the short. Hence the test (shorting the terminals) does not provide any clue, as the current in the short in both cases is identical!

2. Open circuit the terminals, that is, leave the terminals open in vacuum (empty space). In both cases there is no current from node a to b. What is the potential difference between nodes a and b? For the black box on the left,

the potential difference between terminals a and b is V volts. That is because there is no current, hence the the potential difference across the resistor is zero (Ohm's law), thus the potential at node a is identical to the potential at the potential source output. For the black box on the right, the current is i and it is through the resistor R. Ohm's law says that the potential difference across the resistor is $V_{a,b} = iR = \frac{V}{R}R = V$ volts — identical to the box on the left! Yet again this test provides no clue to tell us which circuit is contained in which box.

Thus for these two tests the results were not able to tell which box contains which circuit. In fact there is no test that will reveal to the reader which box contains which circuit. That means that viewed externally, from the point of view of the terminals a and b, there is no difference between the boxes — thus the circuits are equivalent. Why is this useful the reader may ask? It's useful because there are circuits that would yield a simpler solution if a voltage-resistor combination is replaced with an equivalent current-resistor combination. Consider an example, shown in Figure 8.5, where the objective is to compute the potential V_1 (relative to node 0). For this circuit the nodal and mesh method would obviously provide solutions, with moderate effort required. But in this case use will be made of the the idea of an equivalent circuit, to make the circuit analysis simpler. The process is shown in Figure 8.6. Proceed as follows:

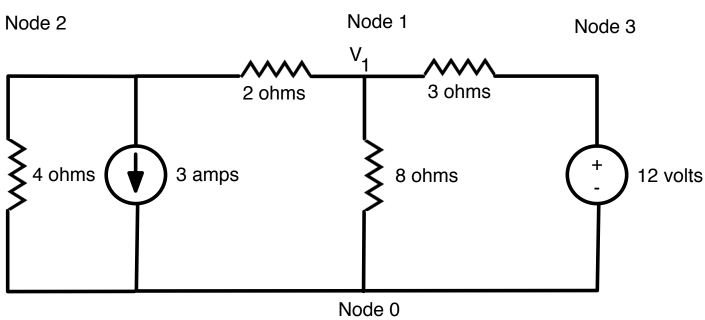

Figure 8.5 Compute the node voltage V_1 as shown in the circuit. This circuit can be simplified by successive source transformations, as shown in Figure 8.6. Node 0 is the reference node. (This figure is a reworking of Figure 4.17 in [1].)

- Make use of a source transformation at node 2 as shown in Figure 8.6.
- Make use of a source transformation at node 3 as shown in Figure 8.6.
- Apply the equivalence again (after merging the 4- and 2-ohm resistors that are in series into a single 6-ohm resistor) to produce the next equivalent circuit as shown in Figure 8.6.
- Apply equivalence again to arrive at the final simplified circuit as shown in Figure 8.6.

The final circuit contains two current sources and a single resistor, and is well suited to superposition. As it has two current sources, turning them off in turn will effectively remove them from the circuit. Hence proceed as follows:

1. Turn off the 2-amp source, which means it becomes an open circuit. The node voltage due to the 4-amp source is thus $V_1^4 = 4\frac{8}{5}$.
2. Now turn off the 4-amp source, thus it becomes an open circuit. The node voltage due to the 2-amp source is thus $V_1^2 = -2\frac{8}{5}$.
3. Thus the node voltage V_1 can be computed through superposition, given by

$$V_1 = V_1^2 + V_1^4 = 4\frac{8}{5} - 2\frac{8}{5} = 3.2 \text{ volts.} \tag{8.2}$$

8.3 Discussion

The reader may agree that the solution presented above making use of the two theorems (equivalence and superposition) is in fact simpler than a direct solution using the nodal or mesh method. However with the availability of powerful computers the modern trend is to use the nodal or mesh methods regardless of the circuit's potential to simplify. Thus the choice of method is really the reader's to make. Most engineers have their own preferences and style, and through using these methods the reader will soon develop his/her own preferences or style when solving circuits.

A final comment pertains to source transforming dependent (controlled) sources. It is recommended that the reader does not source transform dependent sources, as often these sources are dependent on the circuit topology and the transformation may change the topology. For that reason it is recommended that source transformations are only applied to independent sources.

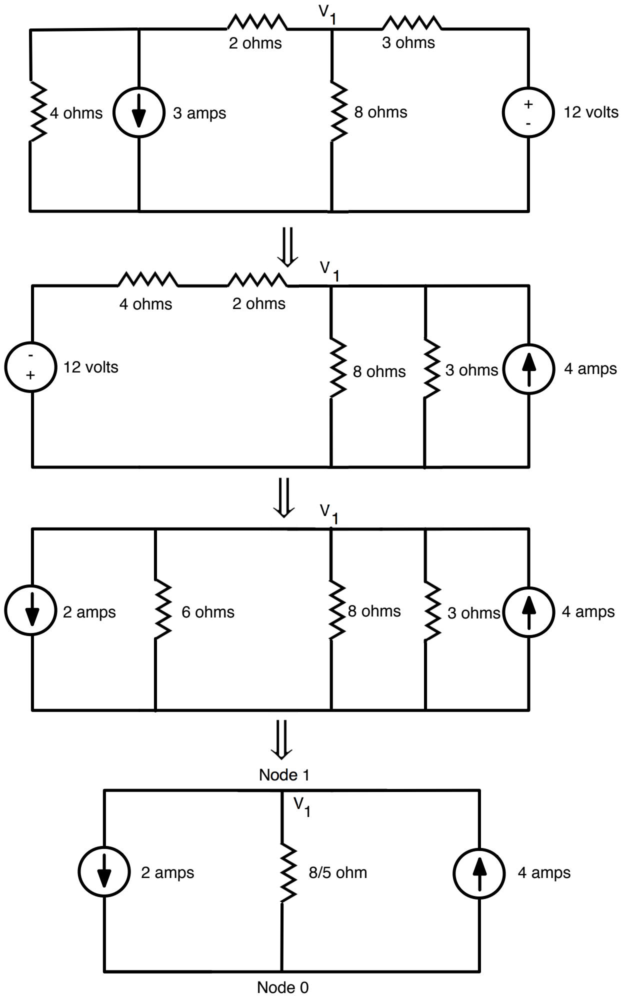

Figure 8.6 Applying equivalence to simplify the circuit in Figure 8.5.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 9

Thevenin and Norton Equivalent Circuits

The equivalence illustrated in the previous chapter can be generalized. It can be argued that it doesn't matter how complex a circuit is (it must be linear though), it can be replaced with an equivalent circuit. If the equivalent circuit is placed inside a black box, and the actual circuit is placed in another identical black box, no test performed at the terminals can tell which box contains the actual circuit.

How complex (or simple) can we make the equivalent circuit? It can be as simple as a potential source and a resistor. This idea is illustrated in Figure 9.1, and can be demonstrated to be correct — it is known as Thevenin's theorem.

According to what procedure would the reader be able to compute R_{TH} and V_{TH}? This book recommends the reader to use the procedure explained below [1]:

9.1 The Thevenin Model

1. To determine V_{TH}: Remove the load resistor, so that the terminals a and b are not connected to any load. Then measure the potential difference of node a relative to node b. It follows that $V_{\mathrm{TH}} = V_{a,b}$.

[1] The procedure proposed in this chapter to determine the Thevenin resistance enables Thevenin equivalence to be applied to any linear circuit. Even if the circuit contains dependent (controlled) sources but no independent sources, the proposed method will provide the correct resistance. Thus the proposed procedure is not vulnerable to instability. Other methods available in the literature, such as those based on using the short circuit current to determine R_{TH}, may fail under certain conditions.

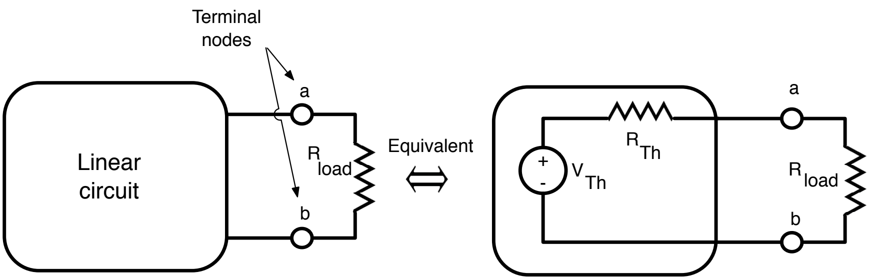

Figure 9.1 The Thevenin equivalent model, using a potential source and a resistor. The two linear circuits are equivalent under all conditions.

2. To determine R_{TH}: Remove the load resistor, so that the terminals a and b of the linear circuit are not connected to the load. Now place an independent current source of 1 amp at the terminals, with current flowing from terminal b to terminal a. Turn all independent sources in the linear circuit off, that is, make independent potential sources zero volts (zero resistance), and independent current sources zero amps (infinite resistance). Do not turn off any dependent (controlled) potential or current sources. Now measure the potential V_a – the value of the potential equals the value of the Thevenin resistance R_{TH}. Hence

$$R_{\mathrm{TH}} = V_a \text{ ohm.} \tag{9.1}$$

9.2 Computing the Thevenin Equivalent Model

As an example illustrating the procedure for determination of the Thevenin equivalent circuit, consider the circuit shown in Figure 9.2. The objective is to replace the entire circuit to the left of the terminals a and b with a single potential source and a resistor – referred to as the Thevenin potential V_{TH} and the Thevenin resistance R_{TH}.

9.2.1 Thevenin Potential V_{TH}

The procedure to find the Thevenin potential V_{TH} explained that the load resistance connected to the terminals a and b must be removed. With reference to Figure 9.2

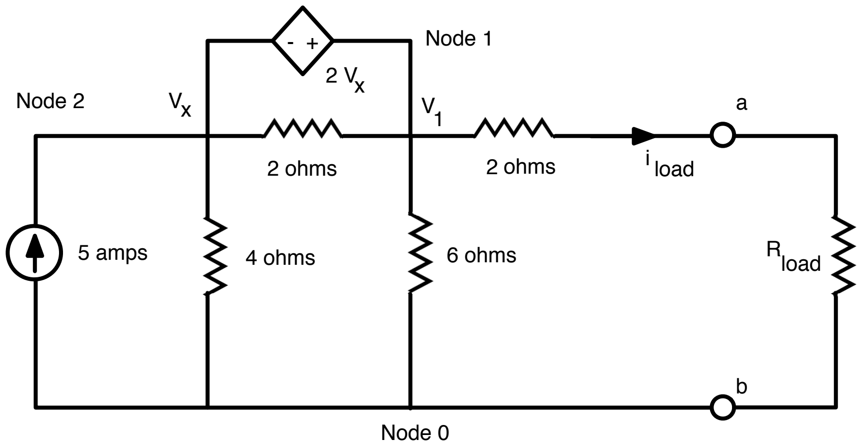

Figure 9.2 A circuit that needs to be simplified through the application of Thevenin's theorem. (This figure is a reworking of Figure 4.31 in [1].)

that means that there can be no current towards terminal a (i.e., $i_{\text{load}} = 0$). What is the reason for this being the case? Because current (moving charges) cannot be destroyed, it means that if there was a current then charges must be accumulating at terminal a — which is physically impossible. With $i_{\text{load}} = 0$, then Ohm's law says that the potential difference across the 2-ohm resistor (connected to node 1 and terminal a) must also be zero.

Thus there is no potential difference between node 1 (which is at potential V_1 relative to node 0) and terminal a. That means that the potential of terminal a relative to terminal b (or node 0) is in fact identical to V_1 (when the load has been removed). Thus since the potential difference between terminal a and b (with the load resistance removed) is the Thevenin potential V_{TH}, all that is required is for V_1 to be computed. This will provide the Thevenin potential V_{TH}.

Either the nodal or mesh methods can now be deployed to find V_1, yielding the Thevenin potential V_{TH}, which in this case is 20 volts. The reader is encouraged to confirm this result.

9.2.2 Thevenin Resistance R_{TH}

The procedure proposed in the previous section for computing the Thevenin resistance requires us to remove the load resistance R_{load}. Then a current source of 1 amp is connected between terminals a and b, and the potential at terminal a relative to b (i.e., $V_{a,b} = V_a$) is computed — the value of the potential equals the Thevenin resistance R_{TH}.

The procedure is shown in Figure 9.3 where the current source has been deployed. Thus it is known that the mesh current $i_1 = 1$ amp. The mesh method produces the following equations:

$$6(i_2 + 1) + 4i_2 + 2i_2 - 2i_3 = 0 \quad (9.2)$$

$$2(i_3 - i_2) - 2V_x = 0 \quad (9.3)$$

$$V_x + 4i_2 = 0. \quad (9.4)$$

The solution of these equations yields the following mesh currents:

$$i_2 = -\frac{1}{3} \quad (9.5)$$

$$i_3 = 1. \quad (9.6)$$

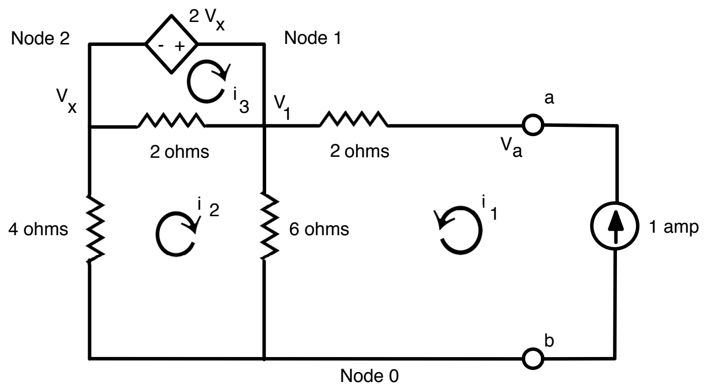

Figure 9.3 The circuit set up to compute the Thevenin resistance. The key is to remove the load and replace it with a 1-amp source. All independent sources are turned off, so that potential sources are shorts and current sources are open circuits.

Thus the potential at node 1 is given by

$$V_1 = 6\,(i_2 + 1) = 6\left(\frac{2}{3}\right) = 4 \text{ volts.} \tag{9.7}$$

To obtain the potential at node a, Kirchhoff's loop law can be applied in the mesh 1 loop starting at node 1 and moving counterclockwise, and yields

$$V_1 - V_a + 2i_1 = 0 \implies 4 - V_a + 2 = 0 \implies V_a = 6 \text{ volts.} \tag{9.8}$$

Thus the Thevenin resistance R_{TH} is given by

$$R_{\text{TH}} = V_a = 6\,\Omega. \tag{9.9}$$

To make sure that the reader appreciates the remarkable implication of these calculations, the final claim (i.e., that the circuit can be significantly simplified) is shown in Figure 9.4.

9.3 The Norton Model

Clearly, a Thevenin model (that is a potential source and resistor) can be converted to an equivalent current source and resistor, as was shown in the previous chapter. The equivalence theorem was shown in Figure 8.4, and can be used to convert the Thevenin model to a current source and resistor, which is referred to as a Norton equivalent model.

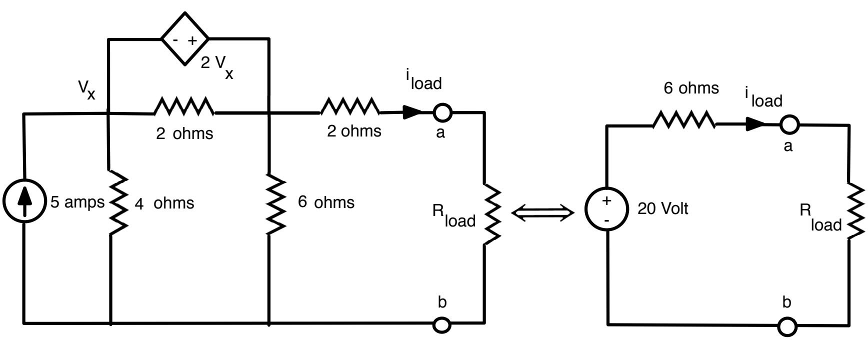

Figure 9.4 The circuit and its Thevenin equivalent model — these two circuits shown are equivalent in all respects from the point of view of the load resister. (This figure is a reworking of Figure 4.31 in [1].)

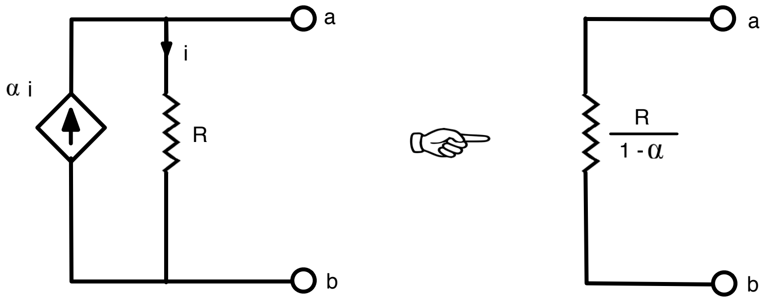

Figure 9.5 A circuit with a dependent source, and its Thevenin equivalent model. Note that the current gain α is a real number and we require $\alpha \neq 1$.

Thus if a Norton model is required, the process followed can be to first compute the Thevenin model, then convert to a Norton model through the equivalence theorem, as shown in Figure 8.4.

9.4 A Second Example

Circuits exist that deploy dependent sources but no independent sources. For example, this is the case when a transistor is modeled in its linear region[2] and for simple models of operational amplifiers as well. We consider a prototype for such circuits shown in Figure 9.5, where the current gain of the circuit is α, and the only requirement is that $\alpha \neq 1$.

Since there are no independent sources, the circuit does not have any source able to provide energy if no load is present. Hence the potential at terminal a relative to b is zero when no load is present, and the Thevenin potential is zero. From the point of view of an observer that has only terminals a and b to perform measurements, what is the resistance of the circuit?

To compute the Thevenin resistance the procedure explained in a previous section recommends adding a 1-amp current source to the two terminals, then computing the potential at terminal a relative to b. Hence a current source is added as shown in Figure 9.6.

[2] For this case use is often made of small signals, that is, signals that do not significantly modify the collector current flowing at the bias point.

In order to compute the potential at terminal a relative to terminal b, make use of the Kirchhoff current law given by

$$\alpha\, i + 1 = i. \tag{9.10}$$

Ohm's law says that the following statement must be true:

$$V_a = R\, i. \tag{9.11}$$

Substituting these equations it follows that

$$\alpha \frac{V_a}{R} + 1 = \frac{V_a}{R} \tag{9.12}$$

and thus it follows that

$$\alpha V_a + R = V_a \implies R = V_a(1-\alpha) \implies V_a = \frac{R}{1-\alpha}. \tag{9.13}$$

Hence the Thevenin resistance is given by

$$R_{\mathrm{TH}} = \frac{R}{1-\alpha} \tag{9.14}$$

as claimed in Figure 9.5.

Thus to answer the original question, measuring the resistance between terminals a and b would yield

$$\frac{R}{1-\alpha}\ \Omega.$$

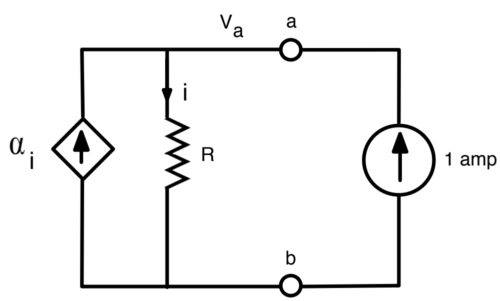

Figure 9.6 Computing the Thevenin resistance seen at terminals a and b. The key step is adding a current source of 1 amp, and thus the Thevenin resistance equals the value of the potential at node a relative to node b. We assume that the current gain $\alpha \neq 1$.

This clearly shows that care should be taken when defining resistance when dependent sources are present in a circuit. Secondly, the value of α is important. A value near 1 will yield a very large effective (Thevenin) resistance, and as stated we require $\alpha \neq 1$.

9.5 Comments

The example above presents one the the key concepts that will be used in other courses where transistors are studied. These devices contain dependent sources and thus resistances that are functions of the properties of the dependent sources. The example thus explains an important concept in electrical engineering.

Secondly it shows that resistors can be active, that is a required resistance value can be generated by deploying a fixed static resistor with a dependent source connected as shown.

We required $\alpha \neq 1$ so that the Thevenin resistance is finite. A value of $\alpha < 1$ provides a positive resistance, that will absorb power under the passive sign convention. However a value of $\alpha > 1$ will provide a negative resistance, which implies that the resistance will deliver power under the passive sign convention. This is often the case for models of circuits with positive feedback such as oscillators. However these aspects are beyond the scope of this text.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 10

Maximum Power Transfer

In previous chapters techniques for the analysis of circuits were introduced along with methods for simplifying circuits. The concept of a Thevenin equivalent circuit plays a key role in simplifying circuits. In this chapter, optimum power delivery is considered, the objective being to maximize the power delivered to a load. In many engineering applications the power that is delivered to a load is a key concept.

10.1 Maximum Power Transfer Theorem

Given that any linear circuit can be reduced to a potential source and a resistor (according to Thevenin's theorem), the question that needs to be addressed is this: what conditions need to be satisfied so that a circuit (or electrical device) is able to deliver maximum power to a load? Thus we consider the setup shown in Figure 10.1, where a circuit or electrical device has been reduced to its Thevenin equivalent and is connected to a load.

To make a beginning towards answering this question, can the reader propose a method to compute the power delivered to the load as shown in Figure 10.1? The answer lies in computing both V_a and i_a, then the power delivered to the load would be given by $P_{\text{load}} = V_a\, i_a$.

Consider the task of computing the load current i_a. Apply Kirchhoff's loop law in the mesh shown in Figure 10.1 then the following equation results

$$V_{\text{TH}} - R_{\text{load}}\, i_a - R_{\text{TH}}\, i_a = 0 \implies i_a = \frac{V_{\text{TH}}}{R_{\text{TH}} + R_{\text{load}}}. \tag{10.1}$$

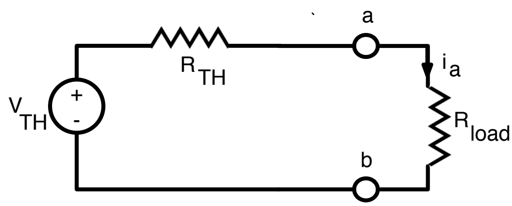

Figure 10.1 The Thevenin equivalent model of a circuit, connected to a load. The maximum power transfer theorem states that maximum power will be delivered to the load when the load equals the Thevenin resistance.

Ohm's law says that the potential at node a relative to node b must be

$$V_a = R_{\text{load}}\, i_a. \tag{10.2}$$

Substituting the equations above into the equation for instantaneous power given by $P_{\text{load}} = V_a\, i_a$ yields

$$P_{\text{load}} = R_{\text{load}} \left(\frac{V_{\text{TH}}}{R_{\text{TH}} + R_{\text{load}}} \right)^2 . \tag{10.3}$$

To form an idea of how the power delivered to the load (i.e., P_{load}) behaves as the load resistance varies, the equation for P_{load} can be plotted as a function of R_{load}, as shown in Figure 10.2. In Figure 10.2, the Thevenin resistance has been normalized so that $R_{\text{TH}} = 1$ ohm, and thus as can be seen the power delivered to the load is maximized when $R_{\text{load}} = R_{\text{TH}}$, regardless of the value of the Thevenin voltage. This finding can be generalized through the use of calculus[1] and it can be shown that it holds in general. This is then known as the maximum power theorem: *Power delivered to a load is maximized when the load resistance equals the Thevenin resistance.*

This theorem has many applications in practice, and is best shown at work through an example.

[1] Differentiate Equation (10.3) with respect to R_{load} and equate to zero. That will show that $R_{\text{load}} = R_{\text{TH}}$ when the power in the load is maximized.

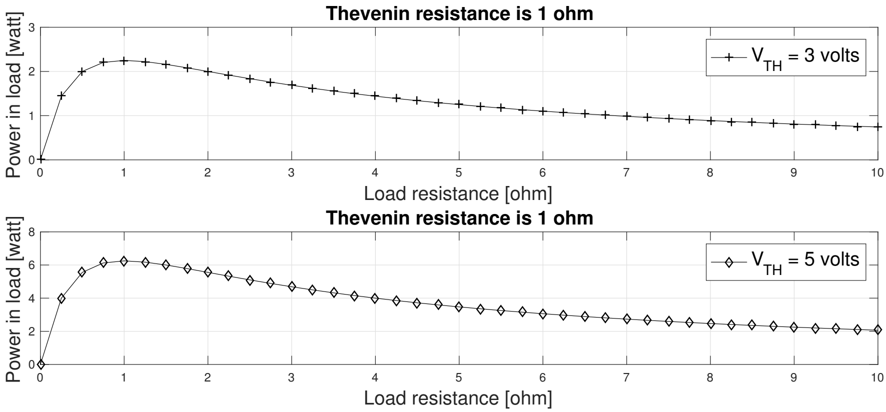

Figure 10.2 The power in the load as a function of the load resistance. The Thevenin resistance is normalized to one ohm. Clearly the power delivered to the load is maximized if the load resistance is equal to the Thevenin resistance.

10.2 An Example: Maximum Power Transfer

Consider the circuit as shown in Figure 10.3, where power is delivered to a load resistor R. The objective is to compute the resistance R that will absorb maximum power.

The maximum power theorem presented in the previous section provides the key to answer the question posed above. In order to maximize the power delivered to the load, the load resistance value R must be chosen to be equal to the Thevenin resistance of the circuit or device that is providing the power. Thus in this case the Thevenin resistance of the circuit to the left of the terminals a and b must be computed.

Following the procedure presented in a previous chapter the load is removed, and a 1-amp current source is attached to the terminals a and b, and the independent sources are turned off. This is shown in Figure 10.4, where the mesh method is deployed. The objective is to compute V_a as the value of potential at node a relative to node b will provide the Thevenin resistance.

The second mesh current i_2 is 1 amp as it must be equal to the current source that is located in mesh 2. Now apply Kirchhoff's loop law in mesh 1, so that the

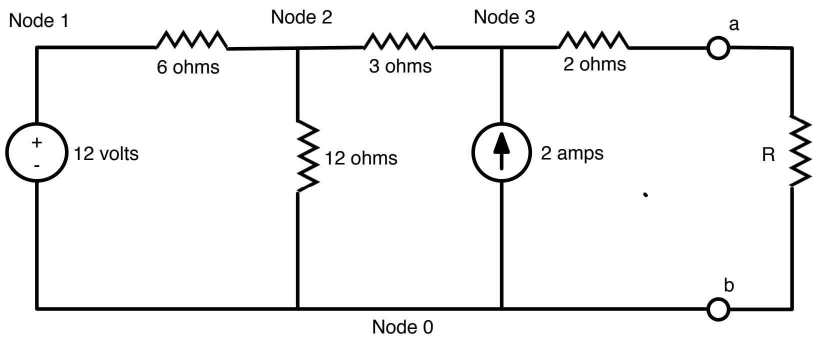

Figure 10.3 A circuit delivering power to the load R. Node 0 is the reference node. (This figure is a reworking of Figure 4.50 in [1].)

following equation is obtained:

$$12i_1 - 12i_2 + 6i_1 = 0 \implies 12i_1 - 12 \times 1 + 6i_1 = 0. \tag{10.4}$$

Thus the solution of the equation above provide a solution as $i_1 = \frac{12}{18}$ amps. To determine V_a apply Kirchhoff's loop law in the outer loop as follows:

$$-V_a + 2 + 3 + 6 \times \frac{12}{18} = 0 \tag{10.5}$$

which yields the potential at node a as $V_a = 9$ volts. Hence the Thevenin resistance is $R_{\text{TH}} = 9\,\Omega$ and hence the optimal value for the load resistance that will maximize the power delivered to the load is

$$R = R_{\text{TH}} = 9 \text{ ohms}. \tag{10.6}$$

The validity of the result above can be tested by varying the load and examining the power delivered to the load. It can be shown that the Thevenin potential is 22 volts, and is used to compute the power delivered to the load. For example the power delivered to a load $R = 8$ ohms is $P_{8\,\Omega} = 13.3979$ watts, and the power delivered to a load $R = 10$ ohms is $P_{10\,\Omega} = 13.4072$ watts. Clearly the power delivered to the load when it is equal to the Thevenin resistance is a maximum, given by $P_{9\,\Omega} = 13.4444$ watts.

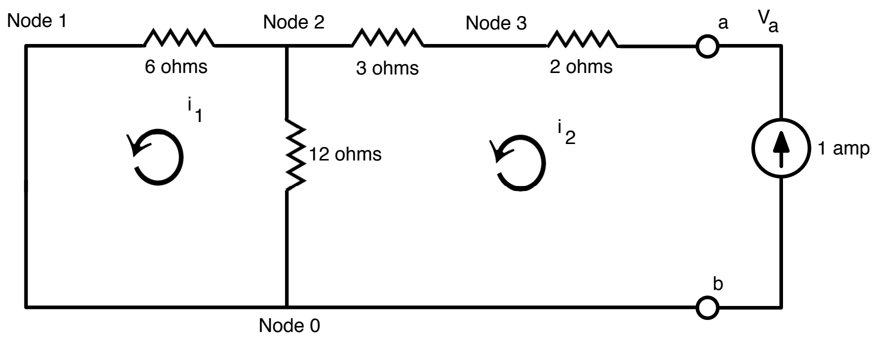

Figure 10.4 The circuit used to compute the Thevenin resistance R_{TH}. Note that an external 1-amp source is connected after the load was removed. The value of the voltage at node a relative to node 0 equals the Thevenin resistance R_{TH}.

10.3 Comments

The reader may know from experience when connecting speakers to a HiFi system that it is important to select a speaker with the correct impedance (i.e., resistance). The manufacturer of the amplifier provides (at the back of the box) the amplifier Thevenin resistance. If this states 8 ohms, then a speaker with an 8-ohm impedance will guarantee maximum power delivery.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Part II

Dynamic Sources, Energy Storage, and Transients

Chapter 11

The Capacitor and the Inductor

11.1 The Capacitor

Resistors are made of materials that do not allow charge to move freely (as a current), thus converting electrical energy into thermal energy (heat). Conductors such as copper and gold do allow current to move freely and no energy is dissipated as heat [1]. Both conductors and resistors are not able to store energy. The closest equivalent phenomenon to a resistor in dynamics would be friction. Friction is a force that opposes movement, and it requires work to make objects move against friction when present. Friction cannot store energy, but converts kinetic energy to thermal energy (heat).

As was shown in a previous chapter an electric field is conservative (as is a gravitational field), and will store energy if a charged particle is moved in a direction against the vector field. The potential difference between two nodes is measured in volts.

A device that is able to store charge and thereby maintain an electric field, is also able to store energy. Such devices exist, and an example of such a device is known as a capacitor. A capacitor is a simple device, made from two conducting plates and separated by a nonconducting material, which is known as a dielectric.

[1] This is a very good approximation. Actually these conductors do have a small resistance at room temperature.

Air is an example of a dielectric, and various forms of plastics and ceramics are also dielectrics. An ideal dielectric will not allow charge to move (a current) through it. A capacitor is shown in Figure 11.1.

The charge is stored on the plates and if the dielectric is ideal (i.e., it does not permit the flow of charge) then the field between the plates will be maintained indefinitely. Thus ideal capacitors can only store energy, it cannot dissipate energy — it cannot convert the stored energy into heat (thermal energy). It can of course release its stored energy to other devices such as a resistor when connected, as will be shown in later chapters.

The amount of charge stored, represented by q, is directly proportional to the potential difference between the plates, say $v_{a,b}$ measured in volts. Each plate is connected to a conducting wire and represents a terminal where other components can be connected in the circuit. The larger the stored charge q for a fixed potential difference, the larger the capacitors capacity or capacitance which is designated with symbol C. This is the constant of proportionality between $v_{a,b}$ and q so that

$$q = C\, v_{a,b}. \tag{11.1}$$

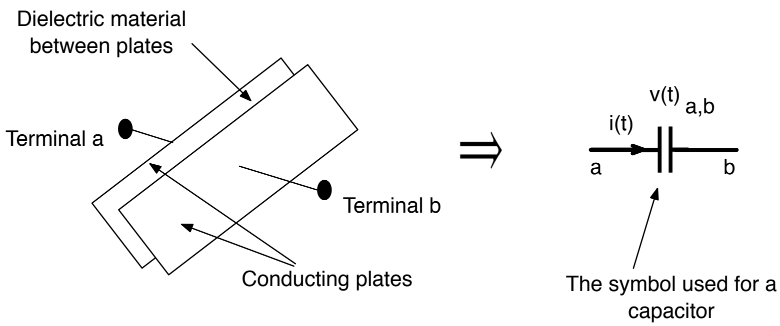

Figure 11.1 The capacitor consists of two conducting plates separated with a dielectric (nonconducting) material. The charge is stored on the plates, thereby maintaining an electric field between the plates. This field therefore stores energy, and thus the capacitor can store energy. The amount of energy stored is a function of the distance between the plates, the area of the plates and the dielectric permittivity between the plates.

The capacitance C has the unit farad (F). One farad is an enormously large capacitance, in electronic circuits typical values for capacitance are milli, micro, or nanofarads[2].

Let us now differentiate the equation above and then we find that

$$\frac{dq}{dt} = C\frac{dv_{a,b}}{dt} \implies i = C\frac{dv_{a,b}}{dt}. \tag{11.2}$$

Thus if the potential across the capacitor terminals (plates) *change with time*, (i.e., it is dynamic), then current does exist (charge flowing)! This current is not a conventional current, as proposed by the great Scottish physicist James Clark Maxwell [1] (see Figure 11.2). He called this current a *displacement current*, which physically comes down to an electric and magnetic field that changes in time. It is not a conventional flow of charge between the plates. However the current through the conductors into the capacitor is a conventional current. These concepts are beyond the scope of this text, and are studied in texts on electrodynamics [2].

At the level of this text the physics of the displacement current will not be further developed, it was mentioned merely to show the reader that the current through a capacitor is not a conventional current consisting of moving charges, and thus there are no inconsistencies.

If the potential across the terminals of a capacitor is a constant, then the derivative is zero. In such a case it shows that there is no current flowing through the ideal capacitor. Thus for static sources the ideal capacitor is indeed an infinite resistance, as no current flows through it for a finite potential (Ohm's law). However when the sources in the circuit are time varying, then the potential across the terminals of the capacitor will be changing in time, the derivative won't be zero, and a displacement current through the capacitor results. All further development on circuits containing capacitors will be based on Equation (11.2). In fact it will be shown that the presence of a derivative causes the equations describing these circuits to become differential equations. The reader may recall that for static sources there were no derivatives, and the models lead to simultaneous linear equations that needed to be solved.

The reader is reminded that this situation also exists for statics and dynamics. In statics forces and moments are constant, and the structure is at equilibrium. This reduces the solution to a system of linear equations. For dynamics the forces and moments are not in equilibrium (in the static sense) and there is displacement of objects and parameters are time varying. An analysis then leads to differential equations to be solved.

[2] Milli means 10^{-3}, micro means 10^{-6}, and nano means 10^{-9}.

Figure 11.2 The Scottish physicist James Clerk Maxwell, who unified electricity and magnetism. (Printed in black and white under license, Dover, 1890, Public Domain, https://commons.wikimedia.org/w/index.php?curid=2311942.)

The instantaneous power delivered to a capacitor is given by

$$p(t) = v(t)\,i(t) = C\,v(t)\,\frac{dv(t)}{dt}. \tag{11.3}$$

Hence the energy stored in the capacitor is given by

$$w = \int_{-\infty}^{t} p(\tau)d\tau = C\int_{-\infty}^{t} v\frac{dv}{d\tau}d\tau = C\int_{v(-\infty)}^{v(t)} v\,dv = \frac{1}{2}Cv^2. \tag{11.4}$$

This follows because $v(-\infty) = 0$ since the capacitor could not have contained charge at $t = -\infty$. This is known as an initial condition or boundary condition. Hence to summarize, the energy stored in a capacitor at time t is given by

$$w = \frac{1}{2}Cv^2. \tag{11.5}$$

w represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy.

11.1.1 Capacitors in Series and Parallel

It won't be proved in this section, but the following two rules can be shown to hold for capacitors:

1. Capacitors in series behave as resistors in parallel: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$.
2. Capacitors in parallel behave as resistors in series: $C_{eq} = C_1 + C_2 + \cdots + C_N$.

11.2 Inductor

In previous chapters it was explained that inside a conducting wire charges are free to move and it does not require energy to do so. Hence a conducting wire is an equipotential plane where the potential does not change. However when current flows through the wire, magnetic and electric fields result, and energy is stored in the fields around the wire. One such a device is known as an inductor as shown in Figure 11.3 (on the left). According to Faraday, a current flowing through a coil with N turns sets up a magnetic flux denoted by ϕ. Magnetic flux is an indication of how much magnetic field passes through an area. Moreover, Faraday discovered that the number of turns and the time rate of change of the flux determines the potential difference induced across the terminals of an inductor, given by

$$v(t) = N\frac{d\phi}{dt}. \tag{11.6}$$

This equation can be rewritten using the rules of calculus as

$$v(t) = N\frac{d\phi}{di}\,\frac{di}{dt}. \tag{11.7}$$

The constant of proportionality between the potential difference across the inductor terminals and the derivative of the current passing through it, is called the inductance $L = N\frac{d\phi}{di}$, and is measured with units henry. Thus the model for the inductor we will use is given by

$$v(t) = L\frac{di(t)}{dt}. \tag{11.8}$$

Energy is stored in the fields around an inductor. Again this stored energy can be released if the inductor is connected to other devices in a circuit, but the ideal inductor on its own cannot convert electrical energy into heat (thermal energy). If the current does not vary in time (is static) then the derivative above is zero, and there is no potential difference across the terminals of the ideal inductor. Thus

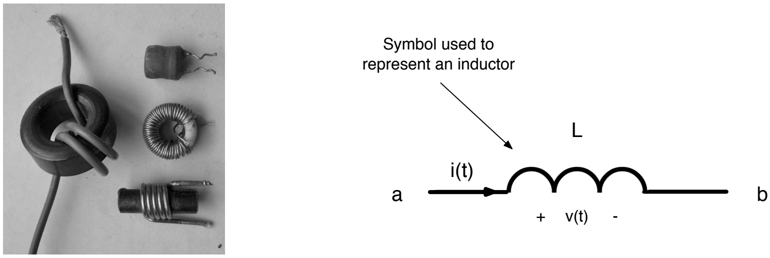

Figure 11.3 The inductor is constructed by winding a conducting wire (that is insulated) to create an interacting magnetic and electric field. This field stores energy, and we model the device as $v(t) = L\frac{di(t)}{dt}$ where L is the inductance measured in henry. (Source: Wikipedia — Printed in black and white under license https://commons.wikimedia.org/w/index.php?curid=1534586.)

for a static current an ideal inductor is a short circuit (just a section of conductor) according to Ohm's law.

The instantaneous power delivered to an inductor is given by

$$p = vi = L\frac{di}{dt}i. \tag{11.9}$$

Hence the energy stored in the inductor is given by

$$w = \int_{-\infty}^{t} p(\tau)d\tau = L\int_{-\infty}^{t} i\frac{di}{d\tau}d\tau = L\int_{i(-\infty)}^{i(t)} idi = \frac{1}{2}Li^2. \tag{11.10}$$

This follows because $i(-\infty) = 0$ since the inductor could not have contained a current at $t = -\infty$. This is known as an *initial condition* or boundary condition. Hence to summarize, the energy stored in an ideal inductor at time t is given by

$$w = \frac{1}{2}Li^2. \tag{11.11}$$

w represents the energy stored in the electric and magnetic fields that exists near the inductor. This energy can be retrieved, since an ideal inductor consisting of a conductor cannot dissipate energy.

11.2.1 Inductors in Series and Parallel

It wont be proved in this section, but the following two rules can be shown to hold for inductors:

1. Inductors in parallel behave as resistors in parallel: $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$.
2. Inductors in series behave like resistors in series: $L_{eq} = L_1 + L_2 + \cdots + L_N$.

11.3 Comments and Summary

This concludes the physics and models deployed for the ideal capacitor and the ideal inductor. The next few chapters will show how circuits containing resistors, capacitors and inductors are analyzed, and how the analysis will lead to differential equations. To summarize, this chapter presented without proof the following results:

1. A capacitor is modeled as $i = C\frac{dv_{a,b}}{dt}$. The capacitance C has units farad (F).
2. Two capacitors C_1 and C_2 in series behave as resistors do in parallel: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$.
3. Two capacitors C_1 and C_2 in parallel behave as resistors do in series: $C_{eq} = C_1 + C_2$.
4. The energy stored in a capacitor at time t is given by $w = \frac{1}{2}Cv^2$.
5. An inductor is modeled as $v(t) = L\frac{di(t)}{dt}$. The inductance L has units henry (H).
6. The energy stored in an ideal inductor at time t is given by $w = \frac{1}{2}Li^2$.
7. Two inductors L_1 and L_2 in parallel behave as resistors do in parallel: $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$.
8. Two inductors L_1 and L_2 in series behave like resistors do in series: $L_{eq} = L_1 + L_2$.

References

[1] https://en.wikipedia.org/wiki/James_Clerk_Maxwell

[2] Jackson, J.D., *Classical Electrodynamics*, Third Edition, Wiley 1998.

Chapter 12

The Source-Free RC Circuit

The first case considered is a circuit containing no independent sources, and any combination of resistors and capacitors. The capacitor is assumed charged at time zero ($t = 0$), hence there is energy stored in the capacitor at time zero. Even though there are no independent sources delivering any energy to the circuit, the stored energy is dissipated over time in the resistors in the circuit. One of the objectives is to study the time evolution of the circuit — in other words how long it takes for the circuit to dissipate the stored energy. Also the potential differences and currents in the circuit will be a function of time, and over time will decay. All these statements will become clear in the next few sections.

12.1 The Source-Free RC Circuit Prototype

The prototype of a source RC circuit is shown in Figure 12.1. In practical source free RC circuits to be studied in later chapters the objective will be to deploy circuit simplification methods to reduce the circuit to this prototype for solution.

The Kirchhoff loop law applied in the mesh yields

$$v_{a,b}(t) + R\,i(t) = 0. \tag{12.1}$$

Deploying the model for the capacitor as given in Equation (11.2) the above equation can be written as

$$v_{a,b}(t) + RC\frac{dv_{a,b}(t)}{dt} = 0 \tag{12.2}$$

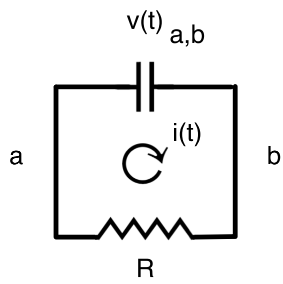

Figure 12.1 The source-free capacitor and resistor (RC) prototype circuit.

which can be rewritten as

$$\frac{dv_{a,b}(t)}{dt} + \frac{1}{RC} v_{a,b}(t) = 0. \tag{12.3}$$

This equation is known as a first order differential equation. Since the units of $v_{a,b}(t)$ are volts, and $\frac{dv_{a,b}(t)}{dt}$ is volts/second, then the units of RC must be seconds. Systematic solution methods exist to solve these differential equations, but in this chapter the solution will be stated, then it will shown that it does solve the differential equation. The solution is

$$v_{a,b}(t) = V_0 \, e^{\frac{-t}{RC}} \tag{12.4}$$

as can be seen by back-substituting it into the differential equation, and verifying that it is a solution. The reader is encouraged to do this. It can be verified that the constant V_0 can take on any value, it will always be a solution. Its value will therefore be determined by knowledge of the initial conditions — that is the value of the potential across the capacitor at $t = 0$.

The solution which is an exponential function decay over time. How fast does it decay? The rate of decay is determined by the value of $\tau = RC$, which is known as the time constant (in seconds). After time has advanced to one time constant (i.e., $t = \tau$) the potential difference across the capacitor terminals has reduced with a factor $\frac{1}{e} \approx 0.37$.

Clearly the smaller τ is the faster the potential will decay. A small resistor value or a small capacitor value (or both) will make the time constant small. If the resistor is large it will take much longer for the capacitor to discharge its energy. Over time the resistor absorbs all the energy that the capacitor stored.

12.2 A First Source-Free RC Circuit Example

As a first example of analyzing a circuit containing a capacitor and resistors, consider the circuit shown in Figure 12.2 (on the left) where we are told that $v(0) = 15$ volts. To compute $v(t)$ for $t \geq 0$ the general idea is to transform the circuit to the prototype studied above, since for the prototype we know the solution. The two resisters to the right of the capacitor are in series — a resistor of 20 ohms. Hence a 20-ohm resistor and the 5-ohm resistor are in parallel, which combines into a 4-ohm resistor. Thus the simplified circuit, in the desired prototype form is shown in Figure 12.2 on the right, where $R_{eq} = 4$ ohms. The time constant for the prototype circuit is thus $\tau = RC = 0.4$ seconds. Hence the general solution is given by

$$v(t) = V_0 e^{\frac{-t}{0.4}}. \tag{12.5}$$

Since it is known that $v(0) = 15$ volts, its clear that $V_0 = 15$ and the potential across the capacitor terminals thus is given by

$$v(t) = 15\, e^{\frac{-t}{0.4}} \quad \forall \;\; t \geq 0. \tag{12.6}$$

The current through the capacitor can be computed using Equation (11.2). The potential as a function of time is shown in Figure 12.3.

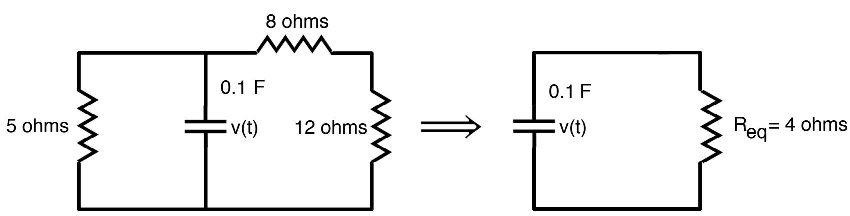

Figure 12.2 A source-free resistor and capacitor circuit. The capacitor is charged and thus at time $t = 0$ it has an initial potential of 15 volts. (This figure is a reworking of Figure 7.5 in [1].)

12.3 A Second Source-Free RC Circuit Example

For the circuit shown in Figure 12.4 the switch is closed for $t < 0$, and then opens at time $t = 0$. Thus for positive time the circuit is source-free, and the prototype above is applicable. This is one of the most important steps in solving these types of circuits — the reader must be able to correctly classify the circuit. The classification is always based on the positive time circuit. In other words, to classify the circuit all that needs to be done is to look at the positive time circuit and decide what prototype describes it.

12.3.1 The Circuit for Negative Time

For $t < 0$ the switch is closed, and thus the source was connected to the capacitor for a very long time (theoretically since the beginning of the universe). Thus by the time the reader decided to look at this circuit it has achieved equilibrium as the source is static. At equilibrium the capacitor is fully charged and the voltage over the capacitor is not changing, which means the current through it is zero (see Equation [11.2]).

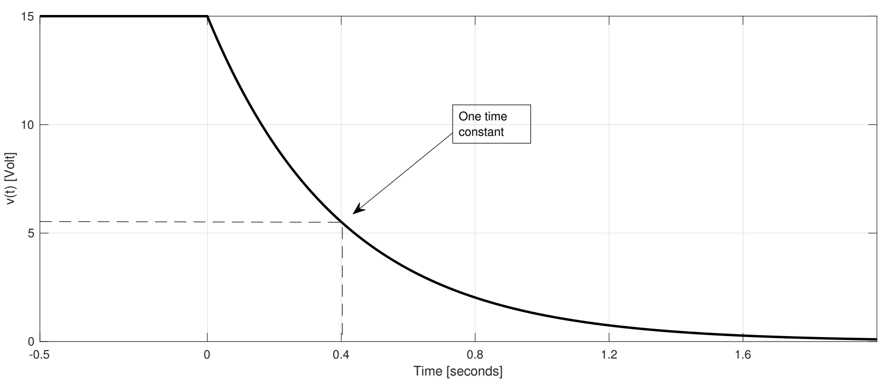

Figure 12.3 The capacitor potential as a function of time. The time constant is $\tau = 0.4$ seconds. After one time constant has elapsed, the voltage over the capacitor terminals is $\frac{15}{e} \approx 5.5$ volts.

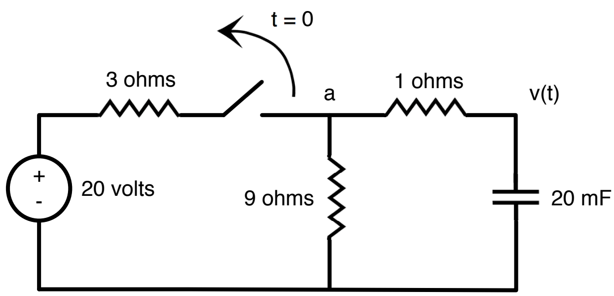

Figure 12.4 A circuit containing a source, resistors, and a capacitor. The switch opens at time $t = 0$, and was closed for $t < 0$. The objective is to compute $v(t)$ for all time t. The notation mF means millifarad or $\frac{F}{1000}$. (This figure is a reworking of Figure 7.8 in [1].)

Thus during negative time there is no current flowing through the 1-ohm resistor. Thus the potential at node a is at a voltage divider point, and thus

$$V_a = 20\left(\frac{9}{3+9}\right) = 15 \text{ volts}, \quad t < 0. \tag{12.7}$$

With no current flowing through the 1-ohm resistor there is no potential difference across the 1-ohm resistor, which means that the potential across the capacitor for $t < 0$ must be 15 volts. Thus we conclude that

$$v(t) = 15 \text{ volts } \forall \ t < 0. \tag{12.8}$$

It is assumed that the potential at a very small negative time (say $-1\ \mu s$[1]) and at $t = 0$ is the same — that is it is assumed the potential across the capacitor is continuous. This is in fact the case in practice as it will take time for stored charges to be transported away from the capacitor and hence potential across a capacitor cannot change instantly. Hence $v(0) = 15$ volts.

12.3.2 The Circuit for Positive Time

The switch opens at time $t = 0$ which means that for positive time the circuit (from the point of view of the capacitor) is as shown in Figure 12.5. The capacitor, which has a potential of 15 volts at time $t = 0$ now sees a circuit that does not contain any sources, hence for positive time the circuit is a source-free RC circuit.

[1] 1 μs means one millionth of a second.

To exploit the prototype solution presented earlier, the circuit needs to be reduced to the prototype form. This can be done by combining the two resistors that are in series into a single equivalent resistor of 10 ohms. Thus the time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ seconds} \tag{12.9}$$

and the potential across the capacitor terminals for positive time is

$$v(t) = 15\, e^{\frac{-t}{0.2}} \quad t \geq 0. \tag{12.10}$$

The voltage across the capacitor for all time is shown in Figure 12.6. Only a small part of all time is shown, but the part shown contains the so-called transient, which is the part that is of interest to the electrical engineer. When time goes to infinity the voltage is zero and the capacitor has discharged its stored charge through the resistor, and all energy that was stored at time zero has been converted to heat.

12.4 A Third Source-Free RC Circuit Example

Consider the circuit shown in Figure 12.7. It contains both independent and dependent sources, and a switch. The switch was closed for a very long time before it opens at time $t = 0$. Thus for positive time the circuit from the capacitor's point of view will not contain an independent source. Thus we can classify the RC circuit as source-free, and it fits the prototype above.

To make a start, consider the circuit for negative time, and then for positive time. The objective during the analysis of the negative time circuit is to find the potential across the terminals of the capacitor for $t \approx 0$ (but negative, that is $t < 0$).

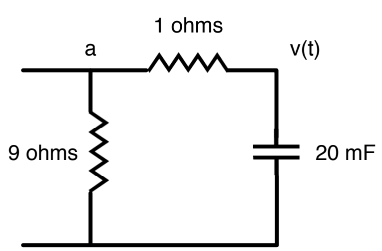

Figure 12.5 A resistor and capacitor circuit for positive time. The objective is to compute $v(t)$ for all time t.

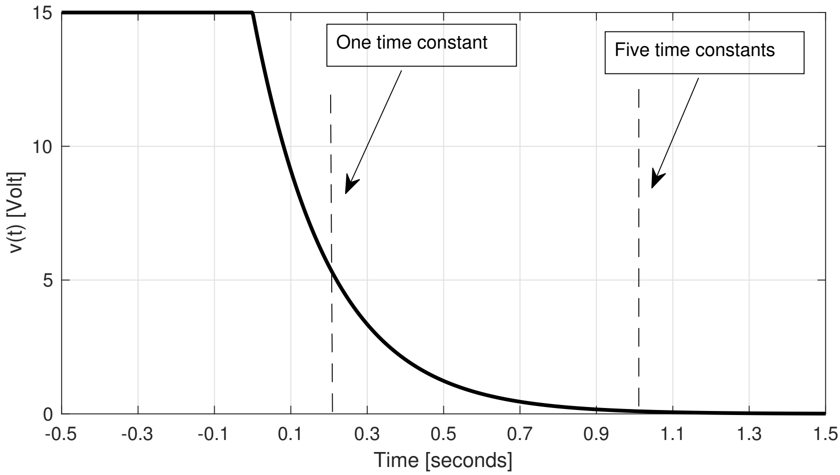

Figure 12.6 The potential $v(t)$ for all time t (only -0.5 to 1.5 seconds is shown). It is clear that the capacitor has all but discharged after five time constants.

That will provide the value of the capacitor's potential at $t = 0$ which will be required for the positive time analysis.

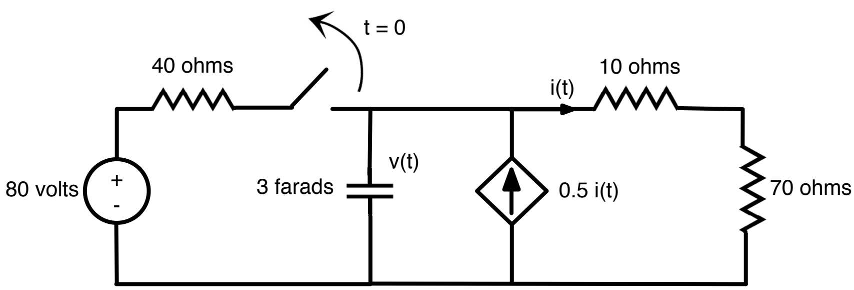

Figure 12.7 An RC circuit containing a dependent source. The switch was closed for a very long time, and then opens at $t = 0$. The objective is to compute $v(t)$ for $t \geq 0$. (The figure is a reworking of Figure 7.110 in [1].)

12.4.1 The Circuit for Negative Time

Consider the circuit for negative time ($t < 0$) as shown in Figure 12.8. The circuit has been in this state for a very long time relative to the time constant (theoretically since the beginning of the universe). Thus it has reached equilibrium, which means that the capacitor is fully charged and hence potential across the capacitor terminals is not a function of time. Thus according to Equation (11.2) the current through capacitor is zero and it acts as an open circuit as shown in Figure 12.8.

For the circuit during negative time ($t < 0$), but close to time zero, what is the potential at node a? It is identical to the potential for all negative time as the circuit (during negative time) is static, and the analysis methods from Part I of this book can be used to compute V_a. Here the nodal method is selected, and thus two equations result given by

$$\frac{80 - v_a}{40} + 0.5\, i_{neg} = i_{neg} \tag{12.11}$$

$$i_{neg} = \frac{v_a}{80}. \tag{12.12}$$

Solving these equations yields $v_a = 64$ volts. Thus it is now known that

$$v(t = 0) = 64 \text{ volts}. \tag{12.13}$$

12.4.2 The Circuit for Positive Time

Consider the circuit for positive time ($t \geq 0$) as shown in Figure 12.9. The part of the circuit indicated with dotted lines can be reduced to (modeled by) a single resistor

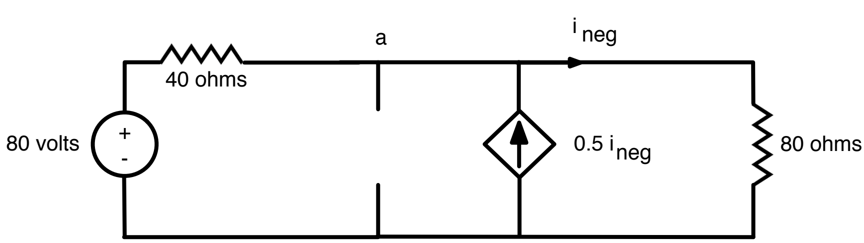

Figure 12.8 The RC circuit of Figure 12.7 for negative time ($t < 0$).

based on Thevenin's theorem. To determine the Thevenin resistance R_{TH} between the two terminals indicated (which is what the capacitor sees), add a current source of 1 amp (from terminal b to a) then compute the voltage at terminal a relative to b (i.e., $V_{a,b}$). This yields two equations to be solved given by

$$1 + 0.5\,i = i \tag{12.14}$$

$$i = \frac{V_{a,b}}{80} \tag{12.15}$$

and its solution yields $V_{a,b} = 160$ volts. Hence

$$R_{\text{TH}} = 160 \text{ ohms}. \tag{12.16}$$

Thus the circuit time constant is given by

$$\tau = R_{\text{TH}}C = 160 \times 3 = 480 \text{ sec} \tag{12.17}$$

and thus the voltage across the capacitor terminals is given by the prototype solution as

$$v(t) = 64\, e^{\frac{-t}{480}}. \tag{12.18}$$

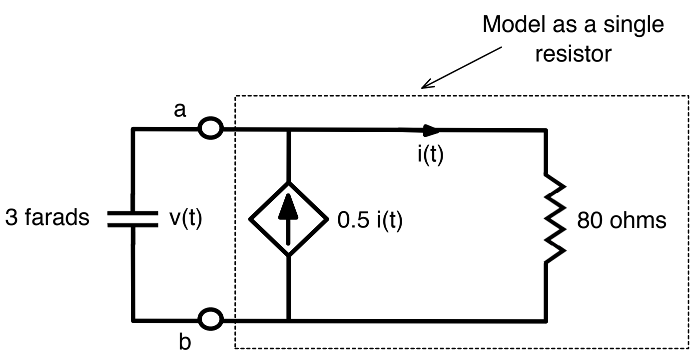

Figure 12.9 The RC circuit of Figure 12.7 for positive time ($t \geq 0$), from the point of view of the capacitor. Thevenin's theorem makes it possible to replace the part of the circuit enclosed with dotted lines with a single resistor.

12.5 Comments

This chapter presented methods for computing the behavior of capacitor potential when a capacitor discharges through a source-free RC circuit. The method is based on the solution of a first order differential equation, and solutions are decaying exponentials. The concept of a time constant was introduced, which regulates the rate of decay of the exponential function. The circuit simplification is based on Thevenin's theorem.

In a subsequent chapter, the positive time circuit and analysis will be generalized and will contain an optional independent source, so that the capacitor can charge or discharge during positive time.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 13

The Source-Free RL Circuit

13.1 The Source-Free RL Circuit Prototype

In the previous chapter the source-free RC circuit was considered. In this chapter the source-free RL circuit is considered. The prototype circuit containing a single inductor and a resistor is shown in Figure 13.1. Applying Kirchhoff's loop law in the mesh as shown yields

$$v(t) + i(t)\,R = 0. \tag{13.1}$$

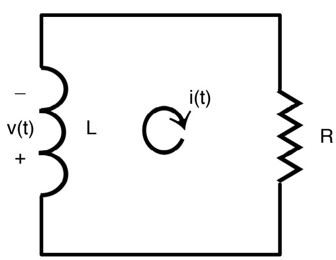

Figure 13.1 The source-free RL prototype circuit.

For an inductor it is known that $v(t) = L\frac{di(t)}{dt}$ and hence the equation above yields

$$L\frac{di(t)}{dt} + i(t)R = 0 \tag{13.2}$$

or

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = 0. \tag{13.3}$$

This equation is known as a first order differential equation. Systematic solution methods exist to solve these differential equations, but in this chapter the solution will be stated then it will shown that it does solve the differential equation. The solution is

$$i(t) = i_0\, e^{\frac{-Rt}{L}} \tag{13.4}$$

as can be seen by back-substituting it into the differential equation, and verifying that it is a solution. It can be verified that the constant i_0 can take on any value, it will always be a solution. Its value will therefore be determined by knowledge of the initial conditions — that is the value of the current through the inductor at $t = 0$.

The solution is an exponential function and will decay over time. How fast does it decay? The rate of decay is determined by the value of $\tau = \frac{L}{R}$, which is known as the time constant. After time has advanced to one time constant (i.e., $t = \tau$) the current through the terminals of the inductor has reduced with a factor $\frac{1}{e} \approx 0.37$.

Clearly the smaller τ is the faster the current will decay. A small inductor value or a large resistor value (or both) will make the time constant small. Over time the resistor absorbs all the energy that the inductor stored at time $t = 0$.

13.2 A First Example of a Source-Free RL Circuit

The circuit shown in Figure 13.2 has no sources connected to the inductor for positive time. Thus the circuit is a source-free RL circuit, and can be dealt with the prototype presented above.

The question we are asked to solve is as follows: The switch in the circuit of Figure 13.2 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t \geq 0$.

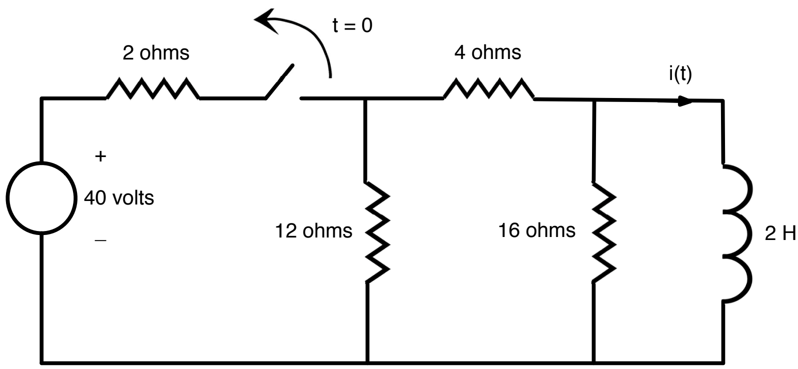

Figure 13.2 A circuit containing resistors and an inductor. The switch indicated has been closed for a long time and then opens at $t = 0$. Calculate $i(t)$ for $t > 0$. (The figure is a reworking of Figure 7.16 in [1].)

13.2.1 The RL Circuit for Negative Time

During the period $t < 0$, the switch is closed and it has been in this state for a very long time. Thus near time zero (but negative) the circuit must be in equilibrium and the potentials and currents are static (as the 40V potential source is static for negative time). This is shown in Figure 13.3. It is known that if the current through an inductor is static, the potential difference across its terminals is zero (see Equation [11.8]). During this phase the inductor is nothing but a piece of copper (i.e., a perfect conductor). That is why, as shown in Figure 13.3, the inductor has been replaced with a conductor and the current flowing through it is retained and indicated as $i(t)$.

The reader can verify that if two resistors are in parallel (see Equation [3.21]), the one with a finite resistance and the other zero, then the parallel combination (of the inductor and the 16-ohm resistor) has a zero resistance. By recognizing that the resistors (4- and 12-ohm) are in parallel, they can be combined into a resistance of 3 ohms. Thus the potential at node a can be computed as a voltage divider point given by Equation (3.14) as

$$V_a = 40\left(\frac{3}{2+3}\right) = 24 \text{ volts.} \tag{13.5}$$

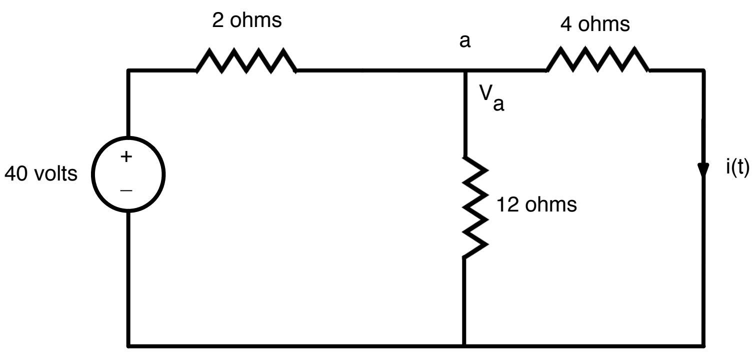

Figure 13.3 The circuit in Figure 13.2 during negative time, $t < 0$. Under these conditions it is static and the inductor is a conductor as indicated.

Hence the current $i(t)$ (for negative time) is given by Ohm's law as

$$i(t) = \frac{24}{4} = 6 \text{ amps.} \tag{13.6}$$

Thus it can be concluded that at $t = 0$ the current through the inductor denoted as i_0 will be 6 amps. This is based on the assumption that the current through an inductor is continuous at $t = 0$. This is in fact the case in practice, as the current in an inductor cannot change instantly.

13.2.2 The RL Circuit for Positive Time

At $t = 0$ the switch opens, and the potential source is cut off from the inductor. Hence from the point of view of the inductor the circuit for $t \geq 0$ is shown in Figure 13.4. The circuit indicated with dotted lines needs to be replaced with a single resistor connected to the terminals a and b. Thus a Thevenin resistance R_{TH} is required, that will transform the circuit to the desired prototype as shown.

The formal procedure to find a Thevenin resistance was explained in a previous chapter. The load is removed from the terminals a and b (in this case the inductor) and a 1-amp current source is connected to the terminals, with current from terminal b to terminal a. Then the potential $V_{a,b}$ is computed, and its value is the value of the Thevenin resistance. Hence $V_{a,b} = 8$ volts and hence

$$R_{\text{TH}} = 8 \text{ ohms.} \tag{13.7}$$

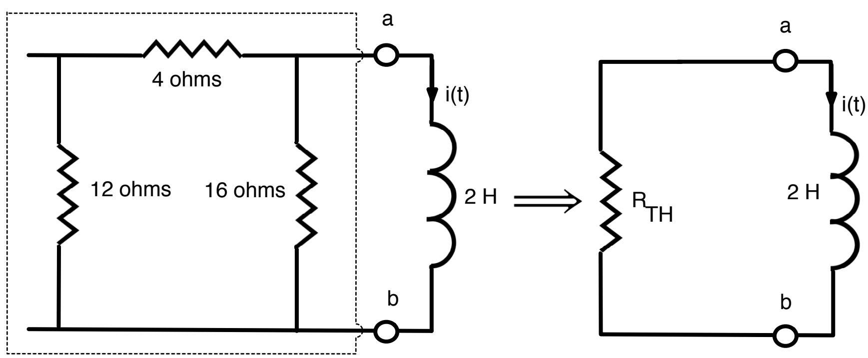

Figure 13.4 The circuit in Figure 13.2 during positive time, $t \geq 0$. The part of the circuit that is indicated in dotted lines is replaced by its Thevenin equivalent circuit as shown.

It is also possible in this case to combine the resistors, yielding directly a single resistor. But in general the Thevenin procedure may be required and will always provide the correct result. Now the next step is to compute the time constant of the prototype circuit as

$$\tau = \frac{L}{R} = \frac{2}{8} = \frac{1}{4} \text{ sec.} \tag{13.8}$$

Thus for positive time the prototype solution is given by

$$i(t) = i_0 e^{\frac{-Rt}{L}} = 6\, e^{-4t} \text{ for } t \geq 0. \tag{13.9}$$

Its clear that the current is decaying exponentially over time, which means that the energy stored in the magnetic and electric fields of the inductor is being dissipated in the resistors as heat. The currents in the resistors will also decay exponentially over time, at the same rate of decay as the current through the inductor. This can be shown through an application of Ohm's law and Kirchhoff's loop and current laws.

13.3 Second Example of a Source-Free RL Circuit

The RL circuit is shown in Figure 13.5 and does not contain any independent sources, and thus can be classified as source-free. The prototype presented in

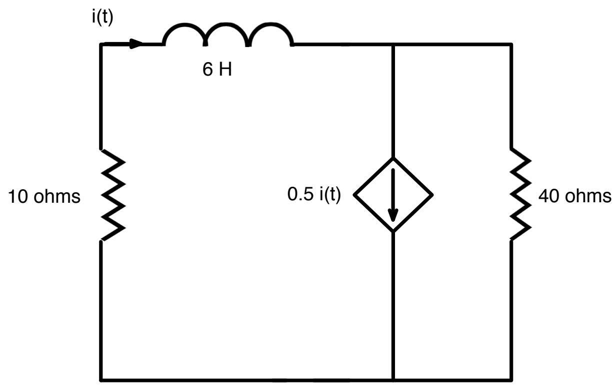

Figure 13.5 A source-free RL circuit where $i(t = 0) = 6$ amps. The objective is to compute $i(t)$ for $t \geq 0$. The circuit contains no independent sources, but it does contain a dependent current source. (The figure is a reworking of Figure 7.99 in [1].)

Section 13.1 can be applied to the circuit. Moreover the initial current in the inductor is provided and thus there is no need for a negative time analysis.

The terminals connected to the inductor (with the inductor seen as the load) and the application of Thevenin's theorem is shown in Figure 13.6. In order to

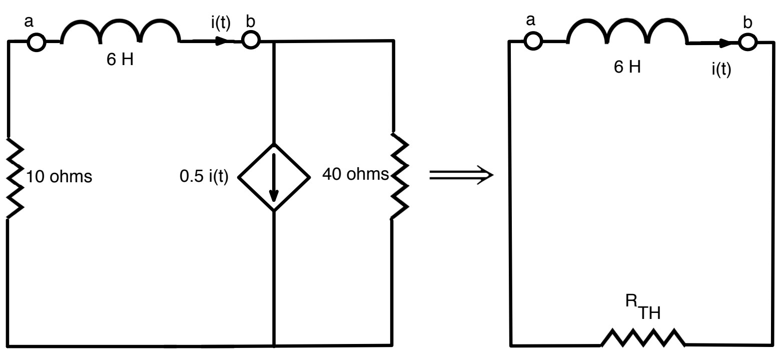

Figure 13.6 A source-free RL circuit where $i(t = 0) = 6$ amps. The Thevenin equivalent circuit is also shown.

compute the Thevenin resistance, the load (i.e., inductor) is removed and a current source is inserted between terminals a and b. The potential $V_{a,b}$ is computed, and this potential equals the Thevenin resistance.

Thus the process for computing the Thevenin resistance is as shown in Figure 13.7, where the mesh method is chosen to compute the potential $V_{a,b}$. The mesh method yields the following equations through application of Kirchhoff's loop and current laws:

$$-V_{a,b} + 10\,i_1 + V_c = 0 \tag{13.10}$$

Adding these two equations yields

$$-V_{a,b} + 10\,i_1 + 40\,i_2 = 0. \tag{13.11}$$

However $i_1 = 1$ as it is fixed by the 1-amp current source, thus the dependent current source yields

$$i_2 - i_1 = 0.5\,i = -0.5\,i_1 \implies i_2 = 0.5\,i_1 = 0.5 \text{ amps.} \tag{13.12}$$

Hence substituting the values for i_1 and i_2 into Equation (13.11) it is clear that

$$V_{a,b} = 10\,i_1 + 40\,i_2 = 30 \text{ volts.} \tag{13.13}$$

Thus it follows that

$$R_{\text{TH}} = 30 \text{ ohms.} \tag{13.14}$$

Hence the time constant for the prototype circuit is given by

$$\tau = \frac{L}{R} = \frac{6}{30} = \frac{1}{5}. \tag{13.15}$$

Finally making use of the prototype solution the current through the conductor is given by

$$i(t) = i_0 e^{\frac{-t}{\tau}} = 6\,e^{-5t} \text{ for } t \geq 0. \tag{13.16}$$

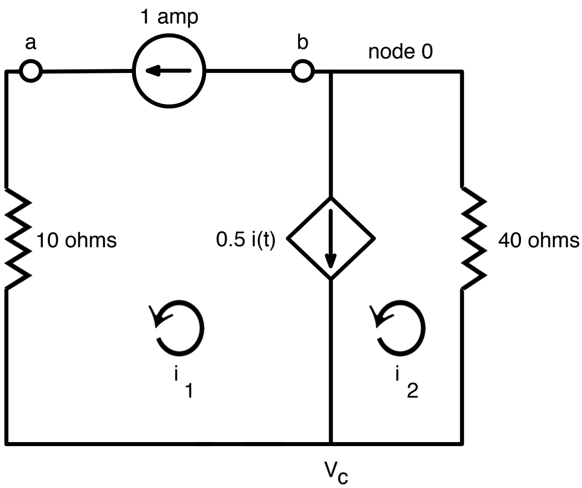

Figure 13.7 The procedure for computing the Thevenin equivalent circuit. The Thevenin resistance is by definition equal to the potential at node a relative to node 0, which was chosen as the reference node.

13.4 Comments

The reader may have noticed that the process to solve the source-free RC and RL circuits is very similar. It can be summarized as follows:

1. Verify the circuit can be classified as source-free for positive time.
2. Determine the initial conditions — that is the potential across the capacitor at $t = 0$, or the current through an inductor at $t = 0$. If these values are not provided, then it will involve computing these with the negative time circuit that is viewed as static.
3. With the capacitor or inductor viewed as the load, compute the Thevenin resistance of the circuit connected to the terminals.
4. With the initial conditions and circuit time constant τ known through the Thevenin resistance, the capacitance C and inductance L (as appropriate

given the circuit), apply the prototype solution to compute the capacitor voltage or inductor current for $t \geq 0$.

All the examples shown in the chapters so far followed this process.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 14

Step Response of an RC Circuit

In the previous chapters of Part II, RC circuits and RL circuits were considered where for positive time there is no independent source present. This meant that stored energy in the capacitor or inductor was dissipated (changed into heat) over time in the resistors of the circuit and caused an exponential decay of potentials and currents in the circuit.

In this chapter, the positive time circuit contains an independent source — thus the circuit is not source-free for positive time. One effect this has is that the positive time circuit may not have currents and potentials that decay to zero over time. The energy provided by the source will cause the circuit to eventually reach a new equilibrium where the potentials and currents are not zero. In other words, the equilibrium that existed before the change in sources (or step) will be disturbed by the change. This will be followed by a transient period where currents and potential move towards the new equilibrium, and eventually the transient will decay away and a new equilibrium will be achieved.

In engineering it is often the case that systems in one equilibrium state move to a new equilibrium state after a change in boundary conditions has occurred. As an example consider two sealed boxes each containing a gas, at a finite pressure and temperature. For negative time ($t < 0$) the boxes are in contact but there is an insulating wall between the boxes that prevent molecules from moving from one box to the other. The two boxes are at an equilibrium point. Now at time $t = 0$ the wall is punctured and a hole results, and a transient phase will now commence where molecules move through this hole from one box to the other. The system moves towards a new equilibrium, where the gasses will be mixed at a new pressure. This process is called diffusion, and can be explained by the kinetic theory of gases.

Typically in circuits the engineer is interested in the equilibrium state that exist before the change, the nature of the transient, the magnitude and duration, and the new equilibrium.

14.1 The Step Response RC Circuit Prototype

Consider a capacitor that is charged — that is, it contains a certain amount of charge q and thus a nonzero potential across its terminals. The capacitor may be connected to a static circuit while in this state and thus the charge remains the same and the potential across its terminals remain unchanged. Now a change or step occurs, which we choose to occur at $t = 0$ for convenience [1]. This change is modeled as an instant step, requiring zero time. After the change a modified circuit is connected to the capacitor — this step or change is typically accomplished with an ideal switch. The step or change causes a transient, and the current and potentials move towards a new equilibrium.

The prototype to model the setup explained above is shown in Figure 14.1, where the change occurs at $t = 0$ due to the use of a switch as indicated. This prototype will be analyzed and its solution will be presented here.

14.2 Negative Time Circuit and Its Analysis

With reference to the circuit shown in Figure 14.1, the negative time $t < 0$ circuit has the capacitor connected to the potential source V_1 and the resistor R_1. This connection has been in this position for a very long time and has achieved equilibrium. Since the potential source V_1 is static and the connection has been stable for such a long time, then near $t = 0$ (but negative) when equilibrium conditions apply, the capacitor must be fully charged, and the potential across its terminals $v_{a,b}$ is not changing. According to Equation (11.2) there is no current flowing through the capacitor as the derivative of the potential is zero. Thus according to Ohm's law there is no potential difference across the resister R_1.

That means that near $t = 0$ (but negative) $v_{a,b} = V_1$. This is a key conclusion that is very important for the reader to understand. The negative time equilibrium

[1] We could choose any time for the change to occur.

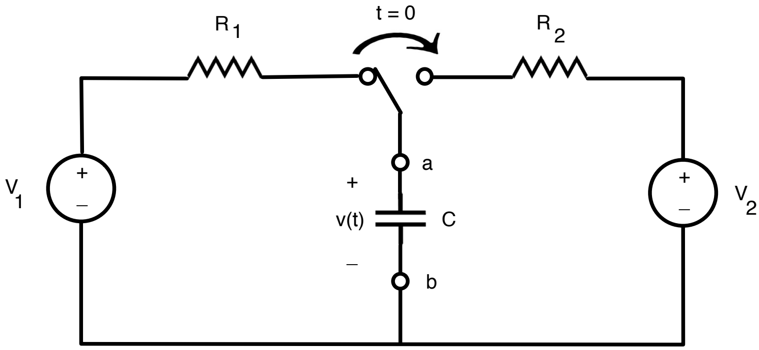

Figure 14.1 The prototype circuit for the step response RC circuit. The switch is in the state shown for a very long time so that the circuit reached an equilibrium state. At $t = 0$ the switch moves into the indicated position instantly, and a transient state occurs where the circuit state moves to a new equilibrium.

conditions for the circuit means there is no current through the resistor, and the charge (and potential) across the capacitor terminals is static.

14.3 Positive Time Circuit and Its Analysis

With reference to the circuit shown in Figure 14.1, the positive time $t \geq 0$ circuit the switch moved and has the capacitor connected to the potential source V_2 and the resistor R_2. The potential difference across the capacitor terminals is continuous as a function of time. Hence the potential difference across the terminals of the capacitor at $t = 0$ is identical to the potential difference across the terminals during negative time. The latter potential we showed above is $v_{a,b}(t = 0) = V_1$, hence

$$v(t = 0) = V_1 \text{ volt.} \tag{14.1}$$

This provides an initial condition to the differential equation that will now be derived. Apply Kirchhoff's law in the loop as shown in Figure 14.2, which yields

$$v(t) - V_2 + R_2 i(t) = 0. \tag{14.2}$$

From Equation (11.2) we know that

$$i(t) = C\frac{dv(t)}{dt} \tag{14.3}$$

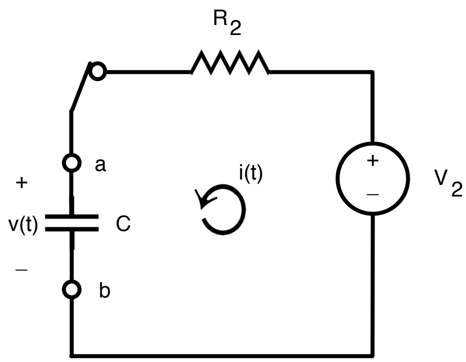

Figure 14.2 The prototype circuit for the step response RC circuit, for positive time. The circuit is moving towards a new equilibrium after the change in switch position occurred.

and substituting this into the previous equation yields

$$v(t) - V_2 + R_2 C \frac{dv(t)}{dt} = 0. \tag{14.4}$$

This is a differential equation of first order with the solution given by

$$v(t) = V_2 + \left[v(t=0) - V_2\right] e^{-\frac{-t}{\tau}}, \quad t \geq 0 \tag{14.5}$$

where

$$\tau = R_2 C. \tag{14.6}$$

The initial condition is $v(t = 0) = V_1$ as was shown above. This equation is the solution to the step RC response prototype.

Consider first the case where $V_1 < V_2$. Then the potential across the terminals of the capacitor will rise exponentially from V_1 at $t = 0$ to V_2 after a long time — with long meaning a time about 5 times the circuit time constant τ. As an example, consider the case where $V_1 = 1$, $V_2 = 2$ volts, and the time constant $\tau = 1$ second. The response of the potential across the capacitor terminals is shown in Figure 14.3.

Consider the prototype equation derived above for positive time $t \geq 0$. Note that the second term $[v(t = 0) - V_2]e^{-\frac{-t}{\tau}} = [V_1 - V_2]e^{-\frac{-t}{\tau}}$ is a transient, meaning it decays over time. The first term (i.e., V_2) is not a transient, it is in fact the asymptote or new equilibrium potential where the capacitor potential will stabilize when the

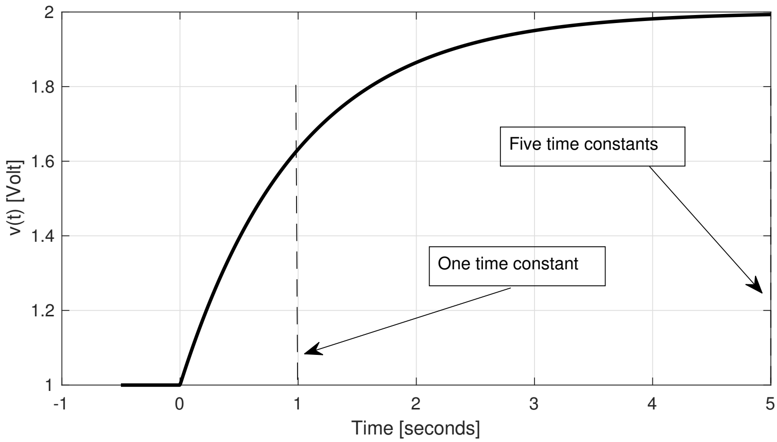

Figure 14.3 The prototype circuit response to the step change in the source potential due to the switch. Here the new equilibrium is at a higher potential than the potential during negative time. The plot shows the transient part and the new equilibrium point at 5 time constants ($\tau = 1$) as shown.

transient is gone. The asymptote and the transient together form the total solution which is shown in Figure 14.3.

Now consider the case where $V_1 > V_2$, as shown in Figure 14.4. Here the potential across the terminals of the capacitor will decay exponentially from V_1 at $t = 0$ to V_2 after a long time — with long meaning a time about 5 times the circuit time constant τ. As an example, consider the case where $V_1 = 2$, $V_2 = 1$ volt, and the time constant $\tau = 1$ second. The response of the potential across the capacitor terminals is shown in Figure 14.4.

14.4 Numerical Solution Based on MATLAB

With the advent of powerful computers it has become possible to solve differential equations numerically. Consider the differential equation for the step response RC circuit, for positive time given by

$$v(t) - V_2 + R_2C\frac{dv(t)}{dt} = 0. \tag{14.7}$$

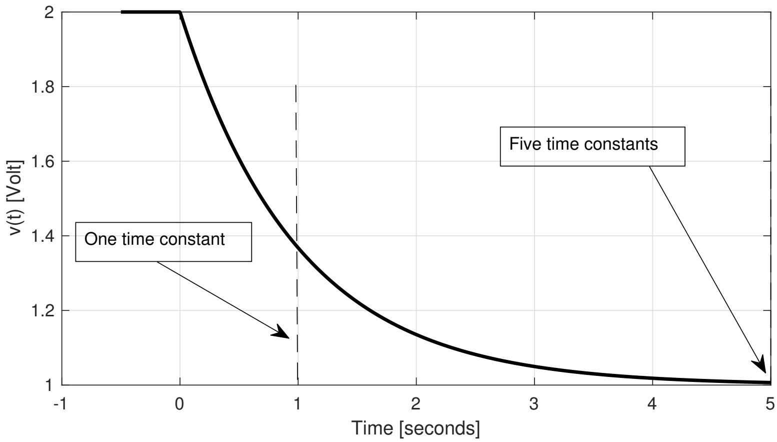

Figure 14.4 The prototype circuit response to the step change in the source potential due to the switch. Here the new equilibrium is at a lower potential than the potential during negative time. The plot shows the transient part and the new equilibrium point at 5 time constants ($\tau = 1$) as shown.

As shown above this is a differential equation of first order with the solution given by

$$v(t) = V_2 + \left[v(t=0) - V_2\right] e^{-\frac{-t}{\tau}}, \quad t \geq 0 \tag{14.8}$$

where

$$\tau = R_2 C. \tag{14.9}$$

The initial condition $v(t = 0) = V_1$ as was shown above. This equation is the solution to the step RC response prototype.

14.4.1 Numerical Solution

There are very sophisticated methods available in mathematics to solve such a differential equation numerically, with many of these methods available as functions in MATLAB [1] [2]. In this case a simple method is used as an example — the finite difference method. The idea in the finite difference solution is to approximate the derivative as

[2] For example see the $ode45$ function in MATLAB.

$$\frac{dv(t)}{dt} \approx \frac{v(t+\delta t) - v(t)}{\delta t}. \tag{14.10}$$

The term δt is a finite time step, known as the time resolution. The smaller the resolution is chosen, the more accurate the solution, but the longer it takes on the computer to obtain the solution using time steps of δt. Using this approximation the differential equation can be written as

$$v(t) - V_2 + R_2 C \frac{v(t+\delta t) - v(t)}{\delta t} = 0 \tag{14.11}$$

which means that

$$v(t+\delta t) = v(t) + \frac{\delta t\, V_2 - \delta t\, v(t)}{R_2 C}. \tag{14.12}$$

This is a recursive equation, that can be solved on a computer. Since we know the value of $v(0)$ we can compute $v(\delta t)$, then use this result to compute the next time instant $v(2\,\delta t)$ and so on. If enough steps are computed, then the entire transient region of the solution can be found up to the new equilibrium potential. The MATLAB code for solving the recursive equation above is presented below:

```
clear all

dt = 0.1;  % time step size in seconds
v0 = 1;  % potential at time 0
v2 = 2;  % positive time independant source potential
C = 1;  % capacitance
R2 = 1;  % resistor
N = 100;  % number of time steps
v(1) = v0; %initial condition
for loop=2:N %recursive
    v(loop) = v(loop-1) + (dt*v2 - dt*v(loop-1))/(R2*C);
end
plot([0:N-1]*dt,v,'-k')
xlabel('Time in seconds')
ylabel('Capacitor voltage')
grid on
hold
time = [0:N-1]*dt;
v_ana = v2 + (v0-v2)*exp(-time/(R2*C));
plot(time,v_ana,'-k')
hold
legend('Numerical solution','Analytical solution')
```

14.4.2 Numerical Results

The MATLAB code above produces the results shown in Figure 14.5. The analytical result is also shown, and it clear that the difference is negligible. This provides the reader with an idea how a circuit can be modeled by setting up a differential equation which is then solved numerically.

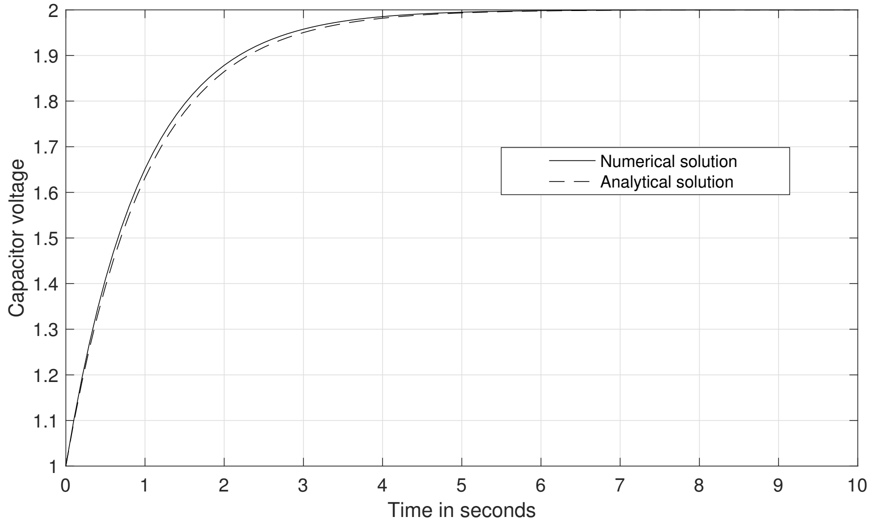

Figure 14.5 The numerical solution compared to the analytical solution. The numerical solution can be improved by making the time step (resolution) smaller.

Reference

[1] https://au.mathworks.com/help/matlab/ref/ode45.html

Chapter 15

Examples: Step Response of an RC Circuit

As always, new concepts are well explained by an example, and this chapter aims to do that. Two examples will be considered, step by step, so that the reader can follow the reasoning used to solve these type of circuits.

15.1 A First Example

Consider the circuit shown in Figure 15.1. Step one is to classify the circuit, that is, what prototype applies to this circuit? To perform a classification always consider the positive time circuit. In this case it is clear that there is an independent source present, thus it cannot be a source-free circuit. Also there is a switch, thus it is a step RC type circuit. Thus the prototype of the previous chapter will apply.

15.1.1 Simplification and the Prototype Form

Now that the circuit has been classified, the objective is to simplify or reduce it to the prototype form, so that the results from the prototype can be applied. The positive time part of the circuit is already in the prototype form and does not need to be simplified. But the negative time circuit is not in the prototype form and needs simplification.

The most general approach to simplify a circuit is to make use of the Thevenin equivalent circuit. This is shown in Figure 15.2 where the Thevenin equivalent

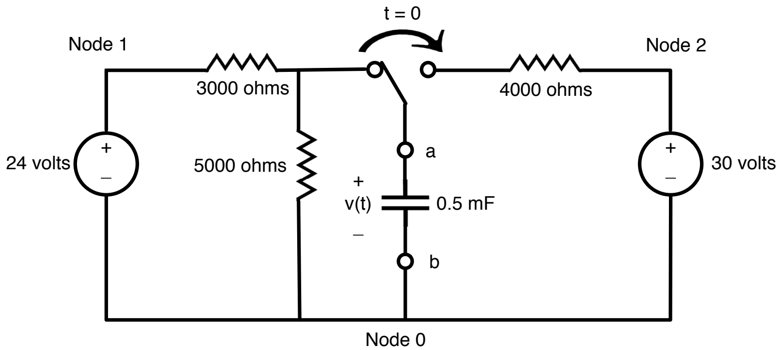

Figure 15.1 A circuit, and the objective is to compute $v(t)$. The switch was in the position shown for a very long time, and then instantly moves to a new position at $t = 0$ as indicated. The capacitor has a value of 0.5 millifarad, and node 0 is the reference node. (The figure is a reworking of Figure 7.43 in [1].)

circuit theorem is applied to force the negative time circuit into the prototype form. The reader can now verify that the following statements are true:

$$V_{TH} = 15 \text{ volts} \quad (15.1)$$

$$R_{TH} = 1875 \text{ ohms}. \quad (15.2)$$

15.1.2 Solution Based on the Simplified Circuit

Consider the simplified circuit shown in Figure 15.2, which is now in prototype form. The switch for negative time has been connecting nodes a and d for a very long time, and thus the capacitor is fully charged and the potential across its terminals is not changing. Thus Equation (11.2) shows that there is no current in the capacitor, so that there is no current flowing in the resistor R_{TH}. Thus there is no potential across the resistor (Ohm's law) and thus $v(t)$ for $t < 0$ must be equal to V_{TH}. Hence we find that

$$v(t = 0) = 15 \text{ volts}. \quad (15.3)$$

This provides the initial condition that is required to apply the prototype solution. At $t = 0$ the switch moves instantly and connects node a to node e. Hence for $t \geq 0$ or positive time the equilibrium potential clearly is 30 volts. Thus we are now in a

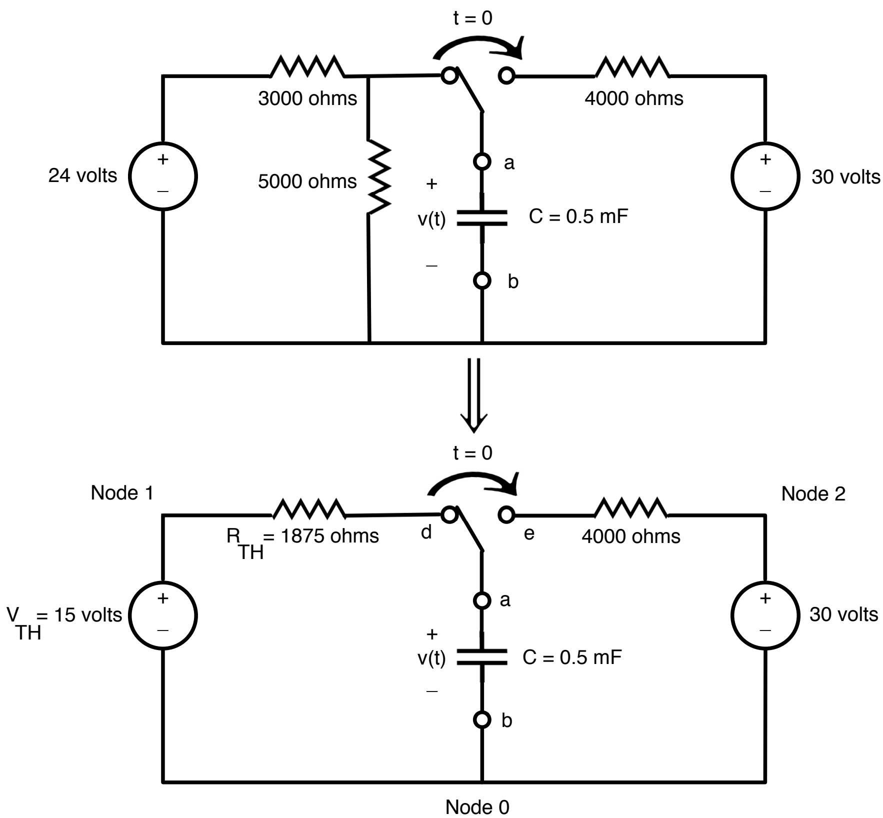

Figure 15.2 Showing how the Thevenin theorem is used to force the negative time circuit into the prototype form (below). The switch connects nodes a and d for a very long time, then moves instantly at $t = 0$ to connect node a to node e for $t \geq 0$. Node 0 is the reference node.

position to apply the prototype solution as

$$v(t) = V_2 + [v(t = 0) - V_2]e^{-\frac{-t}{\tau}} = 30 + [15 - 30]e^{-\frac{-t}{\tau}} \quad t \geq 0 \tag{15.4}$$

where

$$\tau = R_2 C = 4 \times 10^3 \; 0.5 \times 10^{-3} = 2 \text{ sec.} \tag{15.5}$$

The reader is encouraged to plot the functions before time zero, and after time zero using MATLAB to see what the potentials look like over time. In this case the capacitor is at 15 volts for $t < 0$, and charges towards 30 volts when $t \to \infty$.

15.2 A Second Example

Figure 15.3 shows a circuit containing a capacitor and resistors, as well as two independent sources. The objective is to compute $v(t)$ and $i(t)$ for all time. The function $u(t)$ is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{15.6}$$

which means that the potential source at node 1 has a potential of 0 volts for negative time, and thus is a short circuit for $t < 0$. For positive time the source has a potential of 30 volts.

The first step towards an analysis of a circuit containing a switch is to decide which of the prototypes will be used. To do that always examine the positive time circuit. In this case for positive time the switch is open, and the function $u(t) = 1$. Thus for positive time the potential source at node 1 has a potential of 30 volts, and there is thus an independent source connected to the capacitor. Hence the circuit

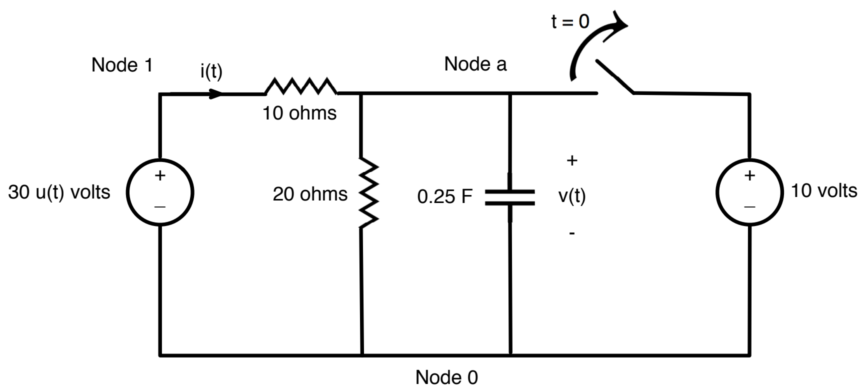

Figure 15.3 An RC circuit, with two independent potential sources. The switch was closed for a very long time, and opens at $t = 0$. The function $u(t)$ is zero for negative time, and one for positive time. (This figure is a reworking of Figure 7.45 in [1].)

is of the step RC type. Armed with this information the reader can now choose the correct prototype which in this case is shown in Figure 14.1. To analyze and solve the circuit consider the negative and positive time circuits in turn.

15.2.1 The Negative Time Circuit

For $t < 0$ the switch is closed and the voltage source at node 1 is 0 volts since $u(t) = 0$ for negative time. Thus the negative time circuit is as shown in Figure 15.4. The circuit has been in this state for a very long time and equilibrium has been reached, thus the capacitor received a full charge and a potential of $v(t < 0) = 10$ volts. The potential across a capacitor is continuous, and thus the potential of the capacitor at $t = 0$ is

$$v(t = 0) = 10 \ \text{volts}. \tag{15.7}$$

Thus the source V_1 in the prototype circuit shown in Figure 14.1 is 10 volts in this case, and the resistor $R_1 = 0$ ohm. The current $i(t)$ for negative time is given by Ohm's law as

$$i(t) = -1 \text{ amp}, \quad t < 0. \tag{15.8}$$

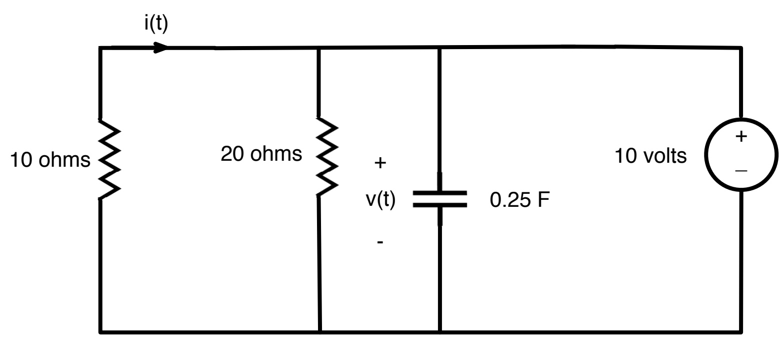

Figure 15.4 The negative time circuit. The capacitor is directly connected to the 10-volt source.

15.2.2 The Positive Time Circuit

For $t \geq 0$ the potential source at node 1 is 30 volts and the switch is open (not connected). Thus the positive time circuit is as shown in Figure 15.5. It is not in prototype form, and the Thevenin theorem is used to reduce it to the prototype form as shown in Figure 15.5. To apply the Thevenin theorem, consider the terminals of the capacitor (i.e., node a and node 0 as the load terminals) and thus compute the Thevenin circuit with the load (in this case the capacitor) removed. The reader can verify that the Thevenin potential is $V_{TH} = 30\frac{20}{30} = 20$ volts, and that $R_{TH} = \frac{20}{3}\ \Omega$.

Hence we are now in a position to make use of the prototype solution shown in Figure 14.1. The values of the circuit elements in the prototype for this case is given by

$$V_1 = 10 \text{ volts} \tag{15.9}$$

$$V_2 = V_{TH} = 20 \text{ volts} \tag{15.10}$$

$$R_2 = R_{TH} = \frac{20}{3}\ \Omega. \tag{15.11}$$

Thus making use of the prototype solution (see previous chapter) the capacitor potential for positive time is given by

$$v(t) = 20 + [10 - 20]\, e^{-\frac{-t}{\tau}} \quad t \geq 0 \tag{15.12}$$

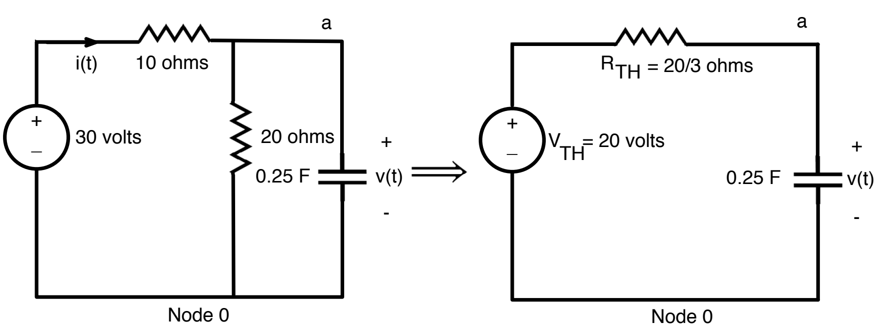

Figure 15.5 The positive time circuit, and its Thevenin equivalent yielding the prototype form.

where

$$\tau = R_2 C = R_{\mathrm{TH}} C = \frac{20}{3}\frac{1}{4} = \frac{5}{3} \text{ sec.} \tag{15.13}$$

Now that the capacitor potential is known, we are in a position to compute the current $i(t)$ which was the query posed for the second example. In Figure 15.3 it is indicated that the capacitor is connected between node a and node 0. Thus it can be deduced that the following statement must be true:

$$v_a(t) = \begin{cases} 10 \text{ volts} & t < 0 \\ 20 - 10e^{-\frac{-t}{\tau}} \text{ volts} & t \geq 0. \end{cases} \tag{15.14}$$

Hence using Ohm's law the current $i(t)$ is given by

$$i(t) = \begin{cases} -1 \text{ amp} & t < 0 \\ \frac{30-\left[20-10e^{-\frac{-t}{\tau}}\right]}{10} \text{ amps} & t \geq 0. \end{cases} \tag{15.15}$$

15.3 Discussion

Consider the expressions for the potential $v_a(t)$ and current $i(t)$ for all time given by

$$v_a(t) = \begin{cases} 10 \text{ volts} & t < 0 \\ 20 - 10e^{-\frac{-t}{\tau}} \text{ volts} & t \geq 0 \end{cases} \tag{15.16}$$

and

$$i(t) = \begin{cases} -1 \text{ amp} & t < 0 \\ \frac{30-\left[20-10e^{-\frac{-t}{\tau}}\right]}{10} \text{ amps} & t \geq 0. \end{cases} \tag{15.17}$$

The asymptote for negative time as $t \to -\infty$ is $v_a(t) = 10$ volts and for the current $i(t) = -1$ amp. The asymptote for positive time as $t \to \infty$ is $v_a(t) = 20$ volts and for the current $i(t) = 1$ amp. Thus the potential of the capacitor rises from 10 volts to 20 volts and the current rises from -1 amp to 1 amp. The current thus reverses direction during its evolution from the negative time asymptote to the positive time asymptote. The reader is encouraged to plot these results to visualize the potential and current's behavior over time.

The region of time where the current rises or falls is known as the transient. The rate at which the transient decays is set by the time constant τ. A small time constant will cause a rapid change and a short transient. Clearly a large time constant will cause the transient to take a long time to decay — that is it will take a long time for the potential and current to move from the negative asymptote towards the positive asymptote.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 16

Step Response of an RL Circuit

Chapter 14 considered the step or transient response of a RC circuit (the prototype circuit) and presented an analysis and a solution. In this chapter the objective is to do the same for the step response of a RL circuit, as indicated in Figure 16.1.

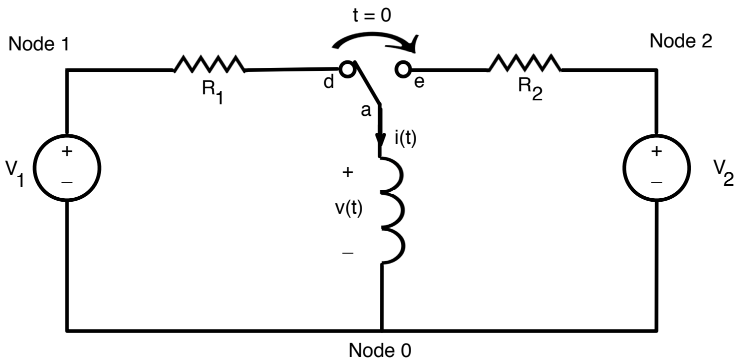

Figure 16.1 The step RL prototype circuit. The switch connects nodes a and d for a very long time, and then at time $t = 0$ instantly connects nodes a and e. The circuit thus moves towards a new equilibrium during $t \geq 0$.

16.1 Negative Time

For negative time the switch connects the inductor to resistor R_1 and potential source V_1. This circuit was connected in this state for a very long time, and thus it achieved equilibrium — the current through the inductor is not changing, and thus the voltage across the inductor is zero. The inductor under these static conditions is merely a conductor. Thus $i(t < 0) = \frac{V_1}{R_1}$, and thus near $t = 0$ (but still negative) the current is $\frac{V_1}{R_1}$.

16.2 Positive Time

Since the current through an inductor is continuous as a function of time it can be concluded that $i(t = 0) = \frac{V_1}{R_1}$. For positive time the switch moves and connects the inductor to the potential V_2 via resistor R_2. The application of Kirchhoff's loop law in the mesh for positive time yields

$$v(t) - V_2 + R_2 i(t) = 0. \tag{16.1}$$

It is known that

$$v(t) = L\frac{di(t)}{dt} \tag{16.2}$$

and substitution into the previous equation yields a differential equation as

$$L\frac{di(t)}{dt} - V_2 + R_2 i(t) = 0 \implies L\frac{di(t)}{dt} + R_2 i(t) = V_2. \tag{16.3}$$

The solution to this differential equation is given by

$$i(t) = \frac{V_2}{R_2} + \left[i(t = 0) - \frac{V_2}{R_2}\right] e^{-\frac{t}{\tau}} \tag{16.4}$$

where

$$\tau = \frac{L}{R_2}. \tag{16.5}$$

16.3 Discussion

Consider the expression for the current for all time given by

$$i(t) = \begin{cases} \frac{V_1}{R_1} \text{ amps} & t < 0 \\ \frac{V_2}{R_2} + \left[\frac{V_1}{R_1} - \frac{V_2}{R_2}\right] e^{-\frac{t}{\tau}} \text{ amps} & t \geq 0. \end{cases} \tag{16.6}$$

The asymptote for negative time as $t \to -\infty$ is $\frac{V_1}{R_1}$. The asymptote for positive time as $t \to \infty$ is $\frac{V_2}{R_2}$. The current either rises or falls exponentially depending on the relative value of $\frac{V_2}{R_2}$ and $\frac{V_1}{R_1}$. For example if $\frac{V_2}{R_2} > \frac{V_1}{R_1}$ then the current rises exponentially. The reader is encouraged to plot some examples to form an idea of the current's behavior.

The region of time where the current rises or falls is known as the transient. The rate at which the transient decays is set by the time constant τ. A small time constant will cause a rapid change and a short transient. Clearly a large time constant will cause the transient to take a long time to decay — that is it will take a long time for the current to move from the negative asymptote towards the positive asymptote.

Chapter 17

Examples: Step Response of RL Circuits

The circuit shown in Figure 17.1 contains an independent potential source, resistors, and an inductor. As always the first step is to classify the circuit. Which of the four prototypes now known to the reader applies to this circuit? To answer this question always consider the positive time circuit. For the circuit above, for positive time the switch is open and thus the two resistors and the potential source can be reduced to a single potential source and a resistor (Thevenin's theorem). Thus clearly the step response RL prototype applies to this circuit.

17.1 A First Example: Step Response RL Circuit

17.1.1 Negative Time Circuit

For negative time $t < 0$ the switch is closed, and thus the 3-ohm resistor is removed from the circuit. This is so because a parallel combination of a resistor and a conductor will have a resistance of zero ohms (the reader can verify this). Thus for negative time the circuit is in the correct form for the prototype to be applied — and the current near $t = 0$ (but negative time) is thus

$$i(t = 0) = \frac{10}{2} = 5 \text{ amps.} \tag{17.1}$$

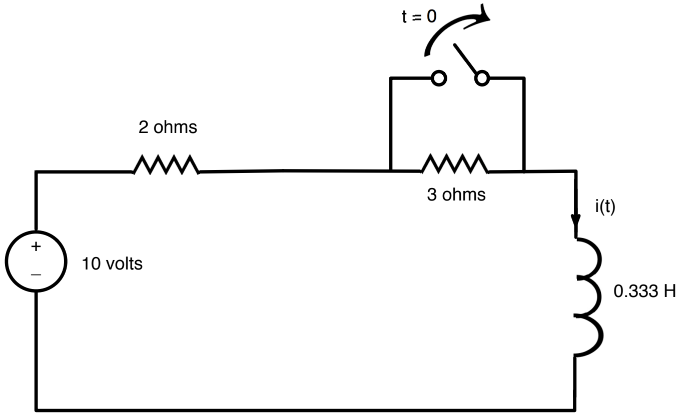

Figure 17.1 A step response RL circuit. The switch is closed for a long time during which the 3-ohm resistor does not play a role in the circuit. Then at time $t = 0$ the switch opens and the 3-ohm resistor changes the time constant. The objective is to compute $i(t)$. (The figure is a reworking of Figure 7.51 in [1].)

17.1.2 Positive Time Circuit

For $t \geq 0$ the switch opens, and the 3-ohm resister now is part of the circuit. To apply the prototype circuit the positive time circuit needs to be reduced to a potential source and a single resister. That can be done by combining the two resistors in series, so $R_2 = 5$ ohms, and the potential $V_2 = 10$ volts.

17.1.3 The Solution for $i(t)$

Substituting all these values into the prototype solution, the solution for the current through the inductor is given by

$$i(t) = \begin{cases} 5 \text{ amps} & t < 0 \\ 2 + [5 - 2]\, e^{-\frac{t}{\tau}} \text{ amps} & t \geq 0. \end{cases} \tag{17.2}$$

The time constant is given by

$$\tau = \frac{0.333}{5} = \frac{L}{R} = \frac{1}{15} \text{ sec.} \tag{17.3}$$

17.2 A Second Example: Step Response RL Circuit

Consider the circuit shown in Figure 17.2. Again the first step is to classify the circuit: which of the four prototypes presented so far applies to this circuit? To answer this question always consider the positive time circuit. For positive time the switch is open, and thus the inductor is connected to an independent source, so that this circuit is suitable for the step response RL circuit prototype.

17.2.1 The Negative Time Circuit

For negative time the switch is closed, and thus the 5-ohm resistor is removed (parallel combination of a conductor and a resistor). This is shown in Figure 17.3. The parallel combination between the current source and the resistor can be converted through a source transformation to a potential source and a resistor in series (see examples from previous chapters). The resistance is 10 ohms, and the potential is 60 volts. Since this is a static circuit at equilibrium, the current through the inductor terminals does not change in time, and hence the potential across the inductor is zero — the inductor is a piece of conductor with no resistance. Hence the current flowing through the inductor which is connected to the 60-volt source through a 10-ohm resistor, is given by $i(t = 0) = 6$ amps.

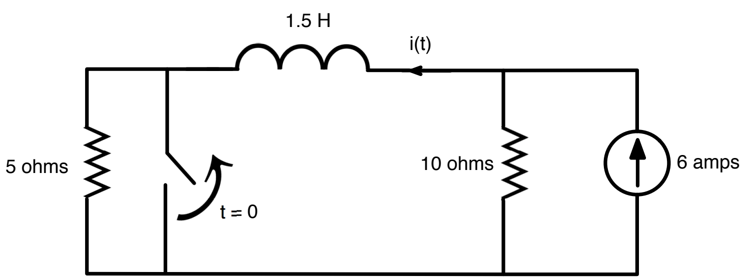

Figure 17.2 A step response RL circuit. The objective is to compute $i(t)$. The switch is closed for a long time during which the 5-ohm resistor plays no role in the circuit. Then at $t = 0$ the switch opens and the 5-ohm resistor modifies the circuit time constant. (The figure is a reworking of Figure 7.52 in [1].)

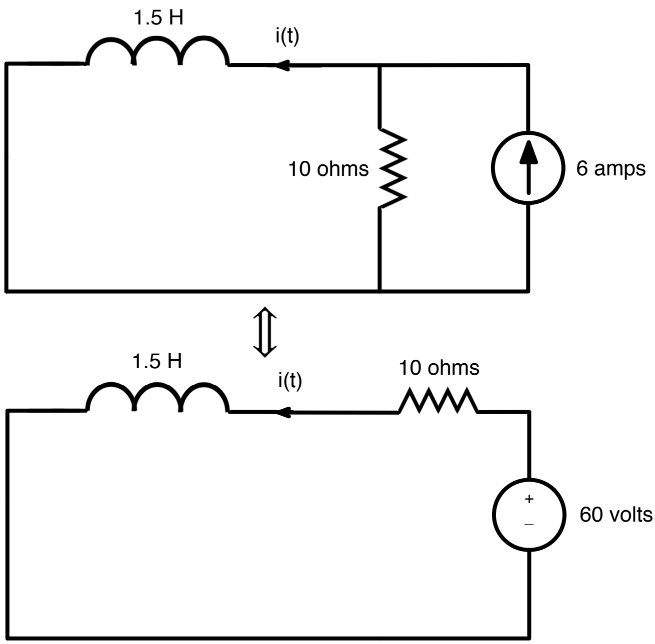

Figure 17.3 The negative time circuit for Figure 17.2.

17.2.2 The Positive Time Circuit

For positive time the switch opens and the 5-ohm resistor is back in the circuit. Again the approach is to reduce the positive time circuit to that of the positive time circuit in the prototype so that the prototype solution can be deployed. This can be accomplished by computing the Thevenin equivalent circuit for the positive time circuit, as shown in Figure 17.4. The Thevenin potential and resistance is given by

$$V_{\mathrm{TH}} = 60 \text{ volts} \tag{17.4}$$
$$R_{\mathrm{TH}} = 15 \text{ ohms.} \tag{17.5}$$

Thus all the information is now available to make use of the prototype solution as

$$i(t) = \begin{cases} 6 \text{ amps} & t < 0 \\ 4 + [6 - 4]\, e^{-\frac{t}{\tau}} \text{ amps} & t \geq 0. \end{cases} \tag{17.6}$$

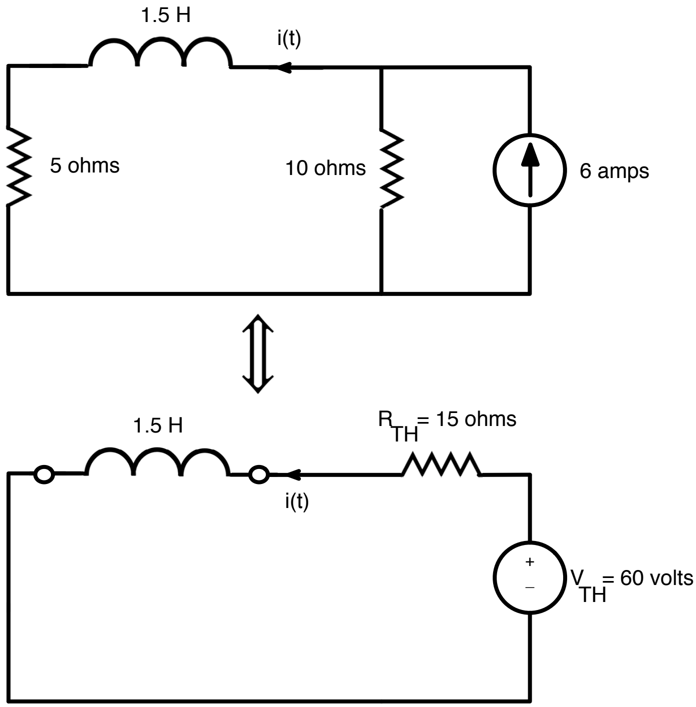

Figure 17.4 The positive time circuit for Figure 17.2. The Thevenin theorem is used to simplify the circuit into the prototype form.

The time constant is given by

$$\tau = \frac{L}{R} = \frac{1.5}{15} = \frac{1}{10} \text{ sec.} \tag{17.7}$$

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 18

Series RLC Source-Free Circuits

In previous chapters circuits were constructed using switches, resistors and capacitors, or using switches, resistors and inductors. Both cases were considered with and without independent sources if $t \geq 0$. This led to four different prototypes to cover all the possibilities.

In this chapter circuits are considered where for positive time there are (1) resistors, (2) capacitors, and (3) inductors, but no independent sources. Thus only source-free RLC circuits are considered.

The physics at work when a circuit contains both capacitors and inductors is quite fascinating. The energy storage mechanisms at work in the two devices are very different. The capacitor stores energy by accumulation of electrical charge and the current through its terminals is proportional to the derivative of the potential. For the inductor it is very different. Its energy storage mechanism involves also the magnetic field and the potential across its terminals is proportional to the derivative of the current. Thus the energy storage or release does not occur at the same time in the capacitor and inductor, and this leads to interesting phenomena explored in what follows.

The behavior of an oscillating pendulum in a gravitational field also has these same considerations — two very different energy storage mechanisms at work. One is the potential energy storage increasing as the mass moves such that vertical displacement is increased. As the total energy is constant, the other storage mechanism, namely the kinetic energy, must be reduced, which means the mass is moving slower. Eventually the mass stops, at a point where the kinetic energy has been been depleted and the potential energy has been maximized. Then the mass starts moving downwards, converting potential energy to kinetic energy. At the lowest point the potential energy is at a minimum, and the kinetic energy has reached a maximum. This conversion of kinetic to potential energy (and vice versa) continues indefinitely if friction is ignored. If friction is considered, then over time the energy is converted to heat, and the motion eventually stops. This chapter shows that these considerations hold also for the RLC circuit.

18.1 Series RLC Prototype

The key to solving these types of circuits is once again using prototype circuits. A given RLC circuit is transformed to one of the prototypes — then the solution is known. The initial conditions for these circuits are more complex than they were for RC and RL circuits. For the RLC circuit there are two energy storage devices that may contain energy at time zero, the inductor and the capacitor. The potential across the capacitor terminals, and the current through the inductor terminals at time zero are both required. These conditions are sufficient to provide a unique solution to the prototype circuit.

The differential equations that result are of second order — that means that there are second order derivatives as well as first order derivatives in the differential equation. The circuit considered contains a resistor, a capacitor and an inductor as shown in Figure 18.1. There is no independent source present, hence the circuit is a source free RLC circuit. There is a single loop (mesh) containing a current $i(t)$. Kirchhoff's loop law can be applied to the circuit and yields

$$v_L(t) + v_R(t) + v(t) = 0. \tag{18.1}$$

For the capacitor we know from Equation (11.2) that

$$i(t) = C\frac{dv(t)}{dt} \tag{18.2}$$

while for the inductor it is known from Equation (11.8) that

$$v_L(t) = L\frac{di(t)}{dt}. \tag{18.3}$$

Differentiation of Equation (18.1) yields

$$\frac{dv_L(t)}{dt} + \frac{dv_R(t)}{dt} + \frac{dv(t)}{dt} = 0 \tag{18.4}$$

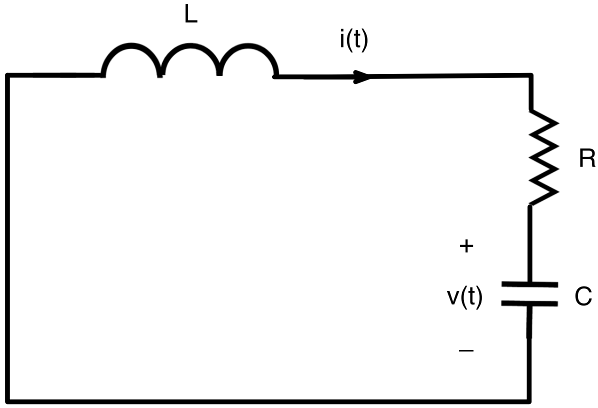

Figure 18.1 The series RLC prototype circuit. It contains no potential source, and the potential across the capacitor and the current through the inductor at $t = 0$ are required as initial conditions.

and substituting the equations above for the capacitor and inductor, while deploying Ohm's law for the resistor yields

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}\, i(t) = 0. \tag{18.5}$$

This is a second order differential equation (DE), and its solution requires two initial conditions, one for the potential across the capacitor at time zero and the other for the current through the inductor at time zero. The solutions are merely stated here, and the reader is referred to the Appendix at the end of this chapter for detail of the solutions.

18.2 Solution of the Second Order DE

Define $\Delta = \sqrt{b^2 - 4c}$ where b and c are given by

$$b = \frac{R}{L} \tag{18.6}$$

$$c = \frac{1}{LC}. \tag{18.7}$$

18.2.1 Δ is Real: The Overdamped Case

Denote the current through the inductor at $t = 0$ as I_0 and the potential across the capacitor terminals at $t = 0$ as V_0. Denote $s_1 = \frac{-b+\Delta}{2}$ and $s_2 = \frac{-b-\Delta}{2}$.

Now solve two simultaneous equations for A_1 and A_2 given by

$$A_1 + A_2 = I_0 \tag{18.8}$$

$$s_1 L A_1 + s_2 L A_2 = -RI_0 - V_0. \tag{18.9}$$

The prototype solution is now known as

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}. \tag{18.10}$$

18.2.2 Δ is Imaginary: The Underdamped Case

Denote the current through the inductor at $t = 0$ as I_0 and the the potential across the capacitor terminals at $t = 0$ as V_0. Denote α and ω_d as

$$\alpha = \frac{R}{2L} \tag{18.11}$$

$$\omega_d = \sqrt{\frac{1}{LC} - \frac{1}{4}\left(\frac{R}{L}\right)^2}. \tag{18.12}$$

Define A_1 and A_2 as

$$A_1 = I_0 \tag{18.13}$$

$$A_2 = \frac{1}{\omega_d}\left(\alpha I_0 - \frac{(RI_0 + V_0)}{L}\right). \tag{18.14}$$

Then the solution to the second order differential equation is

$$i(t) = e^{-\alpha t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right]. \tag{18.15}$$

18.2.3 Δ = 0: The Critically Damped Case

If $\Delta = 0$ then both the solutions offered above will become invalid and the solution to the differential equation is

$$i(t) = (A_1 + A_2 t)e^{-\alpha t} \tag{18.16}$$

where

$$\alpha = \frac{R}{2L} \tag{18.17}$$

$$A_1 = I_0 \tag{18.18}$$

$$A_2 = A_1\alpha - \frac{I_0 R}{L} - \frac{V_0}{L}. \tag{18.19}$$

18.3 Numerical Solution Using MATLAB

In the previous section we presented an analytical solution to the differential equation. However the advent of powerful computers has made possible the numerical solution of the second order differential equation. Based on numerical solutions the effect of varying component values in the circuit can be studied. The serial RLC circuit is described by the second order differential equation given by

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0. \tag{18.20}$$

To show how this differential equation can be solved numerically, we will use the so-called Euler method, which is simple yet effective when the system is damped (contains a resistor with a finite value). The idea is to use a dummy function (or functions) to reduce the high order differential equation to a system of first order differential equations. These can be solved efficiently using simple approximations to the derivative.

Introducing a dummy function $\psi(t) = \frac{di(t)}{dt}$ the differential equation can be written as

$$\frac{d\psi(t)}{dt} + \frac{R}{L}\psi(t) + \frac{1}{LC}i(t) = 0 \tag{18.21}$$

$$\frac{di(t)}{dt} = \psi(t) \tag{18.22}$$

subject to initial conditions

$$i(t=0) = I_0 \tag{18.23}$$

$$\psi(t=0) = \psi_0. \tag{18.24}$$

To represent functions and time in a computer we can sample the variables at a spacing interval of δt seconds, and then approximate the derivatives using a finite difference approximation given by $\frac{df}{dt} \approx \frac{f(t+\delta t)-f(t)}{\delta t}$. Hence assuming δt is made sufficiently small, the differential equations are well approximated as

$$\frac{\psi(t+\delta t)-\psi(t)}{\delta t} + \frac{R}{L}\psi(t) + \frac{1}{LC}\,i(t) \approx 0 \tag{18.25}$$

$$\frac{i(t+\delta t)-i(t)}{\delta t} \approx \psi(t) \tag{18.26}$$

and then can be written as (dropping the $\approx$ under the assumption that δt is very small)

$$\psi(t+\delta t) = \psi(t) - \delta t\,\frac{R}{L}\psi(t) - \delta t\,\frac{1}{LC}\,i(t) \tag{18.27}$$

$$i(t+\delta t) = i(t) + \delta t\,\psi(t). \tag{18.28}$$

These equations are recursive, meaning the values of i and ψ at time $t+\delta t$ can be computed based on knowledge of i and ψ at time t. Since the initial values at $t=0$ were given, we can program the recursive equations in a computer to start at time $t=0$, then advance to any time t we require, in discrete steps of δt, also known as the time resolution. Thus the solution is given by the current i at a resolution determined by δt. The user can select the resolution as fine as is required in practice. MATLAB code for solving the serial RLC circuit is given below.

```
clear all

R = 1; % resistor provided
LC = 0.5*0.02;  % product of L and C  (wd = 10)
%LC = 0.5*0.2;  % product of L and C (wd = 3)
%LC = 0.5*1.9;  % product of L and C (wd = 0.22)
L = 0.5;  % Inductance

i(1) = 1; % initial current at time 0
V0 = 0;  % initial cap voltage at time 0
psi(1) = -V0/L-i(1)*R/L; % initial derivative
N = 200000;  % time steps
dt = 0.00004; % time increment in seconds

for loop = 1:N %numerical integration (recursion)
    i(loop+1) = i(loop) + psi(loop)*dt;
   psi(loop+1)=psi(loop)-1/LC*i(loop)*dt-R/L*psi(loop)*dt;
end

subplot(2,2,1)
t = [0:N]*dt;
```

```
plot(t,i,'k')
xlabel('Time [sec]')
ylabel('Current [Amp]')
legend('Euler numerical solution')
grid on

% analytical solution if delta imaginary
alpha =  R/(2*L);
omega_d = sqrt(1/LC - 1/4*(R/L)^2);
A1 = i(1);
A2 = 1/omega_d*(alpha*i(1) - (R*i(1)+V0)/(L));
i_ana=exp(-alpha*t).*(A1*cos(omega_d*t)+A2*sin(omega_d*t));
subplot(2,2,2)
plot(t,i_ana,'k')
xlabel('Time [sec]')
ylabel('Current [Amp]')
legend('Analytical solution')
grid on
```

18.3.1 Results

We consider a series RLC circuit with $R = 1$ ohm and $L = 0.5$ henry. The current through the inductor at $t = 0$ is 1 amp, and the capacitor has a zero potential at $t = 0$. We consider two cases for C:

1. $C = 0.02$ farad — this yields $\omega_d = 10$ and this case is clearly underdamped.
2. $C = 1.9$ farad - this yields $\omega_d = 0.23$ thus this case is near being critically damped (Δ is small).

The results are shown in Figure 18.2. First it is clear that differences between the analytical solution and the numerical solution are negligible. This confirms the validity of the theory presented in this chapter. Secondly it is clear that the damping is set by R, and the damping for both cases is thus identical. The difference is the frequency ω_d. For the underdamped case the oscillation period $T = \frac{2\pi}{\omega_d}$ is small relative to the damping time ($5\tau = 5$ seconds)[1].

[1] MATLAB has sophisticated built-in functions to solve differential equations. The reader is encouraged to evaluate the *ode*45 function in MATLAB [1].

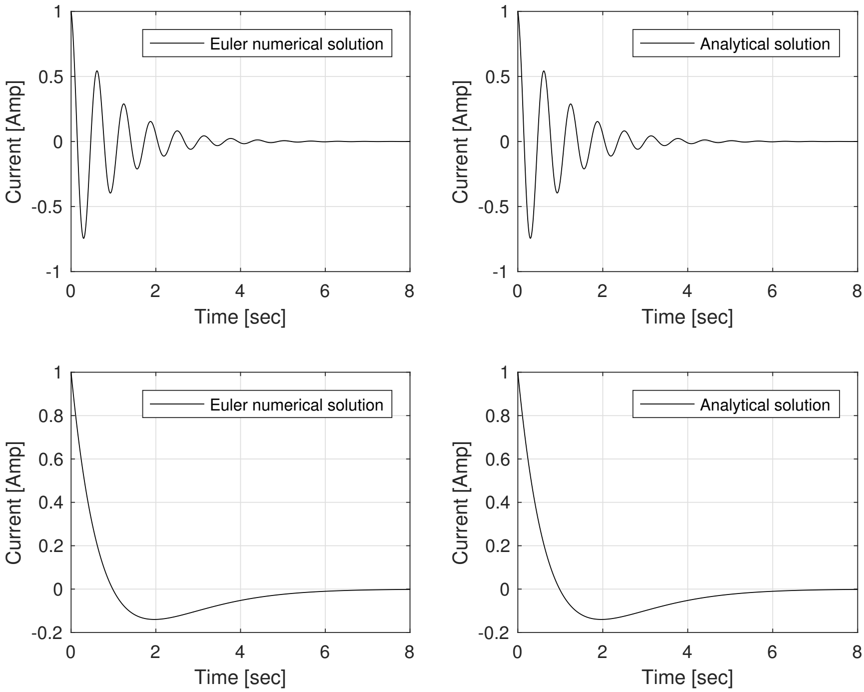

Figure 18.2 Series RLC circuit current using a numerical solution for an underdamped and near critical damping case.

18.4 Appendix: Solution of Second Order DE

The following differential equation is of a general form, and applicable to any of the series and parallel RLC circuits without sources:

$$\frac{d\phi^2}{dt^2} + \lambda \frac{d\phi}{dt} + g\,\phi = 0 \tag{18.29}$$

subject to the initial conditions

$$\phi(t = 0) = \phi_0 \tag{18.30}$$

$$\left.\frac{d\phi}{dt}\right|_{t=0} = 0. \tag{18.31}$$

Thus as a starting point for solving the second order differential equation above, the solution is assumed to be of the form

$$\phi(t) = Ae^{st} \tag{18.32}$$

where through analysis (to follow) the value of s needs to be flushed out. Clearly A will be related to ϕ at $t = 0$ (i.e., $\phi[t = 0]$), but a discussion of initial conditions is delayed so that the nature of s can be considered first. Back-substituting the solution into the differential equations yields

$$\frac{d^2\left[Ae^{st}\right]}{dt^2} + \lambda \frac{d\left[Ae^{st}\right]}{dt} + g\left[Ae^{st}\right] = 0. \tag{18.33}$$

The magnitude of the solution A clearly cancels out, hence

$$\frac{d^2\left[e^{st}\right]}{dt^2} + \lambda \frac{d\left[e^{st}\right]}{dt} + g\left[e^{st}\right] = 0. \tag{18.34}$$

Performing the differentiation yields

$$s^2 e^{st} + \lambda s e^{st} + g e^{st} = 0. \tag{18.35}$$

Now the term e^{st} cancels (assuming it is not zero), hence

$$s^2 + \lambda s + g = 0. \tag{18.36}$$

Thus we succeeded in deriving a quadratic equation that the value of s must satisfy, in order for the solution to be a valid one. Using the solution for the quadratic equation, it is clear that s can take on two possible values, given by

$$s_1 = \frac{1}{2}\left[-b + \sqrt{b^2 - 4c}\right] \text{ and } s_2 = \frac{1}{2}\left[-b - \sqrt{b^2 - 4c}\right] \tag{18.37}$$

where

$$b = \lambda \text{ and } c = g. \tag{18.38}$$

Invoking the concept of superposition in mathematics, the two solutions (because the two values for s provides two different solutions) can be superimposed, and thus the general solution is given by

$$\phi(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}. \tag{18.39}$$

The factor $\Delta = \sqrt{b^2 - 4c}$ has a significant effect on the solution, as it can be real or imaginary. The two cases will be considered separately.

1. Δ is real
   If Δ is real then the solution given above can be used directly. This solution will apply for an overdamped system. The two roots, s_1 and s_2, are both known from the physical system values that are provided. Thus all that is required to obtain the prototype solution is to find the initial conditions so that A_1 and A_2 can be determined.

   The initial conditions are known as

$$\phi(t = 0) = \phi_0 \tag{18.40}$$

$$\left.\frac{d\phi}{dt}\right|_{t=0} = 0 \tag{18.41}$$

   and hence the derivative of the solution is required and given by

$$\frac{d\phi}{dt} = A_1\, s_1\, e^{s_1 t} + A_2\, s_1\, e^{s_2 t}. \tag{18.42}$$

   Thus the initial conditions yield two linear equations with two unknowns that can be solved, and are given by

$$A_1 + A_2 = \phi_0 \tag{18.43}$$

$$A_1\, s_1 + A_2\, s_1 = 0. \tag{18.44}$$

   This yields A_1 and A_2 and hence the solution is now known as

$$\phi(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}. \tag{18.45}$$

2. Δ is imaginary
   With Δ as imaginary, the two roots, s_1 and s_2, are complex. Given Δ is imaginary, then the general solution to the second order differential equation

can be obtained as shown below. First write the two roots as

$$s_1 = -\alpha + j\omega_d \tag{18.46}$$

$$s_2 = -\alpha - j\omega_d \tag{18.47}$$

where α and ω_d are defined as

$$\alpha = \frac{\lambda}{2} \tag{18.48}$$

and

$$\omega_d = \frac{\sqrt{4c - b^2}}{2} = \sqrt{g - \frac{1}{4}(\lambda)^2}. \tag{18.49}$$

Then based on these two roots (s_1 and s_2) the solution to the second order differential equation is

$$\phi(t) = e^{-\alpha t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right]. \tag{18.50}$$

Since it is known that the derivative of ϕ at time zero is 0, an expression for the derivative is required. The term $\frac{d\phi(t)}{dt}|_{t=0}$ can be computed by differentiating Equation (18.50) using the *product rule* in mathematics. This yields

$$\left.\frac{d\phi(t)}{dt}\right|_{t=0} = \omega_d A_2 - \alpha A_1. \tag{18.51}$$

Thus we can solve for the constants, given the initial conditions

$$A_1 = \phi_0 \tag{18.52}$$

and

$$\left.\frac{d\phi(t)}{dt}\right|_{t=0} = 0. \tag{18.53}$$

Thus we find that

$$A_2 = \frac{\alpha A_1}{\omega_d}. \tag{18.54}$$

With the constants A_1 and A_2 now known, the solution for $\phi(t)$ has been obtained, given by

$$\phi(t) = e^{-\alpha t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right]. \tag{18.55}$$

Reference

[1] https://au.mathworks.com/help/matlab/ref/ode45.html

Chapter 19

Examples: Series RLC Source-Free Circuits

19.1 A First Example

The theory and solutions derived in the previous chapter for the prototype will now be applied to representative examples. Figure 19.1 shows a circuit where resistors, independent sources, a capacitor, and an inductor is present, and a switch is used as well. The objective is to compute $i(t)$.

As always, the first step is to classify the circuit. To do that always consider the positive time circuit, which in this case is shown in Figure 19.2. For positive time there is no source present, so it is source-free. However, it contains resistors, an inductor, and a capacitor, hence it is a series RLC circuit. Therefore the source-free series RLC prototype would apply to this circuit. The next step is to compute Δ to determine if it is real or imaginary. To do that use the positive time prototype transformed circuit (see Figure 19.2), which gives

$$\Delta = \sqrt{b^2 - 4c} = \sqrt{\left(\frac{9}{0.5}\right)^2 - \frac{4}{0.5 \times 0.02}} \approx j8.7 \tag{19.1}$$

which is imaginary. Hence the source-free RLC prototype circuit with imaginary Δ applies.

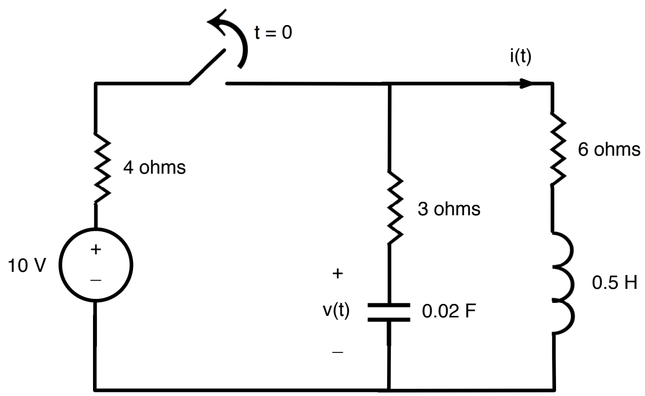

Figure 19.1 The RLC circuit employing a switch. The switch was closed for a very long time and then opens at $t = 0$. The objective is to compute $i(t)$. (The figure is a reworking of Figure 8.10 in [1].)

Hence it follows that

$$\alpha = \frac{9}{2 \times 0.5} = 9 \quad (19.2)$$

$$\omega_d = \sqrt{\frac{1}{0.5 \times 0.02} - \frac{1}{4}\left(\frac{9}{0.5}\right)^2} = 4.4. \quad (19.3)$$

19.1.1 Negative Time Circuit

The next step is to consider the negative time circuit so that the initial conditions (I_0 and V_0) can be determined. For negative time the switch is closed, and thus the circuit is as shown in Figure 19.3. As the circuit has been in this state for a very long time equilibrium has been achieved, and the potential across the capacitor is static, hence from Equation (11.2) the current flowing through the capacitor is zero. It is thus an open circuit as indicated. The inductor current is static, so that Equation (11.8) shows that the potential across the inductor is zero, and hence it is a conductor with zero resistance as indicated.

The current through the capacitor is zero, thus the current through the 3-ohm resistor is zero. Thus there is no potential difference across the 3-ohm resistor, and hence $v(t) = V_a = 10\frac{6}{10} = 6$ volts (voltage divider rule). The current through the

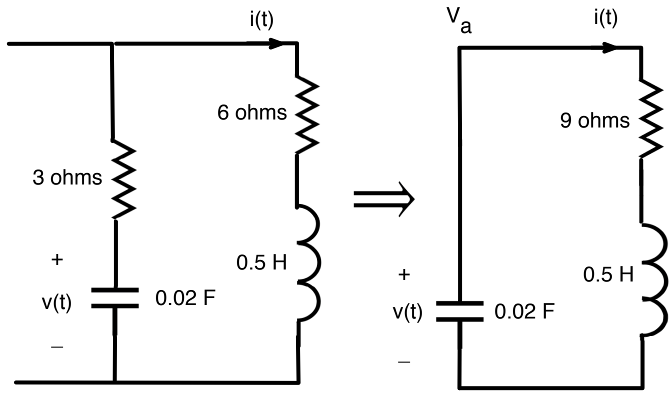

Figure 19.2 The RLC circuit for positive time $t \geq 0$. The prototype form is shown on the right.

6-ohm resistor is 1 amp (Ohm's law). Since the potential across the capacitor and the current through the inductor is continuous as a function of time, it follows that

$$I_0 = i(t = 0) = 1 \text{ amp} \tag{19.4}$$

$$V_0 = -v(t = 0) = -6. \text{ volts} \tag{19.5}$$

V_0 is negative because the definition of the potential in the prototype has an opposite polarity from $v(t)$ as defined here.

19.1.2 Positive Time Circuit

The positive time circuit is shown in Figure 19.2. With the positive time circuit in the prototype form and using all the information computed above the solution for the current can now be determined as

$$A_1 = 1 \tag{19.6}$$

$$A_2 = \frac{1}{4.4}\left(9 - \frac{(9-6)}{0.5}\right) = \frac{1}{4.4}(9-6) = \frac{3}{4.4} \approx 0.68. \tag{19.7}$$

Thus based on the prototype the solution is given by

$$i(t) = \begin{cases} 1 \text{ amp} & t < 0 \\ e^{-9t}\left[\cos(4.4t) + 0.68\sin(4.4t)\right] \text{ amps} & t \geq 0. \end{cases} \tag{19.8}$$

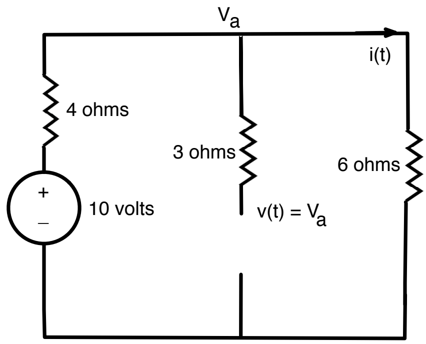

Figure 19.3 The RLC circuit for negative time $t < 0$.

19.1.3 Discussion

The current for $t \geq 0$ for the where Δ is imaginary is oscillating, but decaying over time. The rate of decay is determined by α, and the frequency of oscillation is determined by ω_d. The reason is that the resistor causes damping, while the energy is being exchanged between the inductor and capacitor. Physically, current is flowing from the inductor to the capacitor, then back again. Each time this happens some energy is lost in the resistor (converted to heat), which causes the decay or damping. This type of circuit is known as underdamped.

19.2 A Second Example

Consider the circuit shown in Figure 19.4. The capacitor has a potential of 200 volts at time zero. Compute the current after the switch closes.

Step one is to classify the circuit, and in this case it is source-free series RLC. Thus the correct prototype can be selected and will provide a solution for the circuit. The next step is to determine Δ, and in this case it is real. This case is known as overdamped, and thus the appropriate prototype solution will be used in what follows.

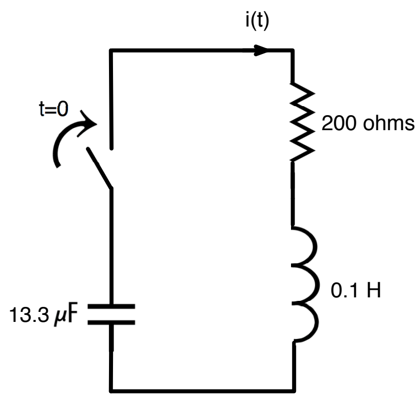

Figure 19.4 The RLC circuit containing a switch that closes at time $t = 0$. The capacitor contains a charge of 200 volts at $t = 0$. (The figure is a reworking of Figure 8.3 in [2].)

19.2.1 The Circuit for Negative Time

Since the switch is open for negative time, there is no current through the inductor, and thus the inductor is not storing any energy. The question provides the information that the capacitor has a potential of 200 volts at time zero. Since current through the inductor and the voltage across a capacitor is continuous, the initial conditions at time zero is given by

$$I_0 = 0 \text{ amps} \quad (19.9)$$

$$V_0 = 200 \text{ volts}. \quad (19.10)$$

19.2.2 The Circuit for Positive Time

For positive time the switch closes and the capacitor discharges its stored energy via the rest of the circuit. As shown above for the prototype solution the solution is given by

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}. \quad (19.11)$$

The roots s_1 and s_2 are given by

$$s_1 = \frac{1}{2}\left[-b + \sqrt{b^2 - 4c}\right] \tag{19.12}$$

$$s_2 = \frac{1}{2}\left[-b - \sqrt{b^2 - 4c}\right] \tag{19.13}$$

where

$$b = \frac{R}{L} = 2000 \tag{19.14}$$

$$c = \frac{1}{LC} = 0.75 \times 10^6. \tag{19.15}$$

Hence

$$s_1 = -500 \tag{19.16}$$

$$s_2 = -1500. \tag{19.17}$$

Thus Δ is real, and to determine A_1 and A_2 the following equations are solved:

$$A_1 + A_2 = 0 \tag{19.18}$$

$$50A_1 + 150A_2 = 200 \tag{19.19}$$

which yields

$$A_1 = -2 \tag{19.20}$$

$$A_2 = 2. \tag{19.21}$$

Thus the solution is given by

$$i(t) = -2e^{-500t} + 2e^{-1500t}. \tag{19.22}$$

19.2.3 Discussion

For this circuit the current contains two terms that decay exponentially over time. In general these terms have different time constants. The term with the smaller time constant decays slower and thus will determine the length of time during which the transient is decaying. This type of circuit is known as overdamped.

References

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

[2] Nahvi, M., and J.A. Edminister, *Electric Circuits*, McGraw-Hill, version 6, 2014.

Chapter 20

Source-Free Parallel RLC Circuits

The previous chapters considered source-free series RLC circuits. Of course it often happens that the resistor, the inductor, and the capacitor are connected in parallel and not in series. The prototype for this circuit is shown in Figure 20.1. For this

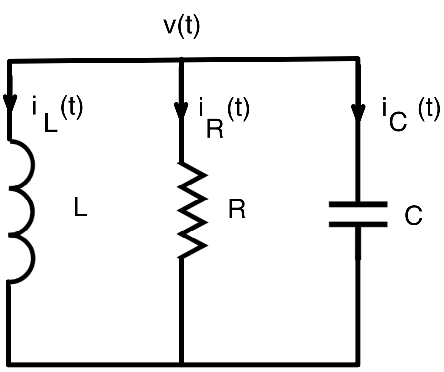

Figure 20.1 The RLC circuit containing a resister, a capacitor and an inductor in parallel. (The figure is a reworking of Figure 8.13 in [1].)

circuit, all the elements have the same potential while for the series RLC circuit the current through the elements were the same. Kirchhoff's current law says that the current flowing into a node and out of the node must be the same (conservation of charge). Since there is no charge flowing into the node then all the currents flowing out of the node has to sum to zero as well. Hence it follows that

$$i_L(t) + i_R(t) + i_C(t) = 0. \tag{20.1}$$

Differentiating this equation, and then deploying Equations (11.2) and (11.8) and Ohm's law it follows that

$$\frac{di_L(t)}{dt} + \frac{di_R(t)}{dt} + \frac{di_C(t)}{dt} = 0 \implies \frac{v(t)}{L} + \frac{1}{R}\frac{dv(t)}{dt} + C\frac{d^2v(t)}{dt^2} = 0 \tag{20.2}$$

which simplifies to

$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0. \tag{20.3}$$

This is a differential equation of second order, and its solution will provide the node potential $v(t)$. The solution is a function of the initial conditions determined by the stored energy, which is the charge (i.e., potential) of the capacitor at $t = 0$ and the current through the inductor at $t = 0$.

Besides the fact that this differential equation has as an independent variable $v(t)$ (as opposed to $i(t)$ for the series RLC circuit) and the coefficients being different than it was for the series RLC circuit, the forms of the differential equations are similar, and the solutions follow from detail provided in Chapter 18.

20.1 Solution of the Second Order DE

The inductor current at time zero is denoted by I_0, and the capacitor potential at time zero denoted by V_0. Define

$$\Delta = \sqrt{b^2 - 4c} \tag{20.4}$$

where b and c are given by

$$b = \frac{1}{RC} \tag{20.5}$$

$$c = \frac{1}{LC}. \tag{20.6}$$

The factor $\Delta = \sqrt{b^2 - 4c}$ has a significant effect on the solution, as it can be real or imaginary. The two cases will be considered separately.

20.1.1 Δ is Real

Define $s_1 = \frac{-b+\Delta}{2}$ and $s_2 = \frac{-b-\Delta}{2}$. Solve the equations for A_1 and A_2 given by

$$A_1 + A_2 = V_0 \quad (20.7)$$

$$s_1 C A_1 + s_2 C A_2 = -\frac{1}{R} V_0 - I_0. \quad (20.8)$$

With this solution available, the prototype solution is know known as

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}. \quad (20.9)$$

20.1.2 Δ is Imaginary

Define α and ω_d as

$$\alpha = \frac{1}{2RC} \quad (20.10)$$

$$\omega_d = \sqrt{\frac{1}{LC} - \frac{1}{4}\left(\frac{1}{RC}\right)^2}. \quad (20.11)$$

A_1 and A_2 are given by

$$A_1 = V_0 \quad (20.12)$$

$$A_2 = \frac{1}{\omega_d}\left(\alpha V_0 - \frac{\left(\frac{V_0}{R} + I_0\right)}{C}\right). \quad (20.13)$$

The prototype solution for the case where Δ is imaginary is given by

$$v(t) = e^{-\alpha t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right]. \quad (20.14)$$

20.2 Discussion

Chapter 18 presented the solution for the current of a series RLC circuit. The series RLC circuit has a resistor, an inductor, and a capacitor in series, and thus all three elements have the same current $i(t)$. We identified two classes of solutions, namely the overdamped and underdamped cases. We provided solutions for both.

This chapter presented the solution for the potential of a parallel RLC circuit. The parallel RLC circuit has a resistor, an inductor, and a capacitor in parallel, and thus all three elements have the same potential $v(t)$. We identified two classes of solutions, namely the overdamped and underdamped cases. We provided solutions for both.

Chapter 18 also considered the numerical solution of the second order differential equation, based on MATLAB. The MATLAB code for solving the series RLC circuit was provided. Numerical results were presented in Chapter 18 where the current $i(t)$ was compared to the analytical solutions for two cases with Δ imaginary: (1) $|\Delta| \gg 0$, and (2) $|\Delta| \approx 0$. It was shown that the latter case behaves much differently from the former, and the reasons for this was explained.

The reader is encouraged to repeat the numerical analysis for the parallel RLC circuit, and to modify the MATLAB code in Chapter 18 to do this. Then compare the results for the underdamped and overdamped cases, verifying that the same concepts apply to the parallel RLC circuit.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 21

Examples for a Parallel RLC Circuit

21.1 A First Example

Consider the parallel RLC circuit shown in Figure 21.1. The objective is to compute $v(t)$ for $t \geq 0$, given $v(0) = 5$ volts, $i(0) = 0$ amps, $L = 1$ henry, and $C = 10$ mF, and $R = 6.25$ ohms[1]. Step one towards a solution is classifying the circuit. In this case it is clear from the definition that the circuit is a parallel RLC circuit that is source-free. Moreover its not necessary to analyze a negative time circuit as the initial conditions are provided.

For the next step it needs to be determined if Δ is real or imaginary. In this case Δ is imaginary and thus the circuit is underdamped and the appropriate solution can be used (from the previous chapter). Hence the solution is

$$v(t) = e^{-8t}\left[A_1\cos(6t) + A_2\sin(6t)\right]. \tag{21.1}$$

Thus the parameters A_1 and A_2 still needs to be determined. From the results in the previous chapter it is known that

$$A_1 = V_0 = 5 \tag{21.2}$$

$$A_2 = \frac{1}{6}\left(8 \times 5 - \frac{(\frac{5}{6.25})}{10 \times 10^{-3}}\right) = -6.6667 \tag{21.3}$$

[1] These settings were taken from Example 8.5 in [1].

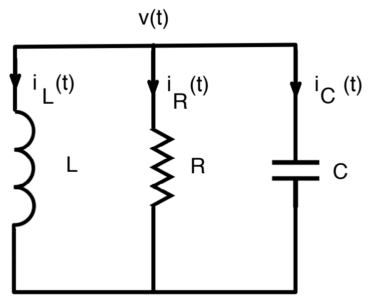

Figure 21.1 A parallel RLC circuit containing a resister, a capacitor, and an inductor in parallel. (The figure is a reworking of Figure 8.13 in [1].)

Thus the complete solution for $t \geq 0$ is given by

$$v(t) = e^{-8t} \left[5\cos(6t) - 6.67\sin(6t)\right]. \tag{21.4}$$

21.2 A Second Example

Consider the circuit shown in Figure 21.2. The objective is to compute $v(t)$ for positive time $t \geq 0$. Step one is to classify the circuit. To do that consider always

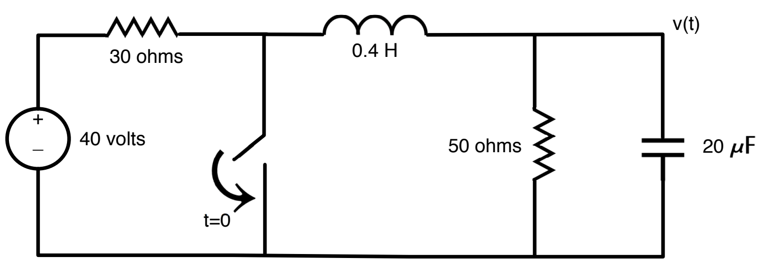

Figure 21.2 A RLC circuit deploying a switch that was open for a very long time, then closes at $t = 0$. The objective is to compute $v(t)$ for $t > 0$. (The figure is a reworking of Figure 8.15 in [1].)

the positive time circuit. In this case for positive time the circuit has the switched closed, and thus the source and resistor is isolated from the rest of the circuit by a conductor (the closed switch). This is shown in Figure 21.4. From the capacitor and inductor point of view these elements are in parallel with the resistor. Thus the circuit is a source-free parallel RLC circuit.

21.2.1 Negative Time Circuit

Consider the negative time circuit shown in Figure 21.3. The reason that the inductor has been replaced with a conductor is because in equilibrium the current is static (the source is static) and thus there is no voltage across the inductor terminals (see Equation [11.8]). Likewise for the capacitor in equilibrium the potential across the capacitor terminals is static and thus no current flows through it (see Equation [11.2]). It is thus an open circuit and can be removed. The potential $v(t)$ is thus at a voltage divider point, and thus

$$v(t < 0) = 40\left(\frac{50}{80}\right) = 25 \text{ volts} \tag{21.5}$$

and the current through the inductor is

$$i(t < 0) = -\frac{40}{30 + 50} = -\frac{1}{2} \text{ amps.} \tag{21.6}$$

The current is negative because the flow direction is the opposite of the definition in the prototype.

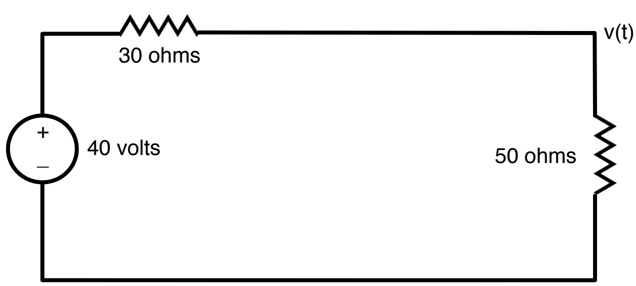

Figure 21.3 The negative time circuit.

21.2.2 Positive Time Circuit

For positive time the switch closes, and the independent source and the 30-ohm resistor become isolated by the conductor introduced by the closed switch. This is indicated with dotted lines in Figure 21.4. Also shown is the fact that the inductor is connected to the reference node by the conductor introduced by the switch. This leads to an equivalent circuit as shown. The equivalent circuit does not need further simplification as it is already in the prototype form.

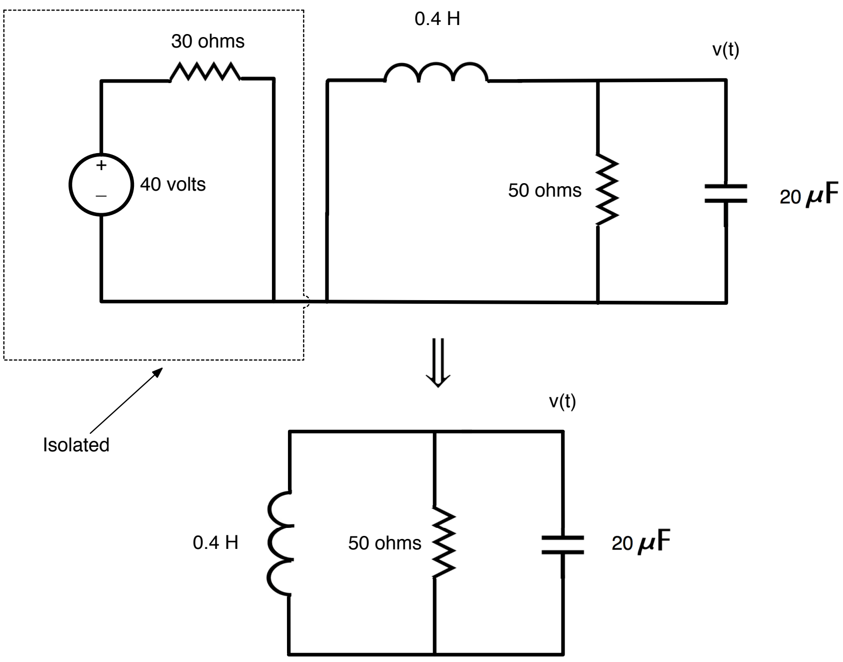

Figure 21.4 The positive time circuit.

Because the current through an inductor is continuous and the voltage across the capacitor is continuous the initial conditions are given by

$$V_0 = 25 \text{ volts} \tag{21.7}$$

$$I_0 = -\frac{1}{2} \text{ amps.} \tag{21.8}$$

The next step is to determine if Δ is real or imaginary. In this case its real, and thus the source-free parallel RLC prototype with a real Δ should be used. The two roots are given by

$$s_1 = -146 \tag{21.9}$$

$$s_2 = -854 \tag{21.10}$$

and the constants A_1 and A_2 can be determined by solving the system of linear equations given by

$$A_1 + A_2 = 25 \tag{21.11}$$

$$-146A_1 - 854A_2 = 0. \tag{21.12}$$

This yields $A_1 \approx 30$ and $A_2 \approx -5$, and hence the solution for $v(t)$ for $t \geq 0$ is given by

$$v(t) = 30\, e^{-146t} - 5\, e^{-854t}. \tag{21.13}$$

21.3 The Use of Circuit Simulators

The use of numerical methods and analysis to solve differential equations and in particular circuits (static and dynamic) is a mature science and commercial software packages are available to use on personal computers [2]. There are many such software packages available today, but one of the most often used software systems is known as SPICE (or PSPICE). SPICE is software that simulates circuits on a computer. Any potential or current waveform in the circuit can be viewed. SPICE performs computations of potential and currents as a function of time (Part I and II of this text) or as a function of frequency (Part III of this text).

SPICE is short for *Simulation Program with Integrated Circuit Emphasis*. Why was this software developed? Probably the advent of powerful mainframe computers in the 1970s coupled with the need to analyze circuit designs before manufacturing using expensive processes. Today, SPICE is available including graphics user interfaces for drawing the circuits and plotting results.

SPICE had its beginning in the 1970s at the University of California at Berkeley. In 1972 Nagel and Pederson released SPICE1 (Simulation Program with IC Emphasis) into the public domain [3]. In retrospect this was a key development as its widespread use and subsequent development let to SPICE becoming an industry standard for circuit simulation.

Early versions were based on nodal analysis and programmed in FORTRAN. There was an updated version released in 1975 (SPICE2), and one of the innovations deployed in this version was dynamic memory allocation. This allowed circuits to grow and made modification of size possible. It also featured adjustable time-step control that speeded up circuit analysis. Version SPICE2G.6 (1983) was the last FORTRAN based version. It is noteworthy that many commercial simulators today are based on SPICE2G.6.

In 1985 SPICE was rewritten in the C programming language and released as SPICE3. These versions featured graphical user interfaces, modeled controlled sources, and included lossy transmission lines and non-ideal switching.

Commercial versions released included HSPICE, IS_SPICE and MICROCAP. MicroSim released PSPICE, the first personal computer version of SPICE.

This book does not include analysis of circuits using SPICE in spite of the widespread use of circuit simulation and analysis. Rather it focuses on the solution of circuits from first principles with analysis based on numerical analysis using MATLAB (as shown in earlier chapters). Understanding the fundamental limitations and methodology behind circuits and their models on which circuit analysis is based (emphasized in this text) is the basis of using circuit simulators such as SPICE. This will make it possible for the reader being able to spot and identify nonsensical results, and generally being able to use SPICE in a proficient manner.

References

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

[2] http://www.ecircuitcenter.com/index.htm

[3] https://en.wikipedia.org/wiki/SPICE

Part III

AC: Sinusoidal Voltage and Current Sources

Chapter 22

Sinusoidal Sources, the Phasor and Impedance

In practice, many potential and current sources are not static (as we assumed in Part I of the book) or dynamic (Part II of the book), but are in fact sinusoidal. These types of sources are known as alternating current (AC). AC sources form the basis of Part III of this book and is the default electrical power standard used throughout the world today.

There is a long and interesting history behind the nature of electrical power sources. The reader would be familiar with the notion that the 110-volt electrical outlet in use today in the United States provides electrical potential, which is an alternating current, or AC. However with the invention and advent of electricity in the nineteenth century, it was not yet clear if potential should be static (DC) or sinusoidal (AC).

Westinghouse corporation supported AC while Edison supported DC. The two opponents sparred in an intense contest for market dominance — it was clear to both that the winning technology will effectively dominate a large market. Westinghouse corporation employed an engineer with exceptional talent, the inventor Nikola Tesla. Tesla's inventions span decades and number in the several hundreds, most notably the polyphase induction motor. His inventions based on AC or sinusoidal potential sources in the end won out. In spite of all his success, Tesla passed away as an impoverished elderly man. Tesla is shown in Figure 22.1.

There are many reasons for AC winning out, but the ease with which AC potential (and current) can be transformed perhaps was a key element of its success.

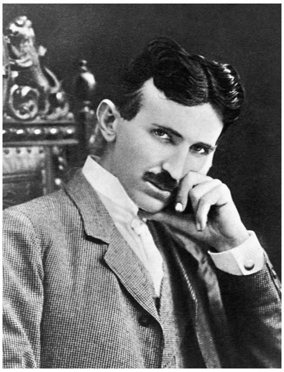

Figure 22.1 Nikola Tesla on the left, and C.P. Steinmetz on the right. (Source: Wikipedia — Printed with permission [PD-1923].)

AC enables high potential to be transformed to low potential and vice versa. AC made it possible to use high voltage for long distance transmission lines, then transform it to lower voltage near areas where consumers need safety. With high voltage potential the currents required to transport power is reduced, which means that less copper could be used that made the transmission lines cheaper. With DC (static) the transmission lines had to be at a lower voltage requiring higher currents, and since thermal losses are related to the square of the current, DC lost out eventually as the price of copper rose.

Advances made in the analysis of AC circuits also played a decisive role. One of the key ideas was due to Steinmetz who is also shown in Figure 22.1. These methods are known as phasor theory, and makes use of the complex numbers that were developed in mathematics.

22.1 The Theory of the Phasor

It is assumed that the reader is familiar with Euler's identity given by

$$e^{j(\omega t+\phi)} = \cos(\omega t + \phi) + j\ \sin(\omega t + \phi) \tag{22.1}$$

which establishes the relationship between the sinusoid, the cosinusoid, the complex exponential function and the number $j^2 = -1$ [1]. The parameter ω is the angular

[1] j is known as the imaginary number.

frequency given by

$$\omega = 2\pi f \tag{22.2}$$

where f is the frequency measured in hertz. The angular frequency is related to the period T as

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{22.3}$$

where T has units seconds. T is the time taken for the complex exponential to make a full revolution on the complex plane, as indicated in Figure 22.2.

The complex exponential $e^{j(\omega t+\phi)}$ can be considered to be a vector in the complex plane, as shown in Figure 22.2. The projection of this vector onto the real axis is indicated, which is $\cos(\omega t + \phi)$ and the projection onto the imaginary axis is $\sin(\omega t + \phi)$ as indicated. The angle that the vector makes with the real axis is $\omega t + \phi$ radians. The angle is a function of time, so that what is shown in Figure 22.2 is a snapshot at time t. For example if $t = 0$ the angle would be smaller, in fact it would be ϕ radians. Thus as time t progresses the vector traces out a circle in the complex plane, and the circle has an amplitude of one, as the magnitude of the complex exponential $e^{j(\omega t+\phi)}$ is one.

The first key idea behind phasor theory is to make use of the Real{} and Imag{} operators. These have the property that they select the real or imaginary part of a complex number, hence

$$\text{Real}\,\{e^{j(\omega t+\phi)}\} = \cos(\omega t + \phi) \tag{22.4}$$

and

$$\text{Imag}\,\{e^{j(\omega t+\phi)}\} = \sin(\omega t + \phi). \tag{22.5}$$

Given the operators defined above, it is clear that a potential source that is cosinusoidal (i.e., AC) can be represented as

$$v(t) = \cos(\omega t + \phi) = \text{Real}\,\{e^{j(\omega t+\phi)}\}. \tag{22.6}$$

22.1.1 A Resistor Using the Phasor Notation

Consider now Ohm's law for a resistor where the potential across its terminals is cosinusoidal, then the following must hold:

$$v(t) = R\,i(t) \implies \text{Real}\,\{V_0\,e^{j(\omega t+\phi)}\} = R\,\text{Real}\,\{I_0\,e^{j(\omega t+\theta)}\}. \tag{22.7}$$

This is a consequence of the fact that the frequency of the potential and the current must be the same (this is because the circuits are linear) — note that it was not

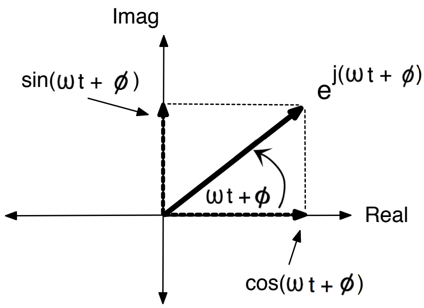

Figure 22.2 The Euler identity shown graphically on the complex plane.

assumed that the phases are the same. Thus it follows that

$$\text{Real}\,\{V_0 e^{j\phi} e^{j(\omega t)}\} = R\,\text{Real}\,\{I_0 e^{j\theta} e^{j(\omega t)}\}. \tag{22.8}$$

This relationship clearly holds for any time t, thus the time dependence can be suppressed, hence it follows that

$$\text{Real}\,\{V_0 e^{j\phi}\} = R\,\text{Real}\,\{I_0 e^{j\theta}\}. \tag{22.9}$$

Defining $\overline{V}$ and $\overline{I}$ as

$$\overline{V} = V_0 e^{j\phi} \tag{22.10}$$

$$\overline{I} = I_0 e^{j\theta} \tag{22.11}$$

then it follows that

$$\frac{\overline{V}}{\overline{I}} = R. \tag{22.12}$$

The terms $\overline{V}$ and $\overline{I}$ are called phasors and the above expression is Ohm's law expressed for cosinusoidal sources in phasor notation. Thus a phasor is the complex representation of a sinusoidal function with the time dependance suppressed. Since Ohm's law in phasor form for a resistor shows that $\frac{\overline{V}}{\overline{I}} = R$ where R is a real number, it means that for a resistor that two phasors $\overline{V}$ and $\overline{I}$ must be in phase (i.e., $\phi = \theta$). That is the only possibility if $\frac{\overline{V}}{\overline{I}}$ is real. Thus Ohm's law in phasor notation is as shown on the complex plane in Figure 22.3, where it is indicated that the potential and current phasors are in-phase, meaning they have the same phase.

The resistor used in Fig 22.3 must have a resistance smaller than 1 ohm. The reader is encouraged to understand why that is the case for the phasors shown.

22.2 Time to Phasor Domain Transformation

The analysis above can be viewed as a time domain to phasor domain transformation, or a phasor domain to time domain transformation — it can be done either way. This concept can be written abstractly as

$$\overline{Z} \iff z(t) \tag{22.13}$$

which means that a transformation exists to obtain either term from the other.

1. Phasor domain to time domain: As an example, imagine being given a phasor, say $\overline{Z}$. The reader is asked to compute the time domain function corresponding to the phasor. To answer this question note that the phase is a complex number, and any complex number can be written in polar form (i.e., $\overline{Z} = Z_0 e^{j\psi}$). The time domain function is obtained by performing the phasor to time domain transformation: re-insert the time dependance then take the real part, given by

$$\begin{aligned} \text{Real}\,\{\overline{Z}e^{j\omega t}\} = \text{Real}\,\{Z_0 e^{j\psi} e^{j\omega t}\} &= \text{Real}\,\{Z_0 e^{j(\omega t+\psi)}\} \\ &= Z_0 \cos(\omega t + \psi) = z(t). \end{aligned} \tag{22.14}$$

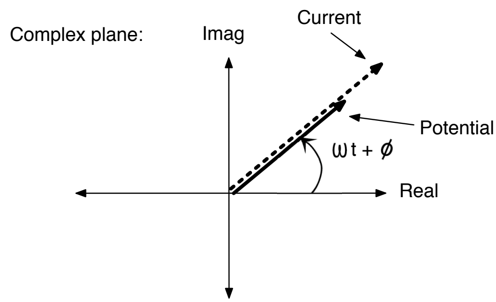

Figure 22.3 A resistor and Ohm's law using phasor notation. The potential across the resistor terminals and the current through the resistor is in-phase.

The equation above shows the path from the phasor to the time domain — where the time dependance is re-inserted.

2. Time domain to phasor domain: We can also transform from the time domain (i.e., $z(t)$) to the phasor form where the time dependance is removed. This is just the reverse proses of the above, given by

$$\begin{aligned} z(t) = \cos(\omega t + \psi) &= \text{Real}\,\{Z_0 e^{j(\omega t+\psi)}\} \\ &= \text{Real}\,\{Z_0 e^{j\psi} e^{j\omega t}\} \implies Z_0 e^{j\psi} = \overline{Z}. \end{aligned} \tag{22.15}$$

The last step removed the time dependance which yields the phasor $\overline{Z}$.

22.3 A Capacitor Using the Phasor: Impedance

The relationship between the potential across a capacitor and the current through the capacitor is given by Equation (11.2) and assuming cosinusoidal sources it is given by

$$i(t) = C\frac{dv(t)}{dt} \implies \text{Real}\,\{I_0 e^{j(\omega t+\phi)}\} = C\,\text{Real}\,\{\frac{d(V_0 e^{j(\omega t+\theta)})}{dt}\}. \tag{22.16}$$

Thus it follows that

$$\text{Real}\,\{I_0 e^{j(\omega t+\phi)}\} = C\,\text{Real}\,\{j\omega V_0 e^{j(\omega t+\theta)}\} \tag{22.17}$$

which simplifies to

$$\text{Real}\,\{I_0 e^{j\phi} e^{j\omega t}\} = C\,\text{Real}\,\{j\omega V_0 e^{j\theta} e^{j\omega t}\}. \tag{22.18}$$

This relationship clearly holds for any time t, thus the time dependence can be suppressed, hence it follows that

$$\text{Real}\,\{I_0 e^{j\phi}\} = \text{Real}\,\{j\omega C\, V_0 e^{j\theta}\}. \tag{22.19}$$

Defining $\overline{V}$ and $\overline{I}$ as

$$\overline{I} = I_0 e^{j\phi} \tag{22.20}$$

$$\overline{V} = V_0 e^{j\theta} \tag{22.21}$$

then it follows that

$$\text{Real}\,\{\overline{I}\} = \text{Real}\,\{j\omega C\overline{V}\}. \tag{22.22}$$

Since the operator Real{ } is deployed on both sides of the equation, it follows that

$$\frac{\overline{V}}{\overline{I}} = \frac{1}{j\omega C} = -jX_C \tag{22.23}$$

and thus the capacitor reactance $X_C = \frac{1}{\omega C}$. The ratio of the phasor potential and phasor current in this case reminds us of resistance, but since its not a resistor that is being analyzed a different word is used in this case — impedance, denoted by the symbol Z. In general impedance is complex, but in the case of the capacitor the impedance is purely imaginary, which is called a reactance. The impedance Z of a capacitor is a function of the angular frequency ω and is given by

$$Z_C(\omega) = \frac{\overline{V}}{\overline{I}} = -jX_C = \frac{1}{j\omega C}. \tag{22.24}$$

This is a powerful result, as it enables us to replace capacitors by an impedance Z_C in a circuit if the sources are cosinusoidal, just as we are able to replace a resistor with R.

To understand the relationship between $\overline{V}$ and $\overline{I}$ for a capacitor, recognize that $j = e^{j\frac{\pi}{2}}$ and thus rewrite the impedance as

$$Z_C(\omega) = \frac{\overline{V}}{\overline{I}} = \frac{1}{e^{j\frac{\pi}{2}}\omega C} = \frac{1}{\omega C}\, e^{-j\frac{\pi}{2}}. \tag{22.25}$$

Thus the magnitude of the capacitor impedance is $|Z_C(\omega)| = X_C = \frac{1}{\omega C}$, and its phase is $-\frac{\pi}{2}$ or -90°. Thus the potential across the capacitor terminals and the current through the terminals are related by

$$\overline{V} = Z_C(\omega)\,\overline{I} = \frac{1}{\omega C}\, e^{-j\frac{\pi}{2}}\,\overline{I}. \tag{22.26}$$

This means that the magnitude of the potential is scaled by the reactance $X_C = \frac{1}{\omega C}$, and the phase is -90° relative to the current — that is lagging the phase of the current by 90°. This is shown in Figure 22.4.

In previous chapters it was explained that for a static potential (in equilibrium) an ideal capacitor is an open circuit (no conductor present). A static source is a cosinusoidal source with a frequency of 0 Hz, and it can be seen that Z_C for $\omega = 0$ is infinite — indeed confirming the comments above. Likewise, if the frequency of a cosinusoidal source is very large (tends to infinity), then $Z_C \to 0$, so a capacitor becomes a short circuit for a very large frequency cosinusoidal potential.

Finally note that the above analysis and models assumed that the capacitor is ideal (is purely reactive). For practical capacitors this is not the case, and they have a small leakage current due to the dielectric material having a finite resistance. Hence the angle between the potential and current for a practical capacitor is not exactly -90°.

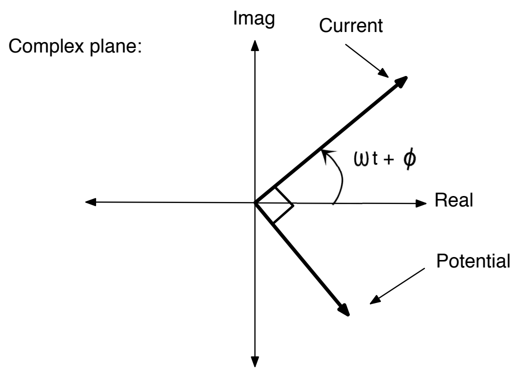

Figure 22.4 A capacitor and the impedance concept using phasor notation. The potential across the capacitor and the current through the capacitor is out of phase by 90°.

22.4 An Inductor Using the Phasor: Impedance

The relationship between the potential across an inductor and the current through the inductor is given by Equation (11.8) and assuming cosinusoidal sources it is given by

$$v(t) = L\frac{di(t)}{dt} \implies \text{Real}\,\{V_0 e^{j(\omega t+\phi)}\} = L\,\text{Real}\,\{\frac{d(I_0 e^{j(\omega t+\theta)})}{dt}\}. \tag{22.27}$$

Thus it follows that

$$\text{Real}\,\{V_0 e^{j(\omega t+\phi)}\} = L\,\text{Real}\,\{j\omega I_0 e^{j(\omega t+\theta)}\} \tag{22.28}$$

which simplifies to

$$\text{Real}\,\{V_0 e^{j\phi} e^{j\omega t}\} = L\,\text{Real}\,\{j\omega I_0 e^{j\theta} e^{j\omega t}\}. \tag{22.29}$$

This relationship clearly holds for any time t, thus the time dependence can be suppressed, hence it follows that

$$\text{Real}\,\{V_0 e^{j\phi}\} = \text{Real}\,\{j\omega L I_0 e^{j\theta}\}. \tag{22.30}$$

Defining $\overline{V}$ and $\overline{I}$ as

$$\overline{V} = V_0 e^{j\phi} \tag{22.31}$$

$$\overline{I} = I_0 e^{j\theta} \tag{22.32}$$

then it follows that

$$\text{Real}\,\{\overline{V}\} = \text{Real}\,\{j\omega L\overline{I}\}. \tag{22.33}$$

Hence we obtain in complex form

$$\frac{\overline{V}}{\overline{I}} = jX_L = j\omega L. \tag{22.34}$$

The ratio of the phasor potential and phasor current is the inductor impedance denoted by the symbol $Z_L = jX_L$. The impedance Z_L of an inductor is a function of the angular frequency ω and is given by

$$Z_L(\omega) = \frac{\overline{V}}{\overline{I}} = j\omega L. \tag{22.35}$$

This result enables us to replace inductors by an impedance Z_L in a circuit if the sources are cosinusoidal, just as we are able to replace a resistor with R.

To understand the relationship between $\overline{V}$ and $\overline{I}$ for an inductor, recognize that $j = e^{j\frac{\pi}{2}}$ and thus rewrite the impedance as

$$Z_L(\omega) = \frac{\overline{V}}{\overline{I}} = \omega L e^{j\frac{\pi}{2}}. \tag{22.36}$$

Thus the magnitude of the inductor impedance is ωL, and its phase is $\frac{\pi}{2}$ or 90°. Hence the potential across the inductor terminals and the current through the terminals are related by

$$\overline{V} = \omega L\, e^{j\frac{\pi}{2}}\, \overline{I}. \tag{22.37}$$

This means that the magnitude of the potential is scaled by ωL, and the phase is $+90^\circ$ relative to the current — that is leading the phase of the current by 90°. This is shown in Figure 22.5.

In previous chapters it was explained that for a static potential (in equilibrium) an ideal inductor is a short circuit (a conductor). A static source is a cosinusoidal source with a frequency of 0 Hz, and it can be seen that Z_L for $\omega = 0$ is zero, confirming the comments above. Likewise, if the frequency of a cosinusoidal source is very large (tends to infinity), then $Z_L \to \infty$, so an inductor becomes an open circuit (non conducting) for a very large frequency cosinusoidal potential.

Finally note that the model of the inductor presented above is ideal, since it assumed that the conductor has no resistance. For practical inductors this is not true since they do have a small resistance. Hence the angle wont be exactly 90°.

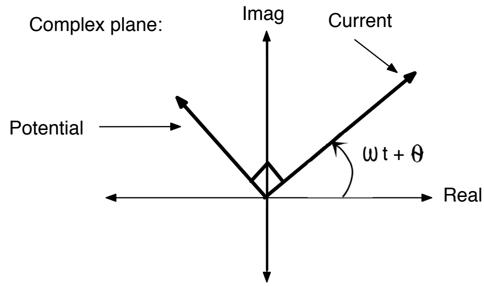

Figure 22.5 An inductor and the impedance concept using phasor notation. The potential across the inductor and the current through the inductor is out of phase by 90°.

22.5 Discussion

Based on the phasor representation of the cosinusoidal sources, we were able to show that on the phasor domain the capacitor, inductor and resistor can be represented by their impedance's given by

$$Z_C(\omega) = -jX_C = \frac{1}{j\omega C} \tag{22.38}$$

$$Z_L(\omega) = jX_L = j\omega L \tag{22.39}$$

$$Z_R(\omega) = R. \tag{22.40}$$

Thus a circuit where the sources are cosinusoidal in the time domain, can be transformed to the phasor domain, where sources are represented by phasors, while the resistors, capacitors and inductors are represented by their impedances given above. This means that the circuit in the time domain can be transformed to the phasor domain, where time is suppressed. This provides a clear advantage as the phasor domain circuit representation is not a function of time. Which in turn means that the circuit is static, and static analysis (Part I) thus applies in the phasor domain.

The algebra used to accomplish the analysis entails complex number theory, but apart from this the methods from Part I applies directly, greatly simplifying the analysis on the phasor domain. The next few chapters will make extensive use of this methodology.

Chapter 23

Circuit Analysis Based on Phasor Domain Representation

23.1 A First Example

Consider the circuit in Figure 23.1 where a potential source is used that is cosinusoidal. The circuit as shown in the figure is in the time domain. The objective is to compute the potential across the resistor and the capacitor.

Step one towards the solution is to convert the circuit to the phasor form or phasor domain, that is perform a time domain to phasor domain transformation of the circuit. This is shown in Figure 23.2. This requires transforming all the circuit elements, and the inductors and capacitors are replaced with their impedances using the phasor method presented in the previous chapter. The transformation was performed as follows:

1. The potential source: $10 \ \cos(4t) = \text{Real}\left\{10e^{j4t}\right\} \implies$ 10 volts on the phasor domain.

2. The resistor remains at 5 ohms as was shown in a previous chapter — a resistor is the only circuit element that is identical in the time and phasor domain.

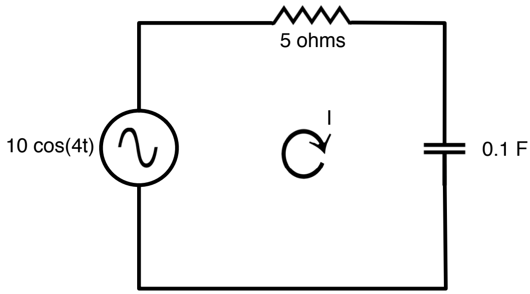

Figure 23.1 A circuit using a cosinusoidal source. The objective is to compute the potential across the resistor and the capacitor. The solution will make use of the phasor domain transformation. (The figure is a reworking of Figure 9.16 in [1].)

3. The capacitor: As shown in a previous chapter the capacitor impedance in the phasor domain is $\frac{\overline{V}}{\overline{I}} = Z_c = \frac{1}{j\omega C} = -j\,2.5$ ohms.

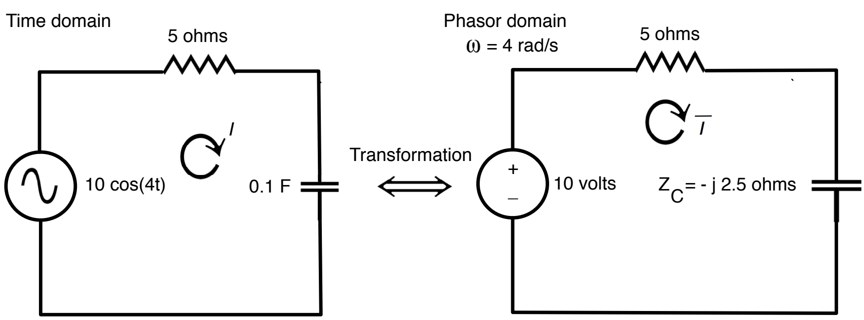

Figure 23.2 The circuit of Figure 23.1 on the time domain and the phasor domain.

23.1.1 Phasor Domain Analysis

Based on the phasor domain representation of the circuit, the Kirchhoff loop law yields

$$5\,\overline{I} - j2.5\,\overline{I} - 10 = 0 \tag{23.1}$$

which yields

$$\overline{I} = \frac{10}{5 - j2.5} \ \text{ amps.} \tag{23.2}$$

Thus according to Ohm's law the potentials across the resistor and capacitor are

$$\overline{V}_R = 5\overline{I} = 5\left(\frac{10}{5 - j2.5}\right) = 8.94\; e^{j0.4636} \ \text{ volts} \tag{23.3}$$

$$\overline{V}_C = -j2.5\overline{I} = -j2.5\left(\frac{10}{5 - j2.5}\right) = 4.47\; e^{-j1.1} \ \text{ volts.} \tag{23.4}$$

These two potentials are *phasors*, while the question asked to compute the potentials $v_R(t)$ and $v_C(t)$ in the time domain. Hence the next step is to transform back to the time domain (i.e., a phasor domain to time domain transformation), from which it follows that the potential across the resistor is

$$v_R(t) = \text{Real}\,\{\overline{V}_R e^{j\omega t}\} = \text{Real}\,\{8.94\; e^{j0.4636} e^{j4t}\} = 8.94\ \cos(4t + 0.4636) \tag{23.5}$$

and across the capacitor it is given by

$$v_C(t) = \text{Real}\,\{\overline{V}_C e^{j\omega t}\} = \text{Real}\,\{4.47\; e^{-j1.1} e^{j4t}\} = 4.47\ \cos(4t - 1.1). \tag{23.6}$$

23.1.2 Discussion

The reader can verify that the potential across the capacitor lags the potential across the resistor by $\frac{\pi}{2}$ radians (90°). This is consistent with the findings in the previous chapter. Also note that the potential across the resistor is in-phase with the current, which is consistent with Ohm's law for a resistor that is real.

The phasors at $t = 0$ are shown in Figure 23.3, which means all the phasors are now relative to the independent potential source. The reader is encouraged to study this figure, and to note the relationship between the phases of the different elements in the circuit. For example, the sum of the potential phasors for the capacitor and the resistor must equal the phasor for the source — which is a requirement of Kirchhoff's loop law.

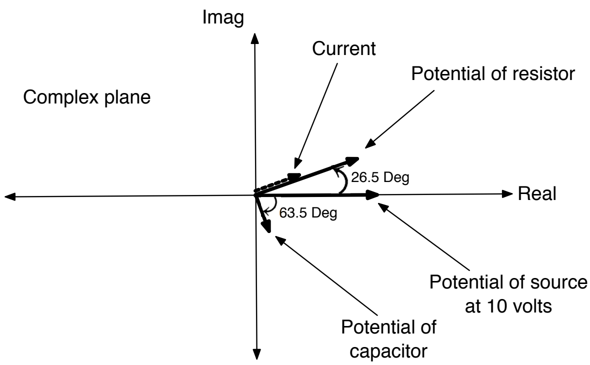

Figure 23.3 The phasors of elements of the circuit in Figure 23.2. Note that the current through the resistor is in phase with the potential across the resistor terminals.

23.2 A Second Example

A circuit is shown in Figure 23.4. The objective is to compute the potential across the capacitor and inductor in the time domain. As the latter elements are in parallel the potential for both would be identical.

Step one is to do a time domain to phasor domain transformation, which is also shown in Figure 23.4. Each element on the phasor domain was computed at $\omega = 4$ rad/s as follows:

1. The potential source: $20\cos(4t - 15^\circ) = \text{Real}\,\{20e^{-j15^\circ}\,e^{j4t}\} \implies 20e^{-j15^\circ}$ volts on the phasor domain.
2. The resistor remains at 60 ohms as was shown in a previous chapter.
3. The capacitor: As shown in a previous chapter the capacitor impedance in the phasor domain is $\frac{\overline{V}}{\overline{I}} = Z_c = \frac{1}{j\omega C} = -j25$ ohms.
4. The inductor: As shown in a previous chapter the inductance impedance in the phasor domain is $\frac{\overline{V}}{\overline{I}} = Z_c = j\omega L = j20$ ohms.

Now the circuit can be solved on the phasor domain. Most circuits can be solved in a variety of ways. For example in this case one possibility is to combine the inductor and the capacitor which are in parallel into a single impedance. Or the

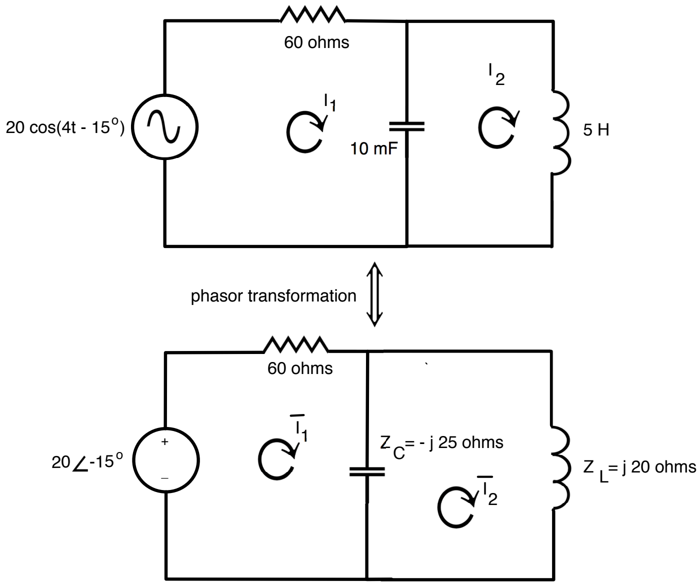

Figure 23.4 A circuit with cosinusoidal source. The objective is to compute the potential across the resistor, capacitor and inductor. The time domain to phasor domain transformation is shown. (The figure is a reworking of Figure 9.25 in [1].)

mesh method can be applied to the circuit as is. In this case the latter approach is used. Hence the loop law applied to the two loops yields

$$(60 - j25)\,\overline{I}_1 + j25\,\overline{I}_2 \quad = \quad 20e^{-j15^\circ} \tag{23.7}$$

$$j25\,\overline{I}_1 - j5\,\overline{I}_2 + \quad = \quad 0 \tag{23.8}$$

which yields

$$\overline{I}_1 \quad = \quad 0.0472 - j\,0.1649 \ \text{ amps} \tag{23.9}$$

$$\overline{I}_2 \quad = \quad 0.2358 - j\,0.8244 \ \text{ amps.} \tag{23.10}$$

Hence the potential across the inductor and capacitor is (in the phasor domain)

$$\overline{V}_L = \overline{V}_C = I_2\,j20 = 17.15\,e^{j15.96^\circ} \tag{23.11}$$

and thus to find the time domain potential a transformation from the phasor domain to the time domain is required. Hence

$$v_L(t) = v_C(t) = \text{Real}\,\{\overline{V}_L e^{j\omega t}\} = \text{Real}\,\{17.15\, e^{j15.96^\circ} e^{j4t}\} \approx 17.2\, \cos(4t + 16^\circ). \tag{23.12}$$

Thus the final results are given by

$$v_L(t) = 17.2\, \cos(4t + 16^\circ) \text{ volts} \tag{23.13}$$

$$v_C(t) = 17.2\, \cos(4t + 16^\circ) \text{ volts} \tag{23.14}$$

$$v_R(t) = 20\, \cos(4t - 15^\circ) - 17.2\, \cos(4t + 16^\circ) \text{ volts.} \tag{23.15}$$

23.3 Comments

This chapter applied phasor theory to circuits containing independent sources, resistors, capacitors, and inductors. Since the sources were AC, the phasor transform enabled us to reduce the circuits to a static circuit (containing complex impedances).

In the next chapter we will show that dependent sources also can be analyzed using the phasor domain.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 24

Dependent and Independent Sources: Phasor Domain

To further emphasize the use of the phasor domain when sources are cosinusoidal (or sinusoidal), additional circuits will be analyzed in this chapter, where there are dependent sources present.

24.1 A First Example

Consider the circuit presented in Figure 24.1, where the sources are cosinusoidal and there is a dependent source. The objective is to compute $v_1(t)$ and $v_2(t)$. The time domain to phasor domain transformation leads to a circuit on the phasor domain as shown. As can be seen, the dependent source now is represented by a dependence on a phasor. Also indicated in dotted lines, is the possibility of a source transformation on the phasor domain. This leads to a phasor domain circuit, as shown in Figure 24.2. The mesh method and Kirchhoff's loop law applied to the circuit yields equations on the phasor domain for $\overline{I}_1$ and $\overline{I}_2$ given by

$$\begin{aligned}(2 - j2.5 + j4)\overline{I}_1 - j4\overline{I}_2 &= 20 \quad (24.1)\\ -j4\overline{I}_1 + (4 + j4)\overline{I}_2 &= -3\overline{V}_1 \quad (24.2)\\ -20 + 2\overline{I}_1 + \overline{V}_1 &= 0 \quad (24.3)\end{aligned}$$

which simplifies to

$$(2 + j1.5)\bar{I}_1 - j4\bar{I}_2 = 20 \tag{24.4}$$

$$-j4\bar{I}_1 + (4 + j4)\bar{I}_2 + 3\bar{V}_1 = 0 \tag{24.5}$$

$$2\bar{I}_1 + \bar{V}_1 = 20. \tag{24.6}$$

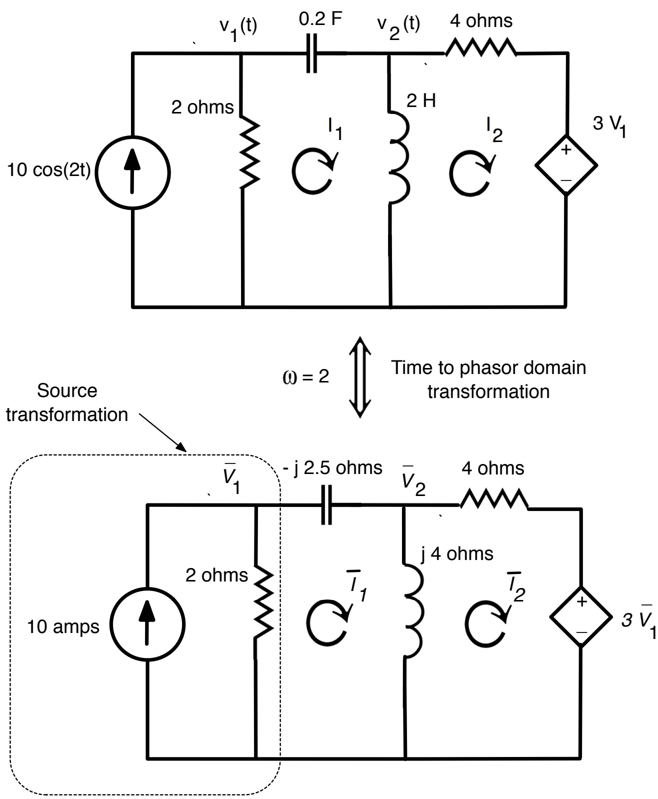

Figure 24.1 A circuit with a dependent source and a cosinusoidal independent source. The source transformation theorem can be deployed to simplify the circuit analysis. (The figure is a reworking of Figure 10.3 in [1].)

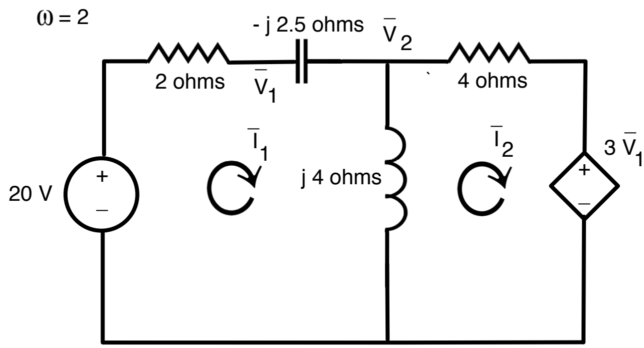

Figure 24.2 The phasor domain circuit simplified for analysis using the source transformation on the phasor domain.

The solution yields the currents and $\overline{V}_1$, and deploying Ohm's law to obtain $\overline{V}_2$ the solution is given by

$$\overline{V}_1 = 11.3\, e^{j60^\circ} \tag{24.7}$$

$$\overline{V}_2 = 33\, e^{j57^\circ}. \tag{24.8}$$

The original question was to compute $v_1(t)$ and $v_2(t)$, and hence a transformation from the phasor domain to the time domain is required. This yields

$$v_1(t) = 11.3\ \cos(2t + 60^\circ) \tag{24.9}$$

$$v_2(t) = 33\ \cos(2t + 57^\circ). \tag{24.10}$$

24.2 A Second Example

Figure 24.3 shows a circuit which is on the phasor domain. The objective is to compute $\overline{I}$. Using mesh analysis with the mesh current of the top mesh known to be 10 amps, the following system of equations is obtained:

$$(8 + j2)\overline{I}_1 - j4\overline{I}_2 = 0 \tag{24.11}$$

$$-j4\overline{I}_1 + (6 + j4)\overline{I}_2 = -50e^{j30^\circ} - 60. \tag{24.12}$$

This provides a solution for $\overline{I}_1$ given by

$$\overline{I}_1 = 6\ e^{-j114^\circ} \tag{24.13}$$

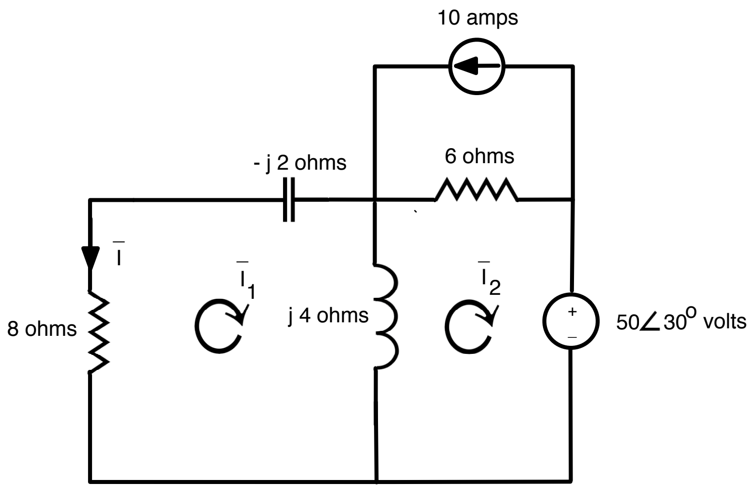

Figure 24.3 A phasor domain circuit. (The figure is a reworking of Figure 10.8 in [1].)

and since $\overline{I} = -\overline{I}_1$ it means that $180°$ are added to the phase so that

$$\overline{I} = 6\, e^{j65.44°} = 6 \angle 65.44°. \tag{24.14}$$

24.3 Comments

This chapter showed that dependent and independant sources can be directly transformed to the phasor domain. On the phasor domain the circuit can be reduced to a system of simultaneous linear equations, making use of complex numbers. The solution provides the phasor form of the currents and potentials, which can be transformed back to the time domain if required.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 25

Superposition: Phasors

In a previous chapter where the concept of a phasor was introduced it was shown that for a resistor (with cosinusoidal sources) it follows that

$$\text{Real}\left\{V_0 e^{j\phi} e^{j\omega t}\right\} = R\ \text{Real}\left\{I_0 e^{j\phi} e^{j\omega t}\right\}. \tag{25.1}$$

Since this relationship holds for any time t, the time dependence can be suppressed and it follows that

$$\text{Real}\left\{V_0 e^{j\phi}\right\} = R\ \text{Real}\left\{I_0 e^{j\phi}\right\}. \tag{25.2}$$

This enabled the phasors to be defined as $\overline{V}$ and $\overline{I}$ as

$$\overline{V} = V_0 e^{j\phi} \tag{25.3}$$

$$\overline{I} = I_0 e^{j\phi}. \tag{25.4}$$

If several sources are present and it is desirable to use superposition, then it would be possible if the frequency was the same for all sources. Then superposition holds in the phasor domain and its application is identical as it was for circuits analyzed in the time domain.

However if there are multiple sources with different frequencies, then superposition *holds only in the time domain.* Potentials or currents with different frequencies can be linearly combined in the time domain. Hence we compute the circuit potentials and currents (in the phasor domain) due to each source independently, and convert the results back to the time domain (for each source). Then the time domain results due to each source are combined through superposition. This approach will also work if the sources have the same frequency.

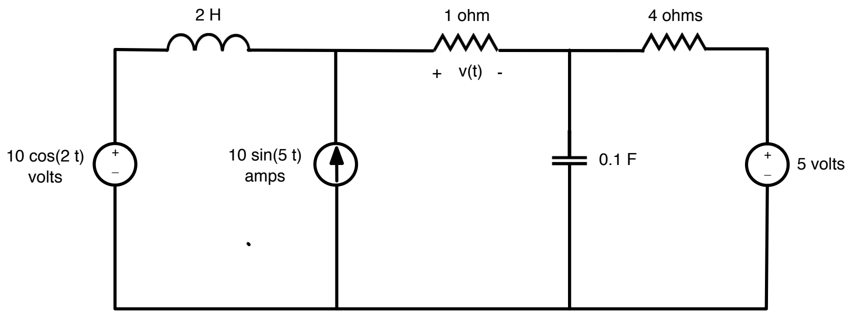

Figure 25.1 A time domain circuit with multiple sources at different frequencies. It is not possible to transform this circuit directly to the phasor domain, as the sources have different frequencies. Thus use is made of superposition. (The figure is a reworking of Figure 10.13 in [1].)

The method is best described through an example. Consider the circuit in Figure 25.1, which contains three sources, all at different frequencies. In fact one of the sources is static with a zero frequency. The objective is to compute $v(t)$.

25.1 Analysis Based on the AC Potential Source

Using superposition we are able to retain only one source. In this case the $10\ \cos(2t)$-volt source is retained. That means the current source is an open circuit (infinite resistance) and the 5-volt static source is a short circuit (zero resistance). Thus the time domain circuit and the transformation to the phasor domain is shown in Figure 25.2. The capacitor and the 4-ohm resistor are in parallel, and thus when in series with the 1-ohm and $j4$-ohm elements it yields a potential across the 1-ohm resistor (using Ohm's law) as[1]

$$\overline{V}_{source1} = \frac{10}{(1 + j4) + (4||(-j5))} = 2.5e^{-j30.8^{\circ}}. \tag{25.5}$$

Hence transforming back to the time domain, the voltage across the 1-ohm resistor in the time domain is

$$v_{source1}(t) = 2.5\ \cos(2t - 30.8^{\circ})\text{ volts.} \tag{25.6}$$

[1] The notation $a||b$ denotes a is in parallel with b.

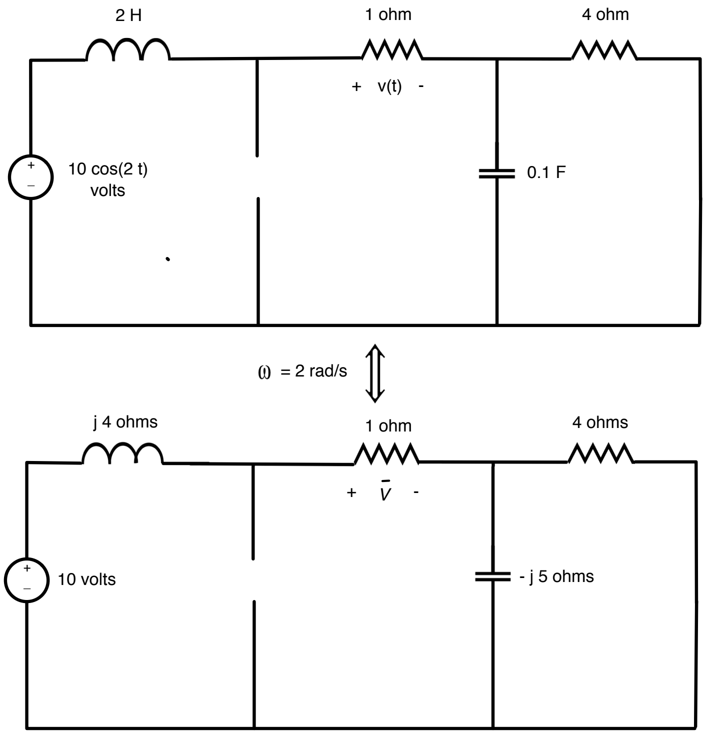

Figure 25.2 A time domain circuit with the current source set to 0 amps and the static source set to 0 volts. Thus the current source is an infinite resistance and the DC potential source is a short circuit. The phasor transformation at $\omega = 2$ rad/s is now possible.

25.2 Analysis Based on the AC Current Source

Next the AC current source is retained. Thus the potential source (at 2 rad/s) is replaced with a short circuit and the static potential source is also replaced with a short circuit. The time domain circuit and the phase domain transformation is shown in Figure 25.3. The current source and the $j10$-ohm element can be source transformed to yield a potential source of 20 volts in series with a $j10$-ohm impedance. Hence the current flowing through the 1-ohm resistor, and also

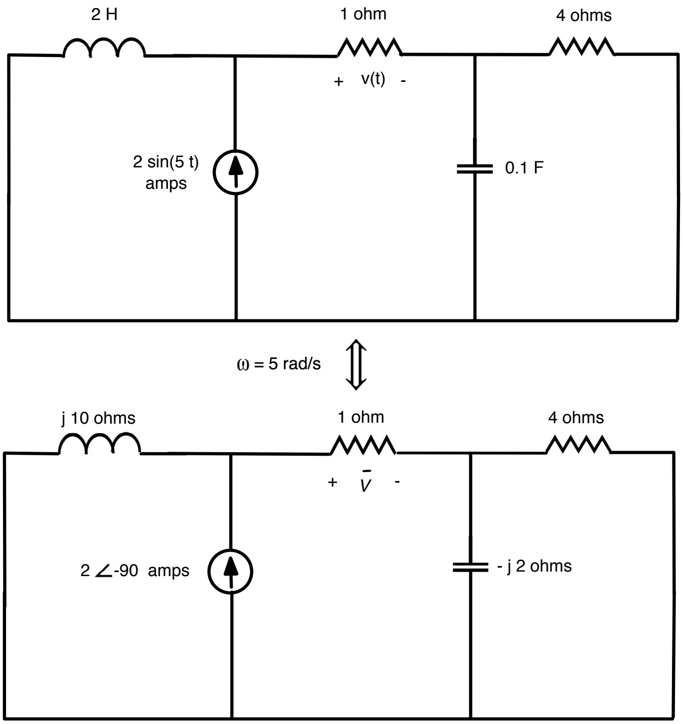

Figure 25.3 A time domain circuit with cosinusoidal voltage source and static source replaced with short circuits. Hence the time to phasor domain transformation is now possible at $\omega = 5$ rad/s.

the potential across that resistor (by Ohm's law) is given by

$$\overline{V}_{source2} = \frac{20}{(1 + j10) + (4||(-j2))} = 2.32e^{-j80^\circ}. \quad (25.7)$$

Hence transforming back to the time domain, the voltage across the 1-ohm resistor in the time domain is

$$v_{source2}(t) = 2.32\ \cos(5t - 80^\circ)\ \text{volts}. \quad (25.8)$$

25.3 Analysis Based on the Static Voltage Source

The last case the static voltage is retained, with the current source an open circuit and the cosinusoidal potential source a short circuit. Thus the circuit in the phasor domain is as shown in Figure 25.4. The current through the 1-ohm resistor is 1 amp and hence the potential across the 1-ohm resistor is -1 volts. Hence in the time domain

$$v_{source3}(t) = -1 \text{ volt}. \tag{25.9}$$

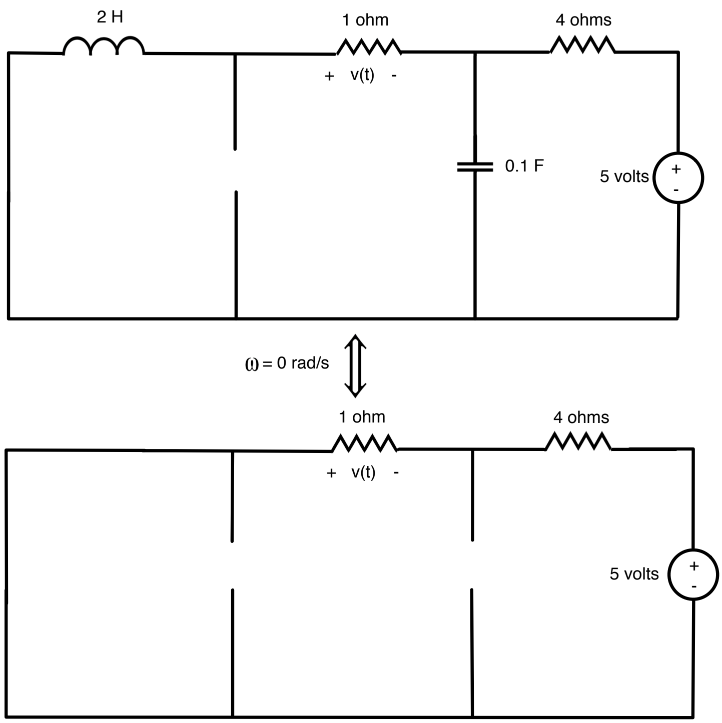

Figure 25.4 A time domain circuit with only static source. The frequency for the phasor transformation is $\omega = 0$ rad/s. Hence the analysis can be done directly on the time domain according to methods presented in Part I (static circuits) of the text.

25.4 Applying Superposition in the Time Domain

The final step is applying superposition in the time domain. Summing (i.e., superposition of) all the potentials obtained above in the time domain, yields the total potential across the 1-ohm resistor as

$$\begin{aligned} v(t) &= v_{source1}(t) + v_{source2}(t) + v_{source3}(t) \\ &= 2.5\ \cos(2t - 30.8^\circ) + 2.32\ \cos(5t - 80^\circ) - 1\ \text{volts}. \end{aligned} \tag{25.10}$$

25.5 Comments

We showed that superposition holds also in the phasor domain. If there are sources with different frequencies, then superposition must be performed only in the time domain. The response of the circuit to each source can however be computed in the phasor domain, then transformed to the time domain.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 26

The Maximum Power Transfer Theorem and Resonance

In a previous chapter the Thevenin theorem was presented, along with the conditions under which maximum power is delivered to a load. There the Thevenin circuit contained real sources and elements (resistors) and thus the Thevenin resistance was real also. Then it was shown that the load resistance must be equal to the Thevenin resistance for maximum power transfer to occur.

In this chapter circuits powered by cosinusoidal sources containing capacitors, inductors and resistors are considered. Under what conditions will maximum power be delivered to a load? This is clearly an important problem in practice, where electrical loads ranging from resistors to electrical motors are often driven using AC power sources.

The second property of AC circuits considered in this chapter is that of resonance. Resonance plays an important role in the design of frequency selective filters and oscillators where frequency selectivity is required.

26.1 Thevenin's Theorem for AC Circuits

The procedure for finding the Thevenin circuit is as before, except that all the methods introduced before are performed on the phasor domain. This means that the Thevenin potential and the Thevenin resistance in general will be complex. The procedure is best illustrated via an example. Consider the circuit shown in Figure 26.1, which is shown on the phasor domain.

26.1.1 Thevenin Potential

The Thevenin potential is defined as the potential of terminal *a* relative to terminal *b*, with the load removed. With the load removed there is no current flowing through the capacitor. Thus there is no potential difference across the capacitor

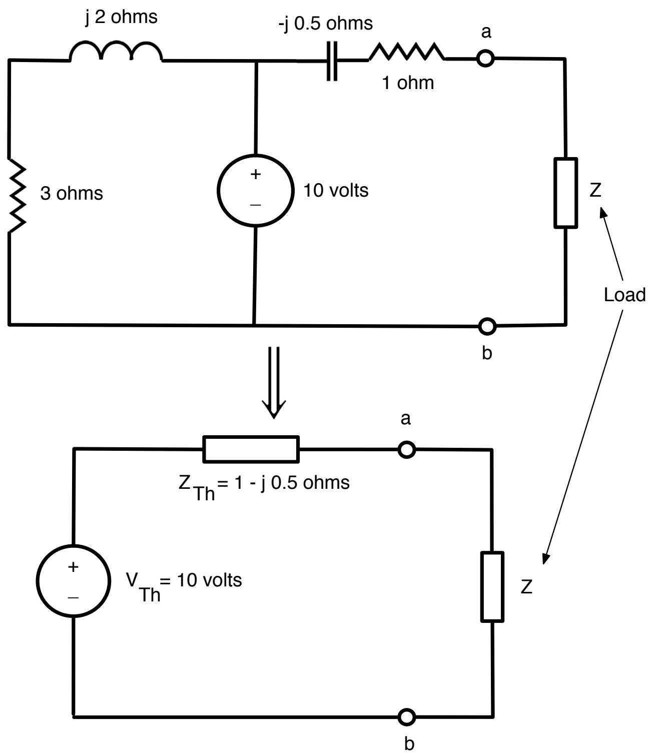

Figure 26.1 A phasor domain circuit and the Thevenin model. Both the Thevenin potential and the Thevenin impedance in general are complex numbers. The procedure for computing these are as before but performed on the phasor domain.

and resistor, and hence the potential at terminal a relative to terminal b will be 10 volts. Hence the Thevenin potential is given by

$$V_{\mathrm{TH}} = 10 \text{ volts.} \tag{26.1}$$

26.1.2 Thevenin Impedance Z_{TH}

The reader may note that the title of this section is Thevenin *impedance*. The reason for the name is the fact that for the phasor domain the Thevenin impedance in general is complex. It is not a pure resistive (real) value as it was for static circuits.

To compute this impedance, turn off all independent sources. There is only one, thus the 10-volt source is turned off, which implies it is replaced with a conductor. Thus the left side of the capacitor is connected to the reference node (terminal b). Next step is to add a 1-amp source, and compute V_{ab}. According to Ohm's law this yields

$$V_{ab} = 1 \times (1 - j0.5) = 1 - j0.5 \text{ volts} \tag{26.2}$$

and thus the Thevenin impedance is

$$Z_{\mathrm{TH}} = 1 - j0.5 \text{ ohms.} \tag{26.3}$$

26.2 Maximum Power Transfer

Now that the circuit of Figure 26.1 has been reduced to a phasor domain Thevenin potential source and a Thevenin impedance, the next question concerns the nature of the load Z that will guarantee maximum power delivery to the load. It can be shown through the use of calculus [1] that the optimum transfer of power occurs when

$$Z = Z^{*}_{\mathrm{TH}} \tag{26.4}$$

where * denotes the complex conjugate operation. Thus for the circuit shown above the maximum power to the load will occur when $Z = 1 + j0.5$ ohms. Intuitively this result makes sense, as it is saying that the reactive part of the load $Z = 1 + j0.5$ ohms will cancel out the reactive component of the Thevenin impedance, and the real part will equal the real part of the Thevenin impedance which is what we showed before is required for maximum power delivery.

26.3 Resonance

26.3.1 Series Resonance

Consider the series combination of a resistor, a capacitor and an inductor as shown in Figure 26.2. What is the Thevenin equivalent circuit? The Thevenin equivalent potential is clearly zero as there is no independent source, and the Thevenin impedance can be computed by adding a 1-amp source and computing V_{ab}, which yields

$$Z_{\mathrm{TH}} = 1 + \left(j0.1\omega + \frac{1}{j1 \times 10^{-3}\omega} \right). \tag{26.5}$$

Under what conditions will the impedance between terminals a and b be purely real? Clearly when

$$\left(j0.1\omega + \frac{1}{j1 \times 10^{-3}\omega} \right) = 0 \implies \omega = \frac{1}{\sqrt{0.1 \times 1 \times 10^{-3}}} = 100 \ \text{rad/s}. \tag{26.6}$$

For this frequency the impedances of the capacitor and inductor cancels each other out and the combined impedance seen at the terminals a and b is purely real. This is known as resonance. The resonance frequency is the frequency that enables the energy stored and released by the inductor and capacitor to be equal and out of phase, thus simply passed from the one to the other, so that the overall reactive (imaginary) component disappears.

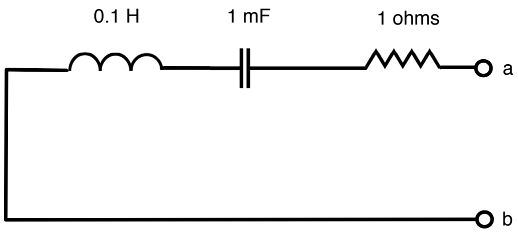

Figure 26.2 A circuit used for studying the so-called series resonance phenomenon. Resonance occurs when the impedance of the inductor and the capacitor cancels, and the load seen between terminals a and b is real.

For the general case where the capacitor is C farad and the inductor in L henry, resonance will occur when

$$\left(j\omega L + \frac{1}{j\omega C}\right) = 0 \implies \omega = \frac{1}{\sqrt{LC}} \text{ rad/s.} \tag{26.7}$$

26.3.2 Parallel Resonance

A circuit deploying parallel resonance is shown in Figure 26.3. The reader is encouraged to perform the calculation when the inductor and the capacitor are in parallel with a resistor, and show that the resonant frequency (for the circuit connected to nodes a and b) is also $\frac{1}{\sqrt{LC}}$ rad/s. The circuit for this case is shown below in Figure 26.3.

Some insight into the effect of the resistance R on resonance can be obtained by plotting the magnitude of the potential $\overline{V}_a$ versus frequency ω. The magnitude of the potential $\overline{V}_a$ is given by

$$|\overline{V}_a| = \frac{|\overline{I}|}{\left|\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right|} = \frac{I_m}{\left|\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right|}. \tag{26.8}$$

For convenience let us assume that $I_m = 1$, $L = 1$ and $C = 1$, so that the resonant frequency is $\omega_{res} = 1$ rad/sec. Figure 26.4 plots the magnitude of $\overline{V}_a$ versus frequency. It is clear that the voltage magnitude peaks at the resonant frequency, and is always given by $I_m R$.

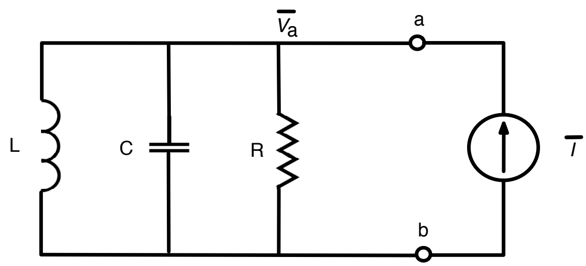

Figure 26.3 A parallel circuit used for studying the so-called resonance phenomenon. Resonance occurs when the impedance of the inductor and the capacitor cancels, and the impedance seen between terminals a and b is real.

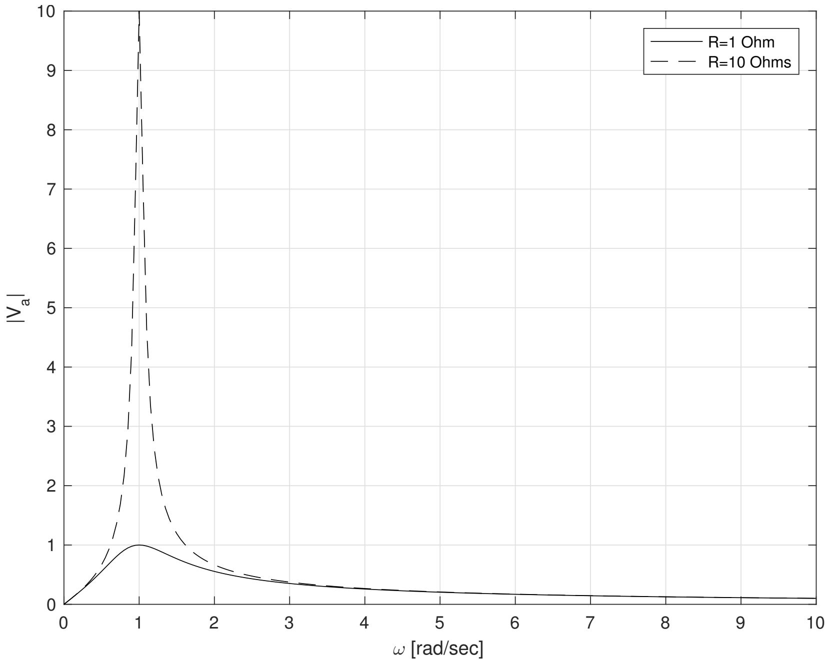

Figure 26.4 The magnitude of the node voltage $\overline{V}_a$ of the parallel RLC circuit versus frequency. The current source has magnitude $I_m = 1$. The resonant frequency is 1 rad/s as $C = 1$ farad and $L = 1$ henry.

Also it is evident that the sharpness of the voltage versus frequency curve is proportional to the resistance R, and is known as the quality or Q factor. The circuit in fact filters the input current, with only frequencies near resonance having significant amplitude. Thus in theory, if we could construct a parallel circuit with ideal components so that the only resistance is that of R, then a circuit with $Q \to \infty$ could be constructed by letting $R \to \infty$. Of course in real circuits the components are not ideal and the resistance cannot be infinite. Thus practical parallel resonance circuits have finite Q factors. Filters are used in many electronic circuits to reject unwanted frequencies and pass wanted frequencies.

26.4 Summary

This chapter demonstrated that maximum power transfer for AC sources require the load to be the complex conjugate of the Thevenin impedance. The phasor domain is well suited to the application of Thevenin's theorem, and directly provides the Thevenin impedance which is a complex number.

Secondly we showed that circuits containing inductors and capacitors can resonate, and the resonance frequency was defined. We also provided examples of series and parallel resonance circuits, and briefly explained the concept of a quality (Q) factor in resonant circuits.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 27

AC Power Concepts

Electrical power generation and its delivery to a consumer contributes greatly towards raising the standard of living of people worldwide. It's a technology that provides energy to billions and makes a difference to the lives of people. In terms of analysis it requires some of the most subtle concepts in engineering, and often confuses readers who are not used to ideas such as complex power and the like. This chapter aims to correct this situation, and present a clear analysis of electrical power.

27.1 Energy, Instantaneous and Average Power

By definition and as shown in the first few chapters, the instantaneous power absorbed by a circuit element at time t is given by

$$p(t) = i(t)\, v(t) \tag{27.1}$$

with $i(t)$ the current through the element and $v(t)$ the potential across the terminals of the element. It is very important that the reader recall that power is the rate of change of energy. Put differently, power is energy absorbed (or delivered) per second, or joules/second.

If the objective is to heat water, energy is required, measured in joules. For example to heat one liter of water 1℃, approximately 4.2 megajoules of energy is required. Consider a resistor placed in one liter of water with the objective to heat it, specifically to increase its temperature with 1℃. It is known that to accomplish this task 4.2 megajoules of energy will be required. If the current through the resistor is $i(t)$ amps and the potential across the terminals of the resistor is $v(t)$ volts, how long will it take for the 1℃ rise in temperature to occur? Denote the time taken as T_0 seconds, then assuming the time measurement starts at time $t = 0$, T_0 can be computed as

$$E = 4.2 \times 10^6 = \int_0^{T_0} p(t)\, dt = \int_0^{T_0} i(t)\, v(t)\, dt. \tag{27.2}$$

Recall from calculus that an integral is the area under a function. Hence if the function is the instantaneous power $p(t)$ measured in joules per second then its integral (area under function) is joule. This integral yields the total energy delivered in T_0 seconds. If it is required to state what was the average rate of change of energy or average power during that time, then we have to divide the energy delivered by the time taken — in other words we define the average power over T_0 seconds denoted by P as

$$P = \frac{E}{T_0} = \frac{1}{T_0} \int_0^{T_0} p(t)\, dt = \frac{1}{T_0} \int_0^{T_0} i(t)\, v(t)\, dt. \tag{27.3}$$

For static sources, the above integrals are redundant, as the instantaneous power is constant in time, and the relation between energy and power becomes straightforward. But for AC (cosinusoidal) sources the instantaneous power varies in time, and the integrals above become a useful tool to define average power, as shown in the next section.

27.2 Average Power for AC Circuits and Electrical Systems: Time Domain Analysis

Consider an element in a circuit or electrical system containing AC sources, with the current through the element denoted $i(t)$ and the potential across the element denoted as $v(t)$. Since the circuit is an AC circuit, then it follows that

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{27.4}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{27.5}$$

where V_m and I_m denote the maximum (or peak) values respectively. Thus the instantaneous power $p(t)$ is given by

$$p(t) = v(t)\, i(t) = V_m \cos(\omega t + \theta_v)\, I_m \cos(\omega t + \theta_i) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i). \tag{27.6}$$

The expression above can be simplified through trigonometric identities[1], and yields

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + (\theta_v + \theta_i)). \quad (27.7)$$

The instantaneous power for AC thus has two components:

1. The static or time independent term: $\frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$. If this term is averaged over a period of the AC source (i.e., $T = \frac{2\pi}{\omega}$) to yield the average power, it yields the same term as it is not a function of time.
2. The time varying term: $\frac{1}{2} V_m I_m \cos(2\omega t + (\theta_v + \theta_i))$. This term is cosinusoidal at twice the frequency of the AC source. Thus it averages to zero over a period $T = \frac{2\pi}{\omega}$ and thus has a zero average power. Thus it does not contribute towards the average power. Only the first term contributes to the average power.

The AC source has average power given by

$$P = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} p(t)\, dt = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i). \quad (27.8)$$

Since it is the average power that does work, such as warming water or accelerate a mass, the second term (with a zero average) has no effect (no contribution) on the work being done by an AC source. To visualize why this is so, it is instructive to plot the total instantaneous power over time, and to view visually the superposition of the two terms. This is shown in Figure 27.1 where its clear that the average power is given by term one (i.e., Equation [27.8]). Note that the average power P is a function of the angle $\delta\theta = \theta_v - \theta_i$. For example if the angle $\delta\theta$ is zero, then the average power P is a maximum, and the instantaneous power curve will touch the zero line but will not cross it.

27.3 Average Power for AC: Phasor Domain

The analysis of the instantaneous and average power for AC circuit elements were presented in the previous section. In this section the analysis will be extended using phasors, and it will be shown that phasor theory offers an equivalent but more compact representation.

[1] $2 \cos A \cos B = [\cos(A - B) \cos(A + B)]$.

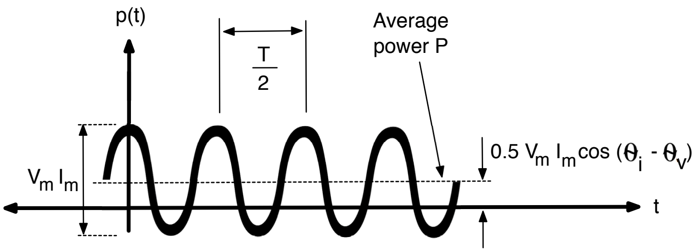

Figure 27.1 AC instantaneous power as a function of time. (This figure is a reworking of Figure 11.2 in [1].)

First of all recall that in any element in a circuit driven by an AC source the time domain potential across the terminals and the current through the element is given by

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{27.9}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{27.10}$$

and the average power P is given by

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i). \tag{27.11}$$

Based on the time domain formulation, the phasors for the potential and current are given by

$$\overline{V} = V_m e^{j\theta_v} \tag{27.12}$$

$$\overline{I} = I_m e^{j\theta_i}. \tag{27.13}$$

Hence it follows that

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \text{Real}\,\{\frac{1}{2} V_m I_m e^{j(\theta_v - \theta_i)}\} = \text{Real}\,\{\frac{1}{2} V_m e^{j\theta_v} I_m e^{-j\theta_i)}\}. \tag{27.14}$$

Hence substituting the phasors into the previous equation yields the average power as

$$P = \text{Real}\,\{\frac{1}{2} V_m e^{j\theta_v} I_m e^{-j\theta_i)}\} = \text{Real}\,\{\frac{1}{2} \overline{V}\,\overline{I}^*\} \tag{27.15}$$

where * denotes the complex conjugate operation. This shows that the average power (i.e., the power that does work) in a circuit element powered by an AC source is simply half the real part of the potential phasor multiplied by the complex conjugate of the current phasor.

In the next chapter these results will form the basis of the further analysis of AC power.

27.3.1 Root Mean Square Power

The effective or root mean square (RMS) value of a potential or current followed from the need to measure the effectiveness to deliver power to a load. The effective value of a periodic current with period T is defined as the DC current that will deliver the same average power to a resistor as the periodic current. It is defined as

$$I_{\mathrm{RMS}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}\,. \tag{27.16}$$

When the current is cosinusoidal then the relation between the peak value I_m and the RMS value I_{RMS} is given by

$$I_{\mathrm{RMS}} = \frac{I_m}{\sqrt{2}}. \tag{27.17}$$

Using the RMS notation the average power of a power source can be written as

$$P = \mathrm{Real}\,\{V_{\mathrm{RMS}}\, e^{j\theta_v}\, I_{\mathrm{RMS}}\, e^{-j\theta_i)}\} = \mathrm{Real}\,\{\frac{V_m}{\sqrt{2}}\, e^{j\theta_v}\, \frac{I_m}{\sqrt{2}}\, e^{-j\theta_i)}\} = \mathrm{Real}\,\{\frac{1}{2}\,\overline{V}\,\overline{I}^*\}. \tag{27.18}$$

The power industries tend to specify phasor magnitudes by their RMS values rather than peak values. As an example the 110 volts available at every household in the United States is the RMS value of the voltage. Thus the peak voltage is actually higher by a factor of $\sqrt{2}$. When designing components to withstand a certain voltage care should be taken that it is the peak value that is being used, not the RMS.

27.4 Summary

This chapter introduced the concept of average power in AC circuits. It was shown that it is average power the does work — that is heat or move objects. It was shown that the phasor domain provides a compact representation of average power. Average power was also defined and demonstrated in the time domain, and the relation between average power on the time and phasor domains was also provided.

Finally the concept of root mean square (RMS) power was defined. It was shown that for AC power the relationship between RMS and peak voltage and currents is given by a scaling constant.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 28

Power Factor and Power Factor Correction

In the previous chapters both a time domain and phasor domain analysis of power concepts were presented. To illustrate the concepts further, an example of an AC circuit is analyzed in terms of the power absorbed and delivered in each element. An AC circuit is as shown in Figure 28.1, where each element in the circuit is shown on the phasor domain.

28.1 Power Absorbed by the Resistor

Consider the case where $R = 1$ ohm, $Z_C = -j2$ ohms, and $Z_L = j5$ ohms, and assume that $\overline{V} = 1$ volt. Hence the phasor for the current through each of the three elements (that are in series) is given by Ohm's law as

$$\overline{I} = \frac{1}{1 + j3} = 0.32\, e^{-j71.6^\circ}. \tag{28.1}$$

Thus the potential difference across the terminals of the resistor is given by Ohm's law as

$$\overline{V}_R = R\,\overline{I} = 0.32\, e^{-j71.6^\circ} \tag{28.2}$$

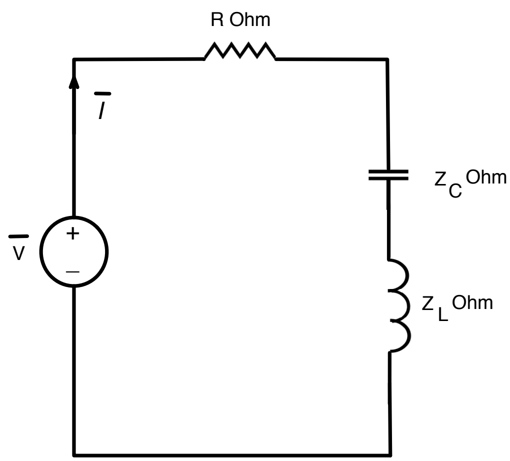

Figure 28.1 AC circuit on the phasor domain.

since the resistor has a resistance of 1 ohm. The average power absorbed by the resistor is given by Equation (27.8) or making use of the compact phasor form as

$$P_R = \text{Real}\{\frac{1}{2}\,\overline{V}_R\,\overline{I}^*\} = 0.5\,0.32\,e^{-j71.6^\circ}\,0.32\,e^{j71.6^\circ} = 0.5\,0.32^2 = 0.05 \;\text{ watts.} \tag{28.3}$$

Note that the current through the resistor and the potential across the resistors terminals are in phase. The reader can verify that for a resistor the same result will be obtained if the power is computed as $\frac{1}{2}|\overline{I}|^2R$.

28.2 Power Absorbed by the Capacitor

The phasor form of the capacitor current is given by

$$\overline{I} = 0.32\,e^{-j71.6^\circ}, \tag{28.4}$$

and thus the potential across its terminals is given by Ohm's law as

$$\overline{V}_C = -j2\,0.32\,e^{-j71.6^\circ} = 0.63\,e^{-j161.6^\circ}. \tag{28.5}$$

Thus the average power absorbed by the capacitor is given by

$$P_C = \text{Real}\{\frac{1}{2}\,\overline{V}_C\,\overline{I}^*\} = \text{Real}\{0.1\,e^{-j-161.6^\circ}\,e^{j71.6^\circ}\} = \text{Real}\{0.1\,e^{-j90^\circ}\} = 0 \;\text{ watts.} \tag{28.6}$$

The reader is encouraged to verify this fundamental result. The average power absorbed by an ideal capacitor is always zero. This result is not unique to the capacitors in the circuit above, it is generally true for the ideal capacitor. The reason is that the phase between the potential across the ideal capacitor terminals and the capacitor current is always 90° out of phase. This is a result we have shown to be valid in a previous chapter — see Figure 22.4. In practice the capacitors are not ideal and there is a small leakage current due to the dielectric, and thus for a practical capacitor the phase will not be exactly equal to 90°.

In the ideal device the 90° phase difference between the potential and current always leads to a $\text{Real}\{e^{-j90^\circ}\}$ term, that is identically zero. How can we explain this finding? The reason is that for an AC circuit the energy is stored in one half of a cycle, and then released in the other half of the cycle within a single period of the cosinusoid. Thus the energy is never lost, but the average power absorbed is identically zero.

To model a practical capacitor a large resistor can be placed in parallel with the ideal capacitor. The resistor will model the leakage effects due to non-ideal dielectric.

28.3 Power Absorbed by the Inductor

The current flowing through the inductor is given by

$$\overline{I} = 0.32\,e^{-j71.6^\circ}, \tag{28.7}$$

and thus the potential across its terminals is given by Ohm's law as

$$\overline{V}_L = j5\;0.32\,e^{-j71.6^\circ} = 1.58\,e^{j18.43^\circ}. \tag{28.8}$$

Thus the average power absorbed by the inductor is given by

$$P_L = \text{Real}\{\frac{1}{2}\,\overline{V}_L\,\overline{I}^*\} = \text{Real}\{0.25\,e^{j18.43^\circ}\,e^{j71.6^\circ}\} = \text{Real}\{0.25\,e^{j90^\circ}\} = 0 \;\text{ watts.} \tag{28.9}$$

Again the reader is encouraged to verify this fundamental result. The average power absorbed by an ideal inductor is always zero. This result is not unique to the circuit above, it is true in general for all ideal inductors. The reason is that the phase between the potential across the inductor terminals and the current through it is always 90° out of phase. This is a result we have shown to be valid in a previous chapter (see Figure 22.5).

The 90° phase difference between the potential and current for the ideal inductor always leads to a $\text{Real}\{e^{j90^\circ}\}$ term, that is identically zero. How is this possible? The reason is that for AC circuits the inductor stores energy in one half of a cycle, and then releases the same energy the other half of a cycle of the cosinusoid. Thus the energy is never lost, and the average power absorbed is identically zero.

Of course in practice the conductors used to construct the inductor has finite resistance and the phase difference is not exactly 90°. The ideal inductor model proposed above can be augmented by adding a small resistor in series to model the operation of practical inductors.

28.4 Power Factor

In previous chapters it was shown that

$$P = \frac{1}{2} V_m\, I_m\, \cos(\theta_v - \theta_i). \tag{28.10}$$

Define the power factor for any element in an AC circuit as

$$\text{PF} = \cos(\theta_v - \theta_i). \tag{28.11}$$

For a purely reactive element that has an impedance that is purely imaginary (such as an ideal capacitor or an ideal inductor) the angle between the potential across its terminals and current through it is always 90°, and thus its PF = 0. If the PF = 0, the average power absorbed is zero.

For an element with a purely real impedance (or resistance) such as a resistor the angle between the potential across its terminals and the current through it is zero, and have PF = 1.

Practical elements have complex impedance Z and the angle between the potential and current is the angle of the complex number Z. This makes it possible to compute the PF for a circuit component.

28.5 Power Factor Correction

A major application of the concepts presented in previous sections is the idea of power factor correction. Many electrical devices in practice have complex impedance Z, and are not purely resistive or purely reactive. Examples are induction motors that are widely deployed in industry. Given the complex load impedance is $Z = |Z|\, e^{j\phi}$ the power factor is given by

$$\mathrm{PF} = \cos(\phi) \tag{28.12}$$

and so if the load has a significant imaginary component then PF can be small. This means that the real power absorbed by the load is reduced. Thus to deliver a high amount of average power in a load with a low PF requires a larger current amplitude. Since ohmic losses in practical conductors (that are not perfect) are proportional to the square of the current, this is not desirable. Ideally to deliver power to a load the PF should be close to one — then the current amplitude required is as small as possible. This is the case for a resistor of course.

For example an industrial plant (acting as a load to the electricity generator) typically requires a large amount of power, and hence high current levels. One way to reduce the current levels is to use higher potential but that is unsafe. So without that option it is important to have the PF as close to one as is possible, so that current amplitude is as small as is possible. Many electrical power providers will require clients to not have a PF below a certain level. If the PF of the plant falls below this level penalties may apply and thus most plants monitor the PF continuously.

Hence there is a need to increase the power factor PF if it is too low, so that it exceeds the required levels. This is known as PF correction, considered in this section. Consider an inductive load as shown on the left in Figure 28.2. The impedance of this load is $Z = R + jZ_L$, and thus the power factor can be computed (it can also be measured in practice). The PF may be too low, and the electricity provider may force the operator of the load to improve the power factor.

One way that is popular in practice is to add a corrective capacitor as shown on the right of Figure 28.2. The ideal capacitor does not consume any power (it was shown in a previous chapter that the average power absorbed by an ideal capacitor is zero). Thus adding the capacitor will not increase real power consumption. But it will affect the PF as shown in Figure 28.3, where the behavior of the load with and without the correction capacitor is shown. The use of vectors enables a compact and efficient representation.

The potential across the terminals of the inductive load is $\overline{V}_{a,b}$. This vector is used as a reference shown on the real axis — the reader is reminded that all potentials are relative.

- Kirchhoff's current law says that $\overline{I} = \overline{I}_L + \overline{I}_C$. This is thus a vector addition as shown in Figure 28.3.
- The power factor for the inductive load (without the capacitor) and current $\overline{I}_L$ is $\cos(\theta_1)$.

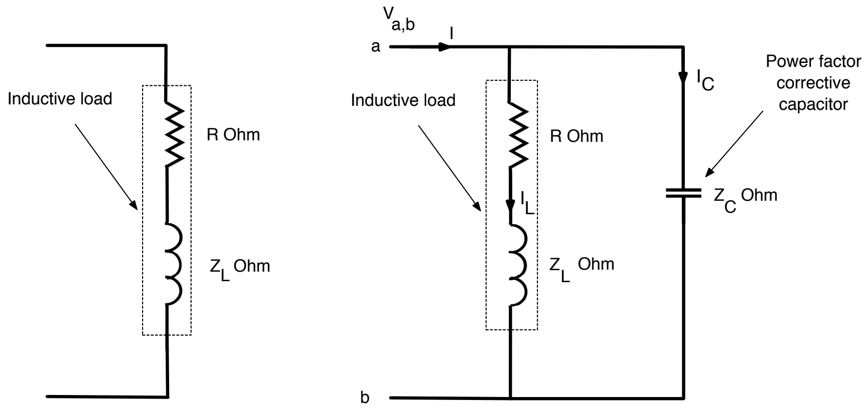

Figure 28.2 The concept of power factor correction. On the left the load of a plant is shown that is inductive (as is often the case). On the right the PF correcting capacitor is shown. This connection has the advantage that the capacitor that is in parallel will not need to pass the plant current as a capacitor in series would have to. (The figure is a reworking of Figure 11.27 in [1].)

- After adding $\overline{I}_C$ to $\overline{I}_L$, the new angle (after correction due to the capacitor) is θ_2 as shown in Figure 28.3. The power factor is now improved (smaller), as clearly $\cos(\theta_2) > \cos(\theta_1)$.

Adding a capacitor in series and selecting the value of C such that resonance occur will render the PF one, since at resonance the total impedance is real. However then the plant (load) current will be through the capacitor also, which will increase ohmic losses (as the conductors are not perfect). For this reason the parallel configuration is preferred.

In all cases power factor correction can be performed by a graphical analysis using vectors as shown in Figure 28.3. If a minimum PF is specified by the electricity provider the capacitor current magnitude can be computed. Thus given the frequency of the AC potential source the capacitance then follows.

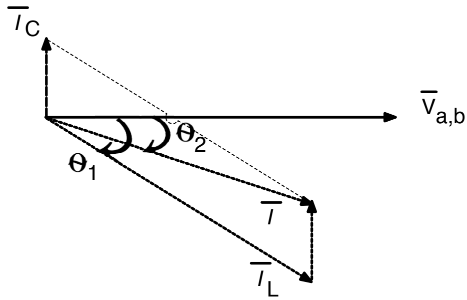

Figure 28.3 The concept of power factor correction. The current after correction is denoted $\overline{I} = \overline{I}_C + \overline{I}_L$. This has the effect of reducing the angle θ_2 which increases the power factor. (The figure is a reworking of Figure 11.28 in [1].)

28.5.1 Power Factor Correction Example

In this example we consider an example for power correction through adding a parallel capacitor to improve the power factor. The following information is provided: An inductive load is absorbing 4 kilowatts with a power factor (PF) equal to 0.8. The AC supply voltage is 120 volts RMS ($V_m = 120\sqrt{2}$), at a frequency of 60 Hz. We are required to compute the value of a capacitor that is able to improve the load PF to 0.95 [1].

Solution: Since we are told that the load is absorbing 4 kilowatts, we know that

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = 4 \times 10^3 \tag{28.13}$$

where θ_v and θ_i represent the phases of the potential across the load and current through the load respectively. We are also told that the PF is currently 0.8, hence

$$4 \times 10^3 = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2} 120\sqrt{2}\, I_m\, 0.8. \tag{28.14}$$

Thus we find

$$I_m = 60 \text{ amps}. \tag{28.15}$$

[1] These settings are from Figure 11.28 in [1].

Since PF $= 0.8 = \cos(\theta_v - \theta_i) = \cos(\Delta\theta)$ we find that $\Delta\theta = 37°$ and using the applied voltage $\overline{V}$ as the reference, we can visually represent this information as shown in Figure 28.4. The real and imaginary projections are

$$|I_r| = I_m \cos(\Delta\theta) = 48 \text{ amps} \quad (28.16)$$
$$|I_i| = I_m \sin(\Delta\theta) = 36 \text{ amps.} \quad (28.17)$$

A visualization after correction is possible by noting that the post correction angle must be $\Delta\theta = 18°$, as we are told that the PF after correction must be 0.95. This is shown in Figure 28.5. We know that in Figure 28.5 the following must be true

$$I_m \cos(18°) = 48. \quad (28.18)$$

This is because the addition of the parallel capacitor cannot change the real part of the current. Thus the magnitude of the current after the correction must be

$$I_m = 50.5 \text{ amps.} \quad (28.19)$$

Hence we find that the imaginary projection after correction must be

$$|I_i| = 50.5 \sin(18°) = 15.6. \quad (28.20)$$

This can be accomplished by adding a vector $\overline{I}_c$ which is the capacitor current, $90°$ ahead of the supply voltage as shown in Figure 28.3. Thus this current adds directly

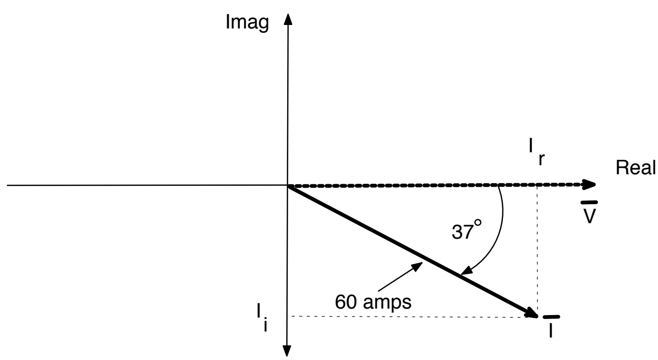

Figure 28.4 The phasor diagram before PF correction. The reference at 0° is the potential $\overline{V}$. The projection of the current vector onto the real and imaginary axis are indicated. The real component of the current I_r cannot be changed by adding a capacitor in parallel.

to the imaginary projection before correction, thus subtracting to reduce it. Thus we find

$$15.6 = 36 - |\bar{I}_c| \tag{28.21}$$

so that

$$|\bar{I}_c| = 20.4 \ \text{amps}. \tag{28.22}$$

Thus using the impedance formula for the capacitor we find

$$\frac{120\sqrt{2}}{|\bar{I}_c|} = \frac{1}{\omega C} \tag{28.23}$$

which yields (at a frequency of 60 Hz)

$$C = 319 \ \mu F. \tag{28.24}$$

Thus adding a capacitor of $C = 319\ \mu F$ in parallel to the given load the PF would improve the PF to 0.95 as required.

It is clear that the magnitude of the current after correction has been reduced, which is the objective of power factor correction. A reduced current magnitude reduces thermal losses in the power supply lines, which are proportional to the

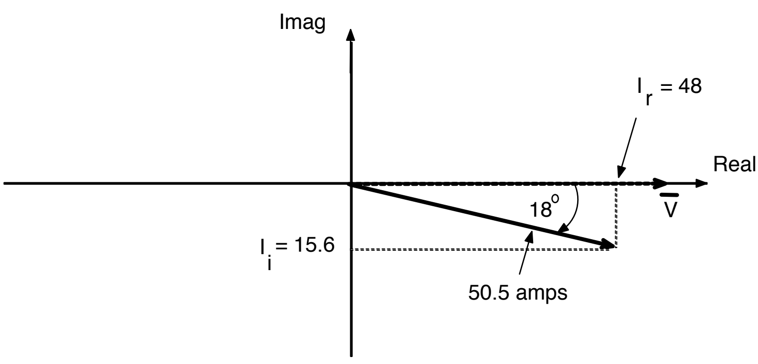

Figure 28.5 The phasor diagram after PF correction by adding a capacitor in parallel. The reference at 0° is the potential $\bar{V}$. The projection of the current vector onto the real and imaginary axis are indicated. However adding the capacitor reduces the imaginary component of the current vector as is indicated. This causes the PF to improve.

square of the current amplitude. Hence any reductions in the current amplitude will lead to savings in energy that otherwise will be lost (converted to heat).

28.6 Summary of Power Terminology

We are now in a position to summarize the terminology often used for power in AC systems:

1. Complex power $\overline{S}$: $\overline{S} = \frac{1}{2}\overline{V}\,\overline{I}^*$
2. Apparent power S: $S = |\overline{S}| = \frac{1}{2}|\overline{V}|\,|\overline{I}|$
3. Real power P: $P = \text{real}\{\overline{S}\}$
4. Imaginary power Q: $Q = \text{imag}\{\overline{S}\}$
5. Power factor PF: $\text{PF} = \frac{P}{S} = \cos(\theta_v - \theta_i)$

Note that work done by electrical power (heating water or accelerating a mass) requires the use of the real power P. However the other quantities are often useful to understand the energy storage and power factor of electrical systems.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 29

Magnetically Coupled Circuits

Field coupling by two or more different coils in close proximity is arguably one of the most important methods of energy and power transfer available to us. It makes possible the transfer of power without physical contact, which lead to the development of very efficient devices such as the polyphase AC induction motor.

The ideas behind mutual induction are relatively easy to explain, and are based on observational or experimental facts, making use of mathematical models that are able to model the effects observed. Consider two coils with self inductance, L_1 and L_2 respectively, however these coils are in close proximity as is shown in Figure 29.1. This causes the magnetic fields to couple the two coils, even though they are not in physical contact. It can be shown experimentally that the current $i(t)$ flowing through coil one (that is time varying) induces a potential $v_1(t)$ across coil one through a mechanism known as self induction — hence based on its own self inductance L_1 henry (see Equation [11.8]). But due to coupling of the magnetic fields, the second coil also develops a potential $v_2(t)$, induced into coil two by the current $i(t)$. This is known as a *mutual inductance*.

Moreover, it has been found experimentally that both potentials induced are proportional to the derivative of the current $i(t)$, so that constants of proportionality can be used to relate the potentials and current, given by

$$v_1(t) = L_1 \frac{di(t)}{dt} \tag{29.1}$$

$$v_2(t)_{a,b} = M \frac{di(t)}{dt}. \tag{29.2}$$

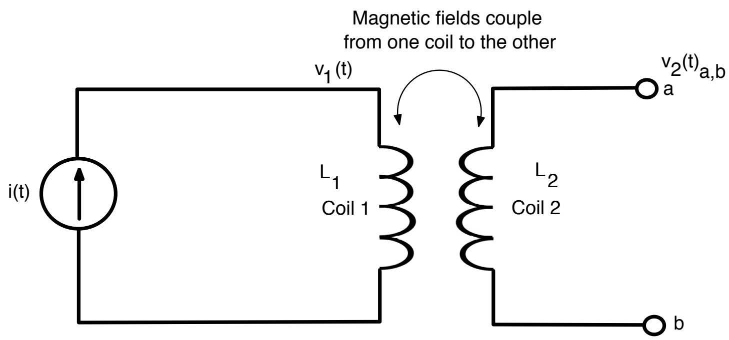

Figure 29.1 Two coupled coils. Note that the two coils do not have to share the same reference point. Hence the potential in coil two is indicated as the potential at node *a* relative to node *b*. (This figure is a reworking of Figure 13.2 in [1].)

The inductance M designates the mutual inductance in henries. The manner in which these coils are physically wound and orientated in space, and hence interact makes it possible for the potential induced in coil two to be negative. Thus it is common practice to use the dot convention to indicate the polarity of the mutual coupling, as shown in Figure 29.2.

The convention is that if current enters from the dotted side, then the potential induced will be positive at the dot in the coupled coil.

29.1 An Example: Circuit Making Use of Mutual Coupling

The concepts are best illustrated via an example of its use in a circuit, as shown in Figure 29.2. Because of the dot's positions the mesh currents were chosen as indicated which is consistent with the convention that a positive potential will be induced. There are two sources causing the currents to flow as indicated. Thus each mesh has an induced component due to the mutual coupling. This may be most easily be appreciated by applying superposition, and the reader can verify that each source causes an induced potential in the coupled mesh. If the two coils are placed far away from each other, the mutual coupling term M will vanish, and

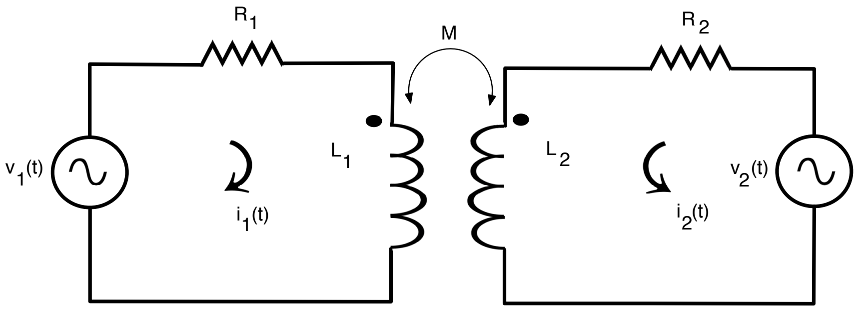

Figure 29.2 A circuit making use of magnetic coupling. (The figure is a reworking of Figure 13.7 in [1].)

the two meshes will become independent. Thus the mutual coupling causes the two mesh currents to become codependent.

With all the choices correctly made consistent with the conventions adopted, it is a mere formality to apply Kirchhoff's loop law in both meshes, which yields

$$R_1\, i_1(t) + L_1 \frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt} = v_1(t) \tag{29.3}$$

$$R_2\, i_2(t) + L_2 \frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt} = v_2(t). \tag{29.4}$$

With the sources specified, we can solve for the currents. For the specific case where the sources are cosinusoidal, then the time to phase domain transformation applies, and for AC sources the two equations above become

$$R_1\, \overline{I}_1 + L_1 j\omega \overline{I}_1 + Mj\omega \overline{I}_2 = \overline{V}_1 \tag{29.5}$$

$$R_2\, \overline{I}_2 + L_2 j\omega \overline{I}_2 + Mj\omega \overline{I}_1 = \overline{V}_2. \tag{29.6}$$

The reader should understand that the way the circuit in Figure 29.2 was drawn is arbitrary. For example the same circuit with identical equations and solution is shown in Figure 29.3.

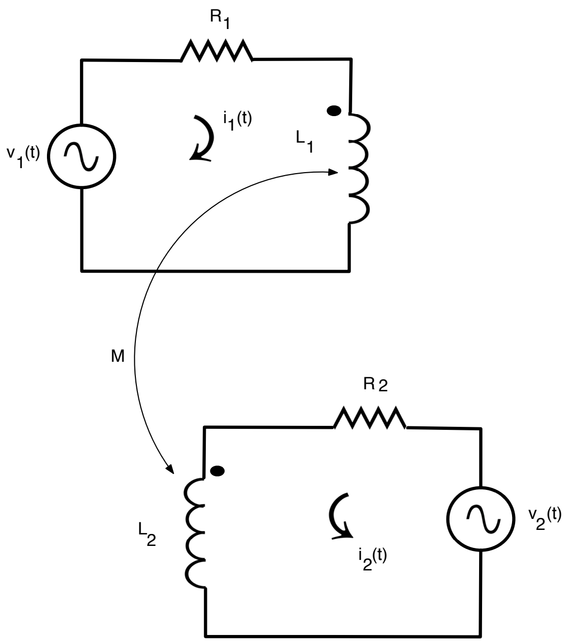

Figure 29.3 The circuit of Figure 29.2 drawn in a different way.

29.2 A Second Example: Mutual Coupling

The second example considered is shown in Figure 29.4, where the mutual coupling is indicated as between the two coils shown. The circuit is shown on the phasor domain, with a source of 100 volts AC. The reader should notice that the mesh currents were chosen to be consistent with the dot's as indicated, so that all mutual potentials induced have positive polarity[1].

To be specific, there are two induced mutual potentials:

[1] If a circuit is considered where this is not possible, then the induced polarity of the mutual potential would be made negative.

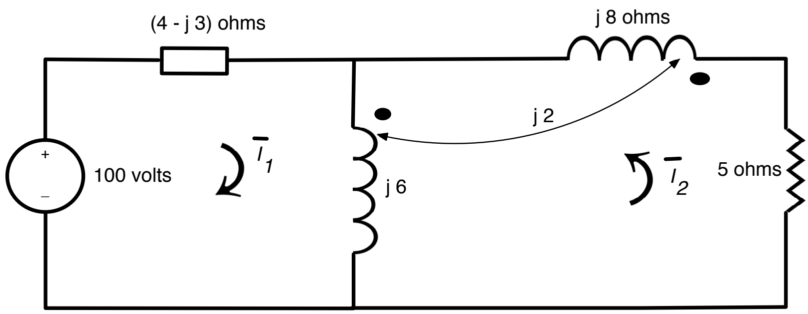

Figure 29.4 A circuit on the phasor domain, where mutual coupling between two coils modifies the mesh currents. (The figure is a reworking of Figure 13.11 in [1].)

1. Mesh current $\overline{I}_2$ enters the $j8$-ohm coil from the dotted side, and hence induces a potential of $j2 \times \overline{I}_2$ volts across the $j6$-coil. For both meshes this potential will show up in the application of Kirchhoff's loop law.
2. The mesh currents add up to form a total current $\overline{I}$ through the $j6$-coil, so that the induced potential across the $j8$-coil in mesh two is given by $(\overline{I}_1 + \overline{I}_2) \times j2$. This potential is visible only when Kirchhoff's law is applied in mesh two.

With these observations in mind, let us now turn to applying Kirchhoff's loop law in setting up the mesh equations for the circuit, given by

$$-100 + (4 - j3)\overline{I}_1 + j6\,(\overline{I}_1 + \overline{I}_2) + j2\,\overline{I}_2 = 0 \quad (29.7)$$

$$5\overline{I}_2 + j8\overline{I}_2 + j6\,(\overline{I}_1 + \overline{I}_2) + j2\,\overline{I}_2 + j2\,(\overline{I}_1 + \overline{I}_2) = 0. \quad (29.8)$$

This can be written in matrix form given by

$$\begin{bmatrix} 4 + j3 & j8 \\ j8 & 5 + j18 \end{bmatrix} \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \quad (29.9)$$

and the solution for the mesh currents follow.

It's important that the reader spot the contributions made by the mutual coupling term $j2$. If there was no mutual coupling in the circuit, then the two mesh equations would loose the terms due to the mutual coupling (these all involve the $j2$ factor). Thus mutual coupling is a powerful method for inducing currents in various parts of a circuit, and is a key idea behind the transformer. Also note that the mutual inductance alters the effect of the self-inductance of the inductors in the circuit[2].

For this circuit, the power dissipated in the 5-ohm resistor (which is proportional to the square of the magnitude of $\overline{I}_2$) is a function of the mesh current $\overline{I}_1$ as well because the magnitude of $\overline{I}_2$ is a function of $\overline{I}_1$ due to mutual coupling that modified the matrix above. The reader is encouraged to repeat the solution of this circuit with a different value of the mutual coupling, and verifying that the power delivered to the 5-ohm resister changes.

Finally, note that the coupling can be tight or loose. Tight coupling means the mutual inductance is on the order of the self inductance, and conversely loose coupling means that the mutual inductance is small relative to the self inductance. In fact, it can be shown based on arguments of stored energy that the mutual inductance is limited through a relation given by

$$M^2 \leq L_1 L_2 \tag{29.10}$$

where L_1 and L_2 denote the self inductance of the coils.

29.3 Linear Transformer

A linear transformer is based on the concept of mutual induction, and the model is shown in Figure 29.5. These are linear because the output potential (or secondary potential measured at the load) is directly proportional to the input potential $\overline{V}$ (also known as the primary potential). The transformer makes possible the delivery of power to a load without any physical contact, which is useful in practice. Also, as will be shown below, the load impedance can be transformed through the properties of the coupled coils.

Applying Kirchhoff's loop law to the two meshes two coupled equations are obtained given by

$$-\overline{V} + \overline{I}_1 \left(Z_1 + j\omega L_1\right) - \overline{I}_2 \left(j\omega M\right) = 0 \tag{29.11}$$

$$\overline{I}_2 \left(Z_{\text{load}} + j\omega L_2\right) - \overline{I}_1 \left(j\omega M\right) = 0. \tag{29.12}$$

The reason the induced potential is negative is because the current through the load is flowing into the non-dotted side of the coil. These equations can be solved

[2] If inductors are in series or parallel, the total inductance can be modified through use of the mutual inductance.

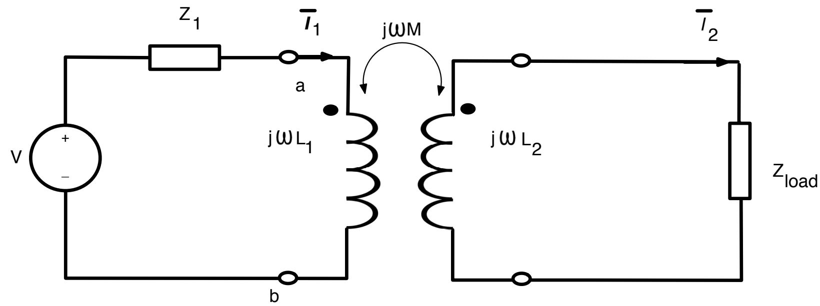

Figure 29.5 A model of a linear transformer connecting a load Z_{load} to the terminals a and b of the input source (represented by its Thevenin model). It is clear that the load is connected to the source without any physical contact with the source. All power is delivered to the load via mutual induction. (This figure is a reworking of Figure 13.19 in [1].)

analytically to determine the currents, given by

$$\overline{I}_1 = \frac{(Z_{\text{load}} + j\omega L_2)\,V}{(Z_1 + j\omega L_1)(Z_{\text{load}} + j\omega L_2) + \omega^2 M^2} \tag{29.13}$$

$$\overline{I}_2 = \frac{(j\omega M)\,V}{(Z_1 + j\omega L_1)(Z_{\text{load}} + j\omega L_2) + \omega^2 M^2}. \tag{29.14}$$

If the load was physically connected between terminals a and b then the source $\overline{V}$ would see a load $Z_1 + Z_{\text{load}}$. However since the load is coupled to the terminals through the coupled coils (or transformer) we can compute the *effective* impedance seen by the source V as

$$Z_{\text{eff}} = Z_1 + Z_{\text{transformed}} = \frac{\overline{V}}{\overline{I}_1} = \frac{(Z_1 + j\omega L_1)(Z_{\text{load}} + j\omega L_2) + \omega^2 M^2}{(Z_{\text{load}} + j\omega L_2)} \tag{29.15}$$

which can be simplified as

$$Z_{\text{eff}} = Z_1 + Z_{\text{transformed}} = Z_1 + \left(j\omega L_1 + \frac{\omega^2 M^2}{(Z_{\text{load}} + j\omega L_2)} \right). \tag{29.16}$$

Thus we conclude that the impedance seen by the source $\overline{V}$ is a series combination of the impedance Z_1 and a transformed load impedance given by

$$Z_{\text{transformed}} = \left(j\omega L_1 + \frac{\omega^2 M^2}{(Z_{\text{load}} + j\omega L_2)}\right). \tag{29.17}$$

Let us consider the effect of varying the mutual inductance M:

1. We move coil two far away so that it is loosely coupled, until $M \rightarrow 0$. Then the transformed load impedance is reduced to the self impedance term, or

$$\lim_{M\rightarrow 0} Z_{\text{transformed}} = j\omega L_1. \tag{29.18}$$

This we call the input (primary) impedance.

2. As coil two is moved closer to coil one the flux couples the two coils and M increases. The load impedance now effects the impedance that the source sees by modifying $Z_{\text{transformed}}$. It does this by adding a *coupled impedance* term to $Z_{\text{transformed}}$. The coupled impedance term is

$$Z_{\text{coupled}} = \frac{\omega^2 M^2}{(Z_{\text{load}} + j\omega L_2)} \tag{29.19}$$

so that

$$Z_{\text{transformed}} = \left(j\omega L_1 + Z_{\text{coupled}}\right) = \left(j\omega L_1 + \frac{\omega^2 M^2}{(Z_{\text{load}} + j\omega L_2)}\right). \tag{29.20}$$

29.4 The Ideal Transformer

The ideal transformer has ideal coupling with no leakage flux, so that $M = \sqrt{L_1 L_2}$. The effect this has on the coupled coils making up the ideal transformer, as shown in Figure 29.6, can be understood by analyzing the transformer. First apply Ohm's law in both coils which yields

$$\overline{V}_1 = j\omega L_1 \overline{I}_1 + j\omega M \overline{I}_2 \tag{29.21}$$

$$\overline{V}_2 = j\omega M \overline{I}_1 + j\omega L_2 \overline{I}_2. \tag{29.22}$$

We can find $\overline{I}_1$ from the first of the above equations as

$$\overline{I}_1 = \frac{\overline{V}_1 - j\omega M \overline{I}_2}{j\omega L_1} \tag{29.23}$$

and substituting into the second equation we find that

$$\overline{V}_2 = j\omega L_2 \overline{I}_2 + \frac{M\overline{V}_1}{L_1} - \frac{j\omega M^2 \overline{I}_2}{L_1}. \tag{29.24}$$

However for the ideal transformer we know that $M = \sqrt{L_1 L_2}$ and substituting this relation into the above equation we find

$$\overline{V}_2 = \sqrt{\frac{L_2}{L_1}}\,\overline{V}_1. \tag{29.25}$$

The self inductances can be written in terms of the number of turns in each coil, thus we find that

$$\overline{V}_2 = \sqrt{\frac{L_2}{L_1}}\,\overline{V}_1 = n\overline{V}_1 \tag{29.26}$$

where n is known as the turns ratio. Thus the turns ratio n is given by

$$n = \sqrt{\frac{L_2}{L_1}} = \frac{N_2}{N_1} = \frac{\overline{V}_2}{\overline{V}_1}. \tag{29.27}$$

Here N_1 and N_2 are the number of turns in the two coils respectively. Thus for the ideal transformer the output voltage is directly related to the input voltage as

$$\overline{V}_2 = n\,\overline{V}_1. \tag{29.28}$$

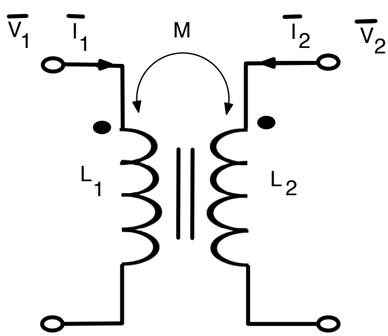

Figure 29.6 A model of an ideal transformer. Here coupling is ideal so that $M = \sqrt{L_1 L_2}$. For this transformer the output voltage V_2 and the input voltage V_1 are related by the ratio of turns in each coil.

This makes it possible to change the voltage using a transformer based on mutual induction.

29.5 Comments

Using mutual coupling and the transformer it is possible to add a load to a source without any physical contact. The load is thus isolated from the source. Power will be dissipated in the load, if the real part of the load is not zero. Through mutual coupling power is literally transferred through the vacuum of space, making use of the electromagnetic force. Since the load impedance is transformed, transformers are often used to alter the impedance of a load to suit the impedance of the system delivering the power. We showed before that for the power delivered to a load to be maximized, the load impedance must be the complex conjugate of the Thevenin impedance of the source. Transformers make it possible to achieve maximum power delivery through impedance matching — matching the load impedance to the source impedance [2].

References

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

[2] Abrie, P.L.D., *Design of RF and Microwave Amplifiers and Oscillators*, Artech House, 2009.

Chapter 30

Frequency Response and System Transfer Function

The word response is used here to mean the effect (of a system) that follows the cause. The system in this case is a linear circuit, viewed as a system or black box as shown in Figure 30.1. The system comes with four terminals, two terminals used as an input by the user, and two terminals as an output provided to the user.

To define the frequency response of a system, the user provides as an input a cosinusoidal potential source at a frequency ω, given by

$$v_{in}(t) = A\cos(\omega t + \phi). \tag{30.1}$$

The phase is arbitrary as the system is linear, and we can in fact consider the case where $\phi = 0$. Transforming the input and output, as well as the system (or circuit) to the phasor domain yields an equivalent model, as a function of frequency ω, as shown in Figure 30.2. Let us define the system transfer function as

$$H(\omega) = \frac{\overline{V}_{out}(\omega)}{\overline{V}_{in}(\omega)}. \tag{30.2}$$

It's important that the reader appreciates that this definition is on the phasor domain, and is a function of frequency, and that it does not hold in the time domain, that is

$$h(t) \neq \frac{v_{out}(t)}{v_{in}(t)}. \tag{30.3}$$

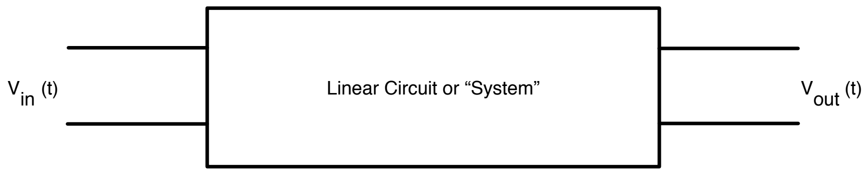

Figure 30.1 A model of a linear system on the time domain, with input $v_{in}(t)$ and output $v_{out}(t)$.

30.1 Example: Transfer Function for an RC Circuit

Consider an RC circuit on the phasor domain shown in Figure 30.3. The potential at output terminal a relative to terminal b is $\overline{V}_a(\omega)$ and is given by the voltage divider rule as

$$\overline{V}_a(\omega) = \overline{V}(\omega)\,\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{30.4}$$

which can be written as

$$H(\omega) = \frac{\overline{V}_a(\omega)}{\overline{V}(\omega)} = \frac{1}{1 + j\omega RC}. \tag{30.5}$$

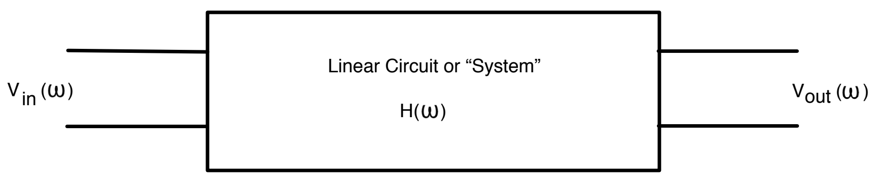

Figure 30.2 A model of a linear system after transformation to the phasor domain, with input $V_{in}(\omega)$ and output $V_{out}(\omega)$, and a transfer function defined by $H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)}$.

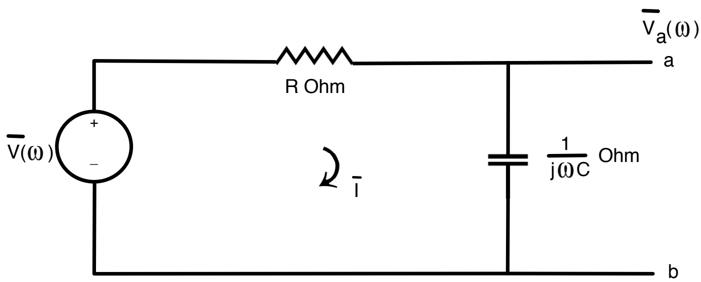

Figure 30.3 An RC circuit on the phasor domain. The transfer function $H(\omega)$ is defined as the output phasor divided by the input phasor. For this circuit the time constant is inversely related to the frequency where $|H(\omega)|^2 = \frac{1}{2}$.

The transfer function $H(\omega)$ is clearly a complex function, and can be written as

$$H(\omega) = |H(\omega)|\, e^{j\,\angle H(\omega)}. \tag{30.6}$$

Of specific interest is the behavior of the magnitude of the transfer function as a function of frequency. The absolute value of the numerator is not a function of frequency and is unity, and the magnitude of the denominator is given by

$$D(\omega) = \sqrt{1 + (\omega R C)^2} \tag{30.7}$$

so that

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega R C)^2}}. \tag{30.8}$$

Clearly at $\omega = 0$ rad/s the transfer function magnitude is one. As the frequency increases the magnitude is less than one and decreases. In the limit of infinite frequency the transfer function magnitude is zero. A specific point that is of interest is where $|H(\omega_c)|^2 = \frac{1}{2}$. At this frequency half the power is available at the output of the system, and that occurs where $\omega_c R C = 1$ or

$$\omega_c = \frac{1}{RC} = \frac{1}{\tau}. \tag{30.9}$$

The reader may recall that the time constant for the RC circuit is $\tau = RC$, and thus we now see that the inverse of the time constant provides the frequency where half the power is available at the output of the circuit.

30.2 Example: Third Order Low Pass Filter

This section presents the frequency response or transfer function $H(\omega)$ of a third order low pass filter as shown in Figure 30.4. However we are not going to make use of the phasor formulation of the previous section, but rather solve the differential equations numerically and directly in the time domain using MATLAB. By selecting the input waveform to be a sinusoid $v_i(t) = \sin(\omega t)$ [1] we can compute the output $v_o(t) = B_0 \sin(\omega t + \phi)$, at a frequency ω. The magnitude of the transfer function or frequency response at ω, given $v_i(t) = \sin(\omega t)$, follows as

$$|H(\omega)| = B_0. \tag{30.10}$$

This is because the input waveform has unity amplitude, and B_0 is the amplitude of the output waveform, thus this definition is consistent with the transfer function definition given in (30.2).

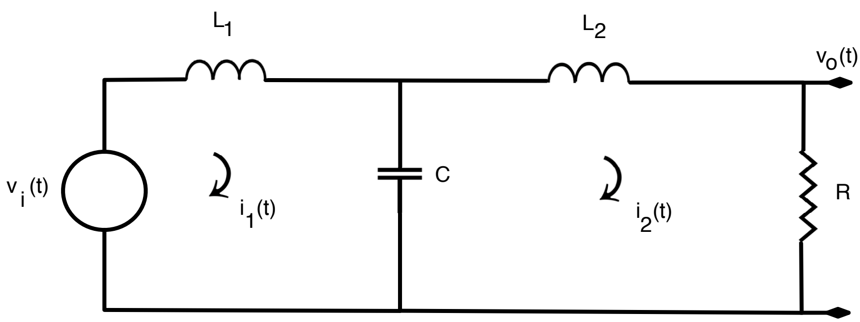

Figure 30.4 A third order low pass filter, where the objective is to determine $H(\omega)$ directly in the time domain through numerical solution.

[1] We assume $v_i(t)$ is zero for negative time.

30.2.1 Analysis in the Time Domain

Kirchhoff's laws yields a system of differential equations for the circuit given by

$$-v_i(t) + L_1\frac{di_1(t)}{dt} + \frac{q(t)}{C} = 0 \tag{30.11}$$

$$-\frac{q(t)}{C} + L_2\frac{di_2(t)}{dt} + R\,i_2(t) = 0 \tag{30.12}$$

$$\frac{dq(t)}{dt} - \{i_1(t) - i_2(t)\} = 0 \tag{30.13}$$

where $q(t)$ is the charge of the capacitor at time t. This can be written compactly in matrix form as

$$\begin{bmatrix} \frac{i_1(t)}{dt} \\ \frac{i_2(t)}{dt} \\ \frac{dq(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1C} \\ 0 & \frac{-R}{L_2} & \frac{1}{L_2C} \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} \frac{v_i(t)}{L_1} \\ 0 \\ 0 \end{bmatrix} \tag{30.14}$$

and we designate the matrix by the symbol $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1C} \\ 0 & \frac{-R}{L_2} & \frac{1}{L_2C} \\ 1 & -1 & 0 \end{bmatrix}. \tag{30.15}$$

We use the notation where δt is the time resolution of the numerical solution, and a sample of a function $f(t)$ at time $f(n\,\delta t)$ is designated as $f[n]$. Thus the above equation can be approximated by a recursive equation as a function of discrete time n given by

$$\begin{bmatrix} i_1[n+1] \\ i_2[n+1] \\ q[n+1] \end{bmatrix} = \begin{bmatrix} i_1[n] \\ i_2[n] \\ q[n] \end{bmatrix} + \delta t\,\mathbf{A} \begin{bmatrix} i_1[n] \\ i_2[n] \\ q[n] \end{bmatrix} + \delta t \begin{bmatrix} \frac{v_i[n]}{L_1} \\ 0 \\ 0 \end{bmatrix}. \tag{30.16}$$

Since the circuit is assumed at rest the initial currents and charge is zero at $t = 0$, and the above recursive equation can be numerically integrated in a straightforward manner. The MATLAB code for performing the numerical integration and obtaining the results that will follow is given below.

```
clear all

omega_low = 0.05;  % low end of frequency range
omega_high = 3.05; % high end of frequency range
omega_inc = 0.1; % frequency resolution

% circuit parameters
L1 = 1.5;
L2 = .5;
C = 1.33;
R = 1;

% numerical solution parameters
samples = 200; % samples per period
N = 10*samples; % run for 10 periods at each frequency

A = [0  0  -1/L1/C % transfer matrix
     0 -R/L2 1/L2/C
     1  -1  0];
count = 1;
for omeg = omega_low:omega_inc:omega_high % freq sweep
    dt = 2*pi/omeg/samples; % time resolution
    X(:,1) = [0 0 0].';  % zero initial conditions
    for l=2:N % time sweep
        vi = sin(omeg*(l-2)*dt);
        V = [vi/L1 0 0].';
        X(:,l)=X(:,l-1)+dt*A*X(:,l-1)+dt*V; %Euler
    end
    H_omega(count) = R*max(abs(X(2,N-samples:N)));
    count = count + 1;
end

semilogx([omega_low:omega_inc:omega_high],...
20*log10(H_omega),'-+k')
grid on
xlabel('Angular frequency [rad/sec]')
ylabel('H(\omega) [dB]')
legend('Transfer function')
```

At a frequency of $\omega = 3.05$ the MATLAB program provides the input and output in the time domain as

$$v_i(t) = \sin(3.05\,t) \tag{30.17}$$

$$v_o(t) = 0.0357\,\sin(3.05\,t + \phi)\ \ \forall\ \ t \gg 0 \tag{30.18}$$

The output is shown in Figure 30.5, and is characterized by an initial erratic period known as the transient, which over time gives way to the steady state given above. The MATLAB code measures the output amplitude (transfer function magnitude) B_0 during the last period when the transient has decayed to negligible values.

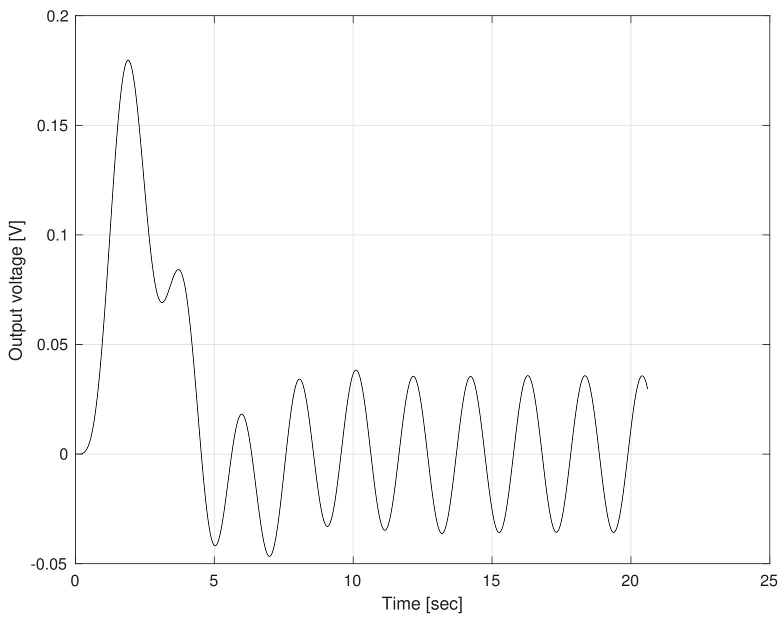

Figure 30.5 The output of the circuit in the time domain with $\omega = 3.05$ rad/sec. The erratic period known as the transient decays and the output eventually becomes a sinusoidal function.

30.2.2 Bode Plot

Based on the MATLAB code above, the transfer function for the circuit shown in Figure 30.4 was computed and is shown in Figure 30.6. The figure shows the transfer function $|H(\omega)|$ as a function of ω, but it plots $|H(\omega)|$ on the y axis as

$$10\,log_{10}\,|H(\omega)|^2 \tag{30.19}$$

which is designated as decibel or dB. This type of plot is known as a magnitude Bode plot, and the frequency ω_c where the transfer function satisfies $|H(\omega_c)|^2 = \frac{1}{2}$ (or half power) is known as the *corner frequency*. At the corner frequency we can see that

$$10\,log_{10}\,|H(\omega_c)|^2 = 10\,log_{10}\left(\frac{1}{2}\right) = -3.01\ \ dB. \tag{30.20}$$

For this circuit $\omega_c = 1$ rad/second (see Figure 30.6). From equation (30.9) we can infer that this circuit (of third order) will have a time constant of $\tau \approx \frac{2\pi}{\omega_c}$. Thus in practice we can expect that the output signal in the time domain will have stabilized (transients decayed) after $5\tau \approx 10\pi$ seconds. This is can be confirmed by examining the time domain response shown in Figure 30.5.

In terms of the circuit functionality, it is clear that the circuit passes sinusoidal signals with frequencies below the corner frequency, and tends to suppress frequencies above the corner frequency. Hence the name low pass filter used to describe the circuit.

30.3 Summary

This chapter introduced the concept of a frequency response of an AC circuit. We defined the transfer function on the phasor domain, and we showed that there is a direct inverse relationship between the half power or corner frequency and the time domain time constant. This was demonstrated via an analysis of a first order RC circuit and a third order low pass filter.

We also introduced the concept of direct time domain numerical analysis using MATLAB. This has the advantage that an arbitrary circuit with arbitrary input waveforms can be analyzed. However this chapter only considered AC input and output waveforms, so that the transfer function definition is valid.

We also introduced the so-called Bode plot and corner frequency. The Bode plot makes use of decibel (dB) on the y axis, and makes for an exceptionally simple visualization of the frequency response.

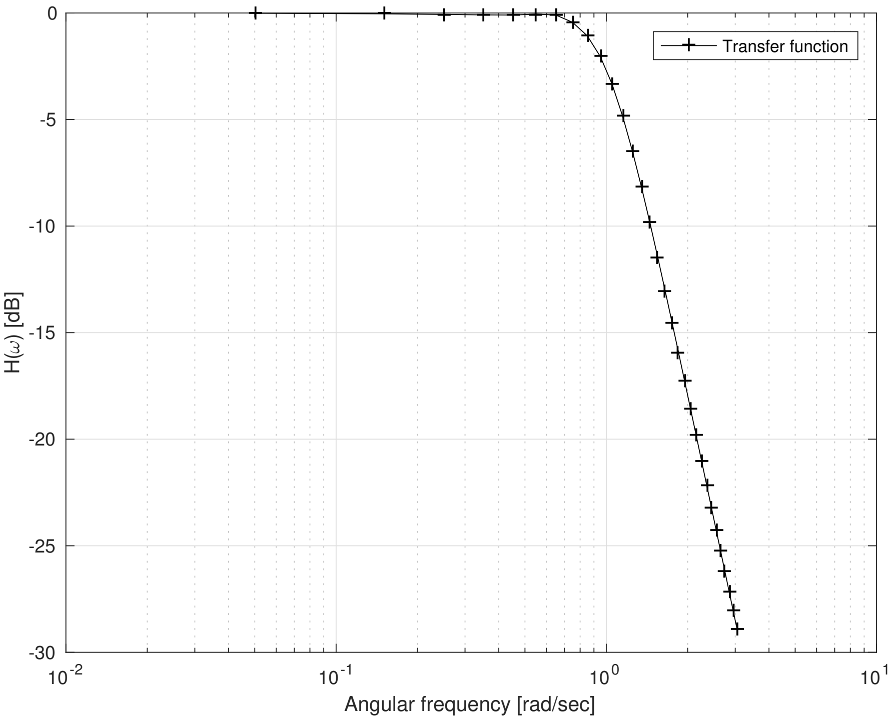

Figure 30.6 A magnitude Bode plot of the transfer function magnitude $|H(\omega)|$ as a function of the angular frequency ω. The phase response $\angle H(\omega)$ is not shown. The circuit is known as a low pass filter, as it passes sinusoidal signals with a low frequency.

Chapter 31

Three Phase Systems: An Introduction

One of the key developments in establishing the dominance of electrical power and the AC paradigm was the invention of the three phase system by Tesla. The combination of the three phase (or polyphase) electrical system, as well as the key ideas behind Tesla's polyphase induction motor, achieved efficiencies and reliability that could not be matched by the DC systems.

The reader is assumed to be proficient in the analysis of cosinusoidal sources both on the time domain and the phasor domain — this material was studied over several chapters and should be reviewed before considering the material in this chapter if required.

31.1 Balanced Three Phase System

The essential idea behind three phase is shown in Figure 31.1. Polyphase means more than one phase, in this case there are 3 sources deployed in the three phase source, each with a different phase.

There are three sources, each has a phase that is $120°$ shifted in the phasor domain. To visualize what the three nodal voltages at a, b, and c looks like in the time domain, consider Figure 31.2 where the potential of the three sources (relative to neutral defined in Figure 31.1) is shown. The phase shift on the phasor domain is of course a time shift in the time domain as is clearly shown in Figure 31.2. The

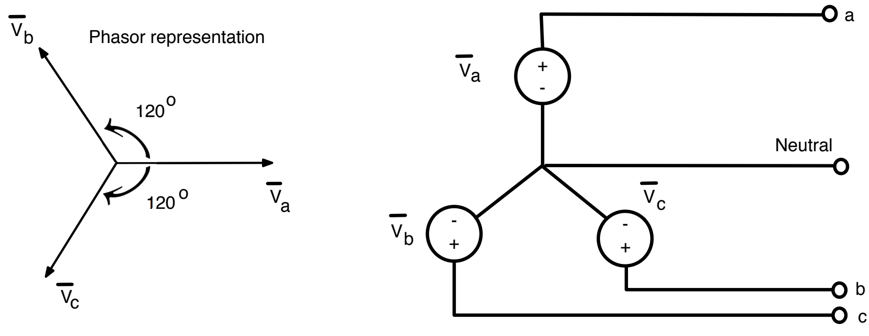

Figure 31.1 The concept of a three phase source. The reference node is the neutral node.

potential relative to neural of each source making up the three phase polysource is 120° shifted in phase, which is exactly a third of a period in the time domain. By viewing the potentials in the phasor domain and in the time domain, it may not be obvious why this arrangement is elegant. To see why this is the case consider the instantaneous power in the time domain. The instantaneous power per ohm (i.e.,

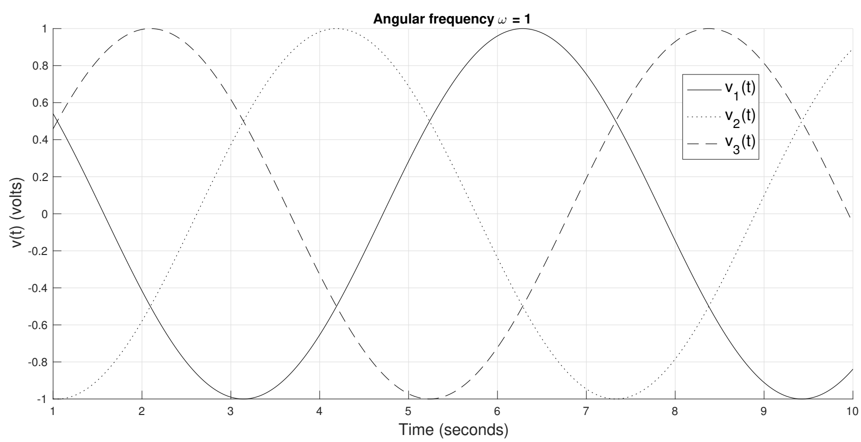

Figure 31.2 A three phase polysource in the time domain. Each source is 120° phase shifted relative to the other sources. The frequency is chosen (for convenience) as $\omega = 1$.

it is assumed each phase has a load of 1 Ω [balanced]) is defined as

$$p(t) = v_a^2(t) + v_b^2(t) + v_c^2(t). \tag{31.1}$$

The time domain instantaneous power delivered to a 1-Ω per phase load is shown in Figure 31.3, for single, two, and three sources. The reader can verify that the instantaneous power at any time t is exactly a constant if all three sources deliver power to a balanced load. An example of such a load is shown in Figure 31.4. A three phase induction motor will have three windings, identical, such that it presents three identical loads to the polyphase source.

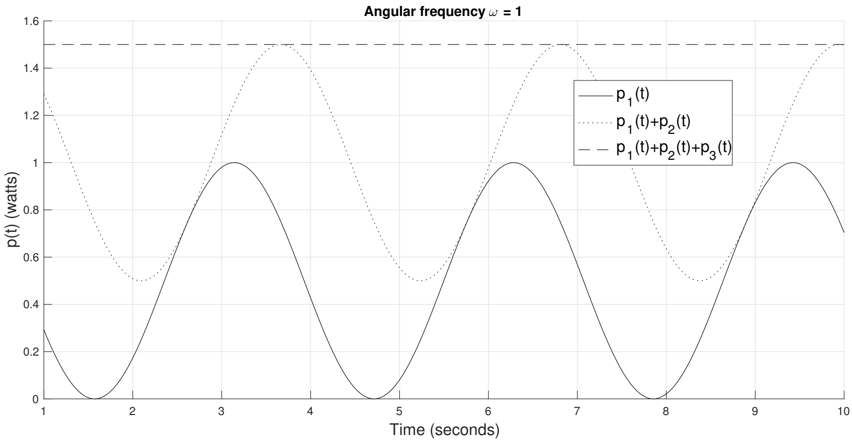

Figure 31.3 The instantaneous power (each phase has a 1-ohm load) in the time domain for a polyphase source system (three phase). The magnitude of each phase is one volt. The frequency is chosen (for convenience) as $\omega = 1$. The total instantaneous power due to all three sources combined is constant in time.

In this way the power delivered is absolutely constant, and does not vary over time, even though the sources do vary over time! This provides smooth power and torque when converting electrical energy to kinetic energy.

Assuming each phase has a 1-Ω load, the MATLAB code below computes the instantaneous power over time. The reader is encouraged to convince himself/herself of the validity of the claim of constant power when all three sources are used to deliver power to a balanced load. The MATLAB code used to generate Figure 31.3 is provided below:

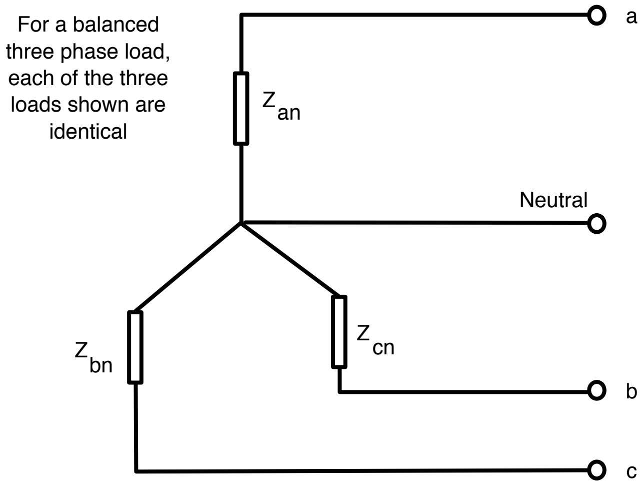

Figure 31.4 A three phase polysource load. Each individual load is complex, and $Z_{an} = Z_{bn} = Z_{cn}$ for a balanced load. (The figure is a reworking of Figure 12.8 in [1].)

```
clear all
t = [1:0.01:10];
v1 = cos(t);
v2 = cos(t + 120/180*pi);
v3 = cos(t - 120/180*pi);
p1 = v1.^2;
p2 = v1.^2 + v2.^2;
p3 = v1.^2 + v2.^2 + v3.^2; % each phase has a 1 ohm load
hold
plot(t,p1,'-k')
plot(t,p2,':k')
plot(t,p3,'-k')
hold
xlabel('Time (seconds)')
ylabel('p(t) (Watt)')
legend('p_1(t)','p_1(t)+p_2(t)','p_1(t)+p_2(t)+p_3(t)')
grid on
title('Angular frequency \omega = 1')
```

31.2 Analysis of a Y-Y Balanced System

The source and the load as shown in Figure 31.1 and Figure 31.4 respectively represent a so called Y-Y balanced connection if they are connected as shown in Figure 31.5. As an example for the analysis of the balanced Y-Y polyphase system, consider the case where the sources are given by

$$\overline{V}_a = 110\, e^{j0^\circ} \text{ volts} \quad (31.2)$$

$$\overline{V}_b = 110\, e^{-j120^\circ} \text{ volts} \quad (31.3)$$

$$\overline{V}_c = 110\, e^{-j240^\circ} \text{ volts.} \quad (31.4)$$

In this case the three loads (contained in the polyload) are identical so that

$$Z_{an} = Z_{bn} = Z_{cn} = 5 + j6 \text{ ohms.} \quad (31.5)$$

The objective is to compute the current through each load, and the power absorbed by each load.

The first step is to compute the currents and the mesh method is selected. There are three meshes as indicated, and the reader should note that the neutral

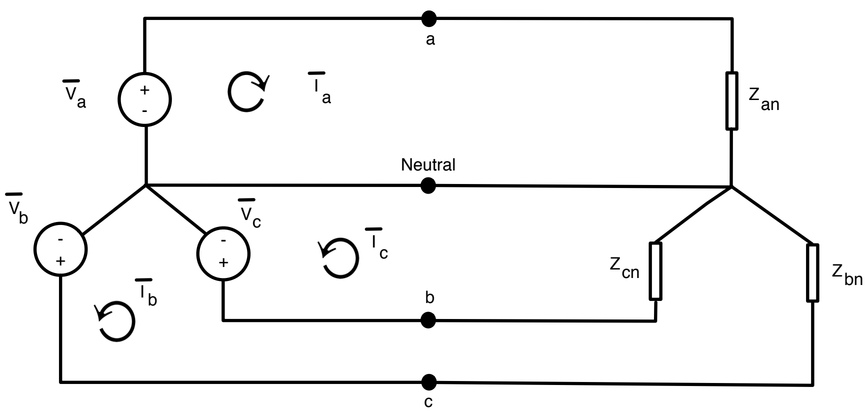

Figure 31.5 A three phase polysource in the four wire Y configuration connected to a polyphase balanced load also in the Y configuration. Each component of the load is identical (and in general complex) so that $Z_{an} = Z_{bn} = Z_{cn}$. (The figure is a reworking of Figure 12.10 in [1].)

wire does not have any resistance (it is a perfect conductor). Hence the potential across the neutral wire is zero, regardless of the current through it. In practice this of course is only an approximation, but typically the potential is very small. Thus assuming the neutral wire is a perfect conductor, the meshes decouple and we can apply the Kirchhoff loop law directly to each mesh and obtain three equations given by

$$-\overline{V}_a + \overline{I}_a Z_{an} = 0 \quad (31.6)$$
$$-\overline{V}_b + \overline{I}_b Z_{bn} = 0 \quad (31.7)$$
$$-\overline{V}_c + \overline{I}_c Z_{cn} = 0 \quad (31.8)$$

Since the meshes are decoupled (the neutral wire is a perfect conductor) the equations can be solved independently, and since the system is balanced, power in each phase is identical. Thus we will just use phase a to perform the required calculations. Firstly the current is given by

$$\overline{I}_a = \frac{110}{Z_{an}} = \frac{110}{5 + j6} = 14e^{-j50^\circ}. \quad (31.9)$$

The power absorbed by each phase in the load is thus

$$P = \frac{1}{2}\text{real}\{\overline{V}\,\overline{I}^*\} = \frac{1}{2}\text{real}\{110 \times 14e^{j50^\circ}\} = \frac{1}{2}\,(110)\,(14)\,\cos(50^\circ) = 495\,\text{watts}. \quad (31.10)$$

31.3 Analysis of a Δ-Δ Balanced System

The Y configuration used above is not the only possibility of achieving the three phase system. Another possibility is to use the so-called Δ configuration, which is shown in Figure 31.6. The polyphase source is a so-called positive sequence generator, which means that the three potential sources must be given by

$$\overline{V}_{ab} = V_m e^{-j\,0^\circ} \quad (31.11)$$
$$\overline{V}_{bc} = V_m e^{-j\,120^\circ} \quad (31.12)$$
$$\overline{V}_{ca} = V_m e^{j\,120^\circ}. \quad (31.13)$$

The objective is to compute the line currents $\overline{I}_a$, $\overline{I}_b$ and $\overline{I}_c$.

First of all notice that the choice of the three sources coupled as shown is valid as they satisfy Kirchhoff's loop law, given by

$$V_{ab} + V_{bc} + V_{ca} = 0. \quad (31.14)$$

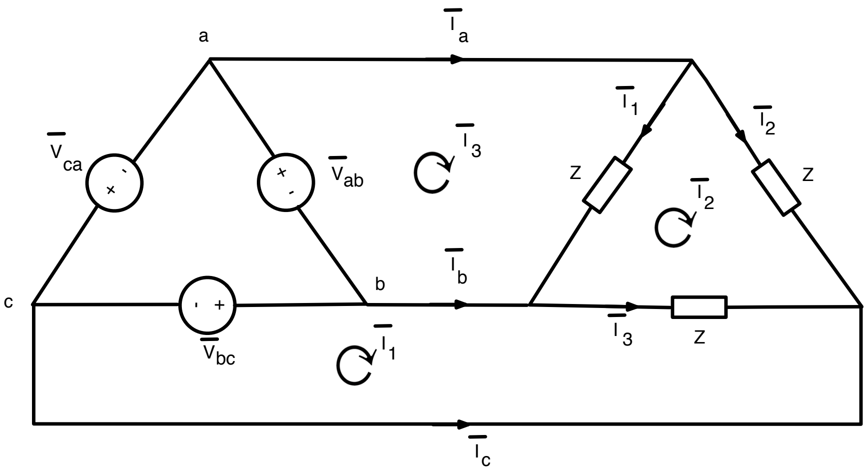

Figure 31.6 A three phase polysource in the Δ configuration connected to a polyphase balanced load also in the Δ configuration. Each component of the load is identical, and in general complex. (The figure is a reworking of Figure 12.17 in [1].)

Solving for the three mesh currents as indicated enables us to calculate any current in the system. Thus applying Kirchoff's loop law in the three meshes yields

$$-\overline{I}_2 Z + \overline{I}_3 Z - \overline{V}_1 = 0 \quad (31.15)$$

$$3Z\overline{I}_2 - Z\overline{I}_3 - \overline{I}_1 Z = 0 \quad (31.16)$$

$$Z\overline{I}_1 - Z\overline{I}_2 - \overline{V}_2 = 0. \quad (31.17)$$

We can write these equations into matrix form as

$$\begin{bmatrix} 0 & -Z & Z \\ -Z & 3Z & -Z \\ Z & -Z & 0 \end{bmatrix} \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} = \begin{bmatrix} \overline{V}_{ab} \\ 0 \\ \overline{V}_{bc} \end{bmatrix}. \quad (31.18)$$

It's solution yields the mesh currents, from which all currents can be computed.

As an example, consider computing $\overline{I}_a$ as indicated in Figure 31.6. It is clear that $\overline{I}_a = \overline{I}_3$. Consider the case where $\overline{V}_{ab} = 330\angle 0^\circ$ and $\overline{V}_{bc} = 330\angle - 120^\circ$. We select the impedance of the load phases Z as $Z = 2 - j15$ ohms. Solving the matrix equation above yields

$$\bar{I}_3 = 22.86\,\angle 6.9^{\circ}. \tag{31.19}$$

From knowledge of the mesh currents, the real power absorbed by the load can be computed. The MATLAB code for obtaining this result is given below.

```
clear all
Z = 20 -j*15; % single phase load
V1 = 330;
V2 = 330*exp(-j*120/180*pi);

A = [0  -Z  Z
     -Z 3*Z -Z
     Z  -Z  0];

 V = [V1 0 V2].';
 I = inv(A)*V;
 abs(I(3)) % magnitude of I_3
 angle(I(3))/pi*180 % angle of I_3 in degrees
```

31.4 Y and Δ Transformations

It is possible to transform a Y connected circuit to an equivalent Δ connected circuit (and vice versa) as shown in Figure 31.7. This is known as the Y to Δ (or Δ to Y) transformation.

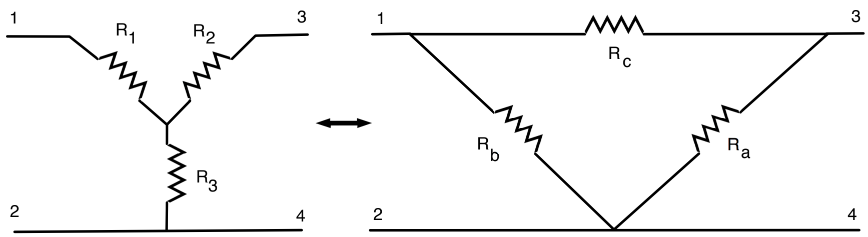

Figure 31.7 A Y connected load on the left, and a Δ connected load on the right.

31.4.1 Y to Δ Transformation

This transformation can be accomplished as follows:

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \quad (31.20)$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \quad (31.21)$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}. \quad (31.22)$$

31.4.2 Δ to Y Transformation

This transformation can be accomplished as follows:

$$R_1 = \frac{R_bR_c}{R_a + R_b + R_c} \quad (31.23)$$

$$R_2 = \frac{R_cR_a}{R_a + R_b + R_c} \quad (31.24)$$

$$R_3 = \frac{R_aR_b}{R_a + R_b + R_c}. \quad (31.25)$$

31.5 Comments

In this chapter we considered only balanced three phase systems, and then only the $Y - Y$ and $\Delta - \Delta$ configurations. It is possible to connect Y and Δ sources/loads, but this goes beyond the scope of this text.

The phasor domain and the Kirchhoff laws were applied to these systems. It was shown that circuit analysis yields the potentials, currents and power delivered to a load, as it did for the single phase systems we studied in previous chapters.

Reference

[1] Alexander, A. and M. Sadiku, *Fundamentals of Electric Circuits*, 5th Edition, McGraw-Hill.

Chapter 32

Concluding Remarks and Further Reading

The development of electrical engineering (EE) from its beginnings in the nineteenth century until the time of writing is arguably one of the most remarkable accomplishments of humankind. The technology made possible by EE is pervasive, and our daily lives make use of products based on EE such as personal computers, Wi-Fi, cell phones, the internet, and electrical power.

Many areas such as the generation and transmission of power, control theory, electronics, communications, and sensor technology are based on electric circuit theory. Circuit theory is thus an important part of the education of an engineer and a physicist.

Circuit theory is based on models of the real world, and the subsequent application of mathematics to produce practical results. It deals with electrical current and the electromotive force, phenomena that are observable via experimentation. The storage of energy, the delivery of power, and the conservation laws that must be satisfied in any circuit all contribute to this fascinating topic.

This book was written as an introduction to circuit theory, or as a primer for the professional worker who needs a refresher course. It is assumed that the reader is proficient in the application of matrix theory to the solution of simultaneous linear equations, complex number theory, algebra, and calculus. Also it is assumed that the reader has some knowledge and prior exposure to differential equations. Knowledge of electricity and electromagnetic fields are not assumed and the essential concepts are developed starting from basic experimental facts. Readers who need a more detailed and in-depth treatment of circuits after working through this text is referred to the books by Alexander et al. [1] and Dorf et al. [2]. The book by Nahvi et al. [3] contains many additional worked examples.

Because of the intended audience the decision was taken to present the material on the time domain, and on the phasor domain. No mention of the Laplace and Fourier transforms were made. The interested reader wishing to read further and learn these methods is referred to some excellent texts on this topic. For example the book by Papoulis [4] on the Fourier transformation is a classic and is a good option for the reader wanting to fully understand and comprehend the transform and its applications.

Papoulis also wrote a book on linear systems and circuit theory, where the Laplace and Z transform used for sampled (discrete) systems analysis are treated [5]. The book by Hsu [6] provides an introduction to the material, and it includes many worked examples. References [1] and [2] also introduce the Laplace transform. This book mentioned the applications of circuit theory to electronics. There are numerous texts available on the application of circuit theory to electronics. The book by Millman et al. [7] is comprehensive.

32.1 Digital Simulation and Design

The availability of powerful computers has changed the way circuits are designed and analyzed. Circuit simulators such as SPICE are mature and very powerful, and the book by Vladimirescu may be of interest to the reader [8]. The use of MATLAB (and Octave) to analyze circuits is pervasive and there are many excellent texts available to provide the reader with further material to study [9]. Many of these resources are online as well, and the world wide web contains interesting and informative material on this topic [10, 11].

32.2 History of EE

The history of EE (which includes the development of circuit theory) and the electronic computer may be of interest to the reader for further reading. The reader may find the following readings useful:

1. The book [12] is informative. The role played in the development of the programmable electronic computer by women and the mathematicians Alan Turing [13] and John Von Neumann [14] is noteworthy.

2. The role played by innovation and the market economy is insightful [15]. This book also gives insight into the development of the integrated circuit.
3. One of the pioneers of electrical science is Oliver Heaviside, and the reader may find [16] informative.
4. Nikola Tesla was one of the pioneers of the electrical age. The reader is referred to his autobiography available in [17].

References

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JC Olivier received his PhD from the University of Pretoria, South Africa, in 1990. After graduation, he was with Daimler-Benz Aerospace, Bell Northern Research (BNR), and Nortel Networks, Ottawa, Canada. From 1999 until 2003 he was with Nokia Research Center in the United States as a principle scientist. He became a professor of electrical engineering at the University of Pretoria in 2003, and then joined the University of Tasmania in Hobart, Australia as professor of electrical engineering in 2011.

Index